

Fokker–Planck equations on homogeneous time scales: well-posedness, stability, and applications to stochastic processes

S. Sutrima*, S. Wibowo, S. Sutanto

Sebelas Maret University, Surakarta, Indonesia

(E-mail: sutrima@mipa.uns.ac.id, supriyadi_w@staff.uns.ac.id, sutanto@staff.uns.ac.id)

We study one-dimensional Fokker–Planck equations on homogeneous time scales with spatially dependent drift and diffusion coefficients. The model is formulated under reflective boundary conditions and is transformed, through the stationary density, into an abstract Cauchy problem governed by the backward Fokker–Planck operator in a weighted Hilbert space. Using semigroup methods on time scales, we prove the existence and uniqueness of mild solutions and characterize when such solutions are classical. The relation between the forward Fokker–Planck operator and the backward operator is clarified through a conjugacy with the stationary profile, which also preserves the spectral structure. Under the stated assumptions, the backward operator is positive, self-adjoint, and has a discrete spectrum. These properties yield well-posedness of the time-scale Fokker–Planck problem and asymptotic stability of solutions toward the normalized stationary density. The homogeneous time-scale setting covers both continuous and discrete dynamics in a unified framework and allows spectral representations of the resulting evolution. The applicability of the theory is illustrated by the Ornstein–Uhlenbeck process and the geometric Brownian motion, showing how classical stochastic models can be represented within the proposed time-scale formulation.

Keywords: Fokker–Planck equations, homogeneous time scales, reflective boundary conditions, abstract Cauchy problem, backward Fokker–Planck operator, asymptotic stability, Ornstein–Uhlenbeck process, geometric Brownian motion.

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Introduction

The Fokker–Planck equations (FPEs) are central in the study of stochastic dynamics, providing deterministic descriptions of probability density evolution for diffusion processes. Classically, FPEs arise from stochastic differential equations driven by Wiener processes and provide a fundamental framework for stochastic dynamics [1], with applications in control and spectral analysis [2,3], nonlinear dynamics [4], entropy production [5–7], plasma and kinetic models [8,9], biology [10], and economic and financial modeling [11,12]. For one-dimensional FPEs, Bluman [13] initiated the analysis of exact solutions. In general, the transition density $u(x, t)$ on a bounded interval with drift $f(x)$ and diffusion $D(x) > 0$ satisfies the forward form

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}[f(x)u] + \frac{\partial^2}{\partial x^2}\left[\frac{D(x)}{2}u\right], \quad (1)$$

and the backward form

$$\frac{\partial u}{\partial t} = -f(x)u_x + \frac{D(x)}{2}u_{xx}$$

[14]. Both are parabolic PDEs, whose well-posedness follows classical results [15] and which relate closely to advection–diffusion equations [11,16]. This makes it natural to reformulate them as abstract Cauchy problems and apply C_0 -semigroup methods [17,18].

*Corresponding author. E-mail: sutrima@mipa.uns.ac.id

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In the literature, FPEs are treated either in continuous time, yielding PDEs on $\mathbb{R}^+$, or in discrete time, leading to recurrence equations on $\mathbb{Z}^+$. A general framework that unifies both cases, and potentially hybrid dynamics, remains underdeveloped. Time-scale calculus, introduced by Hilger [19], provides precisely such a framework by allowing dynamical equations on arbitrary closed subsets of $\mathbb{R}$ that may combine continuous and discrete intervals. Time-scale methods have been developed extensively for dynamic equations and differential or difference equations [20–22], variational problems [23], and controllability of dynamic equations [24, 25]. However, their application to stochastic processes, and to the FPE in particular, remains largely unexplored.

The extension of C_0 -semigroup theory from $\mathbb{R}^+$ to time scales has been investigated in recent works [26–28]. In particular, Hamza and Oraby [26, 27] proposed semigroup formulations under strong assumptions. In analogy with classical semigroup theory for infinite-dimensional systems [29], abstract dynamic equations on time scales have been developed through operator-semigroup and Cauchy-problem frameworks [27, 28, 30], with related models including Gompertz-type and heat equations on time scales [31, 32]; stability and controllability aspects are treated in [24, 26, 33].

This paper develops a unified time-scale theory of FPEs, covering continuous, discrete, and mixed temporal domains. Under suitable assumptions on the drift and diffusion coefficients, we prove existence and uniqueness of mild solutions, establish stability, and show consistency with the classical PDE formulation when $\mathbb{T} = \mathbb{R}$ and with difference-type FPEs when $\mathbb{T} = \mathbb{Z}$ or $h\mathbb{Z}$. The framework also reveals intermediate dynamics on hybrid grids. Its applicability is illustrated through the Ornstein–Uhlenbeck process (OUP) and the geometric Brownian motion (GBM), for which explicit or semi-explicit time-scale FPE solutions and numerical experiments show how nonuniform temporal domains modify distributional evolution, with implications for financial modeling, diffusion approximation, and systems subject to nonuniform interventions.

1 Preliminaries

The calculus on time scales unifies continuous and discrete analysis by allowing dynamic equations on closed subsets $\mathbb{T} \subset \mathbb{R}$. For $t \in \mathbb{T}$, the forward and backward jump operators are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The graininess function $\mu(t) := \sigma(t) - t$ measures the local step size. A time scale is called homogeneous when $\mu(t)$ is constant. Throughout this paper, we consider homogeneous time scales with constant graininess μ and assume $0 \in \mathbb{T}$. This setting provides a tractable framework for studying the associated Fokker–Planck operator and its C_0 -semigroup.

For $a, b \in \mathbb{T}$, let $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$. If X is a Banach space, a function $f : \mathbb{T} \rightarrow X$ is called rd-continuous if it is continuous at right-dense points and has finite left-sided limits at left-dense points. The space $C_{\text{rd}}([a, b]_{\mathbb{T}}, X)$, equipped with the supremum norm, is a Banach space [20–22].

For $f : \mathbb{T} \rightarrow X$, write $f^\sigma = f \circ \sigma$. The Δ -derivative of f at $t \in \mathbb{T}^\kappa$ is the element $f^\Delta(t) \in X$, if it exists, such that

$$\|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)\| \leq \varepsilon|\sigma(t) - s|$$

for all s sufficiently close to t . Equivalently [21, Theorem 1.16],

$$f^\Delta(t) = \begin{cases} \frac{f^\sigma(t) - f(t)}{\sigma(t) - t}, & \sigma(t) > t, \\ \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}, & \sigma(t) = t. \end{cases}$$

Basic rules mirror classical calculus:

$$(f + g)^\Delta = f^\Delta + g^\Delta, \quad (\alpha f)^\Delta = \alpha f^\Delta, \quad (fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = f g^\Delta + f^\Delta g^\sigma.$$

The Laplace transform $\mathcal{L}$ on time scales [20, 21] extends the classical case. For $f : \mathbb{T} \rightarrow \mathbb{R}$ rd-continuous,

$$\mathcal{L}\{f\}(z) := \int_0^\infty f(t) e_{-z}(\sigma(t), 0) \Delta t,$$

with $e_z(t, s)$ is the generalized exponential. These standard properties remain valid on $\mathbb{T}$.

Similarly to the classical case, if an operator A generates a C_0 -semigroup T on the time scale $\mathbb{T}$, then T solves the abstract Cauchy problem

$$u^\Delta(t) = Au(t), \quad t \in \mathbb{T}, \quad u(0) = u_0,$$

with solution $u(t) = T(t)u_0$ [24]. The Laplace transform connects to the resolvent

$$\mathcal{L}\{T(\cdot)u_0\}(z) = (zI - A)^{-1}u_0,$$

valid for $\Re(z)$ large. Analogues of the Hille-Yosida theorem provide generation criteria for semigroups on $\mathbb{T}$ [34]. These tools form the foundation for analyzing well-posedness of Fokker-Planck equations on time scales.

2 Fokker-Planck equations on time scales

In the classical Fokker-Planck framework, boundary conditions may be absorbing, reflective, or periodic. We focus on reflective conditions, as they ensure conservation of probability and fit naturally with the time-scale setting. The dynamic FPEs (1) can be generalized with initial data and reflective boundaries, with $x \in [a, b]$ and $t \in \mathbb{T}_0^+$:

$$\begin{aligned} u^\Delta(x, t) &= -\frac{\partial}{\partial x}[f(x)u] + \frac{\partial^2}{\partial x^2}\left[\frac{D(x)}{2}u\right], \quad x \in (a, b), \quad t \in \mathbb{T}_0^+ \setminus \{0\}, \\ J(u)(a) &= J(u)(b) = 0, \\ u(0, x) &= u_0(x), \quad x \in [a, b], \end{aligned} \tag{2}$$

where

$$J(u)(x) := f(x)u(x) - \frac{1}{2}\frac{d}{dx}(D(x)u(x)).$$

The well-posedness and long-time behavior of (2) will be studied via semigroup methods.

Problem (2) can be written as a time-scale Cauchy problem

$$\begin{aligned} u^\Delta(t) &= \mathcal{A}u(t), \quad t \geq 0, \\ u(0) &= u_0, \end{aligned}$$

where $\mathcal{A}$ is the Fokker-Planck operator defined by

$$\mathcal{A}u := -\frac{d}{dx}[f(x)u] + \frac{d^2}{dx^2}\left[\frac{D(x)}{2}u\right]$$

on a domain $\mathcal{D}(\mathcal{A}) := \{u \in C^2(a, b) : J(u)(a) = J(u)(b) = 0\}$.

To simplify boundary conditions, we instead use the backward Kolmogorov operator A , after introducing the stationary solution u_s of (2), obtained from

$$-\frac{d}{dx}\left(f(x)u_s(x) - \frac{1}{2}\frac{d}{dx}(D(x)u_s(x))\right) = 0.$$

Integrating and imposing $J(u_s) = 0$ gives

$$u_s(x) = N^{-1} \frac{1}{D(x)} \exp \left(2 \int_0^x \frac{f(\tau)}{D(\tau)} d\tau \right), \quad (3)$$

with N the normalization constant given by

$$N = \int_a^b \frac{1}{D(x)} \exp \left(2 \int_0^x \frac{f(\tau)}{D(\tau)} d\tau \right) dx.$$

Writing $u(x, t) = u_s(x)v(x, t)$ leads to

$$v^\Delta(t) = Av(t), \quad v(0) = v_0 = u_s^{-1}u_0, \quad (4)$$

where

$$Av := f(x) \frac{dv}{dx} + \frac{D(x)}{2} \frac{d^2v}{dx^2} \quad (5)$$

on the domain $\mathcal{D}(A) = \{v \in C^2(a, b) : v'(a) = v'(b) = 0\}$.

Existence and uniqueness. Let $X = L^2(a, b)$ with inner product

$$\langle v, w \rangle_{u_s} = \int_a^b v(x)w(x)u_s(x) dx.$$

A mild solution of (4) is defined as

$$v(t) = v_0 + A \int_0^t v(s) \Delta s, \quad t \in \mathbb{T}_0^+, \quad (6)$$

where $\int_0^t v(s) \Delta s \in \mathcal{D}(A)$. Under sufficient regularity assumptions, classical and mild solutions coincide [28], as stated in the following theorem.

Theorem 1. A mild solution v of (4) is a classical solution if and only if $v \in C_{\text{rd}}^1(\mathbb{T}_0^+, X)$.

The following auxiliary results, adapted from [35], describe the relationship between the operators $\mathcal{A}$ and A , and summarize several basic properties of A in the Hilbert space X .

Lemma 1. If $u(x) = v(x)u_s(x)$ for $v \in \mathcal{D}(A)$, then $\mathcal{A}u = u_s Av$ and both have common eigenvalues.

Here, we write $u(x)$ and $v(x)$ in place of $u(x, \cdot)$ and $v(x, \cdot)$, respectively, since the operator A acts only on the spatial variable $x \in [a, b]$ and is independent of the time variable t .

Lemma 2. Operator A has properties:

- (a) $-A$ is a positive operator on X .
- (b) The null space of A consists of constants.
- (c) A is self-adjoint with a discrete spectrum in the Hilbert space X .

By Lemma 2, A is a closed self-adjoint operator on X with eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$, whose associated eigenfunctions $\{\phi_n\}_{n \geq 0}$ form an orthonormal basis of X , with $\phi_0 = 1$. Hence,

$$Av = \sum_{n=0}^{\infty} -\lambda_n \langle v, \phi_n \rangle_{u_s} \phi_n, \quad x \in \mathcal{D}(A). \quad (7)$$

Moreover, operator A generates a C_0 -semigroup

$$T(t)v = \sum_{n=0}^{\infty} e_{-\lambda_n}^\sigma(t, 0) \langle v, \phi_n \rangle_{u_s} \phi_n, \quad v \in X. \quad (8)$$

On the other hand, if $\lambda_n \in \mathcal{R}$, by combining Theorem 3.3 in [28] with the remark following Theorem 2.3 in [24], we arrive at

$$|e_{-\lambda_n}^\sigma(t, 0)| \leq \frac{1}{1 + \lambda_n t}, \quad t \in \mathbb{T}_0^+. \quad (9)$$

Theorem 2. Let $T(t)$ be the C_0 -semigroup on X given in (8). Then:

- (a) $v(t) = T(t)v_0$ is a mild solution of (4) for all $v_0 \in X$;
- (b) $v(t) = T(t)v_0$ is a classical solution if and only if $v_0 \in \mathcal{D}(A)$.

Proof. (a) Let $w_n := \langle v_0, \phi_n \rangle_{u_s}$, $n \in \mathbb{N}$. We can easily verify that the scalar equation

$$v_n^\Delta(t) = -\lambda_n v_n(t), \quad v_n(0) = w_n, \quad t \in \mathbb{T}_0^+$$

has solution $v_n(t) = e_{-\lambda_n}^\sigma(t, 0)w_n$ satisfying, see (9),

$$|v_n(t)| \leq \frac{1}{1 + \lambda_n t} |w_n|. \quad (10)$$

Moreover, by virtue of the Fundamental Theorem of Calculus, we obtain

$$-\lambda_n \int_0^t v_n(\tau) \Delta\tau = v_n(t) - w_n, \quad t \in \mathbb{T}_0^+$$

which implies that

$$\left| \int_0^t v_n(\tau) \Delta\tau \right| \leq \frac{1}{\lambda_n} \left(1 + \frac{1}{1 + \lambda_n t} \right) |w_n|, \quad t \in \mathbb{T}_0^+. \quad (11)$$

We define

$$v(t) := \sum_{n=0}^{\infty} v_n(t) \phi_n, \quad t \in \mathbb{T}_0^+. \quad (12)$$

Equation (10), together with the convergence of series (8), ensures that this definition is well-defined. Moreover, the uniform convergence of the series (12) implies that

$$\int_0^t v(\tau) \Delta\tau = \sum_{n=0}^{\infty} \left(\int_0^t v_n(\tau) \Delta\tau \right) \phi_n, \quad t \in \mathbb{T}_0^+.$$

In addition, using (11), we obtain

$$\sum_{n=0}^{\infty} \lambda_n^2 \left| \int_0^t v_n(\tau) \Delta\tau \right|^2 \leq \sum_{n=0}^{\infty} \left(1 + \frac{1}{1 + \lambda_n t} \right)^2 |w_n|^2 \leq 4 \sum_{n=0}^{\infty} |w_n|^2 < \infty.$$

Therefore

$$\int_0^t v(\tau) \Delta\tau \in \mathcal{D}(A).$$

By (7) and the orthonormality of $\{\phi_n\}$, we have

$$A \int_0^t v(\tau) \Delta\tau = \sum_{n=0}^{\infty} -\lambda_n \left(\int_0^t v_n(\tau) \Delta\tau \right) \phi_n = \sum_{n=0}^{\infty} (v_n(t) - w_n) \phi_n = v(t) - v_0.$$

Thus, v is a mild solution of (4) and (12) yields $v(t) = T(t)v_0$ for $t \in \mathbb{T}_0^+$.

- (b) Let $v_0 \in \mathcal{D}(A)$. Since A is closed, the mild solution $v(t) = T(t)v_0$ in (6) can be expressed as

$$T(t)v_0 = v_0 + \int_0^t T(s)Av_0 \Delta s, \quad t \in \mathbb{T}_0^+.$$

Hence, it is clear that $T \in C_{\text{rd}}^1(\mathbb{T}_0^+, X)$, and the assertion follows from Theorem 1. Conversely, suppose that $v(t) = T(t)v_0$ is a classical solution. By definition, $v_0 = T(0)v_0 \in \mathcal{D}(A)$. $\square$

We see that if v_0 is the one referred to in (4), then w_n defined earlier in the proof to be $\langle u_0, \phi_n \rangle$, the usual inner product.

Theorem 3 (Stationary solution and stability). The Fokker–Planck problem (2) is well-posed. Moreover, for any u_0 , the solution satisfies

$$\lim_{t \rightarrow \infty} u(x, t) = w_0 u_s(x), \quad w_0 = \int_a^b u_0(x) dx. \quad (13)$$

Proof. For $u_0 \in X$ there is v_0 as in (4). For this v_0 , the function $v(t) = T(t)v_0$ is the solution of (4) giving

$$v(t) = \sum_{n=0}^{\infty} w_n e_{-\lambda_n}^{\sigma}(t, 0) \phi_n,$$

where $w_n = \langle u_0, \phi_n \rangle$. Lemma 1 concludes that the function

$$u(x, t) = w_0 u_s(x) + u_s(x) \sum_{n=1}^{\infty} w_n e_{-\lambda_n}^{\sigma}(t, 0) \phi_n(x), \quad t \in \mathbb{T}_0^+, \quad a \leq x \leq b,$$

is a unique solution of the Fokker–Planck problem (2) on the time scales $\mathbb{T}_0^+$. Lemma 2.18 of [28] gives $\lim_{t \rightarrow \infty} e_{-\lambda_n}^{\sigma}(t, 0) = 0$, and (13) follows. $\square$

Remark 1 (Mass conservation on time scales). Assume $u(\cdot, t) \in H^2(a, b)$ for each $t \in \mathbb{T}_0^+$. The regularity assumptions allows the interchange of the Δ -derivative in t with the spatial integral. Hence, integrating PDE (2) over $[a, b]$ and using the zero-flux boundary conditions yields

$$\frac{\Delta}{\Delta t} \int_a^b u(x, t) dx = \int_a^b u \Delta(x, t) dx = \int_a^b \mathcal{A}u(x, t) dx = J(u)(x, t) \Big|_{x=a}^{x=b} = 0.$$

Thus, the total mass is conserved on the time scale:

$$\int_a^b u(x, t) dx = \int_a^b u_0(x) dx =: w_0, \quad \forall t \in \mathbb{T}_0^+.$$

This identity shows that the total mass is conserved on the time scale, i.e.,

$$M(t) := \int_a^b u(x, t) dx, \quad t \in \mathbb{T}_0^+,$$

remains constant. In particular, the conservation holds both at right-dense and right-scattered points, ensuring that the total probability is preserved regardless of whether time evolves continuously, discretely, or in a hybrid manner.

Theorem 4 (Uniqueness of the stationary profile). Under the hypotheses of Theorem 3, any nontrivial stationary solution u_s is unique up to multiplication by a constant. In particular, if u_s is normalized so that $\int_a^b u_s(x) dx = 1$, then u_s is unique.

Proof. Stationary states solve the elliptic equation $\mathcal{A}u = 0$ with zero-flux boundary conditions. Under the hypotheses imposed on $f(x), D(x)$ (regularity and $D(x) > 0$ on $[a, b]$) the operator $-\mathcal{A}$ admits a discrete real spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lambda_n \rightarrow \infty,$$

see Lemma 1, and an orthonormal basis of eigenfunctions $\{\phi_n\}$ in the weighted Hilbert space (as used in Theorem 3). The eigenfunction corresponding to $\lambda_0 = 0$ is precisely the (nonnegative) steady profile u_s . Moreover, Lemma 2 provides that the kernel $\ker(\mathcal{A})$ is one-dimensional, hence any stationary solution is a scalar multiple of u_s . Normalizing the mass fixes the scalar and gives uniqueness. $\square$

Remark 2 (Rate of convergence). From the eigen-expansion used in Theorem 3,

$$u(\cdot, t) - w_0 u_s = u_s(\cdot) \sum_{n=1}^{\infty} w_n e_{-\lambda_n}^{\sigma}(t, 0) \phi_n(\cdot).$$

Moreover, since $\lambda_1 > 0$ and $|e_{-\lambda_n}^{\sigma}(t, 0)| \leq |e_{-\lambda_1}^{\sigma}(t, 0)|$ for all $n \in \mathbb{N}$, we can estimate the decay in terms of $e_{-\lambda_1}^{\sigma}(t, 0)$, so that there exists $C > 0$ with

$$\|u(\cdot, t) - w_0 u_s\|_X \leq C \sup_{n \geq 1} |w_n| e_{-\lambda_1}^{\sigma}(t, 0),$$

showing exponential-type decay governed by the spectral gap λ_1 .

3 Applications to stochastic processes

For Brownian motion in $\mathbb{R}$, there are several problems of the stochastic processes in one dimension corresponding to the FPEs that can be solved analytically. The Ornstein–Uhlenbeck process and geometric Brownian motion are two well-known examples, modeled as SDEs. In this section, we will discuss both processes when the Brownian motion is allowed to operate on the time scales.

3.1 The Ornstein–Uhlenbeck process

The Ornstein–Uhlenbeck process (OUP) is a fundamental stochastic model describing the velocity of a Brownian motion subject to friction. It is governed by the SDE:

$$dX_t = -\alpha X_t dt + \sqrt{2D} dW_t, \quad X(0) = x_0, \quad (14)$$

where $\alpha > 0$ denotes the drift (or relaxation) coefficient, and $D > 0$ represents the diffusion coefficient. The initial condition x_0 may be specified deterministically or randomly from a distribution independent of the Brownian motion W_t .

The OUP (14) can also be characterized through its associated probability density function $p(x, t)$, which describes the likelihood of the process occupying state x at time t . If t represents the time scale and $x \in \mathbb{R}$, OUP (14) is equivalently formulated in terms of the time-scale FPE

$$\begin{aligned} p^{\Delta}(x, t) &= \alpha \frac{\partial}{\partial x} [xp] + D \frac{\partial^2 p}{\partial x^2}, \quad x \in \mathbb{R}, \quad t \in \mathbb{T}_0^+ \setminus \{0\}, \\ \lim_{x \rightarrow -\infty} J(x, t) &= \lim_{x \rightarrow +\infty} J(x, t) = 0, \\ p(x, 0) &= p_0(x). \end{aligned} \quad (15)$$

In this case, we consider $f(x) = -\alpha x$ and $D(x) = 2D$, with the system constrained by reflective boundary conditions. By employing the eigenvalues and eigenfunctions of the generator of OUP, the time-dependent problem can be solved, leading to an explicit expression for the probability density function. It can be shown that the eigenfunctions of OUP are the Hermite polynomials. Referring to (3), we find the stationary solution of (15):

$$p_s(x) = \sqrt{\frac{\alpha}{2D\pi}} e^{-\frac{\alpha}{2D}x^2}.$$

The operator A in (5) has the form

$$Av = -\alpha x \frac{dv}{dx} + D \frac{d^2 v}{dx^2}$$

with $\mathcal{D}(A) = \{v \in C^2(\mathbb{R}) : \lim_{x \rightarrow \mp\infty} v'(x) = 0\}$. This operator is the generator of the OUP (14). Theorem 4.2 [1] implies that the operator $-A$ has the eigenvalues

$$\lambda_n = \alpha n, \quad n = 0, 1, 2, \dots$$

and the corresponding eigenfunctions are the normalized Hermite polynomials:

$$\phi_n(x) = \frac{1}{\sqrt{n!}} H_n \left(\sqrt{\frac{\alpha}{D}} x \right), \quad (16)$$

where

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} \left(e^{-\frac{x^2}{2}} \right).$$

We see that the orthonormality of the modified Hermite polynomials ϕ_n defined in (16) implies that

$$\int_{\mathbb{R}} \phi_n(x) \phi_m(x) p_s(x) dx = \delta_{nm}.$$

Therefore, the associated Cauchy problem (4) has the solution

$$v(x, t) = \sum_{n=0}^{\infty} w_n e^{\sigma_{-\lambda_n}(t, 0)} \phi_n(x), \quad w_n = \int_{\mathbb{R}} p_0(x) \phi_n(x) dx$$

for any initial distribution $p_0 \in L^2(\mathbb{R})_{p_s^{-1}}$. We note that w_n is equivalent to the one in the proof of Theorem 2. Finally, the solution of the problem (15) is given by

$$p(x, t) = p_s(x) \sum_{n=0}^{\infty} w_n e^{\sigma_{-\lambda_n}(t, 0)} \phi_n(x). \quad (17)$$

It is evident that, as the friction coefficient α goes to 0, the transition probability given in (17) converges to the transition probability of the Brownian motion. Moreover, we find that

$$\lim_{t \rightarrow \infty} p(x, t) = p_s(x), \quad t \in \mathbb{T}_0^+.$$

This corroborates the conclusion of Theorem 3. As anticipated, the OUP, being an ergodic Markov process with a Gaussian invariant distribution p_s , naturally leads to this outcome. The result closely mirrors the corresponding case for Brownian motion on $\mathbb{R}$.

The analysis reveals that the relaxation rate of the OUP is governed by the first nonzero eigenvalue $\lambda_1 = \alpha$, implying a universal convergence time scale $1/\alpha$, which represents the characteristic time required for the solution to approach equilibrium. In particular, the dominant decay mode behaves as $e^{-\alpha t}$, so that larger values of α produce faster relaxation. This convergence scale is independent of the underlying time domain. The effect of the time scale enters only through the decay factors $e^{\sigma_{-\lambda_n}(t, 0)}$, which are exponential in the continuous case, polynomial-like in the discrete case, and intermediate for hybrid domains. Consequently, finer graininess μ accelerates convergence while coarser μ slows it down. Nevertheless, the invariant distribution p_s remains Gaussian for all time scales, demonstrating that the ergodicity of the OUP is robust under continuous, discrete, and hybrid temporal structures.

We present simple illustrations of the dynamics of p on several representative time scales, namely $\mathbb{R}^+$, $\mathbb{Z}^+$ and $h\mathbb{Z}^+$. The corresponding C_0 -semigroups on these time scales are:

$$T(t)v = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} \phi_n, \quad t \in \mathbb{R}^+, \quad (18)$$

$$T(t)v = \sum_{n=0}^{\infty} c_n \frac{1}{(1 + \lambda_n)^{t+1}} \phi_n, \quad t \in \mathbb{Z}^+, \quad (19)$$

$$T(t)v = \sum_{n=0}^{\infty} c_n \frac{1}{(1 + \lambda_n h)^{t/h+1}} \phi_n, \quad t \in h\mathbb{Z}^+, \quad (20)$$

where $c_n := \langle v, \phi_n \rangle_{u_s}$ for $v \in L^2(\mathbb{R})$.

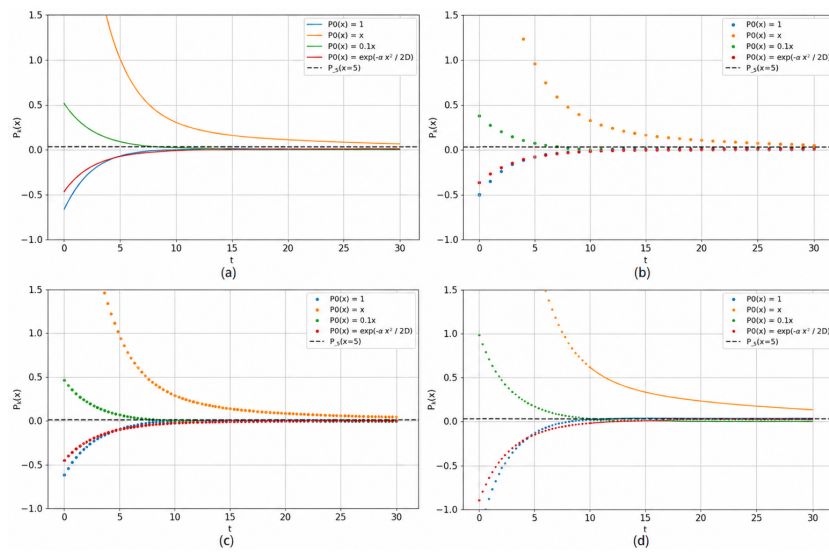


Figure 1. Dynamics of p for $\alpha = 0.05$, $D = 0.6$, $h = 0.2$ and $x = 5$ on: (a) $\mathbb{T} = \mathbb{R}$; (b) $\mathbb{T} = \mathbb{Z}^+$; (c) $\mathbb{T} = h\mathbb{Z}^+$; (d) $\mathbb{T} = h\mathbb{Z}^+ \cup \mathbb{R}^+$

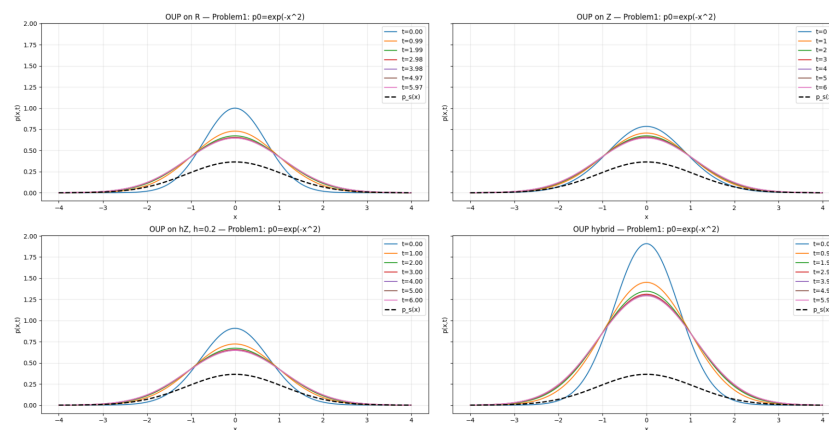


Figure 2. Dynamical behavior of p relative to p_s , under the initial distribution $p_0(x) = e^{-x^2}$

Figures 1–2 illustrate the implications of Theorem 3. Figure 1 shows the behavior on selected time scales $\mathbb{T}_0^+$ with respect to t for fixed x , while Figure 2 displays the spatial variation in x over its prescribed interval for fixed t . In particular, Figure 2 demonstrates that the OUP rapidly converges

to the Gaussian invariant distribution for large $|x|$, whereas the convergence is significantly slower in a small neighborhood of 0.

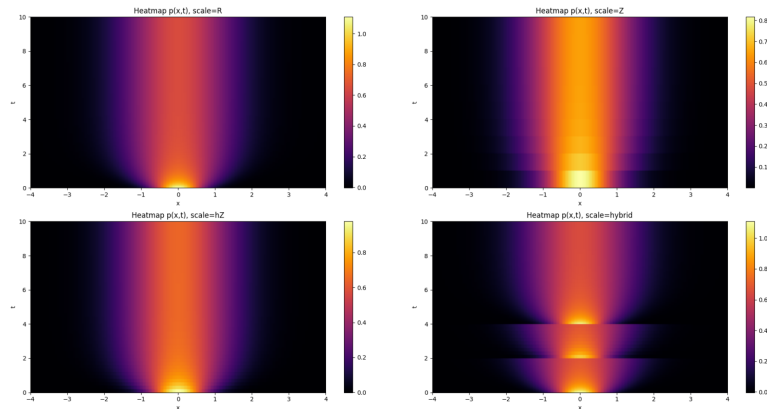


Figure 3. Heatmaps show how time scales shape convergence: smooth on $\mathbb{R}^+$, stepwise on $\mathbb{Z}^+$, nearly continuous on $h\mathbb{Z}^+$, and mixed freezing–stepwise behavior on the hybrid scale

Figure 3 shows that the bright region near $x = 0$ at $t = 0$ corresponds to the initial profile $p_0(x) = e^{-x^2}$. As time evolves, the density relaxes according to the underlying time scale toward the stationary Gaussian distribution $p_s(x)$. The symmetric contraction of the heatmap reflects the mean-reverting nature of the OUP, which progressively drives the probability mass toward equilibrium.

3.2 Geometric Brownian motion

Geometric Brownian Motion (GBM) is a well-known stochastic process, particularly in mathematical finance, for describing the dynamics of asset prices such as stocks. It is a continuous-time model in which the logarithm of the asset price evolves as a Brownian motion with a constant drift. This formulation captures both the inherent randomness of market movements and their tendency to fluctuate around an underlying trend. The model is governed by two key parameters: the drift α , representing the average rate of return, and the volatility β , representing the magnitude of fluctuations. Mathematically, GBM arises as the solution to a scalar linear SDE with multiplicative noise

$$dX_t = \alpha X_t dt + \beta X_t dW_t, \quad X(0) = x_0. \quad (21)$$

This represents a basic framework for describing population dynamics under environmental fluctuations.

In an alternative formulation, when we take $f(x) = \alpha x$ and $D(x) = \frac{\beta^2 x^2}{2}$ confined to the interval $[1, b]$ with decay at the boundaries, the process (21) simplifies to the FPE subject to absorbing boundary conditions. Furthermore, if the time is allowed to evolve over the time scale $\mathbb{T}_0^+$ rather than being purely continuous, this framework naturally extends to the FPE on the time scale $\mathbb{T}_0^+$ with absorbing boundary conditions

$$\begin{aligned} u^\Delta(x, t) &= -\frac{\partial}{\partial x} [\alpha x u] + \frac{\partial^2}{\partial x^2} \left[\frac{\beta^2 x^2}{2} u \right], \quad x \in [1, b], \quad t \in \mathbb{T}_0^+ \setminus \{0\}, \\ u(1, t) &= u(b, t) = 0, \quad t \in \mathbb{T}_0^+, \\ u(x, 0) &= u_0(x), \quad x \in [1, b]. \end{aligned} \quad (22)$$

The generator of this process is

$$\mathcal{A}u = -\frac{d}{dx} [\alpha x u] + \frac{d^2}{dx^2} \left[\frac{\beta^2 x^2}{2} u \right]$$

the linear operator defined on the domain

$$\mathcal{D}(\mathcal{A}) = \{u \in L^2(1, b) : u, u' \text{ are absolute continuous, } u'' \in L^2(1, b), u(1) = u(b) = 0\}.$$

We see that this operator is not uniformly elliptic due to the diffusion coefficient vanishes at $x = 0$. This is the reason why we consider the domain to be the interval $[1, b]$.

By applying the Dirichlet condition, we can easily find out the stationary solution of the forward FPE (22):

$$u_s(x) = 0, \quad 1 \leq x \leq b.$$

By reducing the eigenproblem $\mathcal{A}u = -\lambda u$ to a Cauchy-Euler equation, we obtain the eigenvalues and the corresponding normalized eigenfunctions of $\mathcal{A}$ as

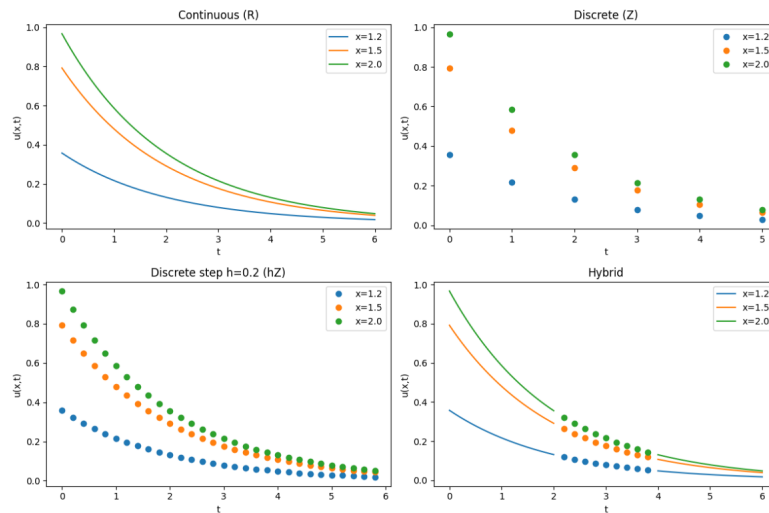


Figure 4. Dynamics of u with parameters $\alpha = 0.3$, $\beta = 0.6$, step size $h = 0.2$, for $u_0(x) = \sin\left(\frac{\pi(x-1)}{e-1}\right)$ on time scales: (a) $\mathbb{T} = \mathbb{R}^+$; (b) $\mathbb{T} = \mathbb{Z}^+$; (c) $\mathbb{T} = h\mathbb{Z}^+$; and (d) $\mathbb{T} = [0, 2] \cup (h\mathbb{Z}^+ \cap [2, 4]) \cup [4, 6]$

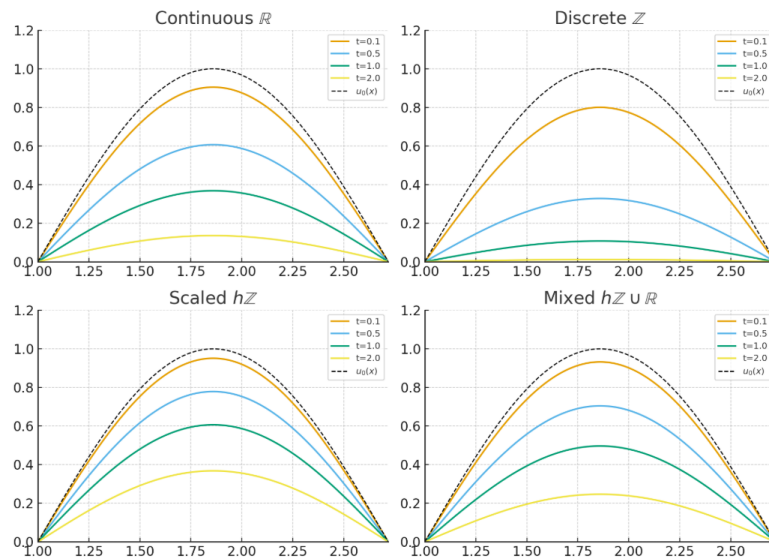


Figure 5. Dynamics of u for $\alpha = 0.3$ and $\beta = 0.6$, starting from $u_0(x) = \sin\left(\frac{\pi(x-1)}{e-1}\right)$, imposed with absorbing boundary conditions

$$\lambda_n = \frac{\beta^2}{2} \left[\left(\frac{n\pi}{\ln b} \right)^2 + \left(\frac{\alpha}{\beta^2} - \frac{1}{2} \right)^2 \right],$$

$$\phi_n(x) = \sqrt{\frac{2}{\ln b}} x^{r-1} \sin \left(\frac{n\pi}{\ln b} \ln x \right), \quad r = \frac{\alpha}{\beta^2} - \frac{1}{2}, \quad n \in \mathbb{N},$$

where the eigenfunctions are orthonormal with respect to the weight function $\rho(x) = x^{1-2r}$. Therefore, for an arbitrary initial condition $u_0 \in L^2(1, b)$, the mild solution of (22) is given by

$$u(x, t) = \sum_{n=1}^{\infty} w_n e^{-\lambda_n t} \phi_n(x), \quad (23)$$

where $w_n = \int_1^b u_0(x) \phi_n(x) \rho(x) dx$. It is evident that the presence of noise in equation (21) can fundamentally change the qualitative behavior of the solution. In particular, if $\alpha > 0$, the solution to the deterministic equation ($\beta = 0$) exhibits exponential growth as $t \rightarrow \infty$. In contrast, for the stochastic formulation, equation (23) demonstrates that the solution converges to 0 with probability one. This phenomenon is consistent with the fact that all Brownian particles are eventually absorbed at the boundary.

Figures 4–5 illustrate that under absorbing boundary conditions, the geometric Brownian motion $u(x, t)$ converges to the stationary state $u_s \equiv 0$ on all considered time scales. The convergence is smooth on the continuous time scale $\mathbb{R}^+$, stepwise on the discrete scales $\mathbb{Z}^+$ or $h\mathbb{Z}^+$, and exhibits intermediate behavior on the hybrid scale $\mathbb{T} = h\mathbb{Z}^+ \cup \mathbb{R}^+$. The eigenfunction expansion (23) shows that higher modes decay faster, so the long-time dynamics are dominated by the slowest-decaying mode, providing an estimate of the convergence rate. Moreover, the C_0 -semigroup simulations reveal that transient dynamics strongly depend on the time-scale structure, even though the ultimate absorption at the boundaries is time-scale independent. These findings highlight the influence of continuous, discrete, and hybrid temporal representations on the evolution of stochastic processes with multiplicative noise.

Referring to the semigroups in (18), (19), and (20), and replacing u_s with ρ , we perform simulations of the geometric Brownian motion on the time scales $\mathbb{R}^+$, $\mathbb{Z}^+$, and $h\mathbb{Z}^+$ for $x \in [1, e]$.

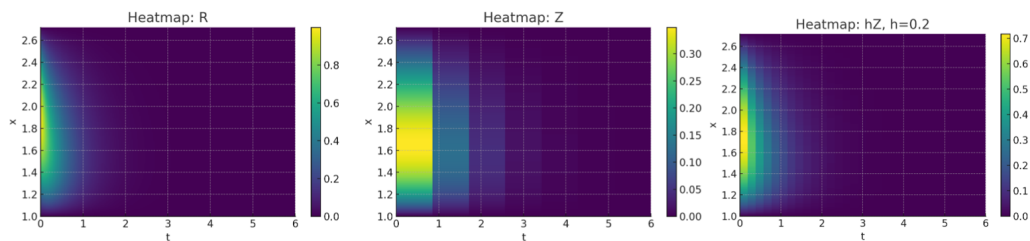


Figure 6. Heatmaps of solution decay to zero across the time scales $\mathbb{R}^+$, $\mathbb{Z}^+$, and $h\mathbb{Z}^+$, consistent with the asymptotic behavior of GBM with absorbing boundary conditions. The temporal evolution appears smooth in the continuous case and stepwise in the discrete cases

Figure 6 illustrates the temporal decay of the GBM solution on different time scales. In all cases, the solution converges toward zero as time increases, reflecting the dissipative nature of the system under absorbing boundary conditions. For the continuous time scale $\mathbb{R}^+$, the decay is smooth and exponential-like, corresponding to the classical continuous semigroup evolution. On the discrete time scales $\mathbb{Z}^+$ and $h\mathbb{Z}^+$, the evolution exhibits a stepwise structure induced by the discrete temporal dynamics. Moreover, the finer graininess in the $h\mathbb{Z}^+$ case produces a smoother transition and faster relaxation compared with the standard discrete case. These observations confirm that the qualitative asymptotic behavior of the GBM remains stable across different time-scale structures, while the temporal regularity and convergence profile are influenced by the underlying graininess.

Conclusion

We have shown the well-posedness of the one-dimensional Fokker–Planck equations on homogeneous time scales with reflective boundary conditions by employing the backward Fokker–Planck operator framework. The analysis ensures existence, uniqueness, and asymptotic stability of solutions, converging to a unique stationary distribution as $t \rightarrow \infty$. This unified time-scale formulation naturally incorporates continuous, discrete, and hybrid dynamics, as illustrated through the Ornstein–Uhlenbeck process and the geometric Brownian motion, and thus provides a versatile foundation for further studies of stochastic processes.

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Author contributions

S. Sutrima developed the theoretical framework and main proofs. S. Wibowo contributed to the stability analysis, asymptotic behavior, and OUP application. S. Sutanto worked on the GBM example, numerical simulations, and presentation of results. All authors discussed the results, revised the manuscript, and approved its submission.

Conflict of interest

The authors declare that there are no conflicts of interest related to this work. The funding sponsor had no role in the design of the study, analysis, interpretation of results, or in the decision to publish the manuscript.

References

- 1 Pavliotis, G.A. (2014). *Stochastic processes and applications*. New York: Springer. <https://doi.org/10.1007/978-1-4939-1323-7>
- 2 Breiten, T., Kunisch, K., & Pfeiffer, K. (2018). Control strategies for the Fokker–Planck equation. *ESAIM: Control, Optimisation and Calculus of Variations*, 24(2), 741–763. <https://doi.org/10.1051/cocv/2017046>
- 3 Liberzon, D., & Brockett, R.W. (2000). Spectral analysis of Fokker–Planck and related operators arising from linear stochastic differential equations. *SIAM Journal on Control and Optimization*, 38(5), 1453–1467. <https://doi.org/10.1137/S0363012998338193>
- 4 Heninger, J.M., Lippolis, D., & Cvitanović, P. (2018). Perturbation theory for the Fokker–Planck operator in chaos. *Communications in Nonlinear Science and Numerical Simulation*, 55, 16–28. <https://doi.org/10.1016/j.cnsns.2017.06.025>
- 5 Chandrasegaran, J., & Sharma, P. (2020). A short review on Fokker–Planck equations, entropy production and entropy generation. *Bioinformatics Proteomics and Imaging Analysis*, 4(1), 21–24.
- 6 Lucia, U. (2014). Entropy generation and the Fokker–Planck equation. *Physica A: Statistical Mechanics and Its Applications*, 393, 256–260. <https://doi.org/10.1016/j.physa.2013.09.028>

- 7 Tomé, T. (2006). Entropy production in nonequilibrium systems described by a Fokker–Planck equation. *Brazilian Journal of Physics*, 36(4A), 1285–1289. <https://doi.org/10.1590/S0103-97332006000700029>
- 8 Flandoli, F., Leocata, M., & Ricci, C. (2021). The Navier–Stokes–Vlasov–Fokker–Planck system as a scaling limit of particles in a fluid. *Journal of Mathematical Fluid Mechanics*, 23, Article 40. <https://doi.org/10.1007/s00021-021-00570-6>
- 9 Peeters, A.G., & Strintzi, D. (2008). The Fokker–Planck equation and its application in plasma physics. *Annalen der Physik*, 17(2–3), 142–157.
- 10 Philipp, L., & Shizgal, B.D. (2019). A pseudospectral solution of a bistable Fokker–Planck equation that models protein folding. *Physica A: Statistical Mechanics and Its Applications*, 522, 158–166. <https://doi.org/10.1016/j.physa.2019.01.146>
- 11 Chilarescu, C., & Vaneecloo, N. (2007). A stochastic approach to the Cobb–Douglas production function. *Economics Bulletin*, 3(9), 1–9.
- 12 Brigo, D., & Mercurio, F. (2000). Option pricing impact of alternative continuous-time dynamics for discretely observed stock prices. *Finance and Stochastics*, 4(2), 147–159. <https://doi.org/10.1007/s007800050009>
- 13 Bluman, G.W. (1971). Similarity solutions of the one-dimensional Fokker–Planck equation. *International Journal of Non-Linear Mechanics*, 6(2), 143–153. [https://doi.org/10.1016/0020-7462\(71\)90051-5](https://doi.org/10.1016/0020-7462(71)90051-5)
- 14 Risken, H. (1989). *The Fokker–Planck equation: Method of solution and applications*. Springer. <https://doi.org/10.1007/978-3-642-61544-3>
- 15 Friedman, A. (1983). *Partial Differential Equations of Parabolic Type*. Malabar, FL: Robert E. Krieger Publishing Company.
- 16 Largier, J.L. (2003). Considerations in estimating larval dispersal distances from oceanographic data. *Ecological Applications*, 13(1), 71–89. [https://doi.org/10.1890/1051-0761\(2003\)013\[0071:CIELDD\]2.0.CO;2](https://doi.org/10.1890/1051-0761(2003)013[0071:CIELDD]2.0.CO;2)
- 17 Sutrima, S., Mardiyana, M., Setiyowati, R., & Respatiwan, R. (2021). A new approach on the well-posedness of nonautonomous Cauchy problems: An application in population growth. *Journal of Nonlinear Functional Analysis*, 2021, Article 29.
- 18 Sutrima, S., Wibowo, S., & Setiyowati, R. (2024). Well-posedness of boundary control system of nonlinear chemical reaction. *Nonlinear Dynamics and Systems Theory*, 24(6), 648–662.
- 19 Hilger, S. (1988). *Ein Masskettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten [A measure chain calculus with application to center manifolds]* (dissertation). Universität Würzburg [in German].
- 20 Bohner, M., & Peterson, A. (2001). *Dynamic equations on time scales: An introduction with applications*. Birkhäuser. <https://doi.org/10.1007/978-1-4612-0201-1>
- 21 Bohner, M., & Peterson, A. (2003). *Advances in dynamic equations on time scales*. Birkhäuser. <https://doi.org/10.1007/978-0-8176-8230-9>
- 22 Bohner, M., & Peterson, A. (2001). First and second order linear dynamic equations on time scales. *Journal of Difference Equations and Applications*, 7(6), 767–792. <https://doi.org/10.1080/10236190108808302>
- 23 Bohner, M. (2004). Calculus of variations on time scales. *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, 13, 339–349.
- 24 Bohner, M., Duque, C., Leiva, H., & Sivoli, Z. (2024). A lemma on C_0 -semigroups on the time scales and approximate controllability of the heat dynamic equation. *Quaestiones Mathematicae*, 47(9), 1807–1826. <https://doi.org/10.2989/16073606.2024.2345845>

- 25 Bohner, M., Duque, C., & Leiva, H. (2022). Controllability of dynamic equations with memory. *Nonlinear Dynamics and Systems Theory*, 22(5), 489–502.
- 26 Hamza, A.E., & Oraby, K.M. (2012). Stability of abstract dynamic equations on time scales. *Advances in Difference Equations*, 2012, Article 143. <https://doi.org/10.1186/1687-1847-2012-143>
- 27 Hamza, A.E., & Oraby, K.M. (2015). Semigroups of operators and abstract dynamic equations on time scales. *Applied Mathematics and Computation*, 270, 334–348. <https://doi.org/10.1016/j.amc.2015.07.110>
- 28 Henríquez, H.R., Lizama, C., & Mesquita, J.G. (2020). Semigroups on time scales and applications to abstract Cauchy problems. *Topological Methods in Nonlinear Analysis*, 56(1), 83–115. <https://doi.org/10.12775/TMNA.2019.114>
- 29 Curtain, R., & Zwart, H. (1995). *An introduction to infinite-dimensional linear system theory*. Springer. <https://doi.org/10.1007/978-1-4612-4224-6>
- 30 Montenegro, S., Moya, A.T., & Leiva, H. (2024). Quasi-semigroups of operators on homogeneous time scales. *Nonautonomous Dynamical Systems*, 11(1), Article 20240005. <https://doi.org/10.1515/msds-2024-0005>
- 31 Cuchta, T., & Wintz, N. (2022). Periodic functions related to the Gompertz difference equation. *Mathematical Biosciences and Engineering*, 19(9), 8774–8785. <https://doi.org/10.3934/mbe.2022407>
- 32 Cuchta, T., & Ferreira, R.A.C. (2023). The heat equation on time scales. *Opuscula Mathematica*, 43(4), 475–491. <https://doi.org/10.7494/OpMath.2023.43.4.475>
- 33 Pötzsche, C., Siegmund, S., & Wirth, F. (2003). A spectral characterization of exponential stability for linear time-invariant systems on time scales. *Discrete and Continuous Dynamical Systems*, 9(5), 1223–1241. <https://doi.org/10.3934/dcds.2003.9.1223>
- 34 Gibson, A.G. (1972). A discrete Hille–Yosida–Phillips theorem. *Journal of Mathematical Analysis and Applications*, 39, 761–770. [https://doi.org/10.1016/0022-247X\(72\)90196-5](https://doi.org/10.1016/0022-247X(72)90196-5)
- 35 Sutrima, S., Siswanto, S., Wibowo, S., & Indrati, C.R. (2025). Fokker–Planck equation and its application in production function. *Nonlinear Dynamics and Systems Theory*, 25(6), 691–704.

*Author information**

Sutrima Sutrima (*corresponding author*) — Doctor of Mathematics, Professor, Lecturer at the Department of Mathematics, Sebelas Maret University, Ir. Sutami 36 A, Surakarta 57126, Indonesia; e-mail: sutrima@mipa.uns.ac.id; <https://orcid.org/0000-0001-9668-5670>

Supriyadi Wibowo — Doctor of Physics and Mathematics, Associate Professor, Lecturer at the Department of Mathematics, Sebelas Maret University, Ir. Sutami 36 A, Surakarta 57126, Indonesia; e-mail: supriyadi_w@staff.uns.ac.id; <https://orcid.org/0000-0001-8131-7373>

Sutanto Sutanto — Doctor of Mathematics, Associate Professor, Lecturer at the Department of Mathematics, Sebelas Maret University, Ir. Sutami 36 A, Surakarta 57126, Indonesia; e-mail: sutanto@staff.uns.ac.id; <https://orcid.org/0000-0002-8072-1216>

*Authors' names are presented in the following order: first name, middle name (if any), last name.