

Asymptotics of solutions of a reaction-diffusion system with nonlinear boundary conditions

Z.R. Rakhmonov¹, A.R. Zaripova^{2,*}

¹National University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan;

²Karshi State University, Karshi, Uzbekistan

(E-mail: zraxmonov@inbox.ru, azizazaripova1990@mail.ru)

This paper investigates the asymptotic properties of a system of parabolic equations with nonlinear boundary conditions. Systems of parabolic equations describe the behavior of various processes over time, such as heat propagation and material diffusion. Nonlinear boundary conditions involve complex properties that differ from the usual local conditions at the boundaries of the system, which influence the solutions of the system. In this article, a system of parabolic differential equations associated with nonlinear boundary conditions is reduced to a system of self-similar equations by means of certain transformations using the self-similar approach, and the asymptotic behavior of solutions with compact support is found. In addition, the reaction-diffusion system is approximated using implicit difference schemes and an iterative process for the numerical solution is constructed. It is known that the main problem for the iterative process is the choice of the best initial approximation. The article proposes to use the constructed asymptotic formula to solve this problem. Computational experiments were carried out with different values of numerical parameters and the results are presented graphically. The numerical results obtained in the case of slow diffusion $p > 2$ of a nonlinear reaction-diffusion system show that the reaction-diffusion process proceeds at a finite rate.

Keywords: system of nonlinear equations, reaction-diffusion, global solution, asymptotic, compact support, self-similar solution, sweep method, numerical solution.

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Introduction

In this paper, we investigate the asymptotics of self-similar solutions for the following system of perturbed parabolic-type equations in the domain $Q = \{(x, t) \mid x \in \mathbb{R}_+, 0 < t < T\}$

$$\partial_t u_i = \left(|\partial_x u_i|^{p_i-2} \partial_x u_i \right)_x, \quad i = 1, 2, \dots, k, \quad x > 0, \quad 0 < t < T, \quad (1)$$

with nonlinear boundary conditions

$$- |\partial_x u_i|^{p_i-2} \partial_x u_i(0, t) = u_{i+1}^{q_i}(0, t), \quad i = 1, 2, \dots, k, \quad u_{k+1} = u_1, \quad 0 < t < T, \quad (2)$$

and initial conditions

$$u_i(x, 0) = u_{i0}(x), \quad i = 1, 2, \dots, k, \quad x > 0. \quad (3)$$

The problem (1)–(3) is considered under the conditions $p_i > 2$, $q_i > 0$, where p_i is associated with the governing equation (1), and q_i arises in the nonlinear boundary condition (2). The initial functions $u_{i0}(x)$ ($i = 1, 2, \dots, k$) are nonnegative continuous functions with compact support in $\mathbb{R}_+$.

*Corresponding author. E-mail: azizazaripova1990@mail.ru

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The system of nonlinear parabolic equations (1) is widely used in the mathematical modeling of processes such as turbulent diffusion, population dynamics, chemical reactions, and heat transfer [1, 2]. Furthermore, system (1)–(3) describes many other physical processes [3–5].

Throughout the paper, we write u_i where the index i is assumed to range over $\{1, \dots, k\}$ unless otherwise specified. It is known that in the region where $u_i = 0$, the system of equations (1) is degenerate and does not have a solution in the classical sense. In such cases, we consider generalized solutions that satisfy the property $|\partial_x u_i|^{p_i-2} \partial_x u_i \in C(Q)$, $0 \leq u_i \in C(Q)$ and these solutions satisfy the system of equations (1) in the sense of an integral identity [1, 2, 6].

Unbounded (blow-up) and global solutions of nonlinear parabolic systems of equations, as well as the conditions for their existence, have been widely studied in [7–9].

In [10], the critical global existence curve was established for fast diffusive polytropic filtration equations with nonlinear boundary fluxes, and a Fujita-type critical curve was conjectured.

S.N. Zheng, X.F. Song, and Z.X. Jiang [11] considered the following problem with a nonlinear boundary conditions:

$$\begin{aligned} \partial_t u &= \partial_x (\partial_x u^m), \quad \partial_t \nu = \partial_x (\partial_x \nu^n), \quad x > 0, \quad 0 < t < T, \\ -\partial_x u^m(0, t) &= u^\alpha(0, t) \nu^\beta(0, t), \quad -\partial_x \nu^n(0, t) = u^\theta(0, t) \nu^\phi(0, t), \quad 0 < t < T, \\ u(x, 0) &= u_0(x), \quad \nu(x, 0) = \nu_0(x), \quad x > 0. \end{aligned} \quad (4)$$

It was proved that the solutions of problem (4) are global if the conditions $\alpha < \frac{1}{2}(n+1)$, $\beta < \frac{1}{2}(n+1)$, $pq \leq (\frac{1}{2}(m+1) - \alpha)(\frac{1}{2}(n+1) - \beta)$ are satisfied.

If, however $\alpha > \frac{1}{2}(m+1)$ or $\beta > \frac{1}{2}(n+1)$, then the solutions become unlimited (blow up) in finite time.

In the case when $\alpha > \frac{1}{2}(m+1)$, $\beta > \frac{1}{2}(n+1)$, $pq > (\frac{1}{2}(m+1) - \alpha)(\frac{1}{2}(n+1) - \beta)$ if at least one of the conditions $l_1 < k_1$, $l_2 < k_2$ or $l_1 = k_1$, $l_2 = k_2$ is being carried out, then any nonnegative nontrivial solution blows up in finite time.

On the other hand, $l_1 < k_1$, $l_1 > k_1$ it has been shown that for sufficiently large initial data the solutions blow up, whereas for small initial data the solutions remain global.

$$\begin{aligned} k_1 &= \frac{2p + n + 1 - 2\beta}{4pq - (n + 1 - 2\alpha)(m + 1 - 2\beta)}, \quad l_1 = \frac{1 - k_1(m - 1)}{2}, \\ k_2 &= \frac{2p + m + 1 - 2\beta}{4pq - (n + 1 - 2\alpha)(m + 1 - 2\beta)}, \quad l_2 = \frac{1 - k_2(n - 1)}{2}. \end{aligned}$$

Z.Y. Xiang, Q. Chen, and C.L. Mu [12] investigated the following system:

$$\begin{cases} \partial_t u_i = \partial_x (|\partial_x u_i|^{m_i-1} \partial_x u_i), \quad (i = 1, 2, \dots, k), \quad x > 0, \quad 0 < t < T, \\ -|\partial_x u_i|^{m_i-1} \partial_x u_i(0, t) = u_{i+1}^p(0, t), \quad (i = 1, 2, \dots, k), \quad u_{k+1} = u_1, \quad 0 < t < T, \\ u_i(x, 0) = u_{i0}(x), \quad (i = 1, 2, \dots, k), \quad x > 0. \end{cases} \quad (5)$$

Here the parameters are specified as follows: $k \geq 2$, $m > 1$, $p_i > 0$, u_{i0} ($i = 1, 2, \dots, k$) are continuous and nonnegative functions. The authors obtained the conditions for the global existence of solutions and the values of the Fujita-type critical exponent. In [13], the properties of solutions to problem (5) in the case of fast diffusion were studied.

Jiang Zheng [14] investigated the following equation:

$$\begin{cases} \partial_t u = \left(|\partial_x u|^\beta \partial_x u^m \right)_x, \quad x > 0, \quad 0 < t < T, \\ -|\partial_x u|^\beta \partial_x u^m(0, t) = u^p(0, t), \quad 0 < t < T, \\ u(x, 0) = u_0(x), \quad x > 0. \end{cases}$$

Here $m \geq 1$, $p > 0$, $\beta > 0$. They showed that $p_0 = \frac{2\beta+m+1}{\beta+2}$ is the critical exponent for global existence, and when $p_c = 2\beta + m + 1$ is the Fujita critical exponent. These results represent a generalization of the works of Galaktionov and Levine [15].

In [16–18], the main focus is on determining the critical conditions for the solutions of systems of degenerate parabolic equations, with particular emphasis on deriving theoretical results that characterize the threshold between global existence and blow-up phenomena, especially in the presence of nonlinear boundary fluxes.

In [19, 20], conditions for the existence of global-in-time and blow-up solutions were established for a model of heat and moisture transfer in a two-component medium, described by a cross-diffusion system with nonlinear boundary conditions.

In addition, in [21], the properties of solutions to the following diffusion problem with a source were investigated:

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x} \left(\left| \frac{\partial u_i^k}{\partial x} \right|^{m-1} \frac{\partial u_i^k}{\partial x} \right) + u_i^{p_i}, \quad x \in \mathbb{R}_+, \quad t > 0, \quad i = 1, 2, \quad (6)$$

$$\begin{aligned} - \left| \frac{\partial u_i^k}{\partial x} \right|^{m-1} \frac{\partial u_i^k}{\partial x} \Big|_{x=0} &= u_{3-i}^{q_i}(0, t), \quad t > 0, \quad i = 1, 2, \\ u_i|_{t=0} &= u_{i0}(x), \quad i = 1, 2. \end{aligned} \quad (7)$$

Here $m > 1$, $k \geq 1$ and $q_i, p_i > 0$ are given numerical parameters, while $u_{i0}(x)$, $i = 1, 2$ are functions defined on $\mathbb{R}_+$, nonnegative and continuous.

The authors proved that if the inequalities $q_1 q_2 > \left(\frac{m(k+1)}{m+1} \right)^2$, $\frac{1}{m} < p_i \leq 1$, ($i = 1, 2$) hold and if $\xi_i \rightarrow \left(\frac{d_i}{D_i} \right)^{\frac{m}{m+1}} = \xi_{i0}$, ($i = 1, 2$) then the solutions of problem (6)–(7) with compact support possess the following asymptotics:

$$u_i(x, t) = (T + t)^{\alpha_i} \bar{\varphi}_i(\xi_i) (1 + o(1)), \quad (i = 1, 2),$$

here

$$\alpha_i = \frac{1}{1 - p_i}, \quad \beta_i = \frac{p_i - km}{(p_i - 1)(m + 1)}, \quad \xi_i = x(T + t)^{-\beta_i},$$

$$\bar{\varphi}_i(\xi_i) = \left(d_i - D_i \xi_i^{\frac{m+1}{m}} \right)^{\frac{m}{mk-1}}, \quad d_i = \text{const} \geq 0, \quad D_i = \frac{\beta_i^{\frac{1}{m}} (mk - 1)}{k(m + 1)}.$$

Charkaoui A. and El Guermai G. [22] studied the following system and its particular cases:

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = f_i(u_1, \dots, u_m) & \text{on } (0, T) \times \Omega, \\ \alpha_i \frac{\partial u_i}{\partial n} + (1 - \alpha_i) u_i = \beta_i & \text{on } (0, T) \times \partial\Omega, \\ u_i(0, \cdot) = u_{i0}, \end{cases} \quad (8)$$

where the $f_i : \mathbb{R}^m \rightarrow \mathbb{R}$ are C^1 functions of $u = (u_1, \dots, u_m)$, $d_i \in (0, \infty)$, $\alpha_i \in [0, 1]$, $0 \leq \beta_i \in C^2([0, T] \times \bar{\Omega})$, $T > 0$, $i = 1, \dots, m$, $\Omega \subset \mathbb{R}^N$.

The authors proved the existence of a global classical solution for the reaction-diffusion system (8) in the case $\alpha_i = 0$, under the assumptions of quasipositivity and at most polynomial growth for the function f . They utilized a lower triangular invertible matrix P with nonnegative entries that satisfies a specific linear growth constraint, along with L^∞ initial data.

They have extended upon the results of Pierre M. and Schmitt D. [23], showing that a blow-up phenomenon can occur under some conditions in the case $\alpha_i = 0$, $m = 2$. To support their findings, they used the following forms of solutions:

$$u(t, x) = \frac{a_1(T - t) + b_1|x|^2}{(T - t + |x|^2)^\gamma}, \quad v(t, x) = \frac{a_2(T - t) + b_2|x|^2}{(T - t + |x|^2)^\gamma}, \quad a_j, b_j > 0, \quad \gamma > 1, \quad j = 1, 2.$$

The authors further demonstrated the existence of global weak supersolutions for bounded L^1 -nonlinearities by reformulating the original problem (8) as an approximate problem. Additionally, they reviewed relevant literature and highlighted open problems related to specific cases of the system (8).

In [24], the authors reviewed the global existence of solutions to reaction-diffusion systems with positivity preservation and uniformly bounded mass. The paper summarizes existing theoretical results, highlights unresolved problems, and discusses conditions under which global solutions exist.

In [25], Charkaoui et al. propose a nonlinear parabolic system with variable growth conditions for image denoising and restoration. The authors establish the existence and uniqueness of weak solutions within the framework of Musielak–Orlicz spaces using the Faedo–Galerkin method. Numerical experiments on grayscale and MRI images demonstrate the efficiency and robustness of the proposed approach in noise removal compared with several existing methods.

The work in [26] investigates the asymptotic behavior of stochastic Allen–Cahn-type reaction-diffusion equations with time white noise. The authors show that the long-term dynamics are governed by curvature-driven interface motion under stochastic perturbations. Using the theory of stochastic solutions and generalized front propagation, they establish global-in-time results without imposing regularity assumptions on the moving interface.

In [27, 28], recent studies have focused on the qualitative properties of nonlinear parabolic and reaction-diffusion systems, including global existence, blow-up phenomena, periodic weak solutions, and problems with nonlinear boundary conditions. In particular, the existence and uniqueness of weak periodic solutions for quasilinear parabolic systems have been established using topological and variational approaches, while several works investigated global solvability together with mass control and dissipation effects in reaction-diffusion models. These studies provide important theoretical foundations for the analysis of nonlinear evolution equations and are methodologically related to the problem considered in the present work.

In the p -Laplacian model, the case $p_i > 2$ corresponds to the degenerate (slow) diffusion regime, whereas the range $1 < p_i < 2$ leads to singular (fast) diffusion. In this work, we consider the case $p_i > 2$, which is consistent with the physical and analytical framework of the problem under consideration.

The main contributions of this paper are summarized as follows. First, we investigate a system of p -Laplacian type equations coupled through nonlinear boundary conditions, generalizing existing models in the literature to a multi-component system of k equations. Second, we focus on the degenerate diffusion regime ($p_i > 2$) and establish novel asymptotic properties of the corresponding self-similar solutions. Finally, we develop a comprehensive analytical framework complemented by numerical experiments that characterize the qualitative behavior of the solutions.

1 Research methodology

In this work, we construct a self-similar solution to problem (1)–(3) and derive the asymptotics of the self-similar solution. We seek the self-similar solution in the following form:

$$\bar{u}_i(x, t) = (T + t)^{-\alpha_i} \varphi_i(\xi_i), \quad \xi_i = x(T + t)^{-\lambda_i}, \quad (9)$$

here $T > 0$, $\lambda_i = \frac{1-\alpha_i(p_i-2)}{p_i}$, $\alpha_{i+1} = \frac{(p_i-1)(2\alpha_i+1)}{p_i q_i}$, ($i = 1, 2, \dots, k$).

Substitution of (9) into (1)–(2) leads to the following system:

$$\begin{cases} \left(|\varphi'_i|^{p_i-2} \varphi'_i \right)' + \lambda_i \xi_i \varphi'_i + \alpha_i \varphi_i = 0, \\ -|\varphi'_i|^{p_i-2} \varphi'_i|_{\xi_i=0} = \varphi_{i+1}^{q_i}(0), \quad t > 0, \quad \varphi_{k+1} = \varphi_1, \quad i = 1, 2, \dots, k. \end{cases} \quad (10)$$

Let us consider the following function:

$$\overline{\varphi}_i = \left(a_i - b_i \xi_i^{\frac{p_i-1}{p_i-2}} \right)_+^{\frac{p_i-1}{p_i-2}},$$

here $a_i > 0$, $b_i = \frac{p_i-2}{p_i} \lambda_i^{\frac{1}{p_i-1}} > 0$, $(y)_+ = \max(0, y)$.

Theorem 1. For the solution of problem (10) with compact support, when $\xi_i \rightarrow \left(\frac{a_i}{b_i} \right)^{\frac{p_i-1}{p_i}}$ the following asymptotics holds:

$$\varphi_i(\xi_i) = \overline{\varphi}_i(1 + o(1)). \quad (11)$$

Proof. We seek the solution of system (10) in the following form:

$$\varphi_i(\xi_i) = \overline{\varphi}_i(\xi_i) w_i(\tau_i), \quad (12)$$

where $\tau_i = -\ln \left(a_i - b_i \xi_i^{\frac{p_i-1}{p_i-2}} \right)$ and the function $w_i(\tau_i)$ is nonnegative and bounded.

We first perform the basic calculations required for the transformation of arguments:

$$\begin{aligned} \frac{d\tau_i}{d\xi_i} &= \frac{p_i}{p_i-1} b_i^{\frac{p_i-1}{p_i}} e^{\tau_i} (a_i - e^{-\tau_i})^{1/p_i}, \\ \frac{d\varphi_i}{d\xi_i} &= \frac{d}{d\tau_i} \left(e^{-\frac{p_i-1}{p_i-2}\tau_i} w_i \right) \frac{d\tau_i}{d\xi_i} = \frac{p_i}{p_i-1} b_i^{\frac{p_i-1}{p_i}} e^{-\frac{1}{p_i-2}\tau_i} (a_i - e^{-\tau_i})^{1/p_i} \left(\frac{dw_i}{d\tau_i} - \frac{p_i-1}{p_i-2} w_i \right). \end{aligned} \quad (13)$$

Applying the relations (12)–(13) to (10), we obtain the following expression:

$$\frac{d}{d\tau_i} (L_i w_i) + a_{i1} L_i w_i + a_{i2} w_i^* + a_{3i} w_i = 0, \quad (14)$$

where $a_{i1} = \frac{p_i-1}{p_i} a_{i0} - \frac{p_i-1}{p_i-2}$, $a_{i0} = \frac{e^{-\tau_i}}{a_i - e^{-\tau_i}}$, $a_{i2} = \frac{\lambda_i p_i}{p_i-1} b_i^*$, $b_i^* = \left| \frac{p_i-1}{p_i} b_i^{\frac{1-p_i}{p_i}} \right|^{p_i}$, $a_{3i} = \alpha_i a_{i0} b_i^*$, $w_i^* = \frac{dw_i}{d\tau_i} - \frac{p_i-1}{p_i-2} w_i$, $L_i w_i = |w_i^*|^{p_i-2} w_i^*$.

The study of the solutions of the last equation is equivalent to the study of the solutions of equation (1), and each of them, on the interval $[\tau_{i0}, +\infty)$, satisfies the following inequalities [29]:

$$w_i > 0, \quad w_i^* \neq 0.$$

Let us verify that the solution $w_i(\tau_i)$ of equation (14) either has a finite limit w_{i0} or becomes unbounded as $\tau_i \rightarrow +\infty$.

Suppose that $\nu_i(\tau_i) = L_i w_i$. Then, for the derivative of the function $\nu_i(\tau_i)$ we get the following:

$$\nu'_i = -(a_{i1} \nu_i + a_{i2} w_i^* + a_{3i} w_i).$$

To analyze the solutions of the last equation, we introduce an auxiliary function

$$\theta_i(\tau_i, \mu_i) = -(a_{i1} \mu_i + a_{i2} w_i^* + a_{3i} w_i),$$

where μ_i is a real number. From this, it is easy to see that for each value of μ_i , the function $\theta_i(\tau_i, \mu_i)$ preserves its sign on some interval $[\tau_{i1}, +\infty) \subset [\tau_{i0}, +\infty)$, and for all $\tau_i \in [\tau_{i1}, +\infty)$, one of the inequalities holds:

$$\nu'_i(\tau_i) > 0, \quad \nu'_i(\tau_i) < 0.$$

Then, analyzing (14) with account of Bolzano's theorem, it follows that the function $\nu_i(\tau_i)$ has a limit as $\tau \rightarrow +\infty$. Now let us perform a limiting transition. Taking into account the last limit and the fact that $w'_i = 0$ from (14), for w_i we obtain the following results:

$$\lim_{\tau_i \rightarrow \infty} a_{i0} = \lim_{\tau_i \rightarrow \infty} a_{i3} = 0, \quad a_{i1}^0 = \lim_{\tau_i \rightarrow \infty} a_{i1} = -\frac{p_i - 1}{p_i - 2}, \quad a_{i2}^0 = \lim_{\tau_i \rightarrow \infty} a_{i2} = \frac{\lambda_i p_i}{p_i - 1} b_i^*,$$

$$w_i^* = \lim_{\tau_i \rightarrow \infty} w_i^* = -\frac{p_i - 1}{p_i - 2} w_i^0, \quad \mu_i^0 = \lim_{\tau_i \rightarrow \infty} L_i w_i = -\left| \frac{p_i - 1}{p_i - 2} \right|^{p_i - 1} (w_i^0)^{p_i - 1}, \quad w_i^0 = \lim_{\tau_i \rightarrow \infty} w_i.$$

Consequently, we obtain the following result:

$$a_{i1}^0 \mu_i^0 + a_{i2}^0 w_i^* = 0,$$

which is equivalent to this

$$w_i^0 \left[(w_i^0)^{p_i - 2} - \lambda_i \left| \frac{p_i - 2}{b_i p_i} \right|^{p_i - 1} \right] = 0,$$

whose solution is $w_i^0 = \left[\lambda_i \left| \frac{p_i - 2}{b_i p_i} \right|^{p_i - 1} \right]^{\frac{1}{p_i - 2}} = 1$, and by virtue of (12) $\varphi_i(\xi_i) = \bar{\varphi}(\xi_i)(1 + o(1))$.

□

2 Computational experiment and visualization results

In this section, we construct a numerical scheme to solve (1)–(3) for the case $k = 2$. In the domain $\Omega = \{(x, t) : 0 \leq x \leq b, 0 \leq t < T\}$, we discretize time and space into a grid as follows: $\tau = \frac{T}{n}$, $h = \frac{b}{n}$. For the numerical solution of problem (1)–(3), we use an implicit scheme:

$$V_\tau = \{t_j = j\tau, \tau > 0, j = 1, 2, \dots, m; m\tau = T, T > 0\}; S_h = \{x_\iota = \iota h; h > 0; \iota = 1, 2, \dots, n; nh = b\}$$

$$\begin{cases} \frac{y_\iota^{j+1} - y_\iota^j}{\tau} = \frac{1}{h^2} \left[D_{\iota+1}^j(y_{\iota+1}^j) (y_{\iota+1}^{j+1} - y_\iota^{j+1}) - D_\iota^j(y_\iota^j) (y_\iota^{j+1} - y_{\iota-1}^{j+1}) \right], \\ \frac{z_\iota^{j+1} - z_\iota^j}{\tau} = \frac{1}{h^2} \left[M_{\iota+1}^j(z_{\iota+1}^j) (z_{\iota+1}^{j+1} - z_\iota^{j+1}) - M_\iota^j(z_\iota^j) (z_\iota^{j+1} - z_{\iota-1}^{j+1}) \right], \\ \iota = 2, \dots, n-1; \quad j = 0, 1, \dots, m-1, \end{cases} \quad (15)$$

$$y_\iota^0 = y(x_\iota, 0), \quad z_\iota^0 = z(x_\iota, 0), \quad \iota = 0, 1, \dots, n,$$

$$\begin{cases} -D_1(y_1) \cdot \frac{y_1^{j+1} - y_0^{j+1}}{h} = (z_0^j)^{q_1}, \\ -M_1(z_1) \cdot \frac{z_1^{j+1} - z_0^{j+1}}{h} = (y_0^j)^{q_2}, \end{cases}$$

$$y_n^j = \varphi_1(t_j), \quad z_n^j = \varphi_2(t_j), \quad j = 2, 3, \dots, m.$$

a)

$$\begin{cases} D_t^j(y_t^j) = D_t^j\left(\frac{y_t^j + y_{t-1}^j}{2}\right), \\ M_t^j(z_t^j) = M_t^j\left(\frac{z_t^j + z_{t-1}^j}{2}\right), \end{cases} \quad (16)$$

b)

$$\begin{cases} D_t^j(y_t^j) = \frac{D_t^j(y_t^j) + D_{t-1}^j(y_{t-1}^j)}{2}, \\ M_t^j(z_t^j) = \frac{M_t^j(z_t^j) + M_{t-1}^j(z_{t-1}^j)}{2}. \end{cases} \quad (17)$$

Here, (16)–(17) are written using the Samarskii-Sobol scheme. If we take according to formula (16), then

$$\begin{cases} D_t^j(y_t^j) = \left| \frac{y_t^j - y_{t-1}^j}{h} \right|^{p_1-2}, \\ M_t^j(z_t^j) = \left| \frac{z_t^j - z_{t-1}^j}{h} \right|^{p_2-2}, \end{cases}$$

if we take according to formula (17), then

$$\begin{cases} D_t^j(y_t^j) = \frac{|y_t^j - y_{t-1}^j|^{p_1-2} + |y_{t-1}^j - y_{t-2}^j|^{p_1-2}}{2h^{p_1-2}}, \\ M_t^j(z_t^j) = \frac{|z_t^j - z_{t-1}^j|^{p_2-2} + |z_{t-1}^j - z_{t-2}^j|^{p_2-2}}{2h^{p_2-2}}, \end{cases}$$

system (15) is a system of nonlinear equations with respect to y^{j+1} and z^{j+1} . The system (18) is solved numerically by freezing the nonlinear coefficients at the s -th iteration to linearize the equations for the $(s+1)$ -th step.

$$\begin{cases} \frac{y_{t+1}^{s+1,j+1} - y_t^{s,j}}{\tau} = \frac{1}{h^2} \left[D_{t+1}^j(y_{t+1}^j) \left(y_{t+1}^{s+1,j+1} - y_t^{s+1,j+1} \right) - D_t^j(y_t^j) \left(y_t^{s+1,j+1} - y_{t-1}^{s+1,j+1} \right) \right], \\ \frac{z_{t+1}^{s+1,j+1} - z_t^{s,j}}{\tau} = \frac{1}{h^2} \left[M_{t+1}^j(z_{t+1}^j) \left(z_{t+1}^{s+1,j+1} - z_t^{s+1,j+1} \right) - M_t^j(z_t^j) \left(z_t^{s+1,j+1} - z_{t-1}^{s+1,j+1} \right) \right], \end{cases} \quad (18)$$

here, $s = 0, 1, 2, \dots$

The choice of the initial approximation determines the efficiency of the iterative method, since it ensures rapid convergence to the solution and preserves the physical meaning of the problem. As such an initial approximation, it is reasonable to use the asymptotic solutions of the problem [30]. At each time step, the initial values of the iterations $y_{t,0}^{s+1}$, $z_{t,0}^{s+1}$ are taken from the previous time step: $y_{t,0}^{j+1} = y_t^j$, $z_{t,0}^{j+1} = z_t^j$. In computations using the iterative scheme, the accuracy of the iterations is determined, and at the same time

$$\begin{cases} \max_{0 \leq \iota \leq n} \left| \bar{y}_{\iota}^{s+1} - \bar{y}_{\iota}^s \right| < \varepsilon, \\ \max_{0 \leq \iota \leq n} \left| \bar{z}_{\iota}^{s+1} - \bar{z}_{\iota}^s \right| < \varepsilon. \end{cases} \quad (19)$$

We introduce the following notation: $\bar{y}_{\iota} = y_{\iota}^{j+1}$, $\bar{z}_{\iota} = z_{\iota}^{j+1}$. Then the system (18) can be written in the following form:

$$\begin{cases} \bar{A}_{1\iota}^s \bar{y}_{\iota-1}^{s+1} - \bar{C}_{1\iota}^s \bar{y}_{\iota}^{s+1} + \bar{B}_{1\iota}^s \bar{y}_{\iota+1}^{s+1} = -\bar{F}_{1\iota}^s, \\ \bar{A}_{2\iota}^s \bar{z}_{\iota-1}^{s+1} - \bar{C}_{2\iota}^s \bar{z}_{\iota}^{s+1} + \bar{B}_{2\iota}^s \bar{z}_{\iota+1}^{s+1} = -\bar{F}_{2\iota}^s, \end{cases} \quad (20)$$

here

$$\bar{A}_{1\iota}^s = \frac{\tau}{h^2} D_{\iota}^j(y_{\iota}^j), \quad \bar{A}_{2\iota}^s = \frac{\tau}{h^2} M_{\iota}^j(z_{\iota}^j),$$

$$\bar{B}_{1\iota}^s = \frac{\tau}{h^2} D_{\iota+1}^j(y_{\iota+1}^j), \quad \bar{B}_{2\iota}^s = \frac{\tau}{h^2} M_{\iota+1}^j(z_{\iota+1}^j),$$

$$\bar{C}_{1\iota}^s = \frac{\tau}{h^2} \left(D_{\iota}^j(y_{\iota}^j) + D_{\iota+1}^j(y_{\iota+1}^j) \right) + 1, \quad \bar{C}_{2\iota}^s = \frac{\tau}{h^2} \left(M_{\iota}^j(z_{\iota}^j) + M_{\iota+1}^j(z_{\iota+1}^j) \right) + 1,$$

$$\bar{F}_{1\iota}^s = y_{\iota}^j, \quad \bar{F}_{2\iota}^s = z_{\iota}^j.$$

The values of the required quantities at the end of the interval $0 \leq x \leq b$ can be obtained using Milne's formula of second-order accuracy.

$$\begin{aligned} \left. \frac{\partial y}{\partial x} \right|_0 &\approx \frac{-y_2 + 4y_1 - 3y_0}{2h}, \quad \left. \frac{\partial y}{\partial x} \right|_n \approx \frac{3y_n - 4y_{n-1} + y_{n-2}}{2h}, \\ \left. \frac{\partial z}{\partial x} \right|_0 &\approx \frac{-z_2 + 4z_1 - 3z_0}{2h}, \quad \left. \frac{\partial z}{\partial x} \right|_n \approx \frac{3z_n - 4z_{n-1} + z_{n-2}}{2h}. \end{aligned} \quad (21)$$

To obtain numerical solutions of the system of linear algebraic equations (20), we use the sweep method (Thomas algorithm). According to the sweep method:

$$\begin{cases} \bar{y}_{\iota} = \alpha_{1\iota} \bar{y}_{\iota+1} + \beta_{1\iota}, \\ \bar{z}_{\iota} = \alpha_{2\iota} \bar{z}_{\iota+1} + \beta_{2\iota}. \end{cases} \quad (22)$$

Here, $\alpha_{1\iota}, \alpha_{2\iota}, \beta_{1\iota}, \beta_{2\iota}$ are coefficients that are computed using the following formulas:

$$\begin{cases} \alpha_{1\iota+1} = \frac{B_{1\iota}}{C_{1\iota} - \alpha_{1\iota} A_{1\iota}} \\ \beta_{1\iota+1} = \frac{A_{1\iota} \beta_{1\iota} + F_{1\iota}}{C_{1\iota} - \alpha_{1\iota} A_{1\iota}} \end{cases}, \quad \begin{cases} \alpha_{2\iota+1} = \frac{B_{2\iota}}{C_{2\iota} - \alpha_{2\iota} A_{2\iota}} \\ \beta_{2\iota+1} = \frac{A_{2\iota} \beta_{2\iota} + F_{2\iota}}{C_{2\iota} - \alpha_{2\iota} A_{2\iota}} \end{cases}. \quad (23)$$

Here $\iota = 1, 2, \dots, n$. The values $\alpha_{10}, \alpha_{20}, \beta_{10}, \beta_{20}$ are obtained from the boundary conditions (21).

It is known that inequality,

$$\left| C_{1l}^s \right| = \left| A_{1l}^s + B_{1l}^s + 1 \right| \geq \left| A_{1l}^s + B_{1l}^s \right|, \quad \left| C_{2l}^s \right| = \left| A_{2l}^s + B_{2l}^s + 1 \right| \geq \left| A_{2l}^s + B_{2l}^s \right|,$$

holds for system (20). Therefore, since the condition of diagonal dominance is satisfied for system (20), the applicability of the sweep method (22) and the stability of system (23) are ensured [31, Chapter 8], [32, Chapter 4], that is

$$\left| C_{1l}^s \right| \geq \left| A_{1l}^s + B_{1l}^s \right|, \quad \left| C_{2l}^s \right| \geq \left| A_{2l}^s + B_{2l}^s \right|.$$

The iterative process continues until condition (19) is satisfied. Using the numerical scheme constructed above, a computational experiment was carried out. We present some of its results. For the computations, the grid step was chosen as $h = 0.05$, the number of nodes $n = 300$ and the iteration accuracy $\varepsilon = 10^{-5}$. Formulas (9) and (11) were used as the initial approximation for the iterative process.

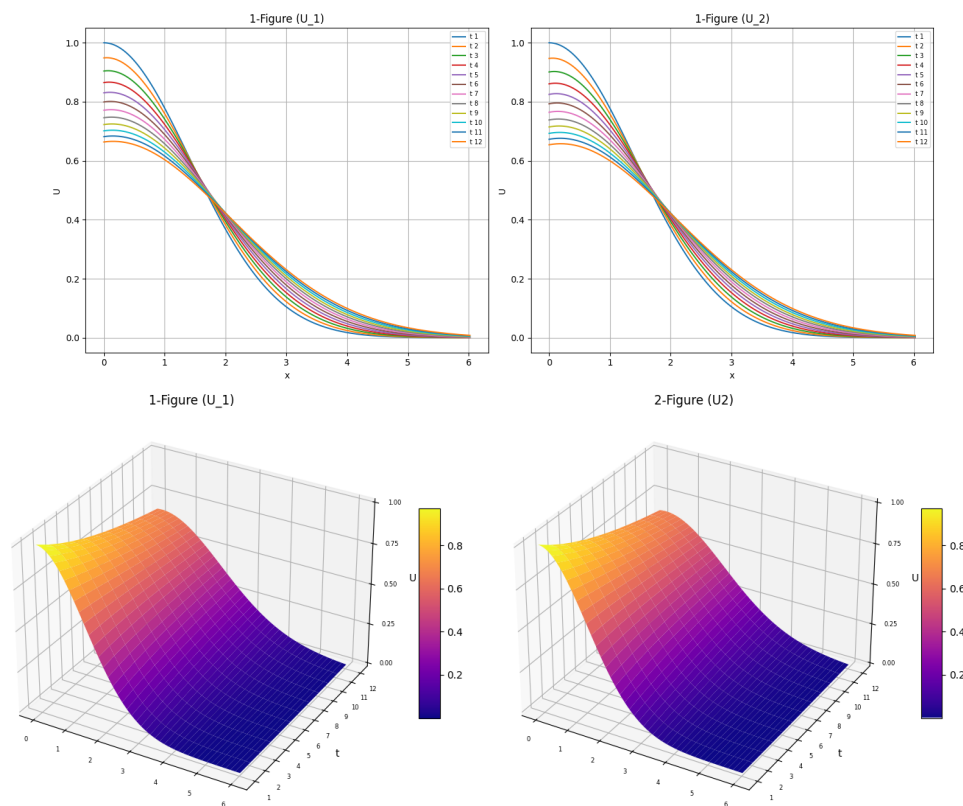


Figure 1. Numerical solution of problem (1)–(3).
 $p_1 = 2.0101$, $p_2 = 2.0105$, $q_1 = 1.85$, $q_2 = 1.95$

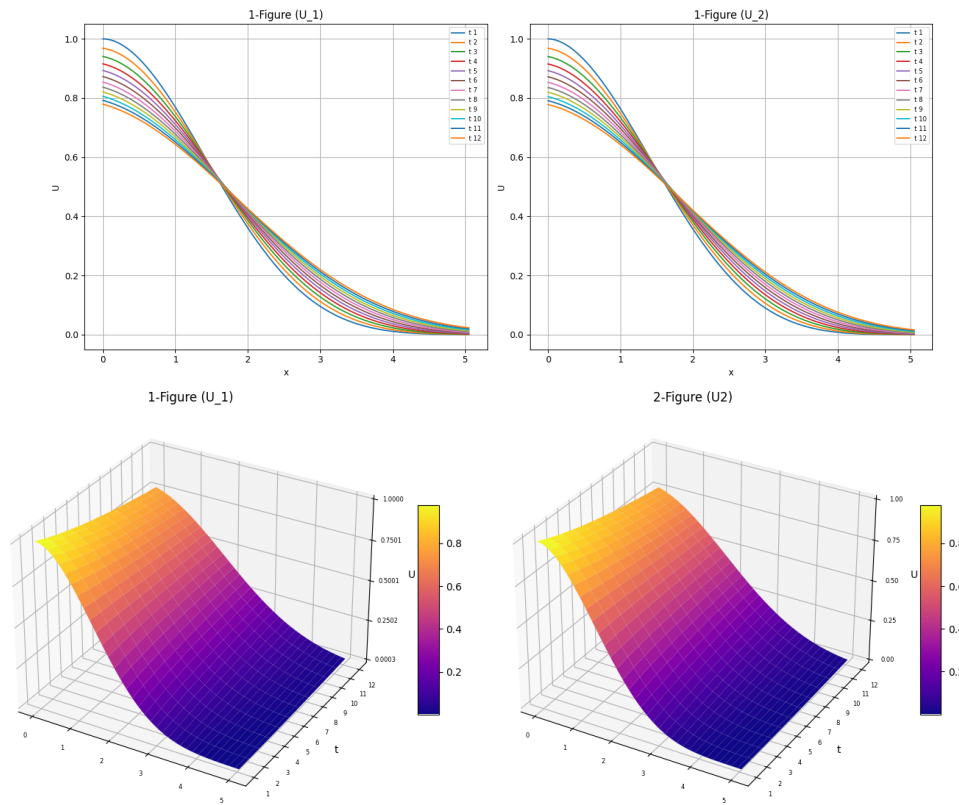


Figure 2. Numerical solution of problem (1)–(3). $p_1 = 2.11$, $p_2 = 2.18$, $q_1 = 2.6$, $q_2 = 2.7$

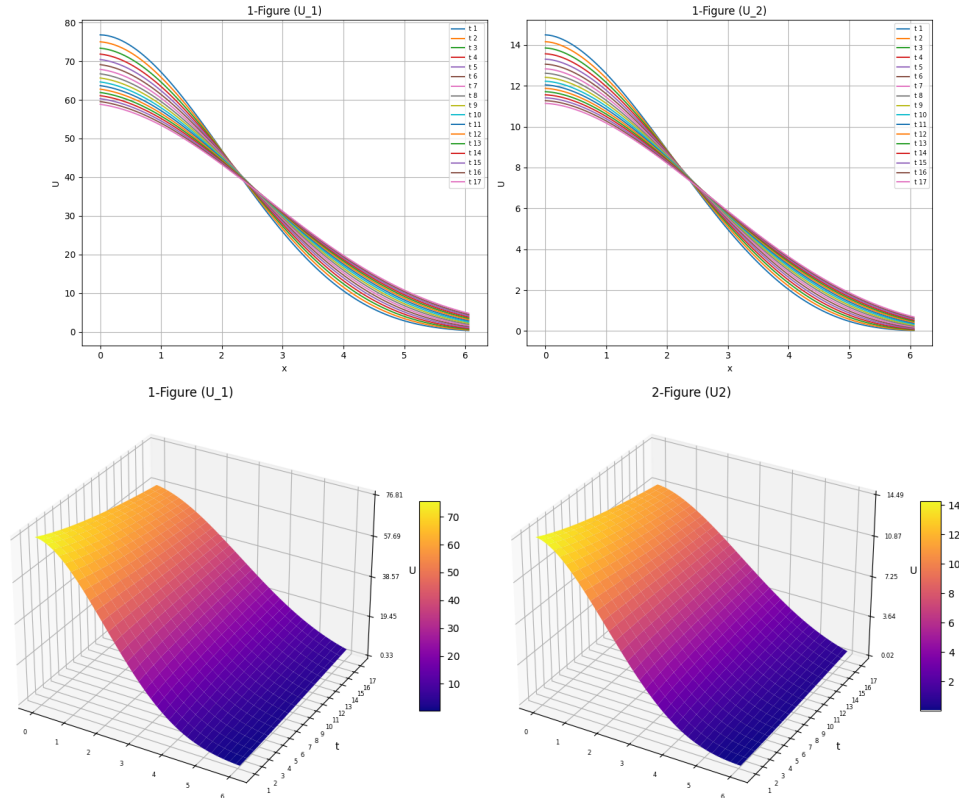


Figure 3. Numerical solution of problem (1)–(3). $p_1 = 2.19$, $p_2 = 2.35$, $q_1 = 2.6$, $q_2 = 2.7$

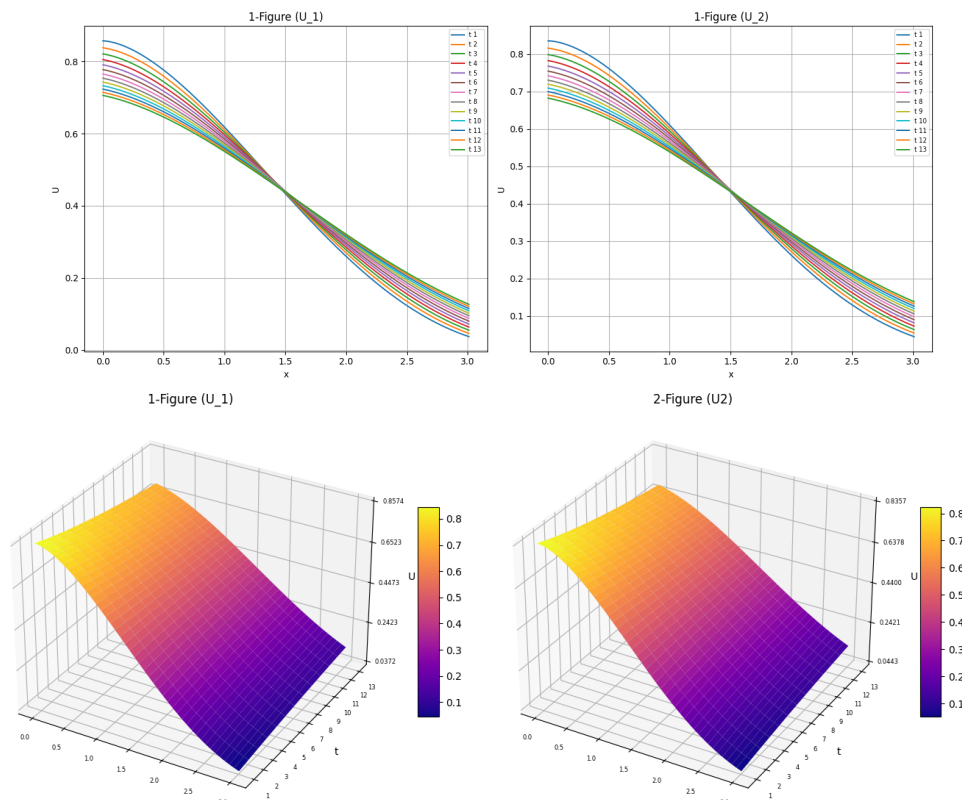


Figure 4. Numerical solution of problem (1)–(3).
 $p_1 = 2.5$, $p_2 = 2.4$, $q_1 = 3.201$, $q_2 = 3.302$

Figures 1–4 show the results of the numerical solution of the problem (1)–(3) for $p_i > 2$, corresponding to the case of slow diffusion. (9), (11) were taken as the initial approximation for the iterative process. For $p_i > 2$, as follows from the asymptotic formulas (11) of the graphs, the heat propagation occurs at a finite rate. The penetration depth of a thermal wave depends on the time and the wave front (the point at which $u_i(x, t)$ becomes zero) for each medium located at the final point:

$$x_{\phi_{u_i}} = \left(\frac{a_i}{b_i} \right)^{\frac{p_i-1}{p_i}} (T + t)^{\lambda_i} < \infty.$$

Numerical experiments have shown that the iterative processes converge within 10–12 steps. The asymptotics of self-similar solutions with compact support, taken as the initial approximation, played an important role in the convergence of the iterative process. The graphs presented above demonstrate that, for the given values of the numerical parameters, the reaction-diffusion processes occur at a finite speed.

Conclusion

It is well known that in the numerical solution of nonlinear problems, one of the main issues is the choice of the initial approximation, which ensures rapid convergence of the iterative process to the exact solution while preserving the qualitative properties of nonlinear processes. This problem was resolved by employing the asymptotic formulas constructed above, depending on the values of the numerical parameters, as the initial approximation.

The results of several computational experiments on the problem, the corresponding graphs, and

the analysis of the numerical solutions are presented. The outcomes of the computational experiments demonstrated the efficiency of the proposed methods, while the numerical solutions reflected the nonlinear features of the problem.

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Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

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*Author information**

Zafar Ravshanovich Rakhmonov — Doctor of Physical and Mathematical Sciences, Professor, National University of Uzbekistan named after Mirzo Ulugbek, 4 University St., Tashkent 100174, Uzbekistan; e-mail: zraxmonov@inbox.ru; <https://orcid.org/0000-0002-4190-7069>

Aziza Rakhmonovna Zaripova (*corresponding author*) — PhD student, Karshi State University, 17 Kuchabag St., Karshi 180119, Uzbekistan; e-mail: azizazaripova1990@mail.ru; <https://orcid.org/0009-0008-1984-5771>

*Authors' names are presented in the following order: first name, middle name (if any), last name.