

On quasivarieties generated by (3-3)-snake lattices

M.A. Arapbay¹, A.M. Asanbekov^{2,*}

¹*L.N. Gumilyov Eurasian National University, Astana, Kazakhstan;*

²*Institute of Mathematics of the National Academy of Sciences, Bishkek, Kyrgyzstan
(E-mail: meruertarapbay@gmail.com, amantai.asanbekov.1@gmail.com)*

This paper addresses two classical and interrelated questions in universal algebra and lattice theory concerning the finite axiomatizability of quasivarieties and the standardness of topological quasivarieties generated by finite lattices. These problems have long attracted significant attention due to their central role in understanding the boundary between algebraic structure and logical definability. Although a general characterization remains elusive, specific families of finite lattices provide valuable insight into the complexity of quasi-equational theories. In this work, we investigate the quasivariety generated by the so-called (3-3)-snake lattice, a finite structure known for its intricate congruence behavior. We demonstrate that the quasivariety determined by this lattice is neither finitely axiomatizable nor standard in the topological sense. This result contributes to the broader program of identifying finite lattices whose quasi-identities inherently resist finite axiomatizability and whose associated Boolean topological algebras fail to exhibit profiniteness. In addition to establishing the main theorems, we present a generalization that expands the scope of the construction, offering an illustrative example and a corollary that highlight the robustness of the method. The findings underscore the continued importance of studying finite lattices with atypical quasi-equational properties and enrich the existing landscape of nonstandard quasivarieties.

Keywords: lattice, modular lattice, quasivariety, splitting pair, semi-splitting pair, profiniteness, finite basis of quasi-identities, non-standard topological quasivariety.

2020 Mathematics Subject Classification: 08C15, 06C99.

Introduction

The following two closely related and well-known problems arise in universal algebra and lattice theory.

The first problem is “Which finite lattices generate finitely based quasivarieties?”

The second one is “Which finite lattices generate standard topological quasivarieties?”

The first problem, often referred to as the finite axiomatizability problem, was first brought to light by V.A. Gorbunov and D.M. Smirnov [1] in 1979. The problem, which concerns the standardness of (quasi)varieties, was formally introduced in 2008 by D.M. Clark, B.A. Davey, M.G. Jackson, and J.G. Pitkethly [2].

Historically, R. McKenzie [3] proved in 1970 that a finite lattice generates a finitely based variety. However, in 1978, V.P. Belkin [4] provided a finite lattice that generates not finitely based quasivariety. Belkin’s example was later extended in 1984 by V.I. Tumanov [5], who identified a broad class of finite lattices that generate not finitely axiomatizable quasivarieties. In that same article he also proposed a conjecture asserting that a finite (modular) lattice has an infinite quasi-identity basis precisely when the quasivariety it generates is proper. This claim, however, does not hold universally. In 1989, W. Dziobiak [6] constructed a counterexample: a finite non-modular lattice whose generated

*Corresponding author. E-mail: amantai.asanbekov.1@gmail.com

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23485395).

Received: 23 September 2025; Accepted: 28 April 2026.

© 2026 The Authors. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>)

quasivariety is proper yet admits a finite quasi-identity basis. The modular case, however, remains open.

This problem is also connected with the second problem on standardness of topological quasivarieties. In [2] was established, in particular, that the topological quasivariety generated by Belkin's lattice is non-standard. The fact that a finite lattice generates a standard topological variety was proved in [7] (with a shorter proof given in [8]). More recently, in [9] was shown that the Tumanov's lattices generate non-standard topological quasivarieties. Further results related to finite axiomatizability and standardness for quasivarieties of lattices were obtained in [10–12]. Sufficient conditions implying the existence of continuum many non-standard subquasivarieties, along with several applications of this result, were established in [13].

The primary focus of the present work is to establish that the quasivariety generated by (3-3)-snake lattice is neither standard nor finitely axiomatizable. We conclude the paper by presenting a generalization of this result, along with an illustrative example and a corresponding corollary.

1 Preliminaries and basic concepts

We start with a short review of the fundamental notions and facts about quasivarieties that will be required in the sequel. Further background on the concepts of universal algebra and lattice theory employed here can be found in [14, 15].

A class of algebras axiomatized by a set of quasi-identities is called a *quasivariety*. By *quasi-identity* we mean a sentence of the form

$$(\forall \bar{x})[p_1(\bar{x}) \approx q_1(\bar{x}) \wedge \dots \wedge p_n(\bar{x}) \approx q_n(\bar{x}) \rightarrow p(\bar{x}) \approx q(\bar{x})],$$

where $\bar{x} = (x_1, \dots, x_m)$, and $p, q, p_1, q_1, \dots, p_n, q_n$ are terms of the algebraic signature. By the Birkhoff-like theorem (see [15]), a class of algebras is a quasivariety if and only if it is closed under subalgebras, direct products, and ultraproducts.

A *variety* is a class of algebras axiomatized by a set of identities, where an *identity* is a sentence of the form $(\forall \bar{x})[s(\bar{x}) \approx t(\bar{x})]$ for some terms $s(\bar{x})$ and $t(\bar{x})$. By Birkhoff's theorem [15], a variety is exactly a quasivariety that is closed under homomorphisms.

Let $\mathbf{Q}(A)$ ($\mathbf{V}(A)$) be a quasivariety (variety) generated by an algebra A .

Let $\mathcal{K}$ be a class of algebras quasivariety. A congruence α on an algebra A is called a $\mathcal{K}$ -congruence if A/α belongs to $\mathcal{K}$. It is easy to see that $A \in \mathcal{K}$ if and only if the diagonal relation Δ_A on A is a $\mathcal{K}$ -congruence. The set $\text{Con}_{\mathcal{K}}A$ of all $\mathcal{K}$ -congruences of A forms an algebraic lattice with respect to inclusion $\subseteq$. By $\theta_{\mathcal{K}}(A)$ we denote the least congruence with property $A/\theta_{\mathcal{K}}(A) \in \mathcal{K}$. Let an algebra $A \in \mathcal{K}$ and $(a, b) \in A \times A$. The least $\mathcal{K}$ -congruence on algebra A containing the pair (a, b) is called a *principal $\mathcal{K}$ -congruence*, and denoted by $\theta_{\mathcal{K}}(a, b)$. If $\mathcal{K}$ is a variety, the congruence $\theta_{\mathcal{K}}(a, b)$ is the usual principal congruence that we denote $\theta(a, b)$.

For a quasivariety $\mathcal{K}$, an algebra $A \in \mathcal{K}$ is *subdirectly $\mathcal{K}$ -irreducible* if the least congruence Δ_A is completely meet-irreducible in $\text{Con}_{\mathcal{K}}A$. By the Birkhoff-like theorem for quasivarieties, every algebra of a quasivariety $\mathcal{K}$ is a subdirect product of subdirectly $\mathcal{K}$ -irreducible algebras [15]. It is known that for a finite algebra A , every subdirectly $\mathbf{Q}(A)$ -irreducible algebra is isomorphic to some subalgebra of A .

A one-element algebra is called *trivial*. A congruence θ on algebra A is trivial if it is the diagonal relation Δ_A on A or $A \times A$. A subalgebra B of algebra A is *proper* if it is not trivial and $B \neq A$.

A triple $\langle A, \sigma, \tau \rangle$ is called a *topological algebra* if $\langle A, \sigma \rangle$ is an algebra of the signature σ , and $\langle A, \tau \rangle$ is a topological space such that each operation belonging to σ is continuous under the topology τ .

A topological space is *Boolean* if it is compact, Hausdorff, and totally disconnected. A topological algebra $A_{\tau} = \langle A, \sigma, \tau \rangle$ is *Boolean* if its topological reduct $\langle A, \tau \rangle$ is Boolean.

For a finite algebra A equipped with the discrete topology, the class of topological algebras consisting of all topologically closed subalgebras of non-zero direct powers of A , endowed with the product topology is called a *topological quasivariety* $\mathbf{Q}_\tau(A)$. An algebra is called *profinite* if it is isomorphic to the inverse limit of finite algebras. An algebra is *profinite with respect to class of algebras* $\mathcal{K}$ if it is isomorphic to the inverse limit of finite algebras from $\mathcal{K}$. It is well-known that profinite algebras are equipped with Boolean topologies. A quasivariety $\mathcal{K}$ is *profinite* if every profinite algebra in $\mathcal{K}$ is profinite with respect to $\mathcal{K}$.

A quasivariety $\mathbf{Q}(A)$ is *standard* if every Boolean topological algebra with its algebraic reduct in $\mathbf{Q}(A)$ is profinite with respect to $\mathbf{Q}(A)$, that is, belongs to $\mathbf{Q}_\tau(A)$. In this case, we say that the algebra A generates a standard (topological) quasivariety.

2 (3-3)-snake lattices and their basic properties

The lattices M_3 , $M_{3,3}$ and M_{3-3} are illustrated in Figure 1.

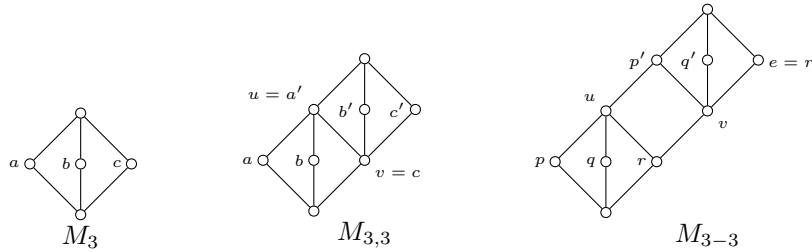


Figure 1. Lattices M_3 , $M_{3,3}$, M_{3-3}

Let L be a lattice and let $a \in L$. We write $a \downarrow$ ($a \uparrow$) for the principal ideal (filter) of a lattice L , that is, $a \downarrow = \{x \in L \mid x \leq a\}$ ($a \uparrow = \{x \in L \mid x \geq a\}$). A pair $(a, b) \in L \times L$ is said to be *splitting* (*semi-splitting*) whenever $L = a \downarrow \cup b \uparrow$ and $a \downarrow \cap b \uparrow = \emptyset$ ($L = a \downarrow \cup b \uparrow$ and $a \downarrow \cap b \uparrow \neq \emptyset$). For example, by notation of Figure 1, the pair (u, v) is a semi-splitting pair in $M_{3,3}$, and a splitting pair in M_{3-3} .

The following construction was suggested in [5].

Let L be a finite lattice with semi-splitting pair (u, v) . Let P be a finite modular lattice with coatom e . Define $L_{uv}(P, e)$ as the subset of $L \times P$ given by

$$L_{uv}(P, e) = \{(x, p), (y, e), (z, 1), (v, q) \mid x, y, z \in L, p, q \in P, \\ p < e, x \leq v, y \leq u, v \leq z, q \not\leq e, q < 1\}.$$

We show that $L_{uv}(P, e)$ is a sublattice of $L \times P$. One can see that $L_{uv}(P, e) = L_{xp} \cup L_{ye} \cup L_{z1} \cup L_{vq}$, where

$$\begin{aligned} L_{xp} &= \{(x, p) \mid p < e, x \leq v\}, \\ L_{ye} &= \{(y, e) \mid y \leq u\}, \\ L_{z1} &= \{(z, 1) \mid v \leq z\}, \\ L_{vq} &= \{(v, q) \mid q \not\leq e, q < 1\}. \end{aligned}$$

Suppose $(a, s), (b, t) \in L_{uv}(P, e)$. And let $(g, h) = (a, s) \wedge (b, t) = (a \wedge b, s \wedge t)$, and $(g', h') = (a, s) \vee (b, t) = (a \vee b, s \vee t)$. We show that $(g, h), (g', h') \in L_{uv}(P, e)$.

Let $(a, s) \in L_{xp}$. Since $a \leq v$, $s < p$, then $g = a \wedge b \leq v$ and $h = s \wedge t < p$. Hence $(g, h) \in L_{xp} \subseteq L_{uv}(P, e)$.

Let $(a, s) \in L_{ye}$. Since $a \leq u$, $s = e$, then $g = a \wedge b \leq u$ and $h = e \wedge t \leq e$. If $h = e$, then $(g, h) \in L_{ye}$. If $h < e$, then $(g, h) \in L_{vq}$. Hence $(g, h) \in L_{uv}(P, e)$.

Let $(a, s) \in L_{z1}$. Since $a \geq v$, $s = 1$, then $g = a \wedge b \geq v$ and $h = s \wedge t = t \leq 1$. If $h = 1$, then $(g, h) \in L_{z1}$. If $h < 1$, then $(b, t) \in L_{vq}$, whence $t \not\leq e$. Hence $(g, h) = (v, t) \in L_{uv}(P, e)$.

At last let $(a, s) \in L_{vq}$ and $(b, t) \in L_{vq}$. Thus $g = v$ and $h < 1$. If $h = e$, then $(g, h) \in L_{ye}$. If $h < e$, then $(g, h) \in L_{xp}$. If $h > e$, then $(g, h) \in L_{vq}$. Hence $(g, h) \in L_{uv}(P, e)$.

Therefore, $L_{uv}(P, e)$ is closed under intersection $\wedge$.

By the same arguments one can check that $L_{uv}(P, e)$ is closed under union $\vee$.

It means that $L_{uv}(P, e)$ forms a sublattice of $L \times P$. Moreover $L \leq_s L \times P$.

Let L and P be finite modular lattices. We say that a lattice K is obtained *by splitting* if $K \cong L_{uv}(P, e)$ for some semi-splitting pair (u, v) of L and a coatom e of P .

We provide an example of splitting.

Example 1. The lattice $L_{uv}(M_{3-3}, c')$ depicted in Figure 2 is obtained by splitting the lattice $M_{3,3}$ from Figure 1 by M_{3-3} . Note that the pair $((u, e), (v, 0))$ is a semi-splitting pair and the pair $((0, u), (v, v))$ is a splitting pair.

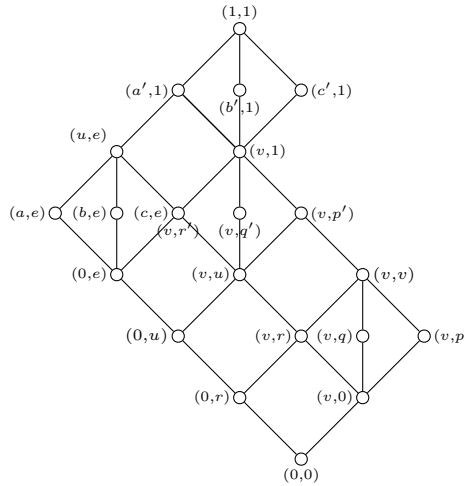


Figure 2. Lattices $L_{uv}(M_{3-3}, c')$

We note if P is a two-element lattice, then $L_{uv}(P, e)$ is a partial case of Day's doubling construction.

Let L be a modular lattice, and (u, v) is a splitting pair, and $\theta = \theta(u \wedge v, v)$. Since L is modular, then $(u/\theta, v/\theta)$ is a semi-splitting pair in L/θ . For a splitting pair (u, v) , we assume

$$L_{uv}(P, e) = L_{u/\theta v/\theta}(P, e).$$

Let L and P be finite modular lattices. We say that a lattice K is obtained *by gluing* the lattice P to L if $K \cong L_{uv}(P, e)$ for some splitting pair (u, v) of L and a coatom e of P .

Example 2. The lattice $L_{uv}(M_{3-3}, c')$ depicted in Figure 2 is obtained by gluing the lattice M_{3-3} to M_{3-3} .

We define the modular lattice S_n inductively by gluing M_{3-3} to S_{n-1} as follows:

1. $n = 0$. $S_0 \cong M_{3-3}$.
2. $n = 1$. $S_1 = L_{u_1 v_1}(M_{3-3}, e)$, where $L = S_0$ and $(u_1, v_1) = (a_1 \vee b_1, a_0 \wedge a'_1)$.
3. $n = 2$. $S_2 = L_{u_2 v_2}(M_{3-3}, e)$, where $L = S_1$ and $(u_2, v_2) = (a_2 \vee b_2, a_1 \wedge a'_2)$.
4. $n > 2$. $S_n = L_{u_n v_n}(M_{3-3}, e)$, where $L = S_{n-1}$ and $(u_n, v_n) = (a_n \vee b_n, a_{n-1} \wedge a'_n)$.

For $n = 0, 1, 2, 3$, the lattice S_n is depicted in Figures 3 and 4.

We refer to the lattice S_n as the $(3-3)$ -snake lattice. Based on its construction, it is clear that S_n is generated by its double irreducible elements, and the total number of these elements equals $s = 3n + 4$.

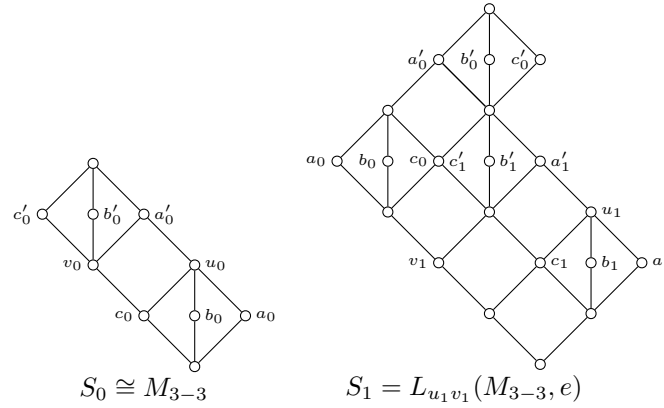


Figure 3. Lattices S_0, S_1

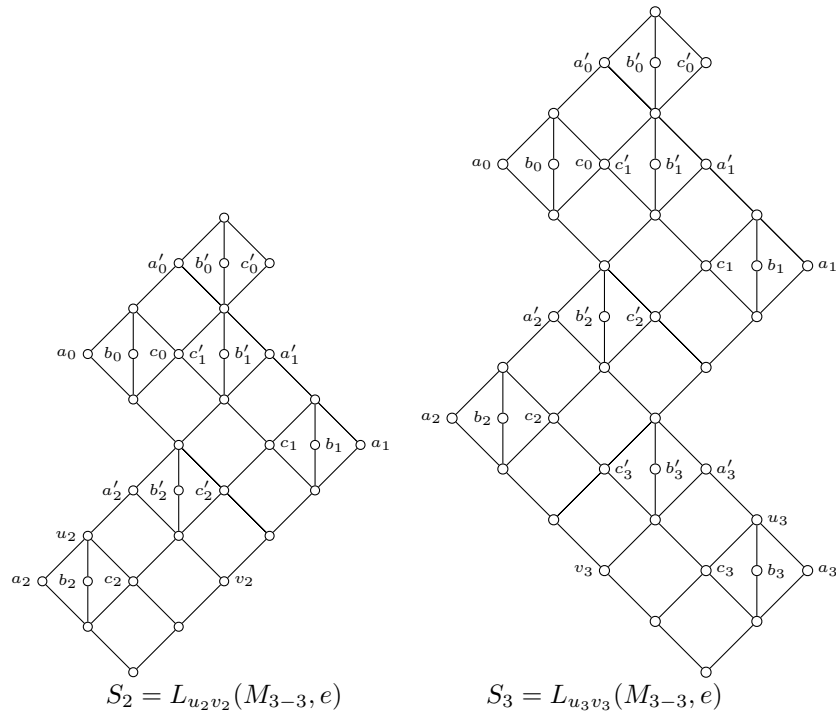


Figure 4. Lattices S_2, S_3

Lemma 1. Let $\mathcal{R}$ be a quasivariety not containing $M_{3,3}$. If $S_n \in \mathcal{R}$ for some $n \geq 0$, then $(a_0, b_0) \in \theta$ for every non-trivial $\mathcal{R}$ -congruence θ on S_n .

Proof. We proceed by induction on n .

For $n = 0$: $S_0 \cong M_{3-3}$. The lattice M_{3-3} has exactly two non-trivial congruences, let us call them α and β , whose quotients are isomorphic with $M_{3,3}$ and $\mathbf{2}$. Since $M_{3,3} \notin \mathcal{R}$, α cannot be an

$\mathcal{R}$ -congruence. Therefore, the lattice of $\mathcal{R}$ -congruences of M_{3-3} has exactly three congruences: the diagonal Δ , β , and the universal congruence ∇ . Any non-trivial $\mathcal{R}$ -congruence θ must therefore contain β , and since $(a_0, b_0) \in \beta$, the conclusion clearly holds.

For $n > 0$: Assume the statement holds for S_n . Let θ be a non-trivial $\mathcal{R}$ -congruence on S_{n+1} . Since θ is non-trivial, there exists a covering pair $x \prec y$ in S_{n+1} such that $(x, y) \in \theta$.

Recall that S_{n+1} is constructed by gluing a new M_{3-3} block, which we denote as $M^{(n+1)}$, to S_n . Based on the placement of x and y , we consider four exhaustive cases:

1. $x, y \in S_n$. In this case, both elements belong to the base lattice. The restriction $\theta|_{S_n}$ is a non-trivial $\mathcal{R}$ -congruence on S_n . By the induction hypothesis, we immediately obtain $(a_0, b_0) \in \theta|_{S_n} \subseteq \theta$.
2. $x, y \in M^{(n+1)}$. The restriction $\theta|_{M^{(n+1)}}$ is a non-trivial $\mathcal{R}$ -congruence on $M^{(n+1)} \cong M_{3-3}$. As shown in the base case, any non-trivial $\mathcal{R}$ -congruence on M_{3-3} must contain the congruence β , which collapses the elements of the gluing interval. Since the gluing interval belongs to S_n , θ inevitably identifies distinct elements within S_n . This makes the restriction $\theta|_{S_n}$ non-trivial. By the induction hypothesis, $(a_0, b_0) \in \theta$.
3. $y \in M^{(n+1)}$ and $x \notin M^{(n+1)}$. By taking the join with the zero element of S_n (denoted 0_{S_n}), the congruence yields $(x \vee 0_{S_n}, y \vee 0_{S_n}) \in \theta$. This operation projects the pair such that $y \vee 0_{S_n} = v_n \wedge u_n$ and $x \vee 0_{S_n} = 0_{S_n}$ where both resulting elements $0_{S_n}, v_n \wedge u_n$ lie strictly in S_n and are distinct. This implies that the restriction $\theta|_{S_n}$ is non-trivial. Thus, by the induction hypothesis, $(a_0, b_0) \in \theta$.
4. $x, y \notin M^{(n+1)}$. By taking the join with the zero element of M_{3-3} (denoted $0_{M_{3-3}}$), we get $(x \vee 0_{M_{3-3}}, y \vee 0_{M_{3-3}}) \in \theta$. These projected elements are distinct and both lie entirely within $M^{(n+1)}$. This means the congruence non-trivially restricts to $M^{(n+1)}$, which reduces our problem to Case 2. Hence, $(a_0, b_0) \in \theta$.

The case $y \notin S_n$ implies $x \notin S_n$. The case where $y \in S_n$ and $x \notin S_n$ implies $y = 0_{S_n}$ or $y \in M^{(n+1)}$. These both cases are covered by cases 2, 3 and 4. This strictly exhausts all possibilities for the covering pair $x \prec y$, completing the proof. $\square$

We construct a modular lattice K_m for S_n by induction on m .

1. $m = 0$. $K_0 = S_n$.
2. $m = 1$. K_1 is a modular lattice generated by $K_0 \cup \{a_{n+1}, b_{n+1}, c_{n+1}\}$ satisfying $\langle \{a_{n+1}, b_{n+1}, c_{n+1}\} \rangle \cong M_3$ and $c_{n+1} = (a_n \vee c_n) \wedge a_{n-1}$, $a_{n+1} \vee b_{n+1} = a'_n \wedge a_{n-1}$ and $a_{n+1} \wedge u, b_{n+1} \wedge u \notin K_0$ for any $u \in K_1$.
3. $m > 1$. K_m is a modular lattice generated by $K_{m-1} \cup \{a_{n+m}, b_{n+m}, c_{n+m}\}$ satisfying $\langle \{a_{n+m}, b_{n+m}, c_{n+m}\} \rangle \cong M_3$ and $c_{n+m} = (a_n \vee c_n) \wedge a_{n+m-1}$, $a_{n+m} \vee b_{n+m} = a_{n+m-1} \wedge c_{n+m-1}$ and $a_{n+m} \wedge u, b_{n+m} \wedge u \notin K_0$ for any $u \in K_{m-1}$.

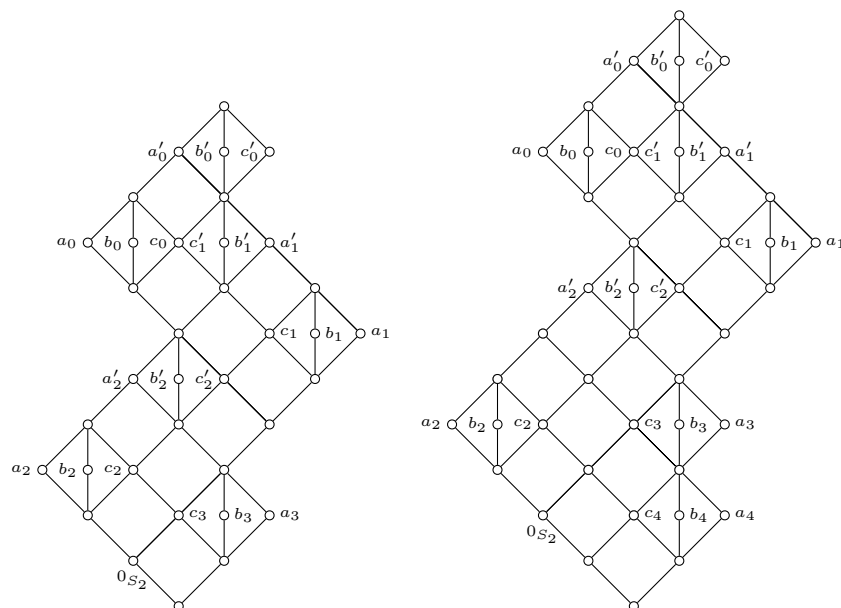
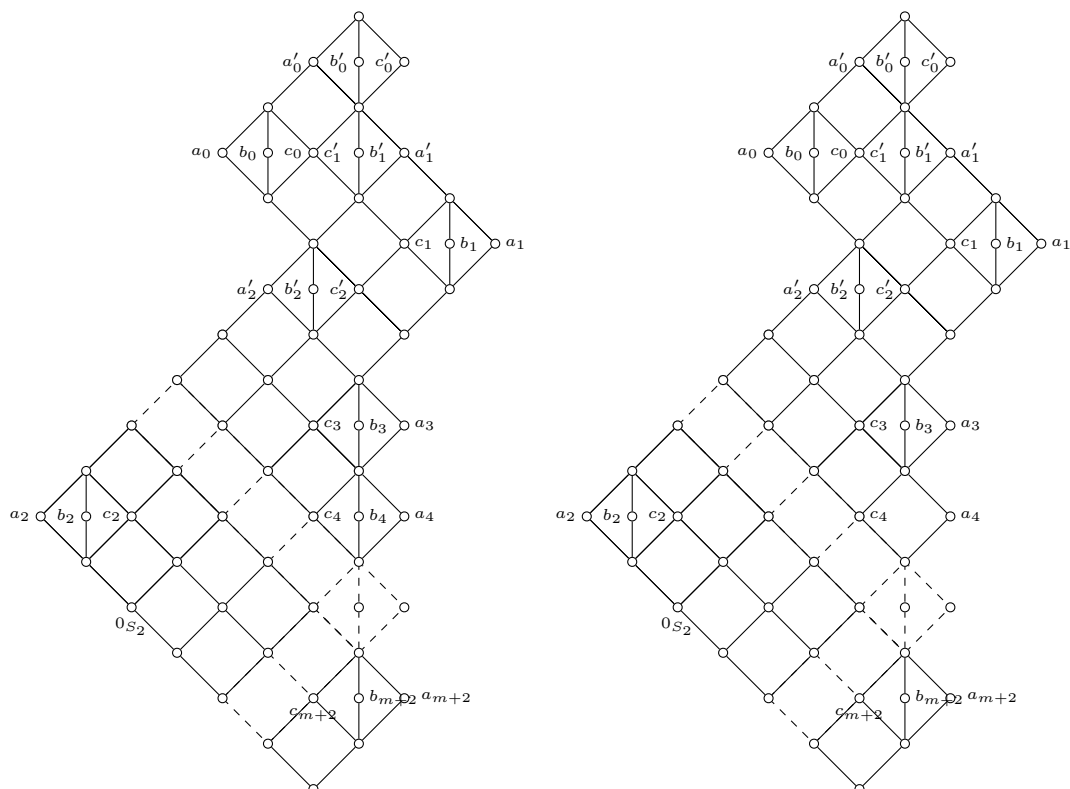
Equivalently, the lattice K_m may be described inductively by splitting the lattice K_{m-1} by M_3 as follows:

1. $m = 0$. $K_0 = S_n$.
2. $m = 1$. $K_1 = L_{u_1 v_1}(M_3, c)$ with $L = K_0$, where c is a coatom of M_3 and $(u_1, v_1) = (a_{n+1} \wedge c_{n+1}, a_n \vee c_n)$.
3. $m > 1$. $K_m = L_{u_m v_m}(M_3, c)$ with $L = K_{m-1}$ and $(u_m, v_m) = (a_{n+m} \wedge c_{n+m}, a_n \vee c_n)$.

For instance, the lattices K_1, K_2 for S_2 are depicted in Figure 5 and the lattice K_m for S_2 is depicted in Figure 6. We observe that (u_1, v_1) is a splitting pair in S_n , whereas (u_i, v_i) , $i > 1$ is a semi-splitting pair in K_{m-1} .

Lemma 2. Let $\mathcal{R}$ be the quasivariety generated by S_n . Then, for every $m \geq 1$, we have:

- i) $(a_0, b_0) \in \theta_{\mathcal{R}}(K_m)$ and $K_m \notin \mathcal{R}$;
- ii) Every subalgebra generated by m elements of K_m belongs to $\mathcal{R}$.


 Figure 5. Lattices K_1, K_2 for S_2

 Figure 6. Lattices $K_m, K_m \setminus \{b_4\}$ for S_2

Proof. i) We proceed by induction on m .

For $m = 1$: $K_1 = L_{u_1 v_1}(M_3, c)$ with $L = K_0$, where c is a coatom of M_3 and $(u_1, v_1) = (a_{n+1} \wedge$

$c_{n+1}, a_n \vee c_n$). Since K_1 is obtained by gluing M_3 , which we denote as $M^{(1)}$, to S_n , then $S_n \leq K_m$. Let θ be an $\mathcal{R}$ -congruence on K_1 and $(x, y) \in \theta$ with $x \prec y$.

Based on the placement of x and y , we consider two exhaustive cases:

1. If $x \in S_n$, then $y \in S_n$, and $(a_0, b_0) \in \theta$ by Lemma 1.
2. If $x \notin S_n$, then $x \in \{a_{n+1}, b_{n+1}, a_{n+1} \wedge b_{n+1}, 0_{K_1}\}$.

If $y \in M^{(1)}$, then θ collapses the elements of the gluing interval. If $y \notin M^{(1)}$, then θ collapses the elements 0_{S_n} and 0_{K_1} , hence two elements from $M^{(1)}$. It follows θ collapses the gluing interval. and we are in Case 1. Thus we get that $(a_0, b_0) \in \theta$ for any $\mathcal{R}$ -congruence θ .

Since $(a_0, b_0) \in \theta$ for every $\mathcal{R}$ -congruence on K_1 , then K_1 is subdirectly $\mathcal{R}$ -irreducible. Since every subdirectly $\mathcal{R}$ -irreducible lattice is a sublattice of S_n , it would follow $K_1 \leq S_n$. It's impossible because of the size of K_1 is strictly greater than the size of S_n . Therefore $K_1 \notin \mathcal{R}$.

For $m > 1$: We show that $(a_0, b_0) \in \theta$, where $\theta = \theta_{\mathcal{R}}(x, y)$ with $x \prec y$.

It is straightforward to verify that $\varphi : S_n \rightarrow K_m$ defined by $\varphi(x) = x$ for $x \in \{a_i, a'_i, b_i, b'_i, c_i, c'_i\}$, $i = 0, 1, \dots, n$, is an embedding.

Based on the placement of x and y relatively to S_n , we have the following four exhaustive cases:

1. Let $x, y \in S_n$. Then Lemma 1 yields $(a_0, b_0) \in \theta$.
2. Let $x \in S_n$ and $y \notin S_n$. Then $x = \{a_n \vee b_n, c_n, a_{n-1} \wedge c_n\}$ and $y = x \vee a_{n+m}$. It follows that $(a_{n+m}, a_{n+m} \wedge b_{n+m}) \in \theta$ as $x \wedge a_{n+m} = a_{n+m} \wedge b_{n+m}$ and $y \wedge a_{n+m} = a_{n+m}$. Hence, for $\theta' = \theta(x, y)$, we have $K_{m+1}/\theta' \cong K_m$ and $\theta_{\mathcal{R}}(x/\theta', y/\theta') = \theta_{\mathcal{R}}(\Delta) = \theta_{\mathcal{R}}(K_m)$, since $\theta' \subseteq \theta$. By induction $(a_0, b_0) \in \theta_{\mathcal{R}}(K_m)$. Therefore $(a_0, b_0) \in \theta$.
3. Let $x \notin S_n$ and $y \in S_n$. Then $y \in \{a'_n, a'_n \wedge b'_n, a'_n \wedge a_{n-1}, c_n \wedge a_{n-1}, a_n \wedge a_{n-1}\}$.
 - (a) If $y = a'_n$ or $y = a'_n \wedge b'_n$, then $x = a_n \vee c_{n+1}$ or $x = c_n \vee c_{n+1}$ respectively. It follows that $(a_{n+1}, a_{n+1} \wedge b_{n+1}) \in \theta$ as $x \wedge a_{n+1} = a_{n+1} \wedge b_{n+1}$ and $y \wedge a_{n+1} = a_{n+1}$.
 - (b) If $y = a'_n \wedge a_{n-1} = a_{n+1} \vee b_{n+1}$, then $x \in \{a_{n+1}, b_{n+1}, c_{n+1}\}$. It follows that $(a_{n+1}, a_{n+1} \wedge b_{n+1}) \in \theta$ as M_3 is a simple lattice.
 - (c) If $y \in \{c_n \wedge a_{n-1}, a_n \wedge a_{n-1}\}$, then $x = y \wedge a_{n+1}$. It follows that $(a_{n+1}, a_{n+1} \vee b_{n+1}) \in \theta$ as $x \vee a_{n+1} = a_{n+1}$ and $y \vee a_{n+1} = a_{n+1} \vee b_{n+1}$.
 Hence $K_{m+1}/\theta' \cong K_m$ and $\theta_{\mathcal{R}}(x/\theta', y/\theta') = \theta_{\mathcal{R}}(\Delta) = \theta_{\mathcal{R}}(K_m)$. By induction $(a_0, b_0) \in \theta_{\mathcal{R}}(K_m)$. Therefore $(a_0, b_0) \in \theta$.
4. Let $x, y \notin S_n$. Since K_{i+1} is obtained by splitting K_i by M_3 , we denote this lattice M_3 by $M^{(i)}$. There are two options. The first one – θ gluings two distinct elements in S_n . Then by Lemma 1 θ collapses (a_0, b_0) . The second one – θ gluings two distinct elements a_{n+i}, b_{n+i} in $M^{(i)}$ for some $i > 0$, then for $\theta' = \theta(a_{n+i}, b_{n+i})$ we have $K_{m+1}/\theta' \cong K_m$ and $\theta_{\mathcal{R}}(a_{n+i}/\theta', b_{n+i}/\theta') = \theta_{\mathcal{R}}(\Delta) = \theta_{\mathcal{R}}(K_m)$. By induction $(a_0, b_0) \in \theta_{\mathcal{R}}(K_m)$. Therefore $(a_0, b_0) \in \theta$.

Since $(a_0, b_0) \in \theta$ for every $\mathcal{R}$ -congruence on K_{m+1} , then K_{m+1} is subdirectly $\mathcal{R}$ -irreducible. Since each subdirectly $\mathcal{R}$ -irreducible lattice is a sublattice of S_n , this would imply $K_{m+1} \leq S_n$. However, it's impossible because of the size of K_{m+1} is strictly greater than the size of S_n . Therefore $K_{m+1} \notin \mathcal{R}$.

ii) The set of generators of the lattice K_m consists of

$$\{a_i, b_i, c_i, a'_i, b'_i, c'_i \mid 0 \leq i \leq n\} \cup \{a_i, b_i, c_i \mid n < i \leq n+m\}.$$

Here $\{a_i, b_i, c_i, a'_i, b'_i, c'_i \mid 0 \leq i \leq n\}$ generates S_n and $\{a_i, b_i, c_i\}$ generates M_3 for each $n < i \leq n+m$.

Let L_m be a sublattice of K_m generated by m elements. Because a_{n+i}, b_{n+i} are double irreducible elements, one necessarily has $L_m \leq K_m \setminus \{a_{n+i}\}$ or $L_m \leq K_m \setminus \{b_{n+i}\}$ for some index $0 < i \leq m$.

We show that $K_m \setminus \{a_{n+i}\}$ is a sublattice of $S_n \times M_3^{m-1}$.

Let α be a congruence on K_m generated by $\bigcup_{j \neq i} M^{(j)} \cup \{(0_{M^{(i)}}, c_{n+i})\}$.

Then $(K_m \setminus \{a_{n+i}\})/\alpha \cong S_n$.

Let β be a congruence on K_m generated by gluing all elements in S_n but asd

$$\beta = \theta\left(\bigcup_{i \leq n} \{(a_i, b_i) \mid i \leq n\}\right).$$

Then $(K_m \setminus \{a_i\})/\beta \leq \prod_{0 < j \leq m} M^{(j)}$.

We show that $\alpha \cap \beta = \Delta$. Let $x \prec y$ and $(x, y) \in \alpha \cap \beta$. By $(x, y) \in \alpha$, we get $x = a_n \vee z$ and $y = 0_{S_n} \vee z$ or $x = a_{n-1} \wedge z$ and $y = (a_{n-1} \vee b_{n-1}) \wedge z$ for some z on $K_m \setminus \{a_i\}$.

By $x \in \beta$, we get $x = 0_{k+n} \vee z'$, $y = a_{k+n} \vee z'$.

So $a_n \vee z = 0_{k+n} \vee z'$ and $0_{S_n} \vee z = a_{k+n} \vee z'$. One can see that it's impossible.

Thus $\alpha \cap \beta = \Delta$ and $K_m \setminus \{a_i\} \leq_s S_n \times M_3^{m-1}$. So $K_m \setminus \{a_i\} \in \mathcal{R}$. Since $L_m \leq K_m \setminus \{a_i\}$, then $L_m \in \mathcal{R}$. □

3 Main result

The following two theorems are the main results of this paper.

Theorem 1. A quasivariety generated by finite (3-3)-snake lattice is not finitely axiomatizable.

Proof. To establish non-finite axiomatizability, we use the following folklore lemma (see [15]):

A locally finite quasivariety $\mathcal{K}$ is not finitely axiomatizable whenever, for each positive integer n there exists a finite algebra X_n such that $X_n \notin \mathcal{K}$, and every subalgebra generated by n elements of X_n belongs to $\mathcal{K}$.

Let $\mathcal{R} = \mathbf{Q}(S_n)$. We show that for $\mathcal{R}$ the lattice $X_m = K_{m+n}$, $m > 0$, satisfies the condition of the above mentioned folklore lemma. Indeed by Lemma 2(i) $X_m \notin \mathcal{R}$ for all $m \geq 1$, by lemma 2(ii) every subalgebra generated by m elements of X_m belongs to $\mathcal{R}$. Consequently, the quasivariety $\mathcal{R}$ generated by finite (3-3)-snake lattice is not finitely axiomatizable. □

To demonstrate that the topological quasivariety generated by S_n is non-standard, we use the lemma below, derived from [2, Lemma 3.3].

Lemma 3. Let $\mathcal{K}$ be a quasivariety, and let $X = \varprojlim \{X_n \mid n \in \mathbb{N}\}$ be a surjective inverse limit of finite algebras. Assume that $X \in \mathcal{K}$ and there exist $a, b \in X$ with $a \neq b$ and $\varphi(a) = \varphi(b)$ for any homomorphism $\varphi : X \rightarrow M$, where $M \in \mathcal{K}$ and M is finite. Then $\mathcal{K}$ is not standard.

Theorem 2. The topological quasivariety generated by finite (3-3)-snake lattice is not standard.

Proof. Let $\mathcal{R} = \mathbf{Q}(S_n)$ for some (3-3)-snake lattice. Let $\varphi_{n,n-1}$ be a homomorphism from K_n to K_{n-1} such that $\ker \varphi_{n,n-1} = \theta(a_n, b_n)$, and let $\varphi_{n,n}$ be an identity map on K_n for all $n > 1$. For $m < n$, let

$$\varphi_{n,m} = \varphi_{m+1,m} \circ \cdots \circ \varphi_{n,n-1}.$$

Note that $\{K_n; \varphi_{n,m}, \mathbb{N}\}$ forms an inverse family where $\mathbb{N}$ is the linearly ordered set of positive integers. We denote $K = \varprojlim \{K_n \mid n \in \mathbb{N}\}$.

We first establish that $K \in \mathcal{R}$. Our reasoning follows the approach of [16] (cf. [11]). Consider a quasi-identity ξ of the form

$$\&_{i \leq r} p_i(x_0, \dots, x_{n-1}) \approx q_i(x_0, \dots, x_{n-1}) \rightarrow p(x_0, \dots, x_{n-1}) \approx q(x_0, \dots, x_{n-1}).$$

Suppose that ξ is valid on $\mathcal{R}$ and $K \models p_i(d_0, \dots, d_{n-1}) = q_i(d_0, \dots, d_{n-1})$ for all $i \leq r$ and some $d_0, \dots, d_{n-1} \in K$. By definition of the inverse limit, $K \leq_s \prod_{n \in \mathbb{N}} K_n$. Hence

$$K_s \models p_i(d_0(s), \dots, d_{n-1}(s)) = q_i(d_0(s), \dots, d_{n-1}(s)) \quad \text{for all } s > 0, i \leq r.$$

By Lemma 2(ii), every subalgebra generated by m elements of K_m belongs to $\mathcal{R}$ for all $s > n$. Therefore, ξ is true in K_s . Consequently,

$$K_s \models p(d_0(s), \dots, d_{n-1}(s)) = q(d_0(s), \dots, d_{n-1}(s)).$$

Since $d_i(m) = \varphi_{s,m}(d_i(s))$ for all $0 \leq i \leq n$ and $m < s$, it follows that

$$K_m \models p(d_0(m), \dots, d_{n-1}(m)) = q(d_0(m), \dots, d_{n-1}(m)) \quad \text{for all } m < s.$$

Hence, $K \models p(d_0, \dots, d_{n-1}) = q(d_0, \dots, d_{n-1})$. Thus $K \models \xi$, for every ξ that is valid on $\mathcal{R}$. Consequently, $K \in \mathcal{R}$.

From the definition of $\varphi_{n,n-1}$ it follows that $\varphi_{n,m}(a_0) = a_0$ and $\varphi_{n,m}(b_0) = b_0$. By the definition of inverse limit,

$$a = (a_0, \dots, a_0, \dots), \quad b = (b_0, \dots, b_0, \dots) \in K.$$

Let $\varphi : K \rightarrow M$ be a homomorphism, $M \in \mathcal{R}$ and M finite. By the universal property of inverse limit, there exists $n > 2$ and homomorphism $\psi_M : K_n \rightarrow M$ such that $\varphi = \varphi_n \circ \psi_M$ for some surjective homomorphism $\varphi_n : K \rightarrow K_n$.

Since $M \in \mathcal{R}$, we have $\ker \psi_M$ is an $\mathcal{R}$ -congruence on K_n .

By Lemma 2(i), we have $(a_0, b_0) \in \theta_{\mathcal{R}}(K_n)$. Since $\theta_{\mathcal{R}}(K_n) \in \ker \psi_M$, it follows that $\psi_M(a) = \psi_M(b)$, and hence $\varphi(a) = \varphi(b)$.

Applying Lemma 3, we conclude that $\mathcal{R}$ is not standard. $\square$

4 Generalization, examples, and corollary

By $h(L)$ we denote the height of a lattice L , that is, the maximum length among all chains in L .

Theorem 3. Let L be a finite lattice with $h(L) = k$. If there is a (3-3)-snake lattice $S_n \leq L$ and K_1 for S_n such that $h(K_1) \geq h(L)$ and $K_1 \notin \mathbf{Q}(L)$, then $\mathbf{Q}(L)$ is not finitely axiomatizable as well as not standard.

Sketch of the proof. It is straightforward to check that the lattices K_m from Lemma 2 satisfy the following conditions:

- i) $K_m \notin \mathbf{Q}(L)$ and $(a_0, b_0) \in \theta_{\mathbf{Q}(L)}(K_m)$;
- ii) Every subalgebra generated by m elements of K_m belongs to $\mathbf{Q}(S_n) \subseteq \mathbf{Q}(L)$.

Using arguments analogous to those in Theorems 1 and 2, the required statement follows.

Example 3. Let L be the lattice shown in Figure 7. It is clear that S_1 is a proper sublattice of L and $M_{3,3} \not\leq L$. Moreover, K_1 for S_1 does not belong to $\mathbf{Q}(L)$. Therefore, by Theorem 3, $\mathbf{Q}(L)$ is neither finitely axiomatizable nor standard. Note also that $\mathbf{Q}(S_1) \subset \mathbf{Q}(L)$, and, in addition, $\mathbf{V}(S_1) \subset \mathbf{V}(L)$.

The reader can construct many other examples that satisfy the conditions of Theorems 1 and 2.

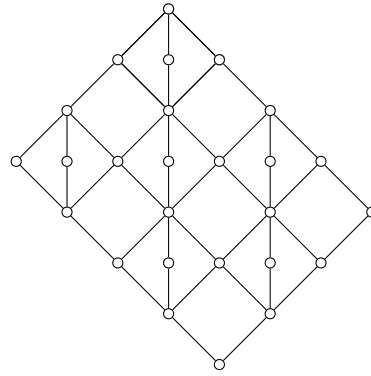
A direct corollary from Theorem 1 is V. P. Belkin's result in [4]:

Corollary 1. [4] A quasivariety generated by M_{3-3} is not finitely axiomatizable.

The reader can easily construct $(m-n)$ -snake lattices and adapt the proofs of the theorems to such lattices.

Conclusion

The results of this work contribute to the study of finite lattices whose quasi-identities resist finite axiomatizability and whose associated Boolean topological algebras do not exhibit profiniteness. We established the principal results, provided a generalization of the construction, and illustrated its applicability through an example and a corollary. These findings emphasize the ongoing significance

Figure 7. Lattice L

of investigating finite lattices with atypical quasi-equational behavior and show how such examples continue to shape and refine our understanding of nonstandard quasivarieties. The results presented here reinforce the view that even comparatively small finite lattices may encode complex logical properties, motivating further research into the connection between algebraic structure, axiomatizability, and topological properties.

Acknowledgments

The authors are grateful to the referee for the careful reading of the manuscript and for the valuable comments and suggestions which significantly improved the presentation of the paper. The authors are also grateful to A. M. Nurakunov for his attention and useful remarks.

Author contributions

All authors contributed equally to this work.

Conflict of interest

The authors declare no conflict of interest.

References

- 1 Gorbunov, V.A., & Smirnov, D.M. (1979). Finite algebras and the general theory of quasivarieties. *Colloquia Mathematica Societatis Janos Bolyai*, 28, Finite Algebra and Multiple-Valued Logic, Szeged (Hungary), 325–332.
- 2 Clark, D.M., Davey, B.A., Jackson, M.G., & Pitkethly, J.G. (2008). The axiomatizability of topological prevarieties. *Advances in Mathematics*, 218(5), 1604–1653. <https://doi.org/10.1016/j.aim.2008.03.020>
- 3 McKenzie, R. (1970). Equational bases for lattice theories. *Mathematica Scandinavica*, 27, 24–38. <https://doi.org/10.7146/math.scand.a-10984>
- 4 Belkin, V.P. (1978). Quasi-identities of finite rings and lattices. *Algebra and Logic*, 17, 171–179. <https://doi.org/10.1007/BF01670283>
- 5 Tumanov, V.I. (1984). Finite lattices having no independent bases of quasi-identities. *Mathematical Notes*, 36, 811–815. <https://doi.org/10.1007/BF01139925>

- 6 Dziobiak, W. (1989). Finitely generated congruence distributive quasivarieties of algebras. *Fundamenta Mathematicae*, 133, 47–57. <https://doi.org/10.4064/fm-133-1-47-57>
- 7 Clark, D.M., Davey, B.A., Freese, R.S., & Jackson, M. (2005). Standard topological algebras: syntactic and principal congruences and profiniteness. *Algebra Universalis*, 52(2–3), 343–376. <https://doi.org/10.1007/s00012-004-1917-6>
- 8 Nurakunov, A.M., & Stronkowski, M.M. (2018). Profiniteness in finitely generated varieties is undecidable. *The Journal of Symbolic Logic*, 83(4), 1566–1578. <https://doi.org/10.1017/jsl.2017.89>
- 9 Nurakunov, A.M., & Schwidefsky, M.V. (2024). Profinite locally finite quasivarieties. *Studia Logica*, 112, 835–859. <https://doi.org/10.1007/s11225-023-10077-y>
- 10 Basheyeva, A.O., & Lutsak, S.M. (2023). On quasi-identities of finite modular lattices. II. *Bulletin of the Karaganda University. Mathematics Series*, 2(110), 45–52. <https://doi.org/10.31489/2023M2/45-52>
- 11 Lutsak, S.M., Basheyeva, A.O., Asanbekov, A.M., & Voronina, O.A. (2023). Some non-standard quasivarieties of lattices. *Bulletin of the Karaganda University. Mathematics Series*, 3(111), 72–80. <https://doi.org/10.31489/2023m3/72-80>
- 12 Lutsak, S.M., & Voronina, O.A. (2022). On some properties of quasivarieties generated by specific finite modular lattices. *Bulletin of L.N. Gumilyov ENU. Mathematics. Computer Science. Mechanics Series*, 140(3), 6–14. <https://doi.org/10.32523/2616-7182/bulmathenu.2022/3.1>
- 13 Kravchenko, A., Nurakunov, A., & Schwidefsky, M. (2021). Structure of quasivariety lattices. IV. Nonstandard quasivarieties. *Siberian Mathematical Journal*, 62, 850–858. <https://doi.org/10.33048/smzh.2021.62.507>
- 14 Burris, S., & Sankappanavar, H.P. (1980). *A course in universal algebra*. Graduate Texts in Mathematics (Vol. 78). New York: Springer.
- 15 Gorbunov, V.A. (1998). *Algebraic theory of quasivarieties*. New York: Consultants Bureau.
- 16 Basheyeva, A.O., Mustafa, M., & Nurakunov, A.M. (2020). Properties not retained by pointed enrichments of finite lattices. *Algebra Universalis*, 81(4), Article 56. <https://doi.org/10.1007/s00012-020-00692-4>

*Author information**

Meruyert Alimbay kyzy Arapbay — PhD student, L.N. Gumilyov Eurasian National University, Astana 010008, Kazakhstan; e-mail: meruertarapbay@gmail.com; <https://orcid.org/0009-0008-7407-4498>

Amantai Mukanmedievich Asanbekov (*corresponding author*) — Researcher, Institute of Mathematics of the National Academy of Sciences, Bishkek 720048, Kyrgyz Republic; e-mail: amantai.asanbekov.1@gmail.com; <https://orcid.org/0009-0009-6254-3071>

*Authors' names are presented in the following order: first name, middle name (if any), last name.