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Qualitative analysis of a system of non-homogeneous doubly nonlinear parabolic equations

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We consider the qualitative properties of solutions to a coupled system of nonhomogeneous doubly nonlinear parabolic equations on the whole line with an exponentially varying density. The characteristic features of degeneracy at vanishing values and gradients are analyzed, and the need for weak solutions and reliable comparison estimates is identified and justified. Using a nonlinear splitting method, we construct explicit comparison functions and, on this basis, apply a comparison principle to obtain global existence of nonnegative solutions for sufficiently small initial data in the slow-diffusion regime. In addition, a self-similar reduction is performed via a nonlinear change of variables, which converts the problem into an auxiliary system for similarity profiles. An asymptotic representation of these self-similar solutions is derived, and the dependence of the solution behaviour on the governing parameters is clarified. It is shown how the parameters affect spatial localization and finite-speed propagation, and a Fujita-type criterion is obtained that provides conditions for the existence and nonexistence of global solutions. To support the analytical results, numerical simulations implemented in Python produce solution profiles and graphical illustrations of the nonlinear diffusion dynamics. The computations agree with the qualitative predictions and help visualize the transition between parameter regimes.

Keywords: doubly nonlinear parabolic system, strong-coupling diffusion, exponential density, weak solutions, self-similar solution, comparison principle, Barenblatt profile, global solvability, asymptotic behaviour, numerical illustrations.

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Introduction

We investigate a system of parabolic partial differential equations expressed in divergence form, defined on the domain $Q = \{(t,x)\,|\,t>0, x\in\mathbb{R}\}$:

$$\begin{cases} \rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{\sigma_1}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + \rho(x)v^{q_1}, \\ \rho(x)\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(v^{\sigma_2}\left|\frac{\partial v}{\partial x}\right|^{p-2}\frac{\partial v}{\partial x}\right) + \rho(x)u^{q_2}, \end{cases} \tag{1}$$

$$\begin{cases} u\big|_{t=0} = u_0(x) \geq 0, \\ v\big|_{t=0} = v_0(x) \geq 0, \end{cases} \qquad \forall x\in\mathbb{R}, \tag{2}$$

where $p\geq 2,\ \sigma_i\geq 0,\ q_i>0\ (i=1,2),\ \ q_1q_2\neq 1,\ \alpha\in\mathbb{R},\ \rho(x)=e^{\alpha x}$ are the numerical parameters and $u=u(t,x)\geq 0,\ v=v(t,x)\geq 0$.

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The system (1)–(2) arises in the mathematical modelling of several nonlinear processes observed in applied sciences, including heat propagation [1–3], nonlinear diffusion in a two-component medium [4–6], gas filtration through porous structures [7, 8], and fluid dynamics in heterogeneous domains. Furthermore, system (1)–(2) describes many physical processes [9–11].

H. Murakawa [12] investigates the connection between cross-diffusion and reaction-diffusion systems, examining their structure and the ways in which these two classes of systems are interconnected. The author offers new considerations on how to view cross-diffusion terms in mathematical models of species interactions or chemical processes as a special case of reaction-diffusion equations. This is beneficial to understand how complex dynamics behave in systems with multiple components.

X. Xu and T. Cheng studied a strongly coupled nonlinear filtration system with nonlocal source terms [13], arising in non-Newtonian fluid flow and a polytropic filtration system:

$$\begin{cases} u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) + \int_\Omega u^\alpha\,dx \int_\Omega v^r\,dx, & (x,t) \in \Omega \times (0,\infty), \\ v_t = \operatorname{div}(|\nabla v^n|^{q-2}\nabla v^n) + \int_\Omega u^s\,dx \int_\Omega v^\beta\,dx, & (x,t) \in \Omega \times (0,\infty), \end{cases}$$

where $p, q > 1$, $m, n, r, s > 0$, $\alpha, \beta \geq 0$, $m(p-1), n(p-1) < 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and the non-negative non-trivial initial data (u_0, v_0) satisfy $(u_0^m, v_0^n) \in (L^\infty(\Omega) \cap W_0^{1,p}(\Omega), L^\infty(\Omega) \cap W_0^{1,q}(\Omega))$. The authors establish explicit conditions ensuring finite-time extinction of solutions. That is, if the parameters satisfy a certain balance condition between diffusion and reaction, the solution satisfies:

$$u(x,t) = v(x,t) = 0, \quad \forall\, t \geq T_{\text{ext}},$$

for some finite T_{ext}. If extinction does not occur, the authors derive the long-time decay rate for solutions:

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} \leq Ct^{-\gamma},$$

where γ depends on the exponents $m, n, p, q, \alpha, r, s, \beta$.

X. Sun, B. Liu, and F. Li [14] considered the following system of parabolic equations with a time-dependent source

$$\begin{cases} u_t = \Delta u + t^{\sigma_1}\left(1+|x|^2\right)^{n/2} u^\alpha v^p, \\ v_t = \Delta v + t^{\sigma_2}\left(1+|x|^2\right)^{m/2} u^q v^\beta, \ x \in \Omega \subset \mathbb{R}^N, \ 0 < t < T. \\ u = v = 0, \ x \in \partial\Omega, \ 0 < t < T, \\ u\,(x,0) = u_0\,(x) \geq 0, \ v\,(x,0) = v_0\,(x) \geq 0, \ x \in \Omega, \end{cases}$$

where p, q, α, β, m, n, σ_1, σ_2 are non-negative numbers, and T is the limit of the existence time of classical solutions of the problem. X. Sun et al. have shown that the problem admits global solutions when the conditions $pq \leq (1-\alpha)(1-\beta)$, $\alpha \leq 1$, $\beta \leq 1$ are satisfied; They further proved that solutions blow up in finite time when the inequality $(1-\alpha)(1-\beta) < pq \leq (pq)_c$ or $1 < \alpha < \alpha_c$ or $1 < \beta \leq \beta_c$ is satisfied; $pq > (pq)_c$, $\alpha > \alpha_c$, $\beta > \beta_c$ and proved that both global and non-global solutions exist under the conditions where

$$\begin{aligned} &(pq)_c = (1-\alpha)(1-\beta) + \frac{2}{N}\max\left\{\sigma\,(p, 1-\beta), \sigma\,(1-\alpha, q), 0\right\}, \\ &\alpha_c = 1 + \frac{n + 2\,(\sigma_1+1)}{N}, \beta_c = 1 + \frac{n + 2\,(\sigma_2+1)}{N}, \\ &\sigma\,(a,b) = a(1+\frac{m}{2})(1+\sigma_2) + b(1+\frac{n}{2})(1+\sigma_1). \end{aligned}$$

R. Castillo, et al. [15] considered the following system in a time-dependent, heterogeneous environment

$$\begin{cases} u_t = \operatorname{div}\left(w(x)\nabla u\right) + t^r v^p, \\ v_t = \operatorname{div}\left(w(x)\nabla v\right) + t^s v^p,\ x \in \mathbb{R}^N,\ 0 < t < T, \\ u\left(x,0\right) = u_0\left(x\right),\ v\left(x,0\right) = v_0\left(x\right),\ x \in \mathbb{R}^N, \end{cases}$$

where $0 \le u_0,\ v_0 \in L^\infty\left(\mathbb{R}^N\right)$,

$$p, q > 0,\ pq > 1,\ r, s > -1$$

and $w(x) = |x|^a$, $a > 0$. R. Castillo et al. showed that the Cauchy problem has local and global solutions, and that for the solution

$$||u(.,t)||_\infty \le C(1+t)^{-\frac{N}{(2-\alpha)r_1}}, \quad ||v(.,t)||_\infty \le C(1+t)^{-\frac{N}{(2-\alpha)r_2}}$$

proved that the bounds are valid, where $r_1 = \frac{N(pq-1)}{(2-\alpha)(r+1+p(s+1))}$, $r_2 = \frac{N(pq-1)}{(2-\alpha)(s+1+q(r+1))}$.

X. Tao and Z.B. Fang [16], L.E. Payne, and G.A. Philippin [17], and the following system with a time-dependent function was considered

$$\begin{cases} u_t = \Delta u + k_1(t) u^p v^q, \\ v_t = \Delta v + k_2(t) u^s v^r,\ x \in \Omega \subset \mathbb{R}^N,\ 0 < t < T, \\ u = v = 0,\ x \in \partial\Omega,\ 0 < t < T, \\ u\left(x,0\right) = u_0\left(x\right),\ v\left(x,0\right) = v_0\left(x\right),\ x \in \Omega, \end{cases}$$

where $p,\ q,\ r,\ s \ge 0,\ k_1(t), k_2(t) \in C^1$ are positive, and $u_0(x), v_0(x) \in C^1$ are non-negative functions. The authors showed that there exist global solutions to the Cauchy problem for sufficiently small initial functions under the conditions $p + q \le 1,\ r + s \le 1$ or $p > 1,\ r + s \ge 1$, that for sufficiently large initial functions under the conditions $p > 1,\ r + s \ge 1$ the solutions to the problem tend to infinity in a finite time, and that Sobolev estimates were obtained.

In [18], self-similar and approximate self-similar solutions to a nonlinear reaction-diffusion problem were studied by Sh.A. Sadullaeva:

$$\begin{cases} \frac{\partial(\rho(x)u)}{\partial t} = \operatorname{div}\left(|x|^n v^{m_1-1}|\nabla u|^{p-2}\nabla u\right) + \rho(x)\gamma(t)u^{\beta_1}, \\ \frac{\partial(\rho(x)v)}{\partial t} = \operatorname{div}\left(|x|^n u^{m_2-1}|\nabla v|^{p-2}\nabla v\right) + \rho(x)\gamma(t)v^{\beta_2}, \\ u(0,x) = u_0(x) \ge 0, \quad v(0,x) = v_0(x) \ge 0, \quad x \in \mathbb{R}^N, \end{cases}$$

where $m_i, n \in \mathbb{R}$, $\beta_i \ge 1,\ p \ge 2$ were given positive numbers, and $u_0(x),\ v_0(x) \ge 0$, $\rho(x) = |x|^{-l}$, $l > 0,\ 0 < \gamma(t) \in C(\mathbb{R}_+),\ i = 1, 2$.

In [19], D.B. Nigmanova examined the Fujita-type [20] global existence and blow-up conditions for a nonlinear parabolic system with initial conditions and variable density. Moreover, the author studied solution estimates and the asymptotic behaviour of self-similar solutions under slow, fast, and critical diffusion cases, highlighting the role of spatially varying density:

$$\begin{cases} |x|^{-l}\frac{\partial u}{\partial t} = \nabla\left(|x|^n|\nabla u^k|^{p-2}\nabla u^{l_1}\right) + \varepsilon|x|^{-l}u^{p_1}v^{q_1}, \\ |x|^{-l}\frac{\partial v}{\partial t} = \nabla\left(|x|^n|\nabla v^k|^{p-2}\nabla v^{l_2}\right) + \varepsilon|x|^{-l}u^{p_2}v^{q_2}, \\ u(0,x) = u_0(x) \ge 0, \quad v(0,x) = v_0(x) \ge 0, \quad x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon = \pm 1,\ k,\ m_i,\ q_i,\ l_i \ge 1,\ i = 1, 2,\ p \ge 2,\ n,\ l$ are given parameters.

In [21, 22], a nonlinear parabolic equation involving a source term and non-uniform density was investigated by the authors in the following form:

$$\rho(x)\frac{\partial u}{\partial t} = \mathrm{div}\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right) + u^{\beta},$$

and

$$\rho(x)\frac{\partial u}{\partial t} = \mathrm{div}\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right) + \rho(x)u^{\beta}, \quad x \in \mathbb{R}^N,$$

where $\rho(x) = |x|^{-n}$ or $\rho(x) = (1+|x|)^{-n}$, $n \geq 0$.

The authors established criteria for finite-time blow-up of solutions to the Cauchy problem.

D. Aronson and J. Graveleau focus on the porous medium equation and derive self-similar solutions to describe the hole-filling phenomenon [23]. The authors rewrite the radial dynamics using the logarithmic transformation $s = \log r$. This substitution converts the governing system into a weighted porous-medium equation

$$\rho(x)u_t = (u^m)_{xx} \tag{3}$$

defined on $Q = \mathbb{R}_+ \times \mathbb{R}^N$, characterized by the exponential density function $\rho(x) = e^{2x}$. This exponential structure is fundamental for determining support of the solution, $[-a, \infty)$, and analyzing the finite-time loss of the inner interface, where a is a free positive parameter describing the initial left endpoint of the support in the x variable or $r_0 = e^{-a} > 0$.

For the nonlinear equation, the density functions of the medium encountered in the $\rho(x) = \{|x|^{-\alpha}, e^{-x}, e^{-x^2}\}$ forms in the work of V. Galaktionov and J. King [24] relate to the asymptotic behaviour of blow-up solutions of the equation (3) with the Cauchy problem.

D. Andreucci and A. Tedeev take the density function as $\rho(x) = e^{g(|x|)}$ and proved for the solutions sup estimates or the decay rate at infinity, the property of finite speed of propagation and support estimates for the following equation [25]:

$$\rho(x)\frac{\partial u}{\partial t} = \mathrm{div}\big(\rho(x)u^{m-1}|\nabla u|^{p-2}\nabla u\big),$$

with the initial condition and under some assumptions for $g(|x|)$ function.

In [26], self-similar solutions to the Cauchy problem for a doubly nonlinear equation incorporating exponential effects were investigated. The equation is given by:

$$\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{\sigma}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right), \quad (t,x) \in Q,$$

$$u\Big|_{t=0} = u_0(x) \geq 0, \quad x \in \mathbb{R},$$

where $Q = \{(t,x) : t > 0,\, x \in \mathbb{R}\}$, $p \geq 2$, $\sigma \in \mathbb{R}_+$, and $\rho(x) = e^x$.

They established the conditions for the existence of Fujita-type global solutions and identified the criteria under which a sub-solution exists for the equation.

The system (1)–(2) may be degenerate at the points where $u = 0$ or $\frac{\partial u}{\partial x} = 0$ and $v = 0$ or $\frac{\partial v}{\partial x} = 0$ [27–29]. Given that classical solutions may not exist in general, we focus on non-negative weak solutions defined by the following weak formulation.

Definition 1. A non-negative function $u(t,x)$ and $v(t,x)$ are a weak solution to the problem (1)–(2) in Q if for any compact subset $\Omega \subset \mathbb{R}$ and any sub-interval $[t_1, t_2] \subset (0,T)$:

$$0 \leq u, v \in C\left(0, T : L_2(\Omega)\right), \; u^{\frac{p+\sigma_1}{p}}, \, v^{\frac{p+\sigma_2}{p}} \in L_p\left(0, T : W^1_{p,0}(\Omega)\right),$$

$$\int_0^T\int_{\Omega}\rho(x)v^{q_1+1}dxdt < \infty, \quad \int_0^T\int_{\Omega}\rho(x)u^{q_2+1}dxdt < \infty,$$

and for any test functions $\phi, \psi \in W_2^1\left(0, T : L_2(\Omega)\right) \cap L_p\left(0, T : W_{p,0}^1(\Omega)\right)$:

$$\int_\Omega u\phi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \left(-u\phi_t + u^{\sigma_1}\left|u_x\right|^{p-2}u_x\phi_x\right)dxdt = \int_{t_1}^{t_2}\int_\Omega v^{q_1}\phi dxdt,$$
$$\int_\Omega v\psi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \left(-v\psi_t + v^{\sigma_2}\left|v_x\right|^{p-2}v_x\psi_x\right)dxdt = \int_{t_1}^{t_2}\int_\Omega u^{q_2}\psi dxdt,$$

and take the initial data as follows

$$\lim_{t\to 0}\int_\Omega u(t,x)\varphi(x)dx = \int_\Omega u_0(x)\varphi(x)dx,$$
$$\lim_{t\to 0}\int_\Omega v(t,x)\chi(x)dx = \int_\Omega v_0(x)\chi(x)dx,$$

for any smooth compactly supported functions φ and χ.

A self-similar equation refers to a differential equation whose solution can be expressed as a function of a combination of independent variables, reflecting the scaling invariance of the process being considered. Self-similar solutions to differential equations are characterized by the fact that the solution depends on a particular combination of the independent variables, such as $\xi = xt^{-\alpha}$, rather than on each variable separately [30, 31]. This property allows the problem involving partial differential equations to be reduced to an ordinary differential equation, significantly simplifying its analysis.

1 Formulas and theorems

1.1 Formulation of a self-similar system of equations

By applying the following transformation, system (1)–(2) is reduced to the auxiliary system (4):

$$\begin{cases} u(t,x) = (t+T)^{-\alpha_1}f(\xi) \\ v(t,x) = (t+T)^{-\alpha_2}\varphi(\xi) \end{cases}, \quad \xi = (t+T)^{-\gamma}\cdot\left(\frac{p}{\alpha}e^{\frac{\alpha x}{p}}\right), \tag{4}$$

where $\alpha_i = \frac{q_i+1}{q_1q_2-1}$, $\gamma p = 1 + \alpha_i(\sigma_i + p - 2)$, $i = 1, 2$, $T > 0$, for $f(\xi)$ and $\varphi(\xi)$, we obtain the system of ordinary differential equations:

$$\begin{cases} \frac{1}{\xi^{p-1}}\frac{d}{d\xi}\left(\xi^{p-1}f^{\sigma_1}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \alpha_1 f + \gamma\xi\frac{df}{d\xi} + \varphi^{q_1} = 0, \\ \frac{1}{\xi^{p-1}}\frac{d}{d\xi}\left(\xi^{p-1}\varphi^{\sigma_2}\left|\frac{d\varphi}{d\xi}\right|^{p-2}\frac{d\varphi}{d\xi}\right) + \alpha_2\varphi + \gamma\xi\frac{d\varphi}{d\xi} + f^{q_2} = 0. \end{cases} \tag{5}$$

Based on the initial formulation of the problem, our goal is to find a non-trivial, non-negative solution to equation (5) that satisfies the following condition:

$$\begin{cases} f'(0) = 0, & f(d_1) = 0, & 0 < d_1 < \infty, \\ \varphi'(0) = 0, & \varphi(d_2) = 0, & 0 < d_2 < \infty. \end{cases}$$

1.2 Slow diffusion case: $\sigma_i + p - 2 > 0$ $(i = 1, 2)$. A global solution to the problem

A comparison principle, as presented in [32, chapter I]; [33, p. 21] is employed to prove the global existence of weak solutions to system (1)–(2). Accordingly, a new system of equations is formulated based on the standard method outlined in [32, p. 19], [34,35] with the Barenblatt profile [7]:

$$\begin{cases} u_+(t,x) = (t+T)^{-\alpha_1}\,\overline{f}(\xi), \\ v_+(t,x) = (t+T)^{-\alpha_2}\,\overline{\varphi}(\xi), \end{cases} \tag{6}$$

$$\begin{cases} \overline{f}(\xi) = A_1\left(a - \xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{\sigma_1+p-2}}, \\ \overline{\varphi}(\xi) = A_2\left(a - \xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{\sigma_2+p-2}}, \end{cases} \tag{7}$$

where $a > 0,\ A_i = \left|\frac{\gamma(\sigma_i+p-2)^{p-1}}{p^{p-1}}\right|^{\frac{1}{\sigma_i+p-2}}, \quad i = 1,2, \quad (k)_+ = \max(0,k).$

For convenience, we introduce the following notation:

$$l_i = \frac{(p-1)q_i}{\sigma_{3-i}+p-2} - \frac{p-1}{\sigma_i+p-2}, \quad m_i = A_i^{-1}A_{3-i}^{q_i}, \quad i = 1,2.$$

Theorem 1. Let $\sigma_i + p - 2 > 0,\ q_i > \dfrac{\sigma_{3-i}+p-2}{\sigma_i+p-2}, \alpha_i(\sigma_i+p-2) + m_i a^{l_i} \le 1,\ i = 1,2,$

$$u_+(0,x) \ge u_0(x), \quad v_+(0,x) \ge v_0(x), \quad x \in \mathbb{R}.$$

If the initial functions $u_0(x)$ and $v_0(x)$ are sufficiently small, then the following property holds:

$$u(t,x) \le u_+(t,x), \quad v(t,x) \le v_+(t,x) \quad \text{in } Q,$$

where $u_+(x)$ and $v_+(x)$ above-mentioned functions.

Proof. Theorem 1 is established using the solution comparison method with $u_+(x)$ and $v_+(x)$ taken as auxiliary comparison functions. By substituting expression (6) into system (1)–(2), we derive the following inequality:

$$\begin{cases} \frac{1}{\xi^{p-1}}\frac{d}{d\xi}\left(\xi^{p-1}f^{\sigma_1}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \alpha_1 f + \gamma\xi\frac{df}{d\xi} + \varphi^{q_1} & \le 0, \\ \frac{1}{\xi^{p-1}}\frac{d}{d\xi}\left(\xi^{p-1}\varphi^{\sigma_2}\left|\frac{d\varphi}{d\xi}\right|^{p-2}\frac{d\varphi}{d\xi}\right) + \alpha_2\varphi + \gamma\xi\frac{d\varphi}{d\xi} + f^{q_2} & \le 0, \end{cases} \tag{8}$$

With the specific expressions for $\overline{f}(\xi)$ and $\overline{\varphi}(\xi)$ given in (7), inequality (8) simplifies to the following form:

$$\begin{cases} -1 + \frac{1+q_1}{q_1q_2-1}(\sigma_1+p-2) + m_1\big(a - \xi^{\frac{p}{p-1}}\big)_+^{l_1} \le 0, \\ -1 + \frac{1+q_2}{q_1q_2-1}(\sigma_2+p-2) + m_2\big(a - \xi^{\frac{p}{p-1}}\big)_+^{l_2} \le 0. \end{cases}$$

It can be readily verified that

$$m_1\big(a - \xi^{\frac{p}{p-1}}\big)_+^{l_1} \le m_1 a^{l_1}, \quad m_2\big(a - \xi^{\frac{p}{p-1}}\big)_+^{l_2} \le m_2 a^{l_2}.$$

Then, taking into account the assumptions of Theorem 1, applying the comparison principle, and the initial functions satisfied the following inequality:

$$u_+(0,x) \ge u_0(x), \quad v_+(0,x) \ge v_0(x), \quad x \in \mathbb{R}.$$

We obtain the following result:

$$u(t,x) \le u_+(t,x), \quad v(t,x) \le v_+(t,x).$$

This completes the proof of the theorem. □

1.3 Asymptotic behaviour of the self-similar solutions

We now investigate the asymptotic behaviour of self-similar solutions corresponding to system (5). A self-similar solution is considered in the form:

$$f(\xi) = \overline{f}(\xi)\, y(\eta), \quad \varphi(\xi) = \overline{\varphi}(\xi)\, z(\eta), \tag{9}$$

where

$$\eta = -\ln\left(a - \xi^{\frac{p}{p-1}}\right), \quad \overline{f}(\xi) = \left(a - \xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{\sigma_1+p-2}}, \quad \overline{\varphi}(\xi) = \left(a - \xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{\sigma_2+p-2}}, \quad a > 0.$$

Under the assumption that $y(\eta) > 0$ and $z(\eta) > 0$, substitution of expression (9) into system (5) yields the following nonlinear system of equations:

$$\begin{cases} \frac{d}{d\eta}(L_1 y) + b_{11}(\eta) L_1 y + b_{12}(\eta)\left(\frac{dy}{d\eta} + b_{10}(\eta) y\right) + b_{13}(\eta) z^{q_1} + b_{14}(\eta) y = 0, \\ \frac{d}{d\eta}(L_2 z) + b_{21}(\eta) L_2 z + b_{22}(\eta)\left(\frac{dz}{d\eta} + b_{20}(\eta) z\right) + b_{23}(\eta) y^{q_2} + b_{24}(\eta) z = 0, \end{cases} \tag{10}$$

here

$$b_{i0}(\eta) = -\frac{p-1}{\sigma_i + p - 2}, \; b_{i1}(\eta) = (p-1)\Big(\frac{e^{-\eta}}{a - e^{-\eta}} - \frac{p-1}{\sigma_i + p - 2}\Big), \; b_{i2} = \gamma\Big(\frac{p-1}{p}\Big)^p,$$

$$b_{i3} = \Big(\frac{p-1}{p}\Big)^p \frac{e^{-s_i \eta}}{a - e^{-\eta}}, \; b_{i4} = \alpha_i \Big(\frac{p-1}{p}\Big)^p \cdot \frac{e^{-\eta}}{a - e^{-\eta}}, \; s_i = 1 + \frac{p-1}{\sigma_{3-i} + p - 2} q_i - \frac{p-1}{\sigma_i + p - 2} \quad (i = 1, 2),$$

$$L_1 y = y^{\sigma_1} \Big|\frac{dy}{d\eta} + b_{10}(\eta) y\Big|^{p-2} \Big(\frac{dy}{d\eta} + b_{10}(\eta) y\Big), \; L_2 z = z^{\sigma_2} \Big|\frac{dz}{d\eta} + b_{20}(\eta) z\Big|^{p-2} \Big(\frac{dz}{d\eta} + b_{20}(\eta) z\Big).$$

There was supposed to be a domain $\xi \in [\xi_0, \xi_1)$, where $0 < \xi_0 < \xi_1$, and $\xi_1 = a^{\frac{p-1}{p}}$.
Therefore, the function $\eta(\xi)$ satisfies the following properties [36, 37]:

$$\eta'(\xi) > 0 \quad \text{at } \xi \in [\xi_0, \xi_1), \quad \eta_0 = \eta(\xi_0) > 0, \quad \lim_{\xi \to \xi_1^-} \eta(\xi) = +\infty.$$

The auxiliary system of equations (10) is considered below under the following conditions:

$$\lim_{\eta \to +\infty} b_{ij}(\eta) = b_{ij}^0 \quad (i = 1, 2; \; j = 0, 1, 2, 3, 4),$$

are assumed to exist, be finite and non-negative, that is:

$$0 \le \left|b_{ij}^0\right| < +\infty.$$

To explore the behaviour of system solutions (1) as $\eta \to +\infty$, we first apply the transformations given in equations (4) and (9). This reformulates the original problem into the study of system (10), under the assumption that its solutions satisfy certain conditions in the vicinity of $+\infty$ [38, 39]. Specifically, the functions must remain positive and obey the inequalities:

$$y(\eta) > 0, \quad y' + b_{10}(\eta) y \neq 0,$$

$$z(\eta) > 0, \quad z' + b_{20}(\eta) z \neq 0.$$

Our focus now turns to examining the asymptotic properties of such positive solutions to system (10), particularly those that converge to a finite, nonzero value as $\eta \to +\infty$.

2 The main theoretical results

To simplify the presentation, we define the following notation:

$$c_{i1}=\Big(\frac{p-1}{\sigma_i+p-2}\Big)^p,\quad c_{i2}=-\gamma\cdot\frac{p-1}{\sigma_i+p-2}\cdot\Big(\frac{p-1}{p}\Big)^p,\quad c_{i3}=\Big(\frac{p-1}{p}\Big)^p\cdot\frac{1}{a}\quad (i=1,2).$$

Let $y(\eta)=y^0+o(1)$, $z(\eta)=z^0+o(1)$, $0<y^0,z^0<+\infty$, as $\eta\to+\infty$, and suppose the following equality holds:

$$(1+q_1)(\sigma_1+p-2)=(1+q_2)(\sigma_2+p-2).$$

Then the following theorems are valid.

Theorem 2. Let $s_1=s_2=0$, then the self-similar solution of equations (1) has the following asymptotic form:

$$\begin{cases} u_A(t,x)\simeq y^0(T+t)^{\frac{1+q_1}{1-q_1q_2}}\left(a-\left(\left(\frac{p}{\alpha}\cdot\frac{e^{\frac{\alpha x}{p}}}{(t+T)^\gamma}\right)^{\frac{p}{p-1}}\right)\right)^{\frac{p-1}{\sigma_1+p-2}}(1+o(1)),\\ v_A(t,x)\simeq z^0(T+t)^{\frac{1+q_2}{1-q_1q_2}}\left(a-\left(\left(\frac{p}{\alpha}\cdot\frac{e^{\frac{\alpha x}{p}}}{(t+T)^\gamma}\right)^{\frac{p}{p-1}}\right)\right)^{\frac{p-1}{\sigma_2+p-2}}(1+o(1)),\end{cases}\qquad(11)$$

as $x\to\frac{p}{\alpha}\ln\left(\frac{\alpha}{p}a^{\frac{p-1}{p}}(T+t)^\gamma\right)$, where w_1,w_2 solutions of the following system:

$$c_{i1}w_i^{\sigma_i+p-1}+c_{i2}w_i+c_{i3}w_{3-i}^{q_i}=0\quad(i=1,2).$$

Theorem 3. Let $s_1=0$ and $s_2>0$, then the self-similar solution of system (1)–(2) has the asymptotic expansion of the form of (11) as $x\to\frac{p}{\alpha}\ln\left(\frac{\alpha}{p}a^{\frac{p-1}{p}}(T+t)^\gamma\right)$, where w_1,w_2 solutions of the following system:

$$c_{11}w_1^{\sigma_1+p-1}+c_{12}w_1+c_{13}w_2^{q_1}=0,$$
$$c_{21}w_2^{\sigma_2+p-1}+c_{22}w_2=0.$$

Theorem 4. Let $s_1>0$ and $s_2=0$, then the self-similar solution of system (1)–(2) has the asymptotic expansion of the form of (11) as

$$x\to\frac{p}{\alpha}\ln\left(\frac{\alpha}{p}a^{\frac{p-1}{p}}(T+t)^\gamma\right),$$

where w_1,w_2 solutions of the following system:

$$c_{11}w_1^{\sigma_1+p-1}+c_{12}w_1=0,$$
$$c_{21}w_2^{\sigma_2+p-1}+c_{22}w_2+c_{23}w_1^{q_2}=0.$$

Theorem 5. Let $s_1>0$ and $s_2>0$, then the self-similar solution of system (1)–(2) has the asymptotic expansion of the form of (11) as

$$x\to\frac{p}{\alpha}\ln\left(\frac{\alpha}{p}a^{\frac{p-1}{p}}(T+t)^\gamma\right),$$

where w_1,w_2 solutions of the following system:

$$c_{i1}w_i^{\sigma_i+p-1}+c_{i2}w_i=0,\quad i=1,2.$$

Proof. Assuming that the system (10) takes the form

$$\mathfrak{g}_1(\eta) = L_1 y, \qquad \mathfrak{g}_2(\eta) = L_2 z, \tag{12}$$

we obtain the following identities:

$$\begin{cases} \mathfrak{g}_1'(\eta) & \equiv -b_{11}(\eta)\mathfrak{g}_1(\eta) - b_{12}(\eta)\mathfrak{g}_1(\eta)\eta^{\frac{1}{p-1}} y^{-\frac{\sigma_1}{p-1}} - b_{13}(\eta)z^{q_1} - b_{14}(\eta)y, \\ \mathfrak{g}_2'(\eta) & \equiv -b_{21}(\eta)\mathfrak{g}_2(\eta) - b_{22}(\eta)\mathfrak{g}_2(\eta)\eta^{\frac{1}{p-1}} z^{-\frac{\sigma_2}{p-1}} - b_{23}(\eta)y^{q_2} - b_{24}(\eta)z. \end{cases} \tag{13}$$

Now, consider the auxiliary functions

$$\begin{cases} h_1(\chi_1, \eta) & \equiv -b_{11}(\eta)\chi_1 - b_{12}(\eta)\chi_1\eta^{\frac{1}{p-1}} y^{-\frac{\sigma_1}{p-1}} - b_{13}(\eta)z^{q_1} - b_{14}(\eta)y, \\ h_2(\chi_2, \eta) & \equiv -b_{21}(\eta)\chi_2 - b_{22}(\eta)\chi_2\eta^{\frac{1}{p-1}} z^{-\frac{\sigma_2}{p-1}} - b_{23}(\eta)y^{q_2} - b_{24}(\eta)z, \end{cases}$$

where $\chi_i \in \mathbb{R}$ $(i = 1, 2)$.

Assume initially that $s_i = 0$ $(i = 1, 2)$. In this case, the functions $h_i(\chi_i, \eta)$ $(i = 1, 2)$ retain a constant sign on the interval $[\eta_1, +\infty) \subseteq [\eta_0, +\infty)$ for each fixed value of χ_i $(i = 1, 2)$, provided it differs from the value that satisfies the system

$$\begin{cases} -b_{11}^0\chi_1 - b_{12}^0\chi_1^{\frac{1}{p-1}} \left(y^0\right)^{\frac{-\sigma_1}{p-1}} - b_{13}^0 \left(z^0\right)^{q_1} - b_{14}^0 y^0 & = 0, \\ -b_{21}^0\chi_2 - b_{22}^0\chi_2^{\frac{1}{p-1}} \left(z^0\right)^{\frac{-\sigma_2}{p-1}} - b_{23}^0 \left(y^0\right)^{q_2} - b_{24}^0 z^0 & = 0. \end{cases}$$

Suppose $s_i > 0$ for $i = 1, 2$. Then, for every fixed $\chi_i \neq \chi_i^*$ not satisfying the corresponding system, the behaviour of the functions $h_i(\chi_i, \eta)$ $(i = 1, 2)$ differs from the special case $\chi_i = \chi_i^*$, where χ_i satisfies the following system of equations:

$$\begin{cases} -b_{11}^0\chi_1 - b_{12}^0\chi_1^{\frac{1}{p-1}} \left(y^0\right)^{\frac{-\sigma_1}{p-1}} - b_{14}^0 y^0 & = 0, \\ -b_{21}^0\chi_2 - b_{22}^0\chi_2^{\frac{1}{p-1}} \left(z^0\right)^{\frac{-\sigma_2}{p-1}} - b_{24}^0 z^0 & = 0. \end{cases}$$

□

The functions $h_i(\chi_i, \eta)$ $(i = 1, 2)$ preserve their sign throughout the interval $[\eta_2, +\infty) \subseteq [\eta_0, +\infty)$. When they can be expressed in an alternative form as follows:

$$\begin{cases} h_1(\chi_1, \eta) & \equiv -b_{11}(\eta)\chi_1 - b_{12}(\eta)\chi_1\, \eta^{\frac{1}{p-1}}\, y^{-\frac{\sigma_1}{p-1}} - b_{13}(\eta)y \left(y^{-1}z^{q_1} + b_{14}(\eta)b_{13}^{-1}(\eta)\right), \\ h_2(\chi_2, \eta) & \equiv -b_{21}(\eta)\chi_2 - b_{22}(\eta)\chi_2\, \eta^{\frac{1}{p-1}}\, z^{-\frac{\sigma_2}{p-1}} - b_{23}(\eta)z \left(z^{-1}y^{q_2} + b_{24}(\eta)b_{23}^{-1}(\eta)\right). \end{cases}$$

From here, we find

$$\lim_{\eta\to+\infty} b_{i1}(\eta) = -\frac{p-1}{\sigma_i + p - 2}, \quad \lim_{\eta\to+\infty} b_{i2}(\eta) = \gamma\Big(\frac{p-1}{p}\Big)^p,$$

$$\lim_{\eta\to+\infty} b_{i3}(\eta) = \begin{cases} (1 - 1/p)^p \frac{1}{a}, & s_i = 0, \\ 0, & s_i > 0, \end{cases} \quad \lim_{\eta\to+\infty} b_{i4}(\eta) = 0 \quad (i = 1, 2),$$

implies that the functions $h_i(\chi_i, \eta)$ $(i = 1, 2)$ maintain a constant sign on the interval $[\eta_2, +\infty) \subseteq [\eta_0, +\infty)$, where $\chi_i \neq 0$ $(i = 1, 2)$. That means the functions $h_i(\chi_i, \eta)$ $(i = 1, 2)$ for all $\eta \in [\eta_i, +\infty)$ $(i = 1, 2)$ satisfy one of the inequalities:

$$h_i(\chi_i, \eta) > 0 \quad \text{or} \quad h_i(\chi_i, \eta) < 0 \quad (i = 1, 2). \tag{14}$$

Assume now that the functions $\mathfrak{g}_i(\eta)$ $(i = 1, 2)$ do not have a limit as $\eta \to +\infty$. Let us consider the case where at least one of the inequalities in (14) holds.

Given the oscillatory nature of the functions $\mathfrak{g}_i(\eta)$ $(i = 1, 2)$, their graphs must intersect the straight line $\overline{\mathfrak{g}}_i(\eta)$ $(i = 1, 2)$ infinitely many times within the interval $[\eta_i, +\infty)$ $(i = 1, 2)$.

Then

$$\begin{cases} \mathfrak{g}_1(\eta) & = y^{\sigma_1}\left|\frac{dy}{d\eta} + b_{10}(\eta)y\right|^{p-2}\left(\frac{dy}{d\eta} + b_{10}(\eta)y\right) = \left(y^0\right)^{\sigma_1} |b_{10}^0 y^0|^{p-2} b_{10}^0 y^0 + o(1), \\ \mathfrak{g}_2(\eta) & = z^{\sigma_1}\left|\frac{dz}{d\eta} + b_{20}(\eta)z\right|^{p-2}\left(\frac{dz}{d\eta} + b_{20}(\eta)z\right) = \left(z^0\right)^{\sigma_1} |b_{20}^0 z^0|^{p-2} b_{20}^0 z^0 + o(1), \end{cases}$$

as $\eta \to +\infty$. And according to relation (13), the derivative of the functions $\mathfrak{g}_i(\eta)$ $(i = 1, 2)$ tends to a finite limit as $\eta \to +\infty$, and this limit is zero.

As a result, it is important to

$$\begin{cases} \lim_{\eta\to+\infty}\left(b_{11}(\eta)\mathfrak{g}_1(\eta) + b_{12}(\eta)y^{-\frac{\sigma_1}{p-1}}\eta^{\frac{1}{p-1}}\mathfrak{g}_1(\eta)\right) + \lim_{\eta\to+\infty}\left(b_{13}(\eta)z^{q_1} + b_{14}(\eta)y\right) & = 0, \\ \lim_{\eta\to+\infty}\left(b_{21}(\eta)\mathfrak{g}_2(\eta) + b_{22}(\eta)z^{-\frac{\sigma_2}{p-1}}\eta^{\frac{1}{p-1}}\mathfrak{g}_2(\eta)\right) + \lim_{\eta\to+\infty}\left(b_{23}(\eta)y^{q_2} + b_{24}(\eta)z\right) & = 0. \end{cases}$$

It readily follows from this that at $s_i < 0$ $(i = 1, 2)$, system (12) cannot have solutions $(y(\eta), z(\eta))$ with a finite non-zero limit as $\eta \to +\infty$, and at $s_i \geq 0$ $(i = 1, 2)$. For such solutions to exist, the conditions stated in Theorems 2, 3, 4, and 5 must be fulfilled.

Consequently, due to the transformations introduced in (4) and (9), the self-similar solution of system (1)–(2) exhibits the following asymptotic behaviour as $x \to \frac{p}{\alpha}\ln\left(\frac{\alpha}{p}a^{\frac{p-1}{p}}(T+t)^{\gamma}\right)$:

$$u_A(t,x) \simeq y^0(T+t)^{\frac{1+q_1}{1-q_1q_2}}\left(a - \left(\left(\frac{p}{\alpha}\cdot\frac{e^{\frac{\alpha x}{p}}}{(t+T)^{\gamma}}\right)^{\frac{p}{p-1}}\right)\right)^{\frac{p-1}{\sigma_1+p-2}},$$

$$v_A(t,x) \simeq z^0(T+t)^{\frac{1+q_2}{1-q_1q_2}}\left(a - \left(\left(\frac{p}{\alpha}\cdot\frac{e^{\frac{\alpha x}{p}}}{(t+T)^{\gamma}}\right)^{\frac{p}{p-1}}\right)\right)^{\frac{p-1}{\sigma_2+p-2}}.$$

The theorems are proved.

3 *Numerical experiments*

To complement the qualitative analysis, we computed representative solution profiles for system (1)–(2) with the exponential density $\rho(x) = e^{\alpha x}$. The computations follow the scheme and implementation principles described in [40] and are written in Python. The diffusion terms are discretized in conservative form, which is natural for divergence operators and helps control the support propagation [32, pp. 258–264]. The degeneracy at $u = 0$ and $u_x = 0$ is handled by a small regularization of the nonlinear mobility, which is a standard practice for degenerate parabolic equations and helps avoid spurious oscillations near interfaces [32, pp. 525–542], [41, pp. 332–356]. Non-negativity is enforced by the update choice and by using non-negative initial data. Figures 1–4 show typical solution profiles for several parameter sets. They illustrate finite-speed propagation, localization driven by the exponential density, and the change of growth rate, in agreement with the comparison estimates and the Fujita-type threshold discussed in the theoretical part [7, 41]. Furthermore, we listed the graphics of the solution in some particular cases:

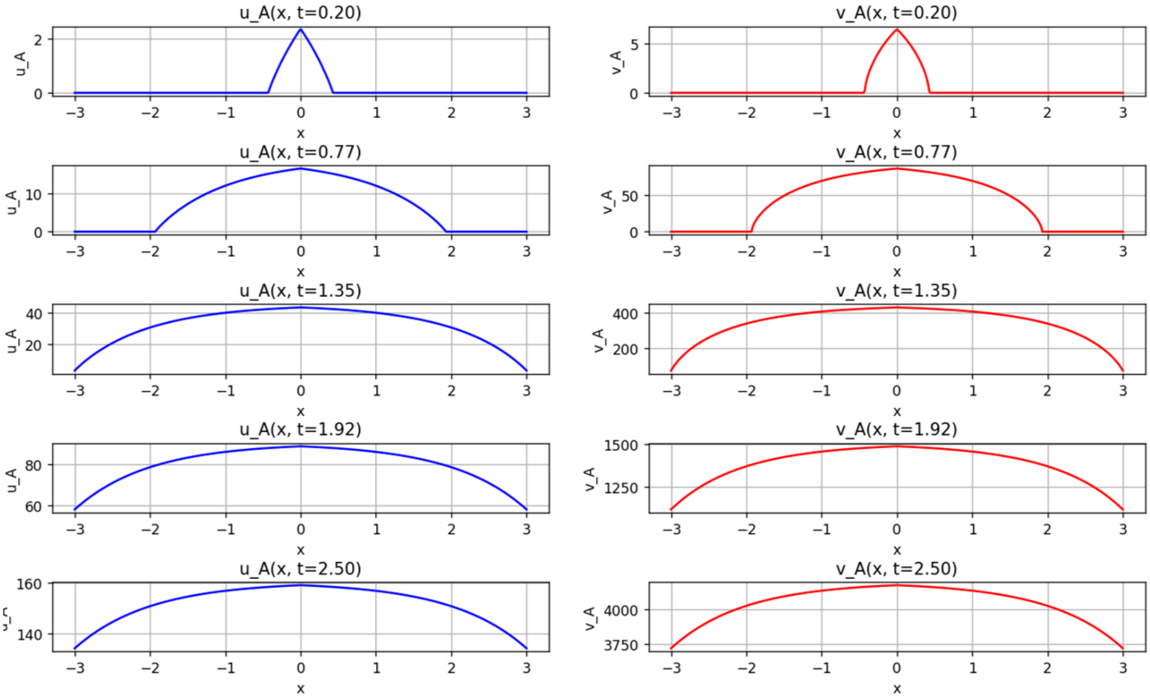

Figure 1. $y_0 = 1.5$, $z_0 = 1.0$, $a = 1.0$, $p = 2.2$, $T = 1.0$, $\alpha = 1.2$, $q_1 = 0.4$, $q_2 = 1.5$, $\sigma_1 = 1.1$, $\sigma_2 = 1.7$

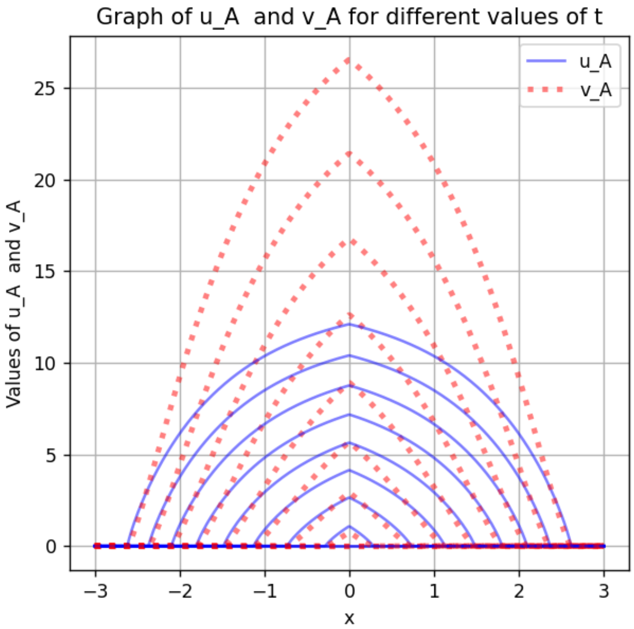

Figure 2. $y_0 = 1.5$, $z_0 = 1.5$, $a = 1.0$, $p = 2.2$, $T = 1.1$, $\alpha = 1.0$, $q_1 = 0.3$, $q_2 = 0.8$, $\sigma_1 = 1.2$, $\sigma_2 = 0.7$

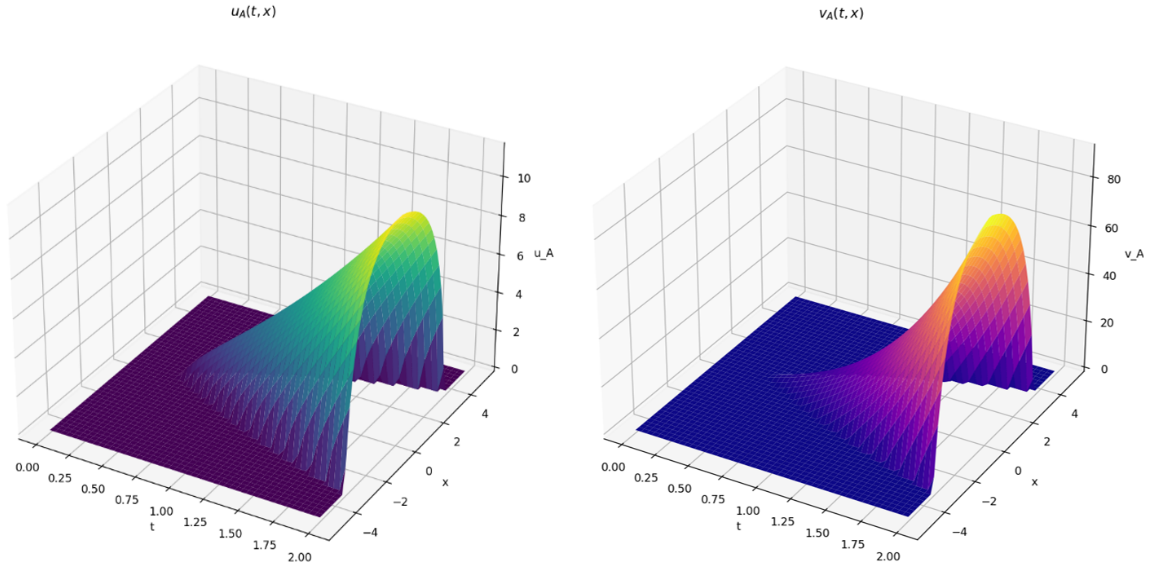

Figure 3. $y_0 = 1.5$, $z_0 = 1.3$, $T = 1.0$, $a = 1.0$, $p = 2.1$, $\alpha = 1.0$, $q_1 = 0.4$, $q_2 = 1.5$, $\sigma_1 = 1.1$, $\sigma_2 = 1.7$

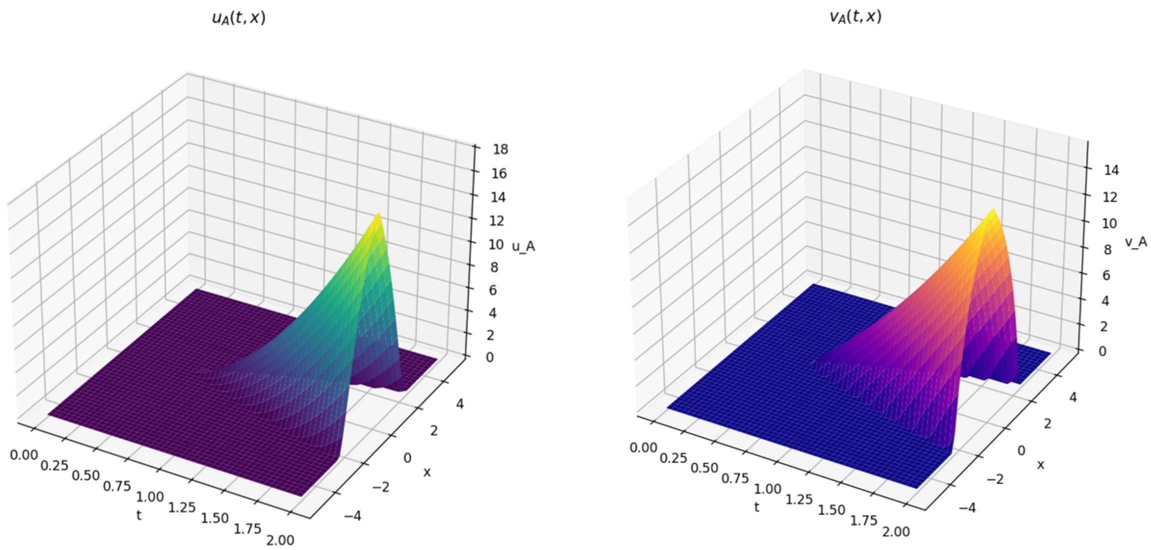

Figure 4. $y_0 = 1.1$, $z_0 = 1.3$, $T = 1.0$, $a = 1.2$, $p = 2.5$, $\alpha = 1.0$, $q_1 = 0.7$, $q_2 = 0.5$, $\sigma_1 = 0.5$, $\sigma_2 = 1.3$

Conclusion

The main novelty of this work is a unified qualitative analysis for a non-homogeneous, doubly nonlinear coupled system under an exponential density $\rho(x) = e^{\alpha x}$. that links three components: comparison estimates, self-similar structure, and computation. The comparison part is based on explicit compactly supported super-solutions constructed from Barenblatt-type profiles and yields global solvability for small initial data in the slow-diffusion range. The self-similar change of variables consistent with the weight converts the PDE system into an auxiliary profile system, which allows us to classify the leading asymptotics of self-similar solutions and to reveal how the parameters control localization and propagation. The resulting Fujita-type criterion provides a clear borderline between the global existence and non-existence of global solutions. Numerical experiments in Python reproduce finite-speed propagation and spatial localization induced by the weight and illustrate the change of qualitative behaviour, in agreement with the theoretical thresholds.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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