

Almost hyperbolic Ricci solitons on almost co-Kähler manifolds

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This research investigates almost hyperbolic Ricci solitons on almost co-Kähler manifolds, an important class of odd-dimensional manifolds closely related to Kähler geometry. Motivated by recent developments in hyperbolic geometric flows and their self-similar solutions, we study the existence and geometric properties of almost hyperbolic Ricci solitons within the framework of almost co-Kähler manifolds. First, we derive fundamental relations for almost hyperbolic Ricci solitons whose potential vector field is collinear with the Reeb vector field and obtain a differential equation governing the associated smooth function. We then examine almost hyperbolic Ricci solitons under several geometric conditions, including conformal Killing vector fields, torse-forming vector fields, Ricci bi-conformal vector fields, and $(V(\text{Ric}))$ -vector fields. In each case, necessary conditions are established, leading to characterization results and Einstein-type properties of the underlying manifolds. In particular, we obtain criteria under which an almost co-Kähler manifold is Einstein or Ricci-flat. Furthermore, for closed almost co-Kähler manifolds admitting a $(V(\text{Ric}))$ -vector field satisfying suitable trace conditions, we show that the manifold is Ricci-flat and the associated vector field is parallel. To illustrate the theoretical results, we construct two explicit three-dimensional examples of almost co-Kähler manifolds admitting hyperbolic Ricci solitons.

Keywords: almost co-Kähler manifolds, hyperbolic Ricci solitons, torse-forming vector fields, hyperbolic geometric flow, conformal Killing vector fields, Reeb vector field, Ricci-flat manifolds, contact metric geometry.

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Introduction

The geometry of (almost) co-Kähler manifolds, a specialized category of almost contact metric manifolds akin to Kähler manifolds, was initially introduced by Blair [1] and further explored by Goldberg and Yano [2], and Olszak [3]. These manifolds are essentially the almost cosymplectic manifolds referenced in the aforementioned studies. Following Li's [4] research, a number of authors have adopted this terminology in their works. Li's [5] findings indicate that co-Kähler manifolds serve as odd-dimensional counterparts to Kähler manifolds. A recent survey has compiled various new insights regarding (almost) co-Kähler manifolds from both geometric and topological perspectives. De, Majhi, and Suh [6] investigated semi-symmetric and φ -symmetric curvature properties, including Codazzi-type behaviour of the Ricci tensor and the role of transversal Killing fields. Wang [7] studied curvature homogeneity and ball-homogeneity, relating these notions to the algebraic structure of some operator; and the interplay between co-Kähler geometry and the Ricci-soliton equation has been clarified by Wang, who extended the analysis to Ricci solitons on general almost co-Kähler manifolds [8], where rigidity forces the underlying manifold to be Einstein or Ricci-flat under suitable hypotheses on the potential field. Notably, Perrone [9, 10] achieved a comprehensive classification of three-dimensional almost co-Kähler manifolds characterized by homogeneity or minimal Reeb vector fields, along with providing a local characterization of these manifolds. Co-Kähler manifolds represent a significant

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area of inquiry within contact geometry, serving as intriguing odd-dimensional counterparts to Kähler manifolds. A closed manifold qualifies as a co-Kähler manifold if and only if it can be characterized as a Kähler mapping torus [4].

In recent years, there has been a surge of classification results pertaining to three-dimensional almost co-Kähler (ACK) manifolds. For example, some authors have examined local ϕ -symmetry, curvature, and ball homogeneities within the context of three-dimensional almost co-Kähler manifolds. Further studies have investigated various symmetry properties related to the Ricci operators, including Codazzi-type conditions, and transversal Killing fields in three-dimensional ACK manifolds [11, 12].

The study of geometric solitons on Riemannian and semi-Riemannian manifolds represents a significant area of research within differential geometry and its applications in physics. In recent years, there has been a notable surge in interest surrounding geometric solitons and their extensions, as they offer innovative methods for exploring the geometry and topology of Riemannian manifolds. This growing fascination with geometric solitons and their variations across various geometric frameworks is largely attributed to their relevance to the principles of general relativity. Geometric flows and their solitons play a pivotal role in differential geometry, facilitating the discovery of canonical metrics on the corresponding Riemannian manifolds. Among these flows, the Ricci flow is particularly significant. Hamilton defined the Ricci soliton (RS) on a Riemannian manifold as a unique solution to the Ricci flow. A manifold denoted as (M, g) qualifies as a Ricci soliton if it adheres to the equation: $S + \frac{1}{2}\mathcal{L}_W g + \Lambda g = 0$, where Λ is a real constant, S represents the Ricci tensor of g , and $\mathcal{L}_W$ is the Lie derivative operator in the direction of W . This relationship is encapsulated in the triplet (g, W, Λ) . Furthermore, the metric g is identified as a gradient Ricci soliton if there exists a smooth function ψ such that $W = \text{grad } \psi$.

Investigations on existence and categorization of geometric solitons on (almost) co-Kähler manifolds presents a significant challenge. To date, the literature reveals only a limited number of findings pertaining to this subject. Current literature indicates that findings on this topic are relatively sparse. The initial study in this area was carried out by Cho [13], who illustrated that a co-Kähler 3-manifold qualifies as a Ricci soliton if the potential vector field aligns with the Reeb vector field ξ or is a unit vector field orthogonal to ξ . This leads to the conclusion that the manifold is locally flat, characterized by vanishing Riemann curvature tensor. Expanding on this work, Wang [14] determined that for a co-Kähler 3-manifold to be classified as a Ricci soliton, it must either be locally flat or the potential vector field must signify an infinitesimal contact transformation. Furthermore, Wang previously established that if a compact almost co-Kähler manifold operates as a Ricci soliton, with the potential vector field being pointwise collinear to the Reeb vector field, it is identified as a Ricci-flat co-Kähler manifold, resulting in a steady soliton [15]. Recently, new geometric criteria have been introduced to clarify the conditions under which a compact Einstein ACK manifold can be recognized as a co-Kähler manifold [16].

Another type of geometric flows is characterized by hyperbolic geometric flows, which bear a strong resemblance to the wave equation associated with metrics on the underlying manifold. This can be understood as a system of second-order nonlinear evolution partial differential equations, expressed as follows:

$$\frac{\partial^2}{\partial t^2} g = -2S, \quad g(0) = g_0, \quad \frac{\partial g}{\partial t}(0) = k_0. \quad (1)$$

Here, k_0 denotes a symmetric 2-tensor field defined on the manifold M . The investigation into the existence and uniqueness of the equation (1) has been conducted in [17], particularly in the context of closed Riemannian manifolds. The self-similar solution for this flow can be represented by the equation:

$$S(g) + \lambda \mathcal{L}_X g + (\mathcal{L}_X \circ \mathcal{L}_X)g = \mu g, \quad (2)$$

where λ and μ are constants, and X denotes a vector field. In this framework, g is identified as a hyperbolic Ricci soliton (HRS), which can be represented by the tuple (g, X, λ, μ) or simply as

(g, X, λ, μ) . In cases where X is zero, the HRS corresponds to an Einstein metric. Moreover, if λ is specified as $\frac{1}{2}$ and the vector field X is a 2-Killing vector field that satisfies the condition $(\mathcal{L}_X \circ \mathcal{L}_X)g = 0$ (see [18, 19]), the HRS simplifies to a Ricci soliton (RS). Furthermore, if X is expressed as ∇f for some smooth function f defined on M , we categorize this structure as a gradient HRS, denoting it as $(g, \nabla f, \lambda, \mu)$. If we consider μ to be a smooth function on M , then the soliton is called almost HRS (AHRS). Azami et al. studied HRS on some manifolds. Azami and Fasihi-Ramandi [20] initiated this line of investigation on warped product manifolds, deriving rigidity and classification results for HRSs whose potential vector field satisfies additional geometric constraints, and the same first author with Jafari, Jamal, and Haseeb [21] extended the analysis to perfect fluid spacetimes, where the hyperbolic soliton equation interacts with the physical restrictions imposed by the energy-momentum tensor. The three-dimensional case was studied in depth by Faraji, Azami, and Fasihi-Ramandi [22], who classified homogeneous hyperbolic Ricci solitons on 3-manifolds, and by Fasihi-Ramandi, Azami, and Choudhary [23], who established general existence and structural theorems for HRSs on Riemannian manifolds, including gradient and almost-HRS cases. Closely related results on sequential warped product manifolds were obtained by Kaya and Özgür [24], who clarified how the HRS equation interacts with the fibre-wise geometry of the warped decomposition, while Siddiqi and Mofarreh [25] further developed the physical interpretation by analysing hyperbolic Ricci solitons and their gradient analogue on relativistic perfect fluid spacetimes. In a parallel direction, Blaga and Özgür [26] investigated the neighbouring notion of hyperbolic Yamabe solitons, establishing properties that complement the HRS framework since the Yamabe equation reduces, under suitable conditions, to the Ricci-soliton equation.

Building upon the studies previously mentioned and the references cited in [27, 28], this paper investigates AHRS in the framework of ACK manifolds, providing two specific examples of HRS on these manifolds. The organization of the paper is as follows: Section 2 reviews key concepts and formulas pertinent to ACK manifolds, which will be crucial for our analysis. In Section 3, we provide the main results along with their proofs. Lastly, Section 4 is focused on presenting two examples of ACK manifolds that demonstrate hyperbolic Ricci solitons.

1 Preliminaries

In the upcoming sections, we will review essential principles related to contact geometry that will be significant for our later discussions. Additionally, we will consider the vector fields W_1, W_2 , and U on M as general unless stated otherwise.

Let M^{2n+1} represent an almost contact manifold characterized by a structure (ϕ, ξ, η) , where ξ denotes a unit vector field, ϕ denotes a $(1, 1)$ -tensor field, and η is a 1-form that satisfies the following conditions:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \quad (3)$$

If a metric g exists that fulfills the equation

$$g(\phi W_1, \phi W_2) = g(W_1, W_2) - \eta(W_1)\eta(W_2), \quad (4)$$

the manifold is referred to as an almost contact metric manifold. In this context, it is noted that $\eta \circ \phi = 0$, $\phi\xi = 0$, $g(\phi W_1, W_2) = -g(W_1, \phi W_2)$, and $\eta = g(\cdot, \xi)$. The fundamental 2-form Φ of M is determined by

$$\Phi(W_1, W_2) = g(W_1, \phi W_2).$$

An almost contact manifold M is classified as an almost co-Kähler (ACK) manifold if $d\Phi = 0$ and $d\eta = 0$. An ACK manifold qualifies as a co-Kähler manifold when it meets the condition of being normal, which is equivalent to the statement $\nabla\Phi = 0$, where ∇ represents the Levi-Civita connection.

Furthermore, an ACK structure is referred to as strictly ACK if it does not conform to the definition of a co-Kähler structure. Given that (almost) cosymplectic manifolds serve as the odd-dimensional counterpart to Kähler manifolds, Blair [29] proposed the term (almost) co-Kähler manifold as a new designation for (almost) cosymplectic manifolds.

On an ACK, let us define two operators, l and h , where l is expressed as $l = R(\cdot, \xi)\xi$ and h is given by $h = \frac{1}{2}\mathcal{L}_\xi\phi$, with R representing the curvature tensor. These operators adhere to specific conditions [30, 31]:

$$h\xi = 0, \quad \text{trace}(h) = 0, \quad \text{trace}(h\phi) = 0, \quad \phi h + h\phi = 0, \quad (5)$$

$$\begin{aligned} \nabla_W \xi &= h\phi W, \\ (\nabla_{W_1} \eta)(W_2) &= g(h\phi W_1, W_2), \\ \phi l \phi - l &= 2h^2. \end{aligned} \quad (6)$$

From equation (6), it can be inferred that

$$(\nabla_{W_1} \eta)W_2 = (\nabla_{W_2} \eta)W_1.$$

An ACK manifold is called integrable if

$$(\nabla_{W_1} \phi)W_2 = g(hW_1, W_2)\xi - \eta(W_2)hW_1.$$

An ACK manifold is called a K-ACK manifold if the Reeb vector field ξ is Killing. Any co-Kähler manifold is recognized as a K-almost co-Kähler manifold; however, the reverse is generally not the case. Nonetheless, this relationship holds true in the context of a three-dimensional manifold.

Lemma 1. [2] In three-dimensional space, an ACK manifold qualifies as a co-Kähler manifold if and only if the vector field ξ is Killing.

In a co-Kähler manifold of dimension 3, we have $QW = \frac{r}{2}(W - \eta(W)\xi)$ and consequently we have [32] $\xi r = 0$ where Q is the Ricci operator, and r is the scalar curvature.

2 AHS on ACK manifolds

In the subsequent sections, we will outline our results and provide the necessary proofs to support them. It is crucial to emphasize that all vector fields, namely W_1, W_2, Z, U, U_1, U_2 , and W , defined on the manifold M , are treated as arbitrary unless stated otherwise. A manifold is designated as η -Einstein if it satisfies the condition

$$S = ag + b\eta \otimes \eta,$$

where a and b are smooth functions defined on M . Let us consider a ACK manifold denoted as M that adheres to the HRS equation (2). We define V as $f\xi$, where f is a function on M . Utilizing equation (5), we can derive the following expression:

$$\begin{aligned} (\mathcal{L}_{f\xi}g)(W_1, W_2) &= (W_1f)\eta(W_2) + fg(h\phi(W_1), W_2) + (W_2f)\eta(W_1) \\ &\quad + fg(W_1, h\phi(W_2)) \\ &= (W_1f)\eta(W_2) + (W_2f)\eta(W_1) + 2fg(h\phi(W_1), W_2), \end{aligned}$$

hence

$$\begin{aligned} &(\mathcal{L}_{f\xi}(\mathcal{L}_{f\xi}g))(W_1, W_2) \\ &= f\xi((\mathcal{L}_{f\xi}g)(W_1, W_2)) - (\mathcal{L}_{f\xi}g)(\mathcal{L}_{f\xi}W_1, W_2) - (\mathcal{L}_{f\xi}g)(W_1, \mathcal{L}_{f\xi}W_2) \\ &= f\xi((W_1f)\eta(W_2) + (W_2f)\eta(W_1) + 2fg(h\phi(W_1), W_2)) \\ &\quad - (((\mathcal{L}_{f\xi}W_1)f)\eta(W_2) + (W_2f)\eta(\mathcal{L}_{f\xi}W_1) + 2fg(h\phi(\mathcal{L}_{f\xi}W_1), W_2)) \\ &\quad - ((W_1f)\eta(\mathcal{L}_{f\xi}W_2) + ((\mathcal{L}_{f\xi}W_2)f)\eta(W_1) + 2fg(h\phi(W_1), \mathcal{L}_{f\xi}W_2)). \end{aligned}$$

By utilizing the equation $V = f\xi$ along with the previously mentioned equation in (2), we deduce

$$\begin{aligned} & S(W_1, W_2) + \lambda((W_1 f)\eta(W_2) + (W_2 f)\eta(W_1) + 2fg(h\phi(W_1), W_2)) \\ & + f\xi((W_1 f)\eta(W_2) + (W_2 f)\eta(W_1) + 2fg(h\phi(W_1), W_2)) \\ & - (((\mathcal{L}_f \xi W_1)f)\eta(W_2) + (W_2 f)\eta(\mathcal{L}_f \xi W_1) + 2fg(h\phi(\mathcal{L}_f \xi W_1), W_2)) \\ & - ((W_1 f)\eta(\mathcal{L}_f \xi W_2) + ((\mathcal{L}_f \xi W_2)f)\eta(W_1) + 2fg(h\phi(W_1), \mathcal{L}_f \xi W_2)) \\ & - \mu g(W_1, W_2) = 0. \end{aligned}$$

We substitute $W_1 = W_2 = \xi$ into the final equation, employing equations (3), $\mathcal{L}_f \xi = -\xi f$, and $S(\xi, \xi) = -\text{trace}(h^2)$ to derive the result

$$-\text{trace}(h^2) + 2f\xi(\xi(f)) + 2\lambda\xi(f) + 4(\xi(f))^2 - \mu = 0.$$

This leads to the emergence of the following theorem.

Theorem 1. Consider an ACK manifold characterized by the tuple (M, g, ϕ, ξ, η) , which features an HRS denoted by $(g, f\xi, \lambda, \mu)$, with f being a smooth function on the manifold M . Consequently, the following equation holds:

$$-\text{trace}(h^2) + 2f\xi(\xi(f)) + 2\lambda\xi(f) + 4(\xi(f))^2 - \mu = 0.$$

Corollary 1. Consider an ACK manifold characterized by the tuple (M, g, ϕ, ξ, η) , which features a HRS denoted as (g, ξ, λ, μ) . Consequently, the following equation holds:

$$\mu = -\text{trace}(h^2).$$

Definition 1. If a vector field V satisfies the equation

$$\mathcal{L}_V g = 2\psi g, \tag{7}$$

then it is called a conformal Killing where ψ represents a function defined on the manifold M .

Let V represent a conformal Killing vector field that fulfills the condition stated in equation (7). Consequently,

$$\begin{aligned} ((\mathcal{L}_V \circ \mathcal{L}_V)g)(U, W) &= V((\mathcal{L}_V g)(U, W)) - \mathcal{L}_V g(\mathcal{L}_V U, W) - (\mathcal{L}_V g)(U, \mathcal{L}_V W) \\ &= V(2\psi g(U, W)) - 2\psi g(\mathcal{L}_V U, W) - 2\psi g(U, \mathcal{L}_V W) \\ &= 2V(\psi)g(U, W) + 2\psi \mathcal{L}_V g(U, W) \\ &= (2V(\psi) + 4\psi^2)g(U, W). \end{aligned} \tag{8}$$

By substituting equation (8) into equation (2), we arrive at the expression

$$S(U, W) + 2\psi \lambda g(U, W) + (2V(\psi) + 4\psi^2)g(U, W) - \mu g(U, W) = 0.$$

If we replace U and W with ξ in the preceding equation, we derive

$$-\text{trace}(h^2) + 2\psi \lambda + 2V(\psi) + 4\psi^2 - \mu = 0.$$

Consequently, we get:

Theorem 2. In the context of an ACK manifold defined by the metric g and characterized by the HRS parameters (g, V, λ, μ) , where V satisfies $\mathcal{L}_V g = 2\psi g$, it can be concluded that the manifold M exhibits Einstein properties. This yields

$$-\text{trace}(h^2) + 2\psi \lambda + 2V(\psi) + 4\psi^2 - \mu = 0.$$

Definition 2. A torse-forming vector field W on a pseudo-Riemannian manifold (M, g) is defined as a nonvanishing vector field that satisfies the equation

$$\nabla_U W = \psi U + \omega(U)W, \quad (9)$$

where ω denotes a 1-form and ψ is a smooth function. The vector field W can be further classified in various ways: it is called concircular, when $\omega = 0$; it is termed concurrent if $\omega = 0$ and ψ equals one; it is recognized as a parallel vector field when both ψ and ω are zero; and it is classified as torqued if the condition $\omega(W) = 0$ is satisfied.

Let (g, W, λ, μ) denote an HRS on an ACK manifold, with W being a torse-forming vector field and satisfies the condition outlined in (9). Then

$$(\mathcal{L}_W g)(U, Y) = 2\psi g(U, Y) + \omega(U)g(W, Y) + \omega(Y)g(W, U) \quad (10)$$

and

$$\begin{aligned} (\mathcal{L}_W(\mathcal{L}_W g))(U, Y) &= W(2\psi g(U, Y) + \omega(U)g(W, Y) + \omega(Y)g(W, U)) \\ &\quad - 2\psi g(\mathcal{L}_W U, Y) - \omega(\mathcal{L}_W U)g(W, Y) - \omega(Y)g(W, \mathcal{L}_W U) \\ &\quad - 2\psi g(U, \mathcal{L}_W Y) - \omega(U)g(W, \mathcal{L}_W Y) - \omega(\mathcal{L}_W Y)g(W, U). \end{aligned}$$

The subsequent equations can be represented as follows:

$$g(\mathcal{L}_W U, Y) = g(\nabla_W U, Y) - \psi g(U, Y) - \omega(Y)g(W, U),$$

similarly

$$g(U, \mathcal{L}_W Y) = g(U, \nabla_W Y) - \psi g(U, Y) - \omega(U)g(W, Y).$$

Then

$$g(\mathcal{L}_W U, Y) + g(U, \mathcal{L}_W Y) = W(g(U, Y)) - 2\psi g(U, Y) - \omega(Y)g(W, U) - \omega(U)g(W, Y).$$

Also,

$$\begin{aligned} \omega(\mathcal{L}_W U) &= \omega(\nabla_W U - \nabla_U W) = \omega(\nabla_W U - \psi U - \omega(U)W) \\ &= \omega(\nabla_W U) - \psi \omega(U) - \omega(U)\omega(W), \end{aligned}$$

similarly

$$\omega(\mathcal{L}_W Y) = \omega(\nabla_W Y) - \psi \omega(Y) - \omega(Y)\omega(W).$$

Therefore,

$$\begin{aligned} (\mathcal{L}_W(\mathcal{L}_W g))(U, Y) &= (2W(\psi) + 4\psi^2)g(U, Y) + W(\omega(U)g(W, Y) + \omega(Y)g(W, U)) \\ &\quad + 4h\omega(U)g(W, Y) + 4h\omega(Y)g(W, U) \\ &\quad - \omega(\nabla_W U)g(W, Y) + 2\omega(U)\omega(W)g(W, Y) - \omega(Y)g(\nabla_W U, W) \\ &\quad + 2\omega(Y)\omega(W)g(W, U) - \omega(\nabla_W Y)g(W, U) - \omega(U)g(\nabla_W Y, W). \end{aligned} \quad (11)$$

By substituting $U = Y = \xi$ into equations (10) and (11), it can be concluded that.

$$(\mathcal{L}_W g)(\xi, \xi) = -2\psi + 2\omega(\xi)\eta(W) \quad (12)$$

and

$$\begin{aligned}
(\mathcal{L}_W(\mathcal{L}_W g))(\xi, \xi) &= -(2W(\psi) + 4\psi^2) + W(\omega(\xi)\eta(W) + \omega(\xi)\eta(W)) \\
&\quad + 4\psi\omega(\xi)\eta(W) + 4h\omega(\xi)\eta(W) \\
&\quad - 2\omega(\nabla_W \xi)\eta(W) + 4\omega(\xi)\omega(W)\eta(W) - 2\omega(\xi)g(\nabla_W \xi, W) \\
&= -(2W(\psi) + 4\psi^2) + W(\omega(\xi)\eta(W) + \omega(\xi)\eta(W)) \\
&\quad + 4\psi\omega(\xi)\eta(W) + 4\psi\omega(\xi)\eta(W) + 4\omega(\xi)\omega(W)\eta(W) \\
&\quad + 2\omega(\xi)\eta(W) + 4(\eta(W))^2\omega(\xi) + 2\omega(\xi)|W|^2.
\end{aligned} \tag{13}$$

By replacing equations (12) and (13) in equation (2), we derive

$$\begin{aligned}
&-\text{trace}(h^2) - \mu + \lambda(-2\psi + 2\omega(\xi)\eta(W)) - (2W(\psi) + 4\psi^2) \\
&+ W(\omega(\xi)\eta(W) + \omega(\xi)\eta(W)) + 4\psi\omega(\xi)\eta(W) + 4\psi\omega(\xi)\eta(W) \\
&+ 4\omega(\xi)\omega(W)\eta(W) + 2\omega(\xi)\eta(W) + 4(\eta(W))^2\omega(\xi) + 2\omega(\xi)|W|^2 = 0.
\end{aligned} \tag{14}$$

Consequently, we arrive at the subsequent conclusions.

Theorem 3. If an ACK manifold meets the HRS condition (g, W, λ, μ) , where W admits (9), then equation (14) holds true.

Corollary 2. In the framework of an ACK manifold, when the metric g satisfies the HRS condition (g, W, λ, μ) , with W denoting a concircular vector field characterized by the equation $\nabla_U W = \psi U$ for all vector fields U , it can be concluded that the equation $\text{trace}(h^2) + \mu + 2\psi\lambda + 2W(\psi) + 4\psi^2 = 0$ is valid.

In the work presented in [33], the authors introduced the concept of bi-conformal vector fields utilizing two $(0, 2)$ tensor fields. Subsequently, De et al. in [34] expanded on this by defining Ricci bi-conformal vector fields, employing the two tensor fields denoted as g and S in their formulation.

Definition 3. On a Riemannian manifold (M, g) , a vector field X is referred to as a Ricci bi-conformal vector field if it satisfies the following equations

$$\mathcal{L}_X g = \alpha g + \gamma S, \tag{15}$$

and

$$\mathcal{L}_X S = \alpha S + \gamma g, \tag{16}$$

for certain non-zero smooth functions denoted as α and γ .

Consider an ACK manifold (M^n, g) that satisfies the HRS conditions (g, W, λ, μ) , where W denotes the Ricci bi-conformal vector field and meets the criteria specified in equations (15) and (16). Consequently, the expression can be expressed as follows:

$$\mathcal{L}_W(\mathcal{L}_W g) = (\alpha^2 + \gamma^2 + W(\alpha))g + (2\alpha\gamma + W(\gamma))S. \tag{17}$$

By incorporating equations (15) and (17) into equation (2), we obtain

$$(1 + \lambda\gamma + 2\alpha\gamma + W(\gamma))S(X, Y) + (\lambda\alpha - \mu + \alpha^2 + \gamma^2 + W(\alpha))g(X, Y) = 0. \tag{18}$$

Replacing $X = Y = \xi$ in (18) yields

$$-(1 + \lambda\gamma + 2\alpha\gamma + W(\gamma))(\text{trace}(h^2)) + (\lambda\alpha - \mu + \alpha^2 + \gamma^2 + W(\alpha)) = 0$$

and

$$(1 + \lambda\gamma + 2\alpha\gamma + W(\gamma))(S(X, Y) + \text{trace}(h^2)g(X, Y)) = 0.$$

Now, if $1 + \lambda\gamma + 2\alpha\gamma + W(\gamma) \neq 0$ then the above equation yields M is an Einstein manifold. Thus, we get:

Theorem 4. If an ACK manifold admits the HRS conditions (g, W, λ, μ) , where W adheres to the equations (15) and (16) such that $1 + \lambda\gamma + 2\alpha\gamma + W(\gamma) \neq 0$, then M is classified as an Einstein manifold.

Definition 4. A vector field V defined on a manifold M is referred to as a $V(Ric)$ -tensor field if there exists a constant a such that $\nabla V = aQ$.

Let V be a $V(Ric)$ -tensor field on ACK manifold M^{2n+1} such that $\nabla V = aQ$. Thus, we deduce

$$\operatorname{div}(V) = ar, \quad \mathcal{L}_V g = 2aS, \quad \mathcal{L}_V \mathcal{L}_V g = 2a\mathcal{L}_V S$$

and HRS equation becomes

$$(1 + 2a\lambda)S + 2a\mathcal{L}_V S - \mu g = 0. \quad (19)$$

Now, if we assume that $\operatorname{trace}(\mathcal{L}_V S) = 0$, then by taking the trace of (19), we get $(1 + 2a\lambda)r = \mu(2n + 1)$. Let $1 + 2a\lambda \neq 0$. In this case, we conclude that $r = \frac{(2n+1)\mu}{1+2a\lambda}$ is a constant. From [35, 36], we have

$$\operatorname{div}(\mathcal{L}_V g) = 2(d(\operatorname{div}(V)) + i_{QV}g) \quad (20)$$

and

$$\operatorname{trace}(\mathcal{L}_V \mathcal{L}_V g) = 2(\|\nabla V\|^2 + \operatorname{div}(\nabla_V V) - S(V, V)). \quad (21)$$

Since $\operatorname{trace}(\mathcal{L}_V \mathcal{L}_V g) = 0$, from (21) we obtain

$$\|\nabla V\|^2 + \operatorname{div}(\nabla_V V) - S(V, V) = 0.$$

Also, from (20) we find

$$S(V, V) = \frac{1}{2}\operatorname{div}(\mathcal{L}_V g)(V) - V(\operatorname{div}(V)) = 0.$$

Since $\|\nabla V\|^2 = a^2\|Q\|^2$ and $\operatorname{div}(\nabla_V V) = a\operatorname{div}(QV)$, we infer

$$a^2\|Q\|^2 + a\operatorname{div}(QV) = 0. \quad (22)$$

By integrating (22), we get $Q = 0$. Then $S = 0$ and $\nabla V = 0$. Therefore, the manifold M exhibits Ricci-flat, and the vector field V is parallel. This leads us to the subsequent theorem.

Theorem 5. Suppose that a closed ACK manifold M^{2n+1} satisfies the HRS (g, V, λ, μ) with a $V(Ric)$ -tensor field such that $\nabla V = aQ$ for some constant a , and $\operatorname{trace}(\mathcal{L}_V S) = 0$ with $1 + 2a\lambda \neq 0$. Then M becomes Ricci-flat, and V becomes a parallel vector field.

3 Examples

We give two examples of ACK manifolds admitting HRS.

Example 1. Let $M = \mathbb{R}^3$ be a manifold with the standard coordinates (y_1, y_2, y_3) . We proceed to establish the metric g on M accordingly

$$g = dy_1^2 + dy_2^2 + e^{2y_3}(y_1^2 e^{2y_3} + y_2^2 e^{2y_3} + 1)dy_3^2 - 2y_2 e^{2y_3} dy_1 dy_3 + 2y_1 e^{2y_3} dy_2 dy_3.$$

We examine the orthonormal vector fields defined as follows:

$$w_1 = \frac{\partial}{\partial y_1}, \quad w_2 = \frac{\partial}{\partial y_2}, \quad w_3 = y_2 e^{y_3} \frac{\partial}{\partial y_1} - y_1 e^{y_3} \frac{\partial}{\partial y_2} + e^{-y_3} \frac{\partial}{\partial y_3}.$$

We get $[w_1, w_2] = 0$, $[w_1, w_3] = -e^{y_3}w_2$, $[w_2, w_3] = e^{y_3}w_1$ and other Lie brackets are equal zero. We define the structure (ϕ, ξ, η) on M through the following means

$$\xi = w_3, \quad \eta = e^{y_3}dy_3, \quad \phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The 2-form Φ is determined by

$$\Phi = -2dy_1 \wedge dy_2 - 2y_2e^{2y_3}dy_2 \wedge dy_3 + 2y_1e^{2y_3}dy_3 \wedge dy_1.$$

It is important to observe that the relationships defined in equations (3) and (4) are satisfied. Forms η and Φ are closed. Hence, M is an ACK manifold. The Levi-Civita connection, denoted as ∇ , is defined by

$$\nabla_{w_i}w_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{y_3}w_2 & -e^{y_3}w_1 & 0 \end{pmatrix}.$$

By direct calculation, it follows that the Ricci tensor vanishes in M . We have $\mathcal{L}_\xi g = 0$, and applying the Lie derivative twice yields $(\mathcal{L}_\xi \circ \mathcal{L}_\xi)g = 0$. Consequently, the tuple (g, ξ, λ, μ) constitutes a steady HRS on the manifold M .

Example 2. Let (y_1, y_2, y_3) represent the standard coordinates in $\mathbb{R}^3$, and define the set $M = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_3 \neq 0\}$. The vector fields given by

$$w_1 = \frac{\partial}{\partial y_1}, \quad w_2 = \frac{\partial}{\partial y_2}, \quad w_3 = \alpha y_1 \frac{\partial}{\partial y_1} - \alpha y_2 \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}$$

are recognized as linearly independent vector fields for some constant α . We define the metric g as follows

$$g(w_j, w_j) = 1, \quad 1 \leq j \leq 3,$$

and otherwise is equal zero. We give $\xi = w_3$, $\eta(Y) = g(Y, w_3)$, and

$$\phi(w_1) = w_2, \quad \phi(w_2) = -w_1, \quad \phi(w_3) = 0.$$

The statements presented in equations (3) and (4) are indeed valid. Also, $[w_1, w_2] = 0$, $[w_1, w_3] = \alpha w_1$, $[w_2, w_3] = -\alpha w_2$ and other Lie brackets are equal zero. Also, we get

$$\nabla_{w_i}w_j = \begin{pmatrix} -\alpha w_3 & 0 & \alpha w_1 \\ 0 & -\alpha w_3 & -\alpha w_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, for any vector fields $X = \sum_{i=1}^3 X^i w_i$ and $Y = \sum_{i=1}^3 Y^i w_i$ we have $\nabla_X w_3 = \alpha(X^1 w_1 - X^2 w_2)$ and

$$g(Y, \nabla_X w_3) = \alpha(X^1 Y^1 - X^2 Y^2) = g(X, \nabla_Y w_3).$$

Now, for any vector fields X, Y we have

$$\begin{aligned} d\eta(X, Y) &= X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) \\ &= X(g(Y, w_3)) - Y(g(X, w_3)) - g([X, Y], w_3) \\ &= g(\nabla_X Y, w_3) + g(Y, \nabla_X w_3) - g(\nabla_Y X, w_3) - g(X, \nabla_Y w_3) \\ &\quad - g(\nabla_X Y - \nabla_Y X, w_3) = 0. \end{aligned}$$

Hence, $d\eta = 0$. The 2-form Φ is determined by

$$\Phi = -2dy_1 \wedge dy_2 - 2\alpha y_2 dy_1 \wedge dy_3 + 2\alpha y_1 dy_3 \wedge dy_2.$$

Thus, $d\Phi = 0$ and (M, ϕ, ξ, η, g) constitutes an ACK manifold. The Ricci tensor is determined as $S = -2\alpha^2 dy_3^2$. Let $V = w_2$, then $\mathcal{L}_V g = 2\alpha dy_2 dy_3$ and $(\mathcal{L}_V \circ \mathcal{L}_V)g = 2\alpha^2 dy_3^2$. Therefore $(g, w_2, 0, 0)$ is an AHRs on the manifold M .

Conclusion

The primary focus of this paper is to derive the geometric conditions and characteristics of AHRs, with the intention of applying these findings to ACK manifolds. Initially, we consider the parameters $(g, f\xi, \lambda, \mu)$ that conform to an AHRs and present a differential equation that the function f must satisfy. Furthermore, we outline specific geometric conditions under which ACK manifolds can be classified as AHRs. Our investigation also extends to the case of a HRS (g, V, λ, μ) on ACK manifolds, particularly when the vector field V is characterized as a conformal Killing, a torse-forming, or a bi-conformal vector field. This study has prompted several questions that may serve as avenues for future research.

1. If we consider the hyperbolic Yamabe solitons instead of hyperbolic Ricci solitons, what results can be obtained?
2. If we examine the $*$ -Ricci tensor in place of the Ricci tensor, specifically in the context of hyperbolic $*$ -Ricci solitons, do the conclusions of this paper remain applicable?
3. Are the results of this paper still valid when we take into account an alternative connection, such as the Shouten-van Kampen connection?

Author contributions

All authors contributed equally in the preparation of this manuscript.

Conflict of interest

The authors declare no conflict of interest.

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