MATHEMATICS

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Fractional differential equations with nonlocal boundary conditions involving initial and final segments of the given domain

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We explore the existence and uniqueness criteria for solutions of a Liouville–Caputo fractional differential equation with the nonlinearity containing the unknown function as well as its lower order fractional derivative, and supplemented with a set of nonlocal fractional boundary data with respect to initial and final segments of the given domain. Integral boundary conditions offer an effective approach to model the flow and drag phenomena in arbitrary shaped vessels, heat conduction, biomedical computational fluid dynamics, engineering problems, etc. The notion of segmental type nonlocal fractional integral boundary conditions introduced in this paper is novel and specializes to periodic/anti-periodic boundary data under a suitable choice of the parameters involved in these conditions (see the second last paragraph of Introduction). We apply Krasnosel'skiĭ's fixed point theorem and Leray-Schauder's nonlinear alternative to prove two existence results for the problem at hand, while the uniqueness of its solutions is established via Banach's contraction mapping principle. Examples are constructed for illustrating the obtained results. Our work is useful in the given configuration as it leads to a new direction for research on fractional boundary value problems. The paper concludes with some interesting observations.

Keywords: fractional differential equations, boundary value problems, Caputo fractional derivative operator, periodic/anti-periodic segmental boundary data, existence, uniqueness, nonlocal boundary conditions, fixed point.

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Introduction

We introduce a novel concept of nonlocal boundary conditions with respect to the segments $(0,\xi)$ and (η,T) of the domain $[0,T]$ and solve a Liouville–Caputo fractional differential equation with the nonlinearity depending upon the unknown function together with its lower order fractional derivative complemented with these conditions. When it is assumed that ξ is close to 0 and η is close to T, the

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periodic/anti-periodic phenomenon with respect to the segments $(0,\xi)$ and (η,T) can be observed. In precise terms, we consider the fractional boundary value problem given by

$$\begin{cases} {}^cD^{\alpha}x(t)=\varphi(t,x(t),\,{}^cD^{\kappa}x(t)),\ 1<\alpha\leq 2,\ \ 0<\kappa\leq 1,\ \ t\in[0,T], \\ \displaystyle\int_0^{\xi}x(s)ds=\delta_1\int_{\eta}^{T}x(s)ds, \int_0^{\xi}({}^cD^{p_1}x(s))ds=\delta_2\int_{\eta}^{T}({}^cD^{p_2}x(s))ds,\ 0<p_1,p_2<1, \end{cases} \tag{1}$$

where ${}^cD^{\varrho}$ denotes the Caputo fractional derivative operator of order ϱ, where $\varrho\in\{\alpha,\kappa,p_1,p_2\}$, δ_1 and δ_2 are real constants, $0<\xi<\eta<T$ and $\varphi:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Let us now dwell on some recent work on nonlinear nonlocal fractional boundary value problems. Fractional differential equations arise in a variety of disciplines of applied sciences, for example, physics and engineering [1], financial economics [2], relaxation filtration processes [3], chaos synchronization [4], etc. For theoretical aspects of fractional calculus, for example, see [5].

Nonlocal boundary conditions can model the physical phenomena experiencing the changes happening at arbitrary positions (nonlocal points and segments) inside the domain. Nonlocal integral boundary conditions can describe non-uniformities on the curved structures. For application details, see fluid flow problems [6], biomedical sciences [7], etc. One can find the engineering applications of strip type integral boundary conditions in the article [8]. For a variety of recent results on nonlocal fractional boundary value problems, we refer the reader to the book [9] and articles [10, 11].

Periodic/anti-periodic boundary value problems constitute a special form of non-separated (Sturm-Liouville) type boundary value problems. Anti-periodic fractional boundary conditions appear in a variety of applications and have been extensively studied in the literature. For a detailed description of anti-periodic boundary value problems involving different types of fractional derivative operators, for instance, see the articles [12, 13]. A new concept of dual anti-periodic boundary conditions was introduced in [14]. The authors in [15] studied p-Laplacian systems with rotating periodic boundary conditions.

The present work is motivated by the fact that a periodic event or pattern within the distinct initial and final sections of a specified range helps to understand its characteristics, verify its periodicity, or examine its behavior at critical points. A tiny segment of the sound wave reveals the pattern of changing air pressure. A brief recording of an electrocardiogram (EKG) will display the electrical activity associated with each heartbeat [16]. The daily temperature fluctuations often follow a somewhat periodic pattern and analyzing temperature data for a brief interval can reveal a part of this daily cycle. Some other examples include temporary oscillations that can occur in response to a sudden change in a system, and occurrence of spectral edges for periodic operators inside the $\mathbb{Z}$-periodic media [17].

The aim of the present study is to develop the existence theory for a fractional differential equation complemented with newly introduced segmental type nonlocal fractional boundary conditions. When ξ is close to 0 and η is close to T, the segmental fractional boundary conditions in (1) can be regarded as periodic and anti-periodic ones for $\delta_1=\delta_2=1$ and $\delta_1=\delta_2=-1$, respectively. On the other hand, the mixed periodic and anti-periodic boundary conditions follow by taking $\delta_1=1,\delta_2=-1$ or $\delta_1=-1,\delta_2=1$ (or vice versa) in (1).

We organize the rest of the article as follows. Section 1 contains some basic definitions and a subsidiary lemma. The two existence results for the problem (1), based on Krasnosel'skiĭ fixed point theorem and Leray-Schauder's nonlinear alternative, are derived in Section 2. We also prove a uniqueness result for the given problem by applying Banach's contraction mapping principle in this section. Illustrative examples for the main results are constructed in Section 3. In the last section, we describe some interesting observations.

1 Preliminaries

We begin this section by recalling some basic definitions.

Definition 1. [5] We define the (left) Riemann–Liouville fractional integral of order $\sigma > 0$ for the function $\vartheta \in L_1[a,b]$, denoted by $I^{\sigma}_{a^+}\vartheta$, as

$$I^{\sigma}_{a^+}\vartheta\,(v) = \int\limits_a^v \frac{(v-\widehat{v})^{\sigma-1}}{\Gamma(\sigma)}\vartheta\,(\widehat{v})\,d\widehat{v},$$

where Γ represents the Euler gamma function.

Definition 2. [5] Let $\vartheta, \vartheta^{(m)} \in L_1[a,b]$, $a,b \in \mathbb{R}$. The Riemann–Liouville fractional derivative of order $\sigma \in (m-1,m)$, $m \in \mathbb{N}$, denoted by $D^{\sigma}_{a^+}\vartheta$, is given by

$$D^{\sigma}_{a^+}\vartheta\,(v) = \frac{d^m}{dv^m} I^{m-\sigma}_{a^+}\vartheta\,(v) = \frac{1}{\Gamma(m-\sigma)}\frac{d^m}{dv^m}\int\limits_a^v (v-\widehat{v})^{m-1-\sigma}\,\vartheta\,(\widehat{v})d\widehat{v},$$

while the Caputo fractional derivative ${}^cD^{\sigma}_{a^+}\vartheta$ of order σ is defined by

$${}^cD^{\sigma}_{a^+}\vartheta\,(v) = D^{\sigma}_{a^+}\left[\vartheta\,(v) - \sum_{p=0}^{m-1}\vartheta^{(p)}(a)\frac{(v-a)^p}{p!}\right].$$

Remark 1. The (left) Caputo fractional derivative for a function $\vartheta \in AC^m[a,b]$ of order σ can also be defined as

$${}^cD^{\sigma}_{a^+}\vartheta(v) = \int_a^v \frac{(v-\widehat{v})^{m-\sigma-1}}{\Gamma(m-\sigma)}\vartheta^{(m)}(\widehat{v})d\widehat{v}.$$

In our article, we write the Riemann–Liouville fractional integral operator I^{σ} and the Caputo fractional derivative operator ${}^cD^{\sigma}$ instead of $I^{\sigma}_{0^+}$ and ${}^cD^{\sigma}_{0^+}$, respectively.
The following lemma deals with the linear version of the problem (1).

Lemma 1. Let $g \in C[0,T]$ and

$$\omega_1 = \frac{\xi^{2-p_1}}{\Gamma(3-p_1)} - \frac{\delta_2(T^{2-p_2} - \eta^{2-p_2})}{\Gamma(3-p_2)} \neq 0, \quad \omega_3 = \xi - \delta_1(T-\eta) \neq 0, \tag{2}$$

then the unique solution of the linear fractional differential equation

$${}^cD^{\alpha}x(t) = g(t),\ 1 < \alpha \leq 2, \tag{3}$$

equipped with the boundary conditions in (1) is

$$\begin{aligned} x(t) \;&=\; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds + \frac{1}{\omega_3}\int_\eta^T\int_0^s\left[\frac{\delta_1(s-u)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\delta_2(t\omega_3-\omega_2)}{\omega_1\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1}\right]g(u)duds \\ &-\frac{1}{\omega_3}\int_0^\xi\left[\frac{(\xi-s)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(t\omega_3-\omega_2)}{\omega_1\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1}\right]g(s)ds, \end{aligned} \tag{4}$$

where

$$\omega_2 = \frac{\xi^2 - \delta_1(T^2-\eta^2)}{2}. \tag{5}$$

Proof. Operating the Riemann–Liouville fractional integral operator I^α on both sides of (3) and using the formula (3.5.13) in [5], we find that

$$x(t) = I^\alpha g(t) + c_0 + c_1 t, \tag{6}$$

where $c_0, c_1 \in \mathbb{R}$ are unknown arbitrary constants, and

$$^cD^{p_i}x(t) = \int_0^t \frac{(t-s)^{\alpha-p_i-1}}{\Gamma(\alpha-p_i)} g(s)ds + c_1 \frac{t^{1-p_i}}{\Gamma(2-p_i)}, \ i=1,2. \tag{7}$$

Using (6) in the first boundary condition of (1), it is found that

$$\begin{aligned} c_0 &= \frac{1}{\xi-\delta_1(T-\eta)}\left[\delta_1 \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} g(u)duds - \int_0^\xi \frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} g(s)ds\right] \\ &\quad - c_1 \frac{(\xi^2-\delta_1(T^2-\eta^2))}{2(\xi-\delta_1(T-\eta))}, \end{aligned}$$

which, on using the notation (2) and (5), takes the form

$$c_0 = \frac{1}{\omega_3}\left(\delta_1 \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} g(u)duds - \int_0^\xi \frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} g(s)ds\right) - c_1 \frac{\omega_2}{\omega_3}. \tag{8}$$

Substituting (7) in the second boundary condition of (1) together with the notation (2), we obtain

$$c_1 = \frac{1}{\omega_1}\left[\delta_2 \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-p_2-1}}{\Gamma(\alpha-p_2)} g(u)duds - \int_0^\xi \frac{(\xi-s)^{\alpha-p_1}}{\Gamma(\alpha-p_1+1)} g(s)ds\right]. \tag{9}$$

Inserting the value of c_1 from (9) into (8), we find that

$$\begin{aligned} c_0 &= \frac{1}{\omega_3}\left(\delta_1 \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} g(u)duds - \int_0^\xi \frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} g(s)ds\right) \\ &\quad - \frac{\omega_2}{\omega_1\omega_3}\left[\delta_2 \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-p_2-1}}{\Gamma(\alpha-p_2)} g(u)duds - \int_0^\xi \frac{(\xi-s)^{\alpha-p_1}}{\Gamma(\alpha-p_1+1)} g(s)ds\right], \end{aligned}$$

where ω_1 and ω_3 are defined in (2), while ω_2 is given in (5). Substituting the above values of c_0 and c_1 in (6) yields the solution (4). The converse of the lemma follows by direct computation. □

2 *Existence and uniqueness results*

Let $\mathcal{U} = \{x : x, \, ^cD^\kappa x \in C([0,T],\mathbb{R})\}$ be a Banach space of all continuous functions defined on $[0,T]$ and equipped with the norm $\|x\|_{\mathcal{U}} = \max\{|x(t)| + |^cD^\kappa x(t)|, t \in [0,T], 0 < \kappa \leq 1\}$.

By Lemma 1, we introduce a fixed point problem $x = \mathcal{H}x$, which is equivalent to the problem (1), where $\mathcal{H} : \mathcal{U} \to \mathcal{U}$ is given by

$$\begin{aligned} (\mathcal{H}x(t)) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s, x(s), \, ^cD^\kappa x(s))ds \\ &\quad + \frac{1}{\omega_3} \int_\eta^T \int_0^s \left[\frac{\delta_1(s-u)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\delta_2(t\omega_3-\omega_2)}{\omega_1\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1}\right] \varphi(u, x(u), \, ^cD^\kappa x(u))duds \\ &\quad - \frac{1}{\omega_3} \int_0^\xi \left[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} + \frac{(t\omega_3-\omega_2)}{\omega_1\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1}\right] \varphi(s, x(s), \, ^cD^\kappa x(s))ds, \ t \in [0,T]. \end{aligned} \tag{10}$$

Next, we set

$$\tau_1 = \max_{t\in[0,T]}\left\{\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{|t\omega_3-\omega_2|}{|\omega_1\omega_3|}\left(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)}+\frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)}\right)\right\}$$
$$+\frac{1}{|\omega_3|}\left(\frac{|\delta_1(T^{\alpha+1}-\eta^{\alpha+1})|+\xi^{\alpha+1}}{\Gamma(\alpha+2)}\right), \tag{11}$$

$$\tau_2 = \frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)}+\frac{T^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)}\left[\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)}+\frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)}\right]. \tag{12}$$

In our first existence result for the problem (1), we make use of Krasnosel'skiĭ's fixed point theorem [18].

Theorem 1. Let $\varphi \in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and the following assumptions hold:

(A_1) for all $t\in[0,T]$, x_i, $y_i\in\mathbb{R}$, $i=1,2$, there exists a positive constant $\mathcal{L}$ such that

$$|\varphi(t,x_1,x_2)-\varphi(t,y_1,y_2)|\le\mathcal{L}(|x_1-y_1|+|x_2-y_2|);$$

(A_2) there exists a function $\psi\in C([0,T],\mathbb{R}^+)$ such that

$$|\varphi(t,x(t),\,{}^cD^\kappa x(t))|\le\psi(t),\quad (t,x,\,{}^cD^\kappa x)\in[0,T]\times\mathbb{R}\times\mathbb{R}.$$

Then, at least one solution to the problem (1) exists on $[0,T]$ if $\mu\mathcal{L}<1$, where

$$\mu=(\tau_1+\tau_2)-\left(\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)}\right), \tag{13}$$

τ_1 and τ_2 are given in (11) and (12), respectively.

Proof. We complete the proof by verifying the hypotheses of Krasnosel'skiĭ's fixed point theorem [18] in several steps. Let us first define the operators $\mathcal{H}_1,\mathcal{H}_2: B_\epsilon\to\mathcal{U}$ as follows:

$$(\mathcal{H}_1x)(t) = \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s,x(s),\,{}^cD^\kappa x(s))ds,$$
$$(\mathcal{H}_2x)(t) = \frac{1}{\omega_3}\int_\eta^T\int_0^s\left[\frac{\delta_1(s-u)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\delta_2(t\omega_3-\omega_2)}{\omega_1\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1}\right]\varphi(u,x(u),\,{}^cD^\kappa x(u))duds$$
$$-\frac{1}{\omega_3}\int_0^\xi\left[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)}+\frac{(t\omega_3-\omega_2)}{\omega_1\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1}\right]\varphi(s,x(s),\,{}^cD^\kappa x(s))ds,$$

where $B_\epsilon=\{x\in\mathcal{U}:\|x\|\le\epsilon\}$ with $\epsilon\ge\|\psi\|(\tau_1+\tau_2)$. Observe that $\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2$.

Step I. Setting $\max_{t\in[0,T]}|\psi(t)|=\|\psi\|$, and taking any $x,y\in B_\epsilon$, we find that

$$\|(\mathcal{H}_1x)+(\mathcal{H}_2y)\| = \max_{t\in[0,T]}|(\mathcal{H}_1x)(t)+(\mathcal{H}_2y)(t)|$$
$$\le \max_{t\in[0,T]}\left\{\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\right.$$
$$+\frac{1}{|\omega_3|}\int_\eta^T\int_0^s\left[\frac{|\delta_1|(s-u)^{\alpha-1}}{\Gamma(\alpha)}+\frac{|\delta_2||t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1}\right]$$
$$\times|\varphi(u,y(u),\,{}^cD^\kappa y(u))|duds$$
$$+\frac{1}{|\omega_3|}\int_0^\xi\left[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)}+\frac{|t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1}\right]$$
$$\left.\times|\varphi(s,y(s),\,{}^cD^\kappa y(s))|ds\right\}\le\|\psi\|\tau_1,$$

and

$$\begin{aligned}\|({}^cD^\kappa\mathcal{H}_1x)+({}^cD^\kappa\mathcal{H}_2y)\| &= \max_{t\in[0,T]}|({}^cD^\kappa\mathcal{H}_1x)(t)+({}^cD^\kappa\mathcal{H}_2y)(t)| \\ &\le \max_{t\in[0,T]}\bigg\{\int_0^t\frac{(t-s)^{\alpha-\kappa-1}}{\Gamma(\alpha-\kappa)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds \\ &\quad+\frac{t^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)}\bigg(|\delta_2|\int_\eta^T\int_0^s\frac{(s-u)^{\alpha-p_2-1}}{\Gamma(\alpha-p_2)}|\varphi(u,y(u),\,{}^cD^\kappa y(u))|duds \\ &\quad+\int_0^\xi\frac{(\xi-s)^{\alpha-p_1}}{\Gamma(\alpha-p_1+1)}|\varphi(s,y(s),\,{}^cD^\kappa y(s))|ds\bigg)\bigg\} \\ &\le \|\psi\|\tau_2.\end{aligned}$$

Thus, $\|\mathcal{H}_1x+\mathcal{H}_2y\|_{\mathcal{U}}\le\|\psi\|(\tau_1+\tau_2)\le\epsilon$. Therefore, $\mathcal{H}_1x+\mathcal{H}_2y\in B_\epsilon$.

Step II. We show that $\mathcal{H}_2$ is a contraction. By the assumption (A_1) and (13), for $x,y\in B_\epsilon$, we have

$$\begin{aligned}\|(\mathcal{H}_2x)-(\mathcal{H}_2y)\| &= \max_{t\in[0,T]}|(\mathcal{H}_2x)(t)-(\mathcal{H}_2y)(t)| \\ &\le \max_{t\in[0,T]}\bigg\{\frac{1}{|\omega_3|}\int_\eta^T\int_0^s\bigg[\frac{|\delta_1|(s-u)^{\alpha-1}}{\Gamma(\alpha)}+\frac{|\delta_2||t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1}\bigg] \\ &\quad\times|\varphi(u,x(u),\,{}^cD^\kappa x(u))-\varphi(u,y(u),\,{}^cD^\kappa y(u))|duds \\ &\quad+\frac{1}{|\omega_3|}\int_0^\xi\bigg[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)}+\frac{|t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1}\bigg] \\ &\quad\times|\varphi(s,x(s),\,{}^cD^\kappa x(s))-\varphi(s,y(s),\,{}^cD^\kappa y(s))|ds\bigg\} \\ &\le \mathcal{L}\bigg(\tau_1-\frac{T^\alpha}{\Gamma(\alpha+1)}\bigg)\|x-y\|,\end{aligned}$$

and

$$\begin{aligned}&\|({}^cD^\kappa\mathcal{H}_2x)-({}^cD^\kappa\mathcal{H}_2y)\| \\ =\ &\max_{t\in[0,T]}|({}^cD^\kappa\mathcal{H}_2x)(t)-({}^cD^\kappa\mathcal{H}_2y)(t)| \\ \le\ &\max_{t\in[0,T]}\bigg\{\frac{t^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)}\bigg(|\delta_2|\int_\eta^T\int_0^s\frac{(s-u)^{\alpha-p_2-1}}{\Gamma(\alpha-p_2)}|\varphi(u,x(u),\,{}^cD^\kappa x(u)) \\ &-\varphi(u,y(u),\,{}^cD^\kappa y(u))|duds+\int_0^\xi\frac{(\xi-s)^{\alpha-p_1}}{\Gamma(\alpha-p_1+1)}|\varphi(s,x(s),\,{}^cD^\kappa x(s)) \\ &-\varphi(s,y(s),\,{}^cD^\kappa y(s))|ds\bigg)\bigg\}\le\mathcal{L}\bigg(\tau_2-\frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)}\bigg)\|x-y\|.\end{aligned}$$

From the foregoing inequalities, we find that

$$\|(\mathcal{H}_2x)-(\mathcal{H}_2y)\|_{\mathcal{U}}=\|(\mathcal{H}_2x)-(\mathcal{H}_2y)\|+\|({}^cD^\kappa\mathcal{H}_2x)-({}^cD^\kappa\mathcal{H}_2y)\|\le\mathcal{L}\mu\|x-y\|.$$

Therefore, $\mathcal{H}_2$ is a contraction according to the assumption $\mu\mathcal{L}<1$.

Step III. Here, it will be shown that $\mathcal{H}_1$ is completely continuous.

Note that continuity of $\varphi(t,x(t),\,{}^cD^\kappa x(t))$ implies that of the operator $\mathcal{H}_1$. For $x\in B_\epsilon$, we obtain

$$\|(\mathcal{H}_1x)\|=\max_{t\in[0,T]}|(\mathcal{H}_1x)(t)|\ \le\ \max_{t\in[0,T]}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\le\|\psi\|\frac{T^\alpha}{\Gamma(\alpha+1)},$$

and

$$\|({}^cD^\kappa\mathcal{H}_1x)\| = \max_{t\in[0,T]}|({}^cD^\kappa\mathcal{H}_1x)(t)| \le \max_{t\in[0,T]}\int_0^t \frac{(t-s)^{\alpha-\kappa-1}}{\Gamma(\alpha-\kappa)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds \le \|\psi\|\frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)}.$$

Hence, $\mathcal{H}_1$ is uniformly bounded.

Next, the operator $\mathcal{H}_1$ is shown to be equicontinuous. Let $\max\limits_{t\in[0,T]}|\varphi(t,x(t),\,{}^cD^\kappa x(t))| = \widehat{\varphi} < \infty$ for $(t,x,\,{}^cD^\kappa x) \in [0,T]\times {B_\epsilon}^2$. Then, for $0<\nu_1<\nu_2<T$, we have

$$\begin{aligned}|(\mathcal{H}_1x)(\nu_2)-(\mathcal{H}_1x)(\nu_1)| &\le \int_0^{\nu_1}\frac{|(\nu_2-s)^{\alpha-1}-(\nu_1-s)^{\alpha-1}|}{\Gamma(\alpha)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\\ &\quad+\int_{\nu_1}^{\nu_2}\frac{|(\nu_2-s)^{\alpha-1}|}{\Gamma(\alpha)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\\ &\le \widehat{\varphi}\left(\frac{2|(\nu_2-\nu_1)^\alpha|+|\nu_2^\alpha-\nu_1^\alpha|}{\Gamma(\alpha+1)}\right)\to 0 \text{ as } \nu_2\to\nu_1,\end{aligned}$$

independently of $x\in B_\epsilon$. Likewise, we have

$$\begin{aligned}|({}^cD^\kappa\mathcal{H}_1x)(\nu_2)-({}^cD^\kappa\mathcal{H}_1x)(\nu_1)| &\le \int_0^{\nu_1}\frac{|(\nu_2-s)^{\alpha-\kappa-1}-(\nu_1-s)^{\alpha-\kappa-1}|}{\Gamma(\alpha-\kappa)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\\ &\quad+\int_{\nu_1}^{\nu_2}\frac{|(\nu_2-s)^{\alpha-\kappa-1}|}{\Gamma(\alpha-\kappa)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\\ &\le \widehat{\varphi}\left(\frac{2|(\nu_2-\nu_1)^{\alpha-\kappa}|+|\nu_2^{\alpha-\kappa}-\nu_1^{\alpha-\kappa}|}{\Gamma(\alpha-\kappa+1)}\right)\to 0\end{aligned}$$

as $\nu_2\to\nu_1$ independently of $x\in B_\epsilon$. Therefore, the operator $\mathcal{H}_1$ is relatively compact on B_ϵ. In view of the foregoing steps, we deduce by the Arzelá-Ascoli theorem [18] that the operator $\mathcal{H}_1$ is compact on B_ϵ. As the hypotheses of Krasnosel'skiĭ's fixed point theorem [18] are verified, it follows by its conclusion that there exists at least one solution to the problem (1) on $[0,T]$. □

Our second existence result for the problem (1) is based on Leray-Schauder's nonlinear alternative [18].

Theorem 2. Let $\varphi\in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and the following assumptions hold:

(A_3) there exist continuous nondecreasing functions $\Psi:[0,\infty)\to(0,\infty)$ and a function $\upsilon\in C([0,T],\mathbb{R}^+)$ satisfying $|\varphi(t,x,\,{}^cD^\kappa x)|\le\upsilon(t)\Psi(\|x\|_\mathcal{U})$, for each $(t,x,\,{}^cD^\kappa x)\in[0,T]\times\mathbb{R}\times\mathbb{R}$;

(A_4) there exists a constant $\mathcal{K}>0$ such that

$$\frac{\mathcal{K}}{\Psi(\mathcal{K})\|\upsilon\|(\tau_1+\tau_2)}>1,$$

where τ_1 and τ_2 are respectively given in (11) and (12).

Then, at least one solution to the problem (1) exists on $[0,T]$.

Proof. We verify the assumptions of Leray-Schauder's nonlinear alternative [18] in different steps.
Step 1. The operator $\mathcal{H}$ maps bounded sets (balls) into bounded sets in $C([0,T],\mathbb{R})$.

Consider a bounded closed ball $B_R = \{x \in C([0,T],\mathbb{R}) : \|x\|_{\mathcal{U}} \leq R\}$ in $C([0,T],\mathbb{R})$. Then, for any $x \in B_R$, we obtain

$$\begin{aligned}
\|(\mathcal{H}x)\| &= \max_{t\in[0,T]} |\mathcal{H}x(t)| \\
&\leq \max_{t\in[0,T]} \Bigg\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(s,x(s),\, {}^cD^\kappa x(s))| ds \\
&\quad + \frac{1}{|\omega_3|} \int_\eta^T \int_0^s \left[\frac{|\delta_1|(s-u)^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\delta_2||t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_2)} (s-u)^{\alpha-p_2-1} \right] \\
&\quad \times |\varphi(u,x(u),\, {}^cD^\kappa x(u))| duds \\
&\quad + \frac{1}{|\omega_3|} \int_0^\xi \left[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} + \frac{|t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_1+1)} (\xi-s)^{\alpha-p_1} \right] |\varphi(s,x(s),\, {}^cD^\kappa x(s))| ds \Bigg\} \\
&\leq \|\upsilon\|\Psi(R) \Bigg[\max_{t\in[0,T]} \Bigg\{ \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{|t\omega_3-\omega_2|}{|\omega_1\omega_3|} \Bigg(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} \\
&\quad + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)} \Bigg) \Bigg\} + \frac{1}{|\omega_3|} \Bigg(\frac{|\delta_1(T^{\alpha+1}-\eta^{\alpha+1})| + \xi^{\alpha+1}}{\Gamma(\alpha+2)} \Bigg) \Bigg] \\
&= \|\upsilon\|\Psi(R)\tau_1.
\end{aligned}$$

Similarly, one can find that

$$\begin{aligned}
\|({}^cD^\kappa \mathcal{H}x)\| &= \max_{t\in[0,T]} |({}^cD^\kappa \mathcal{H}x)(t)| \\
&\leq \max_{t\in[0,T]} \Bigg\{ \int_0^t \frac{(t-s)^{\alpha-\kappa-1}}{\Gamma(\alpha-\kappa)} |\varphi(s,x(s),\, {}^cD^\kappa(s))| ds \\
&\quad + \frac{t^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)} \Bigg[|\delta_2| \int_\eta^T \int_0^s \frac{(s-u)^{\alpha-p_2-1}}{\Gamma(\alpha-p_2)} |\varphi(u,x(u),\, {}^cD^\kappa x(u))| duds \\
&\quad + \int_0^\xi \frac{(\xi-s)^{\alpha-p_1}}{\Gamma(\alpha-p_1+1)} |\varphi(s,x(s),\, {}^cD^\kappa x(s))| ds \Bigg] \Bigg\} \\
&\leq \|\upsilon\|\Psi(R) \Bigg[\frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)} + \frac{T^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)} \Bigg(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} \\
&\quad + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)} \Bigg) \Bigg] \\
&= \|\upsilon\|\Psi(R)\tau_2.
\end{aligned}$$

From the last two inequalities, we have

$$\|\mathcal{H}x\|_{\mathcal{U}} = \|\mathcal{H}x\| + \|{}^cD^\kappa \mathcal{H}x\| \leq \|\upsilon\|\Psi(R)(\tau_1+\tau_2),$$

which shows that the operator $\mathcal{H}$ maps bounded sets (balls) into bounded sets in $C([0,T],\mathbb{R})$.

Step 2. The operator $\mathcal{H}$ maps bounded set into equicontinuous set of $C([0,T],\mathbb{R})$.

Letting $\gamma_1, \gamma_2 \in [0,T]$ with $\gamma_1 < \gamma_2$ and $x \in B_R$, we get

$$\begin{aligned}
|(\mathcal{H}x)(\gamma_2) - (\mathcal{H}x)(\gamma_1)| &\leq \|\upsilon\|\Psi(R) \Bigg\{ \frac{2|(\gamma_2-\gamma_1)^\alpha| + |\gamma_2^\alpha - \gamma_1^\alpha|}{\Gamma(\alpha+1)} \\
&\quad + \frac{|\gamma_2-\gamma_1|}{|\omega_1|} \Bigg(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)} \Bigg) \Bigg\},
\end{aligned}$$

and

$$|({}^cD^\kappa\mathcal{H}x)(\gamma_2) - ({}^cD^\kappa\mathcal{H}x)(\gamma_1)| \leq \|\upsilon\|\Psi(R)\Bigg\{\frac{2|(\gamma_2-\gamma_1)^{\alpha-\kappa}| + |\gamma_2^{\alpha-\kappa} - \gamma_1^{\alpha-\kappa}|}{\Gamma(\alpha-\kappa+1)} + \frac{|\gamma_2^{1-\kappa} - \gamma_1^{1-\kappa}|}{|\omega_1|\Gamma(2-\kappa)}\left[\frac{|\delta_2(T^{\alpha-p_2+1} - \eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)}\right]\Bigg\}.$$

Since the right-hand sides of the last two inequalities tend to zero independently of $x \in B_R$ as $\gamma_2 \to \gamma_1$, so, the operator $\mathcal{H} : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is completely continuous by an application of the Arzelá–Ascoli theorem [18].

Step 3. Suppose that there exists $x \in C([0,T],\mathbb{R})$ with $x = \zeta\mathcal{H}x$, $\zeta \in (0,1)$. As in the first part of the proof, we can obtain that

$$\frac{\|x\|_{\mathcal{U}}}{\|\upsilon\|\Psi(\|x\|_{\mathcal{U}})(\tau_1+\tau_2)} \leq 1.$$

By the condition (A_4), there exists $\mathfrak{J} > 0$ satisfying $\|x\|_{\mathcal{U}} \neq \mathfrak{J}$. Define $\mathcal{M} = \{x \in \mathcal{U} : \|x\|_{\mathcal{U}} < \mathfrak{J}\}$ and observe that $\mathcal{H} : \overline{\mathcal{M}} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. By the definition of the set $\mathcal{M}$, there does not exist any $x \in \partial\mathcal{M}$ satisfying $x = \zeta\mathcal{H}x$ for some $\zeta \in (0,1)$. In consequence, it follows by Leray–Schauder's nonlinear alternative [18] that the operator $\mathcal{H}$ has a fixed point $x \in \overline{\mathcal{M}}$. Hence, the problem (1) has at least one solution on $[0,T]$. □

Finally, we accomplish a uniqueness result for the problem (1) by applying Banach's fixed point theorem.

Theorem 3. Suppose that $\varphi \in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ satisfies the condition (A_1). If $\mathcal{L}(\tau_1+\tau_2) < 1$, where τ_1 and τ_2 are respectively given by (11) and (12), then there exists a unique solution to the problem (1) on $[0,T]$.

Proof. We first show that $\mathcal{H}B_\vartheta \subset B_\vartheta$, where $\mathcal{H} : \mathcal{U} \to \mathcal{U}$ is defined in (10) and $B_\vartheta = \{x \in \mathcal{U} : \|x\| \leq \vartheta\}$ with $\vartheta \geq \dfrac{N(\tau_1+\tau_2)}{1-\mathcal{L}(\tau_1+\tau_2)}$ and $\max_{t\in[0,T]}|\varphi(t,0,0)| = N < \infty$. For $x \in B_\vartheta$, $t \in [0,T]$, by the assumption (A_1) and notation (11), we get

$$\begin{aligned}
\|(\mathcal{H}x)\| &= \max_{t\in[0,T]}|(\mathcal{H}x)(t)| \\
&\leq \max_{t\in[0,T]}\Bigg\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds \\
&\quad + \frac{1}{|\omega_3|}\int_\eta^T\int_0^s\left[\frac{|\delta_1|(s-u)^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\delta_2||t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_2)}(s-u)^{\alpha-p_2-1}\right]|\varphi(u,x(u),\,{}^cD^\kappa x(u))|duds \\
&\quad + \frac{1}{|\omega_3|}\int_0^\xi\left[\frac{(\xi-s)^\alpha}{\Gamma(\alpha+1)} + \frac{|t\omega_3-\omega_2|}{|\omega_1|\Gamma(\alpha-p_1+1)}(\xi-s)^{\alpha-p_1}\right]|\varphi(s,x(s),\,{}^cD^\kappa x(s))|ds\Bigg\} \\
&\leq (\mathcal{L}\vartheta+N)\Bigg[\max_{t\in[0,T]}\Bigg\{\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{|t\omega_3-\omega_2|}{|\omega_1\omega_3|}\Bigg(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} \\
&\quad + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)}\Bigg)\Bigg\} + \frac{1}{|\omega_3|}\left(\frac{|\delta_1(T^{\alpha+1}-\eta^{\alpha+1})| + \xi^{\alpha+1}}{\Gamma(\alpha+2)}\right)\Bigg] = (\mathcal{L}\vartheta+N)\tau_1,
\end{aligned}$$

and

$$\begin{aligned}
\|({}^cD^\kappa\mathcal{H}x)\| &\leq (\mathcal{L}\vartheta+N)\Bigg[\frac{T^{\alpha-\kappa}}{\Gamma(\alpha-\kappa+1)} + \frac{T^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)}\Bigg(\frac{|\delta_2(T^{\alpha-p_2+1}-\eta^{\alpha-p_2+1})|}{\Gamma(\alpha-p_2+2)} \\
&\quad + \frac{\xi^{\alpha-p_1+1}}{\Gamma(\alpha-p_1+2)}\Bigg)\Bigg] = (\mathcal{L}\vartheta+N)\tau_2.
\end{aligned}$$

In consequence, we obtain

$$\|\mathcal{H}x\|_{\mathcal{U}} = \|\mathcal{H}x\| + \|{}^cD^{\kappa}\mathcal{H}x\| \leq (\mathcal{L}\vartheta + N)(\tau_1 + \tau_2) \leq \vartheta.$$

Therefore, $\mathcal{H}B_\vartheta \subset B_\vartheta$ as $x \in B_\vartheta$ is an arbitrary element. Next, we accomplish that $\mathcal{H}$ is a contraction. For $x, y \in B_\vartheta$ and $t \in [0,T]$, we get

$$\begin{aligned}
\|(\mathcal{H}x) - (\mathcal{H}y)\| &= \max_{t\in[0,T]} |(\mathcal{H}x)(t) - (\mathcal{H}y)(t)| \\
&\leq \max_{t\in[0,T]} \Bigg\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(s, x(s),\, {}^cD^{\kappa}x(s)) - \varphi(s, y(s),\, {}^cD^{\kappa}y(s))| ds \\
&\quad + \frac{1}{|\omega_3|} \int_\eta^T \int_0^s \left[\frac{|\delta_1|(s-u)^{\alpha-1}}{\Gamma(\alpha)} + \frac{|\delta_2||t\omega_3 - \omega_2|}{|\omega_1|\Gamma(\alpha - p_2)} (s-u)^{\alpha - p_2 - 1} \right] \\
&\quad \times |\varphi(u, x(u),\, {}^cD^{\kappa}x(u)) - \varphi(u, y(u),\, {}^cD^{\kappa}y(u))| du ds \\
&\quad + \frac{1}{|\omega_3|} \int_0^\xi \left[\frac{(\xi - s)^\alpha}{\Gamma(\alpha+1)} + \frac{|t\omega_3 - \omega_2|}{|\omega_1|\Gamma(\alpha - p_1 + 1)} (\xi - s)^{\alpha - p_1} \right] \\
&\quad \times |\varphi(s, x(s),\, {}^cD^{\kappa}x(s)) - \varphi(s, y(s),\, {}^cD^{\kappa}y(s))| ds \Bigg\} \\
&\leq \mathcal{L}\|x-y\| \Bigg[\max_{t\in[0,T]} \Bigg\{ \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{|t\omega_3 - \omega_2|}{|\omega_1\omega_3|} \Bigg(\frac{|\delta_2(T^{\alpha - p_2 + 1} - \eta^{\alpha - p_2 + 1})|}{\Gamma(\alpha - p_2 + 2)} \\
&\quad + \frac{\xi^{\alpha - p_1 + 1}}{\Gamma(\alpha - p_1 + 2)} \Bigg) \Bigg\} + \frac{1}{|\omega_3|} \Bigg(\frac{|\delta_1(T^{\alpha+1} - \eta^{\alpha+1})| + \xi^{\alpha+1}}{\Gamma(\alpha+2)} \Bigg) \Bigg] \\
&= \tau_1 \mathcal{L}\|x-y\|,
\end{aligned}$$

and

$$\begin{aligned}
\|({}^cD^{\kappa}\mathcal{H}x) - ({}^cD^{\kappa}\mathcal{H}y)\| &= \max_{t\in[0,T]} |({}^cD^{\kappa}\mathcal{H}x)(t) - ({}^cD^{\kappa}\mathcal{H}y)(t)| \\
&\leq \mathcal{L}\|x-y\| \Bigg[\frac{T^{\alpha-\kappa}}{\Gamma(\alpha - \kappa + 1)} + \frac{T^{1-\kappa}}{|\omega_1|\Gamma(2-\kappa)} \Bigg(\frac{|\delta_2(T^{\alpha - p_2 + 1} - \eta^{\alpha - p_2 + 1})|}{\Gamma(\alpha - p_2 + 2)} \\
&\quad + \frac{\xi^{\alpha - p_1 + 1}}{\Gamma(\alpha - p_1 + 2)} \Bigg) \Bigg] = \tau_2 \mathcal{L}\|x-y\|.
\end{aligned}$$

Combining the foregoing inequalities, we have $\|\mathcal{H}x - \mathcal{H}y\|_{\mathcal{U}} \leq \mathcal{L}(\tau_1 + \tau_2)\|x-y\|_{\mathcal{U}}$, which shows that $\mathcal{H}$ is a contraction since $\mathcal{L}(\tau_1 + \tau_2) < 1$. Hence, by Banach's contraction mapping principle, there exists a unique fixed point for the operator $\mathcal{H}$. In consequence, a unique solution to the problem (1) exists on $[0,T]$. □

3 Examples

In this section, we construct examples to illustrate the results derived in the last two sections.

Example 1. Consider a segmental fractional boundary value problem given by

$$\begin{aligned}
&{}^cD^{\frac{3}{2}}x(t) = \varphi(t, x(t),\, {}^cD^{\frac{1}{3}}x(t)),\ t \in [0,1], \\
&\int_0^{\frac{1}{9}} x(s)ds = \frac{1}{6}\int_{\frac{9}{10}}^1 x(s)ds,\ \int_0^{\frac{1}{9}} ({}^cD^{\frac{1}{4}}x(s))ds = \frac{1}{5}\int_{\frac{9}{10}}^1 ({}^cD^{\frac{1}{2}}x(s))ds.
\end{aligned} \tag{14}$$

Here, $\alpha = \frac{3}{2}, T = 1$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{2}$, $\delta_1 = \frac{1}{6}$, $\delta_2 = \frac{1}{5}$, $\xi = \frac{1}{9}$, $\eta = \frac{9}{10}$, $\kappa = \frac{1}{3}$. Using the given data, it is found that $\omega_1 \simeq -0.00869849$, $\omega_2 \simeq -0.00966049$, $\omega_3 \simeq 0.09444444$, $\tau_1 \simeq 3.65035558$, $\tau_2 \simeq 3.69959683$, where τ_1, τ_2 are given in (11) and (12), respectively.

(i) We illustrate Theorem 1 by considering

$$\varphi(t, x(t), {}^cD^{\frac{1}{3}}x(t)) = \frac{1}{t^2+36}\left(\frac{(1+x)^2}{1+(1+x)^2} + \frac{|{}^cD^{\frac{1}{3}}x(t)|}{1+|{}^cD^{\frac{1}{3}}x(t)|}\right) + \frac{e^{-t}}{\sqrt{t^2+4}}. \tag{15}$$

Observe that

$$|\varphi(t, x(t), {}^cD^{\frac{1}{3}}x(t))| \leq \frac{2}{t^2+36} + \frac{e^{-t}}{\sqrt{t^2+4}} = \psi(t),$$

and $\|\psi\| = \frac{5}{9}$, $\mathcal{L} = \frac{1}{36}$. Furthermore, $\mu \simeq 5.67377488$, and $\mu\mathcal{L} \simeq 0.15760486 < 1$, where μ is given in (13). Thus, by Theorem 1, there exists at least one solution to the problem (14) with $\varphi(t, x(t), {}^cD^{\frac{1}{3}}x(t))$ defined in (15) on $[0, 1]$.

(ii) For illustrating Theorem 2, we consider

$$\varphi(t, x(t), {}^cD^{\frac{1}{3}}x(t)) = \frac{1}{\sqrt{t^6+625}}\left(\frac{|x|^2}{1+|x|} + \sin({}^cD^{\frac{1}{3}}x(t)) + \frac{1}{2}\right), \tag{16}$$

and find that

$$|\varphi(t, x(t), {}^cD^{\frac{1}{3}}x(t))| \leq \frac{1}{\sqrt{t^6+625}}\left(\|x\|_{\mathcal{U}} + \frac{1}{2}\right).$$

Clearly, $\upsilon(t) = \frac{1}{\sqrt{t^6+625}}$ with $\|\upsilon\| = \frac{1}{25}$ and $\Psi(\|x\|_{\mathcal{U}}) = \|x\|_{\mathcal{U}} + \frac{1}{2}$. By the condition (A_4), we find that $\mathcal{K} > \mathcal{K}_1$, where $\mathcal{K}_1 \simeq 0.20821339$. Thus, the assumptions of Theorem 2 hold true and hence, its conclusion implies that the problem (14) with $\varphi(t, x(t), {}^cD^{\frac{1}{3}}x(t))$ given by (16) has at least one solution on $[0, 1]$.

(iii) For explaining Theorem 3, we take

$$\varphi(t, x(t), {}^cD^{\frac{1}{3}}x(t)) = \frac{1}{(t^2+25)}\left(\sqrt{([{}^cD^{\frac{1}{3}}x(t)]^2+4)} + \tan^{-1}x(t)\right) + \frac{3}{5+t^3}, \tag{17}$$

and note that $\varphi(t, x(t), {}^cD^{\frac{1}{3}}x(t))$ satisfies the condition (A_1) with $\mathcal{L} = \frac{1}{25}$ and $\mathcal{L}(\tau_1+\tau_2) \simeq 0.293998096 < 1$. As all the assumptions of Theorem 3 hold true, so its conclusion applies to the problem (14) with $\varphi(t, x(t), {}^cD^{\frac{1}{3}}x(t))$ given in (17).

Conclusion

We introduced a new notion of segmental type nonlocal fractional boundary conditions and obtained existence and uniqueness results for a more general type fractional differential equation complemented with these conditions. Though the standard tools of the fixed point theory are employed to study the problem (1), yet their imposition to the given problem produces new results for it. Moreover, the results for segmental type periodic, anti-periodic and mixed periodic-anti-periodic boundary conditions follow as special cases by fixing the parameters δ_1 and δ_2 in the results accomplished in this article as described in the second last paragraph of Introduction section. In case $p_1 = p_2 = p$, our results correspond to the ones with boundary conditions

$$\int_0^{\xi} x(s)ds = \delta_1 \int_{\eta}^{T} x(s)ds, \int_0^{\xi} ({}^cD^p x(s))ds = \delta_2 \int_{\eta}^{T} ({}^cD^p x(s))ds, \ 0 < p < 1.$$

Thus, our results are not only useful in the given configuration but also give rise a new avenue for research on fractional boundary value problems.

Author Contributions

All authors contributed equally to this work, participated in its revision and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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