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Research article

Statistical properties of neutrosophic n -normed spaces in a non-Archimedean framework

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The notion of statistical convergence is a natural extension of ordinary convergence which has been studied by several authors in different spaces. Recently, several studies have shown a growing interest in convergence theory in neutrosophic spaces. In this research article, we investigate the statistical convergence and related notions of sequences in a non-Archimedean neutrosophic n -normed space. By combining natural density with the three neutrosophic components of truth-membership, indeterminacy-membership, and falsity-membership, we formulate statistical convergence with respect to the neutrosophic n -norm structure. Several equivalent characterizations of statistical convergence are established, including formulations in terms of exceptional index sets of natural density zero and componentwise statistical limits. We prove that the statistical limit, whenever it exists, is unique. Moreover, we show that ordinary convergence with respect to the neutrosophic n -norm implies statistical convergence, and we examine the corresponding behavior of algebraic operations, particularly the preservation of statistical convergence under addition. Statistical Cauchy sequences are also introduced and studied. In particular, every Cauchy sequence in the neutrosophic n -norm sense is shown to be statistically Cauchy. The notion of statistical completeness is then discussed through statistically Cauchy sequences and their statistical limits. Finally, illustrative examples in finite-dimensional real spaces are provided to clarify the abstract definitions and demonstrate sequences that are statistically convergent even when ordinary convergence fails. These results extend classical statistical convergence concepts to a neutrosophic non-Archimedean setting. The developed framework also provides a foundation for investigations of summability, approximation, and nonlinear analytical problems.

Keywords: natural density, statistical convergence, statistical Cauchy sequence, non-Archimedean field, neutrosophic norm, neutrosophic n -normed space, statistical completeness, statistical limit, non-Archimedean neutrosophic n -normed space (NA- Nn NS), neutrosophic Banach space.

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Introduction

The notion of statistical convergence was introduced by Steinhaus [1] and Fast [2], independently. Since then, statistical convergence has been studied in many generalized settings. Suja and Srinivasan considered statistically convergent and statistically Cauchy sequences in non-Archimedean fields [3], while Chu and Ku studied non-Archimedean n -normed spaces in connection with the Mazur–Ulam problem [4].

The development of generalized normed spaces is closely related to the theory of fuzzy sets introduced by Zadeh [5]. Fuzzy n -normed linear spaces were later introduced by Narayanan and Vijayabalaji [6]. Deferred

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statistical convergence in intuitionistic fuzzy normed spaces was studied by Melliani et al. [7], and generalized statistical convergence in non-Archimedean L -fuzzy normed spaces was investigated by Eghbali and Ganji [8]. Intuitionistic fuzzy n -normed linear spaces were developed by Vijayabalaji et al. [9], while statistical convergence in intuitionistic fuzzy n -normed linear spaces was studied by Sen and Debnath [10]. Konwar and Debnath further considered intuitionistic fuzzy n -normed algebra and continuous product structures [11]. Recently, Saranya and Suja examined statistical convergence on intuitionistic fuzzy n -normed spaces over non-Archimedean fields [12].

Further developments in convergence theory include multidimensional sliding window Pringsheim convergence for measurable functions studied by Savaş and Patterson [13], summability approximation theorems of double random variables [14], and multidimensional characterizations of asymptotically I_2^g -equivalent sequences [15]. In the neutrosophic setting, Shankar et al. investigated lacunary Δ^m -statistical convergence in neutrosophic normed spaces [16]. Debnath et al. studied deferred statistical convergence of sequences in neutrosophic normed spaces [17]. Kirişci and Şimşek introduced neutrosophic normed spaces and statistical convergence [18]. Mursaleen developed λ -statistical convergence [19], and Kumar et al. studied neutrosophic n -normed linear spaces [20]. Recently, Kişi studied ideal convergence in neutrosophic normed spaces [21] and also investigated I_θ -convergence in the same setting [22]. Kişi and Gürdal examined triple difference sequences in neutrosophic normed spaces [23]. Khan et al. studied neutrosophic normed spaces via Tribonacci convergent sequence spaces [24], while Khan et al. also considered neutrosophic λ -statistically convergent sequence spaces and their properties [25]. Gönül Bilgin investigated arithmetic statistical convergence [26] and Hilbert I -statistical convergence in neutrosophic normed spaces [27]. Khan et al. studied statistical convergence of sequences of functions in neutrosophic normed spaces [28], and rough lacunary statistical convergence was examined by Khan et al. [29]. Ersan considered ρ -statistical convergence in neutrosophic normed spaces [30]. Rough ideal statistical convergence was studied by Antal et al. [31], and λ - Δ^m -statistical convergence was investigated by Antal and Chawla [32]. Khan et al. introduced neutrosophic fuzzy Tribonacci $\mathcal{I}$ -lacunary statistical convergent sequence spaces [33]. Mursaleen et al. studied rough deferred statistical convergence in neutrosophic normed spaces [34]. Kaur et al. investigated neutrosophic Copson type rough I -statistically convergent sequences [35]. Jenifer et al. examined rough ideal statistical convergence in neutrosophic normed spaces [36]. Khan et al. studied Hilbert ideal convergent sequence spaces in neutrosophic normed spaces [37]. Khan and Arshad applied neutrosophic normed spaces to the convergence of sequences involving neutrosophic operators [38]. Most recently, Kişi et al. discussed recent advances in the convergence of function sequences in neutrosophic normed spaces via ideals [39].

Motivated by the above developments, this paper investigates statistical convergence in non-Archimedean neutrosophic n -normed spaces. We introduce statistical convergence with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$, establish equivalent characterizations, prove the uniqueness of the statistical limit, and study algebraic properties such as preservation under addition. Statistical Cauchy sequences are also introduced, and the relationship between ordinary Cauchy sequences and statistically Cauchy sequences is discussed. The notion of statistical completeness is then considered, and examples are provided to illustrate that statistical convergence may hold even when ordinary convergence fails.

In this study, we work with sequences defined over a non-Archimedean field $\mathcal{K}$. Unless stated otherwise, $\mathcal{K}$ refers to a complete non-Archimedean field equipped with a nontrivial valuation, and n represents a positive integer. The study begins with a brief overview of essential notations and definitions that will be used throughout.

1 Preliminary concepts

Definition 1. [12] Let $\mathcal{U}$ be a vector space of dimension $d \geq n$ over a valued field $\mathcal{K}$, equipped with a non-Archimedean valuation $|\cdot|$. A mapping

$$\|\cdot, \cdot, \dots, \cdot\| : \underbrace{\mathcal{U} \times \mathcal{U} \times \dots \times \mathcal{U}}_{n \text{ times}} \rightarrow [0, \infty)$$

is called a *non-Archimedean n -norm* if it satisfies the following conditions:

- (i) $\|\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\| = 0 \Leftrightarrow \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ are linearly dependent;
- (ii) $\|\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\| = \|\tilde{x}_{\tilde{r}_1}, \tilde{x}_{\tilde{r}_2}, \dots, \tilde{x}_{\tilde{r}_n}\|$ for every permutation $(\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_n)$ of $(1, 2, \dots, n)$;
- (iii) $\|a\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\| = |a| \|\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n\|$;
- (iv) $\|\tilde{x} + \tilde{y}, \tilde{x}_2, \tilde{x}_3, \dots, \tilde{x}_n\| \leq \max\{\|\tilde{x}, \tilde{x}_2, \dots, \tilde{x}_n\|, \|\tilde{y}, \tilde{x}_2, \dots, \tilde{x}_n\|\}$ for all $\alpha \in \mathcal{K}$ and $\tilde{x}, \tilde{y}, \tilde{x}_2, \dots, \tilde{x}_n \in \mathcal{U}$. Then $(\mathcal{U}, \|\cdot, \cdot, \dots, \cdot\|)$ is a non-Archimedean n -normed space.

Example 1. Let $\mathcal{U} = \mathbb{R}^n$ and define the n -norm $\|\cdot, \dots, \cdot\|_{\widetilde{M}}$ for vectors $\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n \in \mathbb{R}^n$ by

$$\|\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n\|_{\widetilde{M}} = \prod_{j=1}^n \max_{1 \leq i \leq n} |\widetilde{x}_{ij}|,$$

where $\widetilde{x}_i = (\widetilde{x}_{i1}, \widetilde{x}_{i2}, \dots, \widetilde{x}_{in})$ for $i = 1, 2, \dots, n$. This n -norm takes the maximum absolute value in each column across all vectors, then multiplies these maximums together. For example, if $n = 2$ and $\widetilde{x}_1 = (1, 3)$, $\widetilde{x}_2 = (4, -2)$, then

$$\|\widetilde{x}_1, \widetilde{x}_2\|_{\widetilde{M}} = \max(|1|, |4|) \cdot \max(|3|, |-2|) = 4 \cdot 3 = 12.$$

This function satisfies the properties of an n -norm and defines a normed structure on $\mathbb{R}^n$.

Definition 2. [17] A binary operation $\triangleleft: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t -norm if it satisfies the following conditions:

- (a) $\triangleleft$ is associative and commutative;
- (b) $\triangleleft$ is continuous;
- (c) $\widetilde{a} \triangleleft 1 = \widetilde{a}$ for all $\widetilde{a} \in [0, 1]$;
- (d) $\widetilde{a} \triangleleft \widetilde{b} \leq \widetilde{c} \triangleleft \widetilde{d}$, whenever $\widetilde{a} \leq \widetilde{c}$ and $\widetilde{b} \leq \widetilde{d}$ for each $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \in [0, 1]$.

Definition 3. [12] A binary operation $\odot: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is considered a continuous t -conorm if

- (a') $\odot$ is associative and commutative;
- (b') $\odot$ is continuous;
- (c') $\widetilde{a} \odot 0 = \widetilde{a}$ for all $\widetilde{a} \in [0, 1]$;
- (d') $\widetilde{a} \odot \widetilde{b} \leq \widetilde{c} \odot \widetilde{d}$ whenever $\widetilde{a} \leq \widetilde{c}$ and $\widetilde{b} \leq \widetilde{d}$ for each $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} \in [0, 1]$.

Definition 4. The neutrosophic n -normed space over $\mathcal{K}$ is defined as follows. The six-duple $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ is said to be an NA-NnN space if $\mathcal{U}$ is a vector space over a field $\mathcal{K}$, $\odot$ is a continuous t -conorm, $\triangleleft$ is a continuous t -norm, and Ω, Υ, Θ are functions of $\mathcal{U} \times \mathbb{R}$ to $[0, 1]$ satisfying the following conditions. For every $(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n) \in \mathcal{U}^n$ and $u, v \in \mathcal{K}$

- (a) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) + \Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) \leq 1$.
- (b) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) > 0$.
- (c) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 1 \iff \widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n$ are linearly dependent.
- (d) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u)$ is invariant under any permutation of $\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n$.
- (e) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, c \cdot \widetilde{x}_n, u) = \Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \frac{u}{|c|})$ for $c \neq 0, c \in \mathcal{K}$.
- (f) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) \triangleleft \Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}'_n, v) \leq \Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n + \widetilde{x}'_n, \max\{u, v\})$.
- (g) $\Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous.
- (h) $\lim_{u \rightarrow \infty} \Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 1$ and $\lim_{u \rightarrow 0} \Omega(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 0$.
- (i) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) < 1$.
- (j) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 0 \iff \widetilde{x}_1, \widetilde{x}_2, \dots, c \cdot \widetilde{x}_n$ are linearly dependent.
- (k) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u)$ is invariant under any permutation of $\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n$.
- (l) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, c \cdot \widetilde{x}_n, u) = \Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \frac{u}{|c|})$ for each $c \neq 0, c \in \mathcal{K}$.
- (m) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) \odot \Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}'_n, v) \geq \Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n + \widetilde{x}'_n, \max\{u, v\})$.
- (n) $\Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous.
- (o) $\lim_{u \rightarrow \infty} \Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 0$ and $\lim_{u \rightarrow 0} \Upsilon(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 1$.
- (p) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) < 1$.
- (q) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 0 \iff \widetilde{x}_1, \widetilde{x}_2, \dots, c \cdot \widetilde{x}_n$ are linearly dependent.
- (r) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u)$ is invariant under any permutation of $\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n$.
- (s) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, c \cdot \widetilde{x}_n, u) = \Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \frac{u}{|c|})$ for each $c \neq 0, c \in \mathcal{K}$.
- (t) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) \odot \Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}'_n, v) \geq \Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n + \widetilde{x}'_n, \max\{u, v\})$.
- (u) $\Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous.
- (v) $\lim_{u \rightarrow \infty} \Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 0$ and $\lim_{u \rightarrow 0} \Theta(\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n, u) = 1$.

Here, $(\Omega, \Upsilon, \Theta)$ is called a non-Archimedean neutrosophic n -norm.

Example 2. Let $(\mathcal{U}, \|\cdot\|)$ be an n -normed space over $\mathbb{R}^n$ with a standard n -norm $\|\cdot\|$ (e.g., determinant-based).

Define the operations and mappings as follows for $\tilde{a}, \tilde{b} \in [0, 1]$: $\tilde{a} \triangleleft \tilde{b} = \min\{\tilde{a}, \tilde{b}\}$, $\tilde{a} \odot \tilde{b} = \max\{\tilde{a}, \tilde{b}\}$,

$$\Omega(\tilde{x}_1, \dots, \tilde{x}_n) = \frac{1}{1 + \|\tilde{x}_1, \dots, \tilde{x}_n\|}, \quad \Upsilon(\tilde{x}_1, \dots, \tilde{x}_n, \tilde{u}) = \frac{\|u, \tilde{x}_2, \dots, \tilde{x}_n\|}{1 + \|\tilde{x}_1, \dots, \tilde{x}_n\|},$$

$$\Theta(\tilde{x}_1, \dots, \tilde{x}_n, \tilde{u}) = 1 - \Omega(\tilde{x}_1, \dots, \tilde{x}_n).$$

Then, the structure $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ forms a Neutrosophic n -Normed Space (NnNS).

For instance, let $\tilde{x}_1 = (1, 0)$, $\tilde{x}_2 = (0, 1)$ in $\mathbb{R}^2$, with $n = 2$ and the 2-norm defined by $\|\tilde{x}_1, \tilde{x}_2\| = |\det(\tilde{x}_1, \tilde{x}_2)| = 1$. Then,

$$\Omega(\tilde{x}_1, \tilde{x}_2) = \frac{1}{1+1} = \frac{1}{2}, \quad \Theta(\tilde{x}_1, \tilde{x}_2, \tilde{x}_1) = 1 - \frac{1}{2} = \frac{1}{2}, \quad \Upsilon(\tilde{x}_1, \tilde{x}_2, \tilde{x}_1) = \frac{|\det(\tilde{x}_1, \tilde{x}_2)|}{1 + |\det(\tilde{x}_1, \tilde{x}_2)|} = \frac{1}{2}.$$

This example satisfies the neutrosophic space structure using bounded norm-based mappings.

Definition 5. [18] Let $\{\tilde{x}_Q\}$ be a sequence in a non-Archimedean neutrosophic n -normed space $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$. The sequence is said to converge to $\ell \in \mathcal{U}$ with respect to $(\Omega, \Upsilon, \Theta)^n$ if, for every $\epsilon > 0$ and $u > 0$, and for all fixed elements $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$, there exists an integer $Q_0 \in \mathbb{N}$ such that for all $Q \geq Q_0$, the following inequalities hold:

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\}.$$

This type of convergence is denoted by $(\Omega, \Upsilon, \Theta)^n$ - $\lim \tilde{x}_Q = \ell$.

Definition 6. [19] A sequence $x = \{\tilde{x}_Q\}$ is said to be *statistically convergent* to a real number ℓ if, for each $\epsilon > 0$, the set $\{Q \in \mathbb{N} : |\tilde{x}_Q - \ell| \geq \epsilon\} = 0$. In other words, the relative frequency of terms that differ from ℓ by at least ϵ approaches zero as $Q \rightarrow \infty$. This is denoted by: $\text{Stat-lim } \tilde{x}_Q = \ell$.

Example 3. Consider the sequence $y = \{\tilde{y}_Q\}$ defined by $\tilde{y}_Q = \begin{cases} \frac{1}{2^Q}, & \text{if } Q \text{ is prime} \\ 0, & \text{otherwise.} \end{cases}$ Now, we consider this

sequence in the 2-adic number field $\mathbb{Q}_2$, using the 2-adic absolute value $|\cdot|_2$.

Observe that for each prime Q , we have $|\tilde{y}_Q|_2 = |\frac{1}{2^Q}|_2 = 2^Q$, which tends to ∞ as $Q \rightarrow \infty$, but recall that in $\mathbb{Q}_2$, higher exponents correspond to smaller values in the usual sense (i.e., $|2^{-Q}|_2 = 2^Q$ is large, but the number itself is small).

In particular, $\tilde{y}_Q \rightarrow 0$ in the 2-adic topology on a sparse set (the primes), and $\tilde{y}_Q = 0$ elsewhere.

The density of the primes among the natural numbers is zero:

$$\lim_{n \rightarrow \infty} \frac{|\{Q \leq n : Q \text{ prime}\}|}{n} = 0.$$

Thus, for any $\epsilon > 0$, the set $A_\epsilon = \{Q \in \mathbb{N} : |\tilde{y}_Q|_2 > \epsilon\}$ is contained in the primes and hence has density 0. Therefore, $\{\tilde{y}_Q\}$ is statistically convergent to 0 in $\mathbb{Q}_2$.

2 Statistical convergence in NA-NnNS

Definition 7. [18] Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space. A sequence $x = \{\tilde{x}_Q\} \subseteq \mathcal{U}$ is said to be *statistically convergent* to $\ell \in \mathcal{U}$ with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$ if, for every $\epsilon > 0$, $u > 0$, and for any $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$, the following condition holds:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or} \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

In this case, we write $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n}$ - $\lim x = \ell$.

Lemma 1. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a neutrosophic n -normed linear space (NnNLS). Then, for any $\epsilon > 0$ and $u > 0$, and for all fixed elements $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{n-1} \in \mathcal{U}$, the following statements are mutually equivalent:

- (a) $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim x = \ell;$
- (b)
$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0;$$
- (c)
$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \text{ and} \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \text{ and} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\} \right| = 1;$$
- (d)
$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\} \right| = 1;$$
- (e)

$\text{Stat-lim } \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 1,$ and $\text{Stat-lim } \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 0,$
and $\text{Stat-lim } \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 0.$

Theorem 1. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space (NA-NnNS). If a sequence $x = \{\tilde{x}_Q\} \subseteq \mathcal{U}$ is statistically convergent with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$, then the statistical limit $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x}_Q$ exists and is uniquely determined.

Proof. Assume that $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim x_1 = \ell_1$ and $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x} = \ell_2$. Let $\epsilon > 0$ be given. Choose $d \in (0, 1)$ such that $(1 - \tilde{d}) \triangleleft (1 - \tilde{d}) > 1 - \epsilon$ and $d \odot d < \epsilon$. For any $u > 0$ and for all $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{n-1} \in \mathcal{U}$, we define the following sets:

$$\begin{aligned} Q_{\Omega,1}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Omega(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \tilde{d} \right\}, \\ Q_{\Omega,2}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Omega(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \tilde{d} \right\}, \\ Q_{\Upsilon,1}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \tilde{d} \right\}, \\ Q_{\Upsilon,2}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \tilde{d} \right\}, \\ Q_{\Theta,1}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Theta(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \tilde{d} \right\}, \\ Q_{\Theta,2}(d, \tilde{u}) &:= \left\{ Q \in \mathbb{N} : \Theta(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \tilde{d} \right\}. \end{aligned}$$

Assuming $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x} = \ell_1$, we deduce from Lemma 1 that

$$\lim_n \frac{1}{n} |\{Q_{\Omega,1}(\epsilon, \tilde{u})\}| = \lim_n \frac{1}{n} |\{Q_{\Upsilon,1}(\epsilon, \tilde{u})\}| = \lim_n \frac{1}{n} |\{Q_{\Theta,1}(\epsilon, \tilde{u})\}| = 1$$

for all $u > 0$.

Moreover, based on the assumption that $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x} = \ell_2$, we conclude

$$\lim_n \frac{1}{n} |\{Q_{\Omega,2}(\epsilon, \tilde{u})\}| = \lim_n \frac{1}{n} |\{Q_{\Upsilon,2}(\epsilon, \tilde{u})\}| = \lim_n \frac{1}{n} |\{Q_{\Theta,2}(\epsilon, \tilde{u})\}| = 1$$

for all $u > 0$.

Let us now consider,

$$Q_{\Omega, \Upsilon, \Theta}(\epsilon, \tilde{u}) = (Q_{\Omega,1}(\epsilon, \tilde{u}) \cap Q_{\Omega,2}(\epsilon, \tilde{u})) \cup (Q_{\Upsilon,1}(\epsilon, \tilde{u}) \cap Q_{\Upsilon,2}(\epsilon, \tilde{u})) \cup (Q_{\Theta,1}(\epsilon, \tilde{u}) \cap Q_{\Theta,2}(\epsilon, \tilde{u})).$$

In the next step, $\lim_n \frac{1}{n} |\{Q_{\Omega, \Upsilon, \Theta}(\epsilon, \tilde{u})\}| = 1$. Suppose $Q \in Q_{\Omega, \Upsilon, \Theta}(\epsilon, \tilde{u})$. Our initial focus is on $Q \in (Q_{\Omega, 1}(\epsilon, \tilde{u}) \cap Q_{\Omega, 2}(\epsilon, \tilde{u}))$. Then we have

$$\begin{aligned} & \Omega(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & \geq \Omega(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \triangleleft \Omega(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & > (1 - \tilde{d}) \triangleleft (1 - \tilde{d}). \end{aligned}$$

Given that $(1 - \tilde{d}) \triangleleft (1 - \tilde{d}) > 1 - \epsilon$, it follows that $\Omega(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude

$$\Omega(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 1 \quad \text{for all } u > 0 \text{ and } \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}.$$

Hence, it must be that $\ell_1 = \ell_2$.

On the other hand, assuming $Q \in Q_{\Upsilon, 1}(\epsilon, \tilde{u}) \cap Q_{\Upsilon, 2}(\epsilon, \tilde{u})$, we have

$$\begin{aligned} & \Upsilon(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & \leq \Upsilon(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \odot \Upsilon(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & < d \odot d < \epsilon. \end{aligned}$$

Since ϵ can be chosen freely, it follows that $\Upsilon(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 0$ for every $u > 0$ and all $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$. This implies that ℓ_1 must be equal to ℓ_2 .

As a result, if $Q \in (Q_{\Upsilon, 1}(\epsilon, \tilde{u}) \cap Q_{\Upsilon, 2}(\epsilon, \tilde{u}))$, then

$$\begin{aligned} & \Theta(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & \leq \Theta(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \odot \Theta(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & < d \odot d < \epsilon. \end{aligned}$$

Since ϵ is arbitrary, it follows that $\Theta(\ell_1 - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, u) = 0$ for all $u > 0$ and $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$, which implies $\ell_1 = \ell_2$. Therefore, the $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n}$ - $\lim \tilde{x}$ is unique. $\square$

Theorem 2. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be an NA-NnNS. If $(\Omega, \Upsilon, \Theta)^n$ - $\lim \tilde{x} = \ell$, then $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n}$ - $\lim x = \ell$.

Proof. Assume that $(\Omega, \Upsilon, \Theta)^n$ - $\lim \tilde{x}_Q = \ell$. Then, for any $\epsilon > 0$, $d > 0$, and fixed elements $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{n-1} \in \mathcal{U}$, there exists a positive integer $n_0 \in \mathbb{N}$ such that for all $Q \geq n_0$, the following conditions are satisfied:

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\}.$$

Thus, the set of indices where these inequalities are violated,

$$\left\{ Q \in \mathbb{N} \left| \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \\ \text{or } \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \\ \text{or } \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right. \right\}$$

is finite, which implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or } \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or } \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

Therefore, $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n}$ - $\lim \tilde{x} = \ell$.

Note: Interestingly, the converse of the above theorem, which is not classically true, is true in NA-NnNS as shown below.

Let $x = \{\tilde{x}_Q\}$ be $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n}$ that converges to $\ell \in \mathcal{U}$. Subsequently, for every $\epsilon > 0$ and $u > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or } \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or } \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

We now prove that $\{\tilde{x}_Q\}$ is $(\Omega, \Upsilon, \Theta)^n$ - convergent to $\ell \in \mathcal{U}$. i.e., to prove

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon \text{ and} \\ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \text{ and} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\}.$$

We assume that

$$\begin{aligned} & \{\Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or } \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ & \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon\} \\ \Rightarrow & \{Q \in \mathbb{N} : \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or } \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ & \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon\} \end{aligned}$$

has infinitely many terms.

$$\lim_n \frac{1}{n} \left| \left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or} \\ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| \neq 0.$$

However, this finding is contradictory. Thus, $\{\tilde{x}_Q\}$ is $(\Omega, \Upsilon, \Theta)^n$, which converges to $\ell \in \mathcal{U}$. $\square$

Theorem 3. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be an NA-NnNS. If $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim x_Q = \tilde{x}$ and $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim y_Q = \tilde{y}$, then $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim (\tilde{x}_Q + \tilde{y}_Q) = \tilde{x} + \tilde{y}$.

Proof. Given any $\epsilon > 0$, choose $d > 0$ such that $(1 - d)^2 > 1 - \epsilon$. Let ℓ_1 and ℓ_2 denote the statistical limit points of the sequences $(\tilde{x}_Q)$ and $(\tilde{y}_Q)$, respectively. Then, for any $\tilde{u} > 0$ and elements $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$, we define the following sets:

$$Q_{\Omega,1}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Omega(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - d\},$$

$$Q_{\Omega,2}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Omega(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - d\},$$

$$Q_{\Upsilon,1}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < d\},$$

$$Q_{\Upsilon,2}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < d\},$$

$$Q_{\Theta,1}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Theta(\tilde{x}_Q - \ell_1, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < d\},$$

$$Q_{\Theta,2}(d, \tilde{u}) := \{Q \in \mathbb{N} : \Theta(\tilde{x}_Q - \ell_2, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < d\}.$$

As

$$\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim x_Q = \tilde{x} \text{ and } \text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim y_Q = \tilde{y},$$

$$\lim_n \frac{1}{n} \left| \{Q_{\Omega,1}(\epsilon, \tilde{d})\} \right| = \lim_n \frac{1}{n} \left| \{Q_{\Upsilon,1}(\epsilon, \tilde{d})\} \right| = \lim_n \frac{1}{n} \left| \{Q_{\Theta,1}(\epsilon, \tilde{d})\} \right| = 1,$$

and

$$\lim_n \frac{1}{n} \left| \{Q_{\Omega,2}(\epsilon, \tilde{t})\} \right| = \lim_n \frac{1}{n} \left| \{Q_{\Upsilon,2}(\epsilon, \tilde{d})\} \right| = \lim_n \frac{1}{n} \left| \{Q_{\Theta,2}(\epsilon, \tilde{d})\} \right| = 1.$$

Let

$$Q_{\Omega,u} = (Q_{\Omega,1}(\epsilon, \tilde{u}) \cap Q_{\Omega,2}(\epsilon, \tilde{u})) \cup (Q_{\Upsilon,1}(\epsilon, \tilde{u}) \cap Q_{\Upsilon,2}(\epsilon, \tilde{u})) \cup (Q_{\Theta,1}(\epsilon, \tilde{u}) \cap Q_{\Theta,2}(\epsilon, \tilde{u})).$$

Then,

$$\lim_n \frac{1}{n} \left| \{Q_{\Omega, \Upsilon, \Theta}(\epsilon, \tilde{u})\} \right| = 1.$$

If $Q \in Q_{\Omega, \Upsilon, \Theta}(\epsilon, \tilde{u})$ at the outset, consider $Q \in (Q_{\Omega,1}(\epsilon, \tilde{u}) \cap Q_{\Omega,2}(\epsilon, \tilde{u}))$. Following that, we arrive at

$$\begin{aligned} & \Omega(\tilde{x}_Q + \tilde{y}_Q - \tilde{x} - \tilde{y}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & \geq \Omega(\tilde{x}_Q - \tilde{x}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \triangleleft \Omega(\tilde{y}_Q - \tilde{y}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ & > (1 - d) \triangleleft (1 - d) \\ & > 1 - \epsilon. \end{aligned}$$

Since, ϵ is arbitrary, $\Omega(\tilde{x}_Q + \tilde{y}_Q - (\tilde{x} + \tilde{y}), \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 1$.

In a similar way,

$$\Upsilon(\tilde{x}_Q + \tilde{y}_Q - (\tilde{x} + \tilde{y}), \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 0, \text{ and also } \Theta(\tilde{x}_Q + \tilde{y}_Q - (\tilde{x} + \tilde{y}), \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) = 0.$$

Thus, $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim(\tilde{x}_Q + \tilde{y}_Q) = \tilde{x} + \tilde{y}$. □

Theorem 4. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be an NA-NnNS over $\mathcal{K}$. If $\left\{ \begin{array}{l} \lim_{Q \rightarrow \infty} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 1, \\ \lim_{Q \rightarrow \infty} \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 0, \text{ and} \\ \lim_{Q \rightarrow \infty} \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 0 \end{array} \right\}$.

Then, $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim_{Q \rightarrow \infty} \tilde{x}_Q = \ell$.

Proof. Assume that the following limits hold:

$$\left\{ \begin{array}{l} \lim_{Q \rightarrow \infty} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 1, \\ \lim_{Q \rightarrow \infty} \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 0, \\ \lim_{Q \rightarrow \infty} \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) = 0 \end{array} \right\}.$$

Then, for any $\epsilon > 0$ and $u > 0$, there exists a natural number $Q_0 \in \mathbb{N}$ such that for all $Q \geq Q_0$, we have

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\}.$$

Hence, the count of elements $Q \leq n$ for which these conditions are violated tends to zero as $n \rightarrow \infty$.

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or} \\ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} = 0$$

is finite. So,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or} \\ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

Thus, $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x}_Q = \ell$. □

3 Statistical cauchy sequence in NA-NnNS

Definition 8. [18] Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space (NA-NnNS). A sequence $\{\tilde{x}_n\} \subset \mathcal{U}$ is said to be a *Cauchy sequence* with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$ if for every $\epsilon > 0$ and $u > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ and for all $\tilde{y}_2, \dots, \tilde{y}_n \in \mathcal{U}$, the following conditions hold:

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_n - \tilde{x}_m, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon \end{array} \right\}.$$

Definition 9. [18] Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space (NA-NnNS). A sequence $x = \{\tilde{x}_Q\} \subset \mathcal{U}$ is called a *statistically Cauchy sequence* if for every $\epsilon > 0$ and $u > 0$, there exists $N \in \mathbb{N}$ such that for all $\ell \geq N$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ \begin{array}{l} \Omega(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \text{ or} \\ Q \in \mathbb{N} : \Upsilon(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \text{ or} \\ \Theta(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

The NA-NnNS is said to be *complete* if every $(\Omega, \Upsilon, \Theta)^n$ -statistically Cauchy sequence converges statistically in the same norm. Such a space is called a non-Archimedean neutrosophic Banach space.

Theorem 5. In the NA-NnNS $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ over $\mathcal{K}$, every Cauchy sequence with respect to $(\Omega, \Upsilon, \Theta)^n$ is statistically Cauchy.

Proof. Assume that $\{\tilde{x}_Q\}$ is a Cauchy sequence with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$. Then for every $\epsilon > 0$ and $u > 0$, there exist $n_0 \in \mathbb{N}$ and a constant $p \in \mathbb{N}$ such that for all $Q \geq n_0$,

$$\left\{ \begin{array}{l} \Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) < \epsilon. \end{array} \right\}.$$

Moreover, the set where these conditions fail has zero natural density:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

Thus, $\{\tilde{x}_Q\}$ is statistically Cauchy with respect to $(\Omega, \Upsilon, \Theta)^n$. $\square$

Example 4. Let $\mathcal{U} = \mathbb{R}^n$ over $\mathcal{K} = \mathbb{R}$ and define the usual n -norm by $\|z_1, z_2, \dots, z_n\|_n = |\det(z_1, z_2, \dots, z_n)|$. For $u > 0$, define as

$$\Omega(z_1, \dots, z_n, u) = \frac{u}{u + \|z_1, \dots, z_n\|_n}, \quad \Upsilon(z_1, \dots, z_n, u) = \frac{\|z_1, \dots, z_n\|_n}{u + \|z_1, \dots, z_n\|_n},$$

and

$$\Theta(z_1, \dots, z_n, u) = \frac{\|z_1, \dots, z_n\|_n}{2u + \|z_1, \dots, z_n\|_n}.$$

Take $\tilde{x}_Q = \frac{1}{Q}e_1$, $Q \in \mathbb{N}$, where $e_1, e_2, \dots, e_n$ are the standard basis vectors of $\mathbb{R}^n$. We show that $\{\tilde{x}_Q\}$ is statistically Cauchy with respect to $(\Omega, \Upsilon, \Theta)^n$.

Fix $p \in \mathbb{N}$, $u > 0$, and let $\tilde{y}_2 = e_2, \tilde{y}_3 = e_3, \dots, \tilde{y}_n = e_n$. Then,

$$\tilde{x}_{Q+p} - \tilde{x}_Q = \left(\frac{1}{Q+p} - \frac{1}{Q} \right) e_1 = -\frac{p}{Q(Q+p)} e_1.$$

Hence,

$$\|\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n\|_n = \frac{p}{Q(Q+p)}.$$

Therefore,

$$\Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) = \frac{u}{u + \frac{p}{Q(Q+p)}} \rightarrow 1$$

as $Q \rightarrow \infty$, while

$$\Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) = \frac{\frac{p}{Q(Q+p)}}{u + \frac{p}{Q(Q+p)}} \rightarrow 0, \text{ and } \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) = \frac{\frac{p}{Q(Q+p)}}{2u + \frac{p}{Q(Q+p)}} \rightarrow 0.$$

Thus, for every $\epsilon > 0$, there exists $Q_0 \in \mathbb{N}$ such that for all $Q \geq Q_0$,

$$\Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) > 1 - \epsilon, \quad \Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) < \epsilon, \text{ and } \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) < \epsilon.$$

Consequently, the set

$$\left\{ Q \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \geq \epsilon \end{array} \right\}$$

is contained in the finite set $\{1, 2, \dots, Q_0 - 1\}$. Hence its natural density is zero. That is,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \left| \left\{ Q \leq m : \begin{array}{l} \Omega(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_{Q+p} - \tilde{x}_Q, e_2, \dots, e_n, u) \geq \epsilon \end{array} \right\} \right| = 0.$$

Therefore, $\{\tilde{x}_Q\}$ is statistically Cauchy with respect to $(\Omega, \Upsilon, \Theta)^n$.

Theorem 6. Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space over the field $\mathcal{K}$. If a sequence $\{\tilde{x}_k\}_{k \in \mathbb{N}}$ is statistically convergent, then its statistical limit is unique.

Proof. Suppose the sequence $\{\tilde{x}_Q\}$ is statistically convergent to ℓ . Then for every $\epsilon > 0$, $\tilde{u} > 0$, and $\tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n \in \mathcal{U}$, it holds that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

Next, consider the set

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Theta(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon \end{array} \right\} = 0.$$

By using the operations $\triangleleft$ and $\odot$, it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ \triangleleft \Omega(\tilde{x}_\ell - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ \odot \Upsilon(\tilde{x}_\ell - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \\ \odot \Theta(\tilde{x}_\ell - \ell, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

□

4 Statistical completeness in NA-NnNS

Definition 10. [20] A non-Archimedean neutrosophic n -normed space $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ over the field $\mathcal{K}$ is called statistically complete if every sequence that is statistically Cauchy with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$ also statistically converges within the same norm.

Theorem 7. Every NA-NnNS $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ over $\mathcal{K}$ is statistically complete with respect to $(\Omega, \Upsilon, \Theta)^n$.

Proof. Assume that $\{\tilde{x}_Q\}$ is statistically Cauchy but not statistically convergent to $\ell \in \mathcal{U}$; then, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \text{at least one of the following holds:} \\ \Omega(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \Theta(\tilde{x}_Q - \tilde{x}_\ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ Q \leq n : \begin{array}{l} \text{at least one of the following holds:} \\ \Omega(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \triangleleft \Omega(\tilde{x}_\ell - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \Upsilon(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \odot \Upsilon(\tilde{x}_\ell - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \Theta(\tilde{x}_Q - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \odot \Theta(\tilde{x}_\ell - \ell, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0. \end{aligned}$$

However, this finding is contradictory. □

Example 5. Let $\mathcal{U} = \mathbb{R}^n$ over $\mathcal{K} = \mathbb{R}$ and define the usual n -norm by $\|z_1, z_2, \dots, z_n\|_n = |\det(z_1, z_2, \dots, z_n)|$. For $u > 0$, define

$$\Omega(z_1, \dots, z_n, u) = \frac{u}{u + \|z_1, \dots, z_n\|_n}, \quad \Upsilon(z_1, \dots, z_n, u) = \frac{\|z_1, \dots, z_n\|_n}{u + \|z_1, \dots, z_n\|_n},$$

and

$$\Theta(z_1, \dots, z_n, u) = \frac{\|z_1, \dots, z_n\|_n}{2u + \|z_1, \dots, z_n\|_n}.$$

Together with $a \triangleleft b = ab$, $a \odot b = \min\{a + b, 1\}$, this gives a standard example of an NA-NnNS.

Now, we define a sequence $\{\tilde{x}_Q\}$ in $\mathbb{R}^n$ by

$$\tilde{x}_Q = \begin{cases} e_1, & Q = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } e_1 = (1, 0, \dots, 0).$$

We claim that $\{\tilde{x}_Q\}$ is statistically convergent to 0 with respect to $(\Omega, \Upsilon, \Theta)^n$. Let $\epsilon > 0$, $u > 0$, and $\tilde{y}_2, \dots, \tilde{y}_n \in \mathbb{R}^n$ be fixed. Consider the set

$$A_{\epsilon, u} = \left\{ Q \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_Q - 0, \tilde{y}_2, \dots, \tilde{y}_n, u) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_Q - 0, \tilde{y}_2, \dots, \tilde{y}_n, u) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_Q - 0, \tilde{y}_2, \dots, \tilde{y}_n, u) \geq \epsilon \end{array} \right\}.$$

If Q is not a perfect square, then $\tilde{x}_Q = 0$. Hence,

$$\Omega(0, \tilde{y}_2, \dots, \tilde{y}_n, u) = 1, \quad \Upsilon(0, \tilde{y}_2, \dots, \tilde{y}_n, u) = 0, \quad \text{and} \quad \Theta(0, \tilde{y}_2, \dots, \tilde{y}_n, u) = 0.$$

Therefore, the above inequalities can fail only when Q is a perfect square. Thus, $A_{\epsilon, u} \subseteq \{k^2 : k \in \mathbb{N}\}$. Since the set of perfect squares has natural density zero, we have

$$\lim_{m \rightarrow \infty} \frac{1}{m} |\{Q \leq m : Q = k^2 \text{ for some } k \in \mathbb{N}\}| = \lim_{m \rightarrow \infty} \frac{\lfloor \sqrt{m} \rfloor}{m} = 0.$$

Hence, $\lim_{m \rightarrow \infty} \frac{1}{m} |A_{\epsilon, u} \cap \{1, 2, \dots, m\}| = 0$. Therefore, $\text{Stat} - \lim_{Q \rightarrow \infty} \tilde{x}_Q = 0$ with respect to $(\Omega, \Upsilon, \Theta)^n$.

Consequently, $\{\tilde{x}_Q\}$ is statistically convergent, and hence it is statistically Cauchy. This example shows how a sequence may fail to be ordinarily convergent but still becomes statistically convergent in an NA-NnNS because its exceptional terms occur only on a set of natural density zero.

5 Statistically continuous in NA-NnNS

Definition 11. [2] Consider the space $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$, which forms a non-Archimedean neutrosophic n -normed space (NA-NnNS) over the field $\mathcal{K}$. A function $f : \mathcal{U} \rightarrow \mathcal{U}$ is said to be continuous at a point $x \in \mathcal{U}$ with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$ if whenever a sequence $\{\tilde{x}_n\}$ converges to $\tilde{x}$ in this space, the image sequence $\{f(\tilde{x}_n)\}$ converges to $f(\tilde{x})$ in the same neutrosophic n -norm.

Definition 12. [2, 18, 20] Let $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ be a non-Archimedean neutrosophic n -normed space (NA-NnNS) over the field $\mathcal{K}$. A mapping $f : \mathcal{U} \rightarrow \mathcal{U}$ is said to be statistically continuous at a point $\tilde{x} \in \mathcal{U}$ if for every sequence $\{\tilde{x}_n\}$ in $\mathcal{U}$,

$$\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x}_n = \tilde{x}$$

implies that

$$\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim f(\tilde{x}_n) = f(\tilde{x}).$$

Theorem 8. Consider $(\mathcal{U}, \Omega, \Upsilon, \Theta, \triangleleft, \odot)$ as a non-Archimedean neutrosophic n -normed space over the field $\mathcal{K}$. A function $f : \mathcal{U} \rightarrow \mathcal{U}$ is said to be statistically continuous if it is continuous with respect to the neutrosophic n -norm $(\Omega, \Upsilon, \Theta)^n$.

Proof. Let $\{\tilde{x}_n\}$ be a sequence in $\mathcal{U}$ such that $\text{Stat}_{(\Omega, \Upsilon, \Theta)^n} - \lim \tilde{x}_n = \tilde{x}$. Then, for every $\epsilon > 0$ and $\tilde{u} > 0$, the following conditions hold:

$$\begin{cases} \Omega(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) > 1 - \epsilon, \\ \Upsilon(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon, \\ \Theta(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \tilde{y}_3, \dots, \tilde{y}_n, \tilde{u}) < \epsilon. \end{cases}$$

These imply:

$$\begin{cases} \Omega(f(\tilde{x}_n) - f(\tilde{x}), f(\tilde{y}_2), f(\tilde{y}_3), \dots, f(\tilde{y}_n), \tilde{u}) > 1 - \epsilon, \\ \Upsilon(f(\tilde{x}_n) - f(\tilde{x}), f(\tilde{y}_2), f(\tilde{y}_3), \dots, f(\tilde{y}_n), \tilde{u}) < \epsilon, \\ \Theta(f(\tilde{x}_n) - f(\tilde{x}), f(\tilde{y}_2), f(\tilde{y}_3), \dots, f(\tilde{y}_n), \tilde{u}) < \epsilon. \end{cases}$$

Due to the continuity of $\mathfrak{f}$, the set

$$\left\{ n \in \mathbb{N} : \begin{array}{l} \Omega(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \geq \epsilon \end{array} \right\}$$

is a subset of

$$\left\{ n \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\}.$$

Since $\text{Stat}(\Omega, \Upsilon, \Theta)^n\text{-}\lim \tilde{x}_n = \tilde{x}$, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ n \in \mathbb{N} : \begin{array}{l} \Omega(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\tilde{x}_n - \tilde{x}, \tilde{y}_2, \dots, \tilde{y}_n, \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ n \in \mathbb{N} : \begin{array}{l} \Omega(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \leq 1 - \epsilon, \\ \text{or } \Upsilon(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \geq \epsilon, \\ \text{or } \Theta(\mathfrak{f}(\tilde{x}_n) - \mathfrak{f}(\tilde{x}), \mathfrak{f}(\tilde{y}_2), \dots, \mathfrak{f}(\tilde{y}_n), \tilde{u}) \geq \epsilon \end{array} \right\} \right| = 0,$$

which means $\text{Stat}(\Omega, \Upsilon, \Theta)^n\text{-}\lim \mathfrak{f}(\tilde{x}_n) = \mathfrak{f}(\tilde{x})$. Hence, $\mathfrak{f}$ is statistically continuous. $\square$

Conclusion

In this study, we investigated statistical convergence in non-Archimedean neutrosophic n -normed spaces by combining natural density with the truth-membership, indeterminacy-membership, and falsity-membership components. Several basic properties were established, including equivalent characterizations, the uniqueness of the statistical limit, preservation under addition, and the relation between ordinary convergence and statistical convergence. Statistical Cauchy sequences and statistical completeness were also discussed. The examples show that this approach is flexible enough to handle sequences that are not ordinarily convergent but are statistically convergent. These results extend the classical theory of statistical convergence to the neutrosophic non-Archimedean n -normed setting and may motivate further studies on ideal convergence, lacunary, and rough statistical convergence in this framework. These results open avenues for a deeper exploration of convergence concepts in more generalized and uncertain environments.

Building on the results of this study, future research may focus on extending the concepts of statistical convergence and completeness to other generalized neutrosophic structures, such as fuzzy or neutrosophic n -normed spaces. Additionally, exploring the relationships between statistical convergence and other types of convergence (e.g., ideal convergence or lacunary statistical convergence) in the non-Archimedean neutrosophic setting could yield deeper insights. Applications of these concepts to functional analysis, fixed point theory, and decision-making problems under uncertainty also present promising directions for further investigation.

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Author contributions

All authors contributed equally to writing this paper. Furthermore, this manuscript was read and approved by all authors.

Conflict of interest

The authors declare that there is no conflict of interest.

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