

## Characterization of weighted inequalities for superpositions of integral and supremal operators

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In this paper, we provide a characterization of the boundedness of positive sublinear operators that are superposition of three operators: Copson, Hardy, and Tandori (supremal) operators defined on the half-axis from a weighted Lebesgue space  $L_p(v)$  to another weighted Lebesgue space  $L_1(w)$ , where  $v$  and  $w$  are weight functions on the half-axis  $(0, \infty)$ , and  $1 < p < \infty$ . Our characterization is entirely different from existing results in the literature. The motivation for investigating such inequalities stems from the problem of finding a minimal rearrangement invariant space that contains the cones of non-increasing rearrangement of the functions represented by generalized fractional maximal function acting on functions from weighted Lorentz function spaces. More specifically, by obtaining two-sided estimates for the best constant in the corresponding inequality, we derive a characterization of the associate space of minimal rearrangement invariant spaces containing cones of non-increasing rearrangement of generalized fractional maximal function. To achieve this goal, we are using discretization and anti-discretization methods. In particular, we extend existing discretization techniques to handle the operators formed by iterating the Copson, Hardy, and Tandori operators. We first establish a discrete characterization in Theorem 3. Then, applying anti-discretization techniques, we derive a continuous characterization in Theorem 1.

*Keywords:* weighted inequality, function spaces, supremal operator, Copson operator, Hardy operator, Tandori operator, best constant, superposition of operators, discretization.

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### Introduction

In this paper, we study weighted one-dimensional inequalities for operators generated by iterating the Copson, Hardy, and Tandori operators. We do this by developing the methods of discretization and anti-discretization from the paper [1].

In the paper [2], it was shown that the boundedness of the fractional maximal function  $M_\gamma$  between classical Lorentz spaces is equivalent to the following weighted inequality

$$\left( \int_0^\infty \left( \operatorname{ess\,sup}_{t < s < \infty} s^{\frac{\gamma}{n}-1} \int_0^s f(y) dy \right)^q w(t) dt \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f^p(t) v(t) dt \right)^{\frac{1}{p}}$$

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for all non-increasing functions  $f$  on  $(0, \infty)$ ,  $0 < p, q < \infty$  and  $\gamma \in (0, n)$ ,  $n \in \mathbb{N}$ . The last estimate can be interpreted as the boundedness of non-linear operators  $T_\gamma$ , defined by

$$(T_\gamma f)(t) = \operatorname{ess\,sup}_{t < s < \infty} s^{\frac{\gamma}{n}-1} \int_0^s f(y) dy$$

on the cone of non-increasing functions defined on  $(0, \infty)$ . The operator  $T_\gamma$  was studied in [3], and it was named as a *Hardy-type operator involving suprema*. Various aspects of these operators were studied in [4–6] and their extensions and applications were further considered in [7–9]. More recent results, including weighted estimates for iterated operators, can be found in [10–12]. Additional results are given in [13, 14], as well as in [15, 16]. A systematic exposition of the theory is presented in the monograph [17].

The weighted inequalities with quasilinear integral operators restricted on the cone of monotone functions arise in connection with the studies of boundedness of the operators of harmonic analysis in Lorentz spaces. In the papers mentioned above, the authors use the reduction method for quasilinear operators of iterated type. The inequality restricted to the cone of monotone functions is reduced to the inequalities on all non-negative measurable functions. In this case, reduced inequalities contain the iteration of three operators.

To find a minimal rearrangement invariant space for generalized Besov, Sobolev, and Caldéron spaces is reduced to finding a minimal rearrangement invariant space for cones consisting of decreasing functions of Riesz and Bessel potential of functions from function spaces (see, [18]). This approach was used in the papers [19] and [20] to obtain a characterization of minimal rearrangement invariant spaces for generalized Sobolev and Bessel spaces. In a recent paper [21], general methods were developed to describe a minimal rearrangement invariant space for the cone of decreasing functions related to the generalized fractional maximal functions  $M_\varphi$ . To achieve this goal, we need to obtain the characterization of iterated inequality containing Copson, Hardy, and Tandori(supremal) operators.

Let  $1 < p < \infty$ . Let  $w$  and  $v$  be weight functions i.e. non-negative measurable functions on  $(0, \infty)$ . Let  $\varphi$  be a non-negative, non-decreasing quasi-concave function on  $(0, \infty)$ . Our goal in this article is to find the conditions on  $v, w$ , and  $\varphi$  under which the inequality

$$\int_0^\infty \left( \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi(s)} \int_0^s \left( \int_\tau^\infty h(y) dy \right) d\tau \right) w(t) dt \leq C \left( \int_0^\infty h^p(t)v(t) dt \right)^{\frac{1}{p}}$$

is satisfied for all non-negative measurable functions  $h$  on  $(0, \infty)$ . Using Fubini’s Theorem, we obtain an equivalent form of this inequality as follows:

$$\int_0^\infty \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi(s)} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right) w(t) dt \leq C \left( \int_0^\infty h^p(t)v(t) dt \right)^{\frac{1}{p}}. \tag{1}$$

As we already mentioned above, the related inequalities

$$\int_0^\infty \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi(s)} \left( \int_0^s \tau h(\tau) d\tau \right) w(t) dt \leq C \left( \int_0^\infty h^p(t)v(t) dt \right)^{\frac{1}{p}}, \tag{2}$$

$$\int_0^\infty \operatorname{ess\,sup}_{t < s < \infty} \frac{s}{\varphi(s)} \left( \int_s^\infty h(\tau) d\tau \right) w(t) dt \leq C \left( \int_0^\infty h^p(t)v(t) dt \right)^{\frac{1}{p}} \tag{3}$$

have been studied intensively over the last twenty years. The characterization of inequality (1) is, in fact, equivalent to the characterization of inequalities (2) and (3). Each of the latter yields two separate conditions, resulting in a total of four conditions. However, such a formulation is not convenient

for future applications, where a single, unified condition is preferable. To achieve this, we employ discretization techniques as developed in [1] and further elaborated in the monograph [17].

Throughout the paper, for  $p > 1$ ,  $p' = p/(p - 1)$ . We always denote by  $c$  or  $C$  a positive constant which is independent of the main parameters, but it may vary from line to line. We write  $A \lesssim B$  (or  $A \gtrsim B$ ) if  $A \leq cB$  (or  $cA \geq B$ ) for some positive constant  $c$  independent of appropriate quantities involved in the expressions  $A$  and  $B$ , and  $A \approx B$  if  $A \lesssim B$  and  $A \gtrsim B$ .

To formulate our main result, we need some definitions.

*Definition 1.* [1, Definition 2.2] Let  $\varphi$  be a continuous strictly increasing function on  $[0, \infty)$  such that  $\varphi(0) = 0$  and  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ . Then we say that  $\varphi$  is admissible.

Let  $\varphi$  be an admissible function. We say that a function  $h$  is  $\varphi$ -quasiconcave if  $h$  is equivalent to an increasing function on  $[0, \infty)$  and  $\frac{h}{\varphi}$  is equivalent to a decreasing function on  $(0, \infty)$ . We say that a  $\varphi$ -quasiconcave function  $h$  is non-degenerate if

$$\lim_{t \rightarrow 0^+} h(t) = \lim_{t \rightarrow \infty} \frac{1}{h(t)} = \lim_{t \rightarrow \infty} \frac{h(t)}{\varphi(t)} = \lim_{t \rightarrow 0^+} \frac{\varphi(t)}{h(t)} = 0.$$

The family of non-degenerate  $\varphi$ -quasi-concave functions will be denoted by  $\Omega_\varphi$ . If  $\varphi(t) = t$  we say that  $h$  is quasi-concave. If  $h$  is a non-degenerate quasi-concave function, we write  $h \in \Omega_1$ .

*Definition 2.* [1, Definition 2.9] Let  $\varphi$  be an admissible function and let  $\nu$  be a non-negative Borel measure on  $[0, \infty)$ . We say that the function  $h$ , defined as

$$h(t) = \varphi(t) \int_{[0, \infty)} \frac{d\nu(s)}{\varphi(s) + \varphi(t)}, \quad t \in (0, \infty),$$

is the fundamental function of the measure  $\nu$  with respect to  $\varphi$ . We will also say that  $\nu$  is a representation measure of  $h$  with respect to  $\varphi$ .

We say that  $\nu$  is a non-degenerate measure with respect to  $\varphi$  if the following conditions are satisfied for every  $t \in (0, \infty)$ :

$$\int_{[0, \infty)} \frac{d\nu(s)}{\varphi(s) + \varphi(t)} < \infty, \quad t \in (0, \infty) \quad \text{and} \quad \int_{[0, 1]} \frac{d\nu(s)}{\varphi(s)} = \int_{[1, \infty)} d\nu(s) = \infty.$$

The main result of this paper is presented in the following theorem. To this end, define

$$H(t) := \varphi^{p'}(t) \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi^{p'}(s)} \left( \int_0^\infty \frac{\tau^{p'} s^{p'}}{s^{p'} + \tau^{p'}} v^{1-p'}(\tau) d\tau \right). \tag{4}$$

Observe that  $H \in \Omega_{\varphi^{p'}}$ . There exist a representation measure  $\nu$  of  $H$  with respect to  $\varphi^{p'}$ , (see, Lemma 1), i.e.,

$$H(t) = \int_0^\infty \frac{\varphi^{p'}(t)}{\varphi^{p'}(s) + \varphi^{p'}(t)} d\nu(s). \tag{5}$$

*Theorem 1.* Let  $1 < p < \infty$ . Assume that  $v$  and  $w$  are non-negative measurable functions on  $(0, \infty)$  and that  $\varphi$  is a quasi-concave admissible function on  $(0, \infty)$ . Then following statements are equivalent:

(i) There exists a constant  $C > 0$  such that inequality (1) holds for all non-negative measurable functions  $h$  on  $(0, \infty)$ .

(ii)  $C_1 < \infty$ , where

$$C_1 := \left( \int_0^\infty \left( \int_0^\infty \frac{w(s) ds}{\varphi(s) + \varphi(t)} \right)^{p'} d\nu(t) \right)^{\frac{1}{p'}},$$

and  $\nu$  is a representation measure of  $H$  (defined in (4)) with respect to  $\varphi^{p'}$ , i.e., satisfies (5).

(iii)  $C_2 < \infty$ , where

$$C_2 := \left( \int_0^\infty \left( \int_0^\infty \frac{w(s)ds}{\varphi(s) + \varphi(t)} \right)^{p'-1} \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi^{p'}(s)} \left( \int_0^\infty \frac{\tau^{p'} s^{p'}}{s^{p'} + \tau^{p'}} v^{1-p'}(\tau) d\tau \right) w(t) \varphi^{p'-1}(t) dt \right)^{\frac{1}{p'}}.$$

Moreover, the best constant  $C$  in (1) satisfies  $C \approx C_1 \approx C_2$ .

The results obtained may have important applications in the theory of integral operators, interpolation theory and nonlinear analysis, especially in problems related to three-weight estimates and non-standard function spaces.

The paper is organized as follows. Section 1 provides several auxiliary definitions and preliminary lemmas. In Section 2, we develop a discrete characterization of inequality (1). Section 3 contains the proof of the main result, namely, the proof of Theorem 1.

### 1 Definitions and Preliminaries

This section provides the necessary definitions and supporting known statements that are used in the proofs of the main theorems.

*Definition 3.* [1, Definition 2.1] Let  $\{a_k\}_{k \in \mathbb{Z}}$  be a sequence of positive numbers. We say that  $\{a_k\}_{k \in \mathbb{Z}}$  is strongly increasing or strongly decreasing and write  $a_k \uparrow\uparrow$  or  $a_k \downarrow\downarrow$  if

$$\inf_{k \in \mathbb{Z}} \frac{a_{k+1}}{a_k} > 1 \quad \text{or} \quad \sup_{k \in \mathbb{Z}} \frac{a_{k+1}}{a_k} < 1,$$

respectively.

*Definition 4.* [1, Definition 2.4] Assume that  $\varphi$  is an admissible function and  $h \in \Omega_\varphi$ . We say that  $\{\mu_k\}_{k \in \mathbb{Z}}$  is a discretization sequence for  $h$  with respect to  $\varphi$  if

- (i)  $\mu_0 = 1$  and  $\varphi(\mu_k) \uparrow\uparrow$ ;
- (ii)  $h(\mu_k) \uparrow\uparrow$  and  $\frac{h(\mu_k)}{\varphi(\mu_k)} \downarrow\downarrow$ ;
- (iii) there is a decomposition  $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2$  such that  $\mathbb{Z}_1 \cap \mathbb{Z}_2 = \emptyset$  and for every  $t \in [\mu_k, \mu_{k+1}]$ ,

$$\begin{aligned} h(\mu_k) &\approx h(t) && \text{if } k \in \mathbb{Z}_1, \\ \frac{h(\mu_k)}{\varphi(\mu_k)} &\approx \frac{h(t)}{\varphi(t)} && \text{if } k \in \mathbb{Z}_2. \end{aligned}$$

By Lemma [1, Lemma 2.7], for every admissible function  $\varphi$  and  $h \in \Omega_\varphi$ , there exists, a discretization sequence for  $h$  with respect to  $\varphi$ .

*Lemma 1.* [1, Lemma 2.8] Let  $\varphi$  be an admissible function. Then the following statements are equivalent:

- (i)  $h \in \Omega_\varphi$ .
- (ii) There exists a non-negative, a non-degenerate Borel measure  $\nu$  on  $[0, \infty)$  with respect to  $\varphi$  such that

$$h(t) \approx \varphi(t) \int_{[0, \infty)} \frac{d\nu(s)}{\varphi(s) + \varphi(t)}, \quad t \in (0, \infty).$$

Let us now present special cases of some well-known results concerning the discretization of integral expressions, as found in [1, 17].

*Lemma 2.* [1, Corollary 2.13] Assume that  $\varphi$  is an admissible function,  $f \in \Omega_\varphi$ ,  $\nu$  is a non-negative non-degenerate Borel measure on  $[0, \infty)$  and  $h$  is the fundamental function of  $\nu$  with respect to  $\varphi$ . If  $\{x_k\}_{k \in \mathbb{Z}}$  is a discretization sequence for  $h$  with respect to  $\varphi$ , then

$$\int_{[0, \infty)} \frac{f(t)}{\varphi(t)} d\nu(t) \approx \sum_{k \in \mathbb{Z}} \frac{f(x_k)}{\varphi(x_k)} h(x_k).$$

*Lemma 3.* [1, Lemma 3.5] Let  $1 < p < \infty$ . Assume that  $\varphi$  is an admissible function,  $f \in \Omega_\varphi$  and  $g \in \Omega_{\varphi^p}$ . If  $\{x_k\}_{k \in \mathbb{Z}}$  is a discretization sequence for  $f$  with respect to  $\varphi$  and  $\{\lambda_\ell\}_{\ell \in \mathbb{Z}}$  is a discretization sequence of  $g$  with respect to  $\varphi^p$ . Then

$$\sum_{k \in \mathbb{Z}} \frac{f(x_k)^{p'}}{g(x_k)^{\frac{p'}{p}}} \approx \sum_{\ell \in \mathbb{Z}} \frac{f(\lambda_\ell)^{p'}}{g(\lambda_\ell)^{\frac{p'}{p}}}$$

and

$$\sup_{t \in (0, \infty)} \frac{f(t)}{g(t)^{\frac{1}{p}}} \approx \sup_{k \in \mathbb{Z}} \frac{f(x_k)}{g(x_k)^{\frac{1}{p}}} \approx \sup_{\ell \in \mathbb{Z}} \frac{f(\lambda_\ell)}{g(\lambda_\ell)^{\frac{1}{p}}}.$$

*Lemma 4.* [1, Lemma 3.6] Let  $0 < r < \infty$ . Assume that  $\varphi$  is an admissible function,  $\nu$  is a non-degenerate positive Borel measure on  $[0, \infty)$  measure with respect to  $\varphi^r$ ,  $h$  is the fundamental function of  $\nu$  with respect to  $\varphi^r$  and  $f$  is a measurable function on  $[0, \infty)$ . If  $\{x_k\}_{k \in \mathbb{Z}}$  is a discretization sequence for  $h$  with respect to  $\varphi^r$ . Then

$$\int_0^\infty \left( \int_0^\infty \frac{|f(t)| dt}{\varphi(t) + \varphi(x)} \right)^r d\nu(x) \approx \sum_{k \in \mathbb{Z}} \left( \int_0^\infty \frac{|f(t)| dt}{\varphi(t) + \varphi(x_k)} \right)^r h(x_k).$$

*Lemma 5.* [1, Lemma 3.7] Assume that  $\varphi$  is an admissible function,  $\nu$  is a non-degenerate non-negative Borel measure on  $[0, \infty)$ ,  $h$  is the fundamental function of  $\nu$  with respect to  $\varphi$  and  $f$  is a measurable function on  $[0, \infty)$ . If  $\{x_k\}_{k \in \mathbb{Z}}$  is a discretization sequence for  $h$  with respect to  $\varphi$ , then

$$\int_{[0, \infty)} \operatorname{ess\,sup}_{y \in (0, \infty)} \frac{|f(y)|}{\varphi(x) + \varphi(y)} d\nu(x) \approx \sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x_k \leq y < x_{k+1}} |f(y)| \varphi^{-1}(y) h(y).$$

*Lemma 6.* [1, Lemma 3.1] (see also [22] for related results). Let  $\{a_k\}_{k \in \mathbb{Z}}$ ,  $\{b_k\}_{k \in \mathbb{Z}}$  and  $\{\sigma_k\}_{k \in \mathbb{Z}}$  be sequences of non-negative numbers.

If  $\tau_k \downarrow\downarrow$ , then

$$\sum_{k \in \mathbb{Z}} \left( \sum_{m=-\infty}^k a_m \right)^q \tau_k \approx \sum_{k \in \mathbb{Z}} a_k^q \tau_k. \tag{6}$$

If  $\sigma_k \uparrow\uparrow$ , then

$$\sum_{k \in \mathbb{Z}} \left( \sum_{m=k}^\infty a_m \right)^q \sigma_k \approx \sum_{k \in \mathbb{Z}} a_k^q \sigma_k. \tag{7}$$

*Lemma 7.* [1, Lemma 3.2] Let  $\{a_k\}_{k \in \mathbb{Z}}$ ,  $\{b_k\}_{k \in \mathbb{Z}}$  and  $\{\sigma_k\}_{k \in \mathbb{Z}}$  be sequences of non-negative numbers. If  $\sigma_k \uparrow\uparrow$ , then

$$\sum_{k \in \mathbb{Z}} \left( \sup_{k \leq m < \infty} a_m \right) \sigma_k \approx \sum_{k \in \mathbb{Z}} a_k \sigma_k. \tag{8}$$

*Lemma 8.* [1, Proposition 4.1, (ii)] Let  $1 < p < \infty$  and  $\{v_k\}_{k \in \mathbb{Z}}$  be a sequence of positive numbers. Then inequality

$$\sum_{k \in \mathbb{Z}} a_k v_k \leq c \left( \sum_{k \in \mathbb{Z}} a_k^p \right)^{1/p}$$

is satisfied for every sequence  $\{a_k\}_{k \in \mathbb{Z}}$  of non-negative numbers if and only if

$$B_1 := \left( \sum_{k \in \mathbb{Z}} v_k^{p'} \right)^{1/p'} < \infty.$$

Moreover, the best constant  $c$  in inequality (8) satisfies  $c \approx B_1$ .

*Theorem 2.* [17, Theorem 2.4.3] Let  $0 < r < \infty$ . Assume that  $\varphi$  is an admissible function and  $f$  is a non-negative measurable function on  $(0, \infty)$ . Then

$$\left( \int_0^\infty \frac{\varphi(t)\varphi(x)|f(t)|dt}{\varphi(t) + \varphi(x)} \right)^r \approx \int_0^\infty \min \left( \varphi^r(s), \varphi^r(x) \right) \left( \int_0^\infty \frac{\varphi(t)|f(t)|dt}{\varphi(t) + \varphi(s)} \right)^{r-1} |f(s)|ds. \quad (9)$$

### 2 Auxiliary results and Discrete characterization

In this section, we present the necessary lemmas, propositions, and proofs to establish the main theorems. The results presented here play a crucial role in the analysis of weighted inequalities and their discretized forms. We rely on known properties of integral operators to establish a connection between the continuous case and its discrete counterpart.

The following lemma provides a discretized formulation of the left-hand side of inequality (1). To formulate the next result, define

$$G(t) := \varphi(t) \int_0^\infty \frac{w(s)ds}{\varphi(t) + \varphi(s)}. \quad (10)$$

Observe that by Lemma 1,  $G \in \Omega_\varphi$ .

*Lemma 9.* Let  $h$  and  $w$  be non-negative measurable functions on  $(0, \infty)$  and  $\varphi$  be a quasi-concave admissible function on  $(0, \infty)$ . Assume that  $G$  is as defined in (10) and let  $\{x_k\}_{k \in \mathbb{Z}}$  be a discretization sequence of  $G$  with respect to  $\varphi$ . Then

$$\begin{aligned} & \int_0^\infty \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi(s)} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right) w(t) dt \\ & \approx \sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x_k < s < x_{k+1}} \frac{G(s)}{\varphi(s)} \left( \int_{x_k}^s \tau h(\tau) d\tau + s \int_s^{x_{k+1}} h(\tau) d\tau \right). \end{aligned} \quad (11)$$

*Proof.* As  $(\int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau)$  is increasing, we have

$$\begin{aligned} & \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi(s)} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right) \\ & = \max \left\{ \frac{1}{\varphi(t)} \operatorname{ess\,sup}_{0 < s < t} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right), \right. \\ & \left. \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi(s)^r} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right) \right\} \\ & = \operatorname{ess\,sup}_{0 < s < \infty} \min \left\{ \frac{1}{\varphi(t)}, \frac{1}{\varphi(s)} \right\} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right). \end{aligned}$$

Moreover, using the elementary inequality

$$\frac{1}{\varphi(s) + \varphi(t)} \leq \min \left\{ \frac{1}{\varphi(t)}, \frac{1}{\varphi(s)} \right\} \leq \frac{2}{\varphi(s) + \varphi(t)}, \tag{12}$$

we get

$$\begin{aligned} & \operatorname{ess\,sup}_{0 < s < \infty} \min \left\{ \frac{1}{\varphi(t)}, \frac{1}{\varphi(s)} \right\} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right) \\ & \approx \operatorname{ess\,sup}_{0 < s < \infty} \frac{1}{\varphi(s) + \varphi(t)} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right). \end{aligned}$$

Therefore,

$$\begin{aligned} & \int_0^\infty \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi(s)} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right) w(t) dt \\ & \approx \int_0^\infty \operatorname{ess\,sup}_{0 < s < \infty} \frac{1}{\varphi(t) + \varphi(s)} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right) w(t) dt. \end{aligned}$$

Let  $\{x_k\}_{k \in \mathbb{Z}}$  be a discretization sequence for  $G$  with respect to  $\varphi$ . Applying Lemma 5 with a function  $f(s) = \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau$ , and measure  $d\nu(x) = w(x) dx$ , we get

$$\begin{aligned} & \int_0^\infty \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi(s)} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right) w(t) dt \\ & \approx \sum_{k \in \mathbb{Z}} \sup_{x_k < s < x_{k+1}} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right) \varphi^{-1}(s) G(s). \end{aligned}$$

Since  $x_k < s < x_{k+1}$ , by decomposing the integrals as  $\int_0^s = \int_0^{x_k} + \int_{x_k}^s$  and  $\int_s^\infty = \int_s^{x_{k+1}} + \int_{x_{k+1}}^\infty$ , we obtain

$$\begin{aligned} & \int_0^\infty \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi(s)} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^\infty h(\tau) d\tau \right) w(t) dt \\ & \approx \sum_{k \in \mathbb{Z}} \sup_{x_k < s < x_{k+1}} \frac{G(s)}{\varphi(s)} \left( \int_{x_k}^s \tau h(\tau) d\tau + s \int_s^{x_{k+1}} h(\tau) d\tau \right) \\ & \quad + \sum_{k \in \mathbb{Z}} \sup_{x_k < s < x_{k+1}} \frac{G(s)}{\varphi(s)} \int_0^{x_k} \tau h(\tau) d\tau + \sum_{k \in \mathbb{Z}} \sup_{x_k < s < x_{k+1}} \frac{G(s)}{\varphi(s)} s \int_{x_{k+1}}^\infty h(\tau) d\tau \\ & =: I_1 + I_2 + I_3. \end{aligned} \tag{13}$$

Since  $\frac{G}{\varphi} \downarrow$  and  $\frac{G(x_k)}{\varphi(x_k)} \downarrow\downarrow$ , according to (6), we have

$$\begin{aligned} I_2 & = \sum_{k \in \mathbb{Z}} \frac{G(x_k)}{\varphi(x_k)} \int_0^{x_k} \tau h(\tau) d\tau \\ & = \sum_{k \in \mathbb{Z}} \frac{G(x_k)}{\varphi(x_k)} \sum_{m=-\infty}^k \int_{x_{m-1}}^{x_m} \tau h(\tau) d\tau \\ & \approx \sum_{k \in \mathbb{Z}} \frac{G(x_k)}{\varphi(x_k)} \int_{x_{k-1}}^{x_k} \tau h(\tau) d\tau \\ & \leq \sum_{k \in \mathbb{Z}} \sup_{x_{k-1} < s < x_k} \frac{G(s)}{\varphi(s)} \int_{x_{k-1}}^s \tau h(\tau) d\tau \\ & \leq I_1. \end{aligned}$$

Next, as  $\frac{x_{k+1}}{\varphi(x_{k+1})} \uparrow$  and  $G(x_{k+1}) \uparrow\uparrow$ , it follows that  $\frac{G(x_{k+1})x_{k+1}}{\varphi(x_{k+1})} \uparrow\uparrow$ . Applying (7), we get

$$\begin{aligned} I_3 &= \sum_{k \in \mathbb{Z}} \frac{G(x_{k+1})}{\varphi(x_{k+1})} x_{k+1} \int_{x_{k+1}}^{\infty} h(\tau) d\tau \\ &= \sum_{k \in \mathbb{Z}} \frac{G(x_{k+1})}{\varphi(x_{k+1})} x_{k+1} \sum_{m=k+1}^{\infty} \int_{x_m}^{x_{m+1}} h(\tau) d\tau \\ &\approx \sum_{k \in \mathbb{Z}} \frac{G(x_{k+1})}{\varphi(x_{k+1})} x_{k+1} \int_{x_{k+1}}^{x_{k+2}} h(\tau) d\tau \\ &\leq \sum_{k \in \mathbb{Z}} \sup_{x_{k+1} < s < x_{k+2}} \frac{G(s)s}{\varphi(s)} \int_s^{x_{k+2}} h(\tau) d\tau \\ &\leq I_1. \end{aligned}$$

Therefore,

$$I_1 + I_2 + I_3 \approx I_1$$

holds, and from (13) follows (11). □

*Proposition 1.* Let  $1 < p < \infty$ ,  $w, v$  be non-negative measurable functions on  $(0, \infty)$  and  $\varphi$  be a quasi-concave admissible function on  $(0, \infty)$ . Assume that  $G$  is as defined in (10) and let  $\{x_k\}_{k \in \mathbb{Z}}$  be a discretization sequence of  $G$  with respect to  $\varphi$ . Then, there exists a positive constant  $C$  such that (1) holds for all non-negative measurable functions  $h$  on  $(0, \infty)$  if and only if there exists a positive constant  $\tilde{C}$  such that

$$\sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x_k < s < x_{k+1}} \frac{G(s)}{\varphi(s)} \left( \int_{x_k}^s \tau h(\tau) d\tau + s \int_s^{x_{k+1}} h(\tau) d\tau \right) \leq \tilde{C} \left( \sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} h^p(\tau) v(\tau) d\tau \right)^{\frac{1}{p}} \quad (14)$$

holds for all non-negative measurable functions  $h$  on  $(0, \infty)$ . Moreover the best constants  $C$  and  $\tilde{C}$ , respectively in (1) and (14) satisfy  $C \approx \tilde{C}$ .

*Proof.* Let  $\{x_k\}_{k \in \mathbb{Z}}$  be a discretization sequence of  $G$ . Applying (11), for the left hand side of (1), we have

$$\begin{aligned} &\int_0^{\infty} \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi(s)} \left( \int_0^s \tau h(\tau) d\tau + s \int_s^{\infty} h(\tau) d\tau \right) w(t) dt \\ &\approx \sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x_k < s < x_{k+1}} \frac{G(s)}{\varphi(s)} \left( \int_{x_k}^s \tau h(\tau) d\tau + s \int_s^{x_{k+1}} h(\tau) d\tau \right), \end{aligned}$$

on the other hand, for the right-hand side of (1), we have

$$\left( \int_0^{\infty} h^p(\tau) v(\tau) d\tau \right)^{\frac{1}{p}} = \left( \sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} h^p(\tau) v(\tau) d\tau \right)^{\frac{1}{p}}.$$

Then, it is clear that there exists a positive constant  $C$  such that inequality (1) holds for all non-negative measurable functions  $h$  on  $(0, \infty)$  if and only if there exists a positive constant  $\tilde{C}$  such that (14) holds for all non-negative measurable functions  $h$  on  $(0, \infty)$ . Moreover,  $C \approx \tilde{C}$ . □

The next proposition establishes an inequality that is equivalent to inequality (11). For this purpose, denote by

$$M(x_k, x_{k+1}) := \sup_{h \geq 0} \frac{\operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G(t)}{\varphi(t)} \left( \int_{x_k}^t \tau h(\tau) d\tau + t \int_t^{x_{k+1}} h(\tau) d\tau \right)}{\left( \int_{x_k}^{x_{k+1}} h^p(\tau) v(\tau) d\tau \right)^{\frac{1}{p}}}, \quad (15)$$

where  $1 < p < \infty$ , and the functions  $\varphi$  and  $G$  are as specified in Proposition 1.

Note that, using the characterizations of weighted iterated Copson and Hardy inequalities (see [17, Theorem 5.3.1]), we have

$$M(x_k, x_{k+1}) \approx \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G(t)}{\varphi(t)} \left( \int_{x_k}^t \tau^{p'} v^{1-p'}(\tau) d\tau + t^{p'} \int_t^{x_{k+1}} v^{1-p'}(\tau) d(\tau) \right)^{\frac{1}{p'}}. \quad (16)$$

*Proposition 2.* Let  $1 < p < \infty$ ,  $w, v$  be non-negative measurable functions on  $(0, \infty)$  and  $\varphi$  be a quasi-concave admissible function on  $(0, \infty)$ . Assume that  $G$  is as defined in (10) and let  $\{x_k\}_{k \in \mathbb{Z}}$  be a discretization sequence of  $G$  with respect to  $\varphi$ . Then, there exist, a positive constant  $\tilde{C}$  such that inequality (14) holds for all non-negative measurable  $h$  on  $(0, \infty)$  if and only if there exists a positive constant  $\mathcal{C}$  such that

$$\sum_{k \in \mathbb{Z}} a_k M(x_k, x_{k+1}) \leq \mathcal{C} \left( \sum_{k \in \mathbb{Z}} a_k^p \right)^{\frac{1}{p}}, \quad (17)$$

holds for every sequence of non-negative numbers  $\{a_k\}_{k \in \mathbb{Z}}$ , where  $M(x_k, x_{k+1})$  is defined in (15). Moreover, the best constants  $\tilde{C}$  and  $\mathcal{C}$  in (14) and (17), respectively satisfy  $\tilde{C} \approx \mathcal{C}$ .

*Proof.* Suppose that inequality (14) holds for all non-negative measurable  $h$  on  $(0, \infty)$ . By the definition of  $M(x_k, x_{k+1})$  in (15), there exist functions  $h_k \geq 0$ , such that

$$\operatorname{supp} h_k \subset [x_k, x_{k+1}], \quad \left( \int_{x_k}^{x_{k+1}} h_k^p(\tau) v(\tau) d\tau \right)^{\frac{1}{p}} = 1 \quad (18)$$

and

$$\operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G(t)}{\varphi(t)} \left( \int_{x_k}^t \tau h_k(\tau) d\tau + t \int_t^{x_{k+1}} h_k(\tau) d\tau \right) \geq \frac{1}{2} M(x_k, x_{k+1}). \quad (19)$$

For any sequence of non-negative numbers  $\{a_k\}_{k \in \mathbb{Z}}$  define

$$h(\tau) = \sum_{j \in \mathbb{Z}} a_j h_j(\tau).$$

Testing inequality (14) with  $h$ , and using (19), we get

$$\begin{aligned} & \sum_{k \in \mathbb{Z}} \sup_{x_k < s < x_{k+1}} \frac{G(s)}{\varphi(s)} \left( \int_{x_k}^s \tau h(\tau) d\tau + s \int_s^{x_{k+1}} h(\tau) d\tau \right) \\ &= \sum_{k \in \mathbb{Z}} a_k \sup_{x_k < s < x_{k+1}} \frac{G(s)}{\varphi(s)} \left( \int_{x_k}^s \tau h_k(\tau) d\tau + s \int_s^{x_{k+1}} h_k(\tau) d\tau \right) \\ &\geq \frac{1}{2} \sum_{k \in \mathbb{Z}} a_k M(x_k, x_{k+1}) \end{aligned}$$

and by (18) we have

$$\left( \sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} h^p(\tau)v(\tau)d\tau \right)^{\frac{1}{p}} = \left( \sum_{k \in \mathbb{Z}} a_k^p \int_{x_k}^{x_{k+1}} h_k^p(\tau)v(\tau)d\tau \right)^{\frac{1}{p}} = \left( \sum_{k \in \mathbb{Z}} a_k^p \right)^{\frac{1}{p}}.$$

Thus, (17) holds with  $\mathcal{C} \lesssim \tilde{\mathcal{C}}$ .

Conversely, observe that for each non-negative measurable  $h$

$$M(x_k, x_{k+1}) \geq \frac{\operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G(t)}{\varphi(t)} \left( \int_{x_k}^t \tau h(\tau)d\tau + t \int_t^{x_{k+1}} h(\tau)d(\tau) \right)}{\left( \int_{x_k}^{x_{k+1}} h^p(\tau)v(\tau)d\tau \right)^{\frac{1}{p}}}$$

holds. Then (14) follows by inserting

$$a_k = \left( \int_{x_k}^{x_{k+1}} h^p(\tau)v(\tau)d\tau \right)^{\frac{1}{p}}$$

in (17) and  $\tilde{\mathcal{C}} \leq \mathcal{C}$ . Therefore, we obtain  $\tilde{\mathcal{C}} \approx \mathcal{C}$ . □

*Theorem 3.* Let  $1 < p < \infty$ ,  $w, v$  be non-negative measurable functions on  $(0, \infty)$  and  $\varphi$  be a quasi-concave admissible function on  $(0, \infty)$ . Assume that  $G$  is as defined in (10) and let  $\{x_k\}_{k \in \mathbb{Z}}$  be a discretization sequence of  $G$  with respect to  $\varphi$ . Then there exists a positive constant  $C$  such that inequality (1) holds for all non-negative measurable functions  $f$  on  $(0, \infty)$  if and only if

$$A := \left( \sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left( \int_{x_k}^t s^{p'} v^{1-p'}(s)ds + t^{p'} \int_t^{x_{k+1}} v^{1-p'}(s)ds \right) \right)^{\frac{1}{p'}} < \infty. \tag{20}$$

Moreover, the best constant  $C$  in (1) satisfies  $C \approx A$ .

*Proof.* Using Proposition 1 and Proposition 2, the best constant in (1) satisfies  $C \approx \mathcal{C}$ , where  $\mathcal{C}$  is the best constant in inequality (17). Therefore, applying Lemma 8 and using (16), the result follows. □

### 3 Proof of Theorem 1

*Proof of Theorem 1* (i)  $\Leftrightarrow$  (ii). By Theorem 3, it is enough to show that  $C_1 \approx A$ . Let  $H$  be defined as in (4) and  $G$  be defined as in (10). Assume that  $\{x_k\}_{k \in \mathbb{Z}}$  is a discretization sequence of  $G$  with respect to  $\varphi$  and  $\{y_k\}_{k \in \mathbb{Z}}$  is a discretization sequence of  $H$  with respect to  $\varphi^{p'}$ .

Using Lemma 4 and the definition of  $G$  (see (10)), we have

$$C_1^{p'} \approx \sum_{k \in \mathbb{Z}} \left( \int_0^\infty \frac{w(s)ds}{\varphi(y_k) + \varphi(s)} \right)^{p'} H(y_k) = \sum_{k \in \mathbb{Z}} \frac{G^{p'}(y_k)}{(\varphi^p(y_k)H(y_k)^{-\frac{p}{p'}})^{\frac{p'}{p}}}.$$

It is easy to check that  $g = \varphi^p H^{-\frac{p}{p'}} \in \Omega_{\varphi^p}$ . As  $\{y_k\}_{k \in \mathbb{Z}}$  is a discretization sequence of  $H$  with respect to  $\varphi^{p'}$ , it is also a discretization sequence for  $g$  with respect to  $\varphi^p$ . Now, applying Lemma 3 for function  $f = G$  and  $g$  and using the definition of  $H$  (see (4)), we obtain

$$C_1^{p'} \approx \sum_{k \in \mathbb{Z}} \frac{G^{p'}(x_k)}{(\varphi^p(x_k)H^{-\frac{p}{p'}}(x_k))^{\frac{p'}{p}}} = \sum_{k \in \mathbb{Z}} G^{p'}(x_k) \operatorname{ess\,sup}_{x_k < t < \infty} \frac{1}{\varphi^{p'}(t)} \left( \int_0^\infty \frac{s^{p'} t^{p'}}{s^{p'} + t^{p'}} v^{1-p'}(s)ds \right).$$

On the other hand, observe that

$$\int_0^\infty \frac{t^{p'} s^{p'}}{s^{p'} + t^{p'}} v^{1-p'}(s) ds \approx \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds.$$

Then,

$$\begin{aligned} C_1^{p'} &\approx \sum_{k \in \mathbb{Z}} G^{p'}(x_k) \operatorname{ess\,sup}_{x_k < t < \infty} \frac{1}{\varphi^{p'}(t)} \left( \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds \right) \\ &= \sum_{k \in \mathbb{Z}} G^{p'}(x_k) \sup_{k \leq m} \operatorname{ess\,sup}_{x_m < t < x_{m+1}} \frac{1}{\varphi^{p'}(t)} \left( \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds \right). \end{aligned}$$

Since  $G^{p'}(x_k) \uparrow \uparrow$ , applying (8), we obtain

$$C_1^{p'} \approx \sum_{k \in \mathbb{Z}} G^{p'}(x_k) \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{1}{\varphi^{p'}(t)} \left( \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds \right).$$

Next, we want to show that

$$\begin{aligned} &\sum_{k \in \mathbb{Z}} G^{p'}(x_k) \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{1}{\varphi^{p'}(t)} \left( \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds \right) \\ &\approx \sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left( \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds \right). \end{aligned} \tag{21}$$

As the function  $G^{p'}$  is increasing, the upper estimate is trivial. Conversely,

$$\begin{aligned} &\sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left( \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds \right) \\ &\approx \sum_{k \in \mathbb{Z}_1} \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left( \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds \right) \\ &\quad + \sum_{k \in \mathbb{Z}_2} \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left( \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds \right) \\ &\approx \sum_{k \in \mathbb{Z}_1} G^{p'}(x_k) \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{1}{\varphi^{p'}(t)} \left( \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds \right) \\ &\quad + \sum_{k \in \mathbb{Z}_2} \frac{G^{p'}(x_{k+1})}{\varphi^{p'}(x_{k+1})} \left( \int_0^{x_{k+1}} s^{p'} v^{1-p'}(s) ds + x_{k+1}^{p'} \int_{x_{k+1}}^\infty v^{1-p'}(s) ds \right) \\ &\lesssim \sum_{k \in \mathbb{Z}} G^{p'}(x_k) \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{1}{\varphi^{p'}(t)} \left( \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds \right). \end{aligned}$$

Therefore (21) holds. Thus

$$C_1^{p'} \approx \sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G^{p'}(t)}{\varphi(t)^{p'}} \left( \int_0^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^\infty v^{1-p'}(s) ds \right).$$

Next, decomposing the integrals as  $\int_0^t = \int_0^{x_k} + \int_{x_k}^t$  and  $\int_t^\infty = \int_t^{x_{k+1}} + \int_{x_{k+1}}^\infty$  and using fact that  $G^{p'}(x_k) \uparrow\uparrow$  and  $\frac{t}{\varphi(t)}$  is increasing and therefore  $\frac{G^{p'}(x_k)x_k^{p'}}{\varphi^{p'}(x_{k+1})} \uparrow\uparrow$  and applying Lemma 6, by (20) we obtain

$$\begin{aligned} C_1^{p'} &\approx \sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left( \int_{x_k}^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^{x_{k+1}} v^{1-p'}(s) ds \right) \\ &\quad + \sum_{k \in \mathbb{Z}} \frac{G^{p'}(x_k)}{\varphi^{p'}(x_k)} \int_0^{x_k} s^{p'} v^{1-p'}(s) ds + \sum_{k \in \mathbb{Z}} \frac{G^{p'}(x_{k+1})x_{k+1}^{p'}}{\varphi^{p'}(x_{k+1})} \int_{x_{k+1}}^\infty v^{1-p'}(s) ds \\ &\approx \sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left( \int_{x_k}^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^{x_{k+1}} v^{1-p'}(s) ds \right) \\ &\quad + \sum_{k \in \mathbb{Z}} \frac{G^{p'}(x_k)}{\varphi^{p'}(x_k)} \int_{x_{k-1}}^{x_k} s^{p'} v^{1-p'}(s) ds + \sum_{k \in \mathbb{Z}} \frac{G^{p'}(x_{k+1})x_{k+1}^{p'}}{\varphi^{p'}(x_{k+1})} \int_{x_{k+1}}^{x_{k+2}} v^{1-p'}(s) ds \\ &\approx \sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x_k < t < x_{k+1}} \frac{G^{p'}(t)}{\varphi^{p'}(t)} \left( \int_{x_k}^t s^{p'} v^{1-p'}(s) ds + t^{p'} \int_t^{x_{k+1}} v^{1-p'}(s) ds \right) \\ &= A^{p'}. \end{aligned}$$

Consequently,  $C_1 \approx A$ , which is the desired estimate.

(ii)  $\Leftrightarrow$  (iii). Using (9) for  $f = \varphi^{-1}w$  and  $r = p'$ , (12), Fubini theorem and (5), we obtain

$$\begin{aligned} C_1^{p'} &\approx \int_0^\infty \left( \int_0^\infty \left( \int_0^\infty \frac{w(s) ds}{\varphi(s) + \varphi(t)} \right)^{p'-1} \frac{\varphi(t)^{p'-1} w(t)}{\varphi(x)^{p'} + \varphi(t)^{p'}} dt \right) d\nu(x) \\ &= \int_0^\infty \left( \int_0^\infty \frac{w(s) ds}{\varphi(s) + \varphi(t)} \right)^{p'-1} \left( \int_0^\infty \frac{d\nu(x)}{\varphi(x)^{p'} + \varphi(t)^{p'}} \right) \varphi(t)^{p'-1} w(t) dt \\ &\approx \left( \int_0^\infty \left( \int_0^\infty \frac{w(s) ds}{\varphi(s) + \varphi(t)} \right)^{p'-1} \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi^{p'}(s)} \left( \int_0^\infty \frac{\tau^{p'} s^{p'}}{s^{p'} + \tau^{p'}} v^{1-p'}(\tau) d\tau \right) w(t) \varphi^{p'-1}(t) dt \right)^{\frac{1}{p'}} \\ &= C_2^{p'}. \end{aligned}$$

The theorem is proved.

*Remark 1.* Note that the case when  $p = 1$  was considered in [23].

### Conclusion

In this paper, we investigate the conditions on the weight functions  $v, w$  ensuring that inequality (1) holds for all non-negative measurable functions. Inequality (1) holds if and only if a certain condition involving the weight functions  $v, w$  and the quasi-concave function  $\varphi$  holds.

The main result, presented in Theorem 1, confirms that the inequality under consideration holds for all non-negative measurable functions  $h$  when the associated integral condition is finite. To derive this result, we first prove Theorem 3, which provides a discrete characterization of inequality (1). This discrete formulation provides a detailed understanding of the best constant  $C$  in inequality (1), and shows conditions under which this constant can be determined. Finally, by applying antidiscretization techniques, we obtain the proof of the main result, Theorem 1.

The results obtained in this paper may be useful for further study of inequalities in weighted spaces, especially in analysing the interactions between weight functions and quasi-concave functions.

In particular, characterizations of the embeddings of the cones generated by the non-increasing rearrangements of the generalized fractional maximal functions [21]. They may also have significant applications in various areas of analysis, where such inequalities play a crucial role in understanding the behaviour of integrals involving multiple weight functions and functional operators.

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#### *Author Contributions*

All authors contributed equally to this work. They also participated in revising the manuscript and approved the final submission.

#### *Conflict of Interest*

The authors declare no conflict of interest.

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