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An analogue of Leibniz's rule for Hadamard derivatives and their application

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This paper explores new analogues of the Leibniz rule for Hadamard and Caputo–Hadamard fractional derivatives. Unlike classical derivatives, fractional ones have a strong nonlocal character, meaning that the value of the derivative at a given point depends on the entire history of the function. Because of this nonlocality, the standard product rule cannot be directly applied. The study develops refined formulas for differentiating the product of two functions, which include additional integral terms representing memory effects inherent to fractional calculus. The paper also establishes a series of inequalities that make it possible to estimate the fractional derivatives of nonlinear expressions, such as powers of a function, through the derivative of the function itself. In particular, it is shown that a specific inequality holds for positive functions that relates the fractional derivative of the function power to the function product and its fractional derivative. These theoretical results are of great importance for the study of linear and nonlinear fractional diffusion equations. They provide useful tools for proving the existence, uniqueness, and stability of their solutions and for deriving a priori estimates that describe the qualitative behavior of such systems.

Keywords: linear and nonlinear diffusion equation, Hadamard-type time fractional derivative, Hadamard time fractional derivative, Mittag-Leffler function, a priori estimates, Leibniz rule, porous medium equation, Gronwall inequality.

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Introduction

In the theory of differential calculus, the Leibniz's rule is one of the most important rules. Leibniz's rule states that: for two differentiable functions $u(x)$ and $v(x)$, the derivative of their product $u(x)v(x)$ is given by

$$\frac{d}{dx}(u(x)v(x)) = u'(x)v(x) + u(x)v'(x). \tag{1}$$

The Leibniz's rule is applied to many problems in PDEs, including a priori estimates for solutions to linear and nonlinear parabolic problems.

However, in the case of fractional derivatives, it is not possible to obtain a simple expression analogous to (1). Tarasov [1] demonstrated that the formula

$$D^\alpha(u(x)v(x)) = D^\alpha u(x)v(x) + u(x)D^\alpha v(x)$$

α is an integer. This limitation arises from the inherently nonlocal nature of fractional derivatives. Nevertheless, various analogues of the classical The Leibniz rule for fractional derivatives have been deve-loped in the literature. In particular, the foundations of fractional calculus and the main properties of fractional operators, including Hadamard-type derivatives, were systematically presented in the

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monographs [2–4]. Further generalizations of the Leibniz formula for fractional derivatives of different types were obtained in [5–8], where both analytical and operator approaches were discussed. The results concerning fractional diffusion equations and applications of fractional Leibniz-type rules to boundary and initial value problems can be found in [9–11]. For example, in [9] Alsaedi, Ahmad and Kirane obtained an analogue of the Leibniz's rule in the following form:

$$\begin{aligned} D^{\alpha}(uv)(t) &= u(t)D^{\alpha}v(t) + v(t)D^{\alpha}u(t) \\ &\quad - \frac{u(t)v(t)}{\Gamma(1-\alpha)t^{\alpha}} - \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{(u(s)-u(t))(v(s)-v(t))}{(t-s)^{1+\alpha}}ds, \end{aligned}$$

where D^{α} is the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$:

$$D^{\alpha}u(t) = \frac{1}{(\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}u(s)ds.$$

Later in [10], Cuesta et al. extended this formula to the Riemann-Liouville fractional derivative of variable order $\alpha(t) \in (0,1)$, $t > 0$. This makes fractional calculus particularly relevant in fields such as physics, biology, materials science, and economics, where traditional approaches are insufficient to describe real-world phenomena. The application of fractional models in continuum mechanics and physical systems was discussed in [12, 13], while the classical foundations of fractional calculus were established in [14, 15]. Further developments related to anomalous diffusion processes and boundary value problems in mathematical physics were presented in [16, 17].

In recent years, there has been a growing interest in the study of both linear and nonlinear differential equations involving Hadamard and Hadamard-type fractional derivatives. Fundamental results on the theory and applications of such derivatives can be found in [15, 18, 19]. Theoretical and numerical studies addressing the well-posedness, regularity, and stability of related equations are provided in [20–22]. Moreover, generalized forms of the Leibniz-type rule for Hadamard fractional operators and their applications to extremum principles have been explored in [23–25]. In [24], it was proved that the Hadamard multi-index fractional diffusion problem has at most one classical solution, and this solution depends continuously on its initial boundary conditions. In [25], Kirane and Torebek obtained new estimates for the fractional Hadamard derivatives of a function at its extreme points, and using the extremum principle, showed that linear and nonlinear fractional diffusion equations with initial-boundary conditions have at most one classical solution, and this solution continuously depends on the initial and boundary conditions. For Hadamard fractional differential equations with initial boundary conditions involving a fractional Laplace operator, Wang, Ren, and Baleanu [24] applied the maximum principle and obtained certain existence and uniqueness results.

In [26], the authors have given a small generalization of the Gronwall inequality, which they used to study a solution to a generalized Cauchy-type problem with a Hilfer–Hadamard-type fractional derivative. The Leibniz's rule for fractional derivatives of constant order was introduced in [9] as an extension of the classical product rule for integer-order derivatives. This differentiation rule (as well as other fractional rules found in the literature) includes additional terms that account for the non-local nature of fractional derivatives, particularly in the case of fractional derivatives of variable order (FDVO). The authors present a contemporary proof of the maximum principle applicable to the linear and nonlinear Riemann–Liouville fractional diffusion equations using the following inequality, for any integer $p \geq 2$ and $u \geq 0$

$$D^{\alpha}_{a+,t}u^p \leq pu^{p-1}D^{\alpha}_{a+,t}u, \quad \begin{cases} \text{for } p \text{ even}, \\ \text{for } p \text{ odd whenever}. \end{cases} \tag{2}$$

In [10], the authors further advance this concept by extending this property to fractional derivatives with a variable order $\alpha(t)$. They derive a Leibniz inequality and an integration by parts formula. They also studied an initial value problem with their time variable order fractional derivative and present a regularity result for it, and study its on the asymptotic behavior.

Motivated by the neet to explore in the context of Hadamard derivatives, we embarked on an investigation of the Leibniz inequality for both linear and nonlinear diffusion equations. After establishing inequality (2) for Hadamard and Hadamard-type fractional derivatives using the Gronwall inequality, we explored a priori decay estimates for the solutions.

Our main results are given in the following form:

Lemma 1. Let u, v satisfy the following condition

$$u \in AC[a,T] \text{ and } v \in AC[a,T],\ 0 < \alpha \leq 1.$$

Then, the following holds true

$$\begin{aligned} D^{\alpha}_{a+,t}[uv](t) &= u(t)D^{\alpha}_{a+,t}v(t) + v(t)D^{\alpha}_{a+,t}u(t) \\ &- \frac{u(t)v(t)}{\Gamma(1-\alpha)\left(\log\frac{t}{a}\right)^{\alpha}} - \frac{\alpha}{\Gamma(1-\alpha)}\int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t}{s}\right)^{1+\alpha}}ds. \end{aligned}$$

This leads to the following cases.

Corollary 1. If u and v have the same signs, then

$$D^{\alpha}_{a+,t}(uv)(t) \leq u(t)D^{\alpha}_{a+,t}v(t) + v(t)D^{\alpha}_{a+,t}u(t). \tag{3}$$

Let $u \in AC[a,T]$ and $0 < \alpha \leq 1$. Applying $u = v$ in inequality (3), we get the following statement

$$2u(t)D^{\alpha}_{a+,t}u(t) \geq D^{\alpha}_{a+,t}u^2(t). \tag{4}$$

Then

$$D^{\alpha}_{a+,t}u^p \leq pu^{p-1}D^{\alpha}_{a+,t}u, \tag{5}$$

where $p \geq 2$ and $u \geq 0$. Using mathematical induction we can prove inequality (5).

Lemma 2. Let u, v satisfy the following condition

$$u \in AC[a,T] \text{ and } v \in AC[a,T],\ 0 < \alpha \leq 1.$$

Then, the following holds true

$$\begin{aligned} {}^{C}_{H}D^{\alpha}_{a+,t}[uv](t) &= u(t){}^{C}_{H}D^{\alpha}_{a+,t}v(t) + v(t){}^{C}_{H}D^{\alpha}_{a+,t}u(t) \\ &- \frac{(u(a)-u(t))(v(a)-v(t))}{\Gamma(1-\alpha)\left(\log\frac{t}{a}\right)^{\alpha}} \\ &- \frac{\alpha}{\Gamma(1-\alpha)}\int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t}{s}\right)^{1+\alpha}}ds. \end{aligned}$$

This leads to the following cases.

Corollary 2. If u and v have the same signs, then

$${}^{C}_{H}D^{\alpha}_{a+,t}(uv)(t) \leq u(t){}^{C}_{H}D^{\alpha}_{a+,t}v(t) + v(t){}^{C}_{H}D^{\alpha}_{a+,t}u(t). \tag{6}$$

Then

$${}^{C}_{H}D^{\alpha}_{a+,t}u^p \leq pu^{p-1}{}^{C}_{H}D^{\alpha}_{a+,t}u, \tag{7}$$

where $p \geq 2$ and $u \geq 0$. Applying mathematical induction we can prove inequality (7).

1 Preliminaries

1.1 The weighted space of continuous functions space

Let us consider the weighted space of continuous functions denoted by $C_{\gamma,\log}[a,b]$, where $0 \leq \gamma < 1$. A function $f : (a,b] \to \mathbb{R}$ belongs to this space if the function $\left(\log \frac{t}{a}\right)^{\gamma} f(t)$ can be continuously extended to the closed interval $[a,b]$. More precisely,

$$C_{\gamma,\log}[a,b] = \left\{ f : (a,b] \to \mathbb{R} \;\middle|\; \left(\log \frac{t}{a}\right)^{\gamma} f(t) \in C[a,b] \right\}.$$

The norm associated with this space is given by

$$\|f\|_{C_{\gamma,\log}[a,b]} = \left\| \left(\log \frac{t}{a}\right)^{\gamma} f(t) \right\|_{C[a,b]}.$$

It is worth noting that for $\gamma = 0$, this space reduces to the classical space of continuous functions, i.e., $C_{0,\log}[a,b] = C[a,b]$.

For any positive integer n, we work within the Banach space $C^{n}_{\delta,\gamma}[a,b]$ of functions possessing continuous δ-derivatives up to order $n-1$ on $[a,b]$, and a δ^{n}-derivative on $(a,b]$ such that $\delta^{n} f \in C_{\gamma,\log}[a,b]$. The dilation operator is defined as $\delta = t\frac{d}{dt}$. Functions in this space satisfy the norm condition

$$\|f\|_{C^{n}_{\delta,\gamma}} = \sum_{k=0}^{n-1} \|\delta^{k} f\|_{C[a,b]} + \|\delta^{n} f\|_{C_{\gamma,\log}[a,b]} < \infty.$$

In the special case of $n = 0$, the space $C^{0}_{\delta,\gamma}[a,b]$ coincides with $C_{\gamma,\log}[a,b]$.

Additionally, we make use of the space $AC^{n}_{\delta}[a,b]$, which consists of functions $f : [a,b] \to \mathbb{C}$ for which the $(n-1)$-th δ-derivative, $\delta^{n-1} f$, belongs to the space of absolutely continuous functions $AC[a,b]$. Explicitly,

$$AC^{n}_{\delta}[a,b] = \left\{ f : [a,b] \to \mathbb{C} \;\middle|\; \delta^{n-1} f \in AC[a,b] \right\}.$$

It is evident that $AC^{1}_{\delta}[a,b]$ coincides with $AC[a,b]$.

These functional spaces and operators provide a natural framework for analyzing differential equations involving weighted logarithmic behaviors and dilation-invariant properties, which are especially relevant in the study of nonlocal models and fractional dynamics (see more details [4, 17] and links therein).

Definition 1. [4, p. 110] Let $f \in L^{1}_{loc}([a,b])$. The Hadamard fractional integral $I^{\alpha}_{a+,t}$ of order $\alpha \in (0,1)$ $(a > 0)$ is defined as

$$I^{\alpha}_{a+,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}.$$

Definition 2. [4, p. 111] Let $a > 0$ and $f \in W^{1}_{2}([a,b])$. The Hadamard fractional derivative of order $\alpha \in (0,1)$ is defined by

$$D^{\alpha}_{a+,t} f(t) = t\frac{d}{dt} I^{1-\alpha}_{a+,t} f(t) = t\frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_{a}^{t} \left(\log \frac{t}{s}\right)^{-\alpha} f(s) \frac{ds}{s}.$$

Property 1. [4, p. 116] Let $0 < \alpha < 1$ and $0 < a, b < \infty$. If $f \in C_{\mu,\log}[a,b]\,(0 \le \mu < 1)$ and $I^{1-\alpha}_{a+,t} f \in C^1_{\delta,\mu}[a,b]$, then

$$\left(I^{\alpha}_{a+,t} D^{\alpha}_{a+,t} f\right)(t) = f(t) - \frac{\left(I^{1-\alpha}_{a+,t} f\right)(a)}{\Gamma(\alpha)} \left(\log \frac{t}{a}\right)^{\alpha-1}, \; t \in [a,b]$$

holds at any point $t \in (a,b]$.

Definition 3. [4, p. 115] The Hadamard-type fractional derivative of order $\alpha \in (0,1)$ with $a > 0$, then for $f(t) \in AC[a,b]$

$${}^{C}_{H}D^{\alpha}_{a+,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{-\alpha} f'(s) ds.$$

Alternatively, for $u \in C^1[a,t]$ an equivalent representation is

$${}^{C}_{H}D^{\alpha}_{a+,t} u(t) = \frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{u(s) - u(t)}{s \log(t/s)^{\alpha}} ds$$

Definition 4. [4, p. 42] The Mittag-Leffler function with two parameters is represented as

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \; (z, \beta \in \mathbb{C}, \; \Re(\alpha) > 0).$$

Lemma 3. [26, Lemma 3.1] Let $\alpha > 0$, $u(t)$, $v(t)$ be nonnegative functions and locally integrable on $0 < a \le t < T \le \infty$, and $\mathcal{M}(t)$ is a nonnegative, nondecreasing continuous function defined on $0 < a \le t < T \le \infty$, $\mathcal{M}(t) \le m\,(constant)$

$$u(t) \le v(t) + \mathcal{M}(t) \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} u(s) \frac{ds}{s},$$

then

$$u(t) \le v(t) + \int_a^t \left[\sum_{k=1}^{\infty} \frac{(\mathcal{M}(t)\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log \frac{t}{s}\right)^{k\alpha-1} \frac{v(s)}{s} \right] ds.$$

Lemma 4. Let a nonnegative absolutely continuous function $y(t)$ satisfy the inequality

$$\partial^{\alpha}_{a,t} y(t) \le \theta y(t) + \mu(t), \quad 0 < \alpha \le 1$$

for almost all t in $[a,T]$, where $\theta > 0$ and $\mu(t)$ is an integrable nonnegative function on $[a,T]$. Then

$$y(t) \le y(a) E_{\alpha,1}\left(\theta \left(\log \frac{t}{a}\right)^{\alpha}\right) + \Gamma(\alpha) E_{\alpha,\alpha}\left(\theta \left(\log \frac{t}{a}\right)^{\alpha}\right) \partial^{-\alpha}_{a,t} \mu(t),$$

where the function $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function.

Remark 1. The case $\alpha = 1$ of Lemma 4 is studied in [17, p. 152].

Proof. Let $\partial^{\alpha}_{a,t} y(t) - \theta y(t) = g(t)$, then

$$y(t) = y(a) E_{\alpha,1}\left(\theta \left(\log \frac{t}{a}\right)^{\alpha}\right) + \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\theta \left(\log \frac{t}{\tau}\right)^{\alpha}\right) \frac{g(\tau)}{\tau} d\tau.$$

By virtue of the inequality $g(t) \le \mu(t)$, the positivity of the Mittag-Leffler function $E_{\alpha,\alpha}\left(\theta\left(\log\frac{t}{\tau}\right)^{\alpha}\right)$ for given parameters, and the growth of the function $E_{\alpha,\alpha}(t)$, from [26], we obtain

$$\begin{aligned} y(t) &\le y(a)E_{\alpha,1}\left(\theta\left(\log\frac{t}{a}\right)^{\alpha}\right) + \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\theta\left(\log\frac{t}{\tau}\right)^{\alpha}\right)\frac{\mu(\tau)}{\tau}d\tau \\ &\le y(a)E_{\alpha,1}\left(\theta\left(\log\frac{t}{a}\right)^{\alpha}\right) + \Gamma(\alpha)E_{\alpha,\alpha}\left(\theta\left(\log\frac{t}{a}\right)^{\alpha}\right)\partial_{a,t}^{-\alpha}\mu(t), \end{aligned}$$

which completes the proof. □

1.2 The proof of the main results

In this subsection, we give a detailed proof of our main results.

The proof of Lemma 1. In view of the expression

$$u(s)v(s) = (u(s)-u(t))(v(s)-v(t)) + u(t)v(s) + u(s)v(t) - u(t)v(t)$$

and the Definition 2

$$D_{a+,t}^{\alpha}[uv](t) = \frac{t}{\Gamma(1-\alpha)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_a^{t+\varepsilon}\frac{u(s)v(s)}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds - \int_a^t \frac{u(s)v(s)}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds\right],$$

we arrive at

$$\begin{aligned} D_{a+,t}^{\alpha}[uv](t) = \frac{t}{\Gamma(1-\alpha)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[&(\mathcal{I}_1(\varepsilon)-\mathcal{I}_1(0)) + u(t)\left(\mathcal{I}_2(\varepsilon)-\mathcal{I}_2(0)\right) \\ &+ v(t)\left(\mathcal{I}_3(\varepsilon)-\mathcal{I}_3(0)\right) - u(t)v(t)\left(\mathcal{I}_4(\varepsilon)-\mathcal{I}_4(0)\right)\Big], \end{aligned} \tag{8}$$

with

$$\begin{aligned} \mathcal{I}_1(\varepsilon) &= \int_a^{t+\varepsilon}\frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds, \quad & \mathcal{I}_2(\varepsilon) &= \int_a^{t+\varepsilon}\frac{v(s)}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds, \\ \mathcal{I}_3(\varepsilon) &= \int_a^{t+\varepsilon}\frac{u(s)}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds, & \mathcal{I}_4(\varepsilon) &= \int_a^{t+\varepsilon}\frac{1}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds. \end{aligned}$$

Hence, $u(t)\left(\mathcal{I}_2(\varepsilon)-\mathcal{I}_2(0)\right)$ and $v(t)\left(\mathcal{I}_3(\varepsilon)-\mathcal{I}_3(0)\right)$ are standard Hadamard derivatives, then

$$\begin{aligned} \frac{tu(t)}{\Gamma(1-\alpha)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_a^{t+\varepsilon}\frac{v(s)}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds - \int_a^t\frac{v(s)}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds\right] &= u(t)D_{a+,t}^{\alpha}v(t), \\ \frac{tv(t)}{\Gamma(1-\alpha)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_a^{t+\varepsilon}\frac{u(s)}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds - \int_a^t\frac{u(s)}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds\right] &= v(t)D_{a+,t}^{\alpha}u(t). \end{aligned}$$

Similarly, for the last term we have

$$\begin{aligned} u(t)v(t)\left(\mathcal{I}_4(\varepsilon)-\mathcal{I}_4(0)\right) &= \frac{tu(t)v(t)}{\Gamma(1-\alpha)}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_a^{t+\varepsilon}\frac{1}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds - \int_a^t\frac{1}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds\right] \\ &= \frac{tu(t)v(t)}{\Gamma(1-\alpha)}\cdot\frac{d}{dt}\int_a^t\frac{1}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds \\ &= \frac{u(t)v(t)}{\Gamma(1-\alpha)\left(\log\frac{t}{a}\right)^{\alpha}}. \end{aligned}$$

Now for the most complex term, we apply differentiation under the integral and use integration by parts, which gives

$$\frac{t}{\Gamma(1-\alpha)} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\int_a^{t+\varepsilon} \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t+\varepsilon}{s}\right)^{\alpha}}ds - \int_a^{t} \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds\right]$$
$$= -\frac{\alpha}{\Gamma(1-\alpha)}\int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t}{s}\right)^{1+\alpha}}ds.$$

The combination of integrals in (8) completes the proof. □

The proof of Lemma 2. Similar to the previous Lemma, we now use the decomposition

$$u(s)v(s)-u(t)v(t) = (u(s)-u(t))(v(s)-v(t)) + u(t)(v(s)-v(t)) + v(t)(u(s)-u(t)).$$

Then taking into account Definition 3, we obtain

$${}_H^C D_{a+,t}^{\alpha}[uv](t) = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,$$

where

$$\mathcal{J}_1 = \frac{t}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds,$$
$$\mathcal{J}_2 = u(t)\cdot\frac{t}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{v(s)-v(t)}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds,$$
$$\mathcal{J}_3 = v(t)\cdot\frac{t}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{u(s)-u(t)}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds.$$

From the Caputo–Hadamard derivative definition

$${}_H^C D_{a+,t}^{\alpha} v(t) = \frac{t}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{v(s)-v(t)}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds,$$

this yields

$$\mathcal{J}_2 = u(t)\cdot {}_H^C D_{a+,t}^{\alpha} v(t),$$
$$\mathcal{J}_3 = v(t)\cdot {}_H^C D_{a+,t}^{\alpha} u(t).$$

We now calculate

$$\mathcal{J}_1 = \frac{t}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds.$$

This term is nonlocal, and it was shown earlier that

$$\frac{t}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t}{s}\right)^{\alpha}}ds$$
$$= -\frac{(u(a)-u(t))(v(a)-v(t))}{\Gamma(1-\alpha)\left(\log\frac{t}{a}\right)^{\alpha}} - \frac{\alpha}{\Gamma(1-\alpha)}\int_a^t \frac{(u(s)-u(t))(v(s)-v(t))}{s\left(\log\frac{t}{s}\right)^{\alpha+1}}ds.$$

Finally, combining the integrals, we complete our proof. □

2 Applications

In this section, due to the obtained results we explored a-priori estimates of the solutions.

2.1 Time-fractional diffusion equations

Let us consider the following time-fractional diffusion equation

$$D^{\alpha}_{a+,t}u = b(t)\Delta_x u + c(t,x)u + f(t,x), \quad (t,x) \in (a,T] \times \Omega := Q, \tag{9}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with regular boundary $\partial\Omega$, and the Dirichlet boundary condition

$$u(t,x) = 0, \quad t > a, \quad x \in \partial\Omega \tag{10}$$

or the Neumann boundary condition

$$\frac{\partial u}{\partial \eta} = 0, \quad t > a, \quad x \in \partial\Omega, \tag{11}$$

where η is the outward normal and the initial condition is

$$\lim_{t\to a} \Gamma(\alpha) \left(\log \frac{t}{a}\right)^{1-\alpha} u(a,x) = u_0(x). \tag{12}$$

Here
(A) $b(t)$ is a nonnegative continuous function;
(B) $||c(t,x)||_{C(([a,T);L^2(\Omega))} = d$;
(C) $||f(t,x)||_{C(([a,T);L^2(\Omega))} = h$.

Theorem 1. Let $u_0 \in L^2(\Omega)$ and statements (A), (B), (C) hold true. If u satisfies (9)–(12) for every $t \in (a,T]$, then

$$||u||_{C_{1-\alpha,\log}((a,T];L^2(\Omega))} \le K_1(T)\,||u_0||_{L^2(\Omega)} + K_2(T)\,||f||_{C((a,T];L^2(\Omega))},$$

where

$$K_1(T) = \left[\frac{1}{\Gamma(\alpha)} + (2d+1)\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha}\left((2d+1)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right]$$

and

$$K_2(T) = \left(\log\frac{T}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)} + (2d+1)\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right].$$

Proof. Multiplying (9) by u and integrating over Ω, we get

$$\int_\Omega (D^{\alpha}_{a+,t}u)u\,dx = b(t)\int_\Omega (\Delta_x u)u\,dx + \int_\Omega c(t,x)u^2 dx + \int_\Omega f(t,x)u\,dx.$$

We begin by integrating by parts and then apply (4) together with Holder's inequality to get

$$\frac{1}{2}D^{\alpha}_{a+,t}\int_\Omega u^2 dx \le b(t)\int_{\partial\Omega} u\frac{\partial u}{\partial n}d\sigma - b(t)\int_\Omega \nabla u \nabla u\,dx$$
$$+ \int_\Omega c(t,x)u^2 dx + \left(\int_\Omega |f(t,x)|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}.$$

Taking into account (B) and using $-b(t)\int_\Omega \nabla u \nabla u dx \le 0$, we have

$$D^{\alpha}_{a+,t}\int_\Omega u^2dx \le 2d\int_\Omega u^2dx + 2\bigg(\int_\Omega |f(t,x)|^2dx\bigg)^{\frac{1}{2}}\bigg(\int_\Omega |u|^2dx\bigg)^{\frac{1}{2}}.$$

At this stage, applying Young's inequality to the last term of the previous inequality, we deduce that

$$D^{\alpha}_{a+,t}\int_\Omega u^2dx \le (2d+1)\int_\Omega u^2dx + \int_\Omega |f(t,x)|^2dx. \tag{13}$$

Let us define $y(t) = ||u(t,\cdot)||^2_{L^2(\Omega)}$ and taking into account (C) in (13), we get the time-fractional differential inequality

$$D^{\alpha}_{a+,t}y(t) \le (2d+1)\,y(t) + h. \tag{14}$$

Applying the integral $I^{\alpha}_{a+,t}$ to both sides of the inequality (14) and using the Property 1, we obtain

$$\begin{aligned} y(t) &\le \frac{\left(I^{1-\alpha}_{a+,t}y\right)(a)}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}[(2d+1)y(s)+h]\frac{ds}{s} \\ &= \frac{2d+1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}y(s)\frac{ds}{s} \\ &+ \underbrace{\frac{\left(I^{1-\alpha}_{a+,t}y\right)(a)}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} + \frac{h}{\Gamma(1+\alpha)}\left(\log\frac{t}{a}\right)^{\alpha}}_{g(t)}. \end{aligned}$$

Using Lemma 3 to the last estimate, it yields

$$\begin{aligned} y(t) &\le g(t) + \int_a^t\left[\sum_{k=1}^{\infty}\frac{(2d+1)^k}{\Gamma(k\alpha)}\left(\log\frac{t}{a}\right)^{k\alpha-1}\frac{g(s)}{s}\right]ds \\ &\overset{j=k+1}{\le} (2d+1)\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}\right]ds. \end{aligned}$$

Consequently, it follows that

$$\begin{aligned} y(t) &\le \frac{\left(I^{1-\alpha}_{a+,t}y\right)(a)}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} + \frac{h}{\Gamma(1+\alpha)}\left(\log\frac{t}{a}\right)^{\alpha} \\ &+ \frac{\left(I^{1-\alpha}_{a+,t}y\right)(a)(2d+1)}{\Gamma(\alpha)}\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{2(\alpha-1)}\right]\frac{ds}{s} \\ &+ \frac{(2d+1)h}{\Gamma(1+\alpha)}\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{t}{a}\right)^{\alpha}\right]\frac{ds}{s}. \end{aligned} \tag{15}$$

Applying formula (2.2.51) from [4, p. 86], we have the following calculations

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)}\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{2(\alpha-1)}\right]\frac{ds}{s} \\ &= \left(\log\frac{t}{a}\right)^{2\alpha-1}E_{\alpha,2\alpha}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right) \end{aligned} \tag{16}$$

and

$$\begin{aligned}\frac{1}{\Gamma(\alpha+1)}\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{t}{a}\right)^{\alpha}\right]\frac{ds}{s}\\ =\left(\log\frac{t}{a}\right)^{2\alpha}E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right).\end{aligned}\tag{17}$$

Substituting (16), (17) in to the inequality (15), we obtain

$$\begin{aligned}y(t)\le\left(I^{1-\alpha}_{a+,t}y\right)(a)\left(\log\frac{t}{a}\right)^{\alpha-1}\left[\frac{1}{\Gamma(\alpha)}+(2d+1)\left(\log\frac{t}{a}\right)^{\alpha}E_{\alpha,2\alpha}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right)\right]\\ +h\left(\log\frac{t}{a}\right)^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)}+(2d+1)\left(\log\frac{t}{a}\right)^{\alpha}E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right)\right].\end{aligned}\tag{18}$$

By multiplying both sides of (18) by $\left(\log\frac{t}{a}\right)^{1-\alpha}$, we get

$$\begin{aligned}\left(\log\frac{t}{a}\right)^{1-\alpha}y(t)&\le\left(I^{1-\alpha}_{a+,t}y\right)(a)\left[\frac{1}{\Gamma(\alpha)}+(2d+1)\left(\log\frac{t}{a}\right)^{\alpha}E_{\alpha,2\alpha}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right)\right]\\ &+h\left(\log\frac{t}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)}+(2d+1)\left(\log\frac{t}{a}\right)^{\alpha}E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right)\right]\\ &\le\left(I^{1-\alpha}_{a+,t}y\right)(a)\left[\frac{1}{\Gamma(\alpha)}+(2d+1)\left(\log\frac{T}{a}\right)^{\alpha}E_{\alpha,2\alpha}\left((2d+1)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right]\\ &+h\left(\log\frac{T}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)}+(2d+1)\left(\log\frac{T}{a}\right)^{\alpha}E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right].\end{aligned}$$

Then, we have

$$||u||_{C_{1-\alpha,\log}((a,T];L^2(\Omega))}\le K_1(T)\,||u_0||_{L^2(\Omega)}+K_2(T)\,||f||_{C((a,T],L^2(\Omega))},$$

which gives the desired result. □

2.2 The porous medium equation

Next, we study the porous medium equation

$$D^{\alpha}_{a+,t}u(t,x)=a(t,x)\Delta u^m(t,x)+f(t,x),\quad (t,x)\in(a,T]\times\Omega:=Q,\tag{19}$$

with the initial condition

$$\lim_{t\to a}\Gamma(\alpha)\left(\log\frac{t}{a}\right)^{1-\alpha}u(t,x)=\lim_{t\to a}\left(I^{1-\alpha}_{a+}u\right)(t,x)=\phi(x),\quad x\in\Omega\tag{20}$$

and the boundary condition

$$u(t,x)=0,\quad t>a,\quad x\in\partial\Omega,\tag{21}$$

where $m>1$ and $a(t,x)$, $f(t,x)$ are nonnegative continuous functions.

Theorem 2. Let $\Omega\subset\mathbb{R}^n$ and $\phi\in L^p(\Omega)$. The function $u\in C_{1-\alpha,\log}((a,T];L^p(\Omega))$ is a solution of problem (19)–(21) and

$$||u||_{C_{1-\alpha,\log}((a,T];L^p(\Omega))}\le K_3(T)\,||\phi)||_{L^p(\Omega)}+K_4(T)\,||f||_{C((a,T];L^p(\Omega))},$$

where

$$K_3(T)=\left[\frac{1}{\Gamma(\alpha)}+M\left(\log\frac{T}{a}\right)^{\alpha}E_{\alpha,2\alpha}\left(M\left(\log\frac{T}{a}\right)^{\alpha}\right)\right],$$

$$K_4(T)=\left(\log\frac{T}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)}+M\left(\log\frac{T}{a}\right)^{\alpha}E_{\alpha,2\alpha+1}\left(M\left(\log\frac{T}{a}\right)^{\alpha}\right)\right].$$

Proof. Multiplying (19) by pu^{p-1} $(p \geq 2)$, and integrating over Ω, we arrive at

$$\int_\Omega pu^{p-1} D^\alpha_{a+} u dx - \int_\Omega a(t,x) pu^{p-1} \Delta u^m dx - \int_\Omega pu^{p-1} f(t,x) dx = 0.$$

In view of the expression

$$\begin{aligned} p\int_\Omega a(t,x)u^{p-1}\Delta u^m dx &= p\int_{\partial\Omega} a(t,x)u^{p-1}u^{m-1}\frac{\partial}{\partial\eta}u d\sigma \\ &\quad - p\int_\Omega (p-1)a(t,x)u^{p-2}u^{m-1}|\nabla u|^2 dx \\ &= -p\int_\Omega (p-1)a(t,x)u^{p-2}u^{m-1}|\nabla u|^2 dx, \end{aligned}$$

it follows that

$$\begin{aligned} &\int_\Omega pu^{p-1} D^\alpha_{a+} u dx + p\int_\Omega (p-1)a(t,x)u^{p-2}u^{m-1}|\nabla u|^2 dx \\ &\quad - p\int_\Omega u^{p-1} f(t,x) dx = 0. \end{aligned} \tag{22}$$

Applying (7) and the Hölder inequality to (22), we obtain

$$\begin{aligned} &\int_\Omega D^\alpha_{a+} u^p dx + \frac{4p(p-1)d_1}{(p+m-1)^2}\int_\Omega \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 dx \\ &\quad - p\left(\int_\Omega |f(t,x)|^p dx\right)^{1/p}\left(\int_\Omega u^p dx\right)^{1-1/p} \leq 0. \end{aligned} \tag{23}$$

Using Young's inequality in the last term of (23), it follows that

$$\begin{aligned} &\int_\Omega D^\alpha_{a+} u^p dx + \frac{4p(p-1)d_1}{(p+m-1)^2}\int_\Omega \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 dx \\ &\quad - \varepsilon^p \int_\Omega |f(t,x)|^p dx - \frac{p-1}{\varepsilon^{\frac{p}{p-1}}}\int_\Omega u^p dx \leq 0, \quad \varepsilon > 0. \end{aligned}$$

Let's make the following notations

$$y(t) = ||u(t,\cdot)||^p_{L^p(\Omega)}, \quad \mathcal{H} = \varepsilon^p ||f(t,\cdot)||^p_{L^p(\Omega)}, \quad M = \frac{p-1}{\varepsilon^{\frac{p}{p-1}}}.$$

Then, we have

$$D^\alpha_{a+} y(t) \leq My(t) + \mathcal{H}. \tag{24}$$

Starting from (24), by performing the same actions as in the proof of the previous theorem, we obtain the following conclusion

$$\begin{aligned} \left(\log\frac{t}{a}\right)^{1-\alpha} y(t) &\leq \left(I^{1-\alpha}_{a+,t} y\right)(a)\left[\frac{1}{\Gamma(\alpha)} + M\left(\log\frac{t}{a}\right)^\alpha E_{\alpha,2\alpha}\left(M\left(\log\frac{t}{a}\right)^\alpha\right)\right] \\ &\quad + \mathcal{H}\left(\log\frac{t}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)} + M\left(\log\frac{t}{a}\right)^\alpha E_{\alpha,2\alpha+1}\left(M\left(\log\frac{t}{a}\right)^\alpha\right)\right] \\ &\leq \left(I^{1-\alpha}_{a+,t} y\right)(a)\left[\frac{1}{\Gamma(\alpha)} + M\left(\log\frac{T}{a}\right)^\alpha E_{\alpha,2\alpha}\left(M\left(\log\frac{T}{a}\right)^\alpha\right)\right] \\ &\quad + \mathcal{H}\left(\log\frac{T}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)} + M\left(\log\frac{T}{a}\right)^\alpha E_{\alpha,2\alpha+1}\left(M\left(\log\frac{T}{a}\right)^\alpha\right)\right]. \end{aligned}$$

Hence, we deduce that

$$||u||_{C_{1-\alpha,\log}((a,T];L^p(\Omega))} \leq K_3(T)\,||\phi||_{L^p(\Omega)} + K_4(T)\,||f||_{C((a,T];L^p(\Omega))}\,,$$

where

$$K_3(T) = \left[\frac{1}{\Gamma(\alpha)} + M\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha}\left(M\left(\log\frac{T}{a}\right)^{\alpha}\right)\right]$$

and

$$K_4(T) = \left(\log\frac{T}{a}\right)\left[\frac{1}{\Gamma(\alpha+1)} + M\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha+1}\left(M\left(\log\frac{T}{a}\right)^{\alpha}\right)\right].$$

Therefore, we have proven the statement. □

2.3 Fractional-order diffusion equation

In the next case, we consider the fractional-order diffusion equation

$${}^{C}_{H}D^{\alpha}_{a+,t}u = \log\left(\frac{t}{a}\right)\Delta_x u + c(t,x)u + f(t,x), \quad (t,x) \in (a,T] \times \Omega, \tag{25}$$

with the Dirichlet boundary condition

$$u(t,x) = 0, \quad t > a > 0, \quad x \in \partial\Omega \tag{26}$$

and with the Cauchy condition

$$u(a,x) = u_0(x), \tag{27}$$

where the functions $c(t,x), f(t,x)$ satisfy

(A) $\|c(t,x)\|_{C(([a,T);L^2(\Omega))} = d, c(t,x) \leq 0;$

(B) $\|f(t,x)\|_{C(([a,T);L^2(\Omega))} = h.$

Theorem 3. Suppose $u_0 \in L^2(\Omega)$ and (A), (B) hold. If the function $u(t,x)$ satisfies the problem (25)–(27) for each $t \in (a,T]$, then the following estimate holds

$$\|u\|_{C((a,T];L^2(\Omega))} \leq K_5(T)\,\|u_0\|_{L^2(\Omega)} + K_6(T)\|f\|_{C((a,T];L^2(\Omega))},$$

where

$$K_5(T) = 1 + (2d+1)\int_a^T \left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\right]\frac{ds}{s}$$

and

$$K_6(T) = \left(\log\frac{T}{a}\right)^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)} + (2d+1)\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right].$$

Proof. Multiplying each term of equation (25) by the function u and integrating over Ω,

$$\int_\Omega \left({}^{C}_{H}D^{\alpha}_{a+,t}u\right) u dx = \log\left(\frac{t}{a}\right)\int_\Omega \left(\Delta_x u\right) u dx + \int_\Omega c(t,x)u^2 dx + \int_\Omega f(t,x)u dx.$$

Taking into account the estimate (6) and using Hölder's inequality for the last term of the previously mentioned inequality, we arrive at

$$\frac{1}{2}{}^{C}_{H}D^{\alpha}_{a+,t}\int_\Omega u^2 dx \leq \log\left(\frac{t}{a}\right)\int_{\partial\Omega} u\frac{\partial u}{\partial n}d\sigma - \log\left(\frac{t}{a}\right)\int_\Omega \nabla u\nabla u dx$$
$$+ \int_\Omega c(t,x)u^2 dx + \left(\int_\Omega |f(t,x)|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}.$$

Applying Young's inequality to the last term of the previous inequality and in view of

$$-\log\left(\frac{t}{a}\right)\int_{\Omega}\nabla u\nabla u dx\leq 0$$

with the notation (A), (B) and $y(t)=\|u(t,x)\|^2_{L^2(\Omega)}$, we obtain

$$^{C}_{H}D^{\alpha}_{a+,t}y(t)\leq(2d+1)y(t)+h. \tag{28}$$

By applying the integral ${}_HI^{\alpha}_{a+,t}$ to both sides of inequality (28) and using Property 1, we derive the following expression

$$\begin{aligned}y(t)&\leq y(a)+\frac{2d+1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}y(s)\frac{ds}{s}+\frac{h}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\&=\underbrace{y(a)+\frac{h}{\Gamma(1+\alpha)}\left(\log\frac{t}{a}\right)^{\alpha}}_{g(t)}+\frac{2d+1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}y(s)\frac{ds}{s}.\end{aligned}$$

According to the result of Lemma 3, we deduce that

$$\begin{aligned}y(t)&\leq g(t)+\int_a^t\left[\sum_{k=1}^{\infty}\frac{[(2d+1)\Gamma(\alpha)]^k}{\Gamma(k\alpha)}\left(\log\frac{t}{a}\right)^{k\alpha-1}\frac{g(s)}{s}\right]ds\\&\overset{j=k+1}{\leq} g(t)+(2d+1)\Gamma(\alpha)\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\Gamma(\alpha)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s)}{s}\right]ds.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}y(t)&\leq y(a)+\frac{h}{\Gamma(1+\alpha)}\left(\log\frac{t}{a}\right)^{\alpha}\\&+y(a)(2d+1)\Gamma(\alpha)\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\Gamma(\alpha)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\right]\frac{ds}{s}\\&+\frac{(2d+1)\Gamma(\alpha)h}{\Gamma(\alpha+1)}\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\Gamma(\alpha)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{2\alpha-1}\right]\frac{ds}{s}.\end{aligned}$$

In view of formula (2.2.51) in [4, p. 86], we arrive at the following:

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\int_a^t\left[E_{\alpha,\alpha}\left((2d+1)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{2\alpha-1}\right]\frac{ds}{s}\\&=\left(\log\frac{t}{a}\right)^{2\alpha}E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{t}{a}\right)^{\alpha}\right).\end{aligned}$$

It implies that

$$\begin{aligned}y(t)&\leq y(a)\left[1+(2d+1)\Gamma(\alpha)\int_a^T\left[E_{\alpha,\alpha}\left((2d+1)\Gamma(\alpha)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\right]\frac{ds}{s}\right]\\&+h\left(\log\frac{T}{a}\right)^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)}+(2d+1)\Gamma(\alpha)\left(\log\frac{T}{a}\right)^{\alpha}E_{\alpha,2\alpha+1}\left((2d+1)\Gamma(\alpha)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right].\end{aligned}$$

Finally, we conclude

$$\|u\|_{C((a,T];L^2(\Omega))} \leq K_5(T)\,\|u_0\|_{L^2(\Omega)} + K_6(T)\|f\|_{C((a,T],L^2(\Omega))},$$

where

$$K_5(T) = 1 + (2d+1)\Gamma(\alpha)\int_a^T \left[E_{\alpha,\alpha}\left((2d+1)\Gamma(\alpha)\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{t}{s}\right)^{\alpha-1}\right]\frac{ds}{s}$$

and

$$K_6(T) = \left(\log\frac{T}{a}\right)^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)} + (2d+1)\left(\log\frac{T}{a}\right)^{\alpha} E_{\alpha,2\alpha+1}\left((2d+1)\left(\log\frac{T}{a}\right)^{\alpha}\right)\right],$$

which completes the proof. □

Conclusion

In this work, we have established new analogues of the Leibniz rule for the Hadamard and Caputo–Hadamard fractional derivatives, taking into account their inherent nonlocal properties. The refined differentiation formulas and derived inequalities provide a deeper understanding of how fractional derivatives interact with nonlinear functions. In particular, the obtained estimates form an analytical foundation for studying fractional diffusion equations of various types. The results can be effectively applied to prove the existence, uniqueness, and stability of solutions, as well as to derive a priori bounds essential for the qualitative analysis of such models. Future research may extend these methods to systems with variable order or to multidimensional fractional operators.

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Conflict of Interest

The authors declare no conflict of interest.

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