Bulletin of the Karaganda University. Mathematics Series, No. 1(121), 2026, pp. 156–168

https://doi.org/10.31489/2026M1/156-168 *Research article*

On estimates for norms of matrix operators: the case $q < p$

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The study of matrix operators acting between weighted sequence spaces $l_{p,v}$ and $l_{q,u}$ has become an important direction in functional analysis, particularly due to its close connection with Hardy-type inequalities and the general theory of linear operators on discrete structures. A key problem in this framework is determining when such operators are bounded and obtaining precise value or sharp estimates for their operator norms. Although considerable attention has been devoted to matrix operators whose entries satisfy the so-called *Oinarov conditions*, including several extensions to broader classes of kernels, the literature still lacks comprehensive norm estimates, especially in the case $1 < q < p < \infty$. In this paper, we establish necessary and sufficient criteria for the boundedness of matrix operators with entries satisfying the Oinarov conditions. Furthermore, we provide both lower and upper estimates for their norms. These results not only refine previously known inequalities but also provide new tools for analyzing the structure and behavior of weighted sequence spaces. Applications of our findings include spectral characterization of matrix operators, investigation of oscillatory and non-oscillatory properties of solutions to higher-order difference equations, and the evaluation of sequences via their discrete derivatives.

Keywords: operator, matrix, sequence, norm, discrete analysis, Hardy-type inequality, weight function, kernel, Lebesgue sequence space, best constant.

2020 Mathematics Subject Classification: Primary 47B37, 47A30; Secondary 26D10, 39A12.

Introduction

Let $1 < p,\, q < \infty$ and let $u = \{u_n\}_{n=1}^{\infty}$ and $v = \{v_n\}_{n=1}^{\infty}$ be positive sequences of real numbers, which we call weights. Denote by $l_{p,v}$ the space of all sequences $f = \{f_n\}_{n=1}^{\infty}$ of real numbers for which the norm

$$\|f\|_{l_{p,v}} = \left(\sum_{n=1}^{\infty} |f_n|^p v_n\right)^{\frac{1}{p}}$$

is finite. Similarly, the space $l_{q,u}$ is defined. Let us consider the following triangular matrix operator

$$A : l_{p,v} \to l_{q,u}, \quad (Af)_n := \sum_{k=1}^{n} a_{n,k} f_k \tag{1}$$

acting between weighted sequence spaces $l_{p,v}$ and $l_{q,u}$. The entries $a := \{a_{n,k}\}_{n,k=1}^{\infty}$, $n \geq k \geq 1$ are nonnegative $a_{n,k} \geq 0$.

Since the 1980s, conditions ensuring the boundedness of the operator, that is, the existence of a constant $C > 0$ such that the inequality

$$\|Af\|_{l_{q,u}} \leq C\|f\|_{l_{p,v}},$$

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Received: 28 May 2025; *Accepted:* 25 December 2025.

i.e., discrete weighted Hardy type inequality (HTI)

$$\left(\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}a_{n,k}f_k\right|^q u_n\right)^{\frac{1}{q}} \le C\left(\sum_{n=1}^{\infty}|f_n|^p v_n\right)^{\frac{1}{p}} \tag{2}$$

holds for all $f \in l_{p,v}$ — for various forms of $a_{n,k}$ have been established.

If $a_{n,k} \equiv 1$, then the inequality takes the form

$$\left(\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}f_k\right|^q u_n\right)^{\frac{1}{q}} \le C\left(\sum_{n=1}^{\infty}|f_n|^p v_n\right)^{\frac{1}{p}}, \tag{3}$$

which is called general weighted inequality (GWI). In 1983, K.F. Andersen and H.P. Heinig provided a characterization of the weight conditions ensuring the validity of inequality (3) in the case $1 < p \le q < \infty$. Subsequently, in 1985, H.P. Heinig established a sufficient condition for the same inequality and derived estimates for its best constant $C = \|A\|_{l_{p,v}\to l_{q,u}}$ in the case $1 < q < p < \infty$, see e.g. [1]. We now present Heinig's result in a form adapted to the GWI

$$A = \left(\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}u_k\right)^{\frac{r}{q}}\left(\sum_{k=1}^{n}v_k^{1-p'}\right)^{\frac{r}{q'}} v_n^{1-p'}\right)^{\frac{1}{r}} < \infty \tag{4}$$

and the corresponding upper estimate

$$C \le q^{\frac{1}{q}}(p')^{\frac{1}{q'}}A, \tag{5}$$

which will be used in the proof of the main theorem of this work, where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$.

If $\{a_{n,k}\}$ is different from constant sequence then the analysis of HTI becomes considerable more complicated. The first result was obtained by K.F. Andersen and H.P. Heinig, who established sufficient conditions for the validity of the inequality in the case $1 < p \le q < \infty$ under a specific choice of the kernel $a_{n,k}$, which was assumed to be non-increasing in k and non-decreasing in n. For further details, see [1] and the references therein.

Later, R. Oinarov, C.A. Okpoti, and L.-E. Persson [2], as well as R. Oinarov, S. Shalginbayeva, T. Temirkhanova and others [3–5] obtained necessary and sufficient conditions for the boundedness of the matrix operator in the case $1 < q < p < \infty$. Their results hold under hypotheses on the entries $a_{n,k}$ that are weaker than those originally imposed by Oinarov. Estimates of the best constant in the discrete Hardy-type inequality for matrix operators satisfying Oinarov condition are given in [6].

Most of the works cited above focus on characterizing boundedness; by contrast, exact values or even two-sided estimates for the operator norm are rarely treated. Recently, several papers have appeared that provide lower and upper bounds for the norm of the integral analogue of this matrix operator (see, for example, [7, 8]).

There are also a number of related contributions concerning iterated discrete Hardy inequalities [9, 10], three-weight inequalities [11, 12], and the boundedness of iterated matrix operators [13, 14], which may be consulted for further reference.

In this paper, we derive lower and upper estimates for the norm of the matrix operator, as well as necessary and sufficient conditions for its boundedness, in the case $1 < q < p < \infty$. We consider the operators whose entries $\{a_{n,k}\}$ are non-decreasing in n, non-increasing in k, and satisfy the condition

$$a_{n,k} \le h(a_{n,l}+a_{l,k}) \quad \text{for all} \quad n \ge l \ge k \ge 1 \tag{6}$$

for some $h \ge 1$. These conditions are known in the theory of Hardy-type inequalities as *Oinarov conditions.*

1 Main results

In this section, we present the main result of the paper along with the auxiliary lemmas needed for its proof. We begin by introducing the following notations:

$$B_1 := \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} a_{k,n}^q u_k \right)^{\frac{r}{q}} \left(\sum_{k=1}^{n} v_k^{1-p'} \right)^{\frac{r}{q'}} v_n^{1-p'} \right)^{\frac{1}{r}}$$

and

$$B_2 := \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} u_k \right)^{\frac{r}{p}} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'} \right)^{\frac{r}{p'}} u_n \right)^{\frac{1}{r}},$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Let us present the auxiliary lemmas, with particular emphasis on the constants appearing in the estimates, as they play a crucial role in determining sharp estimates for the norm of the operator.

Lemma 1. Let $1 < q < \infty$ and $\{u_n\} \in \ell_1$ be a nonnegative sequence. Then for any $m \in \mathbb{N}$ the following estimate holds

$$\sum_{n=m}^{\infty} \left(\sum_{k=n}^{\infty} u_k \right)^{-\frac{1}{q}} u_n \le q' \left(\sum_{k=m}^{\infty} u_k \right)^{\frac{1}{q'}}, \tag{7}$$

where $q' = \frac{q}{q-1}$.

Proof. Let denote

$$x_n = \sum_{k=n}^{\infty} u_k,$$

so that $x_m \ge ... \ge x_n \ge x_{n+1} \ge ... \ge 0$. Then inequality (7) can be written as

$$\sum_{n=m}^{\infty} x_n^{-\frac{1}{q}} (x_n - x_{n+1}) \le q' x_m^{\frac{1}{q'}},$$

where $q' = \frac{q}{q-1}$. To estimate the left hand side, observe that

$$\sum_{n=m}^{\infty} \frac{x_n - x_{n+1}}{x_n^{\frac{1}{q}}} = \sum_{n=m}^{\infty} \left(\int_{x_{n+1}}^{x_n} \frac{1}{x_n^{\frac{1}{q}}} dx \right) \le \sum_{n=m}^{\infty} \left(\int_{x_{n+1}}^{x_n} \frac{1}{x^{\frac{1}{q}}} dx \right) = \int_0^{x_m} \frac{1}{x^{\frac{1}{q}}} dx = q' x_m^{\frac{1}{q'}}.$$

The proof is complete. □

Lemma 2. Let $1 < q < \infty$, and $\{u_n\} \in \ell_1$ be a nonnegative sequence. Suppose the entries $\{a_{n,k}\}$ of the matrix operator satisfy Oinarov conditions. Then, for every nonnegative sequence $\{f_n\} \in l_{p,v}$, the following estimate holds:

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_{n,k} f_k \right)^q u_n \le (2h)^{q-1} q [S_1 + S_2], \tag{8}$$

where

$$S_1 = \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n \right) \left(\sum_{k=1}^{m} f_k \right)^{q-1}$$

and

$$S_2 = \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m} u_n \right) \left(\sum_{k=1}^{m} a_{m,k} f_k \right)^{q-1}.$$

Proof. Using Lagrange's mean value theorem, Fubini's theorem, Oinarov condition (6) and the inequality

$$(a+b)^{q-1} \le 2^{q-1}(a^{q-1}+b^{q-1}) \quad \text{for all} \quad a, b \ge 0,$$

we estimate the left hand side of (8) in the form

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_{n,k} f_k \right)^q u_n = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \left[\left(\sum_{k=1}^{m} a_{n,k} f_k \right)^q - \left(\sum_{k=1}^{m-1} a_{n,k} f_k \right)^q \right] \right) u_n \\
&= q \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} a_{n,m} f_m \left(\sum_{k=1}^{m-1} a_{n,k} f_k + \xi_{n,m} a_{n,m} f_m \right)^{q-1} \right) u_n \\
&\le q \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} a_{n,m} f_m \left(\sum_{k=1}^{m} a_{n,k} f_k \right)^{q-1} \right) u_n \\
&= q \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m} u_n \left(\sum_{k=1}^{m} a_{n,k} f_k \right)^{q-1} \right) \\
&\le q h^{q-1} \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m} u_n \left(a_{n,m} \sum_{k=1}^{m} f_k + \sum_{k=1}^{m} a_{m,k} f_k \right)^{q-1} \right) \\
&\le (2h)^{q-1} q \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n \left(\sum_{k=1}^{m} f_k \right)^{q-1} + \sum_{n=m}^{\infty} a_{n,m} u_n \left(\sum_{k=1}^{m} a_{m,k} f_k \right)^{q-1} \right) \\
&= (2h)^{q-1} q \left[\sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n \right) \left(\sum_{k=1}^{m} f_k \right)^{q-1} + \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m} u_n \right) \left(\sum_{k=1}^{m} a_{m,k} f_k \right)^{q-1} \right] \\
&= (2h)^{q-1} q [S_1 + S_2],
\end{aligned}$$

where $\xi_{n,m} \in (0,1)$, $m = 1, 2, ..., n$.

The proof is complete. □

Theorem 1. Let $1 < q < p < \infty$ and the entries $\{a_{n,k}\}$ of matrix operator (1) satisfy Oinarov conditions. Then the operator acting between weighted sequence spaces $l_{p,v}$ and $l_{q,u}$ is bounded, i.e., inequality (2) holds if and only if

$$B = \max\{B_1, B_2\} < \infty.$$

Moreover, the following estimates are valid for its norm

$$\max\left\{ \left(\frac{p'q}{r} \right)^{\frac{1}{q'}} B_1, \left(\frac{p'q}{r} \right)^{\frac{1}{p}} B_2 \right\} \le \|A\|_{l_p \to l_q} \le \left((2h)^{q-1} q^3 (p')^{q-1} + (2h)^{q(q-1)} q^q \left(q'\right)^{\frac{q^2}{p}} \right)^{\frac{1}{q}} B,$$

where $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$.

Proof. Let us recall that, for inequality (2) to hold, it is necessary and sufficient that the inequality

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_{n,k} f_k\right)^q u_n\right)^{\frac{1}{q}} \leq C\left(\sum_{n=1}^{\infty} f_n^p v_n\right)^{\frac{1}{p}} \tag{9}$$

holds for all nonnegative sequences $\{f_n\} \in l_{p,v}$.

(*Necessity and lower estimate.*) To investigate the necessity of the conditions and to derive a lower estimate for the best constant C, we consider the dual form of inequality (9):

$$\left(\sum_{k=1}^{\infty} v_k^{1-p'}\left(\sum_{n=k}^{\infty} a_{n,k} g_n\right)^{p'}\right)^{\frac{1}{p'}} \leq C\left(\sum_{k=1}^{\infty} u_k^{1-q'} g_k^{q'}\right)^{\frac{1}{q'}}, \tag{10}$$

where $\{g_k\} \in l_{q',u^{1-q'}}$. It is well known that these inequalities are equivalent in the sense that the validity of the dual inequality is both necessary and sufficient for the validity of (9). Moreover, their best constants coincide (see, for example, [2]).

Let us assume that (9) holds for a finite constant C and, for fixed $n \in Z^+$, apply the following test sequence to (9):

$$(f_N)_k := \left(\sum_{n=k}^{N} a_{n,k}^q u_n\right)^{\frac{1}{p-q}}\left(\sum_{n=1}^{k} v_n^{1-p'}\right)^{\frac{q-1}{p-q}} v_k^{1-p'} \chi_{[1,N]}(k), \tag{11}$$

where $\chi_{[1,N]}(k)$ is the characteristic sequence. Putting (11) to the right hand side of (9), we have

$$\begin{aligned}\left(\sum_{k=1}^{\infty}(f_N)_k^p v_k\right)^{\frac{1}{p}} &= \left(\sum_{k=1}^{N}\left(\sum_{n=k}^{N} a_{n,k}^q u_n\right)^{\frac{p}{p-q}}\left(\sum_{n=1}^{k} v_n^{1-p'}\right)^{\frac{p(q-1)}{p-q}} v_k^{1-p'}\right)^{\frac{1}{p}}\\ &= \left(\sum_{k=1}^{N}\left(\sum_{n=k}^{N} a_{n,k}^q u_n\right)^{\frac{r}{q}}\left(\sum_{n=1}^{k} v_n^{1-p'}\right)^{\frac{r}{q'}} v_k^{1-p'}\right)^{\frac{1}{p}}.\end{aligned} \tag{12}$$

For the left hand side of (9), apply Fubini's theorem and we find that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_{n,k}(f_N)_k\right)^q u_n &\geq \sum_{n=1}^{N}\left(\sum_{k=1}^{n} a_{n,k}(f_N)_k\right)^q u_n\\ &= \sum_{n=1}^{N}\left(\sum_{k=1}^{n} a_{n,k}(f_N)_k\left(\sum_{k=1}^{n} a_{n,k}(f_N)_k\right)^{q-1}\right) u_n\\ &\geq \sum_{n=1}^{N}\left(\sum_{k=1}^{n} a_{n,k}(f_N)_k\left(\sum_{l=1}^{k} a_{n,l}(f_N)_l\right)^{q-1}\right) u_n\\ &= \sum_{k=1}^{N}(f_N)_k\left[\sum_{n=k}^{N} a_{n,k}\left(\sum_{l=1}^{k} a_{n,l}(f_N)_l\right)^{q-1} u_n\right]\end{aligned}$$

[using $a_{n,k} \le a_{n,l}$ for $n \ge k \ge l \ge 1$]

$$\ge \sum_{k=1}^{N} (f_N)_k \sum_{n=k}^{N} a_{n,k}^q u_n \left(\sum_{l=1}^{k} (f_N)_l \right)^{q-1}$$

$$= \sum_{k=1}^{N} (f_N)_k \sum_{n=k}^{N} a_{n,k}^q u_n \left(\sum_{l=1}^{k} v_l^{1-p'} \left(\sum_{n=l}^{N} a_{n,l}^q u_n \right)^{\frac{1}{p-q}} \left(\sum_{n=1}^{l} v_n^{1-p'} \right)^{\frac{q-1}{p-q}} \right)^{q-1}$$

[using now $a_{n,k} \le a_{n,l}$ for $n \ge k \ge l \ge 1$ again and then Lemma 1]

$$\ge \sum_{k=1}^{N} (f_N)_k \left(\sum_{n=k}^{N} a_{n,k}^q u_n \right)^{\frac{p-1}{p-q}} \left(\sum_{l=1}^{k} v_l^{1-p'} \left(\sum_{n=1}^{l} v_n^{1-p'} \right)^{\frac{q-1}{p-q}} \right)^{q-1}$$

$$\ge \left(\frac{p-q}{p-1} \right)^{q-1} \sum_{k=1}^{N} (f_N)_k \left(\sum_{n=k}^{N} a_{n,k}^q u_n \right)^{\frac{p-1}{p-q}} \left(\sum_{l=1}^{k} v_l^{1-p'} \right)^{\frac{(p-1)(q-1)}{p-q}}$$

$$= \left(\frac{p'q}{r} \right)^{q-1} \sum_{k=1}^{N} \left(\sum_{n=k}^{N} a_{n,k}^q u_n \right)^{\frac{p}{p-q}} \left(\sum_{l=1}^{k} v_l^{1-p'} \right)^{\frac{p(q-1)}{p-q}} v_k^{1-p'},$$

i.e.,

$$\left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_{n,k} (f_N)_k \right)^q u_n \right)^{\frac{1}{q}} \ge \left(\frac{p'q}{r} \right)^{\frac{1}{q'}} \left(\sum_{k=1}^{N} \left(\sum_{n=k}^{N} a_{n,k}^q u_n \right)^{\frac{r}{q}} \left(\sum_{l=1}^{k} v_l^{1-p'} \right)^{\frac{r}{q'}} v_k^{1-p'} \right)^{\frac{1}{q}}. \tag{13}$$

Using (12) and (13) estimates in inequality (9), we obtain

$$\left(\frac{p'q}{r} \right)^{\frac{1}{q'}} \left(\sum_{k=1}^{N} \left(\sum_{n=k}^{N} a_{n,k}^q u_n \right)^{\frac{r}{q}} \left(\sum_{l=1}^{k} v_l^{1-p'} \right)^{\frac{r}{q'}} v_k^{1-p'} \right)^{\frac{1}{r}} \le C$$

and from this with a constant independent of N and hence, letting $N \to \infty$, we obtain

$$\left(\frac{p'q}{r} \right)^{\frac{1}{q'}} B_1 \le C. \tag{14}$$

Now for fixed $N \in Z^+$ apply the following test sequence to (10) $g_N = \{(g_N)_k\}_{k=1}^{\infty}$, where

$$(g_N)_k := u_k \left(\sum_{n=k}^{N} u_n \right)^{\frac{q-1}{p-q}} \left(\sum_{l=1}^{k} a_{k,l}^{p'} v_l^{1-p'} \right)^{\frac{(p-1)(q-1)}{p-q}} \chi_{[1,N]}(k). \tag{15}$$

Applying (15) to the right hand side of (10), we have

$$\left(\sum_{k=1}^{\infty} u^{1-q'} (g_N)_k{}^{q'} \right)^{\frac{1}{q'}} = \left(\sum_{k=1}^{N} u_k \left(\sum_{n=k}^{N} u_n \right)^{\frac{q}{p-q}} \left(\sum_{l=1}^{k} a_{k,l}^{p'} v_l^{1-p'} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{1}{q'}}$$

$$= \left(\sum_{k=1}^{N} u_k \left(\sum_{n=k}^{N} u_n \right)^{\frac{r}{p}} \left(\sum_{l=1}^{k} a_{k,l}^{p'} v_l^{1-p'} \right)^{\frac{r}{p'}} \right)^{\frac{1}{q'}}. \tag{16}$$

Applying Fubini's theorem for the left-hand side of (10), we obtain

$$\sum_{k=1}^{\infty} v_k^{1-p'} \left(\sum_{n=k}^{\infty} a_{n,k}(g_N)_n\right)^{p'} = \sum_{k=1}^{N} v_k^{1-p'} \left(\sum_{n=k}^{N} a_{n,k}(g_N)_n\right)^{p'}$$

$$= \sum_{k=1}^{N} v_k^{1-p'} \left(\sum_{n=k}^{N} a_{n,k}(g_N)_n \left(\sum_{n=k}^{N} a_{n,k}(g_N)_n\right)^{p'-1}\right)$$

$$\geq \sum_{k=1}^{N} v_k^{1-p'} \left(\sum_{n=k}^{N} a_{n,k}(g_N)_n \left(\sum_{l=n}^{N} a_{l,k}(g_N)_l\right)^{p'-1}\right)$$

$$= \sum_{n=1}^{N} (g_N)_n \left(\sum_{k=1}^{n} a_{n,k} v_k^{1-p'} \left(\sum_{l=n}^{N} a_{l,k}(g_N)_l\right)^{p'-1}\right)$$

$$\geq \sum_{n=1}^{N} (g_N)_n \left(\sum_{l=n}^{N} (g_N)_l\right)^{p'-1} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'}\right)$$

$$\geq \sum_{n=1}^{N} (g_N)_n \left(\sum_{l=n}^{N} u_l \left(\sum_{k=1}^{l} a_{l,k}^{p'} v_k^{1-p'}\right)^{\frac{(p-1)(q-1)}{p-q}} \left(\sum_{k=l}^{N} u_k\right)^{\frac{q-1}{p-q}}\right)^{p'-1} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'}\right)$$

$$\geq \sum_{n=1}^{N} (g_N)_n \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'}\right)^{\frac{p-1}{p-q}} \left(\sum_{l=n}^{N} u_l \left(\sum_{k=l}^{N} u_k\right)^{\frac{q-1}{p-q}}\right)^{p'-1}$$

$$\geq \left(\frac{p-q}{p-1}\right)^{p'-1} \sum_{n=1}^{N} (g_N)_n \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'}\right)^{\frac{p-1}{p-q}} \left(\sum_{l=n}^{N} u_l\right)^{\frac{1}{p-q}}$$

$$= \left(\frac{p-q}{p-1}\right)^{p'-1} \sum_{n=1}^{N} u_n \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'}\right)^{\frac{(p-1)(q-1)}{p-q}} \left(\sum_{l=n}^{N} u_l\right)^{\frac{q-1}{p-q}} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'}\right)^{\frac{p-1}{p-q}} \left(\sum_{l=n}^{N} u_l\right)^{\frac{1}{p-q}}$$

$$= \left(\frac{p-q}{p-1}\right)^{p'-1} \sum_{n=1}^{N} u_n \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'}\right)^{\frac{q(p-1)}{p-q}} \left(\sum_{l=n}^{N} u_l\right)^{\frac{q}{p-q}}$$

$$= \left(\frac{p'q}{r}\right)^{p'-1} \sum_{n=1}^{N} u_n \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'}\right)^{\frac{r}{p'}} \left(\sum_{l=n}^{N} u_l\right)^{\frac{r}{p}},$$

i.e.,

$$\left(\sum_{k=1}^{\infty} v_k^{1-p'} \left(\sum_{n=k}^{\infty} a_{n,k}(g_N)_n\right)^{p'}\right)^{\frac{1}{p'}} \geq \left(\frac{p'q}{r}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} u_n \left(\sum_{l=n}^{N} u_l\right)^{\frac{r}{p}} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'}\right)^{\frac{r}{p'}}\right)^{\frac{1}{p'}}. \tag{17}$$

Using (16) and (17) in inequality (10), we obtain

$$\left(\frac{p'q}{r}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} u_n \left(\sum_{l=n}^{N} u_l\right)^{\frac{r}{p}} \left(\sum_{k=1}^{n} a_{n,k}^{p'} v_k^{1-p'}\right)^{\frac{r}{p'}}\right)^{\frac{1}{r}} \leq C$$

with a constant independent of N and hence, letting $N \to \infty$, we obtain

$$\left(\frac{p'q}{r}\right)^{\frac{1}{p}} B_2 \le C. \tag{18}$$

Thus, in view of (14), (18) and our assumption, we have

$$\max\left\{\left(\frac{p'q}{r}\right)^{\frac{1}{q'}} B_1, \left(\frac{p'q}{r}\right)^{\frac{1}{p}} B_2\right\} \le C.$$

The necessity part is proved.

(*Sufficiency and upper estimate.*) Now, we proceed to estimate the right-hand side of (9). To this end, taking into account Lemma 2, we estimate the terms S_1 and S_2 separately:

$$\begin{aligned}
S_1 &= \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right)\left(\sum_{k=1}^{m} f_k\right)^{q-1} \\
&= \sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right)\left(\sum_{k=1}^{m} f_k - \sum_{k=1}^{m-1} f_k\right)\left(\sum_{k=1}^{m} f_k\right)^{q-1} \\
&\le \sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right)\left[\left(\sum_{k=1}^{m} f_k\right)^{q} - \left(\sum_{k=1}^{m-1} f_k\right)^{q}\right] \\
&= \sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right)\left(\sum_{k=1}^{m} f_k\right)^{q} - \sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right)\left(\sum_{k=1}^{m-1} f_k\right)^{q} \\
&= \sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right)\left(\sum_{k=1}^{m} f_k\right)^{q} - \sum_{m=1}^{\infty}\left(\sum_{n=m+1}^{\infty} a_{n,m+1}^q u_n\right)\left(\sum_{k=1}^{m} f_k\right)^{q} \\
&= \sum_{m=1}^{\infty}\left[\left(\sum_{n=m}^{\infty} a_{n,m}^q u_n\right) - \left(\sum_{n=m+1}^{\infty} a_{n,m+1}^q u_n\right)\right]\left(\sum_{k=1}^{m} f_k\right)^{q} \\
&= \sum_{m=1}^{\infty} \bar{u}_m \left(\sum_{k=1}^{m} f_k\right)^{q} \\
&\le C_{p,q}^q \left(\sum_{m=1}^{\infty} f_m^p v_m\right)^{\frac{q}{p}}.
\end{aligned}$$

To get the last estimate we used inequality (3) with the weight sequence $\bar{u}_m := (\sum_{n=m}^{\infty} a_{n,m}^q u_n) - \left(\sum_{n=m+1}^{\infty} a_{n,m+1}^q u_n\right)$, since the satisfying of which follows from condition (4), i.e.,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} \bar{u}_k\right)^{\frac{r}{q}}\left(\sum_{k=1}^{n} v_k^{1-p'}\right)^{\frac{r}{q'}} v_n^{1-p'} = \\
&= \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\left[\left(\sum_{m=k}^{\infty} a_{m,k}^q u_m\right) - \left(\sum_{m=k+1}^{\infty} a_{m,k+1}^q u_m\right)\right]\right)^{\frac{r}{q}}\left(\sum_{k=1}^{n} v_k^{1-p'}\right)^{\frac{r}{q'}} v_n^{1-p'} \\
&= \sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty} a_{m,n}^q u_m\right)^{\frac{r}{q}}\left(\sum_{k=1}^{n} v_k^{1-p'}\right)^{\frac{r}{q'}} v_n^{1-p'} = B_1^r < \infty.
\end{aligned}$$

Moreover, using (5), we obtain the estimate for the best constant $C_{p,q}$, i.e.,

$$C_{p,q} \le q^{\frac{1}{q}} (p')^{\frac{1}{q'}} B_1.$$

Therefore,

$$S_1^{\frac{1}{q}} \le q^{\frac{1}{q}} (p')^{\frac{1}{q'}} B_1 \|f\|_{p,v},$$

i.e.,

$$S_1 \le q(p')^{q-1} B_1^q \|f\|_{p,v}^q. \tag{19}$$

Now we estimate S_2. Applying Hölder inequality, we get

$$\begin{aligned} S_2 &= \sum_{m=1}^{\infty} f_m \left(\sum_{n=m}^{\infty} a_{n,m} u_n\right) \left(\sum_{k=1}^{m} a_{m,k} f_k\right)^{q-1} \\ &\le \left(\sum_{m=1}^{\infty} f_m^p v_m\right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} v_m^{1-p'} \left(\sum_{n=m}^{\infty} a_{n,m} u_n\right)^{p'} \left(\sum_{k=1}^{m} a_{m,k} f_k\right)^{(q-1)p'}\right)^{\frac{1}{p'}} \\ &= \|f\|_{p,v} \bar{S}_2^{\frac{1}{p'}}, \end{aligned}$$

where

$$\bar{S}_2 = \sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} a_{n,m} u_n\right)^{p'} \left(\sum_{k=1}^{m} a_{m,k} f_k\right)^{(q-1)p'} v_m^{1-p'}.$$

To estimate $\bar{S}_2$, we proceed as follows:

$$\begin{aligned} \bar{S}_2 &= \sum_{m=1}^{\infty} v_m^{1-p'} \left(\sum_{n=m}^{\infty} a_{n,m} u_n\right)^{p'} \sum_{l=1}^{m} \left[\left(\sum_{k=1}^{l} a_{l,k} f_k\right)^{(q-1)p'} - \left(\sum_{k=1}^{l-1} a_{l,k} f_k\right)^{(q-1)p'}\right] \\ &= \sum_{l=1}^{\infty} \left[\sum_{m=l}^{\infty} v_m^{1-p'} \left(\sum_{n=m}^{\infty} a_{n,m} u_n\right)^{p'}\right] \left[\left(\sum_{k=1}^{l} a_{l,k} f_k\right)^{(q-1)p'} - \left(\sum_{k=1}^{l-1} a_{l,k} f_k\right)^{(q-1)p'}\right] \end{aligned}$$

[we apply the generalized Minkowskii inequality to the second bracket]

$$\begin{aligned} &\le \sum_{l=1}^{\infty} \left[\left(\sum_{k=1}^{l} a_{l,k} f_k\right)^{(q-1)p'} - \left(\sum_{k=1}^{l-1} a_{l-1,k} f_k\right)^{(q-1)p'}\right] \left(\sum_{n=l}^{\infty} u_n \left(\sum_{m=1}^{n} a_{n,m}^{p'} v_m^{1-p'}\right)^{\frac{1}{p'}}\right)^{p'} \\ &= \sum_{l=1}^{\infty} \left[\sum_{n=l}^{\infty} \left(\sum_{m=n}^{\infty} u_m\right)^{\frac{1}{p}} \left(\sum_{m=1}^{n} a_{n,m}^{p'} v_m^{1-p'}\right)^{\frac{1}{p'}} u_n^{1-\frac{q}{p}} \left(\sum_{m=n}^{\infty} u_m\right)^{-\frac{1}{p}} u_n^{\frac{q}{p}}\right]^{p'} \\ &\times \left[\left(\sum_{k=1}^{l} a_{l,k} f_k\right)^{(q-1)p'} - \left(\sum_{k=1}^{l-1} a_{l-1,k} f_k\right)^{(q-1)p'}\right] \end{aligned}$$

[applying Hölder inequality with the degrees $\frac{p}{p-q}$ and $\frac{p}{q}$ and according to (7), we have]

$$\leq \sum_{l=1}^{\infty}\left(\sum_{n=l}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{1}{p-q}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{p-1}{p-q}}u_n\right)^{\frac{p-q}{p-1}}\left(\sum_{n=l}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{-\frac{1}{q}}u_n\right)^{\frac{q}{p-1}}$$
$$\times\left[\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{(q-1)p'}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{(q-1)p'}\right]$$
$$\leq (q')^{\frac{q}{p-1}}\sum_{l=1}^{\infty}\left(\sum_{n=l}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{1}{p-q}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{p-1}{p-q}}u_n\right)^{\frac{p-q}{p-1}}$$
$$\times\left(\sum_{m=l}^{\infty}u_m\right)^{\frac{q-1}{p-1}}\left[\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{(q-1)p'}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{(q-1)p'}\right]$$

[again applying Hölder inequality with the degrees $\frac{p-1}{p-q}$ and $\frac{p-1}{q-1}$, we get]

$$\leq (q')^{\frac{q}{p-1}}\left[\sum_{l=1}^{\infty}\left(\sum_{n=l}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{1}{p-q}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{p-1}{p-q}}u_n\right)\right.$$
$$\left.\times\left[\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{(q-1)p'}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{(q-1)p'}\right]^{\frac{1}{p'}}\right]^{\frac{p-q}{p-1}}$$
$$\times\left[\sum_{l=1}^{\infty}\left(\sum_{m=l}^{\infty}u_m\right)\left[\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{(q-1)p'}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{(q-1)p'}\right]^{\frac{q'}{p'}}\right]^{\frac{q-1}{p-1}}$$

[applying the inequality $(x-y)^{\alpha}\leq x^{\alpha}-y^{\alpha}$ $(0<\alpha<1)$, followed by Fubini's theorem on the first and second products, we get]

$$\leq (q')^{\frac{q}{p-1}}\left[\sum_{l=1}^{\infty}\left(\sum_{n=l}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{1}{p-q}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{p-1}{p-q}}u_n\right)\right.$$
$$\left.\times\left[\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{q-1}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{q-1}\right]\right]^{\frac{p-q}{p-1}}$$
$$\times\left[\sum_{l=1}^{\infty}\left(\sum_{m=l}^{\infty}u_m\right)\left(\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{q}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{q}\right)\right]^{\frac{q-1}{p-1}}$$

$$= \left(q'\right)^{\frac{q}{p-1}} \left[\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{1}{p-q}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{p-1}{p-q}} u_n\right.$$

$$\times\left(\sum_{l=1}^{n}\left[\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{q-1}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{q-1}\right]\right)\Bigg]^{\frac{p-q}{p-1}}$$

$$\times\left[\sum_{m=1}^{\infty}u_m\left(\sum_{l=1}^{m}\left(\sum_{k=1}^{l}a_{l,k}f_k\right)^{q}-\left(\sum_{k=1}^{l-1}a_{l-1,k}f_k\right)^{q}\right)\right]^{\frac{q-1}{p-1}}$$

$$= \left(q'\right)^{\frac{q}{p-1}} \left[\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{1}{p-q}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{p-1}{p-q}} u_n^{\frac{1}{q}}u_n^{\frac{1}{q'}}\left(\sum_{k=1}^{n}a_{n,k}f_k\right)^{q-1}\right]^{\frac{p-q}{p-1}}$$

$$\times\left[\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}a_{m,k}f_k\right)^{q}u_m\right]^{\frac{q-1}{p-1}}$$

[applying Hölder inequality with exponents q and q' to the first product, we obtain]

$$= \left(q'\right)^{\frac{q}{p-1}}\left(\left[\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{q}{p-q}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{q(p-1)}{p-q}}u_n\right]^{\frac{1}{q}}\left[\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}a_{n,k}f_k\right)^{q}u_n\right]^{\frac{1}{q'}}\right)^{\frac{p-q}{p-1}} S^{\frac{q-1}{p-1}}$$

$$= \left(q'\right)^{\frac{q}{p-1}}\left[\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{r}{p}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{r}{p'}}u_n\right]^{\frac{p-q}{q(p-1)}}\left[\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}a_{n,k}f_k\right)^{q}u_n\right]^{\frac{p-q}{q'(p-1)}} S^{\frac{q-1}{p-1}}$$

$$= \left(q'\right)^{\frac{q}{p-1}}\left[\sum_{n=1}^{\infty}\left(\sum_{m=n}^{\infty}u_m\right)^{\frac{r}{p}}\left(\sum_{m=1}^{n}a_{n,m}^{p'}v_m^{1-p'}\right)^{\frac{r}{p'}}u_n\right]^{\frac{p'}{r}} S^{\frac{p-q}{q'(p-1)}} S^{\frac{q-1}{p-1}}$$

$$= \left(q'\right)^{\frac{q}{p-1}} B_2^{p'} S^{\frac{p'}{q'}}.$$

Thus, we have

$$\bar{S}_2 \le \left(q'\right)^{\frac{q}{p-1}} B_2^{p'} S^{\frac{p'}{q'}}$$

and then

$$S_2 \le \|f\|_{p,v}\bar{S}_2^{\frac{1}{p'}} \le \|f\|_{p,v}\left(q'\right)^{\frac{q}{p}} B_2 S^{\frac{1}{q'}}. \tag{20}$$

By summing estimates (19) and (20), we obtain

$$\begin{aligned} S &\le (2h)^{q-1}q[S_1+S_2] \\ &\le (2h)^{q-1}q\left[q(p')^{q-1}B_1^q\|f\|_{p,v}^q + \|f\|_{p,v}(q')^{\frac{q}{p}}B_2 S^{\frac{1}{q'}}\right] \\ &= (2h)^{q-1}q^2(p')^{q-1}B_1^q\|f\|_{p,v}^q + (2h)^{q-1}q\|f\|_{p,v}\left(q'\right)^{\frac{q}{p}} B_2 S^{\frac{1}{q'}}. \end{aligned}$$

Using Young's inequality on the second term yields

$$S \le (2h)^{q-1}q^2(p')^{q-1}B_1^q\|f\|_{p,v}^q + \frac{\left((2h)^{q-1}q\|f\|_{p,v}\,(q')^{\frac{q}{p}}\,B_2\right)^q}{q} + \frac{S}{q'}$$

and then

$$S \leq (2h)^{q-1}q^3(p')^{q-1}B_1^q\|f\|_{p,v}^q + \left((2h)^{q-1}q\|f\|_{p,v}\left(q'\right)^{\frac{q}{p}}B_2\right)^q,$$

i.e.,

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}a_{n,k}f_k\right)^q u_n\right)^{\frac{1}{q}} \leq \left((2h)^{q-1}q^3(p')^{q-1}B_1^q + (2h)^{q(q-1)}q^q\left(q'\right)^{\frac{q^2}{p}}B_2^q\right)^{\frac{1}{q}}\left(\sum_{n=1}^{\infty}f_n^p v_n\right)^{\frac{1}{p}}.$$

Finally, we arrive at the following estimate

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}a_{n,k}f_k\right)^q u_n\right)^{\frac{1}{q}} \leq \left((2h)^{q-1}q^3(p')^{q-1} + (2h)^{q(q-1)}q^q\left(q'\right)^{\frac{q^2}{p}}\right)^{\frac{1}{q}}B\|f\|_{p,v}.$$

Hence, the sufficiency of the condition and the upper for the norm are established.

The proof is complete. □

Conclusion

The necessary and sufficient conditions for the boundedness of the matrix operators whose entries satisfy the so-called *Oinarov conditions* are established. In addition, both lower and upper estimates for the norms of such operators are derived. The obtained results can be used in the spectral analysis of matrix operators and in determining the oscillatory or non-oscillatory behavior of solutions to certain difference equations. Furthermore, the derived inequalities can be employed to evaluate sequences via their higher-order differences.

Acknowledgments

The research of the first author was supported by the project AL-9224104685 of the Republic of Uzbekistan.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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