

Linear maps on Banach algebras which are Lie centralizers at a point

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In this paper, we study continuous linear maps that behave as Lie centralizers at specific points, including identity products, zero products, and idempotent products. Our aim is to determine when local Lie centralizer conditions force a linear map to have a global centralizer-like structure. We first show that every continuous linear map satisfying the Lie centralizer condition at identity products is necessarily a commuting map. We then obtain explicit descriptions of such map on C^* -algebras, von Neumann algebras, standard operator algebras, and matrix algebras. For maps acting at zero products, we prove that under suitable assumptions they are Lie centralizers, and we derive corresponding structural characterizations. We also investigate Lie centralizers at idempotent products and establish several characterization theorems for von Neumann algebras and standard operator algebras. These results extend a number of earlier theorems on commuting maps, centralizers, and Lie centralizers, and provide new criteria under which local conditions imply global behavior. Moreover, our approach highlights the strong relationship between local algebraic conditions and the global structure of linear maps. Consequently, this paper contributes to the theory of linear preserver problems and centralizer-type mappings on Banach and operator algebras.

Keywords: centralizer, Lie centralizer, Lie triple centralizer, commuting map, C^* -algebra, von Neumann algebras, standard operator algebras, matrix algebras.

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Introduction

Let A be an associative algebra over $\mathbb{C}$ with center $Z(A)$. A linear map θ from algebra A into itself is called a *left centralizer* (*right centralizer*) if for all $a, b \in A$,

$$\theta(ab) = \theta(a)b, \quad (\theta(ab) = a\theta(b)),$$

and θ is called a *centralizer* if it is both left and right centralizer. In case $\theta : A \rightarrow A$ is a centralizer, then by [1, Theorem 2.3.2] there exists an element $\lambda \in C$ such that $\theta(a) = \lambda a$ for all $a \in A$, where C is the extended centroid of A .

By considering a new product $[a, b] = ab - ba$, called the *Lie product*, A takes on a structure of a Lie algebra. A linear map $\theta : A \rightarrow A$ is called a *Lie centralizer* if

$$\theta([a, b]) = [\theta(a), b], \quad a, b \in A. \quad (1)$$

Note that condition (1) is equivalent to $\theta([a, b]) = [a, \theta(b)]$ for all $a, b \in A$. A simple example of a Lie centralizer $\theta : A \rightarrow A$ is a map of the form

$$\theta(a) = \phi(a) + \xi(a), \quad a \in A,$$

where ϕ is a centralizer, and ξ is a linear map of A into $Z(A)$ vanishing at commutators $[a, b]$ for all $a, b \in A$. In particular, if $\phi(a) = ca$ for some $c \in Z(A)$, then $\theta(a) = ca + \xi(a)$ for all $a \in A$. The goal is to show that in certain cases these are in fact the only examples.

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A linear map θ from algebra A into itself is called *centralizing* if $[\theta(a), a] \in Z(A)$ for every $a \in A$. In the special case where $[\theta(a), a] = 0$ for all $a \in A$, the map θ is said to be a *commuting map*.

It is shown in [2, Theorem 1] that every centralizing map on a prime algebra A is a commuting map. Recall that A is called *prime* if $aAb = 0$ implies that $a = 0$ or $b = 0$.

Obviously, each Lie centralizer is a commuting map, but the converse fails in general; see Example 1 below. Moreover, every centralizer is a special case of a Lie centralizer, however, there exist Lie centralizers which are not centralizers, [3, Example 2.1].

Determining the structure of commuting maps in terms of centralizers or Lie centralizers is a matter of interest. In [4], Brešar proved that every commuting additive map θ on a prime ring A is of the form $\theta(a) = \lambda a + \xi(a)$ for all $a \in A$, where $\lambda \in C$ and ξ is an additive map from A into C . We refer the reader to the survey paper [5] for the background and history of commuting maps.

We say that a map θ is a Lie centralizer at a given point $z \in A$ if

$$a, b \in A, \quad ab = z \quad \implies \quad \theta([a, b]) = [\theta(a), b]. \quad (2)$$

This kind of maps were discussed by several authors. For example, it is shown [6] that if A is a prime ring containing a non-trivial idempotent and θ is an additive map satisfying $\theta(ab) = a\theta(b)$ for all $a, b \in A$ with $ab = 0$, then θ is a right centralizer. The same result was obtained [7] for unital C^* -algebra A and the continuous linear map $T : A \rightarrow X$ where X is a unital Banach A -bimodule. Article [8] investigated this problem for Banach algebras at the identity products.

Lie centralizers at zero products on a class of operator algebras are discussed in [3]. In [9], the authors have studied Lie centralizers on triangular rings. In [10], Ghahramani considered the problem of characterizing linear maps behaving like Lie centralizers at idempotent products. We refer the reader to [8, 11, 12] for a full account of the topic and a list of references.

Taking the ideas from the above, the aim of this paper is to characterize linear maps that satisfy in condition (2) at the identity products, zero products and idempotent products.

1 Lie centralizer at identity

We mention that every Lie centralizer is a commuting map, but the converse fails, in general. The following example illustrates this fact.

Example 1. Let

$$A = \left\{ \begin{bmatrix} 0 & z_1 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{bmatrix} : z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

Define $\theta : A \rightarrow A$ via

$$\theta \left(\begin{bmatrix} 0 & z_1 & z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & z_1 & -z_2 \\ 0 & 0 & z_3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $[\theta(a), a] = 0$ for all $a \in A$, and hence θ is a commuting map, however, θ is not a Lie centralizer.

The following result is the main theorem of [4].

Theorem 1. Let R be a prime ring and $\theta : R \rightarrow R$ be an additive centralizing map. If either $\text{char} R \neq 2$ or θ is a commuting map, then

$$\theta(a) = \lambda a + \xi(a), \quad a \in A,$$

where $\lambda \in C$, the extended centroid of R , and ξ is an additive map from A into C .

Combining Theorem 1 with the fact that the extended centroid of a prime C^* -algebra is isomorphic to the complex numbers, we see that if θ is an additive centralizing map on a prime C^* -algebra A , then there exist $\lambda \in Z(A)$ and an additive map $\xi : A \rightarrow Z(A)$ such that $\theta(a) = \lambda a + \xi(a)$, for all $a \in A$.

The next result states that this conclusion holds in general (not necessarily prime) von Neumann algebras.

Theorem 2 ([13, Theorem 2.1]). Let A be a von Neumann algebra. If $\theta : A \rightarrow A$ is an additive centralizing map, then $\theta(a) = ca + \xi(a)$ for all $a \in A$, where $c \in Z(A)$ and ξ is an additive map from A into $Z(A)$.

The second dual space A'' of a Banach algebra A admits two Banach algebra products known as the first and second Arens products, each extending the product on A . These products, which we denote by $\square$ and $\diamond$, respectively, can be defined as follows:

$$\Phi \square \Psi = \lim_i \lim_j a_i \cdot b_j, \quad \Psi \diamond \Phi = \lim_j \lim_i b_j \cdot a_i, \quad \Phi, \Psi \in A'',$$

where $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in I}$ are nets in A that converge in the w^* -topology to Φ and Ψ , respectively. If these products coincide on A'' , then A is said to be Arens regular. It is shown in [14] that every C^* -algebra A is Arens regular.

We regard A as a closed subalgebra of both $(A'', \square)$ and $(A'', \diamond)$ by identifying A with $k(A)$, where $k : A \rightarrow A''$ is the canonical embedding, that is

$$k(a)(f) = \widehat{a}(f) = f(a), \quad a \in A, f \in A',$$

and A is w^* -dense in A'' by Goldstine's theorem. In this regard, we have

$$\widehat{a} \square \Phi = \widehat{a} \diamond \Phi = \widehat{a} \cdot \Phi, \quad \text{and} \quad \Phi \square \widehat{a} = \Phi \diamond \widehat{a} = \Phi \cdot \widehat{a},$$

for all $a \in A$ and $\Phi \in A''$. One may refer to the monograph of Dales [14] for a full account of Arens product.

Next we generalize Theorem 2 for all C^* -algebras.

Theorem 3. Suppose that A is a C^* -algebra. If $\theta : A \rightarrow A$ is a continuous centralizing map, then $\theta(a) = ca + \xi(a)$ for all $a \in A$, where $c \in Z(A'')$ and ξ is an additive map from A into $Z(A'')$. In particular, θ is a commuting map.

Proof. Since θ is a centralizing map, we have $[\theta(a), a] \in Z(A)$ for all $a \in A$. Replacing a by $a + b$, we obtain

$$[\theta(a), b] + [\theta(b), a] \in Z(A), \quad a, b \in A.$$

Let $\Phi, \Psi \in A''$ and $\widehat{a}_i \rightarrow \Phi, \widehat{b}_j \rightarrow \Psi$ in the w^* -topology. Then for each $f \in A'$,

$$\begin{aligned} (\theta^{**}(\widehat{a}_i))f &= (\widehat{a}_i \circ \theta^*)f \\ &= \widehat{a}_i(\theta^*(f)) \rightarrow \Phi(\theta^*(f)) = (\theta^{**}(\Phi))f. \end{aligned}$$

Therefore, $\theta^{**}(\widehat{a}_i) \rightarrow \theta^{**}(\Phi)$ in the w^* -topology. Consequently,

$$\begin{aligned} [\theta^{**}(\Phi), \Psi] + [\theta^{**}(\Psi), \Phi] &= \lim_i \lim_j [\theta^{**}(\widehat{a}_i), \widehat{b}_j] + [\theta^{**}(\widehat{b}_j), \widehat{a}_i] \\ &= \lim_i \lim_j [\widehat{\theta(a_i)}, \widehat{b_j}] + [\widehat{\theta(b_j)}, \widehat{a_i}] \in Z(A''). \end{aligned}$$

In particular, $[\theta^{**}(\Phi), \Phi] \in Z(A'')$ for all $\Phi \in A''$. This means that θ^{**} is a centralizing map. It is known that the second dual of each C^* -algebra is a von Neumann algebra [14], hence by Theorem 2, we have

$$\theta^{**}(\Phi) = c \square \Phi + \rho(\Phi), \quad \Phi \in A'', \quad (3)$$

where $c \in Z(A'')$ and ρ is an additive map from A'' into $Z(A'')$.

Let ξ be the restriction of ρ on A , so $\xi(a) = \rho(a)$ for all $a \in A$. Therefore, by (3) we obtain

$$\theta(a) = ca + \xi(a),$$

for all $a \in A$, where $c \in Z(A'')$ and ξ is an additive map from A into $Z(A'')$.

Since $c, \xi(a) \in Z(A'')$, by the above equality we conclude that

$$\begin{aligned} [\theta(a), a] &= [ca + \xi(a), a] \\ &= [ca, a] + [\xi(a), a] = 0, \end{aligned}$$

for all $a \in A$. Therefore, θ is a commuting map. This completes the proof. $\square$

As a consequence of Theorem 3, we obtain the following result.

Corollary 1. Let A be a C^* -algebra and $\theta : A \rightarrow A$ be a continuous linear map. Then θ is a Lie centralizer if and only if $\theta(a) = ca + \xi(a)$ for all $a \in A$, where $c \in Z(A'')$ and ξ is a linear map from A into $Z(A'')$ such that $\xi([a, b]) = 0$ for all $a, b \in A$.

Next we prove that every Lie centralizer at the identity product is a commuting map.

Theorem 4. Let A be a unital Banach algebra and $\theta : A \rightarrow A$ be a continuous linear map satisfying

$$a, b \in A, \quad ab = e_A \quad \implies \quad \theta([a, b]) = [\theta(a), b]. \quad (4)$$

Then θ is a commuting map.

Proof. Let $a \in A$ be arbitrary. For $\lambda \in \mathbb{C}$ with $|\lambda| < 1/\|a\|$, $e_A - \lambda a$ is invertible and $(e_A - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. Noticing that $(e_A - \lambda a)(e_A - \lambda a)^{-1} = e_A$, thus by condition (4) we have

$$\theta([e_A - \lambda a, \sum_{n=0}^{\infty} \lambda^n a^n]) = [\theta(e_A - \lambda a), \sum_{n=0}^{\infty} \lambda^n a^n].$$

Obviously,

$$\theta([e_A - \lambda a, \sum_{n=0}^{\infty} \lambda^n a^n]) = 0, \quad a \in A.$$

Therefore,

$$\begin{aligned} 0 &= [\theta(e_A) - \lambda\theta(a), \sum_{n=0}^{\infty} \lambda^n a^n] \\ &= \sum_{n=0}^{\infty} \lambda^n [\theta(e_A), a^n] - \sum_{n=0}^{\infty} \lambda^{n+1} [\theta(a), a^n] \\ &= \sum_{n=0}^{\infty} \lambda^{n+1} [\theta(e_A), a^{n+1}] - \sum_{n=0}^{\infty} \lambda^{n+1} [\theta(a), a^n]. \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} \lambda^{n+1} ([\theta(e_A), a^{n+1}] - [\theta(a), a^n]) = 0,$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| < 1/\|a\|$. Hence

$$[\theta(e_A), a^{n+1}] = [\theta(a), a^n], \quad n = 0, 1, 2, \dots \quad (5)$$

Taking $n = 0$, we obtain $[\theta(e_A), a] = 0$ for all $a \in A$. Now it follows from (5) that

$$[\theta(a), a] = 0, \quad a \in A.$$

Consequently, θ is a commuting map. □

Let $B(X)$ be the operator algebra of all bounded linear operators on Banach space X . A *standard operator algebra* is any subalgebra of $B(X)$ which contains the identity, and the ideal $F(X)$ of all finite rank operators.

We mention that standard operator algebra and $M_n(\mathbb{C})$ are prime algebras, but C^* -algebras are not prime, in general.

Combining Theorem 4 and Theorem 1, we obtain the next result.

Corollary 2. Let A be either standard operator algebra or $M_n(\mathbb{C})$ and let $\theta : A \rightarrow A$ be a continuous linear map satisfying (4). Then θ is a commuting map, and

$$\theta(a) = \lambda a + \xi(a),$$

for all $a \in A$, where $\lambda \in \mathbb{C}$ and ξ is a linear map of A into $\mathbb{C}e_A$.

Corollary 3. Let A be a unital C^* -algebra and let $\theta : A \rightarrow A$ be a continuous linear map such that the condition (4) holds. Then

$$\theta(a) = ca + \xi(a),$$

for all $a \in A$, where $c \in Z(A'')$ and ξ is a linear map of A into $Z(A'')$ such that $\xi([a, b]) = 0$ for all $a, b \in A$ with $ab = e_A$.

Proof. By Theorem 4, θ is a commuting map, so it follows from Theorem 3 that θ is of the form $\theta(a) = ca + \xi(a)$ for all $a \in A$, where $c \in Z(A'')$ and $\xi : A \rightarrow Z(A'')$ is a linear map.

Now let $a, b \in A$ with $ab = e_A$. Then by (4), we have

$$\begin{aligned} \xi([a, b]) &= \theta([a, b]) - c[a, b] \\ &= [\theta(a), b] - [ca, b] = [\xi(a), b] = 0. \end{aligned}$$

This completes the proof. □

A linear map θ on algebra A is called a *Lie triple centralizer* if

$$\theta([a, [b, c]]) = [a, [b, \theta(c)]], \quad a, b, c \in A.$$

We mention that the above identity is equivalent to assuming that

$$\theta([a, [b, c]]) = [a, [\theta(b), c]], \quad a, b, c \in A.$$

This notion introduced in [15] and the general form of Lie triple centralizers on a generalized matrix algebra are characterized.

Clearly, every Lie centralizer is a Lie triple centralizer, but the converse fails, in general. For example, let A and θ be as in Example 1. Then θ is a Lie triple centralizer, but it is not a Lie centralizer.

The following result is an analogous version of Theorem 4 for Lie triple centralizers.

Theorem 5. Let A be a unital Banach algebra and $\theta : A \rightarrow A$ be a continuous linear map satisfying

$$a, b \in A, \quad ab = e_A \implies \theta([a, [b, a]]) = [a, [\theta(b), a]]. \quad (6)$$

Then θ is a centralizing map. Moreover, if A is prime, then θ is a commuting map.

Proof. Let $a \in A$ be arbitrary. Similar to the proof of Theorem 4, we have

$$\theta([e_A - \lambda a, [\sum_{n=0}^{\infty} \lambda^n a^n, e_A - \lambda a]]) = [e_A - \lambda a, [\sum_{n=0}^{\infty} \lambda^n \theta(a^n), e_A - \lambda a]].$$

Obviously, the left hand side of the above equality is zero, thus by a simple calculation we get

$$\sum_{n=0}^{\infty} \lambda^{n+1} ([a, [\theta(a^n), a]]) = 0,$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| < 1/\|a\|$. Hence

$$[a, [\theta(a^n), a]] = 0, \quad n = 0, 1, 2, \dots$$

Taking $n = 1$, we obtain $[\theta(a), a] \in Z(A)$ for all $a \in A$, and so θ is a centralizing map. If A is prime, then by [2, Theorem 1], θ is a commuting map. $\square$

By Theorem 5 and Theorem 1, we conclude the next result.

Corollary 4. Let A be either standard operator algebra or $M_n(\mathbb{C})$ and let $\theta : A \longrightarrow A$ be a continuous linear map satisfying (6). Then θ is a commuting map, and

$$\theta(a) = \lambda a + \xi(a),$$

for all $a \in A$, where $\lambda \in \mathbb{C}$ and ξ is a linear map of A into $\mathbb{C}e_A$.

The corollary below follows from Theorem 5 and Theorem 3.

Corollary 5. Let A be a unital C^* -algebra and $\theta : A \longrightarrow A$ be a continuous linear map such that the condition (6) holds. Then

$$\theta(a) = ca + \xi(a), \quad a \in A,$$

where $c \in Z(A'')$ and $\xi : A \longrightarrow Z(A'')$ is a linear map such that $\xi([a, [b, a]]) = 0$ for every $a, b \in A$ with $ab = e_A$.

2 Lie centralizers at zero point

This section is devoted to characterize linear maps behaving like Lie centralizers at zero point. We begin with the following lemma.

Lemma 1. Let A be a unital algebra and $\theta : A \longrightarrow A$ be a linear map satisfying

$$a, b \in A, \quad ab = 0 \quad \implies \quad \theta([a, b]) = [\theta(a), b]. \quad (7)$$

Then for every idempotent $p, q \in A$,

$$\theta([p, q]) = [\theta(p), q].$$

Proof. From $(p - pq)q = 0$, we have

$$\theta([p - pq, q]) = [\theta(p - pq), q],$$

and hence

$$\theta([p, q]) - \theta([pq, q]) = [\theta(p), q] - [\theta(pq), q]. \quad (8)$$

Similarly, as $pq(q - e_A) = 0$, we get

$$\theta([pq, q - e_A]) = [\theta(pq), q - e_A],$$

and thus

$$\theta([pq, q]) = [\theta(pq), q]. \quad (9)$$

It follows from (8) and (9) that

$$\theta([p, q]) = [\theta(p), q],$$

for every idempotent $p, q \in A$. □

Theorem 6. Let A be a von Neumann algebra and $\theta : A \longrightarrow A$ be a continuous linear map satisfying (7). Then θ is a Lie centralizer.

Proof. By Lemma 1,

$$\theta([p, q]) = [\theta(p), q], \quad (10)$$

for every idempotent $p, q \in A$. Let A_{sa} denote the set of all self-adjoint elements of A and let $x \in A_{sa}$. Then x is the limit of a sequence of linear combinations of projections in A , i.e., self-adjoint idempotents, therefore

$$x = \lim_n \sum_{k=1}^n \lambda_k p_k,$$

and thus from (10), we get $\theta([x, y]) = [\theta(x), y]$ for all self-adjoint elements $x, y \in A$. Now let $a \in A$ be arbitrary. Then $a = x + iy$ for $x, y \in A_{sa}$ and hence

$$\theta([a, b]) = [\theta(a), b], \quad a, b \in A.$$

Consequently, θ is a Lie centralizer. □

Recall that a *bounded approximate identity* for A is a bounded net $\{e_\lambda\}_{\lambda \in I}$ in A such that $e_\lambda a \longrightarrow a$ and $ae_\lambda \longrightarrow a$ for every $a \in A$. It seems appropriate to mention that the group algebra $L^1(G)$ for a locally compact group G and every C^* -algebra has a bounded approximate identity bounded by one, see [14].

The Banach algebra A is called *zero product determined* if each continuous bilinear map $\phi : A \times A \longrightarrow X$, where X is an arbitrary Banach space, satisfying

$$a, b \in A, \quad ab = 0 \quad \implies \quad \phi(a, b) = 0,$$

can be written as

$$\phi(a, b) = T(ab), \quad a, b \in A,$$

for some continuous linear map $T : A \longrightarrow X$.

The concept of zero product determined appeared as a byproduct of the so-called property $(\mathbb{B})$ introduced in [16].

We now generalize Theorem 6 to zero product determined Banach algebras.

Theorem 7. Let A be a zero product determined Banach algebra equipped a bounded approximate identity. If $\theta : A \longrightarrow A$ is a continuous linear map satisfying (7), then θ is a Lie centralizer.

Proof. Define a continuous bilinear map $\phi : A \times A \longrightarrow A$ by

$$\phi(a, b) = \theta([a, b]) - [\theta(a), b], \quad a, b \in A.$$

Then $ab = 0$ implies that $\phi(a, b) = 0$. Since A is zero product determined, there exists a continuous linear map $T : A \longrightarrow A$ such that $\phi(a, b) = T(ab)$ for all $a, b \in A$. Hence,

$$\begin{aligned} \theta([a, bc]) - [\theta(a), bc] &= \phi(a, bc) \\ &= T(a(bc)) \\ &= T((ab)c) \\ &= \phi(ab, c) = \theta([ab, c]) - [\theta(ab), c]. \end{aligned}$$

Letting $c = e_\lambda$ in the above equality, we get $\theta([a, b]) = [\theta(a), b]$ for all $a, b \in A$. Therefore, θ is a Lie centralizer. $\square$

It should be pointed out that every C^* -algebra, as well as every group algebra $L^1(G)$ of locally compact group G , and the matrix algebra $M_n(\mathbb{C})$ are zero product determined Banach algebra [16, 17]. However, standard operator algebras are not zero product determined, in general.

Combining Theorem 7 and Corollary 1, we obtain the following result.

Corollary 6. Let A be a C^* -algebra and $\theta : A \longrightarrow A$ be a continuous linear map such that the condition (7) holds. Then

$$\theta(a) = ca + \xi(a), \quad a \in A,$$

where $c \in Z(A'')$ and $\xi : A \longrightarrow Z(A'')$ is a linear map such that $\xi([a, b]) = 0$ for all $a, b \in A$.

As a consequence of Theorem 7 and Theorem 1, we get the next corollary.

Corollary 7. Let $A = M_n(\mathbb{C})$ and $\theta : A \longrightarrow A$ be a continuous linear map such that condition (7) holds. Then

$$\theta(a) = \lambda a + \xi(a),$$

for all $a \in A$, where $\lambda \in \mathbb{C}$ and ξ is a linear map of A into $\mathbb{C}e_A$.

For a standard operator algebra we have the following result. Note that in the theorem below θ is not continuous, in general.

Theorem 8. Let A be a standard operator algebra, and let $\theta : A \longrightarrow A$ be a linear map such that condition (7) holds. Then $\theta(a) = \lambda a + \xi(a)$ for all $a \in A$, where $\lambda \in \mathbb{C}$ and ξ is a linear map of A into $\mathbb{C}e_A$.

Proof. By [3, Theorem 3.1], θ is a commuting map. Now Theorem 1 applies. $\square$

3 Lie centralizers at idempotents

In this section, we consider the question of characterizing Lie centralizers by acting at idempotents products on von Neumann and standard operator algebras.

The following result is another tool for characterizing linear maps at an idempotent. For its proof, see [18, Theorem 2.1].

Theorem 9. Let A be a unital Banach algebra and X a Banach space. Suppose that $\phi : A \times A \longrightarrow X$ is a continuous bilinear map such that

$$a, b \in A, \quad ab = p \implies \phi(a, b) = 0,$$

where p is an idempotent in A . Then, for all $a \in A$,

$$\phi(a, p) = \phi(e_A, ap), \quad \text{and} \quad \phi(p, a) = \phi(pa, e_A).$$

The following result for $p = 0$ is Theorem 6.

Theorem 10. Let A be a von Neumann algebra and $\theta : A \longrightarrow A$ be a continuous linear map satisfying

$$a, b \in A, \quad ab = p \implies \theta([a, b]) = [\theta(a), b], \quad (11)$$

for every non-trivial idempotent $p \in A$. Then θ is a Lie centralizer.

Proof. Let p be a non-trivial idempotent in A . Define a bilinear map $\phi : A \times A \longrightarrow A$ via

$$\phi(a, b) = \theta([a, b]) - [\theta(a), b], \quad a, b \in A.$$

Then $ab = p$ implies that $\phi(a, b) = 0$. Applying Theorem 9, we obtain

$$\begin{aligned} \theta([p, b]) - [\theta(p), b] &= \phi(p, b) \\ &= \phi(pb, e_A) \\ &= \theta([pb, e_A]) - [\theta(pb), e_A] \\ &= 0. \end{aligned}$$

Therefore,

$$\theta([p, b]) = [\theta(p), b], \quad (12)$$

for all $a \in A$ and every idempotent $p \in A$.

Since the linear span of projections is norm dense in every von Neumann algebra, thus from (12) we arrive at $\theta([a, b]) = [\theta(a), b]$ for all $a, b \in A$. Hence θ is a Lie centralizer. $\square$

From Theorem 10 and Theorem 2 we get the following result.

Corollary 8. Let A be a von Neumann algebra and $\theta : A \longrightarrow A$ be a continuous linear map such that the condition (11) holds. Then

$$\theta(a) = ca + \xi(a), \quad a \in A,$$

where $c \in Z(A)$ and $\xi : A \longrightarrow Z(A)$ is a linear map such that $\xi([a, b]) = 0$ for all $a, b \in A$.

Theorem 11. Let A be a standard operator algebra and let $\theta : A \longrightarrow A$ be a continuous linear map such that condition (11) holds. Then θ is a commuting map.

Proof. Let $p \in A$ be a non-trivial idempotent of rank-one. Define a bilinear map $\phi : A \times A \longrightarrow A$ via

$$\phi(a, b) = \theta([a, b]) - [\theta(a), b], \quad a, b \in A.$$

Then $ab = p$ implies that $\phi(a, b) = 0$. Applying Theorem 9, we get

$$\begin{aligned} \theta([a, p]) - [\theta(a), p] &= \phi(a, p) \\ &= \phi(e_A, ap) \\ &= \theta([e_A, ap]) - [\theta(e_A), ap] \\ &= [ap, \theta(e_A)]. \end{aligned} \quad (13)$$

Replacing a by p in (13), we obtain

$$[p, \theta(p)] = [p, \theta(e_A)].$$

On the other hand, since $pp = p$ the condition (11) implies that

$$[\theta(p), p] = \theta[p, p] = 0.$$

From the above two equalities, we get $[p, \theta(e_A)] = 0$. By [11, Lemma 1.1] each element $u \in F(X)$ is a linear combination of rank-one idempotents. Thus,

$$[u, \theta(e_A)] = 0, \quad u \in F(X),$$

and hence $[au, \theta(e_A)] = 0$ for all $a \in A$. Therefore, from (13) we conclude that

$$\theta(ap) = \theta(pa) + \theta(a)p - p\theta(a),$$

for all $a \in A$ and every idempotent $p \in A$. In particular,

$$\theta(au) = \theta(ua) + \theta(a)u - u\theta(a), \quad a \in A, u \in F(X). \quad (14)$$

Let $a, b \in A$. By applying (14), we obtain

$$\theta(abu) = \theta(uab) + \theta(ab)u - u\theta(ab), \quad u \in F(X).$$

Replacing u by bu in (14), we have

$$\begin{aligned} \theta(abu) &= \theta(bua) + \theta(a)bu - bu\theta(a) \\ &= \theta(uab) + \theta(b)ua - ua\theta(b) + \theta(a)bu - bu\theta(a). \end{aligned}$$

By comparing the two expressions for $\theta(abu)$, we arrive at

$$\theta(ab)u - u\theta(ab) = \theta(b)ua - ua\theta(b) + \theta(a)bu - bu\theta(a),$$

for all $a, b \in A$ and $u \in F(X)$.

Since $F(X)$ is dense in $B(X)$ with respect to the strong operator topology, there is a net u_i in $F(X)$ such that $u_i \rightarrow e_A$. Therefore,

$$\theta(ab)u_i - u_i\theta(ab) = \theta(b)u_i a - u_i a \theta(b) + \theta(a)bu_i - bu_i\theta(a).$$

From the separate continuity of product in $B(X)$, we get

$$\theta(b)a - a\theta(b) + \theta(a)b - b\theta(a) = 0.$$

By letting $a = b$ we get $[\theta(a), a] = 0$ for every $a \in A$. Thus, θ is a commuting map. $\square$

The corollary below follows from Theorem 11 and Theorem 1.

Corollary 9. Let A be a standard operator algebra and let $\theta : A \rightarrow A$ be a continuous linear map such that condition (11) holds. Then

$$\theta(a) = \lambda a + \xi(a), \quad a \in A,$$

where $\lambda \in \mathbb{C}$ and ξ is a linear map of A into $\mathbb{C}e_A$.

The set of idempotents of a given algebra A is denoted by $\mathcal{I}(A)$. A Banach algebra A is generated by idempotents, whenever $A = \text{alg}\mathcal{I}(A)$.

For example, von Neumann algebras, $B(\mathcal{H})$ for infinite-dimensional complex Hilbert space $\mathcal{H}$, topologically simple Banach algebras containing a non-trivial idempotent, the group algebra $L^1(G)$ for a compact group G , and the matrix algebra $M_n(A)$ over a unital Banach algebra A are generated by idempotents. Other classes of Banach algebras with this property can be found in [16].

Now, similar to Theorem 10, we can prove the next result.

Theorem 12. Let A be a unital Banach algebra with the property that $A = \overline{\text{alg}\mathcal{I}(A)}$. If $\theta : A \rightarrow A$ is a continuous linear map satisfying (11), then

$$\theta(a) = ca + \xi(a), \quad a \in A,$$

where $c \in Z(A)$ and $\xi : A \rightarrow Z(A)$ is a linear map such that $\xi([a, b]) = 0$ for all $a, b \in A$.

Conclusion

In this paper, we investigated linear maps behaving like Lie centralizers at identity products, zero products, and idempotent products on several classes of Banach algebras. We proved that under suitable assumptions, such mappings are closely related to commuting maps and Lie centralizers. In particular, we obtained structural characterizations for these maps on C^* -algebras, von Neumann algebras, standard operator algebras, and matrix algebras. These results extend several known characterizations of Lie centralizers and provide new perspectives on local conditions that force a linear map to exhibit global centralizer-like behavior.

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Conflict of interest

The author declares no conflict of interest.

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