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CONTENT

MATHEMATICS

MATHEMATICS

https://doi.org/10.31489/2025M3/4-21 *Research article*

Classification and reduction to canonical form of linear differential equations partial of the sixth-order with non-multiple characteristics

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This paper studies the problems of classification and reduction to canonical form of linear partial differential equations of the sixth-order with non-multiple characteristics and constant coefficients. Considering that with the growth of the order of the equation or the increase in the number of independent variables, the problems of classification and reduction to canonical form become more complicated. The article first provides a general formula for the coefficients of the new equation obtained after the transformation of variables, and then formulates and proves three lemmas that play an important role in finding the canonical form of the equation. The classification problems are considered and the corresponding canonical types of equations are found by a new method in four cases in which the equation with partial derivatives of the sixth-order has: 1) six different real characteristics; 2) four different real roots and two complex-conjugate characteristics; 3) two real roots and four different complex-conjugate characteristics; 4) six different complex-conjugate characteristics and, consequently, the corresponding theorem is proved.

Keywords: a sixth-order partial differential equation, hyperbolic differential operator, elliptic differential operator, classification of differential equations, canonical form of differential equations, non-multiple characteristics, multiple characteristics, real characteristics, complex characteristics, equations of characteristics.

2020 Mathematics Subject Classification: 35G05.

Introduction

In order to achieve meaningful outcomes in the study of boundary or initial value problems for partial differential equations, it is essential to begin by identifying the type of equation and deriving its corresponding canonical form. This classification and transformation play a fundamental role in understanding the general properties of the solutions, ensuring the correct formulation of boundary value problems, informing the selection of suitable solution methods, and facilitating the analysis of both direct and inverse problems. Furthermore, in certain cases, establishing the canonical form may enable the derivation of a general solution or the reduction of the order of the equations .

Therefore, the comprehensive classification, identification of the equation type, and derivation of the corresponding canonical form represent a task of great importance in the theory of differential equations, carrying not only theoretical relevance but also practical significance.

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The classification and determination of the canonical forms of second-order partial differential equations are well known. A comprehensive treatment of the classification and canonical form reduction for third- and fourth-order equations was provided in [1] and [2], respectively. Further investigations into fifth-order equations were conducted in [3], while the study presented in [4] addressed the derivation of canonical forms for n-th order partial differential equations involving two independent variables.

A significant number of studies have been devoted to the investigation of boundary value problems for high-order partial differential equations. For example, in [5] and [6], initial-boundary value problems for high even-order partial differential equations are analyzed. In [7], a completely new numerical method is proposed for solving general linear and nonlinear high-order partial differential equations. In [8], an initial-boundary value problem for a high-order partial differential equation in the multidimensional case is studied.

However, to this day, the issues of complete classification and determination of canonical forms of linear partial differential equations of sixth and higher orders remain unstudied. Although sixth-order partial differential equations do indeed arise in applied problems (for example, wave motion in water with surface tension is described by a sixth-order equation [9]), significant research has also been devoted to the study of boundary value problems for sixth-order partial differential equations. For instance, in [10] and [11], sixth-order partial differential equations are analyzed with respect to the Painlevé property and the behavior of their solutions. In [12], the reduction of equations describing orthotropic bodies to a sixth-order partial differential equation and its analysis is presented. In [13], a nonlocal inverse boundary value problem for a sixth-order partial differential equation with additional integral conditions is investigated.

Therefore, the wide range of applications involving sixth-order partial differential equations underscores the need for a comprehensive investigation into their full classification and reduction to canonical forms.

It should be noted that the classification and determination of canonical forms of partial differential equations are carried out based on the classification of the roots of the corresponding algebraic equations. As the order of the equation increases or the number of independent variables grows, the problems of classification and reduction to canonical form become increasingly complex. The complete classification of second-order partial differential equations and the determination of their corresponding canonical forms have been studied in three cases; for third-order equations — in four cases; and for fourth- and fifth-order partial differential equations — in nine and twelve cases, respectively.

Based on the above analysis, it can be concluded that the classification of sixth-order linear partial differential equations is fundamentally influenced by the quantity and multiplicity of real and complex roots of the corresponding sixth-degree algebraic equations.

For sixth-degree linear algebraic equations, one of the following scenarios invariably applies:

1) six distinct real roots;
2) four distinct real roots accompanied by one pair of complex conjugates;
3) two distinct real roots along with two distinct pairs of complex conjugates;
4) three distinct pairs of complex conjugate roots;
5) one double real root plus four distinct real roots;
6) two double real roots and two distinct real roots;
7) three double real roots;
8) one double root, one triple root, and one simple real root;
9) one triple root together with three distinct real roots;
10) two triple real roots;
11) one double root and one quadruple real root;
12) one quadruple root with two distinct real roots;
13) one quintuple root alongside one simple real root;

14) one sextuple real root;
15) one double root, two distinct real roots, and one pair of complex conjugates;
16) one double real root and two distinct pairs of complex conjugates;
17) one double real root and one double pair of complex conjugates;
18) one triple root, one simple real root, and one pair of complex conjugates;
19) one quadruple real root and one pair of complex conjugates;
20) two distinct real roots and two distinct double pairs of complex conjugates;
21) two distinct pairs of complex conjugates plus two distinct double pairs of complex conjugates;
22) two distinct double real roots and one pair of complex conjugates;
23) three distinct double pairs of complex conjugate roots.

Consequently, the comprehensive classification and reduction to canonical form of sixth-order equations can be systematically explored through exactly 23 distinct cases, each corresponding to one of the possible root structures of sixth-degree algebraic equations.

In this study, within the scope of the article, we focus on the classification and reduction to canonical form of sixth-order linear partial differential equations possessing non-multiple characteristics.

1 Main part

In some domain Ω of the plane xOy, we consider the sixth-order partial differential equation with two independent variables, linear with respect to the highest derivatives:

$$L[u]=\sum_{k=0}^{6} A_k \frac{\partial^6 u}{\partial x^{6-k}\partial y^k}=F, \tag{1}$$

where A_k $(k=\overline{0,6})$ are given constants, and F is a continuous function depending on x, y, u and its partial derivatives with respect to x, y up to the fifth order inclusive, where $\sum\limits_{k=0}^{6} A_k^2 \neq 0$.

Using the transformation of variables $\xi=\xi(x,y)$, $\eta=\eta(x,y)$, allowing for the inverse transformation, that is, fulfilling condition $J=\xi_x\eta_y-\xi_y\eta_x\neq 0$, from (1), we obtain

$$M\left[u\right]=\sum_{k=0}^{6} a_k \frac{\partial^6 u}{\partial \xi^{6-k}\partial \eta^k}=F_1, \tag{2}$$

where F_1 is a function depending on ξ, η, u and its partial derivatives with respect to ξ, η up to the fifth order inclusive, and a_k are new coefficients that are linearly dependent on A_k, $k=\overline{0,6}$.

Taking into account the notation

$$f\left(z_x,z_y\right)=A_0z_x^6+A_1z_x^5z_y+A_2z_x^4z_y^2+A_3z_x^3z_y^3+A_4z_x^2z_y^4+A_5z_xz_y^5+A_6z_y^6,$$

the coefficients a_k $(k=\overline{0,6})$ of equation (2) can be written as

$$a_k=\frac{1}{k!}\left(\eta_x\frac{\partial}{\partial\xi_x}+\eta_y\frac{\partial}{\partial\xi_y}\right)^k f\left(\xi_x,\xi_y\right)\equiv\frac{1}{(6-k)!}\left(\xi_x\frac{\partial}{\partial\eta_x}+\xi_y\frac{\partial}{\partial\eta_y}\right)^{6-k} f\left(\eta_x,\eta_y\right). \tag{3}$$

Let us choose variables ξ and η such that equation (1) has a canonical form and so that the largest coefficients of equation (2) vanish. Since, from formula (3) it is clear that all coefficients of equation (2) are related to the function $f\left(z_x,z_y\right)$ and its partial derivatives with respect to the arguments, we will consider an equation with partial derivatives of the first order:

$$A_0z_x^6+A_1z_x^5z_y+A_2z_x^4z_y^2+A_3z_x^3z_y^3+A_4z_x^2z_y^4+A_5z_xz_y^5+A_6z_y^6=0. \tag{4}$$

Let $z = \varphi(x,y)$ be a particular solution of this equation. If we set $\xi = \varphi(x,y)$, then the coefficient a_0 will obviously be equal to zero. Thus, the above-mentioned problem of choosing new independent variables will be related to the solution of equation (4), and the solution of equation (4) is related by the general integral of the following ordinary differential equation

$$A_0(dy)^6 - A_1(dy)^5dx + A_2(dy)^4(dx)^2 - A_3(dy)^3(dx)^3 + A_4(dy)^2(dx)^4 - A_5dy(dx)^5 + A_6(dx)^6 = 0. \quad (5)$$

Equation (5) is called characteristics equation for equation (1), and its integrals are called characteristics. Dividing both parts of (5) by $(dx)^6$ and introducing the notation $t = dy/dx$, we have the following algebraic equation

$$A_0t^6 - A_1t^5 + A_2t^4 - A_3t^3 + A_4t^2 - A_5t + A_6 = 0. \quad (6)$$

Considering $t = dy/dx$, we can see that finding the general integral of the ordinary differential equation (5) is connected with the roots (algebraic with respect to t ($t = dy/dx$) of the equation (6).

Similarly, as in [4], we will prove the following three lemmas, which play an important role in finding the canonical form of equation (1):

Lemma 1. If the function $z = \varphi(x,y)$ is a solution to equation (4), then the relation $\varphi(x,y) = \text{const}$ is a general integral of the ordinary differential equation (5).

Proof. Since the function $z = \varphi(x,y)$ is a solution to equation (4), then the equality

$$A_0\varphi_x^6 + A_1\varphi_x^5\varphi_y + A_2\varphi_x^4\varphi_y^2 + A_3\varphi_x^3Z_y^3 + A_4\varphi_x^2\varphi_y^4 + A_5\varphi_x\varphi_y^5 + A_6\varphi_y^6 = 0$$

is an identity in the domain where the solution is considered. Dividing both sides of the last equation by φ_y^6, we obtain the following identity:

$$A_0\left(-\frac{\varphi_x}{\varphi_y}\right)^6 - A_1\left(-\frac{\varphi_x}{\varphi_y}\right)^5 + A_2\left(-\frac{\varphi_x}{\varphi_y}\right)^4 - A_3\left(-\frac{\varphi_x}{\varphi_y}\right)^3 + A_4\left(-\frac{\varphi_x}{\varphi_y}\right)^2 - A_5\left(-\frac{\varphi_x}{\varphi_y}\right) + A_6 = 0. \quad (7)$$

It is known that if a function y, determined from an implicit relation $\varphi(x,y) = \text{const}$, satisfies equation (5), then $\varphi(x,y) = \text{const}$ is a general integral of the ordinary differential equation (5). Let $y = f(x,C)$ be this function. Then

$$\frac{dy}{dx} = -\left[\frac{\varphi_x(x,y)}{\varphi_y(x,y)}\right]_{y=f(x,C)}. \quad (8)$$

Here, the square brackets and the index $y = f(x,C)$ indicate that on the righthand side of equality (8) the variable y is not an independent variable, but has a value equal to $f(x,C)$. It follows that $y = f(x,C)$ satisfies equation (5), since

$$A_0\left(\frac{dy}{dx}\right)^6 - A_1\left(\frac{dy}{dx}\right)^5 + A_2\left(\frac{dy}{dx}\right)^4 - A_3\left(\frac{dy}{dx}\right)^3 + A_4\left(\frac{dy}{dx}\right)^2 - A_5\left(\frac{dy}{dx}\right) + A_6 =$$

$$= \Bigg[A_0\left(-\frac{\varphi_x}{\varphi_y}\right)^6 - A_1\left(-\frac{\varphi_x}{\varphi_y}\right)^5 + A_2\left(-\frac{\varphi_x}{\varphi_y}\right)^4 - A_3\left(-\frac{\varphi_x}{\varphi_y}\right)^3 +$$

$$+A_4\left(-\frac{\varphi_x}{\varphi_y}\right)^2 - A_5\left(-\frac{\varphi_x}{\varphi_y}\right) + A_6\Bigg]_{y=f(x,C)} = 0,$$

by virtue of (7) the expression in square brackets is equal to zero for all values of x, y, and not only for $y = f(x,C)$.

Lemma 2. If $\varphi(x, y) = \text{const}$ is a k-fold ($k \leq 6$) general integral of equation (5), then for $z = \varphi(x, y)$ the function $f(z_x, z_y)$ and all its derivatives with respect to z_x, z_y up to and including $(k-1)$ order are equal to zero.

Proof. Let $\varphi(x, y) = \text{const}$ be a k-fold general integral of equation (5), and $t_1, t_2, \ldots, t_6$ be the roots of equation (6), where t_1 ($t_1 = -\varphi_x/\varphi_y$) is the corresponding k-fold root of equation (6). Then, based on the corollary of Bezout's theorem, equation (6) can be written in the form

$$A_0 (t - t_1)^k \prod_{j=k+1}^{6} (t - t_j) = 0. \tag{9}$$

If we consider $t = dy/dx$, the equation (9) takes the form

$$A_0 (dy - t_1 dx)^k \prod_{j=k+1}^{6} (dy - t_j dx) = 0.$$

Taking this into account, the function $f(z_x, z_y)$ and the equation (4) can be written as $f(z_x, z_y) = A_0 (z_x + t_1 z_y)^k \prod\limits_{j=k+1}^{6} (z_x + t_j z_y)$ and $A_0 (z_x + t_1 z_y)^k \prod\limits_{j=k+1}^{6} (z_x + t_j z_y) = 0$, respectively. Therefore, for $z = \varphi(x, y)$, we have

$$f(z_x, z_y) = A_0 (z_x + t_1 z_y)^k \prod_{j=k+1}^{6} (z_x + t_j z_y) = 0.$$

It easily follows from this that all derivatives of the function $f(z_x, z_y)$ with respect to z_x, z_y up to $(k-1)$ order inclusive for $z = \varphi(x, y)$ are equal to zero.

Lemma 3. When transforming variables $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ that allow inverse transformation, the number and multiplicity of real and complex roots of equation (6) are invariant, and the identity holds $\tilde{D}_6 = J^{30} D_6$, where

$$D_6 = A_0^{10} \prod_{6 \geq i > j \geq 1} (t_i - t_j)^2 \tag{10}$$

is the discriminant of the equation (6), and

$$\tilde{D}_6 = a_0^{10} \prod_{6 \geq i > j \geq 1} (\mu_i - \mu_j)^2 \tag{11}$$

is the discriminant of the following equation

$$a_0 \mu^6 - a_1 \mu^5 + a_2 \mu^4 - a_3 \mu^3 + a_4 \mu^2 - a_5 \mu + a_6 = 0 \quad (\mu = d\eta/d\xi), \tag{12}$$

where $t_1, t_2, \ldots, t_6$ and $\mu_1, \mu_2, \ldots, \mu_6$ are the roots of the equations (6) and (12), respectively.

Proof. As shown above, when transforming the variables $\xi = \xi(x, y)$, $\eta = \eta(x, y)$, the equation (1) with the condition $J = \xi_x \eta_y - \xi_y \eta_x \neq 0$, was transformed into equation (2). By introducing the notation $t = dy/dx$, $\mu = d\eta/d\xi$ into the equations of the characteristics for equations (1) and (2), algebraic equations (6) and (12) were found, respectively. Then, taking into account $t = dy/dx$ and $\mu = d\eta/d\xi$, we have the following relation

$$\mu = \frac{d\eta(x, y)}{d\xi(x, y)} = \frac{\eta_x dx + \eta_y dy}{\xi_x dx + \xi_y dy} = \frac{\eta_x + \eta_y (dy/dx)}{\xi_x + \xi_y (dy/dx)} = \frac{\eta_x + \eta_y t}{\xi_x + \xi_y t}. \tag{13}$$

From (13), we find the relation between the roots μ_i and t_i of the equations (6) and (12) in the form $\mu_i = (\eta_x + \eta_y \cdot t_i) / (\xi_x + \xi_y \cdot t_i)$. It follows that the number and multiplicity of the real and complex roots of equations (6) and (12) are the same. That is, when transforming variables $\xi = \xi(x,y), \eta = \eta(x,y)$, allowing for an inverse transformation, the number and multiplicity of the real and complex roots of equation (6) are invariant. In addition, we have

$$\mu_k - \mu_j = J(t_k - t_j)\left[(\xi_x + t_k\xi_y)(\xi_x + t_j\xi_y)\right]^{-1}, \quad k,j = \overline{1,6}.$$

Using these equalities, from (11), we find

$$\tilde{D} = a_0^{10} J^{30} \left[(\xi_x + t_1\xi_y) \cdot (\xi_x + t_2\xi_y) \cdot \ldots \cdot (\xi_x + t_6\xi_y)\right]^{-10} \prod_{6 \geq k > j \geq 1} (t_k - t_j)^2 .$$

From here, opening the brackets inside the square bracket and taking into account equality (10), we obtain

$$\tilde{D} = a_0^{10} J^{30} \cdot D\left[\xi_x^6 + (t_1 + t_2 + \ldots + t_6) \cdot \xi_x^5 \xi_y + \ldots + t_1 \cdot t_2 \cdot \ldots \cdot t_6 \xi_y^6\right]^{-10} A_0^{-10}. \tag{14}$$

On the other hand, according to Vieta's formulas, the following equalities hold:

$$t_1 + t_2 + \ldots + t_6 = \frac{A_1}{A_0}, \quad t_1t_2 + t_1t_3 + \ldots + t_5t_6 = \frac{A_2}{A_0}, \ldots, t_1 \cdot \ldots \cdot t_6 = \frac{A_6}{A_0}. \tag{15}$$

Based on (15), equality (14) takes the form

$$\tilde{D} = a_0^{10} J^{30} D \left(A_0 \xi_x^6 + A_1 \xi_x^5 \xi_y + \ldots + A_6 \xi_y^6\right)^{-10}.$$

Since, according to formula (3), $a_0 = A_0\xi_x^6 + A_1\xi_x^5\xi_y + \ldots + A_6\xi_y^6$, then from the latter it follows that $\tilde{D} = J^{30}D$. From this equality, by virtue of $J \neq 0$, it follows that when transforming variables, the sign of the discriminant D is invariant.

Without loss of generality, we can assume [4] that condition $A_0 > 0$ is also satisfied.

As is well established from the corollary to the Fundamental Theorem of Algebra, any polynomial of degree n over the field of complex numbers possesses exactly n roots, counted with their multiplicities. Accordingly, equation (6) has exactly six roots — real and/or complex conjugates — taking multiplicities into account.

Given that the algebraic equation (6) presents 23 possible root configurations, the corresponding partial differential equation (1) may be analyzed in all these cases. Nevertheless, owing to limitations of space, the present study will concentrate solely on the four cases in which equation (6) has exclusively simple (non-repeated) roots.

1. Let equation (6) have six different real roots $t_1 = \lambda_1$, $t_2 = \lambda_2$, $t_3 = \lambda_3$, $t_4 = \lambda_4$, $t_5 = \lambda_5$, $t_6 = \lambda_6$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 > \lambda_6$. Then, equation (5) has six different real general integrals:

$$\Psi_1(x,y) = y - \lambda_1 x = \text{const}, \quad \Psi_2(x,y) = y - \lambda_2 x = \text{const}, \quad \Psi_3(x,y) = y - \lambda_3 x = \text{const},$$

$$\Psi_4(x,y) = y - \lambda_4 x = \text{const}, \quad \Psi_5(x,y) = y - \lambda_5 x = \text{const}, \quad \Psi_6(x,y) = y - \lambda_6 x = \text{const}.$$

If we take into account (15), then equation (1) can be written as:

$$A_0 \left[\frac{\partial^6 u}{\partial x^6} + (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) \frac{\partial^6 u}{\partial x^5 \partial y} + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_5\lambda_6) \frac{\partial^6 u}{\partial x^4 \partial y^2} + \ldots \right.$$

$$\left. \ldots + (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6) \frac{\partial^6 u}{\partial y^6}\right] = F.$$

Using first-order differential operators of the form $\left(\frac{\partial}{\partial x}+\lambda_i\frac{\partial}{\partial y}\right)$, the last equation can be formally written as:

$$A_0\left[\prod_{k=1}^{6}\left(\frac{\partial}{\partial x}+\lambda_k\frac{\partial}{\partial y}\right)\right]u=F. \tag{16}$$

By introducing the following notations $\frac{(\lambda_1-\lambda_5)}{(\lambda_1-\lambda_6)}=\mu_1$, $\frac{(\lambda_2-\lambda_5)}{(\lambda_2-\lambda_6)}=\mu_2$, $\frac{(\lambda_3-\lambda_5)}{(\lambda_3-\lambda_6)}=\mu_3$, $\frac{(\lambda_4-\lambda_5)}{(\lambda_4-\lambda_6)}=\mu_4$, let us change the variables by

$$\xi=\left(1+\sqrt{\mu_3\mu_4}\right)y-\left(\lambda_5+\lambda_6\sqrt{\mu_3\mu_4}\right)x,\quad \eta=\left(1-\sqrt{\mu_3\mu_4}\right)y-\left(\lambda_5-\lambda_6\sqrt{\mu_3\mu_4}\right)x. \tag{17}$$

Then, taking (17) into account, we have

$$\frac{\partial}{\partial x}=\xi_x\frac{\partial}{\partial\xi}+n_x\frac{\partial}{\partial\eta}=-\left(\lambda_5+\lambda_6\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial\xi}-\left(\lambda_5-\lambda_6\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial\eta},$$
$$\frac{\partial}{\partial y}=\xi_y\frac{\partial}{\partial\xi}+n_y\frac{\partial}{\partial\eta}=\left(1+\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial\xi}+\left(1-\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial\eta}.$$

Substituting these expressions of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ into the equation (16), we obtain

$$\left[\left(\lambda_1-\lambda_5+\sqrt{\mu_3\mu_4}\left(\lambda_1-\lambda_6\right)\right)\frac{\partial}{\partial\xi}+\left(\lambda_1-\lambda_5-\sqrt{\mu_3\mu_4}\left(\lambda_1-\lambda_6\right)\right)\frac{\partial}{\partial\eta}\right]\times$$
$$\times\left[\left(\lambda_2-\lambda_5+\sqrt{\mu_3\mu_4}\left(\lambda_2-\lambda_6\right)\right)\frac{\partial}{\partial\xi}+\left(\lambda_2-\lambda_5-\sqrt{\mu_3\mu_4}\left(\lambda_2-\lambda_6\right)\right)\frac{\partial}{\partial\eta}\right]\times$$
$$\times\left[\left(\lambda_3-\lambda_5+\sqrt{\mu_3\mu_4}\left(\lambda_3-\lambda_6\right)\right)\frac{\partial}{\partial\xi}+\left(\lambda_3-\lambda_5-\sqrt{\mu_3\mu_4}\left(\lambda_3-\lambda_6\right)\right)\frac{\partial}{\partial\eta}\right]\times$$
$$\times\left[\left(\lambda_4-\lambda_5+\sqrt{\mu_3\mu_4}\left(\lambda_4-\lambda_6\right)\right)\frac{\partial}{\partial\xi}+\left(\lambda_4-\lambda_5-\sqrt{\mu_3\mu_4}\left(\lambda_4-\lambda_6\right)\right)\frac{\partial}{\partial\eta}\right]\times$$
$$\times\left[\sqrt{\mu_3\mu_4}\left(\lambda_5-\lambda_6\right)\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right)\right]\left[\left(\lambda_6-\lambda_5\right)\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\right]u=F_1.$$

Let us divide both sides of the last equation by

$$-\sqrt{\mu_3\mu_4}\left(\lambda_1-\lambda_6\right)\left(\lambda_2-\lambda_6\right)\left(\lambda_3-\lambda_6\right)\left(\lambda_4-\lambda_6\right)\left(\lambda_5-\lambda_6\right)^2(\neq 0).$$

Then, we have

$$\left[\left(\mu_1+\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial\xi}+\left(\mu_1-\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial\eta}\right]\times\left[\left(\mu_2+\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial\xi}+\left(\mu_2-\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial\eta}\right]\times$$
$$\times\sqrt{\mu_3}\left[\left(\sqrt{\mu_3}+\sqrt{\mu_4}\right)\frac{\partial}{\partial\xi}+\left(\sqrt{\mu_3}-\sqrt{\mu_4}\right)\frac{\partial}{\partial\eta}\right]\times\sqrt{\mu_4}\left[\left(\sqrt{\mu_4}+\sqrt{\mu_3}\right)\frac{\partial}{\partial\xi}+\left(\sqrt{\mu_4}-\sqrt{\mu_3}\right)\frac{\partial}{\partial\eta}\right]\times$$
$$\times\left(\frac{\partial^2u}{\partial\xi^2}-\frac{\partial^2u}{\partial n^2}\right)=F_2, \tag{18}$$

where $F_2=F_1/\left\{-\sqrt{\mu_3\mu_4}\left(\lambda_1-\lambda_6\right)\left(\lambda_2-\lambda_6\right)\left(\lambda_3-\lambda_6\right)\left(\lambda_4-\lambda_6\right)\left(\lambda_5-\lambda_6\right)^2\right\}$.

And the equation (18) can be rewritten as:

$$\left[\frac{\partial}{\partial\xi}+\frac{\left(\mu_1-\sqrt{\mu_3\mu_4}\right)}{\left(\mu_1+\sqrt{\mu_3\mu_4}\right)}\frac{\partial}{\partial\eta}\right]\times\left[\frac{\partial}{\partial\xi}+\frac{\left(\mu_2-\sqrt{\mu_3\mu_4}\right)}{\left(\mu_2+\sqrt{\mu_3\mu_4}\right)}\frac{\partial}{\partial\eta}\right]\times$$

$$\times\left[\frac{\partial^2}{\partial\xi^2}-\frac{\left(\sqrt{\mu_3}-\sqrt{\mu_4}\right)^2}{\left(\sqrt{\mu_3}+\sqrt{\mu_4}\right)^2}\frac{\partial^2}{\partial\eta^2}\right]\times\left(\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\eta^2}\right)=F_3$$

or

$$\left(\frac{\partial}{\partial\xi}+c_1\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+c_2\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^2}{\partial\xi^2}-b^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\eta^2}\right)=F_3,$$

where $c_1=\dfrac{\left(\mu_1-\sqrt{\mu_3\mu_4}\right)}{\left(\mu_1+\sqrt{\mu_3\mu_4}\right)},\quad c_2=\dfrac{\left(\mu_2-\sqrt{\mu_3\mu_4}\right)}{\left(\mu_2+\sqrt{\mu_3\mu_4}\right)},\quad b^2=\dfrac{\left(\sqrt{\mu_3}-\sqrt{\mu_4}\right)^2}{\left(\sqrt{\mu_3}+\sqrt{\mu_4}\right)^2},$
$F_3=F_2/\left\{\sqrt{\mu_3\mu_4}\left(\mu_1+\sqrt{\mu_3\mu_4}\right)\left(\mu_2+\sqrt{\mu_3\mu_4}\right)\left(\sqrt{\mu_3}+\sqrt{\mu_4}\right)^2\right\}.$

Example 1. Consider the following sixth-order partial differential equation:

$$u_{xxxxxx}+6u_{xxxxxy}-10u_{xxxxyy}-100u_{xxxyyy}-111u_{xxyyyy}+94u_{xyyyyy}+120u_{yyyyyy}=0. \tag{19}$$

The characteristic equation corresponding to the equation (19) has the form

$$(dy)^6-6(dy)^5(dx)-10(dy)^4(dx)^2+100(dy)^3(dx)^3-111(dy)^2(dx)^4-94(dy)(dx)^5+120(dx)^6=0.$$

It is easy to verify that this equation has six different real roots for $t=dy/dx$:

$$t_1=1,\ t_2=-1,\ t_3=2,\ t_4=3,\ t_5=-4,\ t_6=5.$$

Then, equation (19) can be written as follows:

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-4\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+5\frac{\partial}{\partial y}\right)u=0. \tag{20}$$

After the transformation $\xi=(1+\sqrt{7})y+(4-5\sqrt{7})x$, $\eta=(1-\sqrt{7})y+(4+5\sqrt{7})x$, we obtain $\frac{\partial}{\partial x}=\left(4-5\sqrt{7}\right)\frac{\partial}{\partial\xi}+\left(4+5\sqrt{7}\right)\frac{\partial}{\partial\eta}$, $\frac{\partial}{\partial y}=\left(1+\sqrt{7}\right)\frac{\partial}{\partial\xi}+\left(1-\sqrt{7}\right)\frac{\partial}{\partial\eta}$. Considering these, from (20), we have

$$\left(\frac{\partial}{\partial\xi}+c_1\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+c_2\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^2}{\partial\xi^2}-b^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\eta^2}\right)=0,$$

where $c_1=\frac{(5+4\sqrt{7})}{(5-4\sqrt{7})}$, $c_2=\frac{(3+6\sqrt{7})}{(3-6\sqrt{7})}$, $b^2=\left(\frac{28+11\sqrt{7}}{3\sqrt{7}}\right)^2$.

2. Let the equation (6) have four different real roots $t_1=\lambda_1$, $t_2=\lambda_2$, $t_3=\lambda_3$, $t_4=\lambda_4$ and two complex conjugate roots $t_5=\alpha+\beta i$, $t_6=\alpha-\beta i$. Then the equation (5) has four different real and two different complex conjugate general integrals:

$\Psi_1(x,y)=y-\lambda_1 x=\text{const}$, $\Psi_2(x,y)=y-\lambda_2 x=\text{const}$, $\Psi_3(x,y)=y-\lambda_3 x=\text{const}$,

$\Psi_4(x,y)=y-\lambda_4 x=\text{const}$, $\varphi(x,y)=y-\alpha x-i\beta x=\text{const}$, $\varphi^*(x,y)=y-\alpha x+i\beta x=\text{const}$, where $\alpha,\beta\in R$ and $\beta\neq 0$.

If we take into account (15), then the equation (1) can be written as:

$$A_0\left[\frac{\partial^6 u}{\partial x^6}+(\lambda_1+\lambda_2+\lambda_3+\lambda_4+t_5+t_6)\frac{\partial^6 u}{\partial x^5\partial y}+(\lambda_1\lambda_2+\lambda_1\lambda_3+\ldots+t_5t_6)\frac{\partial^6 u}{\partial x^4\partial y^2}+\ldots\right.$$

$$\left....+(\lambda_1\lambda_2\lambda_3\lambda_4t_5t_6)\frac{\partial^6 u}{\partial y^6}\right]=F.$$

Hence, similarly to equation (16), we have

$$A_0\left\{\prod_{k=1}^{4}\left(\frac{\partial}{\partial x}+\lambda_k\frac{\partial}{\partial y}\right)\right\}\left(\frac{\partial^2 u}{\partial x^2}+2\alpha\frac{\partial^2 u}{\partial x\partial y}+\left(\alpha^2+\beta^2\right)\frac{\partial^2 u}{\partial y^2}\right)=F. \tag{21}$$

Let us change the variables by the following formulas

$$\xi=y-\alpha x,\quad \eta=\beta x. \tag{22}$$

Then $\frac{\partial}{\partial x}=\xi_x\frac{\partial}{\partial \xi}+n_x\frac{\partial}{\partial \eta}=-\alpha\frac{\partial}{\partial \xi}+\beta\frac{\partial}{\partial \eta},\quad \frac{\partial}{\partial y}=\xi_y\frac{\partial}{\partial \xi}+n_y\frac{\partial}{\partial \eta}=\frac{\partial}{\partial \xi}$.

Substituting these expressions of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ into the equation (21), we have

$$A_0\beta^2\left\{\prod_{k=1}^{4}\left[\left(\lambda_k-\alpha\right)\frac{\partial}{\partial \xi}+\beta\frac{\partial}{\partial \eta}\right]\right\}\left(\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}\right)=F_1. \tag{23}$$

Assume that $(\lambda_1-\alpha)(\lambda_2-\alpha)(\lambda_3-\alpha)(\lambda_4-\alpha)\neq 0$. Then, dividing both parts of the equality (23) by $A_0\beta^2(\lambda_1-\alpha)(\lambda_2-\alpha)(\lambda_3-\alpha)(\lambda_4-\alpha)$ and introducing the notation

$$\mu_5=\frac{\beta}{\lambda_1-\alpha},\ \ \mu_6=\frac{\beta}{\lambda_2-\alpha},\ \ \mu_7=\frac{\beta}{\lambda_3-\alpha},\ \ \mu_8=\frac{\beta}{\lambda_4-\alpha},$$

we obtain

$$\left(\frac{\partial}{\partial \xi}+\mu_5\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi}+\mu_6\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi}+\mu_7\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi}+\mu_8\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}\right)=F_4,$$

or

$$\left(\frac{\partial^2}{\partial \xi^2}+(\mu_5+\mu_6)\frac{\partial^2}{\partial \xi\partial \eta}+\mu_5\mu_6\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial \xi^2}+(\mu_7+\mu_8)\frac{\partial^2}{\partial \xi\partial \eta}+\mu_7\mu_8\frac{\partial^2}{\partial \eta^2}\right)\times$$
$$\times\left(\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}\right)=F_4, \tag{24}$$

where $F_4=F_1/\left[A_0\beta^2(\lambda_1-\alpha)(\lambda_2-\alpha)(\lambda_3-\alpha)(\lambda_4-\alpha)\right]$.

If $\mu_5=-\mu_6,\ \mu_7=-\mu_8$, then the equation (24) takes the form

$$\left(\frac{\partial^2}{\partial \xi^2}-\mu_5^2\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial \xi^2}-\mu_7^2\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}\right)=F_4.$$

Let $\mu_5\neq-\mu_6,\ \mu_7\neq-\mu_8$, then $\frac{\partial^2}{\partial \xi^2}+(\mu_5+\mu_6)\frac{\partial^2}{\partial \xi\partial \eta}+\mu_5\mu_6\frac{\partial^2}{\partial \eta^2}$ and $\frac{\partial^2}{\partial \xi^2}+(\mu_7+\mu_8)\frac{\partial^2}{\partial \xi\partial \eta}+$ $+\mu_7\mu_8\frac{\partial^2}{\partial \eta^2}$ are hyperbolic differential operators, since $(\mu_5+\mu_6)^2-4\mu_5\mu_6=(\mu_5-\mu_6)^2>0$ and $(\mu_7+\mu_8)^2-4\mu_7\mu_8=(\mu_7-\mu_8)^2>0$.

To further simplify equation (24), we make a change of variables by

$$s=s(\xi,\eta),\quad t=t(\xi,\eta) \tag{25}$$

and $J=s_\xi t_\eta-s_\eta t_\xi\neq 0$, then $\frac{\partial}{\partial \xi}=s_\xi\frac{\partial}{\partial s}+t_\xi\frac{\partial}{\partial t},\frac{\partial}{\partial \eta}=s_\eta\frac{\partial}{\partial s}+t_\eta\frac{\partial}{\partial t}$. Taking this into account, from equation (24), we obtain a new equation in the following form

$$\left((s_\xi+\mu_5 s_\eta)\frac{\partial}{\partial s}+(t_\xi+\mu_5 t_\eta)\frac{\partial}{\partial t}\right)\left((s_\xi+\mu_6 s_\eta)\frac{\partial}{\partial s}+(t_\xi+\mu_6 t_\eta)\frac{\partial}{\partial t}\right)\times$$

$$\times\Bigg(\left(s_\xi^2+(\mu_7+\mu_8)\,s_\xi s_\eta+\mu_7\mu_8 s_\eta^2\right)\frac{\partial^2}{\partial s^2}+\left(t_\xi^2+(\mu_7+\mu_8)\,t_\xi t_\eta+\mu_7\mu_8 t_\eta^2\right)\frac{\partial^2}{\partial t^2}+$$

$$+\left(2\left(s_\xi t_\xi+\mu_7\mu_8 s_\eta t_\eta\right)+(\mu_7+\mu_8)\left(s_\xi t_\eta+s_\eta t_\xi\right)\right)\frac{\partial^2}{\partial s\partial t}\Bigg)\times$$

$$\times\left(\left(s_\xi^2+s_\eta^2\right)\frac{\partial^2 u}{\partial s^2}+2\left(s_\xi t_\xi+s_\eta t_\eta\right)\frac{\partial^2 u}{\partial s\partial t}+\left(t_\xi^2+t_\eta^2\right)\frac{\partial^2 u}{\partial t^2}\right)=F_5, \tag{26}$$

where F_5 is a function depending on s,t,u and its partial derivatives with respect to s,t up to the fifth order inclusive.

To make equation (26) simpler, we take a replacement for (25) as

$$s=\eta+\mu_0\xi,\quad t=\mu_0\eta-\xi, \tag{27}$$

where μ_0 is one of two solutions of the equation

$$\mu^2-\frac{2\,(1-\mu_7\mu_8)}{\mu_7+\mu_8}\mu-1=0, \tag{28}$$

that is, $\mu_0=\dfrac{1-\mu_7\mu_8}{\mu_7+\mu_8}+\sqrt{\dfrac{(1-\mu_7\mu_8)^2}{(\mu_7+\mu_8)^2}+1}>0$ or $\mu_0=\dfrac{1-\mu_7\mu_8}{\mu_7+\mu_8}-\sqrt{\dfrac{(1-\mu_7\mu_8)^2}{(\mu_7+\mu_8)^2}+1}<0$. In this case, $s_\xi=t_n=\mu_0, s_n=-t_\xi=1$ and therefore $J=s_\xi t_n-s_n t_\xi=\mu_0^2+1\neq 0$. In addition, the equalities $2(s_\xi t_\xi+\mu_7\mu_8 s_\eta t_\eta)+(\mu_7+\mu_8)(s_\xi t_\eta+s_\eta t_\xi)=2(-\mu_0+\mu_7\mu_8\mu_0)+(\mu_7+\mu_8)\times(\mu_0^2-$ $-1)=\left[\mu^2-\dfrac{2(1-\mu_7\mu_8)}{\mu_7+\mu_8}\mu-1\right](\mu_7+\mu_8)=0$, $s_\xi t_\xi+s_\eta t_\eta=-\mu_0+\mu_0=0$ are valid. If we take into account these equalities, then equation (26) takes the form

$$\left((\mu_0+\mu_5)\frac{\partial}{\partial s}+(-1+\mu_5\mu_0)\frac{\partial}{\partial t}\right)\left((\mu_0+\mu_6)\frac{\partial}{\partial s}+(-1+\mu_6\mu_0)\frac{\partial}{\partial t}\right)\times$$

$$\times\left[\left(\mu_0^2+(\mu_7+\mu_8)\,\mu_0+\mu_7\mu_8\right)\frac{\partial^2}{\partial s^2}+\left(1-(\mu_7+\mu_8)\,\mu_0+\mu_7\mu_8\mu_0^2\right)\frac{\partial^2}{\partial t^2}\right]\times$$

$$\times\left(\left(\mu_0^2+1\right)\frac{\partial^2 u}{\partial s^2}+\left(1+\mu_0^2\right)\frac{\partial^2 u}{\partial t^2}\right)=F_5.$$

Then, dividing both parts and both sides of the last equation by $\left(\mu_0^2+1\right)(\mu_0+\mu_5)(\mu_0+\mu_6)(\mu_0+\mu_7)(\mu_0+\mu_8)$ and introducing the notations $c_3=\dfrac{(\mu_5\mu_0-1)}{(\mu_0+\mu_5)}$, $c_4=\dfrac{(\mu_6\mu_0-1)}{(\mu_0+\mu_6)}$, $b_1^2=\dfrac{(1-\mu_7\mu_0)(\mu_8\mu_0-1)}{(\mu_0+\mu_7)(\mu_0+\mu_8)}$, we come

$$\left(\frac{\partial}{\partial s}+c_3\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s}+c_4\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial s^2}-b_1^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_6, \tag{29}$$

where $F_6=F_5/\left[\left(\mu_0^2+1\right)(\mu_0+\mu_5)(\mu_0+\mu_6)(\mu_0+\mu_7)(\mu_0+\mu_8)\right]$.

Let us prove that $b_1^2>0$. Introducing the notations $\nu_1=\dfrac{\mu_0\mu_7-1}{\mu_0+\mu_7}$, $\nu_2=\dfrac{\mu_0\mu_8-1}{\mu_0+\mu_8}$ and taking into account that μ_0 is one of the two solutions of equation (28), that is, $\mu_0^2+\mu_0\dfrac{2\,(\mu_7\mu_8-1)}{(\mu_7+\mu_8)}-$ $-1=0$, we have $\nu_1+\nu_2=\dfrac{\mu_0\mu_7-1}{\mu_0+\mu_7}+\dfrac{\mu_0\mu_8-1}{\mu_0+\mu_8}=\dfrac{\mu_0^2+\mu_0\dfrac{2\,(\mu_7\mu_8-1)}{(\mu_7+\mu_8)}-1}{(\mu_0+\mu_7)(\mu_0+\mu_8)}(\mu_7+\mu_8)=0$. Then $b_1^2=-\dfrac{(\mu_0\mu_7-1)(\mu_0\mu_8-1)}{(\mu_0+\mu_7)(\mu_0+\mu_8)}=-\nu_1\nu_2=\dfrac{1}{4}\left[(\nu_1+\nu_2)^2-4\nu_1\nu_2\right]=\dfrac{1}{4}(\nu_1-\nu_2)^2>0.$

It is easy to verify that if one of the expressions $\lambda_k - \alpha (k = 1, 2, 3, 4)$ is equal to zero, that is, for example $\lambda_1 - \alpha = 0$, then the equation (23) takes the form

$$\left\{\prod_{k=2}^{4}\left[(\lambda_k - \alpha)\frac{\partial}{\partial\xi} + \beta\frac{\partial}{\partial\eta}\right]\right\}\left(\frac{\partial^3 u}{\partial\xi^2\partial\eta} + \frac{\partial^3 u}{\partial\eta^3}\right) = F_1/\left(A_0\beta^3\right)$$

and this equation, as in case (29), after changing variables by $s = \eta + \mu_0\xi, t = \mu_0\eta - \xi$, can be brought to the form

$$\left(\frac{\partial}{\partial s} + \mu_0\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s} + c_4\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial s^2} - b_1^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right) = F_7,$$

where $F_7 = F_1/\left[A_0\beta^3(\lambda_2 - \alpha)(\lambda_3 - \alpha)(\lambda_4 - \alpha)\left(\mu_0^2 + 1\right)(\mu_0 + \mu_6)(\mu_0 + \mu_7)(\mu_0 + \mu_8)\right]$.

Example 2. Consider the following sixth-order partial differential equation:

$$u_{xxxxxx} + 24u_{xxxxxy} + 239u_{xxxxyy} + 1264u_{xxxyyy} + 3663u_{xxyyyy} + 5240u_{xyyyyy} + 2625u_{yyyyyy} = 0. \quad (30)$$

The characteristic equation corresponding to the equation (30) has the form

$$(dy)^6 - 24(dy)^5(dx) + 239(dy)^4(dx)^2 - 1264(dy)^3(dx)^3 + 3663(dy)^2(dx)^4 - 5240(dy)(dx)^5 + 2625(dx)^6 = 0.$$

It is easy to verify that this equation has four different real roots and two complex conjugate roots for $t = dy/dx$:

$$t_1 = 5,\ t_2 = 3,\ t_3 = 1,\ t_4 = 7,\ t_5 = 4 + 3i,\ t_6 = 4 - 3i.$$

Then, equation (30) can be written as follows:

$$\left(\frac{\partial}{\partial x} + 5\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + 7\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial x^2} + 8\frac{\partial^2}{\partial x\partial y} + 25\frac{\partial^2}{\partial y^2}\right)u = 0. \quad (31)$$

After the transformation $\xi = y - 4x$, $\eta = 3x$, we obtain: $\frac{\partial}{\partial x} = -4\frac{\partial}{\partial\xi} + 3\frac{\partial}{\partial\eta}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial\xi}$. Considering these from (31), we have

$$\left(\frac{\partial^2}{\partial\xi^2} - 9\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)u = 0.$$

3. Let equation (6) have two different simple real roots and four complex conjugate roots:

$$t_1 = \lambda_1, t_2 = \lambda_2, t_3 = \delta + \gamma i, t_4 = \delta - \gamma i, t_5 = \alpha + \beta i, t_6 = \alpha - \beta i,$$

where $\lambda_1, \lambda_2, \alpha, \beta, \delta, \gamma \in R$ such that $\beta \neq 0, \gamma \neq 0, (\alpha - \delta)^2 + (|\beta| - |\gamma|)^2 \neq 0$.

Then the equation (5) has one real and four different complex conjugate general integrals

$$\Psi_1(x, y) = y - \lambda_1 x = \text{ const}, \Psi_2(x, y) = y - \lambda_2 x = \text{ const},$$
$$\Psi_3(x, y) = y - \delta x - i\gamma x = \text{ const}, \Psi_4(x, y) = y - \delta x + i\gamma x = \text{ const},$$
$$\varphi(x, y) = y - \alpha x - i\beta x = \text{ const}, \ \varphi^*(x, y) = y - \alpha x + i\beta x = \text{ const}.$$

Using the same reasoning as when obtaining equation (21), equation (1) can be written as

$$A_0\left(\frac{\partial}{\partial x} + \lambda_1\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \lambda_2\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial x^2} + 2\delta\frac{\partial^2}{\partial x\partial y} + \left(\delta^2 + \gamma^2\right)\frac{\partial^2}{\partial y^2}\right) \times$$
$$\times\left(\frac{\partial^2 u}{\partial x^2} + 2\alpha\frac{\partial^2 u}{\partial x\partial y} + \left(\alpha^2 + \beta^2\right)\frac{\partial^2 u}{\partial y^2}\right) = F. \quad (32)$$

Let us consider the substitution (22). Then, from equation (32), similarly to (23), we obtain the equation

$$A_0\beta^2\left[(\lambda_1-\alpha)\frac{\partial}{\partial\xi}+\beta\frac{\partial}{\partial\eta}\right]\left[(\lambda_2-\alpha)\frac{\partial}{\partial\xi}+\beta\frac{\partial}{\partial\eta}\right]\times$$

$$\times\left[\left(\alpha^2+\delta^2+\gamma^2-2\alpha\delta\right)\frac{\partial^2}{\partial\xi^2}+2\beta(\delta-\alpha)\frac{\partial^2}{\partial\xi\partial\eta}+\beta^2\frac{\partial^2}{\partial\eta^2}\right]\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}\right)=F_1. \tag{33}$$

Hence, dividing both parts of (33) by $A_0\beta^2\,(\lambda_1-\alpha)\,(\lambda_2-\alpha)\left[(\alpha-\delta)^2+\gamma^2\right](\neq 0)$ and introducing the notations $\mu_5=\dfrac{\beta}{\lambda_1-\alpha}$, $\mu_6=\dfrac{\beta}{\lambda_2-\alpha}$, $\mu_9=\dfrac{\beta(\delta-\alpha)+\gamma\beta i}{(\alpha-\delta)^2+\gamma^2}=\delta_1+\gamma_1 i$, $\mu_{10}=\dfrac{\beta(\delta-\alpha)-\gamma\beta i}{(\alpha-\delta)^2+\gamma^2}=$ $=\delta_1-\gamma_1 i$, $\dfrac{\beta(\delta-\alpha)}{(\alpha-\delta)^2+\gamma^2}=\delta_1$, $\dfrac{\gamma\beta}{(\alpha-\delta)^2+\gamma^2}=\gamma_1$, we have an equation in the form

$$\left(\frac{\partial^2}{\partial\xi^2}+(\mu_5+\mu_6)\frac{\partial^2}{\partial\xi\partial\eta}+\mu_5\mu_6\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2}{\partial\xi^2}+(\mu_9+\mu_{10})\frac{\partial^2}{\partial\xi\partial\eta}+\mu_9\mu_{10}\frac{\partial^2}{\partial\eta^2}\right)\times$$

$$\times\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}\right)=F_8, \tag{34}$$

where $F_8=F_1/A_0\beta^2\left[(\lambda_1-\alpha)\,(\lambda_2-\alpha)\left((\alpha-\delta)^2+\gamma^2\right)\right]$.

Moreover, since $(\mu_9+\mu_{10})^2-4\,(\mu_9\mu_{10})=(\mu_9-\mu_{10})^2=-\gamma_1^2<0$, then $\frac{\partial^2}{\partial\xi^2}+(\mu_9+\mu_{10})\frac{\partial^2}{\partial\xi\partial\eta}+$ $+\mu_9\mu_{10}\frac{\partial^2}{\partial\eta^2}$ is an elliptic differential operator.

If $\mu_9=-\mu_{10}$, then at $\delta_1=0$ equation (34) takes the form

$$\left(\frac{\partial}{\partial\xi}+\mu_5\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+\mu_6\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^2}{\partial\xi^2}+\gamma_1^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}\right)=F_9,$$

where $F_9=F_1/A_0\beta^2\left[(\lambda_1-\alpha)\,(\lambda_2-\alpha)\,\gamma^2\right]$.

To further simplify equation (34) for $\delta_1\neq 0$, we choose the substitution (27) as a change of variables, where μ_0 is one of the two solutions of equation

$$\mu^2-\frac{2\,(1-\mu_9\mu_{10})}{\mu_9+\mu_{10}}\mu-1=0,$$

then, similarly to case 2, we obtain the equation

$$\left(\frac{\partial}{\partial s}+c_3\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s}+c_4\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial s^2}+b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_{10},$$

where $b_2^2=\dfrac{(\mu_0\mu_9-1)\,(\mu_0\mu_{10}-1)}{(\mu_0+\mu_9)\,(\mu_0+\mu_{10})}$, $F_{10}=F_5/\left[\left(\mu_0^2+1\right)(\mu_0+\mu_6)\,(\mu_0+\mu_9)\,(\mu_0+\mu_{10})\right]$.

Let us prove that $b_2^2>0$. Introducing the notations $\nu_3=\dfrac{\mu_0\mu_9-1}{\mu_0+\mu_9}$, $\nu_4=\dfrac{\mu_0\mu_{10}-1}{\mu_0+\mu_{10}}$ and taking into account that is, we have $\nu_3+\nu_4=\dfrac{\mu_0^2+\frac{2\mu_0(\mu_9\mu_{10}-1)}{(\mu_9+\mu_{10})}-1}{(\mu_0+\mu_9)\,(\mu_0+\mu_{10})}(\mu_9+\mu_{10})$, $\mu_0^2+\dfrac{2\mu_0\,(\mu_9\mu_{10}-1)}{(\mu_9+\mu_{10})}-1=0$, $\nu_3+\nu_4=0$. Then $b_2^2=\dfrac{(\mu_0\mu_9-1)\,(\mu_0\mu_{10}-1)}{(\mu_0+\mu_9)(\mu_0+\mu_{10})}=\nu_3\nu_4=\frac{1}{4}\left[4\nu_3\nu_4-(\nu_3+\nu_4)^2\right]=-\frac{1}{4}(\nu_3-\nu_4)^2=$ $=-\dfrac{1}{4}\left(\dfrac{2\gamma_1 i(1+\mu_0)^2}{(\mu_0+\delta_1)^2+\gamma_1^2}\right)^2>0.$

Example 3. Consider the following sixth-order partial differential equation:

$$u_{xxxxxx} + 11u_{xxxxxy} + 60u_{xxxxyy} + 130u_{xxxyyy} - 51u_{xxyyyy} - 781u_{xyyyyy} - 650u_{yyyyyy} = 0. \quad (35)$$

The characteristic equation corresponding equation (35) has the form

$$(dy)^6 - 11(dy)^5(dx) + 60(dy)^4(dx)^2 - 130(dy)^3(dx)^3 - 51(dy)^2(dx)^4 + 781(dy)(dx)^5 - 650(dx)^6 = 0.$$

It is easy to verify that this equation has two different simple real roots and four complex conjugate roots for $t = dy/dx$:

$$t_1 = 1,\ t_2 = -2,\ t_3 = 3+4i,\ t_4 = 3-4i,\ t_5 = 3+2i,\ t_6 = 3-2i.$$

Then, equation (35) can be written as follows:

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial x^2} + 6\frac{\partial^2}{\partial x\partial y} + 25\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + 6\frac{\partial^2}{\partial x\partial y} + 13\frac{\partial^2}{\partial y^2}\right)u = 0. \quad (36)$$

After the transformation $\xi = y - 3x$, $\eta = 2x$, we obtain: $\frac{\partial}{\partial x} = -3\frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}$. Considering these, from (36), we have

$$\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} - \frac{2}{5}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2}{\partial \xi^2} + \frac{1}{4}\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) = 0.$$

4. If equation (6) has six different complex conjugate roots:

$$t_1 = \sigma + \zeta i, \quad t_2 = \sigma - \zeta i, \quad t_3 = \delta + \gamma i, \quad t_4 = \delta - \gamma i, \quad t_5 = \alpha + \beta i, \quad t_6 = \alpha - \beta i,$$

where $\alpha, \beta, \delta, \gamma, \sigma, \zeta \in R$ such that $\beta \neq 0, \gamma \neq 0, \zeta \neq 0$,

$$\left[(\alpha-\delta)^2 + (|\beta| - |\gamma|)^2\right]\left[(\alpha-\sigma)^2 + (|\beta| - |\zeta|)^2\right]\left[(\delta-\sigma)^2 + (|\zeta| - |\gamma|)^2\right] \neq 0,$$

then after replacing (22), from equation (1), similarly to equation (33), we obtain the equation

$$A_0\beta^2\left[\left(\alpha^2 + \sigma^2 + \zeta^2 - 2\alpha\sigma\right)\frac{\partial^2}{\partial \xi^2} + 2\beta(\sigma - \alpha)\frac{\partial^2}{\partial \xi\partial \eta} + \beta^2\frac{\partial^2}{\partial \eta^2}\right]\times$$

$$\times\left[\left(\alpha^2 + \delta^2 + \gamma^2 - 2\alpha\delta\right)\frac{\partial^2}{\partial \xi^2} + 2\beta(\delta - \alpha)\frac{\partial^2}{\partial \xi\partial \eta} + \beta^2\frac{\partial^2}{\partial \eta^2}\right]\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) = F_1. \quad (37)$$

Hence, dividing both parts of (37) by $A_0\beta^2\left[(\alpha-\sigma)^2 + \zeta^2\right]\left[(\alpha-\delta)^2 + \gamma^2\right] (\neq 0)$ and introducing the notations $\mu_{11} = \dfrac{\beta(\sigma-\alpha) + \zeta\beta i}{(\alpha-\sigma)^2 + \zeta^2} = \sigma_1 + \zeta_1 i$, $\mu_{12} = \dfrac{\beta(\sigma-\alpha) - \zeta\beta i}{(\alpha-\sigma)^2 + \zeta^2} = \sigma_1 - \zeta_1 i$, $\mu_9 = \dfrac{\beta(\delta-\alpha) + \gamma\beta i}{(\alpha-\delta)^2 + \gamma^2} = \delta_1 + \gamma_1 i$, $\mu_{10} = \dfrac{\beta(\delta-\alpha) - \gamma\beta i}{(\alpha-\delta)^2 + \gamma^2} = \delta_1 - \gamma_1 i$, $\dfrac{\beta(\sigma-\alpha)}{(\alpha-\sigma)^2 + \zeta^2} = \sigma_1$, $\dfrac{\zeta\beta}{(\alpha-\sigma)^2 + \zeta^2} = \zeta_1$, $\dfrac{\beta\,(\delta-\alpha)}{(\alpha-\delta)^2 + \gamma^2} = \delta_1$, $\dfrac{\gamma\beta}{(\alpha-\delta)^2 + \gamma^2} = \gamma_1$, we have the following equation

$$\left(\frac{\partial^2}{\partial \xi^2} + (\mu_{11} + \mu_{12})\frac{\partial^2}{\partial \xi\partial \eta} + \mu_{11}\mu_{12}\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial \xi^2} + (\mu_9 + \mu_{10})\frac{\partial^2}{\partial \xi\partial \eta} + \mu_9\mu_{10}\frac{\partial^2}{\partial \eta^2}\right)\times$$

$$\times\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) = F_{11}, \quad (38)$$

where $F_{11} = F_1/A_0\beta^2 \left[\left((\alpha-\sigma)^2+\zeta^2\right)\left((\alpha-\delta)^2+\gamma^2\right)\right]$.

Moreover, since $(\mu_{11}+\mu_{12})^2 - 4(\mu_{11}\mu_{12})^2 = (\mu_{11}-\mu_{12})^2 = -\zeta_1^2 < 0$, $(\mu_9+\mu_{10})^2 - 4(\mu_9\mu_{10})^2 = (\mu_9-\mu_{10})^2 = -\gamma_1^2 < 0$, then $\frac{\partial^2}{\partial\xi^2} + (\mu_{11}+\mu_{12})\frac{\partial^2}{\partial\xi\partial\eta} + \mu_{11}\mu_{12}\frac{\partial^2}{\partial\eta^2}$ and $\frac{\partial^2}{\partial\xi^2} + (\mu_9+\mu_{10})\frac{\partial^2}{\partial\xi\partial\eta} + \mu_9\mu_{10}\frac{\partial^2}{\partial\eta^2}$ are elliptic differential operators.

If $\mu_{11} = -\mu_{12}$, $\mu_9 = -\mu_{10}$, that is, when $\sigma_1 = 0, \delta_1 = 0$, equation (38) takes the form

$$\left(\frac{\partial^2}{\partial\xi^2} + \zeta_1^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2}{\partial\xi^2} + \gamma_1^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2}\right) = F_{11}.$$

To further simplify equation (38) for $\sigma_1 \neq 0$ and $\delta_1 \neq 0$, we introduce the substitution (27), where μ_0 is one of the two solutions of equation

$$\mu^2 - \frac{2(1-\mu_9\mu_{10})}{\mu_9+\mu_{10}}\mu - 1 = 0,$$

then, similarly to case 3 , we obtain the equation

$$\left[\frac{\partial^2}{\partial s^2} + (c_5+c_6)\frac{\partial^2}{\partial s\partial t} + c_5c_6\frac{\partial^2}{\partial t^2}\right]\left(\frac{\partial^2}{\partial s^2} + b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right) = F_{12},$$

where $c_5 = [\mu_{11}\mu_0 - 1]/[\mu_{11}+\mu_0]$, $c_6 = [\mu_{12}\mu_0 - 1]/[\mu_{12}+\mu_0]$,
$F_{12} = F_5/\left[\left(\mu_0^2+1\right)(\mu_0+\mu_9)(\mu_0+\mu_{10})(\mu_0+\mu_{11})(\mu_0+\mu_{12})\right]$.

Since $(c_5+c_6)^2 - 4c_5c_6 = (c_5-c_6)^2 = \left(\frac{2\zeta_1 i(1+\mu_0)^2}{(\mu_0+\sigma_1)^2+\zeta_1^2}\right)^2 < 0$ and $b_2^2 > 0$, then the differential operators $\frac{\partial^2}{\partial s^2} + (c_5+c_6)\frac{\partial^2}{\partial s\partial t} + c_5c_6\frac{\partial^2}{\partial t^2}$ and $\frac{\partial^2}{\partial s^2} + b_2^2\frac{\partial^2}{\partial t^2}$ the last equation are elliptic.

Example 4. Consider the following sixth-order partial differential equation:

$$u_{xxxxxx} + 24u_{xxxxxy} + 254u_{xxxxyy} + 1504u_{xxxyyy} + 5233u_{xxyyyy} + 10120u_{xyyyyy} + 8500u_{yyyyyy} = 0. \quad (39)$$

The characteristic equation corresponding to equation (39) has the form

$$(dy)^6 - 24(dy)^5(dx) + 254(dy)^4(dx)^2 - 1504(dy)^3(dx)^3 + 5233(dy)^2(dx)^4 - 10120(dy)(dx)^5 + 8500(dx)^6 = 0.$$

It is easy to verify that this equation has six different complex conjugate roots for $t = dy/dx$:

$$t_1 = 4+i,\ t_2 = 4-i,\ t_3 = 4+2i,\ t_4 = 4-2i,\ t_5 = 4+3i,\ t_6 = 4-3i.$$

Then, equation (39) can be written as follows:

$$\left(\frac{\partial^2}{\partial x^2} + 8\frac{\partial^2}{\partial x\partial y} + 17\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + 8\frac{\partial^2}{\partial x\partial y} + 20\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + 8\frac{\partial^2}{\partial x\partial y} + 25\frac{\partial^2}{\partial y^2}\right)u = 0. \quad (40)$$

After the transformation $\xi = y - 4x$, $\eta = 3x$ we obtain: $\frac{\partial}{\partial x} = -4\frac{\partial}{\partial\xi} + 3\frac{\partial}{\partial\eta}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial\xi}$. Considering these from (40), we have

$$\left(\frac{\partial^2}{\partial\xi^2} + 9\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2}{\partial\xi^2} + \frac{9}{4}\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)u = 0.$$

Thus, we have proved the following

Theorem 1. Let one of the following statements be true with respect to equation (6):
1) has six different real roots;
2) has four different real roots and two complex conjugate roots;
3) has two real roots and four different complex conjugate roots;
4) has six different complex conjugate roots.
Then, in the domain Ω, equation (1) can be reduced to the one of the following canonical forms
1) $\left(\frac{\partial}{\partial\xi}+c_1\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+c_2\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^2}{\partial\xi^2}-b^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\eta^2}\right)=F_3;$
2) $\left(\frac{\partial}{\partial s}+c_3\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s}+c_4\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial s^2}-b_1^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_6;$
3) $\left(\frac{\partial}{\partial s}+c_3\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s}+c_4\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial s^2}+b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_{10};$
4) $\left[\frac{\partial^2}{\partial s^2}+(c_5+c_6)\frac{\partial^2}{\partial s\partial t}+c_5c_6\frac{\partial^2}{\partial t^2}\right]\left(\frac{\partial^2}{\partial s^2}+b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_{12}.$

It should be noted that equations (1), similar to equations of hyperbolic and/or elliptic type, possess simple (non-repeated) real and/or complex characteristics. Consequently, in all cases considered above, the discriminant is non-zero. Moreover, the canonical forms of equation (1) may contain both hyperbolic and/or elliptic differential operators.

Remark 1. The classification and reduction to canonical form of sixth-order linear partial differential equations with multiple real characteristics are studied in ten distinct cases. Analogously, the following theorem can be proven:

Theorem 2. Assume that equation (6) exhibits one of the following root configurations:
1) one double root and four distinct real roots;
2) one double root and one quadruple real root;
3) two triple real roots;
4) one quintuple root and one simple real root;
5) one sextuple real root;
6) two double roots and two distinct real roots;
7) three double real roots;
8) one double root, one triple root, and one simple real root;
9) one triple root and three distinct real roots;
10) one quadruple root and two distinct real roots.

Then, in the domain Ω, equation (1) can be reduced to one of the following canonical forms corresponding to the root structures:
1) $\left(\frac{\partial}{\partial\xi}+c_1\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+c_2\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^4 u}{\partial\xi^4}-\frac{\partial^4 u}{\partial\xi^2\partial\eta^2}\right)=F_3;$
2) $\frac{\partial^6}{\partial\xi^4\partial\eta^2}u=F_1/\left(\lambda_1-\lambda_2\right)^6;$
3) $\frac{\partial^6}{\partial\xi^3\partial\eta^3}u=F_1/\left\{-\left(\lambda_1-\lambda_2\right)^6\right\};$
4) $\frac{\partial^6}{\partial\xi\partial\eta^5}u=F_1/\left\{-\left(\lambda_1-\lambda_2\right)^6\right\};$
5) $\frac{\partial^6 u}{\partial\eta^6}=F_1/\left(\lambda_1-\lambda_2\right)^6;$
6) $\left(\frac{\partial}{\partial\xi}+c_3\frac{\partial}{\partial\eta}\right)^2\left(\frac{\partial^4 u}{\partial\xi^4}-\frac{\partial^4 u}{\partial\xi^2\partial\eta^2}\right)=F_5;$
7) $\left(\frac{\partial}{\partial\xi}+c_4\frac{\partial}{\partial\eta}\right)^2\frac{\partial^4 u}{\partial\xi^2\partial\eta^2}=F_6;$
8) $\frac{\partial^6 u}{\partial\xi^4\partial\eta^2}+c_4\frac{\partial^6 u}{\partial\xi^3\partial\eta^3}=F_7;$
9) $\left(\frac{\partial}{\partial\xi}+c_5\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^5 u}{\partial\xi^2\partial\eta^3}+c_6\frac{\partial^5 u}{\partial\xi\partial\eta^4}\right)=F_8;$

10) $\frac{\partial^6 u}{\partial\xi^2\partial\eta^4} + c_7\frac{\partial^6 u}{\partial\xi\partial\eta^5} = F_9$.

Remark 2. The classification and reduction to canonical form of sixth-order linear partial differential equations with multiple and complex characteristics are studied in nine distinct cases. Analogously, the following theorem can be proven:

Theorem 3. Assume that equation (6) exhibits one of the following root structures:

1) one double real root, two distinct real roots, and one pair of complex conjugate roots;
2) two distinct double real roots and one pair of complex conjugate roots;
3) one triple real root, one simple real root, and one pair of complex conjugate roots;
4) one quadruple real root and one pair of complex conjugate roots;
5) one double real root and two distinct double pairs of complex conjugate roots;
6) two distinct real roots and two distinct double pairs of complex conjugate roots;
7) one double real root and two distinct pairs of complex conjugate roots;
8) two distinct pairs of complex conjugate roots and two distinct double pairs of complex conjugate roots;
9) two distinct triple pairs of complex conjugate roots.

Then, in the domain Ω, equation (1) can be reduced to one of the following canonical forms corresponding to the root structures:

$$1)\ \left(\frac{\partial}{\partial s}+c_1\frac{\partial}{\partial t}\right)^2\left(\frac{\partial^2}{\partial s^2}-b^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_4;\ 2)\ \left(\frac{\partial^2}{\partial s^2}-b_1^2\frac{\partial^2}{\partial t^2}\right)^2\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_6;$$

$$3)\ \left(\frac{\partial}{\partial s}+c_2\frac{\partial}{\partial t}\right)\left(\frac{\partial^5 u}{\partial s^2\partial t^3}+\frac{\partial^5 u}{\partial t^5}\right)=F_8;\ 4)\left(\frac{\partial^6 u}{\partial s^2\partial t^4}+\frac{\partial^6 u}{\partial t^6}\right)=F_{10};\ 5)\ \left(\frac{\partial^3}{\partial s^2\partial t}+\frac{\partial^3}{\partial t^3}\right)^2 u=F_{12};$$

$$6)\ \left(\frac{\partial^2}{\partial s^2}-b_1^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial t^2}\right)^2 u=F_{14};\ 7)\ \left(\frac{\partial}{\partial s}+c_1\frac{\partial}{\partial t}\right)^2\left(\frac{\partial^2}{\partial s^2}+b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_{16};$$

$$8)\ \left(\frac{\partial^2}{\partial s^2}+b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial t^2}\right)^2 u=F_{18};\ 9)\ \left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)^3 u=F_{19}.$$

It should be noted that equation (1), like parabolic-type equations, exhibits multiple characteristics. Consequently, in all the cases discussed above, the discriminant is zero ($D = 0$). Nonetheless, the canonical forms of equation (1) may still involve both hyperbolic and/or elliptic differential operators.

Conclusion

In this paper, we prove a theorem on the canonical forms of equation (1) and three lemmas that play an important role in finding the canonical form of the equation (1).

Arguing similarly, we can find canonical forms of equation (1) in cases with multiple characteristics, provided that the coefficients of the equation (1) are sufficiently smooth functions.

We can give a number of examples when only finding the canonical form of an equation helps to obtain serious results. Considering the canonical form of the equation (1), when studying some boundary value problems, we can use potential theory or the Green or Riman function method. Therefore, the found canonical forms of linear differential equations with partial derivatives of the sixth-order with non-multiple characteristics and with constant coefficients allow us to correctly formulate and systematically study correct boundary value problems for such equations. These problems are the subject of further research.

From the canonical form of the equation (1), obtained in the first case considered above, it is clear that if the function F_3 does not depend on the unknown function u and its derivatives, then it is possible to find a general solution to equation (1).

Based on the proposed method for finding the canonical form of the equation (1), it is possible to study the problems of classification and reduction to canonical form of differential equations of higher-order with partial derivatives.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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On the Convergence of the Approximate Solution to the Optimization Problem for Oscillatory Processes

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This article addresses the non-linear optimization problem of oscillatory processes governed by partial integro-differential equations involving a Fredholm integral operator. A distinctive feature of the problem is that both the objective functional and the functions describing external and boundary influences are non-linear with respect to the vector controls. The integro-differential equation describing the state of the oscillatory process includes Fredholm integral operator, which has a significant impact on the structure and properties of the solutions. The algorithm for constructing the complete solution to this problem, as well as the effect of the Fredholm integral operator on the solution of the corresponding boundary value problem, has been published in previous studies. This article is dedicated to the investigation of the convergence of approximate solutions to the exact solution of the considered non-linear optimization problem. The influence of the Fredholm integral operator on the convergence behavior of the approximations is examined. It is demonstrated that the presence of the integral operator necessitates the construction of three distinct types of approximations of the optimal process: "Resolvent" approximations, based on the resolvent of the kernel of the integral operator; Approximations by optimal controls, constructed through the approximation of control functions; Finite-dimensional approximations.

Keywords: optimal control, optimal process, minimal value of functional, non-linear optimization problem, approximations of complete solution, resolvent approximation, finite-dimensional approximation, convergence.

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Introduction

Optimal control of systems with distributed parameters is one of the intensively developing scientific directions of Optimal control theory. Dynamics of systems with distributed parameters is described by partial differential equations, integral, integro-differential and more complex functional equations. Methods for solving linear optimization problems in programming control of systems with distributed parameters are based on the methods of classical variational calculus, the maximum principle, and they have been developed in studies [1–3]. The mathematical model [4, 5] of many applied problems need to solve non-linear optimization problems, for which methods for solving them are not sufficiently developed [6, 7]. A research group of Kyrgyz mathematicians, led by Professor A. Kerimbekov, is actively investigating the solvability of non-linear optimization problems [8–10] and the convergence of their approximate solutions [11, 12]. The results of the authors' research on solutions to non-linear optimization problems are presented in works [13, 14].

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In the paper [14], we have considered the non-linear optimization problem of oscillation processes described by integro-differential equations in partial derivatives with the integral Fredholm operator and an algorithm was developed for constructing a complete solution to this problem. It is established that the presence of the integral operator significantly affects the solvability of the non-linear optimization problem, in particular, when constructing a generalized solution to the boundary value problem of the controlled process and when proving the existence and uniqueness of a solution to system of non-linear integral equations.

In [14], the problem of non-linear optimization for oscillatory processes described by integro-differential equations with the participation of the Fredholm integral operator was investigated. An algorithm for constructing a complete solution to this problem was developed. It was established that the presence of the Fredholm integral operator has a significant effect on the solvability of the non-linear optimization problem, in particular, on the construction of a generalized solution to the boundary value problem of the controlled process and on the proof of the existence and uniqueness of a solution to a system of non-linear integral equations with respect to optimal controls.

This paper continues the study of the complete solution of the non-linear optimization problem developed in [14], in particular, with the aim of studying the convergence of its approximations. It is shown that the presence of the Fredholm integral operator necessitates constructing three types of approximations of the optimal process: approximation through the resolvent of the kernel of the integral operator, approximation by optimal controls, and finite-dimensional approximation. Accordingly, three types of approximations of the minimum value of the objective functional are also considered. Sufficient conditions are established for the convergence of approximations of both distributed and boundary vector optimal controls, three types of approximations of the optimal process, and approximations of the minimum value of the functional.

1 Formulation of the Non-linear Optimization Problem and Its Complete Solution

Consider the following non-linear optimization problem, where it is required to minimize the quadratic integral functional [14].

$$\begin{aligned} J[\bar{u}(t,x),\bar{\vartheta}(t,x)] = \int_Q [V(T,x)-\xi_1(x)]^2\,dx + \int_Q [V_t(T,x)-\xi_2(x)]^2\,dx + \\ + \int_0^T \left[\alpha\int_Q h^2[t,x,\bar{u}(t,x)]dx + \beta\int_\gamma b^2[t,x,\bar{\vartheta}(t,x)]dx\right]dt, \quad \alpha,\beta>0, \end{aligned} \tag{1}$$

on the set of solutions to the boundary value problem

$$V_{tt}(t,x) - AV(t,x) = \lambda\int_0^T K(t,\tau)V(\tau,x)d\tau + f[t,x,\bar{u}(t,x)], \quad x\in Q\subset\mathbb{R}^n, \quad 0<t<T, \tag{2}$$

$$V(0,x)=\psi_1(x), \quad V_t(0,x)=\psi_2(x), \quad x=(x_1,x_2,\ldots,x_n)\in Q, \tag{3}$$

$$\Gamma V(t,x) \equiv \sum_{i,k=1}^n a_{ik}(x)V_{x_k}(t,x)\cos(\delta,x_i) + a(x)V(t,x) = p[t,x,\bar{\vartheta}(t,x)], \quad x\in\gamma, \quad 0<t<T. \tag{4}$$

It should be noted that the characteristics of the data in problem (1)–(4) are preserved as presented in [14]. It is assumed that the functions describing external and boundary influences satisfy the following monotonicity conditions with respect to the functional variables:

$$f_{u_i}[t,x,\bar{u}(t,x)] \neq 0, \quad i=1,2,\ldots,m, \quad \forall(t,x)\in H(Q_T), \tag{5}$$

$$p_{\vartheta_i}[t,x,\bar{\vartheta}(t,x)] \neq 0, \quad i=1,2,\dots,r, \quad \forall (t,x)\in H(\gamma_T).$$

The conditions stated in (5) guarantee a one-to-one correspondence between the elements of the space of controls $(\bar{u}^0(t,x), \bar{\vartheta}^0(t,x))$ and the space of states $V(t,x)$ the controlled process.

The complete solution of nonlinear optimization problem (1)–(4) is defined in the form of a triple $\big((\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)), V^0(t,x), J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]\big)$ [14], where:

1) the distributed vector optimal control $\bar{u}^0(t,x)$ and the boundary vector optimal control $\bar{\vartheta}^0(t,x)$ are determined by the formulas

$$\bar{u}^0(t,x) = \bar{\varphi}[t,x,\theta_1^0(t,x),\alpha], \quad \theta_1^0(t,x) = \lim_{n\to\infty} \theta_1^{(n)}(t,x), \quad x\in Q, \tag{6}$$

$$\bar{\vartheta}^0(t,x) = \bar{\upsilon}[t,x,\theta_2^0(t,x),\beta], \quad \theta_2^0(t,x) = \lim_{n\to\infty} \theta_2^{(n)}(t,x), \quad x\in \gamma, \tag{7}$$

where functions $\theta_1^{(n)}(t,x)$ and $\theta_2^{(n)}(t,x)$ are defined as solutions of the operator equation

$$\theta^n(t,x) = F\left[\theta^{n-1}(t,x)\right], \quad n=1,2,3,\dots,$$

with

$$\theta^{(n)}(t,x) = \begin{cases} \theta_1^{(n)}(t,x), & x\in Q, \\ \theta_2^{(n)}(t,x), & x\in\gamma, \end{cases}$$

and satisfy the estimate

$$\|\theta^{(0)}(t,x) - \theta^{(n)}(t,x)\|_{H(\bar{Q}_T)} \leq \frac{C^n(\alpha,\beta)}{1-C(\alpha,\beta)} \|F\left(\theta_0(t,x)\right) - \theta_0(t,x)\|_{H(\bar{Q}_T)}, \tag{8}$$

where

$$\theta_0(t,x) = \begin{cases} \theta_{10}(t,x), & x\in Q, \\ \theta_{20}(t,x), & x\in\gamma, \end{cases}$$

is an arbitrary vector function in the space $H(\bar{Q}_T)$, and

$$C(\alpha,\beta) = \sqrt{f_0^2 m\varphi_0^2(\alpha) + p_0^2 r\upsilon_0^2(\beta)}\sqrt{2}E_0G_0T < 1, \tag{9}$$

with constants f_0, p_0, $\varphi_0(\alpha)$, $\upsilon_0(\beta)$, E_0, and G_0 defined appropriately.

2) $V^0(t,x)$ is an optimal process, determined by the following formula

$$V^0(t,x) = \sum_{n=1}^{\infty} \Big(\psi_n(t,\lambda) + \frac{1}{\lambda_n}\int_0^T E_n(t,\eta,\lambda)\Big(\int_Q f[\eta,\xi,\bar{u}^0(\eta,\xi)]z_n(\xi)d\xi + \\ + \int_\gamma p[\eta,\xi,\bar{\vartheta}^0(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x), \tag{10}$$

where

$$\psi_n(t,\lambda) = \psi_{1n}\left[\cos\lambda_n t + \lambda\int_0^T R_n(t,s,\lambda)\cos s ds\right] + \frac{\psi_{2n}}{\lambda_n}\left[\sin\lambda_n t + \lambda\int_0^T R_n(t,s,\lambda)\sin\lambda_n s ds\right],$$

$$E_n(t,\eta,\lambda) = \begin{cases} \sin\lambda_n(t-\eta) + \lambda\int_\eta^T R_n(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & 0\leq\eta\leq t, \\ \lambda\int_\eta^T R_n(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & t\leq\eta\leq T. \end{cases}$$

3) $J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]$ is a minimum value of the functional determined by the following formula

$$J[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)] = \int_Q \Big(\big[V^0(T,x)-\xi_1(x)\big]^2 + \big[V_t^0(T,x)-\xi_2(x)\big]^2\Big)\,dx + \\ + \Big(\alpha\int_Q h^2[t,x,\bar{u}^0(t,x)]dx + \beta\int_\gamma b^2[t,x,\bar{\vartheta}^0(t,x)]dx\Big)\,dt, \quad \alpha>0, \quad \beta>0. \tag{11}$$

2 Approximations of the Complete Solution to a Non-linear Optimization Problem

The main objective of this work is to investigate the construction of approximate solutions to non-linear optimization problem (1)–(4) and to analyze their convergence. Since the complete solution to the problem is represented as a triple $\big((\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)),V^0(t,x),J[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]\big)$ consisting of the optimal control, the optimal process, and the minimum value of the functional, we consider approximations of each of these components separately.

2.1 Convergence of Approximations of Vector Optimal Controls

In formulas (6) and (7), replacing functions $\theta_1^0(t,x)$ and $\theta_2^0(t,x)$ with functions $\theta_1^{(k)}(t,x)$ and $\theta_2^{(k)}(t,x)$, we find the k-th approximation of the vector distributed control by the formula

$$\bar{u}^{(k)}(t,x) = \bar{\varphi}[t,x,\theta_1^{((k))}(t,x),\alpha], \quad x\in Q, \quad k=1,2,3,\dots,$$

and similarly, we find the k-th approximation of the boundary vector control by the formula

$$\bar{\vartheta}^{(k)}(t,x) = \bar{\upsilon}[t,x,\theta_2^{(k)}(t,x),\beta], \quad x\in\gamma, \quad k=1,2,3,\dots,$$

where $\bar{\varphi}[t,x,\theta_1^{(k)}(t,x),\alpha]$ and $\bar{\upsilon}[t,x,\theta_2^{(k)}(t,x),\beta]$ are known vector functions.

Lemma 1. The k-th approximations of the distributed and boundary vector controls for non-linear optimization problem (1)–(4) converge to the optimal distributed and boundary vector controls, respectively, in the norms of the Hilbert spaces $H^m(Q_T)$ and $H^r(\gamma_T)$.

Proof. Let us introduce the notation

$$\bar{U}(t,x) = \begin{cases} \bar{u}(t,x), & x\in Q, \\ \bar{\vartheta}(t,x), & x\in\gamma. \end{cases}$$

Using inequalities (8) and (9), we calculate the following norm:

$$\|\bar{U}(t,x)-\bar{U}^n(t,x)\|^2_{H(\bar{Q}_T)} = \|\bar{u}^0(t,x)-\bar{u}^n(t,x)\|^2_{H^m(\bar{Q}_T)} + \|\bar{\vartheta}^0(t,x)-\bar{\vartheta}^n(t,x)\|^2_{H^k(\bar{\gamma}_T)} \le \\ \le \varphi_0^2(\alpha)\|\theta_1^0(t,x)-\theta_1^n(t,x)\|^2_{H(Q_T)} + \upsilon_0^2(\beta)\|\theta_2^0(t,x)-\theta_2^n(t,x)\|^2_{H(\gamma_T)} \le \Psi^2(\alpha,\beta)\|\theta^0(t,x)-\theta^n(t,x)\|^2_{H(\bar{Q}_T)},$$

$\Psi^2(\alpha,\beta)=\max\{\varphi_0^2(\alpha),\upsilon_0^2(\beta)\}$, from which the assertion of the lemma follows.

2.2 Approximations of the Optimal Process and Their Convergence

The presence of the Fredholm integral operator in boundary value problem (2)–(4), according to formula (10), leads to the construction of the following three types of approximations of the optimal process: approximations based on the resolvent of the kernel of the integral operator; approximations induced by the approximations of the optimal controls; finite-dimensional approximations. Each of these approximation types will be considered separately below.

2.2.1 "Resolvent" Approximations of the Optimal Process and Their Convergence

Functions defined by the formulas

$$V^{(m)}(t,x)=\sum_{n=1}^{\infty}\Big(\psi_n^{(m)}(t,\lambda)+\frac{1}{\lambda_n}\int_0^T E_n^{(m)}(t,\eta,\lambda)\times$$
$$\times\Big(\int_Q f[\eta,\xi,\bar{u}^0(\eta,\xi)]z_n(\xi)d\xi+\int_\gamma p[\eta,\xi,\bar{\vartheta}^0(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x),\quad m=1,2,3,...,$$

are called m-th approximations of the optimal process with respect to the resolvent or "resolvent" approximations of the optimal Process. Here,

$$\psi_n^{(m)}(t,\lambda)=\psi_{1n}\left[\cos\lambda_n t+\lambda\int_0^T R_n^{(m)}(t,s,\lambda)\cos\lambda_n s ds\right]+\frac{\psi_{2n}}{\lambda_n}\left[\sin\lambda_n t+\lambda\int_0^T R_n^{(m)}(t,s,\lambda)\sin\lambda_n s ds\right],$$

$$E_n^{(m)}(t,\eta,\lambda)=\begin{cases}\sin\lambda_n(t-\eta)+\lambda\int_\eta^T R_n^{(m)}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & 0\le\eta\le t,\\ \lambda\int_\eta^T R_n^{(m)}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & t\le\eta\le T,\end{cases}$$

$$R_n^{(m)}(t,s,\lambda)=\sum_{n=0}^{m}\lambda^{i-1}K_{n,i}(t,s),\quad n=1,2,3,\ldots$$

Lemma 2. "Resolvent" approximations $V^{(m)}(t,x)$ of the optimal process under the conditions of non-linear optimization problem (1)–(4) converge to the optimal process $V^0(t,x)$ in the norm of the Hilbert space $H(Q_T)$.

Proof. We evaluate the following norm

$$\|V^0(t,x)-V^{(m)}(t,x)\|^2_{H(Q_T)}\le 2T\frac{2\lambda^2T^2K_0}{\lambda_1^2}\left(|\lambda|\,\frac{T\sqrt{K_0}}{\lambda_1}\right)^{2m}\left(1-\frac{1}{\ln|\lambda|\,\frac{T\sqrt{K_0}}{\lambda_1}}\right)^2\times$$

$$\times\Big(\|\psi_1(x)\|^2_{H(Q)}+\frac{1}{\lambda_1^2}\|\psi_2(x)\|^2_{H(Q)}+\|f[\eta,\xi,\bar{u}^0(\eta,\xi)]\|^2_{H(Q_T)}+\|p[\eta,\xi,\bar{\vartheta}^0(\eta,\xi)]\|^2_{H(\gamma_T)}\Big)\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\to 0,\ m\to\infty,$$

from which, by virtue of the condition $|\lambda|\,\frac{T\sqrt{K_0}}{\lambda_1}<1$, the assertion of the lemma follows.

2.2.2 m,k-th Approximations of the Optimal Process and Their Convergence

Functions defined by following the formula

$$V_k^{(m)}(t,x)=\sum_{n=1}^{\infty}\Big(\psi_n^{(m)}(t,\lambda)+\frac{1}{\lambda_n}\int_0^T E_n^{(m)}(t,\eta,\lambda)\times$$
$$\times\left(\int_Q f[\eta,\xi,\bar{u}^{(k)}(\eta,\xi)]z_n(\xi)d\xi+\int_\gamma p[\eta,\xi,\bar{\vartheta}^{(k)}(\eta,\xi)]z_n(\xi)d\xi\right)d\eta\Big)z_n(x),$$

are called m,k-th approximations of the optimal process with respect to controls, where $\bar{u}^{(k)}(t,x)$ are k-th approximations of the distributed vector control, and $\bar{\vartheta}^{(k)}(t,x)$ are k-th approximations of the boundary vector control.

Lemma 3. m,k-th approximations $V_k^{(m)}(t,x)$ of the optimal process under the conditions of non-linear optimization problem (1)–(4) converge to the "resolvent" approximations $V^{(m)}(t,x)$ when $k\to\infty$ for any $m=1,2,3,\ldots$ in the norm of the space $H(Q_T)$.

Proof. The evaluation of the following norm leads directly to the conclusion of Lemma 3.

$$\|V^{(m)}(t,x) - V_k^{(m)}(t,x)\|^2_{H(Q_T)} \leq 4T\left(1 + \frac{\lambda^2 T^2 K_0}{\left(\lambda_1 - |\lambda|\, T\sqrt{K_0}\right)^2}\right)\times$$

$$\times\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\left(f_0^2\|\bar{u}_0(\eta,\xi) - \bar{u}_{(m)}(\eta,\xi)\|^2_{H(Q_T)} + p_0^2\|\bar{\vartheta}_0(\eta,\xi) - \bar{\vartheta}_{(m)}(\eta,\xi)\|^2_{H(\gamma_T)}\right) \to 0, \quad k \to \infty,$$

which is obtained taking into account the estimate

$$\int_0^T \left(E_n^{(m)}(t,\eta,\lambda)\right)^2 d\eta \leq 2T\left(1 + \lambda^2\frac{T^2 K_0}{\left(\lambda_n - \lambda T\sqrt{K_0}\right)^2}\right).$$

2.2.3 *Finite-Dimensional Approximations of the Optimal Process and Their Convergence*

Functions defined by the following formula

$$\begin{aligned} V_{k,l}^{(m)}(t,x) = &\sum_{n=1}^{l}\Big(\psi_n^{(m)}(t,\lambda) + \frac{1}{\lambda_n}\int_0^T E_n^{(m)}(t,\eta,\lambda)\times \\ &\times\Big(\int_Q f[\eta,\xi,\bar{u}^{(k)}(\eta,\xi)]z_n(\xi)d\xi + \int_\gamma p[\eta,\xi,\bar{\vartheta}^{(k)}(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x), \\ &m = 1,2,3,\ldots,\ \mu_m < \infty, \quad k = 1,2,3,\ldots,\ \mu_k < \infty, \quad l = 1,2,3,\ldots,\ \mu_l < \infty, \end{aligned} \tag{12}$$

are called m,k,l-th approximations or finite-dimensional approximations of the optimal process.

Lemma 4. m,k,l-th approximations $V_{k,l}^{(m)}(t,x)$ or finite-dimensional approximations of the optimal process under the conditions of non-linear optimization problem (1)–(4) converge to m,k-th approximations $V_k^{(m)}(t,x)$ when $l \to \infty$ for any m, k in the norm of the space $H(Q_T)$.

Proof. The assertion of the lemma follows from the following relation:

$$\|V_k^{(m)}(t,x) - V_{k,l}^{(m)}(t,x)\|^2_{H(Q_T)} \leq \sum_{n=l+1}^{\infty}\int_0^T\int_Q\left(V^{(m)}(t,x) - V_k^{(m)}(t,x)\right)^2 dx dt \leq$$

$$\leq 4T\left(1 + \frac{\lambda^2 T^2 K_0}{\left(\lambda_1 - |\lambda|\, T\sqrt{K_0}\right)^2}\right)\times$$

$$\times\sum_{n=l+1}^{\infty}\frac{1}{\lambda_n^2}\left(f_0^2\|\bar{u}_0(\eta,\xi) - \bar{u}_{(m)}(\eta,\xi)\|^2_{H(Q_T)} + p_0^2\|\bar{\vartheta}_0(\eta,\xi) - \bar{\vartheta}_{(m)}(\eta,\xi)\|^2_{H(\gamma_T)}\right) \to 0, \quad l \to \infty,$$

which holds due to the convergence of the remainder terms of the convergent series for each fixed m,k.

2.3 *Approximations of the Generalized Derivative of the Optimal Process and Their Convergence*

Similarly, the convergence of approximations was investigated for the generalized derivative of the optimal process determined by the following formula

$$\begin{aligned} V_t^0(t,x) = &\sum_{n=1}^{\infty}\Big(\psi'_{nt}(t,\lambda) + \frac{1}{\lambda_n}\int_0^T E'_{nt}(t,\eta,\lambda)\Big(\int_Q f[\eta,\xi,\bar{u}^0(\eta,\xi)]z_n(\xi)d\xi + \\ &+ \int_\gamma p[\eta,\xi,\bar{\vartheta}^0(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x), \end{aligned}$$

where

$$\psi'_{nt}(t,\lambda) = \psi_{1n}\left(-\lambda_n \sin\lambda_n t + \lambda\int_0^T R'_{nt}(t,s,\lambda)\cos\lambda_n s\,ds\right) + \frac{\psi_{2n}\lambda_n}{\left(\lambda_n\cos\lambda_n t + \lambda\int_0^T R'_{nt}(t,s,\lambda)\sin\lambda_n s\,ds\right)},$$

$$E'_{nt}(t,\eta,\lambda) = \begin{cases} \lambda_n\cos\lambda_n(t-\eta) + \lambda\int_\eta^T R'_{nt}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & 0\le\eta\le t,\\ \lambda\int_\eta^T R'_{nt}(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & t\le\eta\le T,\end{cases}$$

and it is an element of the space $H(Q_T)$ [14].

2.3.1 "Resolvent" Approximations of the Generalized Derivative of the Optimal Process and Their Convergence

Functions defined by the following formula

$$V_t^m(t,x) = \sum_{n=1}^\infty\Big(\psi'^m_{nt}(t,\lambda) + \frac{1}{\lambda_n}\int_0^T E'^m_{nt}(t,\eta,\lambda)\times$$
$$\times\Big(\int_Q f[\eta,\xi,\bar u^0(\eta,\xi)]z_n(\xi)d\xi + \int_\gamma p[\eta,\xi,\bar\vartheta^0(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x), \qquad m=1,2,3,...,$$

are called m-th approximations or "resolvent" approximations of the generalized derivative of the optimal process.

Lemma 5. "Resolvent" approximations $V'^m_t(t,x)$ of the generalized derivative of the optimal process under the conditions of non-linear optimization problem (1)–(4), converge to the generalized derivative optimal process $V'^0_t(t,x)$ in the norm of the Hilbert space $H(Q_T)$.

Proof. The assertion of Lemma 5 follows from the following relation

$$\begin{aligned}\|V_t^0(t,x) - V_t^m(t,x)\|^2_{H(Q_T)} &\le 4T\lambda^2T^2K_0\left(|\lambda|\sqrt{\frac{K_0T^2}{\lambda_1^2}}\right)^{2m}\left(1-\frac{1}{\ln\left(|\lambda|\,T\sqrt{K_0}\right)}\right)\times\\ &\times\left(\|\psi_1(x)\|^2_{H(Q)} + \frac{1}{\lambda_1^2}\|\psi_2(x)\|^2_{H(Q)} + \right.\\ &+\left.\left(\|f[\eta,\xi,\bar u^0(\eta,\xi)]\|^2_{H(Q_T)} + \|p[\eta,\xi,\bar\vartheta^0(\eta,\xi)]\|^2_{H(Q_T)}\right)\sum_{n=1}^\infty\frac{1}{\lambda_n^2}\right)\to 0, \quad m\to\infty\end{aligned} \tag{13}$$

which holds due to the condition $\frac{|\lambda|\sqrt{K_0T^2}}{\lambda_1} < 1$.

2.3.2 m, k-th Approximations of the Generalized Derivative of the Optimal Process and Their Convergence

Functions defined by the following formula

$$\begin{aligned}V'^m_{tk}(t,x) = &\sum_{n=1}^\infty\Big(\psi'^m_{nt}(t,\lambda) + \frac{1}{\lambda_n}\int_0^T E'^m_{nt}(t,\eta,\lambda)\times\\ &\times\Big(\int_Q f[\eta,\xi,\bar u^{(k)}(\eta,\xi)]z_n(\xi)d\xi + \int_\gamma p[\eta,\xi,\bar\vartheta^{(k)}(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x),\end{aligned} \tag{14}$$

are called m,k-th approximations of the generalized derivative of the optimal process.

Lemma 6. m,k-th approximations $V'^{m}_{tk}(t,x)$ of the generalized derivative of the optimal process under the conditions of non-linear optimization problem (1)–(4) converge to the m-th approximations $V'^{(m)}_{t}(t,x)$ of the generalized derivative of the optimal process when $k \to \infty$ for any value of $m = 1,2,3,\ldots$ in the norm of the space $H(Q_T)$.

Proof. Proof of the lemma follows from the following relation:

$$\|V'^{m}_{t}(t,x) - V^{m}_{tk}(t,x)\|^2_{H(Q_T)} \leq 4T^3\left(1+\frac{\lambda^2 K_0 T}{\lambda_1^2}\right)\cdot\left(f_0^2\left\|\bar{u}^0(t,x)-\bar{u}^{(k)}(t,x)\right\|^2_{H(Q_T)} + \right.$$
$$\left. + p_0^2\left\|\bar{\vartheta}^0(t,x)-\bar{\vartheta}^{(k)}(t,x)\right\|^2_{H(\gamma_T)}\right) \to 0, \quad k\to\infty.$$

2.3.3 Finite-Dimensional Approximations of the Generalized Derivative of the Optimal Process and Their Convergence

Functions defined by the following formula

$$V'^{m}_{tk,l}(t,x) = \sum_{n=1}^{l}\Big(\psi'^{m}_{nt}(t,\lambda) + \frac{1}{\lambda_n}\int_0^T E'^{m}_{nt}(t,\eta,\lambda)\times$$
$$\times\Big(\int_Q f[\eta,\xi,\bar{u}^{(l)}(\eta,\xi)]z_n(\xi)d\xi + \int_\gamma p[\eta,\xi,\bar{\vartheta}^{(l)}(\eta,\xi)]z_n(\xi)d\xi\Big)d\eta\Big)z_n(x),$$

are called m,k,l-th approximations or finite-dimensional approximations of the generalized derivative of the optimal process.

Lemma 7. Finite-dimensional approximations $V'^{m}_{tk,l}(t,x)$ of the generalized derivative of the optimal process under the conditions of non-linear optimization problem (1)–(4) converge to m,k-th approximations $V'^{m}_{tk}(t,x)$ of the generalized derivative of the optimal process when $l\to\infty$ for any value of m,k in the norm of space $H(Q_T)$.

Proof. Proof of the lemma follows from the following inequality

$$\left\|V^{m}_{tk}(t,x) - V^{m}_{tk,l}(t,x)\right\|^2_{H(Q_T)} \leq 8T\left(1+\frac{\lambda^2}{\lambda_n^2}\cdot\frac{K_0T^2\lambda_n^2}{\left(\lambda_n|\lambda|\sqrt{K_0T^2}\right)^2}\right)\cdot\left(\sum_{n=i+1}^{\infty}\lambda_n^2\psi_{1n}^2 + \sum_{n=i+1}^{\infty}\psi_{1n}^2 + \right.$$
$$\left. + \sum_{n=i+1}^{\infty}\int_0^T f_n^2[\eta,\bar{u}^k]\,d\eta + \sum_{n=i+1}^{\infty}\int_0^T p_n^2[\eta,\bar{\vartheta}^k]\,d\eta\right) \to 0, \quad l\to\infty,$$

which hold due to the convergence of the remainder terms of convergent series.

2.4 Approximations of the Minimum Value of the Functional and Their Convergence

The minimum value of functional (11), in accordance with the approximations of the optimal process, has three types of approximations.

Let us first derive the following formula that will be repeatedly used in proving the convergence of approximations of the minimum value of the functional:

$$|J[\hat{u},\hat{\vartheta}] - J[\tilde{u},\tilde{\vartheta}]| \leq \|V(T,x)+W(T,x)-2\xi_1(x)\|_{H(Q)}\cdot\|V(T,x)-W(T,x)\|_{H(Q)}$$
$$+ \|V_t(T,x)+W_t(T,x)-2\xi_2(x)\|_{H(Q)}\cdot\|V_t(T,x)-W_t(T,x)\|_{H(Q)}$$
$$+ \alpha h_0\cdot\|h[t,x,\hat{u}(t,x)]+h[t,x,\tilde{u}(t,x)]\|_{H(Q_T)}\cdot\|\hat{u}(t,x)-\tilde{u}(t,x)\|_{H(Q_T)}$$
$$+ \beta b_0\cdot\|b[t,x,\hat{\vartheta}(t,x)]+b[t,x,\tilde{\vartheta}(t,x)]\|_{H(\gamma_T)}\cdot\|\hat{\vartheta}(t,x)-\tilde{\vartheta}(t,x)\|_{H(\gamma_T)},$$

2.4.1 Finite-dimensional approximations of the functional minimum value and their convergence

According to formulas (12) and (13), finite-dimensional approximations of the functional minimum value are calculated by the formula

$$J_m^{k,j}[\bar{u}^{(k)}(t,x),\bar{\vartheta}^{(k)}(t,x)] = \int_Q \left[\left(V_{k,j}^{(m)}(T,x) - \xi_1(x) \right)^2 + \left(V_{t_{k,j}}(T,x) - \xi_2(x) \right)^2 \right] dx +$$
$$+ \int_0^T \left[\alpha \int_Q h^2(t,x,\bar{u}^{(k)}(t,x))\, dx + \beta \int_\gamma b^2(t,x,\bar{\vartheta}^{(k)}(t,x))\, dx \right] dt.$$

Lemma 8. Finite-dimensional approximations $J_m^k[\bar{u}^0(t,x),\bar{\vartheta}^0(t,x)]$ of the functional minimal value under the conditions of the non-linear optimization problem (1)–(4) converge to the m-th approximations of the functional minimal value when $k\to\infty$ for all fixed values of m,k in the norm of real numbers space R.

Proof. In formula (14), by replacing

$$V(t,x)\to V_{k,l}^{(m)}(t,x),\quad V_t(t,x)\to V_{tk}^{(m)}(t,x),\quad W(t,x)\to V_{k,j}^{(m)}(t,x),\quad W_t(t,x)\to V_{tk,j}^{(m)}(t,x),$$

we obtain the inequality

$$\left| J_m^{(k)}[\bar{u}^k(t,x),\bar{\vartheta}^{(k)}(t,x)] - J_m^{k,j}[\bar{u}^{(k)}(t,x),\bar{\vartheta}^{(k)}(t,x)] \right| \le C^{(2)} \left\| V_k^m(T,x) - V_{k,j}^m(T,x) \right\|_{H(Q)} +$$
$$+ C^{(3)} \left\| V_{tk}^m(T,x) - V_{tk,j}^m(T,x) \right\|_{H(\gamma)} \to 0, \quad k\to\infty,$$

where $C^{(2)}$, $C^{(3)}$ are constants.

3 Main results

Theorem 1. (Convergence of Finite-Dimensional Approximations to the Optimal Process). Let the following conditions be satisfied:

1) Functions of external and boundary influences satisfy the Lipschitz condition for functional variables (for controls):

$$\|f[\eta,\xi,\hat{u}(\eta,\xi)] - f[\eta,\xi,\tilde{u}(\eta,\xi)]\|^2_{H(Q_T)} \le f_0^2 \|\hat{u}(\eta,\xi) - \tilde{u}(\eta,\xi)\|^2_{H(Q_T)},\quad f_0^2 = \text{const},$$

$$\|p[\eta,\xi,\hat{\vartheta}(\eta,\xi)] - p[\eta,\xi,\tilde{\vartheta}(\eta,\xi)]\|^2_{H(Q_T)} \le p_0^2 \|\hat{\vartheta}(\eta,\xi) - \tilde{\vartheta}(\eta,\xi)\|^2_{H(Q_T)},\quad p_0^2 = \text{const}.$$

2) The intermediate vectors $\bar{\varphi}[t,x,\theta_1(t,x),\alpha],\ x\in Q$, and $\bar{\upsilon}[t,x,\theta_2(t,x),\beta],\ x\in\gamma$, of the functions satisfy the Lipschitz condition with respect to functional variables:

$$\|\bar{\varphi}[t,x,\hat{\theta}_1(t,x),\alpha] - \bar{\varphi}[t,x,\tilde{\theta}_1(t,x),\alpha]\|_{H(Q_T)} \le \varphi_0(\alpha)\|\hat{\theta}_1(t,x) - \tilde{\theta}_1(t,x)\|_{H(Q_T)},\quad \varphi_0(\alpha) > 0,$$

$$\|\bar{\upsilon}[t,x,\hat{\theta}_2(t,x),\beta] - \bar{\upsilon}[t,x,\tilde{\theta}_2(t,x),\beta]\|_{H(Q_T)} \le \upsilon_0(\beta)\|\hat{\theta}_2(t,x) - \tilde{\theta}_2(t,x)\|_{H(Q_T)},\quad \upsilon_0(\beta) > 0.$$

3) With respect to the parameters of non-linear optimization problem (1)–(4), the following inequality holds:

$$C(\alpha,\beta) = \sqrt{f_0^2 m \varphi_0^2(\alpha) + p_0^2 r \upsilon_0^2(\beta)}\sqrt{2} E_0 G_0 T < 1.$$

Then finite-dimensional approximations $V_{k,l}^{(m)}(t,x)$ of the optimal process $V^0(t,x)$ under the conditions of the non-linear optimization problem (1)–(4) converge to the optimal process when $m,k,l\to\infty$ in the norm of the space $H(Q_T)$.

Proof. Based on Lemmas 1–4, the assertion of the theorem follows from the inequality:

$$\|V^0(t,x) - V_{k,l}^{(m)}(t,x)\|_{H(Q_T)} \leq \|V^0(t,x) - V^{(m)}(t,x)\|_{H(Q_T)} + \|V^{(m)}(t,x) - V_k^{(m)}(t,x)\|_{H(Q_T)} + \\ + \|V_k^{(m)}(t,x) - V_{k,l}^{(m)}(t,x)\|_{H(Q_T)} \to 0, \quad m,k,l \to \infty.$$

Theorem 2. (Convergence of finite-dimensional approximations of the generalized derivative to the generalized derivative of the optimal process). Let the conditions of Theorem 1 be satisfied. Then Finite-dimensional approximations $V_{tk,l}^{(m)}(t,x)$ of the generalized derivative of the optimal under the conditions of non- linear optimization problem (1)–(4)converge to generalized derivative $V_t^0(t,x)$ of the optimal process when $m,k,l \to \infty$ in the norm of the space $H(Q_T)$.

Proof. Proof of the lemma follows from following inequality

$$\|V_t^0(t,x) - V_{tk,l}^{(m)}(t,x)\|_{H(Q_T} = \|V_t^0(t,x) - V_t^{(m)}(t,x)\|_{H(Q_T)} + \|V_t^{(m)}(t,x) - V_{tk}^{(m)}(t,x)\|_{H(Q_T)} + \\ + \|V_{tk}(t,x) - V_{tk,l}^{(m)}(t,x)\|_{H(Q_T)} \to 0, \quad l \to \infty.$$

Theorem 3. (Convergence of finite-dimensional approximations of the functional minimum value to the minimum value of the functional). Let the conditions of Theorem 1 be satisfied, then Finite-dimensional approximations $J_m^k[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]$ of the functional minimal value under the conditions of non-linear optimization problem (1)–(4) converge to functional minimal value $J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]$ when $m,k,l \to \infty$ in the norm of real numbers space R.

Proof. Proof of Theorem 3 follows from the inequality

$$|J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)] - J_m^{k,j}[\bar{u}^{(k)}(t,x), \bar{\vartheta}^{(k)}(t,x)]| \leq |J[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)] - J_m[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)]| +$$

$$+ |J_m[\bar{u}^0(t,x), \bar{\vartheta}^0(t,x)] - J_m^k[\bar{u}^{(k)}(t,x), \bar{\vartheta}^{(k)}(t,x)]| +$$

$$+ |J_m^k[\bar{u}^{(k)}(t,x), \bar{\vartheta}^{(k)}(t,x)] - J_m^{k,j}[\bar{u}^{(k)}(t,x), \bar{\vartheta}^{(k)}(t,x)]| \to 0, \quad m,k,l \to \infty.$$

Conclusion

In this paper, the influence of the Fredholm integral operator in the integro-differential equation on the convergence of approximate solutions to a nonlinear optimization problem is investigated. It is established that the presence of the Fredholm integral operator leads to the identification of three distinct types of approximations of the optimal process ("Resolvent" approximations, based on the resolvent of the kernel of the integral operator; Approximations by optimal controls, constructed through the approximation of control functions; Finite-dimensional approximations) and corresponding approximations of the minimum value of the functional.

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Author Contributions

Conceptualization and Formal analysis: Elmira Faizuldaevna Abdyldaeva; Methodology: Akylbek Kerimbekov and Elmira Faizuldaevna Abdyldaeva; Preparation and editing of the manuscript, translation into English, and other technical works: Elmira Faizuldaevna Abdyldaeva, Tursun Kamaldinovich Yuldashev, Muhammet Kamali

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Conflict of Interest

The authors declare no conflict of interest.

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On solving the second boundary value problem for the Viscous Transonic Equation

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In a rectangular domain, the second boundary value problem for the Viscous Transonic Equation is considered. The uniqueness of the solution to the problem is proved using the energy integral method. The existence of a solution is proved by the method of separation of variables, i.e. it is sought in the form of a product of two functions $X(x)$ and $Y(y)$. For definition $Y(y)$, an ordinary differential equation of the second order with two boundary conditions on the boundaries of segment $[0, q]$ is obtained. For this problem, the eigenvalues and the corresponding eigenfunctions are found at $n \in N$. For definition $X(x)$, an ordinary differential equation of the third order with three boundary conditions on the boundaries of segment $[0, q]$ is obtained. The solution to this problem is found in the form of an infinite series, uniform convergence, and the possibility of term-by-term differentiation under certain conditions on the given functions is proven. The convergence of the second-order derivative of the solution with respect to variable y is proved using the Cauchy-Bunyakovsky and Bessel inequalities. When substantiating the uniform convergence of the solution, the absence of a “small denominator” is proved.

Keywords: equations with multiple characteristics, boundary value problem, uniqueness, existence, method of separated variables, eigenvalue, eigenfunction, functional series, absolute and uniform convergence.

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Introduction

Third-order partial differential equations are considered when solving problems in the theory of nonlinear acoustics and in the hydrodynamic theory of space plasma and fluid filtration in porous media [1, 2]. Quite often, sharp changes in flow parameters occur in narrow regions adjacent to shock waves. The gradients of flow parameters in them can be so significant that, along with the nonlinear nature of the movement, it becomes necessary to take into account the influence of viscosity and thermal conductivity. Such currents are called short waves. The theory of transonic flows refers to the theory of short waves. It should be noted that recently in the literature this equation is increasingly called the viscous transonic equation, or simply the VT equation.

In [3], taking into account the properties of viscosity and thermal conductivity of the gas, a thirdorder equation with multiple characteristics was obtained from the Navier-Stokes system, containing the second derivative with respect to time

$$u_{xxx} + u_{yy} - \frac{\nu}{y} u_y = u_x u_{xx}, \quad \nu = const.$$

This equation, at $\nu = 1$, describes an axisymmetric flow, while at $\nu = 0$, it describes a plane-parallel flow [4].

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L. Cattabriga in [5] for equation $D_x^{2n+1}u - D_y^2 u = 0$ constructed a fundamental solution in the form of a double improper integral and studied the properties of the potential and solved boundary value problems.

In [6, 7], fundamental solutions for a third-order equation with multiple characteristics were constructed, containing second derivatives with respect to time, expressed through degenerate hypergeometric functions, their properties were studied, and estimates were found for $|t| \to \infty$.

In [8], the Dirichlet problem for third-order hyperbolic equations was investigated, while in [9], an analogue of the Goursat problem for a third-order equation with singular coefficients was studied.

In works [10, 11], nonlocal problems for third-order differential equations were examined, while in works [12–14], the stability of boundary value problems for third-order partial differential equations is studied.

In works [15–17], boundary value problems for third-order partial differential equations were investigated.

1 Formulation of the problem

In the domain $D = \{(x,y): \ 0 < x < p, \ 0 < y < q\}$, consider the equation:

$$L[u] \equiv \frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{1}$$

where $p > 0, \ q > 0$ are given sufficiently smooth functions.

Problem A. Find a function $u(x,y)$ from class $C_{x,y}^{3,2}(D) \cap C_{x,y}^{2,1}(\overline{D})$, that satisfies equation (1) in the domain D and the following boundary conditions:

$$u_y(x,0) = u_y(x,q) = 0, \tag{2}$$

$$\begin{cases} au(0,y) + bu_{xx}(0,y) = \psi_1(y), \\ cu(p,y) + du_{xx}(p,y) = \psi_2(y), \\ u_x(0,y) = \psi_3(y), \end{cases} \tag{3}$$

where a, b, c and d are given constants, and also $a^2+b^2 \neq 0$, $a^2+c^2 \neq 0$, $c^2+d^2 \neq 0$, and $\psi_i(y), \ i = \overline{1,3}$, are given sufficiently smooth functions, and

$$\psi'_i(0) = \psi'_i(q) = 0, \quad i = \overline{1,3}. \tag{4}$$

Note that a similar problem for the adjoint equation was studied in [18–19]. The boundary value problem close to the topic of this work was studied in [20–22]. The case $a = 1, \ b = 0, \ c = 1, \ d = 0$ was considered in work [23].

2 The uniqueness of solution

Theorem 1. If Problem A has a solution, then if conditions $ab \leq 0, \ cd \geq 0$ are met, it is unique.

Proof. Assume the opposite, let Problem A have two solutions $u_1(x,y)$ and $u_2(x,y)$. Then the function $u(x,y) = u_1(x,y) - u_2(x,y)$ satisfies equation (1) with homogeneous boundary conditions. Let us prove that $u(x,y) \equiv 0$ is in $\overline{D}$.

In the domain D the identity

$$uL[u] = uu_{xxx} + uu_{yy} = 0,$$

or

$$u\,L[u] \equiv \frac{\partial}{\partial x}\left(u\,u_{xx} - \frac{1}{2}\,u_x^2\right) + \frac{\partial}{\partial y}\left(u\,u_y\right) - u_y^2 = 0 \tag{5}$$

holds. Integrating identity (5) over the domain D and taking into account homogeneous boundary conditions, we obtain

$$\begin{gathered}\int\limits_0^p\int\limits_0^q \tfrac{\partial}{\partial x}\left[u\left(x,y\right)\,u_{xx}\left(x,y\right) - \tfrac{1}{2}\,u_x^2\left(x,y\right)\right]dxdy + \\ + \int\limits_0^p\int\limits_0^q \tfrac{\partial}{\partial y}\left[u\left(x,y\right)\,u_y\left(x,y\right)\right]dxdy - \int\limits_0^p\int\limits_0^q u_y^2\left(x,y\right)dxdy = 0.\end{gathered}$$

Taking into account homogeneous boundary conditions and requiring $a \neq 0,\ c \neq 0$, we obtain

$$\frac{d}{c}\int\limits_0^q u_{xx}^2\left(p,y\right)dy - \frac{b}{a}\int\limits_0^q u_{xx}^2\left(0,y\right)dy + \frac{1}{2}\int\limits_0^q u_x^2\left(p,y\right)dy + \int\limits_0^p\int\limits_0^q u_y^2\left(x,y\right)dxdy = 0.$$

Taking into account the condition $ab \leq 0,\ cd \geq 0$, we obtain $u_y\left(x,y\right) = 0$. From this, it follows that $u\left(x,y\right) = f\left(x\right)$. Substituting $u\left(x,y\right)$ into equation (1), we get $f'''\left(x\right) = 0$. The solution to this equation is $f\left(x\right) = C_1x^2 + C_2x + C_3$. To satisfy the boundary conditions in equation (3), the constants $C_1,\ C_2,\ C_3$ are determined,

$$\begin{cases} 2bC_1 + aC_3 = \psi_{10}, \\ C_1\left(cp^2 + 2d\right) + cC_3 = \psi_{20}, \\ C_2 = \psi_{30}. \end{cases}$$

The value of the main determinant of this system is as follows:

$$\Delta = \begin{vmatrix} 2b & 0 & a \\ cp^2 + 2d & cp & c \\ 0 & 1 & 0 \end{vmatrix} = acp^2 + 2ad - 2bc.$$

Assume that $\Delta = 0$. In this case, $acp^2 + 2ad - 2bc = 0$, and from this, we derive $p^2 = 2\left(\dfrac{b}{a} - \dfrac{d}{c}\right)$ which represents the uniqueness condition. According to the theorem and the condition $a^2 + c^2 \neq 0$, if we consider $p^2 < 0$, it leads to a contradiction because $p > 0$. Thus, $\Delta \neq 0$. Consequently, this implies $C_1 = C_2 = C_3 = 0$ then $f\left(x\right) = 0$. Hence, the function $u\left(x,y\right) \equiv 0$ for all $\left(x,y\right) \in \overline{D}$. Finally, from the last equation, it follows that $u_1\left(x,y\right) = u_2\left(x,y\right)$.

In cases $b \neq 0,\ d \neq 0;\ a \neq 0,\ d \neq 0;\ c \neq 0,\ b \neq 0$ similarly we obtain the equality of $u\left(x,y\right) \equiv 0$ in $\overline{D}$.

The proof of Theorem 1 is complete.

3 Existence of a solution

Theorem 2. If the functions are $\psi_i\left(y\right) \in C^2\left[0 < y < q\right],\ i = \overline{1,3}$ and conditions (4) are satisfied, then a solution to Problem A exists.

Proof. To prove the existence of a solution to Problem A, we search in the form

$$u\left(x,y\right) = X\left(x\right)Y\left(y\right). \tag{6}$$

Substituting (6) into equation (1) and separating the variables, and taking into account the boundary condition (2), we obtain a Sturm-Liouville type problem with respect to the function $Y(y)$ [24]:

$$\begin{cases} Y'' + \lambda Y = 0, \\ Y'(0) = Y'(q) = 0, \end{cases} \tag{7}$$

where λ is the separation parameter.

We know that the solution to problem (7) is expressed as follows:

$$Y_n(y) = B_n \cos \frac{n\pi}{q} y.$$

It is known that a nontrivial solution to the problem (7) exists only when

$$\lambda_0 = 0, \quad \lambda_n = \left(\frac{n\pi}{q}\right)^2, \quad n \in N.$$

λ_n, with $n \in N \cup \{0\}$, are the eigenvalues, and their corresponding eigenfunctions are as follows:

$$Y_n(y) = \begin{cases} \dfrac{1}{\sqrt{q}}, & n = 0, \\ \sqrt{\dfrac{2}{q}} \cos \dfrac{n\pi}{q} y, & n \in N. \end{cases} \tag{8}$$

Note that the system of eigenfunctions (8) of problem (7) is complete and orthonormal in the $L_2(0, q)$ space [25].

1) When $\lambda_0 = 0$, we get the following problem for the function $X(x)$:

$$\begin{cases} X'''_0 = 0, \\ aX_0(0) + bX''_0(0) = \psi_{10}, \\ cX_0(p) + dX''_0(p) = \psi_{20}, \\ X'_0(0) = \psi_{30}. \end{cases} \tag{9}$$

The solution to the boundary value problem equation (9) is as follows:

$$X_0(x) = C_1 x^2 + C_2 x + C_3,$$

then, taking into account (6) and (9), from equality (7) we search the solution to Problem A in the form:

$$u_0(x) = \frac{1}{\sqrt{q}} \left(C_1 x^2 + C_2 x + C_3\right). \tag{10}$$

Taking into account condition (3), we obtain a system of algebraic equations:

$$\begin{cases} 2bC_1 + aC_3 = \psi_{10}, \\ C_1\left(cp^2 + 2d\right) + cC_3 = \psi_{20}, \\ C_2 = \psi_{30}, \end{cases} \tag{11}$$

where $\psi_{i0},\ i = \overline{1,3}$, are the Fourier coefficients of the function $\psi_i(y),\ i = \overline{1,3}$, i.e.,

$$\psi_{i0} = \sqrt{\frac{1}{q}} \int_0^q \psi_i(y)\, dy, \quad i = \overline{1,3}.$$

Now we find the solution of system (11). To do this, we first calculate the main determinant of system (11), which has the following form

$$\Delta = \begin{vmatrix} 2b & 0 & a \\ cp^2+2d & cp & c \\ 0 & 1 & 0 \end{vmatrix} = acp^2 + 2ad - 2bc.$$

Since $p > 0$, then $\Delta \neq 0$ and system (11) has a solution:

$$C_1 = \frac{-c\psi_{10} + a\psi_{20} - acp\psi_{30}}{acp^2 + 2ad - 2bc},$$

$$C_2 = \psi_{30},$$

$$C_3 = \frac{\psi_{10}\left(cp^2 + 2d\right) - 2b\psi_{20} + 2bcp\psi_{30}}{acp^2 + 2ad - 2bc}.$$

Substituting $C_i,\ i = \overline{1,3}$ into (10), we obtain

$$u_0(x) = \sqrt{\frac{1}{q}}\frac{1}{acp^2 + 2ad - 2bc}\left[\psi_{10}\left(-cx^2 + cp^2 + 2d\right) + \right. \\ \left. + \psi_{20}\left(ax^2 - 2b\right) + \psi_{30}\left(acp^2x + 2adx - 2bcx + 2bcp\right)\right]. \tag{12}$$

In what follows, the maximum value among all positive known numbers found in estimates will be denoted by M. Now we find estimates (12) and $u_0(x)$ in the domain D. From (12) we have

$$|u_0(x)| \leq M\left[|\psi_{10}| + |\psi_{20}| + |\psi_{30}|\right] \leq M,$$

$$\left|u_0{'''}(x)\right| \leq M.$$

2) Now, when $\lambda_n = \left(\frac{n\pi}{q}\right)^2,\ n \in N$, we get the following problem for the function $X(x)$:

$$\left\{ \begin{array}{c} X'''_n - \lambda_n X_n = 0, \\ aX_n(0) + bX''_n(0) = \psi_{1n}, \\ cX_n(p) + dX''_n(p) = \psi_{2n}, \\ X'_n(0) = \psi_{3n}; \end{array} \right. \tag{13}$$

here

$$\psi_{i\,n} = \sqrt{\frac{2}{q}}\int\limits_0^q \psi_i(y)\cos\left(\frac{n\pi}{q}y\right)dy, \quad i = \overline{1,3}, \quad n \in N.$$

The general solution to the equation in problem (13) has the form:

$$X_n(x) = C_{1n}e^{k_n x} + e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\frac{\sqrt{3}}{2}k_n x + C_{3n}\sin\frac{\sqrt{3}}{2}k_n x\right), \tag{14}$$

where

$$k_n = \sqrt[3]{\lambda_n} = \left(\frac{n\pi}{q}\right)^{\frac{2}{3}}, \quad n \in N.$$

Taking into account the boundary conditions of problem (13) for the solution in the form of (14), we obtain the following:

$$\begin{cases} C_{1n}\left(a+bk_n^2\right)+C_{2n}\left(a-\frac{bk_n^2}{2}\right)-C_{3n}\frac{\sqrt{3}bk_n^2}{2}=\psi_{1n}, \\ C_{1n}e^{k_np}\left(c+dk_n^2\right)+C_{2n}e^{-\frac{1}{2}k_np}\left(c\cos\left(\frac{\sqrt{3}}{2}k_np\right)+dk_n^2\cos\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right)\right)+ \\ +C_{3n}e^{-\frac{1}{2}k_np}\left(c\sin\left(\frac{\sqrt{3}}{2}k_np\right)+dk_n^2\sin\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right)\right)=\psi_{2n}, \\ k_nC_{1n}-\frac{1}{2}k_nC_{2n}+\frac{\sqrt{3}}{2}k_nC_{3n}=\psi_{3n}. \end{cases} \tag{15}$$

So, to determine the coefficients $C_{in},\ i=\overline{1,3}$, we received a system of algebraic equations (15).

Let us introduce the notation:

$$\alpha_n=\cos\left(\frac{\sqrt{3}}{2}k_np\right),\ \beta_n=\sin\left(\frac{\sqrt{3}}{2}k_np\right),\ \gamma_n=\cos\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right),\ \delta_n=\sin\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right).$$

Then (15) has the following form:

$$\begin{cases} C_{1n}\left(a+bk_n^2\right)+C_{2n}\left(a-\frac{bk_n^2}{2}\right)-C_{3n}\frac{\sqrt{3}bk_n^2}{2}=\psi_{1n}, \\ C_{1n}e^{k_np}\left(c+dk_n^2\right)+C_{2n}e^{-\frac{1}{2}k_np}\left(c\alpha_n+dk_n^2\gamma_n\right)+C_{3n}e^{-\frac{1}{2}k_np}\left(c\beta_n+dk_n^2\delta_n\right)=\psi_{2n}, \\ k_nC_{1n}-\frac{1}{2}k_nC_{2n}+C_{3n}\frac{\sqrt{3}}{2}k_n=\psi_{3n}. \end{cases} \tag{16}$$

The main determinant of system (16) has

$$\Delta=\begin{vmatrix} a+bk_n^2 & a-\frac{1}{2}bk_n^2 & -\frac{\sqrt{3}}{2}bk_n^2 \\ e^{k_np}\left(c+dk_n^2\right) & e^{-\frac{k_n}{2}p}\left(c\alpha_n+dk_n^2\gamma_n\right) & e^{-\frac{k_n}{2}p}\left(c\beta_n+dk_n^2\delta_n\right) \\ k_n & -\frac{1}{2}k_n & \frac{\sqrt{3}}{2}k_n \end{vmatrix}=\frac{\sqrt{3}}{2}k_n^5e^{k_np}\overline{\Delta},$$

where

$$\overline{\Delta}=\left(\frac{c}{k_n^2}+d\right)\left(b-\frac{a}{k_n^2}\right)+e^{-\frac{3k_n}{2}p}\left\{\left(\frac{ac}{k_n^4}-\frac{2ad}{k_n^2}+\frac{2bc}{k_n^2}-bd\right)\cos\frac{\sqrt{3}}{2}k_np+\sqrt{3}\left(\frac{ac}{k_n^4}+bd\right)\sin\frac{\sqrt{3}}{2}k_np\right\}.$$

We show that $\Delta\neq 0$. To do this, we prove the following lemma:

Lemma 1. The boundary value problem

$$\begin{cases} X'''_n-\lambda_nX_n=0, \\ aX_n(0)+bX''_n(0)=0, \\ cX_n(p)+dX''_n(p)=0, \\ X'_n(0)=0, \end{cases} \tag{17}$$

has only a trivial solution.

Proof. Let's assume the opposite, let $X_n(x)\neq 0$. Consider the identity

$$X_n\left(X'''_n-\lambda_nX_n\right)=0,$$

or

$$\left(X_nX''_n-\frac{1}{2}\left(X'_n\right)^2\right)'-\lambda_nX_n^2=0,$$

integrating over interval $(0 < x < p)$, and taking into account the boundary conditions, we obtain

$$\frac{d}{c}{X''_n}^2(p) - \frac{b}{a}{X''_n}^2(0) + \frac{1}{2}{X_n'}^2(p) + \lambda_n \int\limits_0^p X_n^2 dx = 0.$$

Since $ab \le 0,\ cd \ge 0,\ \lambda_n > 0$, then $X_n \equiv 0$.

Lemma 1 has been proved.

If there is a number n^* such that $\Delta(n^*) = 0$, then there are constants $C_1^*,\ C_2^*,\ C_3^*$ that are not all equal to zero at the same time, satisfying the system

$$\begin{cases} C^*_{1n^*}\left(a + bk^2_{n^*}\right) + C^*_{2n^*}\left(a - \dfrac{bk^2_{n^*}}{2}\right) - C^*_{3n^*}\dfrac{\sqrt{3}bk^2_{n^*}}{2} = 0, \\ C^*_{1n^*}{}^{k_{n^*}p}\left(c + dk^2_{n^*}\right) + C^*_{2n^*}e^{-\frac{1}{2}k_{n^*}p}\left(c\cos\dfrac{\sqrt{3}}{2}k_{n^*}p + dk^2_{n^*}\cos\left(\dfrac{\sqrt{3}}{2}k_{n^*}p - \dfrac{2\pi}{3}\right)\right) + \\ + C^*_{3n^*}e^{-\frac{1}{2}k_{n^*}p}\left(c\sin\dfrac{\sqrt{3}}{2}k_{n^*}p + dk^2_{n^*}\sin\left(\dfrac{\sqrt{3}}{2}k_{n^*}p - \dfrac{2\pi}{3}\right)\right) = 0, \\ k_{n^*}C^*_{1n^*} - \dfrac{1}{2}k_{n^*}C^*_{2n^*} + \dfrac{\sqrt{3}}{2}k_{n^*}C^*_{3n^*} = 0. \end{cases}$$

From this we have that the function

$$X_{n^*}(x) = C^*_{1n^*}e^{k_{n^*}x} + e^{-\frac{1}{2}k_{n^*}x}\left(C^*_{2n^*}\cos\frac{\sqrt{3}}{2}k_{n^*}x + C^*_{3n^*}\sin\frac{\sqrt{3}}{2}k_{n^*}x\right)$$

is a solution to boundary value problem (17), but according to the proven lemma it should be

$$C^*_{1n^*}e^{k_{n^*}x} + e^{-\frac{1}{2}k_{n^*}x}\left(C^*_{2n^*}\cos\frac{\sqrt{3}}{2}k_{n^*}x + C^*_{3n^*}\sin\frac{\sqrt{3}}{2}k_{n^*}x\right) \equiv 0,$$

but this is impossible due to the linear independence of the functions

$$e^{k_{n^*}x}, \quad e^{-\frac{1}{2}k_{n^*}x}\cos\frac{\sqrt{3}}{2}k_{n^*}x, \quad e^{-\frac{1}{2}k_{n^*}x}\sin\frac{\sqrt{3}}{2}k_{n^*}x.$$

Hence the function in the form:

$$u^*(x,y) = u_0(x) + \sqrt{\frac{2}{q}}\sum_{n^*=1}^{+\infty} X_{n^*}(x)\cos\frac{n^*\pi}{q}y$$

are nontrivial solutions to Problem A, and this contradicts the uniqueness theorem. So $\Delta(n) \ne 0$, $n \in N$.

Note that in case $a = 1,\ b = 0,\ c = 1,\ d = 0$, we obtain the result from [23], as a special case.

$$C_{1n} = \tfrac{2e^{-k_np}}{\sqrt{3}\Delta}\left[\tfrac{\psi_{1n}}{k_n^2}e^{-\frac{k_n}{2}p}\left(\tfrac{c}{k_n^2}\cos\left(\tfrac{\sqrt{3}}{2}k_np - \tfrac{\pi}{6}\right) + d\cos\left(\tfrac{\sqrt{3}}{2}k_np + \tfrac{\pi}{6}\right)\right) - \tfrac{\sqrt{3}}{2}\tfrac{\psi_{2n}}{k_n^2}\left(\tfrac{a}{k_n^2} - b\right) + \right.$$
$$\left. + \tfrac{\psi_{3n}e^{-\frac{k_n}{2}p}}{k_n}\left(\left(\tfrac{a}{k_n^2} - \tfrac{1}{2}b\right)\left(\tfrac{c}{k_n^2} - \tfrac{1}{2}d\right)\sin\tfrac{\sqrt{3}}{2}k_np + \tfrac{\sqrt{3}}{2}\left(\tfrac{ad}{k_n^2} - bd + \tfrac{bc}{k_n^2}\right)\cos\tfrac{\sqrt{3}}{2}k_np\right)\right],$$

$$C_{2n} = \tfrac{2}{\sqrt{3}\Delta}\left[\tfrac{\psi_{1n}}{k_n^2}\left(e^{-\frac{3k_n}{2}p}\left(\tfrac{c}{k_n^2}\sin\tfrac{\sqrt{3}}{2}k_np + d\sin\left(\tfrac{\sqrt{3}}{2}k_np - \tfrac{2\pi}{3}\right)\right) - \tfrac{\sqrt{3}}{2}\left(\tfrac{c}{k_n^2} + d\right)\right) + \tfrac{\sqrt{3}}{2}\tfrac{\psi_{2n}}{k_n^2}e^{-k_np}\left(\tfrac{a}{k_n^2} + 2b\right) - \right.$$
$$\left. - \tfrac{\psi_{3n}}{k_n}\left(e^{-\frac{3k_n}{2}p}\left(\tfrac{a}{k_n^2} + b\right)\left(\tfrac{c}{k_n^2}\sin\tfrac{\sqrt{3}}{2}k_np + d\sin\left(\tfrac{\sqrt{3}}{2}k_np - \tfrac{2\pi}{3}\right)\right) + \tfrac{\sqrt{3}}{2}b\left(\tfrac{c}{k_n^2} + d\right)\right)\right],$$

$$C_{3n}=\frac{2}{\sqrt{3}\Delta}\left[-\frac{\psi_{1n}}{k_n^2}\left(\frac{c}{2k_n^2}+\frac{d}{2}-e^{-\frac{3k_n}{2}p}\left(\frac{c}{k_n^2}\cos\frac{\sqrt{3}}{2}k_np+d\cos\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right)\right)\right)+\frac{3a\psi_{2n}}{2k_n^4}e^{-k_np}+\right.$$
$$\left.+\frac{\psi_{3n}}{k_n}\left(e^{-\frac{3k_n}{2}p}\left(\frac{a}{k_n^2}+\frac{b}{2}\right)\left(\frac{c}{k_n^2}\cos\frac{\sqrt{3}}{2}k_np+d\cos\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right)\right)-\left(\frac{a}{k_n^2}-\frac{b}{2}\right)\left(\frac{c}{k_n^2}+d\right)\right)\right].$$

Then the solution to Problem A is written in the following form:

$$u(x,y)=u_0(x)+\sqrt{\frac{2}{q}}\sum_{n=1}^{+\infty}\left[C_{1n}e^{k_n x}+e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\frac{\sqrt{3}}{2}k_nx+C_{3n}\sin\frac{\sqrt{3}}{2}k_nx\right)\right]\cos\frac{n\pi}{q}y. \quad (18)$$

Now we prove the absolute and uniform convergence of series (18) in the domain $\overline{D}$. From (18) we have

$$|u(x,y)|\leq|u_0(x)|+\sqrt{\tfrac{2}{q}}\sum_{n=1}^{+\infty}\left|\left[C_{1n}e^{k_n x}+e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\tfrac{\sqrt{3}}{2}k_nx+C_{3n}\sin\tfrac{\sqrt{3}}{2}k_nx\right)\right]\cos\tfrac{n\pi}{q}y\right|\leq$$
$$\leq M\sum_{n=1}^{+\infty}\left|\left[|C_{1n}|\,e^{k_n x}+|C_{2n}|+|C_{3n}|\right]\right|. \quad (19)$$

Estimating $C_{in},\ i=\overline{1,3}$, we get

$$|C_{1n}|\leq Me^{-k_np}\left[\frac{|\psi_{1n}|}{k_n^2}+\frac{|\psi_{2n}|}{k_n^2}+\frac{|\psi_{3n}|}{k_n}\right],$$

$$|C_{2n}|\leq M\left[\frac{|\psi_{1n}|}{k_n^2}+\frac{|\psi_{2n}|}{k_n^2}+\frac{|\psi_{3n}|}{k_n}\right],$$

$$|C_{3n}|\leq M\left[\frac{|\psi_{1n}|}{k_n^2}+\frac{|\psi_{2n}|}{k_n^2}+\frac{|\psi_{3n}|}{k_n}\right].$$

Substituting the found estimates for $C_{in},\ i=\overline{1,3}$ into (19), we have

$$|u(x,y)|\leq M\sum_{n=1}^{+\infty}e^{k_n(x-p)}\left[\frac{|\psi_{1n}|}{k_n^2}+\frac{|\psi_{2n}|}{k_n^2}+\frac{|\psi_{3n}|}{k_n}\right]\leq M\sum_{n=1}^{+\infty}\left[\frac{|\psi_{1n}|}{k_n^2}+\frac{|\psi_{2n}|}{k_n^2}+\frac{|\psi_{3n}|}{k_n}\right]. \quad (20)$$

Integrating by parts ψ_{1n}, ψ_{2n}, ψ_{3n} and taking into account condition (4), we obtain

$$\psi_{in}=\left(\frac{q}{\pi}\right)^2\frac{\Psi_{in}}{n^2},\quad i=\overline{1,3},$$

where

$$\Psi_{in}=-\sqrt{\frac{2}{q}}\int_0^q\psi''_i(y)\cos\frac{n\pi y}{q}dy.$$

Then

$$|\psi_{in}|\leq M\frac{|\Psi_{in}|}{n^2},\quad i=1,3,\quad M=const>0.$$

Taking these estimates into account, from (20) we find

$$|u(x,y)|\leq M\sum_{n=1}^{+\infty}\left[\frac{|\Psi_{1n}|}{n^{\frac{10}{3}}}+\frac{|\Psi_{2n}|}{n^{\frac{10}{3}}}+\frac{|\Psi_{3n}|}{n^{\frac{8}{3}}}\right]<\infty.$$

It follows that series (18) converges absolutely and uniformly.

Now we prove that the derivatives of series (18) included in equation (1) also converge absolutely and uniformly in the domain $\overline{D}$. To do this, we calculate the derivatives with respect to y, from (18) we obtain

$$\frac{\partial^2 u}{\partial y^2} = -\sqrt{\frac{2}{q}}\left(\frac{\pi}{q}\right)^2 \sum_{n=1}^{+\infty} n^2 \left[C_{1n}e^{k_n x} + e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\frac{\sqrt{3}}{2}k_n x + C_{3n}\sin\frac{\sqrt{3}}{2}k_n x\right)\right]\cos\frac{n\pi}{q}y,$$

taking into account the estimate $u(x,y)$, we have

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \leq M\sum_{n=1}^{+\infty}\left[\frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{4}{3}}} + \frac{|\Psi_{3n}|}{n^{\frac{2}{3}}}\right].$$

Using the Cauchy-Bunyakovsky and Bessel inequalities, we obtain

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \leq M\left[\sum_{n=1}^{+\infty}\frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \sum_{n=1}^{+\infty}\frac{|\Psi_{2n}|}{n^{\frac{4}{3}}} + \sqrt{\sum_{n=1}^{+\infty}|\Psi_{3n}|^2}\sqrt{\sum_{n=1}^{+\infty}\frac{1}{n^{\frac{4}{3}}}}\right] \leq$$
$$\leq M\left[\sum_{n=1}^{+\infty}\frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \sum_{n=1}^{+\infty}\frac{|\Psi_{2n}|}{n^{\frac{4}{3}}} + \|\psi''_{3n}(y)\|\sqrt{\sum_{n=1}^{+\infty}\frac{1}{n^{\frac{4}{3}}}}\right] < \infty;$$

here

$$\sum_{n=1}^{+\infty}|\Psi_{3n}|^2 \leq \left\|\psi''_{3n}(y)\right\|^2_{L_2[0,q]}.$$

Consequently, the series corresponding to the function $\frac{\partial^2 u}{\partial y^2}$ converges absolutely and uniformly. The absolute and uniform convergence of the third derivative with respect to x of series (18) follows from $\left|\frac{\partial^3 u}{\partial x^3}\right| \leq \left|\frac{\partial^2 u}{\partial y^2}\right|$ and what was proven above.

Theorem 2 is proven.

Conclusion

In this paper, the second boundary value problem for the Viscous Transonic Equation in a rectangular domain is investigated. The uniqueness of the solution to the problem is proved using the energy integral method. The existence of a solution is proved using the method of separation of variables. The solution to the problem is found in the form of an infinite series, uniform convergence, and the possibility of term-by-term differentiation under certain conditions on the given functions is proved. The convergence of the second-order derivative of the solution with respect to the variable is proved using the Cauchy-Bunyakovsky and Bessel inequalities. When substantiating the uniform convergence of the solution, the absence of a “small denominator” is proved.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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On solvability of the initial-boundary value problems for a nonlocal hyperbolic equation with periodic boundary conditions

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In this paper, the solvability of initial-boundary value problems for a nonlocal analogue of a hyperbolic equation in a cylindrical domain is studied. The elliptic part of the considered equation involves a nonlocal Laplace operator, which is introduced using involution-type mappings. Two types of boundary conditions are considered. These conditions are specified as a relationship between the values of the unknown function at points in one half of the lateral part of the cylinder and the values at points in the other part of the cylinder boundary. The boundary conditions specified in this form generalize known periodic and antiperiodic boundary conditions for circular domains. The unknown function is represented in the form $u(x) = v(x) + w(x)$, where $v(x)$ is the even part of the function and $w(x)$ is the odd part of the function with respect to the mapping. Using the properties of these functions, we obtain auxiliary initial-boundary value problems with classical hyperbolic equations. In this case, the boundary conditions of these problems are specified in the form of the Dirichlet and Neumann conditions. Further, using the known assertions for the auxiliary problems, theorems on the existence and uniqueness of the solution to the main problems are proved. The solutions to the problems are constructed as a series in systems of eigenfunctions of the Dirichlet and Neumann problems for the classical Laplace operator.

Keywords: antiperiodic condition, Dirichlet problem, eigenfunctions, eigenvalues, Fourier series, hyperbolic equation, initial-boundary value problem, involution, Neumann problem, nonlocal operator, periodic condition.

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Introduction

This paper considers differential equations that belong to the class of equations containing shifts of arguments. Such equations are widely used in describing various scientific models, for example, in modeling immune processes [1, 2], in various population models [3, 4], in modeling the dynamics of nonlinear optical systems [5, 6], and other systems.

Among equations with shifts of arguments, equations with involution occupy a special place. Boundary value and initial-boundary value problems for analogues of elliptic and parabolic equations with involution have been studied by Al-Salti et al. [7, 8], Ashyralyev and Sarsenbi [9, 10], Baranetskij et al. [11], Borikhanov and Mambetov [12], Kozhanov and Bzheumikhova [13], Mussirepova et al. [14, 15], and Yarka et al. [16].

The analogues of hyperbolic equations with involution were considered in [17–19]. In [17], a nonlocal analogue of a hyperbolic equation with involution with respect to the time variable was examined. In the paper, the initial problem was solved by reducing it to an equivalent initial problem

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for a fourth-order equation without involution. The estimates of stability of the solution and its first- and second-order derivatives of the above problem were established. Similar studies were conducted in [18, 19]. In these works, hyperbolic equations with involution with respect to the spatial variable are considered in the one-dimensional case.

In this paper we investigate the solvability of initial-boundary value problems with periodic and antiperiodic boundary conditions in the multidimensional case. Moreover, periodic and antiperiodic boundary conditions are specified on the boundary of a circular cylinder. Boundary value problems with periodic and antiperiodic boundary conditions in circular domains for the Poisson equation were first studied in [20, 21], and for the nonlocal Poisson equation they were investigated in [22]. Note also that boundary value problems with periodic conditions for a hyperbolic equation in rectangular domains were studied in [23].

Let us turn to the formulation of the problems that are considered in this paper. Let Ω be a unit ball, $\partial\Omega$ be a unit sphere, $Q_T = \Omega \times (0,T)$ be an open cylinder. For any $x = (x_1, x_2, ..., x_n)$ we assign a point $Sx = (-x_1, \alpha_2 x_2, ..., \alpha_n x_n)$, where $\alpha_j, j = 2, 3, ..., n$ takes one of the values ± 1.

Let us introduce the operator

$$L_x v(x) \equiv a_0 \triangle v(x) + a_1 \triangle v(Sx),$$

where a_0, a_1 are real numbers, $\triangle = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator.

Denote

$$\partial\Omega_+ = \{x \in \partial\Omega : x_1 \geq 0\}, \quad \partial\Omega_- = \{x \in \partial\Omega : x_1 \leq 0\}, \quad I = \{x \in \partial\Omega : x_1 = 0\}.$$

In the domain Q_T we consider a following problem:

$$\frac{\partial^2 u(t,x)}{\partial t^2} - L_x u(t,x) = f(t,x), \quad (t,x) \in Q_T, \tag{1}$$

$$u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \quad x \in \overline{\Omega}, \tag{2}$$

$$u(t,x) + (-1)^k u(t,Sx) = 0, \quad 0 \leq t \leq T, \quad x \in \partial\Omega_+, \tag{3}$$

$$\partial_\nu u(t,x) - (-1)^k \partial_\nu u(t,Sx) = 0, \quad 0 \leq t \leq T, \quad x \in \partial\Omega_+, \tag{4}$$

where k takes one of the values ± 1 , $\partial_\nu = \frac{\partial}{\partial r}$ is the normal vector, $r = |x|$, $\varphi(x)$ and $\psi(x)$ are the given functions.

A classical solution to problem (1)–(4) is a function $u(t,x)$ from the class $C^{2,2}_{t,x}(Q_T) \cap C^{1,1}_{t,x}(\overline{Q}_T)$ satisfying conditions (1)–(4) in the usual sense.

1 Initial-boundary value problem with Dirichlet boundary condition

In this section we present the well-known statements from V.A. Ilyin's paper [24] regarding the initial-boundary value problem for the classical wave equation

$$\triangle z(t,x) - \frac{1}{a^2}\frac{\partial^2 z(t,x)}{\partial t^2} = -f(t,x), \quad f(t,x) \in Q_T. \tag{5}$$

For equation (5), problems with initial conditions

$$z(0,x) = \tau(x), \quad z_t(0,x) = \rho(x), \quad x \in \overline{\Omega}, \tag{6}$$

and with the Dirichlet boundary condition

$$z(t,x) = 0, \quad [0,T] \times \partial\Omega, \tag{7}$$

or with the Neumann boundary condition

$$\partial_\nu z(t,x) = 0, \quad [0,T]\times\partial\Omega, \tag{8}$$

were studied.

A classical solution to the problem with conditions (5)–(7) (or with conditions (5), (6) and (8)) is a function $z(t,x)$ from the class $C^{2,2}_{t,x}(Q_T)\cap C^{1,1}_{t,x}(\overline{Q}_T)$ satisfying conditions (5)–(7) (or with conditions (5), (6) and (8)) in the usual sense. The following assertions are proved.

Lemma 1. [24] Let the functions $\tau(x), \rho(x)$ and $f(t,x)$ satisfy the following conditions:

1) the function $\tau(x)$ is continuous in the domain $\overline{\Omega}$ and has continuous derivatives up to order $[n/2]+2$ and square-integrable derivatives of order $[n/2]+3$ in this domain. In addition,

$$\tau(x) = \triangle\tau(x) = ... = \triangle^k\tau(x) = 0, \quad k = [(n+4)/4];$$

2) the function $\rho(x)$ is continuous in the domain and has continuous derivatives up to order $[n/2]+1$ and square-integrable derivatives of order $[n/2]+2$ in this domain. In addition,

$$\rho(x) = \triangle\rho(x) = ... = \triangle^k\rho(x) = 0, \quad k = [(n+2)/4];$$

3) the function $f(t,x)$ is continuous in a closed cylinder $\overline{Q}_T = \overline{\Omega}\times[0,T]$ and has continuous derivatives up to order $[n/2]+1$ and square-integrable derivatives of order $[n/2]+2$ in this cylinder. In addition,

$$f(t,x) = \triangle f(t,x) = ... = \triangle^k f(t,x) = 0, \quad k = [(n+2)/4].$$

Then, a classical solution to problem (5)–(7) exists, is unique, and can be represented as

$$z(t,x) = \sum_{m=1}^{\infty}\left\{\tau_m\cos a\sqrt{\mu_m}t + \frac{\rho_m}{a\sqrt{\mu}_m}\sin a\sqrt{\mu_m}t\right\} z_{m,D}(x)+$$

$$+\sum_{m=1}^{\infty}\left\{\frac{a}{\sqrt{\mu}_m}\int_0^t f_m(s)\sin a\sqrt{\mu_m}t-s\right\} z_{m,D}(x).$$

Here $z_{m,D}(x)$ are normalized eigenfunctions of the Dirichlet problem

$$\triangle z(x) + \mu z(x) = 0, \quad x\in\Omega, \quad z(x) = 0, \quad x\in\partial\Omega, \tag{9}$$

and τ_m, ρ_m, and $f_m(t)$ are Fourier coefficients in the expansion of functions $\tau(x), \rho(x)$ and $f(t,x)$ in the system $z_{m,D}(x)$, i.e., $\tau_m = (\tau_m, z_{m,D}(x))$, $\rho_m = (\rho_m, z_{m,D}(x))$ and $f_m = (f_m, z_{m,D}(x))$.

Lemma 2. [24] Let the functions $\tau(x), \rho(x)$ and $f(t,x)$ satisfy the following conditions:

1) the function $\tau(x)$ is continuous in the domain $\overline{\Omega}$ and has continuous derivatives up to order $[n/2]+2$ and square-integrable derivatives of order $[n/2]+3$ in this domain. In addition,

$$\tau(x) = \triangle\tau(x) = ... = \triangle^k\tau(x) = 0, \quad k = [(n+2)/4];$$

2) the function $\rho(x)$ is continuous in the domain and has continuous derivatives up to order $[n/2]+1$ and square-integrable derivatives of order $[n/2]+2$ in this domain. In addition,

$$\rho(x) = \triangle\rho(x) = ... = \triangle^k\rho(x) = 0, \quad k = [n/4];$$

3) the function $f(t, x)$ is continuous in a closed cylinder $\overline{Q}_T = \overline{\Omega} \times [0, T]$ and has continuous derivatives up to order $[n/2] + 1$ and square-integrable derivatives of order $[n/2] + 2$ in this cylinder. In addition,

$$f(t, x) = \triangle f(t, x) = ... = \triangle^k f(t, x) = 0, \quad k = [n/4].$$

Then, a classical solution to the initial boundary value problem for equation (5) with conditions (6), (8) exists, is unique and can be represented as

$$z(t, x) = \sum_{m=1}^{\infty} \left\{ \tau_m \cos a\sqrt{\mu_m} t + \frac{\rho_m}{a\sqrt{\mu}_m} \sin a\sqrt{\mu_m} t \right\} z_{m,N}(x) +$$

$$+ \sum_{m=1}^{\infty} \left\{ \frac{a}{\sqrt{\mu}_m} \int_0^t f_m(s) \sin a\sqrt{\mu_m}(t - s) ds \right\} z_{m,N}(x).$$

Here $z_{m,N}(x)$ are normalized eigenfunctions of the Neumann problem

$$\triangle z(x) + \mu z(x) = 0, x \in \Omega, z(x) = 0, x \in \partial\Omega, \tag{10}$$

and τ_m, ρ_m and $f_m(t)$ are Fourier coefficients in the expansion of functions $\tau(x), \rho(x)$ and $f(t, x)$ in the system $z_{m,N}(x)$.

Further, we present some properties of eigenfunctions $z_{m,D}(x)$ and $z_{m,N}(x)$. In [21], the following statement is proved.

Lemma 3. All eigenfunctions of the Dirichlet problem (9) and the Neumann problem (10) can be chosen so that they have one of the symmetry properties:

$$z(x) - z(Sx) = 0, \tag{11}$$

or

$$z(x) + z(Sx) = 0. \tag{12}$$

2 The main problem

Let $u(t, x)$ be a solution to problem (1)–(4) in the case $k = 1$. From equation (1) we obtain the system

$$\begin{cases} u_t(t, x) - a_0 \triangle u(t, x) - a_1 \triangle u(t, Sx) = f(t, x), \\ u_t(t, Sx) - a_1 \triangle u(t, x) - a_0 \triangle u(t, Sx) = f(t, Sx). \end{cases} \tag{13}$$

We denote the operator of the type $I_S u(t, x) = u(t, Sx)$ as I_S. In [25] it was proved that if S is an orthogonal matrix, then the operator I_S commutes with the operators $\triangle$ and $\Lambda \equiv r\frac{\partial}{\partial r}$, where $r = |x|$. In our case, the mapping matrix S is orthogonal and therefore from (13) it follows that

$$f(t, x) + f(t, Sx) = u_t(t, x) - a_0 \triangle u(t, x) - a_1 \triangle u(t, Sx) + u_t(t, Sx) - a_1 \triangle u(t, x) - a_0 \triangle u(t, Sx) =$$

$$= \partial_t [u(t, x) + u(t, Sx)] - a_o \triangle [u(t, x) + u(t, Sx)] - a_1 \triangle [u(t, x) + u(t, Sx)] =$$

$$= \partial_t [u(t, x) + u(t, Sx)] - (a_0 + a_1) \triangle [u(t, x) + u(t, Sx)],$$

$$f(t, x) - f(t, Sx) = u_t(t, x) - a_0 \triangle u(t, x) - a_1 \triangle u(t, Sx) - [u_t(t, Sx) - a_1 \triangle u(t, x) - a_0 \triangle u(t, Sx)] =$$

$$= \partial_t [u(t, x) - u(t, Sx)] - a_o \triangle [u(t, x) - u(t, Sx)] - a_1 \triangle [u(t, x) - u(t, Sx)] =$$

$$= \partial_t [u(t, x) - u(t, Sx)] - (a_0 - a_1) \triangle [u(t, x) - u(t, Sx)].$$

Let us introduce the notations

$$v(t,x)=\frac{1}{2}[u(t,x)+u(t,Sx)],\quad w(t,x)=\frac{1}{2}[u(t,x)-u(t,Sx)].$$

It is obvious that $u(t,x)=v(t,x)+w(t,x)$ and for all $x\in\Omega$ the symmetry properties

$$v(t,Sx)=v(t,x),\quad w(t,Sx)=-w(t,x)$$

are satisfied.

Then, for the functions $v(t,x)$ and $w(t,x)$, we obtain the following equations

$$v_{tt}(t,x)-(a_0+a_1)\triangle v(t,x)=f^+(t,x),\quad w_{tt}(t,x)-(a_0-a_1)\triangle w(t,x)=f^-(t,x),$$

where $2f^{\pm}(t,x)=f(t,x)\pm f(t,Sx)$.

From initial conditions (2) for the functions $v(t,x)$ and $w(t,x)$, we obtain

$$v(0,x)=\frac{1}{2}[u(0,x)+u(0,Sx)]=\frac{1}{2}[\varphi(x)+\varphi(Sx)]\equiv\varphi^+(x),$$

$$v_t(0,x)=\frac{1}{2}[u_t(0,x)+u_t(0,Sx)]=\frac{1}{2}[\psi(x)+\psi(Sx)]\equiv\psi^+(x),$$

$$w(0,t)=\frac{1}{2}[u(0,x)-u(0,Sx)]=\frac{1}{2}[\varphi(x)-\varphi(Sx)]\equiv\varphi^-(x),$$

$$w_t(0,x)=\frac{1}{2}[u_t(0,x)-u_t(0,Sx)]=\frac{1}{2}[\psi(x)-\psi(Sx)]\equiv\psi^-(x).$$

Further, from boundary condition (3) it follows that if $0\le t\le T,\ x\in\partial\Omega_+$, then

$$v(t,x)\mid_{t\in[0,T],x\in\Omega_+}=u(t,x)+u(t,Sx)\mid_{t\in[0,T],x\in\partial\Omega_+}=0,$$

and if $x\in\partial\Omega_-$, then $Sx\in\partial\Omega_+$, hence

$$v(t,x)\mid_{t\in[0,T],x\in\partial\Omega_-}=u(t,x)+u(t,Sx)\mid_{t\in[0,T],x\in\partial\Omega_-}=u(t,Sx)+u(t,x)\mid_{t\in[0,T],Sx\in\partial\Omega_+}=0.$$

Thus, for the function $v(t,x)$ for all $t\in[0,T]$ and $x\in\partial\Omega$, we have $v(t,x)=0$.

From the symmetry properties of functions $v(t,x)$ and $w(t,x)$, we get the following equalities:

$$\partial_\nu v(t,Sx)\mid_{\partial\Omega}=\Lambda v(t,Sx)\mid_{\partial\Omega}=\Lambda v(t,x)\mid_{\partial\Omega}=\partial_\nu v(t,x)\mid_{\partial\Omega},$$

$$\partial_\nu w(t,Sx)\mid_{\partial\Omega}=\Lambda w(t,Sx)\mid_{\partial\Omega}=-\Lambda w(t,x)\mid_{\partial\Omega}=-\partial_\nu w(t,x)\mid_{\partial\Omega}.$$

Then from boundary condition (4) for the function $w(t,x)$ for all $t\in[0,T]$ and $x\in\partial\Omega$, we obtain the following edge condition

$$\partial_\nu w(t,x)=0.$$

Hence, if $u(t,x)$ is a solution to problem (1)–(4) for $k=1$, then the function $v(t,x)$ satisfies the conditions of the problem

$$v_{tt}(t,x)-(a_0+a_1)\triangle v(t,x)=f^+(t,x),(t,x)\in Q_T, \tag{14}$$

$$v(0,x)=\varphi^+(x),\quad v_t(0,x)=\psi^+(x),\quad x\in\overline{\Omega}, \tag{15}$$

$$v(t,x)=0,\quad [0,T]\times\partial\Omega. \tag{16}$$

Therefore, the function $w(t,x)$ satisfies the conditions of the problem

$$w_{tt}(t,x)-(a_0-a_1)\triangle v(t,x)=f^-(t,x),\quad (t,x)\in Q_T, \tag{17}$$

$$w(0,x)=\psi^-(x),\quad w_t(0,x)=\psi^-(x),\quad x\in\overline{\Omega}, \tag{18}$$

$$\partial_\nu w(t,x)=0,\quad [0,T]\times\partial\Omega. \tag{19}$$

Thus, we have obtained two auxiliary initial-boundary value problems for the classical wave equation. In the first problem, the boundary condition is specified in the form of the Dirichlet condition, and in the second problem, in the form of the Neumann condition.

Further, we assume that $a_0\pm a_1>0$ and rewrite equations (14) and (17) as

$$\triangle v(t,x)-\frac{1}{\varepsilon_0^2}v_{tt}(t,x)=-\frac{1}{\varepsilon_0^2}f^+(t,x),\quad (t,x)\in Q_T,$$

$$\triangle w(t,x)-\frac{1}{\varepsilon_1^2}w_{tt}(t,x)=-\frac{1}{\varepsilon_1^2}f^-(t,x),\quad (t,x)\in Q_T,$$

where $\varepsilon_0=\sqrt{a_0+a_1},\ \varepsilon_1=\sqrt{a_0-a_1}$.

To study the solvability of problem (14)–(16), we can use the assertion of Lemma 1. If the functions $f^+(t,x)$, $\varphi^+(x)$ and $\psi^+(x)$ satisfy the conditions of this lemma, then the classical solution to problem (14)–(16) exists, is unique, and can be represented as

$$v(t,x)=\sum_{m=1}^{\infty}\left\{\varphi_m^+\cos\varepsilon_0\sqrt{\mu_{m,D}}t+\frac{\psi_m^+}{\varepsilon_0\sqrt{\mu_{m,D}}}\sin\varepsilon_0\sqrt{\mu_{m,D}}t\right\}z_{m,D}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\frac{1}{\varepsilon_0\sqrt{\mu_{m,D}}}\int_0^t f_m^+(s)\sin\varepsilon_0\sqrt{\mu_{m,D}}(t-s)ds\right\}z_{m,D}(x), \tag{20}$$

where $\varphi_m^+=(\varphi^+,z_{m,D}),\ \psi_m^+=(\psi^+,z_{m,D})$ and $f_m^+(t)=(f^+,z_{m,D})$.

Similarly, if the functions $f^-(t,x),\ \varphi^-$ and ψ^- satisfy the conditions of Lemma 2, then the classical solution to problem (17)-(18) exists, is unique, and is represented as

$$w(t,x)=\sum_{m=1}^{\infty}\left\{\varphi_m^-\cos\varepsilon_1\sqrt{\mu_{m,N}}t+\frac{\psi_m^-}{\varepsilon_1\sqrt{\mu_{m,N}}}\sin\varepsilon_1\sqrt{\mu_{m,N}}t\right\}z_{m,N}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\frac{1}{\varepsilon_1\sqrt{\mu_{m,N}}}\int_0^t f_m^-(s)\sin\varepsilon_1\sqrt{\mu_{m,N}}(t-s)ds\right\}z_{m,N}(x), \tag{21}$$

where $\varphi_m^-=(\varphi^-,z_{m,D}),\ \psi_m^+=(\psi^+,z_{m,D})$ and $f_m^-(t)=(f^-,z_{m,D})$.

Let us transform the functions $v(t,x)$ and $w(t,x)$ from equalities (20) and (21). To do this, we use the properties of the eigenfunctions $z_{m,D}(x)$ and $z_{m,N}(x)$ formulated in Lemma 3. In this case, we renumber the eigenfunctions $z_{m,D}(x)$ as follows: we denote the eigenfunctions with property (11) as $z_{2m,D}(x)$, and the eigenfunctions with property (12) as $z_{2m-1,D}(x)$. We will use a similar notation for the eigenfunctions $z_{m,D}(x)$ and $z_{m,N}(x)$.

Then, for the coefficients of the function $\varphi(x)$, we have

$$\varphi_m^+=\frac{1}{2}\int\limits_\Omega[\varphi(x)+\varphi(Sx)]z_{m,D}(x)dx=\frac{1}{2}\int\limits_\Omega\varphi(x)[z_{m,D}(x)+z_{m,D}(Sx)]dx.$$

Further, if $m=2j-1,\ j=1,2,...$, then $z_{2j-1,D}(x)+z_{2j-1,D}(Sx)=0$ and if $m=2j,\ j=1,2,...$, then $z_{2j,D}(x)+z_{2j,D}(Sx)=2z_{2j,D}(x)$, thus

$$\varphi_{2m}^{+}=\int\limits_{\Omega}\varphi(x)z_{m,D}(x)dx=\varphi_{2m},\quad m\geq 1.$$

Similarly, for the coefficients φ_{2m-1}^{-}, we obtain the equalities

$$\varphi_{2m-1}^{-}=\int\limits_{\Omega}\varphi(x)z_{2m-1,N}(x)dx=\varphi_{2m-1},\quad m\geq 1.$$

Similar equalities can be obtained for the coefficients $f_m^{\pm}(t,x)$ and $\psi^{-}(x)$:

$$\psi_{2m}^{+}=\psi_{2m}\equiv(\psi,z_{2m,D}),\quad \psi_{2m-1}^{-}=\psi_{2m-1}\equiv(\psi,z_{2m-1,N}),$$

$$f_{2m}^{+}(t)=f_{2m}\equiv(f,z_{2m,D}),\quad f_{2m-1}^{-}(t)=f_{2m-1}\equiv(f,z_{2m-1,N}).$$

Then, formula (20), or more precisely the solution to problem (14)–(16) can be rewritten as

$$v(t,x)=\sum_{m=1}^{\infty}\left\{\varphi_{2m}\cos\varepsilon_0\sqrt{\mu_{2m,D}}t+\frac{\psi_{2m}}{\varepsilon_0\sqrt{\mu_{2m,D}}}\sin\varepsilon_0\sqrt{\mu_{2m,D}}t\right\}z_{2m,D}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\frac{1}{\varepsilon_0\sqrt{\mu_{2m,D}}}\int_0^t f_{2m}(s)\sin\varepsilon_0\sqrt{\mu_{2m,D}}(t-s)ds\right\}z_{2m,D}(x),\quad (22)$$

and formula (21) as

$$w(t,x)=\sum_{m=1}^{\infty}\left\{\varphi_{2m-1}\cos\varepsilon_1\sqrt{\mu_{2m-1,N}}t+\frac{\psi_{2m-1}}{\varepsilon_1\sqrt{\mu_{2m-1,N}}}\sin\varepsilon_1\sqrt{\mu_{2m-1,N}}t\right\}z_{2m-1,N}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\frac{1}{\varepsilon_1\sqrt{\mu_{2m-1,N}}}\int_0^t f_{2m-1}(s)\sin\varepsilon_1\sqrt{\mu_{2m-1,N}}(t-s)ds\right\}z_{2m-1,N}(x).\quad (23)$$

Now we present the main assertion regarding problem (1)–(4).

Theorem 1. Let $k=1, a_0\pm a_1>0$, functions $f(t,x)$, $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Lemma 1. Then, the classical solution to problem (1)–(4), exists, is unique, and can be represented as

$$u(t,x)=\sum_{m=1}^{\infty}\left\{\varphi_{2m}\cos\sqrt{(a_0+a_1)\,\mu_{2m,D}}t+\frac{\psi_{2m}}{\sqrt{(a_0+a_1)\,\mu_{2m,D}}}\sin(a_0+a_1)\sqrt{\mu_{2m}}t\right\}z_{2m,D}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\varphi_{2m-1}\cos\sqrt{(a_0-a_1)\,\mu_{2m-1,N}}t+\frac{\psi_{2m-1}}{\sqrt{(a_0-a_1)\,\mu_{2m-1,N}}}\sin\sqrt{(a_0-a_1)\,\mu_{2m-1,N}}t\right\}z_{2m-1,N}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\frac{1}{\sqrt{(a_0+a_1)\,\mu_{2m,D}}}\int_0^t f_{2m}(s)\sin\sqrt{(a_0+a_1)\,\mu_{2m,D}}(t-s)ds\right\}z_{2m,D}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\frac{1}{\sqrt{(a_0-a_1)\,\mu_{2m-1,N}}}\int_0^t f_{2m-1}(s)\sin\sqrt{(a_0-a_1)\,\mu_{2m-1,N}}(t-s)ds\right\}z_{2m-1,N}(x).\quad (24)$$

Proof. If the functions $f(t,x)$, $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Lemma 1, then the functions $f^+(t,x)$, $\varphi^+(x)$ and $\psi^+(x)$ satisfy the same conditions. Then, by the assertion of Lemma 1, the solution to problem (14)–(16) exists, is unique, and can be represented in the form (20). If the functions $f^-(t,x)$, $\varphi^-(x)$ and $\psi^-(x)$ satisfy the conditions of Lemma 1, they also satisfy the conditions of Lemma 2. Then, by the assertion of Lemma 2, the solution to problem (17)–(19) with functions exists, is unique, and can be represented in the form (21). Note that the functions $v(t,x)$ and $w(t,x)$ from equalities (22) and (23) have the symmetry properties $v(t,Sx) = v(t,x)$ and $w(t,Sx) = -w(t,x)$. We will show that the function $u(t,x) = v(t,x) + w(t,x)$ will be a classical solution to problem (1)–(4).

Indeed, the following equalities hold for this function

$$u_{tt}(t,x) - L_x u(t,x) =$$
$$= v_{tt}(t,x) - a_0 \triangle v(t,x) - a_1 \triangle v(t,Sx) + w_{tt}(t,x) - a_0 \triangle w(t,x) - a_1 \triangle w(t,Sx) =$$
$$= v_{tt}(t,x) - (a_0 + a_1)\triangle v(t,x) + w_{tt}(t,x) - (a_0 - a_1)\triangle w(t,x) =$$
$$= f^+(t,x) + f^-(t,x) = f(t,x),$$
$$u(0,x) = v(0,x) + w(0,x) = \varphi^+(x) + \varphi^-(x) = \varphi(x), \quad x \in \overline{\Omega},$$
$$u_t(0,x) = v_t(0,x) + w_t(0,x) = \psi^+(x) + \psi^-(x) = \psi(x), \quad x \in \overline{\Omega}.$$

From the symmetry conditions, as well as from boundary conditions (16) and (19) for $x \in \partial\Omega_+$ for $k = 1$, we obtain

$$u(t,x) + u(t,Sx) = v(t,x) + w(t,x) + v(t,Sx) + w(t,Sx) =$$
$$= [v(t,x) + v(t,Sx)] + [w(t,x) + w(t,Sx)] = 2v(t,x) + [w(t,x) - w(t,Sx)] = 0$$

and

$$\partial_\nu u(t,x) - \partial_\nu u(t,Sx) = \partial_\nu \left[v(t,x) - v(t,Sx)\right] + \partial_\nu \left[w(t,x) + w(t,Sx)\right] =$$
$$= \partial_\nu [0] + \partial_\nu w(t,x) = 0.$$

Thus, boundary conditions (3) and (4) are also satisfied. Then, substituting the values of the functions $v(t,x)$ and $w(t,x)$ from equalities (22) and (23) into the left-hand side of the equality $u(t,x) = v(t,x) + w(t,x)$, we obtain representation (24). The theorem is proved.

We conduct similar studies in the case $k = 2$. In this case, if we choose functions $v(t,x)$ and $w(t,x)$ in the form (13), then we obtain a problem with conditions (14), (15) and the Neumann boundary condition $\partial_\nu v(t,x) = 0$, $[0,T] \times \partial\Omega$.

Hence, for the function $w(t,x)$, we obtain a problem with conditions (17), (18) and the Dirichlet boundary condition $w(t,x) = 0, [0,T] \times \partial\Omega$. The main assertion regarding problem (1)–(4) in the case $k = 2$ is the following theorem.

Theorem 2. Let $k = 2, a_0 \pm a_1 > 0$, functions $f(t,x), \varphi(x)$ and $\psi(x)$ the functions and satisfy the conditions of Lemma 1. Then the classical solution to problem (1)–(4) exists, is unique and can be represented in the form

$$u(t,x)=\sum_{m=1}^{\infty}\left\{\varphi_{2m-1}\cos\sqrt{(a_0+a_1)\mu_{2m-1,D}}t+\frac{\psi_{2m-1}}{\sqrt{(a_0+a_1)\mu_{2m-1,D}}}\sin(a_0+a_1)\sqrt{\mu_{2m}}t\right\}z_{2m-1,D}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\varphi_{2m}\cos\sqrt{(a_0-a_1)\mu_{2m,N}}t+\frac{\psi_{2m}}{\sqrt{(a_0-a_1)\mu_{2m,N}}}\sin(a_0-a_1)\sqrt{\mu_{2m,N}}t\right\}z_{2m,N}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\frac{1}{\sqrt{(a_0+a_1)\mu_{2m-1,D}}}\int_0^t f_{2m-1}(s)\sin\sqrt{(a_0+a_1)\mu_{2m-1,D}}(t-s)ds\right\}z_{2m-1,D}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\frac{1}{\sqrt{(a_0-a_1)\mu_{2m,N}}}\int_0^t f_{2m}(s)\sin\sqrt{(a_0-a_1)\mu_{2m,N}}(t-s)ds\right\}z_{2m,N}(x).$$

Conclusion

In this paper, the initial-boundary value problem for an analogue of a hyperbolic equation with involution is studied in a multidimensional circular cylinder. Periodic and antiperiodic conditions are considered as boundary conditions. The unknown function is represented as the sum of an even and odd part with respect to the involution transformation. For auxiliary functions, initial-boundary functions for the classical hyperbolic equation are obtained. Using known assertions for the obtained problems, theorems on the existence and uniqueness of the main problems are proved.

It is planned to study similar problems for analogues of hyperbolic equations with multiple involution.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2025M3/57-67 *Research article*

Mixed problem for a third order parabolic-hyperbolic model equation

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In 1978, the journal *Differential Equations* published an article by A.M. Nakhushev, that presented a method for correctly formulating a boundary value problem for a class of second-order parabolic-hyperbolic equations in an arbitrarily bounded domain with a smooth or piecewise smooth boundary. In that work, a boundary value problem was formulated and investigated using the method of a priori estimates, which is currently called the first boundary value problem for a second-order mixed parabolic-hyperbolic equation. In this work, a boundary value problem for a third-order model parabolic-hyperbolic equation is formulated and investigated in a mixed domain, following the approach of A.M. Nakhushev for second-order mixed parabolic-hyperbolic equations. In one part of the mixed domain, the equation under consideration is a degenerate hyperbolic equation of the first kind of the second order, and in the other part, it is a nonhomogeneous equation of the third order with multiple characteristics and reverse-time parabolic type. For various values of the parameter, existence and uniqueness theorems for a regular solution are proved. The uniqueness theorem is proved using the method of energy integrals combined with A.M. Nakhushev's method. The existence theorem is proved by the method of integral equations. In terms of the Mittag-Leffler function, the solution of the problem is found and written out explicitly. Sufficient smoothness conditions for the given functions are found, which ensure the regularity of the obtained solution.

Keywords: second order degenerate hyperbolic equation of the first kind, third-order equation with multiple characteristics, third-order parabolic-hyperbolic equation, Volterra integral equation, Fredholm integral equation, Tricomi method, method of integral equations, integral equation method, Mittag-Leffler functions.

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Introduction

Boundary value problems for model second order parabolic-hyperbolic equations were first studied in [1,2]. The classification of parabolic-hyperbolic equations into equations with characteristic and non-characteristic lines of type change was carried out by [3]. Moreover, in the work of [1], the problem was studied for a model equation of parabolic-hyperbolic type with a characteristic line of type change, and in the work of [2], the problem was studied for a model equation with a non-characteristic line of type change. In 1978, the journal Differential Equations published an article by A.M. Nakhushev, which provided a method for correctly formulating a boundary value problem for a general second-order parabolic-hyperbolic equation in an arbitrary bounded domain with a smooth or piecewise smooth boundary. The boundary value problem investigated in the aforementioned work by A.M. Nakhushev is currently called the first boundary value problem for a mixed parabolic-hyperbolic equation. The most complete review on boundary value problems for parabolic-hyperbolic equations one can find in monographs [4, 5].

The paper considers one mixed problem for a third-order parabolic-hyperbolic model equation. One part of the mixed domain, involves a third order nonhomogeneous parabolic type equation with multiple characteristics, while the other part involves a second order degenerate hyperbolic type equation of the first kind. The paper presents proofs of the existence and uniqueness theorems for a regular solution.

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The uniqueness proof is based on the method of energy integrals combined with A.M. Nakhushev's method. The existence proof is based on the method of integral equations. In solving the problem, we also used the Mittag-Leffler function and wrote down the solution explicitly.

1 *Formulation of the problem*

On the Euclidean plane with independent variables x and y consider the equation

$$0=\begin{cases} (-y)^m u_{xx}-u_{yy}+\lambda(-y)^{(m-2)/2}u_x, & y<0,\\ u_{xxx}+u_y-f, & y>0,\end{cases} \tag{1}$$

where λ, m are the given numbers, and $m>0$, $|\lambda|\leq\frac{m}{2}$; $f=f(x,y)$ is the given function; $u=u(x,y)$ is the desired function.

Equation (1) is a model third order equation of the parabolic-hyperbolic type. For $y<0$, it is equivalent to the degenerate hyperbolic equation of the first kind

$$(-y)^m u_{xx}-u_{yy}+\lambda(-y)^{\frac{m-2}{2}}u_x=0, \tag{2}$$

and for $y>0$ with the third-order inhomogeneous parabolic type equation with multiple characteristics

$$u_{xxx}+u_y=f(x,y). \tag{3}$$

The paper [6] is devoted to the study of the problem with a shift for a degenerate hyperbolic equation of the first kind of the form (2), and the local first and second boundary value problems for a degenerate hyperbolic equation of the second kind are investigated in the papers [7, 8]. The papers [9, 10] are devoted to the study of nonlocal problems of degenerate hyperbolic equations with singular coefficients. In [11], a nonlocal problem of the Frankl type for a second-order mixed parabolic-hyperbolic equation with a characteristic line of type change is investigated. The papers [12, 13] are devoted to the problems of conjugation of the generalized diffusion equation and degenerate hyperbolic equations. The problem with a shift for one second-order mixed parabolic-hyperbolic equation with two perpendicular lines of type change is studied in the paper [14]. Nonlocal problems with a shift in the conjugation of a third-order equation with multiple characteristics and a degenerate hyperbolic equation of the first kind of the second order are formulated and investigated in [15, 16]. A nonlocal problem for a third-order mixed parabolic-hyperbolic equation is investigated in [17].

Equation (1), in this paper, is considered in the mixed domain $\Omega=\Omega_1\cup\Omega_2\cup I$, where Ω_1 is the domain limited by the characteristics $\sigma_1=AC:\ x-\frac{2}{m+2}(-y)^{(m+2)/2}=0$ and $\sigma_2=CB:\ x+\frac{2}{m+2}(-y)^{(m+2)/2}=r$ of equation (2) for $y<0$, coming out from the point $C=(r/2,y_c)$, $y_c=-\left[\frac{(m+2)\,r}{4}\right]^{\frac{2}{m+2}}$, passing through the points $A=(0,0)$, $B=(r,\,0)$ and the segment $J=AB$ of the straight line $y=0$; and Ω_2 is the rectangular domain with vertices at $A=(0,\,0)$, $A_0=(0,\,h)$, $B_0=(r,\,h)$ and $B=(r,\,0)$, $h=const>0$; $J=\{(x,0):\ 0<x<r\}$ is the interval of AB of the straight line $y=0$.

A regular, in the domain Ω, solution to equation (1) we call the function $u=u(x,y)$ by the class $C\left(\bar{\Omega}\right)\cap C^1(\Omega)\cap C^2(\Omega_1)\cap C^{x,\,y}_{3,\,1}(\Omega_2)$; $u_x(x,0)$, $u_y(x,0)\in L_1(J)$, when substituted, equation (1) becomes an identity.

Problem 1. Find a solution to equation (1) regular in the domain and satisfies the conditions

$$u(0,y)=\varphi_1(y),\quad u(r,y)=\varphi_2(y),\quad u_x(r,y)=\varphi_3(y),\quad 0<y<h, \tag{4}$$

$$u\left[\theta_0(x)\right]=\psi(x),\quad 0\leq x\leq r, \tag{5}$$

where $\theta_0(x) = \left(\frac{x}{2}, -\left(\frac{m+2}{4}\right)^{2/(m+2)} x^{2/(m+2)}\right)$ is the affix of the intersection point of a characteristic emanating from the point $(x, 0) \in J$ with the characteristic AC; $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$ are the functions defined on the segment $0 \le y < h$; $\psi(x)$ is the function given on the segment $0 \le x \le r$ with the matching condition $\varphi_1(0) = \psi(0)$ satisfied.

2 Uniqueness theorem

Let there be a regular solution $u = u(x, y)$ of equation (1) in the domain Ω by the class $C\left(\bar{\Omega}\right) \cap C^1(\Omega)$ and let

$$u(x,0) = \tau(x), \quad 0 \le x \le r, \tag{6}$$

$$u_y(x,0) = \nu(x), \quad 0 < x < r. \tag{7}$$

Then, passing in equation (1) to the limit at $y \to +0$, taking into account the notations (6), (7) and conditions (4), we immediately obtain the first fundamental relationship between the functions $\tau(x)$ and $\nu(x)$, transferred from the parabolic part Ω_2 of the domain Ω to the line of type change J:

$$\tau'''(x) + \nu(x) = f(x,0), \quad 0 < x < r, \tag{8}$$

$$\tau(0) = \varphi_1(0), \quad \tau(r) = \varphi_2(0), \quad \tau'(r) = \varphi_3(0). \tag{9}$$

Next, find the relationship between the functions $\tau(x)$ and $\nu(x)$, brought from the hyperbolic domain Ω_1 of equation (1) to the segment AB of the straight line $y = 0$. To do this, we first note that in the characteristic coordinates $\xi = x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}}$, $\eta = x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}}$, equation (2) becomes the Euler–Darboux–Poisson equation

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta} - \frac{\beta_1}{\eta-\xi}\frac{\partial u}{\partial\eta} + \frac{\beta_2}{\eta-\xi}\frac{\partial u}{\partial\xi} = 0,$$

where $\beta_1 = \frac{m-2\lambda}{2(m+2)}$, $\beta_2 = \frac{m+2\lambda}{2(m+2)}$. Designate additionally: $\beta = \beta_1 + \beta_2 = \frac{m}{m+2}$.

First assume $|\lambda| < \frac{m}{2}$ and then $\tau(x) \in C[0, r] \cap C^2(0, r)$, $\nu(x) \in C^1(0, r) \cap L_1(0, r)$. Hence, the regular solution to problem (6), (7) for equation (2) in Ω_1 is written out by the formula in [18; p. 14]:

$$u(x,y) = \frac{\Gamma(\beta)}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^1 \tau\left[x + (1-\beta)(-y)^{1/(1-\beta)}(2t-1)\right] t^{\beta_2-1}(1-t)^{\beta_1-1}\,dt +$$

$$+ \frac{\Gamma(2-\beta)\,y}{\Gamma(1-\beta_1)\Gamma(1-\beta_2)}\int_0^1 \nu\left[x + (1-\beta)(-y)^{1/(1-\beta)}(2t-1)\right] t^{-\beta_1}(1-t)^{-\beta_2}\,dt, \tag{10}$$

where $\Gamma(p) = \int_0^\infty \exp(-t)\, t^{p-1}\,dt$ is the Euler integral of the second kind (Gamma function).

Satisfying in (10) condition (5), we get:

$$u[\theta_0(x)] = u\left[\frac{x}{2}, -(2-2\beta)^{\beta-1} x^{1-\beta}\right] = \frac{\Gamma(\beta)}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^1 \tau(xt)\, t^{\beta_2-1}(1-t)^{\beta_1-1}\,dt -$$

$$-\frac{(2-2\beta)^{\beta-1} x^{1-\beta}\,\Gamma(2-\beta)}{\Gamma(1-\beta_1)\Gamma(1-\beta_2)}\int_0^1 \nu(xt)\, t^{-\beta_1}(1-t)^{-\beta_2}\,dt = \psi(x).$$

Introducing the new integration variable $z = xt$, rewrite the last equality as

$$\frac{\Gamma(\beta)\, x^{1-\beta}}{\Gamma(\beta_1)\,\Gamma(\beta_2)} \int\limits_0^x \frac{\tau(z)\, z^{\beta_2-1}}{(x-z)^{1-\beta_1}}\, dz - \frac{(2-2\beta)^{\beta-1}\, \Gamma(2-\beta)}{\Gamma(1-\beta_1)\,\Gamma(1-\beta_2)} \int\limits_0^1 \frac{\nu(z)\, z^{-\beta_1}}{(x-z)^{\beta_2}}\, dz = \psi(x).$$

Employing the fractional integro-differentiation operator D^{α}_{cx} (in the Riemann–Liouville sense) [19], rewrite the last equality as follows

$$\frac{\Gamma(\beta)\, x^{1-\beta}}{\Gamma(\beta_2)} D^{-\beta_1}_{0x} \left\{\tau(t)\, t^{\beta_2-1}\right\} - \frac{(2-2\beta)^{\beta-1}\, \Gamma(2-\beta)}{\Gamma(1-\beta_1)} D^{\beta_2-1}_{0x} \left\{\nu(t)\, t^{-\beta_1}\right\} = \psi(x). \tag{11}$$

Inverting equation (11) relative to the function $\nu(x)$, and using the well-known weighted Riemann–Liouville fractional integral and differential operators with the same origins [19; p. 18], find

$$\nu(x) = \gamma_1\, D^{1-\beta}_{0x} \tau(t) - \gamma_2\, x^{\beta_1}\, D^{1-\beta_2}_{0x} \psi(t), \tag{12}$$

where $\gamma_1 = \frac{\Gamma(1-\beta_1)\,\Gamma(\beta)\,(2-2\beta)^{1-\beta}}{\Gamma(\beta_2)\,\Gamma(2-\beta)}$, $\gamma_2 = \frac{\Gamma(1-\beta_1)(2-2\beta)^{1-\beta}}{\Gamma(2-\beta)}$.

Indeed relation (12) is the main fundamental relation between the sought functions $\tau(x)$ and $\nu(x)$ transferred from the domain Ω_1 to the line of type change J when $|\lambda| < \frac{m}{2}$.

In the case if $\lambda = \frac{m}{2}$, the coefficients $\beta_1 = 0$, $\beta_2 = \beta = \frac{m}{m+2}$, $\gamma_1 = \gamma_2 = \frac{(2-2\beta)^{1-\beta}}{\Gamma(2-\beta)}$, and the solution to problem (6), (7) for equation (2) can be written by the formula [18; p. 15]:

$$u(x,y) = \tau\left[x + \frac{2}{m+2}(-y)^{(m+2)/2}\right] + \frac{2y}{m+2} \int\limits_0^1 \nu\left[x + \frac{2}{m+2}(-y)^{(m+2)/2}(2t-1)\right](1-t)^{-\beta}\, dt. \tag{13}$$

Satisfying condition (5) in representation (13), we arrive at the fundamental relationship between the functions $\tau(x)$ and $\nu(x)$ as bellow

$$\nu(x) = \gamma_1 \left[D^{1-\beta}_{0x} \tau(t) - D^{1-\beta}_{0x} \psi(t)\right]. \tag{14}$$

In the case if $\lambda = -\frac{m}{2}$, then $\beta_1 = \beta = \frac{m}{m+2}$, $\beta_2 = 0$, $\gamma_1 = 0$, $\gamma_2 = 2^{1-\beta}(1-\beta)^{-\beta}$. The solution to problem (6), (7) for equation (2) here has the form [18; p. 15]:

$$u(x,y) = \tau\left[x - \frac{2}{m+2}(-y)^{(m+2)/2}\right] + \frac{2y}{m+2} \int\limits_0^1 \nu\left[x - \frac{2}{m+2}(-y)^{(m+2)/2}(2t-1)\right](1-t)^{-\beta}\, dt. \tag{15}$$

By (15) under condition (5), we immediately get:

$$\nu(x) = -2^{1-\beta}(1-\beta)^{-\beta}\, x^{\beta}\, \psi'(x). \tag{16}$$

The following theorem on the unique solution to *Problem 1* is true.

Theorem 1. There cannot be more than one regular solution for *Problem 1* in the domain Ω.

Proof. Let's take a homogeneous problem equivalent to *Problem 1.* For instance, assume that $f(x,y) \equiv 0\ \forall\,(x,y) \in \bar{\Omega}_2$, $\varphi_1(y) = \varphi_2(y) = \varphi_3(y) \equiv 0\ \forall\, y \in [0,h]$ and $\psi(x) \equiv 0\ \forall\, x \in [0,r]$. Moreover, taking into account that $\tau(0) = \psi(0) = 0$ by relations (12), (14), (16) for different λ, obtain the bellow equalities:

$$\nu(x) = \gamma_1\, D^{1-\beta}_{0x} \tau(t) = \gamma_1\, D^{-\beta}_{0x} \tau'(t) = \gamma_1\, \partial^{1-\beta}_{0x} \tau(t), \quad -\frac{m}{2} < \lambda \le \frac{m}{2}, \tag{17}$$

$$\nu(x) \equiv 0, \quad \lambda = -\frac{m}{2}, \tag{18}$$

where $\partial_{0x}^{\alpha}\varphi\,(t)$ is the fractional differential operator (in the sense of Caputo).

To further discuss, make use of the operator $\partial_{0x}^{\alpha}\varphi\,(t)$ following property [20]: for any absolutely continuous function $\varphi = \varphi(x)$ on the segment $[0, r]$ that satisfies the condition $\varphi(0) = 0$, the inequality

$$\varphi(x)\,\partial_{0x}^{\alpha}\varphi\,(t) \geq \frac{1}{2}\,\partial_{0x}^{\alpha}\varphi^2\,(t)\,, \quad 0 < \alpha \leq 1 \tag{19}$$

holds.

Let us consider the integral

$$I = \int\limits_0^r \tau(x)\,\nu(x)\,dx. \tag{20}$$

When $-\frac{m}{2} < \lambda \leq \frac{m}{2}$ by (17) and (20), taking into account inequality (19), we arrive at

$$I = \int\limits_0^r \tau(x)\,\nu(x)\,dx = \gamma_1 \int\limits_0^r \tau(x)\,\partial_{0x}^{1-\beta}\tau\,(t)\,\,dx \geq$$

$$\geq \frac{\gamma_1}{2}\int\limits_0^r \partial_{0x}^{1-\beta}\tau^2\,(t)\,\,dx = \frac{\gamma_1}{2\Gamma\,(\beta)}\int\limits_0^r (r-x)^{\beta-1}\,\tau^2(x)\,dx \geq 0. \tag{21}$$

On the other hand, for a homogeneous problem equivalent to *Problem 1* write, bearing in mind (8), (9), the integral (20) as follows

$$I = \int\limits_0^r \tau(x)\,\nu(x)\,dx = -\int\limits_0^r \tau(x)\,\tau'''(x)\,dx = -\frac{1}{2}\,\left[\tau'(0)\right]^2 \leq 0. \tag{22}$$

By inequalities (21) and (22) it follows that the integral $I = 0$, which as follows from the equality,

$$I = \frac{\gamma_1}{2\Gamma\,(\beta)}\int\limits_0^r (r-x)^{\beta-1}\,\tau^2(x)\,dx = 0$$

may occur if and only if $\tau(x) \equiv 0\;\forall\, x \in [0,\, r]$. Then basing on relations (8) and (17) find out that $\nu(x) \equiv 0$ for all $x \in [0,\, r]$ and any $\lambda \in \left(-\frac{m}{2};\, \frac{m}{2}\right]$.

However, if $\lambda = -\frac{m}{2}$, then by (8), (9), and (18) we come to the homogeneous problem

$$\tau(0) = 0, \quad \tau(r) = 0, \quad \tau'(r) = 0 \tag{23}$$

for equation

$$\tau'''(x) = 0, \quad 0 < x < r. \tag{24}$$

Just like in the case $\lambda \in \left(-\frac{m}{2};\, \frac{m}{2}\right]$, the solution to problem (23) for equation (24) cannot be anything but trivial: $\tau(x) \equiv 0$ and $\nu(x) \equiv 0$ for all $x \in [0,\, r]$.

Consequently, as per formula (10), (13) and (15), the solution $u(x, y) \equiv 0$ in Ω_1 to be considered as the solution to homogeneous Cauchy problem (6), (7) for equation (2) for all $\lambda \in \left[-\frac{m}{2};\, \frac{m}{2}\right]$.

Let's show now that even for the homogenous problem

$$Lu = u_{xxx} + u_y = 0, \quad (x, y) \in \Omega_2, \tag{25}$$

$$u(0,y)=0,\quad u(r,y)=0,\quad u_x(r,y)=0,\quad 0<y<h, \tag{26}$$

$$u(x,0)=0,\quad 0\le x\le r \tag{27}$$

in the domain Ω_2 regular solutions are not possible except for trivial ones.

Indeed, let's assume that problems (25)–(27) have a nontrivial solution $u=u(x,y)\neq 0$. Following the work [4; p. 237], in equation (26) put

$$u(x,y)=\upsilon(x,y)\,\exp\left(\mu\,y\right), \tag{28}$$

where $\mu=const$ is some real number.

In this case, by (25) relative to the function $\upsilon=\upsilon(x,y)$, we arrive at the equation

$$L_\mu\upsilon=\upsilon_{xxx}+\upsilon_y+\mu\,\upsilon=0,\quad (x,y)\in\Omega_2 \tag{29}$$

with initial boundary conditions

$$\upsilon(0,y)=0,\quad \upsilon(r,y)=0,\quad \upsilon_x(r,y)=0,\quad 0<y<h, \tag{30}$$

$$\upsilon(x,0)=0,\quad 0\le x\le r. \tag{31}$$

Since, by assumption $u=u(x,y)\neq 0$, then, as follows from (28), the solutions to problems (29)–(31) will also be non-trivial $\upsilon=\upsilon(x,y)\neq 0$.

Introduce an auxiliary domain $\Omega_{2\varepsilon}=\{(x,y):\ \varepsilon<x<r-\varepsilon,\ \varepsilon<y<h-\varepsilon,\ \varepsilon>0\}$, where the identity

$$2\left(\upsilon,\,L_\mu\upsilon\right)_0=2\int\limits_{\Omega_{2\varepsilon}}\upsilon\,L_\mu\upsilon\,d\Omega_{2\varepsilon}=2\int\limits_{\Omega_{2\varepsilon}}\upsilon\,\left[\upsilon_{xxx}+\upsilon_y+\mu\,\upsilon\right]\,d\Omega_{2\varepsilon}=$$

$$=\int\limits_{\Omega_{2\varepsilon}}\left[\frac{\partial}{\partial x}\left(2\,\upsilon\,\upsilon_{xx}-\upsilon_x^2\right)+\frac{\partial}{\partial y}\left(\upsilon^2\right)+2\mu\,\upsilon^2\right]\,d\Omega_{2\varepsilon}=0$$

is valid.

Applying Green's formula to the latter equality, obtain

$$2\left(\upsilon,\,L_\mu\upsilon\right)_0=\int\limits_{\Gamma_{2\varepsilon}}\left(2\,\upsilon\,\upsilon_{xx}-\upsilon_x^2\right)\,dy-\upsilon^2\,dx+2\,\mu\int\limits_{\Omega_{2\varepsilon}}\upsilon^2(x,y)\,d\Omega_{2\varepsilon}=0, \tag{32}$$

where $\Gamma_{2\varepsilon}$ is the auxiliary boundary for $\Omega_{2\varepsilon}$. Let us pass to the limit in the last equality at $\varepsilon\to 0$. It is easy to see that in this case the auxiliary domain $\Omega_{2\varepsilon}$ goes into the domain Ω_2, and the boundary $\Gamma_{2\varepsilon}$ of the auxiliary domain $\Omega_{2\varepsilon}$ goes into the boundary Γ_2 of the domain Ω_2. Taking into account the homogeneous initial-boundary conditions (26)-(27) and the above circumstances, by (32) we arrive at the equality

$$2\left(\upsilon,\,L_\mu\upsilon\right)_0=\int\limits_0^h\upsilon_x^2(0,y)dy+\int\limits_0^r\upsilon^2\left(x,\,h\right)dx+2\,\mu\int\limits_{\Omega_2}\upsilon^2(x,y)\,d\Omega_2=0. \tag{33}$$

By choosing a positive value for the parameter $\mu>0$, we note that (33) can occur if and only if $\upsilon(x,y)\equiv 0$ in the closure of the domain $\bar{\Omega}_2$, which contradicts the initial assumption that $\upsilon=\upsilon(x,y)\neq 0$. However then $u(x,y)\equiv 0$ in $\bar{\Omega}_2$ as follows by (28). Thus, $u(x,y)\equiv 0$ in $\bar{\Omega}$, that is, the solution to problem (1), (4), (5) is unique in the class of regular functions. The theorem is proved.

3 Existence theorem

Let us move on to the existence of a regular solution in Ω to *Problem 1.*

Theorem 2. Let the given functions $f(x,y)$, $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$, $\psi(x)$ be such that they have the properties

$$\varphi_1(y) \in C\,[0,h] \cap C^2\,(0,h)\,,\ \varphi_2(y) \in C\,[0,h] \cap C^2\,(0,h)\,,\ \varphi_3(y) \in C\,[0,h] \cap C^1\,(0,h)\,; \tag{34}$$

$$\psi(x) \in C^1\,[0,r] \cap C^2\,(0,r)\,; \tag{35}$$

$$f(x,y) \in C^1\left(\bar{\Omega}_2\right). \tag{36}$$

Then there is a regular solution to problem (1), (4), (5) in the domain Ω.

Proof. In fact, following the fundamental relationships (8), (12) and (14) obtained above, with respect to the sought functions $\tau(x)$ and $\nu(x)$ at $\lambda \in \left(-\frac{m}{2};\, \frac{m}{2}\right]$ we arrive at the system of equations

$$\begin{cases} \nu(x) = \gamma_1\, D_{0x}^{1-\beta}\tau\,(t) - \gamma_2\, x^{\beta_1}\, D_{0x}^{1-\beta_2}\psi\,(t)\ , \\ \tau'''(x) + \nu(x) = f(x,0). \end{cases} \tag{37}$$

From system (37) we arrive at the problem of finding a regular solution $\tau = \tau(x)$ of an ordinary differential equation of the third order of the form

$$\tau'''(x) + \gamma_1\, D_{0x}^{1-\beta}\tau\,(t) = f(x,0) + \gamma_2\, x^{\beta_1}\, D_{0x}^{1-\beta_2}\psi\,(t)\,, \quad 0 < x < r, \tag{38}$$

satisfying conditions (9).

Repeating integration of (38) three times from 0 to x, arrive at an integral equation equivalent to the given differential equation:

$$\tau(x) = -\frac{\gamma_1}{\Gamma\,(\beta+2)} \int\limits_0^x (x-t)^{\beta+1}\ \tau\,(t)\ dt + \frac{1}{2}\int\limits_0^x (x-t)^2\ F\,(t)\ dt + c_1 + c_2\,x + \frac{1}{2}\,c_3\,x^2, \tag{39}$$

where $F(x) = f(x,0) + \gamma_2\, x^{\beta_1}\, D_{0x}^{1-\beta_2}\psi\,(t)$, and c_1, c_2, c_3 are still arbitrary constants.

Equation (39) is the Volterra integral equation of the second kind with convolution kernel $K\,(x,\,t) = \frac{(x-t)^{\beta+1}}{\Gamma(\beta+2)}$. The functions $K_n\,(x,\,t) = \frac{(x-t)^{n(\beta+2)+\beta+1}}{\Gamma[n(\beta+2)+\beta+2]}$, $n = 0,\,1,\,2,\,\ldots$ are considered iterated kernels of $K\,(x,\,t)$, and the function

$$R\,(x,\,t;\,\beta) = \sum_{n=0}^{\infty} (-\gamma_1)^n\ K_n\,(x,\,t) = (x-t)^{\beta+1} \sum_{n=0}^{\infty} \frac{\left[(-\gamma_1)\ (x-t)^{(\beta+2)}\right]^n}{\Gamma\,[n\,(\beta+2)+\beta+2]}$$

is the resolving kernel $K\,(x,\,t)$ of equation (39).

With the Mittag-Leffler function, the resolving $R\,(x,\,t;\,\beta)$ of equation (39) kernel $K\,(x,\,t)$ of equation (39) takes the following form

$$R\,(x,\,t;\,\beta) = (x-t)^{\beta+1}\ E_{1/(\beta+2)}\left[-\gamma_1\ (x-t)^{\beta+2}\,;\,\beta+2\right],$$

where $E_\rho\,(z,\,\mu) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho^{-1}\,n+\mu)}$ is the Mittag-Leffler function.

The solution of (39) can be written with the resolving $R\,(x,\,t;\,\beta)$ of $K\,(x,\,t)$ as follows

$$\tau(x) = c_1 + c_2\,x + \frac{1}{2}\,c_3\,x^2 + \frac{1}{2}\int\limits_0^x (x-t)^2\ F\,(t)\ dt - c_1\,\gamma_1 \int\limits_0^x R\,(x,\,t;\,\beta)\ dt-$$

$$- c_2\,\gamma_1 \int\limits_0^x t\,R\,(x,\,t;\,\beta)\,dt - \frac{\gamma_1\,c_3}{2}\int\limits_0^x t^2 R\,(x,\,t;\,\beta)\,dt - \frac{\gamma_1}{2}\int\limits_0^x R\,(x,\,t;\,\beta)\left(\int\limits_0^t (t-s)^2\ F\,(s)\,ds\right)dt. \quad (40)$$

By direct calculation find out that

$$\int\limits_0^x R\,(x,\,t;\,\beta)\ dt = \int\limits_0^x (x-t)^{\beta+1}\ E_{1/(\beta+2)}\left[-\gamma_1\ (x-t)^{\beta+2};\,\beta+2\right] dt =$$

$$= x^{\beta+2}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\ \beta+3\right);$$

$$\int\limits_0^x t\,R\,(x,\,t;\,\beta)\ dt = x^{\beta+3}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\ \beta+4\right);$$

$$\int\limits_0^x t^2\,R\,(x,\,t;\,\beta)\ dt = 2\,x^{\beta+4}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\ \beta+5\right);$$

$$\int\limits_0^x R\,(x,\,t;\,\beta)\left(\int\limits_0^t (t-s)^2\,F\,(s)\,ds\right)dt = \int\limits_0^x \left(\int\limits_s^x (t-s)^2\,R\,(x,\,t;\,\beta)\,dt\right)F\,(s)\,ds =$$

$$= 2\int\limits_0^x (x-t)^{\beta+4}\,E_{1/(\beta+2)}\left[-\gamma_1\ (x-t)^{\beta+2};\,\beta+5\right]F\,(t)\,dt.$$

Considering the above calculations, rewrite representation (40) as follows

$$\tau(x) = \left[1 - \ \gamma_1\,x^{\beta+2}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\ \beta+3\right)\right]c_1 +$$
$$+\left[x - \gamma_1 x^{\beta+3}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\ \beta+4\right)\right]c_2 +$$
$$+\frac{1}{2}\left[x^2 - 2\gamma_1\,x^{\beta+4}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\ \beta+5\right)\right]c_3 +$$
$$+\frac{1}{2}\int\limits_0^x \left\{(x-t)^2 - 2\,\gamma_1\,(x-t)^{\beta+4}\ E_{1/(\beta+2)}\left[-\gamma_1\ (x-t)^{\beta+2};\,\beta+5\right]\right\}F\,(t)\,dt. \quad (41)$$

Satisfying conditions (9) for (41), get to the next system of equation with respect to c_2, c_3:

$$\tau(0) = c_1 = \varphi_1(0),$$

$$\begin{cases} \left[r - \gamma_1\,r^{\beta+3}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\ \beta+4\right)\right]c_2 + \frac{1}{2}\left[r^2 - 2\gamma_1\,r^{\beta+4}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\ \beta+5\right)\right]c_3 = \\ = \varphi_2(0) - \left[1 - \ \gamma_1\,r^{\beta+2}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\ \beta+3\right)\right]\ \varphi_1(0) - \\ -\frac{1}{2}\int\limits_0^r \left\{(r-t)^2 - 2\,\gamma_1\,(r-t)^{\beta+4}\ E_{1/(\beta+2)}\left[-\gamma_1\ (r-t)^{\beta+2};\,\beta+5\right]\right\}F\,(t)\ dt; \\ \\ \left[1 - \gamma_1 r^{\beta+2}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\ \beta+3\right)\right]c_2 + \left[r - \gamma_1\,r^{\beta+3}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\ \beta+4\right)\right]c_3 = \\ = \varphi_3(0) + \gamma_1\,r^{\beta+1}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\ \beta+2\right)\ \varphi_1(0) - \\ -\int\limits_0^r \left\{(r-t) - \gamma_1\,(r-t)^{\beta+3}\ E_{1/(\beta+2)}\left[-\gamma_1\ (r-t)^{\beta+2};\,\beta+4\right]\right\}F\,(t)\ dt\,. \end{cases} \quad (42)$$

The determinant

$$\Delta = \Big[r - \gamma_1\, r^{\beta+3}\, E_{1/(\beta+2)}\left(-\gamma_1\, r^{\beta+2};\, \beta+4\right)\Big]^2 - \frac{1}{2}\,\Big[1 - \gamma_1 r^{\beta+2}\, E_{1/(\beta+2)}\left(-\gamma_1\, r^{\beta+2};\, \beta+3\right)\Big] \times$$
$$\times \Big[r^2 - 2\gamma_1\, r^{\beta+4}\, E_{1/(\beta+2)}\left(-\gamma_1\, r^{\beta+2};\, \beta+5\right)\Big]$$

of system (42) is always different from zero by virtue of the uniqueness theorem proved above, that is, the constants c_1, c_2, c_3 in (41) are uniquely determined by conditions (9) and system (42).

Thus, the unique solution to problem (38), (9) for any $\lambda \in \left(-\frac{m}{2};\, \frac{m}{2}\right]$ is obtained by formula (41), where the constants c_1, c_2, c_3 are uniquely determined by (42).

Next, by relations (8) and (16) for $\lambda = -\frac{m}{2}$ in view of conditions (9), obtain

$$\tau(x) = \frac{1}{2r^2}\Bigg\{2\,(r-x)^2\;\varphi_1(0) + 2\,x\;(2r-x)\;\varphi_2(0) + 2rx\;(x-r)\,\varphi_3(0) +$$
$$+ (r-x)^2 \int\limits_0^r t^2\,\Big[f\,(t,\,0) + 2^{1-\beta}\;(1-\beta)^{-\beta}\;t^\beta\,\psi'\,(t)\Big]\,dt -$$
$$- r^2 \int\limits_x^r (t-x)^2\,\Big[f\,(t,\,0) + 2^{1-\beta}\;(1-\beta)^{-\beta}\;t^\beta\,\psi'\,(t)\Big]\,dt\Bigg\}.$$

Once the function $\tau(x)$ is obtained, the second desired function $\nu(x)$, depending on λ, can be obtained using relations (8), (12), (14) or (16). Then the regular solution to Problem 1 in the domain Ω_1 is defined as the solution to the Cauchy problem (6)-(7) for equation (2) and is written out according to one of the formulas (11), (13) or (15). And in the domain Ω_2 we arrive at the initial-boundary value problem (4), (6) for equation (3), the solution of which is written out in the monograph of T.D. Dzhuraev. Note that the conditions (34), (35), (36) listed in Theorem 2 ensure the regularity of the obtained solution in the domain Ω.

Conclusion

In the work in the mixed domain one boundary value problem for the model equation of parabolic-hyperbolic type of the third order is investigated. Theorems of existence and uniqueness of a regular solution of the problem under study are proved. To prove the uniqueness theorem the method of energy integrals is applied together with the method of A.M. Nakhushev. To prove the existence theorem the method of integral equations is applied. In terms of the Mittag-Leffler function the solution of the problem is found and written out in explicit form.

Conflict of Interest

There is no conflict of interest

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Bipartite Digraphs with Modular Concept Lattices of height 2

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This paper investigates the interaction between Formal Concept Analysis (FCA) and graph theory, with a focus on understanding the structure and representation of concept lattices derived from bipartite directed graphs. We establish connections between the complete formal contexts and their associated bipartite digraphs, providing a foundation for studying modular lattices. Particular attention is given to the structure of concept lattices arising from such contexts and their relationship to the combinatorial properties of the corresponding graphs. The results show that the concept lattice of a complete formal context is isomorphic to a modular lattice of height 2 if and only if its associated bipartite digraph is a disconnected union of bicliques. This establishes a precise correspondence between a specific class of formal contexts and well-studied objects in graph theory. Several examples are presented to illustrate these properties and demonstrate the application of the obtained results. The analysis opens the way for further exploration of lattices associated with more complex graph structures and contributes to a deeper understanding of the relationship between discrete mathematics and formal methods of knowledge representation.

Keywords: formal context, full context, formal concept, concept lattice, context graph, bipartite digraph, biclique, modular lattice.

2020 Mathematics Subject Classification: 05C20, 06B23.

Introduction

Formal Concept Analysis (FCA) is a powerful mathematical framework for data analysis and knowledge representation, based on the duality between objects and attributes within a formal context. FCA was introduced in the early 1980s by Rudolf Wille as a mathematical theory [1, 2]. This framework provides a systematic method for deriving concept lattices, which capture hierarchical relationships between object-attribute pairs. These lattices have applications spanning fields such as data mining, machine learning, and ontology engineering [3, 4].

Graph theory [5], on the other hand, offers a complementary perspective by modelling relationships as vertices and edges. The interplay between FCA and graph theory has been a subject of growing interest, particularly in the study of bipartite graphs. In FCA, the incidence relation of a formal context corresponds naturally to a bipartite graph, establishing a direct link between these domains.

This paper investigates the structural properties of concept lattices derived from bipartite graphs, with an emphasis on modular lattices. By characterizing the graph-theoretic properties of bipartite digraphs corresponding to such lattices, we aim to deepen the understanding of their formation and representation.

The main contributions of this work are as follows:

1) We introduce and formalize the notion of full formal contexts, which simplify the study of concept lattices by reducing redundancy in object-attribute relations.

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2) We establish a bijective correspondence between full formal contexts and bipartite digraphs, showing that the concept lattice of a context is determined by its graph structure (Theorem 1).

3) We prove that the concept lattice of a full formal context is isomorphic to a modular lattice of height 2 if and only if the associated bipartite graph is a disjoint union of complete bipartite graphs (Theorem 2).

4) We provide examples, including the graph of a function and its context lattice, to demonstrate the practical implications of our results.

For more information on the basic notions and results of FCA, lattice theory and graph theory introduced below, and used throughout this paper, we refer the reader to [2, 5, 6].

1 Preliminaries

First, we provide the main definitions.

Definition 1. A *graph* is an algebraic structure $G = (V, E)$ where E is a binary relation on V. The set V is called a set of *vertices* (or *nodes*), and $E \subseteq V \times V$ is a set of *edges*. A graph is called *undirected* if $(a, b) \in E$ then $(b, a) \in E$, and it is called *directed* or a *digraph* if $(a, b) \in E$ then $(b, a) \notin E$.

Definition 2. A digraph $G = (V, E)$ is called *bipartite* if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that:

- Every edge $e \in E$ connects a vertex in V_1 to a vertex in V_2.
- No edge exists between two vertices of the same subset.

A *complete* bipartite digraph (biclique) $G = (V, E)$ is a bipartite digraph in which the vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every vertex in V_1 is connected to every vertex in V_2 and there are no edges within V_1 or within V_2. Remark. Usually, a biclique is a complete bipartite undirected graph (see [5]).

Definition 3. A *formal context* $\mathbb{K} = (G, M, I)$ consists of the set of objects G, the set of attributes M, and the incidence relation $I \subseteq G \times M$.

For a formal context $\mathbb{K} = (G, M, I)$ and $A \subseteq G$, $B \subseteq M$ we put $\alpha_{\mathbb{K}}(\varnothing) = M$, $\beta_{\mathbb{K}}(\varnothing) = G$ and

$$\alpha_{\mathbb{K}}(A) = \{m \in M \mid (\forall g \in A)\,[\,(g, m) \in I\,]\},$$

$$\beta_{\mathbb{K}}(B) = \{g \in G \mid (\forall m \in B)\,[\,(g, m) \in I\,]\}.$$

The mappings $\beta_{\mathbb{K}} \circ \alpha_{\mathbb{K}} : \mathcal{P}(G) \to \mathcal{P}(G)$ and $\alpha_{\mathbb{K}} \circ \beta_{\mathbb{K}} : \mathcal{P}(M) \to \mathcal{P}(M)$ are closure operators. The set $\mathcal{L}_{\mathbb{K}}(G)$ ($\mathcal{L}_{\mathbb{K}}(M)$) of the closed subsets of G (M) with respect to $\beta_{\mathbb{K}} \circ \alpha_{\mathbb{K}}$ ($\alpha_{\mathbb{K}} \circ \beta_{\mathbb{K}}$) forms a lattice under inclusion $\subseteq$ (conclusion $\supseteq$). And $\mathcal{L}_{\mathbb{K}}(G)$ is dually isomorphic to $\mathcal{L}_{\mathbb{K}}(M)$.

If $\mathbb{K}$ is clear from the discussion then we omit the subscript $\mathbb{K}$ — e.g., for example, we write $\alpha(A)$ instead of $\alpha_{\mathbb{K}}(A)$.

Definition 4. A *formal concept* of the context $\mathbb{K}$ is a pair (A, B) such that $A \subseteq G$, $B \subseteq M$, $B = \alpha_{\mathbb{K}}(A)$, and $A = \beta_{\mathbb{K}}(B)$. For a formal concept $\Delta = (A, B)$, A is called the *extent* of Δ, and B is the *intent* of Δ.

The ordering $\preceq$ of the concepts of $\mathbb{K}$ is defined as follows:

$$(A_0, B_0) \preceq (A_1, B_1) \;\Leftrightarrow\; A_0 \subseteq A_1 \;\Leftrightarrow\; B_0 \supseteq B_1.$$

The Basic Theorem on Concept Lattices (see [1]) establishes that ordering $\preceq$ on the set of all concepts of $\mathbb{K}$ induces a complete lattice which is called the *concept lattice* of $\mathbb{K}$, and we denote it by $\mathcal{L}(\mathbb{K})$.

From the definition of the partial order $\preceq$, one can see that for a formal context $\mathbb{K} = (G, M, I)$ the mapping $\varphi : \mathcal{L}(\mathbb{K}) \to \mathcal{L}_{\mathbb{K}}(G)$ defined by $\varphi((A, B)) = A$, establishes an isomorphism between $\mathcal{L}(\mathbb{K})$ and $\mathcal{L}_{\mathbb{K}}(G)$.

For the sets A, B and a binary relation $R \subseteq A \times B$, we put

$$\pi_A(R) = \{a \in A \mid \exists b\,[\,(a,b) \in R\,]\}, \quad \pi_B(R) = \{b \in B \mid \exists a\,[\,(a,b) \in R\,]\}.$$

A formal context $\mathbb{K} = (A, B, I)$ is called *full* if $\pi_A(I) = A$, $\pi_B(I) = B$ and $\alpha_{\mathbb{K}}(A) = \beta_{\mathbb{K}}(B) = \varnothing$.

For a formal context $\mathbb{K} = (A, B, I)$ we define the graph $\mathbf{G}_K = (A \cup B; I)$ that consists of the set of vertices $A \cup B$ and the set of edges $I \subseteq A \times B$. The graph $\mathbf{G}_K = (A \cup B; I)$ is called *a context graph* if $A \cap B = \varnothing$. Such a graph we call a *context graph.* It is easy to see that $\mathbf{G}_K$ is a bipartite digraph. We also note that any bipartite digraph $\mathbf{G} = (A \cup B; I)$ with $I \subseteq A \times B$ defines the formal context $\mathbb{K}_G = (A, B, I)$. Similar constructions occur in many papers (see for example [7, 8]).

For any graph $\mathbf{G} = (G, R)$ we define the formal context $\mathbb{K}_G = (G, G, R)$ and the concept lattice $\mathcal{L}(\mathbb{K}_G)$, respectively.

The next theorem, as the reviewer noted: "Theorem 1 is a simple observation which, seemingly, is a "folklore" assertion. For example, in [7], the definition of a formal context is followed by the remark that "The correspondence to a bipartite graph (network) is at hand", brief description of this correspondence, and the conclusion that "In the following we use the terms network, (bipartite) graph, and formal context interchangeably in the sense above". However, I have not found a published formal proof of the assertion". For convenience we provide the formal proof.

Theorem 1. Let $\mathbb{K} = (A, B, I)$ be a full formal context in which $A \cap B = \varnothing$. And let $\mathbf{G}$ be the corresponding context graph $(A \cup B; I)$. Then $\mathcal{L}(\mathbb{K}) \cong \mathcal{L}(\mathbb{K}_G)$.

Proof. By definition, $\mathbb{K}_G = \{A \cup B, A \cup B, I\}$ and

$$\alpha_{\mathbb{K}_G}(X) = \{m \in A \cup B \mid (\forall g \in X)\,[\,(g,m) \in I\,]\},$$

$$\beta_{\mathbb{K}_G}(Y) = \{g \in A \cup B \mid (\forall m \in Y)\,[\,(g,m) \in I\,]\}.$$

By $\alpha_{\mathbb{K}}(\pi_A(I)) = \varnothing$ and $\beta_{\mathbb{K}}(\pi_B(I)) = \varnothing$, one can see that

$$\alpha_{\mathbb{K}_G}(X) = \begin{cases} \alpha_{\mathbb{K}}(X), & \text{if} \quad X \subseteq A, \\ \varnothing, & \text{otherwise}, \end{cases}$$

$$\beta_{\mathbb{K}_G}(Y) = \begin{cases} \beta_{\mathbb{K}}(Y), & \text{if} \quad Y \subseteq B, \\ \varnothing, & \text{otherwise}. \end{cases}$$

Therefore,

$$\beta_{\mathbb{K}_\mathbb{G}} \circ \alpha_{\mathbb{K}_G}(X) = \begin{cases} \beta_{\mathbb{K}} \circ \alpha_{\mathbb{K}}(X), & \text{if} \quad \alpha_{\mathbb{K}_G}(X) \neq \varnothing, \\ A \cup B, & \text{otherwise}, \end{cases}$$

$$\alpha_{\mathbb{K}_\mathbb{G}} \circ \beta_{\mathbb{K}_G}(Y) = \begin{cases} \alpha_{\mathbb{K}} \circ \beta_{\mathbb{K}}(Y), & \text{if} \quad \beta_{\mathbb{K}_G}(Y) \neq \varnothing, \\ A \cup B, & \text{otherwise}. \end{cases}$$

It means that a pair (X, Y) is a concept of $\mathbb{K}_G$ if and only if (X, Y) is a concept of $\mathbb{K}$ for all $X, Y \neq \varnothing$. Also $(A \cup B, \varnothing)$ and $(\varnothing, A \cup B)$ are the concepts of $\mathbb{K}_G$. Since the context $\mathbb{K}$ is full, $(A, \varnothing)$ and $(\varnothing, B)$ are the concepts of $\mathbb{K}$.

Hence the mapping $\varphi\colon \mathcal{L}(\mathbb{K}_G) \to \mathcal{L}(\mathbb{K})$, defined by

$$\varphi((X, Y)) = \begin{cases} (X, Y), & X, Y \neq \varnothing, \\ (A, \varnothing), & Y = \varnothing, \\ (\varnothing, B), & X = \varnothing, \end{cases}$$

is one to one and onto. It is easy to see that φ preserves partial order $\preceq$. Therefore, φ is an isomorphism.

These allow us to study the concept lattices through the bipartite digraphs. We demonstrate this approach in the next section.

2 *Representation of M_n*

For any $n > 2$, by M_n (M_ω) we denote a modular lattice of height 2 with n (ω) atoms.

Theorem 2. Let $\mathbb{K} = (A, B, I)$ be a full formal context in which $A \cap B = \varnothing$. Then the concept lattice $\mathcal{L}(\mathbb{K})$ is isomorphic to M_n for some $n \leq \omega$ if and only if the context graph $\mathbf{G}_K = (A \cup B; I)$ is a disjoint union of n complete bipartite digraphs.

Proof. $\Rightarrow$ By $\mathcal{L}(\mathbb{K}) \cong \mathcal{L}(A)$, we have $\mathcal{L}(A) \cong M_n$. Since $\beta(B) = \varnothing$, $\varnothing = 0_L$ is the least element of $\mathcal{L}(A)$. By $\alpha(A) = \varnothing$ and $\beta(\varnothing) = A \cup B$, we get that $A \cup B = 1_L$ is the greatest element of $\mathcal{L}(A)$. Let S be the set of all non-empty proper closed subsets of A. Since $\mathcal{L}(A) \cong M_n$, $A_0 \cap A_1 = \varnothing$ and $A_0 \vee A_1 = A \cup B$ for any $A_0, A_1 \in S$ with $A_0 \neq A_1$. Since $\pi_A(I) = A$, $\cup\{C \mid C \in S\} = A$. Hence S is a partition of A.

Let $\alpha(S) = \{\alpha(C) \mid C \in S\}$. By definition, $\alpha(C)$ is a closed subset of B. Since $\beta(B) = \varnothing$, $\beta(\varnothing) = A$, as $\mathbb{K}$ is full, and $\beta\alpha(C) = C$, then $\alpha(C)$ is non-empty proper subset of B for all $C \in S$, as well as $\alpha(C_0) \neq \alpha(C_1)$ for all $C_0, C_1 \in S$ and $C_0 \neq C_1$. Since $\mathcal{L}(B)$ is dual isomorphic to $\mathcal{L}(A)$, $\mathcal{L}(B) \cong M_n$. Let $D = \alpha(C_0) \cap \alpha(C_1)$, $C_0 \neq C_1$. Then, by definition, $\beta(D) \supset C_0$ and is a closed subset of A. Since the height of $\mathcal{L}(A)$ is equal to 2, $\beta(D) = A$. It implies $D = \varnothing$, that is, $\alpha(C_0) \cap \alpha(C_1) = \varnothing$. Thus, $\alpha(C_0) \cap \alpha(C_1) = \varnothing$ for all $C_0, C_1 \in S$, $C_0 \neq C_1$.

Let $D = B \setminus \cup\{\alpha(C) \mid C \in S\}$. By definition of D, $\beta(D) \notin S$. Also $\beta(D) \neq \varnothing$ because in this case $\cup\{\alpha(C) \mid C \in S\}$ is empty. Thus $\beta(D) = A$. It implies D is empty. Hence $B = \cup\{\alpha(C) \mid C \in S\}$. Thus, we establish that $\{\alpha(C) \mid C \in S\}$ is a partition of the set B.

Now we need to show that $\cup\{C \times \alpha(C) \mid C \in S\} = I$. First we note that $\alpha(c) = \alpha(C)$ for any $c \in C$. Indeed, assume that $\alpha(c) \supset \alpha(C)$ for some $c \in C$ and $C \in S$. Since $\alpha(c)$ is a closed subset in B and $\mathcal{L}(B) \cong M_n$ (because $\mathcal{L}(B)$ is dually isomorphic to $\mathcal{L}(A)$), $\alpha(c) = B$. Therefore, $c \in \beta(B)$. Since $\mathbb{K}$ is full, $\beta(B) = \varnothing$. Contradiction. Thus, $\alpha(c) = \alpha(C)$ for any $c \in C$. Hence $\cup\{C \times \alpha(C) \mid C \in S\} \subseteq I$. Let $(a, b) \in I$. Then $a \in \beta(b)$ whence $(a, b) \in C \times \alpha(C)$. Thus $\cup\{C \times \alpha(C) \mid C \in S\} = I$.

$\Leftarrow$ Since the graph $G_K = (A \cup B; I)$ is a disjoint union of n complete bipartite digraphs, $G_K = (\cup_{i \leq n} A_i, \cup_{i \leq n} B_i; \cup_{i \leq n} I_i)$ for some partitions $\{A_i \mid i \leq n\}$, $\{B_i \mid i \leq n\}$ of the sets A and B respectively, and $I = \cup_{i \leq n} I_i$ where $I_i = A_i \times B_i$.

The condition $I = \cup_{i \leq n} I_i = \cup_{i \leq n} A_i \times B_i$ give us that $\pi_A(I) = A$, $\pi_B(I) = B$ and the sets $\{b \in B \mid (a, b) \in I \text{ for all } a \in A\}$ and $\{a \in A \mid (a, b) \in I \text{ for all } b \in B\}$ are empty. These mean that $\alpha_{\mathbb{K}}(\pi_A(I)) = \varnothing$ and $\beta_{\mathbb{K}}(\pi_B(I)) = \varnothing$. Therefore, by Theorem 1, we get $\mathcal{L}(\mathbb{K}) \cong \mathcal{L}(\mathbb{K}_G)$. Thus we need to show that $\mathcal{L}(\mathbb{K}_G) \cong M_n$.

For the formal context $\mathbb{K}_G$ we have

$$\alpha_{\mathbb{K}_G}(X) = \begin{cases} B_i, & \text{if} \quad X \subseteq A_i, \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$\beta_{\mathbb{K}_G}(X) = \begin{cases} A_i, & \text{if} \quad X \subseteq B_i, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Thus, for any A_i and $X \supset A_i$, we have

$$\beta_{\mathbb{K}_{\mathbb{G}}} \circ \alpha_{\mathbb{K}_G}(A_i) = A_i,$$

$$\beta_{\mathbb{K}_{\mathbb{G}}} \circ \alpha_{\mathbb{K}_G}(X) = A \cup B.$$

That is, A_i, $1 \leq i \leq n$, and $A \cup B$ are the closed subsets in $A \cup B$ with respect to closure operator $\beta_{\mathbb{K}_{\mathbb{G}}} \circ \alpha_{\mathbb{K}_G}$. Therefore, $A_i \vee A_j = A \cup B$. Since $\{A_i \mid i \leq n\}$ forms a partition of A then $A_i \cap A_j = \varnothing$ for all $i \neq j \leq n$. It means that $\mathcal{L}(A) \cong M_n$. Hence $\mathcal{L}(\mathbb{K}_G) \cong M_n$ because $\mathcal{L}(\mathbb{K}_G) \cong \mathcal{L}(A)$.

Recall that a *bipartite dimension* of a graph is the minimum number of complete bipartite graphs whose union is the given graph. Thus

Corollary 1. Let $\mathbb{K}$ be a formal context and $\mathcal{L}(\mathbb{K}) \cong M_n$. Then the bipartite dimension of the graph $G_{\mathbb{K}}$ is equal to n.

3 Examples

Here we provide some examples.

Example 1. (cf. [9, 10]) Let $f : A \to B$ be a function from A onto B and

$$gr(f) = \{(x, y) \mid f(x) = y \text{ for all } x \in A \ , \ y \in B\}$$

the graph of function f. Consider a formal context $\mathbb{K} = (A, B, gr(f))$, where A represents objects, B represents attributes, and the incident relation is $gr(f)$. Then the concept $\mathbb{K} = (A, B, gr(f))$ satisfies Theorem 2. Hence $\mathcal{L}(\mathbb{K}) \cong M_{|B|}$ where $|B|$ is the size of B, and the bipartite dimension of the graph $G_{\mathbb{K}}$ is equal to $|B|$.

Indeed, let, for any $b \in B$,

$$A_b = \{x \in A \mid f(x) = b\} \subseteq A.$$

Since f is a function and maps A onto B,

$$A = \bigcup_{b \in B} A_b, \quad A_b \cap A_c = \emptyset,$$

for any $b, c \in B$, $b \neq c$. Moreover, $(A_b \cup \{b\}, gr(f_{|A_b}))$ forms a complete digraph (biclique) (Fig. 1). Thus, $\mathbb{K} = (A, B, gr(f))$ is a disjoint union of the bicliques $(A_b \cup \{b\}, gr(f_{|A_b}))$, $b \in B$. By Theorem 2, $\mathcal{L}(\mathbb{K}) \cong M_{|B|}$.

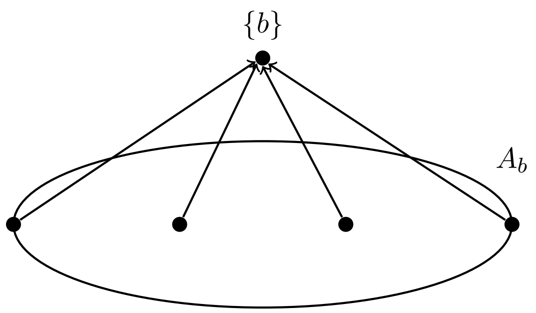

Figure 1. Bipartite digraph $(A_b \cup \{b\}, gr(f_{|A_b}))$

More general

Example 2. Let $f : A \to B$ be a many-valued function from A onto B and $gr(f) = \{(x, y) \mid y \in f(x) \text{ for all } x \in A\}$ the graph of the many-valued function f. And let the set of images of points of A forms a partition of B, that is the set of all proper subsets of B of the form $\{f(a) \subset B \mid a \in A\}$ is a partition of B. Then the concept $\mathbb{K} = (A, B, gr(f))$ satisfies Theorem 2. Hence $\mathcal{L}(\mathbb{K}) \cong M_n$, where n is the bipartite dimension of the graph $G_{\mathbb{K}}$.

Conclusion

In this paper, we explored the interplay between Formal Concept Analysis and graph theory, focusing on the structural representation of concept lattices through bipartite digraphs. The introduction of full formal contexts allowed us to establish a bijective correspondence between these contexts and bipartite digraphs, providing a framework for studying modular lattices. We demonstrated that the concept lattice of a full formal context is isomorphic to a modular lattice of height 2 if and only if its corresponding bipartite digraph is a disjoint union of complete bipartite graphs. This result not only advances the theoretical understanding of FCA but also provides practical tools for analyzing data structures in diverse applications. Future research may investigate the extension of these results to other types of lattices and exploring their computational implications.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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On the solvability of one inverse problem for a fourth-order equation

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In this paper, for a fourth-order equation in a rectangular domain, an inverse problem of finding the unknown right-hand side, which depends on one variable, is considered. Criteria for the uniqueness and existence of a solution to the inverse problem under consideration for a fourth-order equation are established. The solution to the problem is constructed as the sum of a series in eigenfunctions of the corresponding spectral problem. The uniqueness of the solution to the inverse problem follows from the completeness of the system of eigenfunctions. Sufficient conditions are established for the boundary functions that guarantee theorems of existence and stability of the solution to the problem. In a closed domain, absolute and uniform convergence of the found solution to the inverse problem in the form of a series in the class of regular solutions is shown, as well as series obtained by term-by-term differentiation with respect to t and x three and four times, respectively. The stability of the solution of the inverse problem in the norms of the space of square-summable functions and in the space of continuous functions with respect to changes in the input data has also been proven.

Keywords: fourth-order equation, inverse problem, classical solution, method of separation of variables, uniform convergence of the solution, uniqueness, existence, stability of the solution.

2020 Mathematics Subject Classification: 35R30.

Introduction

Boundary value and inverse problems for fourth-order differential equations are widely used in modeling processes in various fields of science and technology: in studying the dynamics of compressible stratified fluid, wave propagation in dispersive media, ship vibrations, oscillations of rods, beams and plates. Such problems are often reduced to studying fourth-order equations with various types of conditions. Numerous studies have been devoted to boundary value problems for fourth-order equations.

In the monograph by Smirnov [1], problems for a model equation of mixed type of the fourth order in various geometric domains are considered. In the work by Amirov and Khojanov [2], the global solvability of initial-boundary value problems for nonlinear analogues of the Boussinesq equation is proved, which expands the range of studied problems of mathematical physics.

The inverse problem for a parabolic equation of the fourth order with a complex-valued coefficient is considered in [3], where a theorem on the existence and uniqueness of a solution is proved. The articles [4–6] consider boundary value problems with local conditions for fourth-order equations in rectangular domains. Thus, in [4] the problems with the third derivative with respect to time are analyzed, and in [5] and [6] – problems with the lowest term and mixed type of equation, respectively.

In the works [7, 8] the boundary value problems with nonlocal conditions are studied. The authors prove that the eigenfunctions and associated functions of the corresponding spectral problem form a Riesz basis, and the solution to the problem is expressed as a biorthogonal series. This is important for constructing analytical solutions in complex domains.

In the works [9, 10] the boundary value problems for fourth-order mixed-type equations are studied.

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Initial-boundary value problems for beam and plate vibrations are studied in the works of Sabitov and co-authors [11–13]. The use of the method of separation of variables allows us to establish solvability conditions and construct explicit representations of solutions for equations taking into account rotational motion and various types of fixings.

Some boundary-value problems for nonhomogeneous biharmonic equation is presented in [14], where the conditions for periodic boundary are studied. In the work of Urinov and Azizov [15], an inverse problem for a fourth-order equation with an unknown right-hand side is considered, a uniqueness theorem is proved, and a constructive solution method is given.

A classification of fourth-order equations with two independent variables is given in the monograph by Dzhuraev and Sopuev [16], where an extensive bibliography on this topic is also presented and various types of boundary value problems are considered.

Inverse problems, as shown in [17–19], have numerous applications in seismology, geophysics, biomedicine, and computed tomography. Here, both problems of restoring the right-hand side and coefficient inverse problems are considered. In particular, Sabitov and Martemyanova [17] investigated a nonlocal inverse problem for a mixed-type equation, and Khojanov [18, 19] proposed methods for restoring special types of right-hand sides in parabolic equations.

General approaches to solving inverse problems and theoretical foundations are presented in classical monographs [20–22], which present regularization methods, a functional-analytical apparatus, and examples of formulations in mathematical physics.

Thus, the present study continues the development of the theory of fourth-order equations, relying on the indicated scientific achievements, and is aimed at formulating and solving new classes of boundary and inverse problems with practical significance.

1 Formulation of the problem

In the domain $\Omega = \{(x,t) : 0 < x < p,\ 0 < t < \beta\}$, we consider the equation

$$Lu \equiv u_{ttt} - u_{xxxx} - b^2 u = f\left(x\right), \tag{1}$$

where $b = const$.

Problem 1. Find functions $u\left(x,t\right)$ and $f\left(x\right)$ in the domain Ω that satisfy the conditions

$$u\left(x,t\right) \in C^{3,1}_{x,t}\left(\bar{\Omega}\right) \cap C^{4,3}_{x,t}\left(\Omega\right), \quad f\left(x\right) \in C\left(0,p\right) \cap L_2\left(0,p\right), \tag{2}$$

$$Lu\left(x,t\right) = f\left(x\right), \tag{3}$$

$$u\left(x,0\right) = \varphi\left(x\right), \quad u_t\left(x,0\right) = \psi\left(x\right), \quad u\left(x,\beta\right) = \xi\left(x\right), \quad u_t\left(x,\beta\right) = \mu\left(x\right), \quad 0 \le x \le p, \tag{4}$$

$$u\left(0,t\right) = u\left(p,t\right) = 0, \quad u_{xx}\left(0,t\right) = u_{xx}\left(p,t\right) = 0, \quad 0 \le t \le \beta, \tag{5}$$

where $\varphi\left(x\right), \xi\left(x\right), \psi\left(x\right), \mu\left(x\right)$ are the given functions, and $\varphi^{(i)}\left(0\right) = \varphi^{(i)}\left(p\right) = 0$, $\xi^{(i)}\left(0\right) = \xi^{(i)}\left(p\right) = 0$, $i = 0, 2$, $\psi\left(0\right) = \psi\left(p\right) = 0$, $\mu\left(0\right) = \mu\left(p\right) = 0$.

By the classical solution of the inverse boundary value problem (2)–(5) we mean a pair $\{u\left(x,t\right), f\left(x\right)\}$ of functions $u\left(x,t\right) \in C^{4,3}_{x,t}\left(\Omega\right)$ and $f\left(x\right) \in C\left(0,\beta\right)$, satisfying conditions (2)–(5) in the usual sense.

2 Uniqueness and existence of a solution to the inverse problem

We solve problems (2)–(5) at $f\left(x\right) \equiv 0$ using the method of separation of variables $u\left(x,t\right) = X\left(x\right)T\left(t\right)$. Then we have the following spectral problem for the function $X\left(x\right)$:

$$\begin{aligned} &X^{IV}\left(x\right) - \eta X\left(x\right) = 0,\ \ 0 < x < p, \\ &X\left(0\right) = X\left(p\right) = X''\left(0\right) = X''\left(p\right) = 0, \end{aligned} \tag{6}$$

where η is the separation constant. Problem (6) has a solution

$$X_k(x) = \sqrt{\frac{2}{p}} \sin \lambda_k x, \quad \lambda_k = \sqrt[4]{\eta_k} = \frac{k\pi}{p}, \quad k = 1, 2, \dots . \tag{7}$$

We look for a solution to problem (2)–(5) in the form

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) X_k(x), \tag{8}$$

$$f(x) = \sum_{k=1}^{\infty} f_k X_k(x), \tag{9}$$

where

$$T_k(t) = \int_0^p u(x,t) X_k(x)\, dx, \tag{10}$$

$$f_k = \int_0^p f(x) X_k(x)\, dx. \tag{11}$$

Based on (10), we introduce the functions

$$T_{k,\varepsilon}(t) = \int_{\varepsilon}^{p-\varepsilon} u(x,t) X_k(x)\, dx, \tag{12}$$

where ε is a fairly small number. We differentiate equalities (12) three times and take into account (1), we have

$$T''_{k,\varepsilon}(t) = \int_{\varepsilon}^{p-\varepsilon} \left[f(x) + u_{xxxx}(x,t) + b^2 u(x,t) \right] X_k(x)\, dx. \tag{13}$$

In integral (13), integrating four times by parts and passing to the limit at $\varepsilon \to 0$ taking into account boundary conditions (5), we have the differential equations:

$$T'''_k(t) - v_k^3 T_k(t) = f_k, \tag{14}$$

where $v_k^3 = \lambda_k^4 + b^2$.

General solutions of equation (14) take the form

$$T_k(t) = a_k e^{v_k t} + e^{-\frac{1}{2} v_k t} \left(b_k \cos \frac{\sqrt{3}}{2} v_k t + c_k \sin \frac{\sqrt{3}}{2} v_k t \right) - v_k^{-3} f_k, \tag{15}$$

where $a_k, \ b_k, \ c_k$ are arbitrary constants.

To determine the coefficients $a_k, \ b_k, \ c_k$ and f_k we use conditions (4), which go over

$$T_k(0) = \varphi_k, \quad T_k{}'(0) = \psi_k, \quad T_k(\beta) = \xi_k, \quad T_k{}'(\beta) = \mu_k, \tag{16}$$

where

$$\begin{aligned} &\varphi_k = \textstyle\int_0^p \varphi(x) X_k(x)\, dx, \quad \psi_k = \int_0^p \psi(x) X_k(x)\, dx, \\ &\xi_k = \textstyle\int_0^p \xi(x) X_k(x)\, dx, \quad \mu_k = \int_0^p \mu(x) X_k(x)\, dx. \end{aligned} \tag{17}$$

Substituting solutions (15) into (16), we obtain a system of equations for determining a_k, b_k, c_k and f_k:

$$\begin{cases} a_k + b_k - v_k^{-3} f_k = \varphi_k, \\ 2a_k - b_k + \sqrt{3} c_k = 2v_k^{-1}\psi_k, \\ a_k e^{\frac{3}{2}v_k\beta} + b_k \cos\frac{\sqrt{3}}{2}v_k\beta + c_k \sin\frac{\sqrt{3}}{2}v_k\beta - v_k^{-3} e^{\frac{1}{2}v_k\beta} f_k = \xi_k e^{\frac{1}{2}v_k\beta}, \\ a_k e^{\frac{3}{2}v_k\beta} - b_k \cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - c_k \sin\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) = v_k^{-1}\mu_k e^{\frac{1}{2}v_k\beta}. \end{cases} \tag{18}$$

The determinant of system (18) takes the form:

$$\Delta_k(\beta) = 2e^{v_k\beta}\left(chv_k\beta - \cos\frac{\sqrt{3}}{2}v_k\beta \cdot ch\frac{1}{2}v_k\beta - \sqrt{3}\sin\frac{\sqrt{3}}{2}v_k\beta \cdot sh\frac{1}{2}v_k\beta\right). \tag{19}$$

Now we represent (19) in the form

$$\Delta_k(\beta) = 2e^{v_k\beta} chv_k\beta \cdot \left[1 - A_k \sin\left(\frac{\sqrt{3}}{2}v_k\beta + \gamma_k\right)\right], \tag{20}$$

where $\gamma_k = \arcsin\frac{ch\frac{1}{2}v_k\beta}{\sqrt{2chv_k\beta - 1}}$, $A_k = \frac{\sqrt{2chv_k\beta - 1}}{chv_k\beta}$.

Lemma 1. For any $\beta > 0$ the following estimate is valid

$$|\Delta_k(\beta)| \geq C_0 e^{2v_k\beta}, \tag{21}$$

where C_0 is a positive constant.

Proof. Taking into account $A_k < 1$, from (20), we have

$$|\Delta_k(\beta)| \geq e^{v_k\beta} \cdot \left(e^{v_k\beta} + e^{-v_k\beta}\right) \cdot \left[1 - A_k\left|\sin\left(\frac{\sqrt{3}}{2}v_k\beta + \gamma_k\right)\right|\right] \geq e^{2v_k\beta} \cdot [1 - A_k] \geq C_0 e^{2v_k\beta}.$$

The lemma is proved.

Then system (18) has a unique solution

$$a_k = \frac{1}{\sqrt{3}v_k\Delta_k(\beta)}\left[2v_k\varphi_k e^{\frac{1}{2}v_k\beta}\sin\frac{\sqrt{3}}{2}v_k\beta + \sqrt{3}\psi_k + 2\psi_k e^{\frac{1}{2}v_k\beta}\sin\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - \right.$$
$$\left. -2v_k\xi_k e^{\frac{1}{2}v_k\beta}\sin\frac{\sqrt{3}}{2}v_k\beta + \sqrt{3}\mu_k e^{v_k\beta} - 2\mu_k e^{\frac{1}{2}v_k\beta}\sin\left(\frac{\sqrt{3}}{2}v_k\beta + \frac{\pi}{3}\right)\right], \tag{22}$$

$$b_k = \frac{e^{\frac{1}{2}v_k\beta}}{\sqrt{3}v_k\Delta_k(\beta)}\left[\sqrt{3}v_k\varphi_k e^{\frac{3}{2}v_k\beta} + 2v_k\varphi_k \sin\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - \right.$$
$$-2\psi_k \sin\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - 2\sqrt{3}\psi_k e^{v_k\beta}\cos\left(\frac{\sqrt{3}}{2}v_k\beta + \frac{\pi}{3}\right) - \sqrt{3}v_k\xi_k e^{\frac{3}{2}v_k\beta} - \tag{23}$$
$$\left. -2v_k\xi_k \sin\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - 2\mu_k \sin\frac{\sqrt{3}}{2}v_k\beta + 2\sqrt{3}\mu_k e^{v_k\beta} sh\frac{1}{2}v_k\beta\right],$$

$$c_k = \frac{e^{\frac{1}{2}v_k\beta}}{\sqrt{3}v_k\Delta_k(\beta)}\left[v_k\varphi_k e^{\frac{3}{2}v_k\beta} - 2v_k\varphi_k\cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) + 2\psi_k e^{\frac{3}{2}v_k\beta} + \right.$$

$$+2\psi_k\cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - 2\sqrt{3}\psi_k e^{v_k\beta}\sin\left(\frac{\sqrt{3}}{2}v_k\beta + \frac{\pi}{3}\right) - v_k\xi_k e^{\frac{3}{2}v_k\beta} +$$

$$\left. +2v_k\xi_k\cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) + \mu_k e^{\frac{3}{2}v_k\beta} + 2\mu_k\cos\frac{\sqrt{3}}{2}v_k\beta - 3\mu_k e^{\frac{1}{2}v_k\beta}\right], \tag{24}$$

$$f_k = \frac{v_k^2}{\Delta_k(\beta)}\left\{v_k\varphi_k\left[2e^{\frac{3}{2}v_k\beta}\cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) - 1\right] + \right.$$

$$+\psi_k\left[-2e^{\frac{3}{2}v_k\beta}\cos\left(\frac{\sqrt{3}}{2}v_k\beta + \frac{\pi}{3}\right) + 1\right] + v_k\xi_k\left[2e^{\frac{1}{2}v_k\beta}\cos\left(\frac{\sqrt{3}}{2}v_k\beta + \frac{\pi}{3}\right) - e^{2v_k\beta}\right] + \tag{25}$$

$$\left. +\mu_k\left[-2e^{\frac{1}{2}v_k\beta}\cos\left(\frac{\sqrt{3}}{2}v_k\beta - \frac{\pi}{3}\right) + e^{2v_k\beta}\right]\right\}.$$

So we obtained a solution of problem (2)–(5) in the form (8)–(9), where $X_k(x)$, $T_k(t)$ and f_k are determined from (7), (15) and (25), respectively, and the coefficients a_k, b_k, c_k are determined from (22)–(24).

Now we will prove the uniqueness of the solution of problem (2)–(5). Let $\xi(x) \equiv 0$, $\mu(x) \equiv 0$, $\varphi(x) \equiv 0$, $\psi(x) \equiv 0$ on $[0, p]$. Then $\varphi_k \equiv 0$, $\psi_k \equiv 0$, $\xi_k \equiv 0$, $\mu_k \equiv 0$ from (15) and (25) it follows that $T_k(t) \equiv 0$ on $[0, \beta]$ and $f_k \equiv 0$ for all $k \in N$. Then from equalities (10)–(11) we have that for all $t \in [0, \beta]$

$$\int_0^p u(x,t)X_k(x)\,dx \equiv 0, \quad \int_0^p f(x)X_k(x)\,dx \equiv 0, \ k \in N.$$

From here, due to the completeness of (7) in space $L_2[0, p]$ and the continuity of the function $u(x,t)$ and $f(x)$ respectively, on the domain $\bar{\Omega}$ and $(0, p)$, it follows that $u(x,t) \equiv 0$ in $\bar{\Omega}$ and $f(x) \equiv 0$ on $(0, p)$. So it's proven.

Theorem 1. If there is a solution of the problem (2)–(5), then it is unique.

Lemma 2. For large natural k, the following estimates are valid:

$$\begin{gathered} |T_k(t)| \le C_1\left[|\varphi_k| + k^{-1\frac{1}{3}}|\psi_k| + |\xi_k| + k^{-1\frac{1}{3}}|\mu_k|\right], \\ \left|T_k{}'(t)\right| \le C_2\left[k^{1\frac{1}{3}}|\varphi_k| + |\psi_k| + k^{1\frac{1}{3}}|\xi_k| + |\mu_k|\right], \\ \left|T_k{}''(t)\right| \le C_3\left[k^{2\frac{2}{3}}|\varphi_k| + k^{1\frac{1}{3}}|\psi_k| + k^{2\frac{2}{3}}|\xi_k| + k^{1\frac{1}{3}}|\mu_k|\right], \\ \left|T_k{}'''(t)\right| \le C_4\left(k^4|\varphi_k| + k^{2\frac{2}{3}}|\psi_k| + k^4|\xi_k| + k^{2\frac{2}{3}}|\mu_k|\right). \end{gathered} \tag{26}$$

Here and below C_i are positive constants.

Proof. From (22)–(25), taking Lemma 1 into account, we obtain the following estimates:

$$\begin{aligned} &|a_k| \le C_5 e^{-2k\beta}\left(|\varphi_k| + k^{-1\frac{1}{3}}|\psi_k| + |\xi_k| + k^{-1\frac{1}{3}}|\mu_k|e^{\frac{2}{3}k\beta}\right), \\ &|b_k| \le C_6\left(|\varphi_k| + k^{-1\frac{1}{3}}|\psi_k| + |\xi_k| + k^{-1\frac{1}{3}}|\mu_k|\right), \\ &|c_k| \le C_7\left(|\varphi_k| + k^{-1\frac{1}{3}}|\psi_k| + |\xi_k| + k^{-1\frac{1}{3}}|\mu_k|\right), \\ &|f_k| \le C_8\left(k^4|\varphi_k| + k^{2\frac{2}{3}}|\psi_k| + k^4|\xi_k| + k^{2\frac{2}{3}}|\mu_k|\right). \end{aligned} \tag{27}$$

From (15) we have

$$|T_k(t)| \le |a_k| e^{v_k t} + |b_k| e^{-\frac{1}{2}v_k t} + |c_k| e^{-\frac{1}{2}v_k t} + v_k^{-3} |f_k|. \tag{28}$$

We substitute (27) into (28). This estimate implies the validity of the first estimate required in the lemma. The proof of the validity of the remaining estimates is shown similarly. The lemma is proved.

Lemma 3. Let $\varphi(x), \xi(x) \in C^5[0,p]$, $\varphi^{(2i)}(0) = \varphi^{(2i)}(p) = 0$, $\xi^{(2i)}(0) = \xi^{(2i)}(p) = 0$, $i = 0,1,2$; $\psi(x), \mu(x) \in C^4[0,p]$, $\psi^{(2i)}(0) = \psi^{(2i)}(p) = 0$, $\mu^{(2i)}(0) = \mu^{(2i)}(p) = 0$, $i = 0,1$. Then the representations are valid

$$\varphi_k = \frac{1}{\lambda_k^5}\bar{\varphi}_k^{(5)}, \quad \psi_k = \frac{1}{\lambda_k^4}\bar{\psi}_k^{(4)}, \quad \xi_k = \frac{1}{\lambda_k^5}\bar{\xi}_k^{(5)}, \quad \xi_k = \frac{1}{\lambda_k^4}\bar{\xi}_k^{(4)}, \tag{29}$$

where

$$\bar{\varphi}_k^{(5)} = \sqrt{\tfrac{2}{p}}\int_0^p \varphi^{(5)}(x)\cos\lambda_k x dx, \quad \bar{\psi}_k^{(4)} = \sqrt{\tfrac{2}{p}}\int_0^p \psi^{(4)}(x)\sin\lambda_k x dx,$$

$$\bar{\xi}_k^{(5)} = \sqrt{\tfrac{2}{p}}\int_0^p \xi^{(5)}(x)\cos\lambda_k x dx, \quad \bar{\mu}_k^{(4)} = \sqrt{\tfrac{2}{p}}\int_0^p \mu^{(4)}(x)\sin\lambda_k x dx.$$

Integrating the first and third integrals in (17) by parts five times, and the second and fourth integrals by parts four times, taking into account the conditions of the lemma, we obtain representations (29).

Theorem 2. Let the functions $\varphi(x)$, $\psi(x)$, $\xi(x)$ and $\mu(x)$ satisfy the conditions of Lemma 3. Then there is a unique solution of problem (2)–(5), which is determined by the series (8)–(9).

Proof. We formally differentiate series (8) term by t three times and by x four times and have

$$u_{ttt}(x,t) = \sum_{k=1}^{\infty} T_k{}'''(t) X_k(x), \tag{30}$$

$$u_{xxxx}(x,t) = \sum_{k=1}^{\infty} \lambda_k^4 T_k(t) X_k(x). \tag{31}$$

From (26) we have

$$C_4\sqrt{\frac{2}{p}}\sum_{k=1}^{\infty}\left(k^4|\varphi_k| + k^{2\frac{2}{3}}|\psi_k| + k^4|\xi_k| + k^{2\frac{2}{3}}|\mu_k|\right). \tag{32}$$

Based on (29), the convergence of series (32) is proved, i.e., the following series

$$\bar{C}_4\sum_{k=1}^{\infty}\left(\frac{1}{k}\left|\bar{\varphi}_k^{(5)}\right| + \frac{1}{k^{1\frac{1}{3}}}\left|\bar{\psi}_k^{(4)}\right| + \frac{1}{k}\left|\bar{\xi}_k^{(5)}\right| + \frac{1}{k^{1\frac{1}{3}}}\left|\bar{\mu}_k^{(4)}\right|\right)$$

converges. From convergence (32), due to the Weierstrass criterion, series (8), (30), (31) uniformly converge in the domain $\bar{\Omega}$ and series (9) on [0, p]. The theorem is proved.

3 Stability of the solution

Let us introduce the following norms:

$$\|u(x,t)\|_{L_2[0,p]} = \left(\int_0^p |u(x,t)|^2 dx\right)^{\frac{1}{2}}, \quad \|u(x,t)\|_{C(\bar{\Omega})} = \max_{\bar{\Omega}}|u(x,t)|,$$

$$\|f(x)\|_{W_2^n[0,p]} = \left(\int_0^p \left(\sum_{k=0}^{n}\left|f^{(k)}(x)\right|^2 dx\right)\right)^{\frac{1}{2}}, \quad n \in N.$$

Theorem 3. Let the conditions of Theorem 2 be satisfied, then for solution (8), (9) of Problem 1 the following estimates are valid:

$$\|u(x,t)\|_{L_2[0,p]} \leq C_9 \left[\|\varphi\|_{L_2} + \|\psi\|_{L_2} + \|\xi\|_{L_2} + \|\mu\|_{L_2}\right], \tag{33}$$

$$\|f(x)\|_{L_2[0,p]} \leq C_{10} \left[\|\varphi\|_{W_2^4} + \|\psi\|_{W_2^3} + \|\xi\|_{W_2^4} + \|\mu\|_{W_2^3}\right], \tag{34}$$

$$\|u(x,t)\|_{C(\bar{\Omega})} \leq C_{11} \left[\|\varphi\|_{W_2^1} + \|\psi\|_{W_2^0} + \|\xi\|_{W_2^1} + \|\mu\|_{W_2^0}\right], \tag{35}$$

$$\|f(x)\|_{C[0,p]} \leq C_{12} \left[\|\varphi\|_{W_2^5} + \|\psi\|_{W_2^4} + \|\xi\|_{W_2^5} + \|\mu\|_{W_2^4}\right]. \tag{36}$$

Proof. From (8), (21) and the first inequality of Lemma 2 we have

$$\begin{aligned} \|u(x,t)\|_{L_2}^2 &= \textstyle\sum_{k=1}^{\infty} T_k^2(t) \leq C_1^2 \sum_{k=1}^{\infty} \left[|\varphi_k| + k^{-1\frac{1}{3}} |\psi_k| + |\xi_k| + k^{-1\frac{1}{3}} |\mu_k|\right]^2 \leq \\ &\leq 4C_1^2 \textstyle\sum_{k=1}^{\infty} \left[|\varphi_k|^2 + k^{-2\frac{2}{3}} |\psi_k|^2 + |\xi_k|^2 + k^{-2\frac{2}{3}} |\mu_k|^2\right] \leq \\ &\leq C_9^2 \left[\|\varphi\|_{L_2}^2 + \|\psi\|_{L_2}^2 + \|\xi\|_{L_2}^2 + \|\mu\|_{L_2}^2\right]. \end{aligned} \tag{37}$$

From inequality (37) estimate (33) follows:

$$\begin{aligned} \|f(x)\|_{L_2[0,p]}^2 &= \textstyle\sum_{k=1}^{\infty} |f_k|^2 \leq C_8^2 \sum_{k=1}^{\infty} \left(k^4 |\varphi_k| + k^{2\frac{2}{3}} |\psi_k| + k^4 |\xi_k| + k^{2\frac{2}{3}} |\mu_k|\right)^2 \leq \\ &\leq 4C_8^2 \textstyle\sum_{k=1}^{\infty} \left[\left(k^4 |\varphi_k|\right)^2 + \left(k^{2\frac{2}{3}} |\psi_k|\right)^2 + \left(k^4 |\xi_k|\right)^2 + \left(k^{2\frac{2}{3}} |\mu_k|\right)^2\right]. \end{aligned}$$

The coefficients φ_k, ψ_k, ξ_k and μ_k can be represented in the form

$$\varphi_k = \frac{1}{\lambda_k^4} \bar{\varphi}_k^{(4)}, \quad \psi_3 = \frac{1}{\lambda_k^3} \bar{\psi}_k^{(3)}, \quad \xi_k = \frac{1}{\lambda_k^4} \bar{\xi}_k^{(4)}, \quad \mu_3 = \frac{1}{\lambda_k^3} \bar{\mu}_k^{(3)},$$

where

$$\bar{\varphi}_k^{(4)} = \sqrt{\tfrac{2}{p}} \textstyle\int_0^p \varphi^{(4)}(x) \sin \lambda_k x dx, \quad \bar{\psi}_k^{(3)} = -\sqrt{\tfrac{2}{p}} \int_0^p \psi^{(3)}(x) \cos \lambda_k x dx,$$

$$\bar{\xi}_k^{(4)} = \sqrt{\tfrac{2}{p}} \textstyle\int_0^p \xi^{(4)}(x) \sin \lambda_k x dx, \quad \bar{\mu}_k^{(3)} = -\sqrt{\tfrac{2}{p}} \int_0^p \mu^{(3)}(x) \cos \lambda_k x dx.$$

Then

$$\begin{aligned} \|f(x)\|_{L_2[0,p]}^2 &\leq 4\bar{C}_8^2 \textstyle\sum_{k=1}^{\infty} \left[\left|\bar{\varphi}_k^{(4)}\right|^2 + \left|\bar{\psi}_k^{(3)}\right|^2 + \left|\bar{\varphi}_k^{(4)}\right|^2 + \left|\bar{\mu}_k^{(3)}\right|^2\right] \leq \\ &\leq 4\bar{C}_8^2 \left[\left\|\varphi^{(4)}\right\|_{L_2}^2 + \left\|\psi^{(3)}\right\|_{L_2}^2 + \left\|\xi^{(4)}\right\|_{L_2}^2 + \left\|\mu^{(3)}\right\|_{L_2}^2\right] \leq \\ &\leq C_{10}^2 \left[\|\varphi\|_{W_2^4}^2 + \|\psi\|_{W_2^3}^2 + \|\xi\|_{W_2^4}^2 + \|\mu\|_{W_2^3}^2\right]. \end{aligned} \tag{38}$$

The validity of estimate (34) follows from (38).

Let $(x,\, t)$ be an arbitrary point from the domain $\bar{\Omega}$. From the first estimate (26) we have

$$|u(x,t)| \leq C_1 \sum_{k=1}^{\infty} \left(|\varphi_k| + k^{-1\frac{1}{3}} |\psi_k| + |\xi_k| + k^{-1\frac{1}{3}} |\mu_k|\right). \tag{39}$$

The coefficients φ_k, ξ_k are presented in the form

$$\varphi_k = \frac{1}{\lambda_k}\bar{\varphi}_k^{(1)}, \quad \xi_k = \frac{1}{\lambda_k}\bar{\xi}_k^{(1)},$$

where

$$\bar{\varphi}_k^{(1)} = \sqrt{\frac{2}{p}}\int_0^p \varphi'(x)\cos\lambda_k x dx, \quad \bar{\xi}_k^{(1)} = \sqrt{\frac{2}{p}}\int_0^p \xi'(x)\cos\lambda_k x dx.$$

From (39) we have

$$|u(x,t)| \le C_1 \sum_{k=1}^{\infty}\left(k^{-1}\left|\bar{\varphi}_k^{(1)}\right| + k^{-1\frac{1}{3}}|\psi_k| + k^{-1}\left|\bar{\xi}_k^{(1)}\right| + k^{-1\frac{1}{3}}|\mu_k|\right) \le$$

$$\le C_{12}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}\left|\bar{\varphi}_k^{(1)}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}|\psi_k|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\left|\bar{\xi}_k^{(1)}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}|\mu_k|^2\right)^{\frac{1}{2}}\right] \le$$

$$\le C_{13}\left[\|\varphi'\|_{L_2} + \|\psi\|_{L_2} + \|\xi'\|_{L_2} + \|\mu\|_{L_2}\right] \le C_{11}\left[\|\varphi\|_{W_2^1} + \|\psi\|_{W_2^0} + \|\xi\|_{W_2^1} + \|\mu\|_{W_2^0}\right].$$

This implies estimate (35). Based on the last estimate (27), we have

$$|f(x)| \le C_8 \sum_{k=1}^{\infty}\left(k^4|\varphi_k| + k^{2\frac{2}{3}}|\psi_k| + k^4|\xi_k| + k^{2\frac{2}{3}}|\mu_k|\right) \le$$

$$\le C_{14}\sum_{k=1}^{\infty}\frac{1}{k}\left(\left|\bar{\varphi}_k^{(5)}\right| + \left|\bar{\psi}_k^{(4)}\right| + \left|\bar{\xi}_k^{(5)}\right| + \left|\bar{\mu}_k^{(4)}\right|\right) \le$$

$$\le C_{14}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}\left|\bar{\varphi}_k^{(5)}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\left|\bar{\psi}_k^{(4)}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\left|\bar{\xi}_k^{(5)}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\left|\bar{\mu}_k^{(4)}\right|^2\right)^{\frac{1}{2}}\right] \le$$

$$\le C_{15}\left(\left\|\varphi^V\right\|_{L_2} + \left\|\psi^{IV}\right\|_{L_2} + \left\|\xi^V\right\|_{L_2} + \left\|\mu^{IV}\right\|_{L_2}\right) \le C_{12}\left(\|\varphi\|_{W_2^5} + \|\psi\|_{W_2^4} + \|\xi\|_{W_2^5} + \|\mu\|_{W_2^4}\right).$$

From this inequality follows (36). The theorem is proved.

Conclusion

In this paper, the inverse problem for a fourth-order equation is considered. The solution is constructed as a series. The uniqueness of the solution to the inverse problem follows from the completeness of the system of eigenfunctions. The stability of the solution to the inverse problem is proven. The results obtained can be used for further development of various direct and inverse problems for a fourth-order equation.

Conflict of Interest

The authors declare no conflict of interest.

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Some Generalized Fractional Hermite-Hadamard-Type Inequalities for $m-$Convex Functions

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Fractional Hermite-Hadamard-type inequalities represent a significant area of study in convex analysis due to their extensive applications in mathematical and applied sciences. These inequalities provide powerful tools for estimating the integral mean of a convex function in terms of its values at the endpoints of a given interval. In this paper, we focus on the development and refinement of fractional Hermite-Hadamard-type inequalities for the class of twice differentiable m-convex functions. Utilizing advanced analytical techniques, such as Hölder's inequality and the power mean integral inequality, we derive new bounds that generalize existing results in the literature. These findings not only extend classical inequalities to a broader class of convex functions but also provide sharper and more versatile estimations. The presented results are expected to have significant implications in various mathematical domains, including fractional calculus, optimization, and mathematical modeling. This work contributes to the ongoing efforts to generalize and refine integral inequalities by incorporating fractional operators and broader convexity assumptions, offering a deeper understanding of the behavior of m-convex functions under fractional integration.

Keywords: integral inequality, fractional Hermite-Hadamard inequality, convex functions, $m-$convex functions, twice differentiable functions, Euler Beta function, Hölder's integral inequality, power mean integral inequality.

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"All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."

– Hardy

Introduction

Let $\xi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function. Then

$$\xi\left(\frac{\varpi_1+\varpi_2}{2}\right) \leq \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2} \xi(\zeta)d\zeta \leq \frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}$$

is known in the literature as Hermite-Hadamard dual inequality [1]. If ξ is concave, then both inequalities hold in the reserved direction. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it easily follows from well-known Jensen's inequality.

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Hermite-Hadamard type inequalities play a significant role in the study of convex functions and have attracted considerable attention in mathematical analysis and its applications. These inequalities provide valuable estimates for the average value of a convex function in terms of its endpoint evaluations. Over the years, various generalizations and extensions have been developed to encompass broader classes of functions, including s-convex, h-convex, and m-convex functions and many more. For further study related to the topic we refer [2–4] to the interested readers.

The concept of m-convexity, introduced as a generalization of classical convexity, is particularly useful in optimization theory, economics, and applied analysis [5]. In 1984, Toader defined the class of $m-$convex functions [6] as:

Definition 1. A function $\xi : [0, \varpi_2] \to \mathbb{R}$ is called $m-$convex, if ξ satisfies

$$\xi(v\zeta_1 + m(1-v)\zeta_2) \leq v\xi(\zeta_1) + m(1-v)\xi(\zeta_2),$$

for all $\zeta_1, \zeta_2 \in [0, \varpi_2]$ and $m,\, v \in [0,1]$.

Remark 1. If we put $m = 0$ and $m = 1$ in the above definition then $m-$convexity changes into Star-shaped [1] and classical convexity [7], respectively.

In parallel, the development of fractional calculus the study of integrals and derivatives of arbitrary (non-integer) order – has led to new avenues for generalizing classical inequalities. Fractional integrals, such as the Riemann–Liouville and Hadamard fractional integrals, have proven to be powerful tools in extending integral inequalities to fractional settings (for example see [8–10]).

By combining the frameworks of fractional calculus and m-convexity, researchers have established fractional Hermite-Hadamard type inequalities for m-convex functions, which provide sharper and more generalized bounds than their classical counterparts. These inequalities not only refine existing results but also open up possibilities for applications in diverse fields such as control theory, mathematical physics, signal processing, and differential equations (for further study see [11] and [12]).

Theorem 1. If $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $g \in L_q$, then $\xi g \in L_1$ and

$$\int |\xi(\zeta)g(\zeta)|d\zeta \leq \|\xi\|_p \|g\|_q, \tag{1}$$

where $\xi \in L_p$ if $\|\xi\|_p = \left(\int |\xi(\zeta)|^p d\zeta\right)^{\frac{1}{p}} < \infty$.

The above inequality is known as Hölder's inequality [13].

Remark 2. Note that Cauchy$-$Schwarz inequality would be obtained by taking $p = q = 2$. Also, if we put $q = 1$ and let $p \to \infty$, then we attain,

$$\int |\xi(\zeta)g(\zeta)|d\zeta \leq ||\xi||_\infty ||g||_1,$$

where $||\xi||_\infty$ stands for essential supremum of $|\xi|$, i.e.,

$$||\xi||_\infty = ess \sup_{\forall \zeta} |\xi(\zeta)|.$$

Another representation of Hölder's inequality is known in literature as Power mean integral inequality [14], defined as:

Theorem 2. If ξ and g are real valued functions defined on I with $|\xi|$ and $|\xi||g|^q$ are integrable on I then for $q \geq 1$, we have:

$$\int_{\varpi_1}^{\varpi_2} |\xi(\zeta)||g(\zeta)|d\zeta \leq \left(\int_{\varpi_1}^{\varpi_2} |\xi(\zeta)|d\zeta\right)^{1-\frac{1}{q}} \left(\int_{\varpi_1}^{\varpi_2} |\xi(\zeta)||g(\zeta)|^q d\zeta\right)^{\frac{1}{q}}. \tag{2}$$

Now, we are going to give some necessary definitions and mathematical results related to fractional calculus which will be used further in this article.

Definition 2. [15] Let $\xi \in L[\varpi_1, \varpi_2]$. The Riemann−Liouville integrals $J^{\alpha}_{\varpi_1+}\xi(\zeta)$ and $J^{\alpha}_{\varpi_2-}\xi(\zeta)$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{\varpi_1+}\xi(\zeta) = \frac{1}{\Gamma(\alpha)}\int\limits_{\varpi_1}^{\zeta}(\zeta - v)^{\alpha-1}\xi(v)dv, \quad \zeta > \varpi_1$$

and

$$J^{\alpha}_{\varpi_2-}\xi(\zeta) = \frac{1}{\Gamma(\alpha)}\int\limits_{\zeta}^{\varpi_2}(v - \zeta)^{\alpha-1}\xi(v)dv, \quad \zeta < \varpi_2,$$

respectively, where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u}u^{\alpha-1}du$ is the Gamma function.

Remark 3. Note that if we take $\alpha = 0$, then $J^{0}_{\varpi_1+}\xi(\zeta) = J^{0}_{\varpi_2-}\xi(\zeta) = \xi(\zeta)$ and if we take $\alpha = 1$, then the fractional integrals reduce to the classical one.

In 2013 Bhatti et al. proved the following three distinct results related to fractional Hermite-Hadamard-type inequality for the class of twice differentiable convex functions [16].

Theorem 3. Let $\xi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $|\xi''|$ is a convex function on I. Suppose that $\varpi_1, \varpi_2 \in I^{\circ}$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$, then the below stated inequality for fractional integrals with $\alpha > 0$ holds:

$$\begin{aligned} &\left|\frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2 - \varpi_1)^{\alpha}}[J^{\alpha}_{\varpi_1+}\xi(\varpi_2) + J^{\alpha}_{\varpi_2-}\xi(\varpi_1)]\right| \\ &\leq \frac{\alpha(\varpi_2 - \varpi_1)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{|\xi''(\varpi_1)| + |\xi''(\varpi_2)|}{2}\right] \\ &\leq \frac{(\varpi_2 - \varpi_1)^2}{(\alpha+1)}\beta(2, \alpha+1)\left[\frac{|\xi''(\varpi_1)| + |\xi''(\varpi_2)|}{2}\right], \end{aligned}$$

where β is the Euler Beta function.

Theorem 4. Let $\xi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $p \in \mathbb{R}$, $p > 1$ such that $|\xi''|^{\frac{p}{p-1}}$ is a convex function on I. Suppose that $\varpi_1, \varpi_2 \in I^{\circ}$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$, then the below stated inequality for fractional integrals with $\alpha > 0$ holds:

$$\begin{aligned} &\left|\frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2 - \varpi_1)^{\alpha}}[J^{\alpha}_{\varpi_1+}\xi(\varpi_2) + J^{\alpha}_{\varpi_2-}\xi(\varpi_1)]\right| \\ &\leq \frac{(\varpi_2 - \varpi_1)^2}{(\alpha+1)}\beta^{\frac{1}{p}}(p+1, \alpha p+1)\left[\frac{|\xi''(\varpi_1)|^q + |\xi''(\varpi_2)|^q}{2}\right]^{\frac{1}{q}}, \end{aligned}$$

where β is the Euler Beta function.

Theorem 5. Let $\xi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $q \geq 1$, $p > 1$ such that $|\xi''|^q$ is a convex function on I. Suppose that $\varpi_1, \varpi_2 \in I^{\circ}$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$, then the below stated inequality for fractional integrals with $\alpha > 0$ holds:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{\alpha(\varpi_2-\varpi_1)^2}{4(\alpha+1)(\alpha+2)}\left[\left(\frac{2\alpha+4}{3\alpha+9}|\xi''(\varpi_1)|^q + \frac{\alpha+5}{3\alpha+9}|\xi''(\varpi_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{\alpha+5}{3\alpha+9}|\xi''(\varpi_1)|^q + \frac{2\alpha+4}{3\alpha+9}|\xi''(\varpi_2)|^q\right)^{\frac{1}{q}}\right].$$

The structure of this article unfolds as follows: In the subsequent section, we aim to establish three unique outcomes concerning fractional Hermite-Hadamard-type inequalities applicable to the category of twice differentiable $m-$convex functions. Our approach will leverage diverse techniques, encompassing Hölder's and power mean integral inequalities. These findings are anticipated to exhibit a broader scope compared to those presented in [16]. The third section will provide a concluding statement, while the final section will offer insights and future prospects for readers interested in further exploration.

1 Various Estimations of Right Bound of Fractional Hermite-Hadamard-type Inequalities for Twice Differentiable $m-$Convex Functions

In order to prove our main results we need to recall following lemma from [16].

Lemma 1. Let $\xi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, the interior of I. Assume that $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$, then the below stated identity for fractional integrals with $\alpha > 0$ holds:

$$\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]$$
$$= \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int_0^1 v(1-v^\alpha)[\xi''(v\varpi_1+(1-v)\varpi_2) + \xi''((1-v)\varpi_1+v\varpi_2)]dv,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$.

Now, we are going to state and prove of our new results related to fractional Hermite-Hadamard-type inequalities for twice differentiable $m-$convex functions.

Theorem 6. Let $\xi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I°, the interior of I. Assume that $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$. If $|\xi''|$ is $m-$convex on I for some $m \in (0,1]$, then the below stated inequality for fractional integrals with $\alpha > 0$ holds:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right| \leq \frac{\alpha(\varpi_2-\varpi_1)^2}{6(\alpha+1)(\alpha+3)}$$
$$\times\left[|\xi''(\varpi_1)| + |\xi''(\varpi_2)| + m\frac{(\alpha+5)}{2(\alpha+2)}\left(\left|\xi''\left(\frac{\varpi_1}{m}\right)\right| + \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|\right)\right].$$

Proof. By using Lemma 1 and the property of absolute value, we have,

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int_0^1 |v(1-v^\alpha)|[|\xi''(v\varpi_1+(1-v)\varpi_2)| + |\xi''((1-v)\varpi_1+v\varpi_2)|]dv. \tag{3}$$

As we have $|\xi''|$ is a $m-$convex function, so we can take

$$\left|\xi''\left(v\varpi_1+(1-v)\varpi_2\right)\right| \le v\left|\xi''\left(\varpi_1\right)\right| + m(1-v)\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|$$

and

$$\left|\xi''\left((1-v)\varpi_1+v\varpi_2\right)\right| \le m(1-v)\left|\xi''\left(\frac{\varpi_1}{m}\right)\right| + v|\xi''(\varpi_2)|.$$

Utilizing the above two results, (3) becomes

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\le \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int_0^1 \left[v^2(1-v^\alpha)|\xi''(\varpi_1)| + mv(1-v)(1-v^\alpha)\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|\right.$$
$$\left. + mv(1-v)(1-v^\alpha)\left|\xi''\left(\frac{\varpi_1}{m}\right)\right| + v^2(1-v^\alpha)|\xi''(\varpi_2)|\right] dv.$$

After arranging and using the following facts the result of Theorem 6 is accomplished.

$$\int_0^1 v^2(1-v^\alpha)dv = \frac{\alpha}{3(\alpha+3)}$$

and

$$\int_0^1 v(1-v^\alpha)(1-v)dv = \frac{\alpha(\alpha+5)}{6(\alpha+2)(\alpha+3)}.$$

Remark 4. The following well-known results would be captured as special cases of our obtained result by varying different values of m and α:

1. If we choose $m=1$ in Theorem 6, then we get first inequality of Theorem 3.
2. If we choose $\alpha=m=1$ in Theorem 6, then we get Hermite-Hadamard-type inequality for twice differentiable convex function [17].

Corollary 1. If we choose $\alpha=1$ in Theorem 6, then we get the following Hermite-Hadamard-type inequality for twice differentiable $m-$convex function:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right|$$
$$\le \frac{(\varpi_2-\varpi_1)^2}{48}\left[|\xi''(\varpi_1)|+|\xi''(\varpi_2)| + m\left(\left|\xi''\left(\frac{\varpi_1}{m}\right)\right| + \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|\right)\right].$$

Theorem 7. Let $\xi : I\subset[0,\infty)\to\mathbb{R}$ be a twice differentiable function on I°, the interior of I. Assume that $\varpi_1,\varpi_2\in I^\circ$ with $\varpi_1<\varpi_2$ and $\xi''\in L[\varpi_1,\varpi_2]$. If $|\xi''|^q$ is $m-$convex on I for some

$m \in (0,1]$ and $q \geq 1$ then the following inequality for fractional integrals with $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ holds:

$$\left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2 - \varpi_1)^\alpha} [J^\alpha_{\varpi_1+} \xi(\varpi_2) + J^\alpha_{\varpi_2-} \xi(\varpi_1)] \right|$$
$$\leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha+1)} \beta^{\frac{1}{p}} (p+1, \alpha p + 1)$$
$$\times \left[\left(\frac{|\xi''(\varpi_1)|^q + m \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{m \left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + |\xi''(\varpi_2)|^q}{2} \right)^{\frac{1}{q}} \right],$$

where β is the Euler Beta function.

Proof. By using Lemma 1 and the property of absolute value, we have

$$\left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2 - \varpi_1)^\alpha} [J^\alpha_{\varpi_1+} \xi(\varpi_2) + J^\alpha_{\varpi_2-} \xi(\varpi_1)] \right|$$
$$\leq \frac{(\varpi_2 - \varpi_1)^2}{2(\alpha+1)} \int_0^1 |v(1-v^\alpha)| [|\xi''(v\varpi_1 + (1-v)\varpi_2)| + |\xi''((1-v)\varpi_1 + v\varpi_2)|] dv. \quad (4)$$

Applying (1) to $\int_0^1 |v(1-v^\alpha)||\xi''(v\varpi_1 + (1-v)\varpi_2)| dv$ and $\int_0^1 |v(1-v^\alpha)||\xi''((1-v)\varpi_1 + v\varpi_2)| dv$ implies

$$\int_0^1 |v(1-v^\alpha)||\xi''(v\varpi_1 + (1-v)\varpi_2)| dv$$
$$\leq \left(\int_0^1 |v(1-v^\alpha)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |\xi''(v\varpi_1 + (1-v)\varpi_2)|^q dv \right)^{\frac{1}{q}}$$

and

$$\int_0^1 |v(1-v^\alpha)||\xi''((1-v)\varpi_1 + v\varpi_2)| dv$$
$$\leq \left(\int_0^1 |v(1-v^\alpha)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |\xi''((1-v)\varpi_1 + v\varpi_2)|^q dv \right)^{\frac{1}{q}}.$$

As we have $|\xi''|^q$ is a $m-$convex function, so we can take

$$\left|\xi''(v\varpi_1 + (1-v)\varpi_2)\right|^q \leq v \left|\xi''(\varpi_1)\right|^q + m(1-v) \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q$$

and

$$\left|\xi''((1-v)\varpi_1 + v\varpi_2)\right|^q \leq m(1-v) \left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + v|\xi''(\varpi_2)|^q.$$

Utilizing the above four results, (4) becomes

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^{\alpha}}[J^{\alpha}_{\varpi_1^+}\xi(\varpi_2)+J^{\alpha}_{\varpi_2^-}\xi(\varpi_1)]\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\left(\int\limits_0^1 v^p(1-v^{\alpha})^p dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(|\xi''(\varpi_1)|^q\int\limits_0^1 vdv+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int\limits_0^1(1-v)dv\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q\int\limits_0^1(1-v)dv+|\xi''(\varpi_2)|^q\int\limits_0^1 vdv\right)^{\frac{1}{q}}\right].$$

After using the following facts, the result of Theorem 7 is accomplished.

$$\int\limits_0^1 vdv=\int_0^1(1-v)dv=\frac{1}{2}$$

and

$$\int\limits_0^1 v^p(1-v^{\alpha})^p dt\leq\int\limits_0^1 v^p(1-v)^{\alpha p}dv=\beta(p+1,\alpha p+1).$$

Remark 5. Following well-known results would be captured as special cases of our obtained result by varying different values of m and α:

1. If we choose $m=1$ in Theorem 7, then we get Theorem 4.
2. If we choose $\alpha=m=1$ in Theorem 7, then we get Theorem 10 of [18].

Corollary 2. Under the assumptions of the Theorem 7,

1. If we put $p=q=2$, then we get the result obtained by using Cauchy− Schwarz integral inequality as:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^{\alpha}}[J^{\alpha}_{\varpi_1^+}\xi(\varpi_2)+J^{\alpha}_{\varpi_2^-}\xi(\varpi_1)]\right|$$
$$\leq\frac{(\varpi_2-\varpi_1)^2}{2^{\frac{3}{2}}(\alpha+1)}\beta^{\frac{1}{2}}(3,2\alpha+1)$$
$$\times\left[\left(|\xi''(\varpi_1)|^2+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^2\right)^{\frac{1}{2}}+\left(m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^2+|\xi''(\varpi_2)|^2\right)^{\frac{1}{2}}\right],$$

   where β is the Euler Beta function.
2. If we put $q=1$ and $p=\infty$, then we get the result involving essential supremum norm as:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^{\alpha}}[J^{\alpha}_{\varpi_1^+}\xi(\varpi_2)+J^{\alpha}_{\varpi_2^-}\xi(\varpi_1)]\right|$$
$$\leq\frac{(\varpi_2-\varpi_1)^2}{4(\alpha+1)}||X||_{\infty}\left[|\xi''(\varpi_1)|+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|+m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|+|\xi''(\varpi_2)|\right],$$

where $||X||_\infty = ess\sup\limits_{v\in[0,1]} \int\limits_0^1 v(1-v)^\alpha$.

3. If we choose $\alpha = 1$, then we get the following Hermite-Hadamard-type inequality for twice differentiable $m-$convex function:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int\limits_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right| \le \frac{(\varpi_2-\varpi_1)^2}{4}\beta^{\frac{1}{p}}(p+1,p+1)$$
$$\times\left[\left(\frac{|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + |\xi''(\varpi_2)|^q}{2}\right)^{\frac{1}{q}}\right].$$

Theorem 8. Let $\xi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I°, the interior of I. Assume that $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$. If $|\xi''|^q$ is $m-$convex on I for some $m \in (0,1]$ and $q \ge 1$ then the following inequality for fractional integrals with $\alpha > 0$ holds:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\le \frac{\alpha(\varpi_2-\varpi_1)^2}{4(\alpha+1)(\alpha+2)(3(\alpha+3))^{\frac{1}{q}}}\left[\left(2(\alpha+2)|\xi''(\varpi_1)|^q + m(\alpha+5)\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(m(\alpha+5)\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + 2(\alpha+2)|\xi''(\varpi_2)|^q\right)^{\frac{1}{q}}\right].$$

Proof. By using Lemma 1 and the property of absolute value, we have

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\le \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int\limits_0^1 |v(1-v^\alpha)|[|\xi''(v\varpi_1+(1-v)\varpi_2)| + |\xi''((1-v)\varpi_1+v\varpi_2)|]dv. \quad (5)$$

Applying (2) to $\int\limits_0^1 |v(1-v^\alpha)||\xi''(v\varpi_1+(1-v)\varpi_2)|dv$ and $\int\limits_0^1 |v(1-v^\alpha)||\xi''((1-v)\varpi_1+v\varpi_2)|dv$ implies

$$\int\limits_0^1 |v(1-v^\alpha)||\xi''(v\varpi_1+(1-v)\varpi_2)|dv$$
$$\le \left(\int\limits_0^1 v(1-v^\alpha)dv\right)^{1-\frac{1}{q}}\left(\int\limits_0^1 v(1-v^\alpha)|\xi''(v\varpi_1+(1-v)\varpi_2)|^q dv\right)^{\frac{1}{q}}$$

and

$$\int\limits_0^1 |v(1-v^\alpha)||\xi''((1-v)\varpi_1+v\varpi_2)|dv$$
$$\le \left(\int\limits_0^1 v(1-v^\alpha)dv\right)^{1-\frac{1}{q}}\left(\int\limits_0^1 v(1-v^\alpha)|\xi''((1-v)\varpi_1+v\varpi_2)|^q dv\right)^{\frac{1}{q}}.$$

Since $|\xi''|^q$ is an $m-$convex function, so we can take

$$\left|\xi''\left(v\varpi_1+(1-v)\varpi_2\right)\right|^q \leq v\left|\xi''\left(\varpi_1\right)\right|^q + m(1-v)\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q$$

and

$$\left|\xi''\left((1-v)\varpi_1+v\varpi_2\right)\right|^q \leq m(1-v)\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + v|\xi''(\varpi_2)|^q.$$

Utilizing the above four results, (5) becomes

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1^+}\xi(\varpi_2)+J^\alpha_{\varpi_2^-}\xi(\varpi_1)]\right|$$

$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\left(\int_0^1 v(1-v^\alpha)dv\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(|\xi''(\varpi_1)|^q\int_0^1 v^2(1-v^\alpha)dv + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int_0^1 v(1-v)(1-v^\alpha)dv\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q\int_0^1 v(1-v^\alpha)dv + |\xi''(\varpi_2)|^q\int_0^1 v^2(1-v)dv\right)^{\frac{1}{q}}\right].$$

After arranging and using the following facts the result of Theorem 8 is accomplished.

$$\int_0^1 v(1-v^\alpha)dv = \frac{\alpha}{2(\alpha+2)},$$

$$\int_0^1 v^2(1-v^\alpha)dv = \frac{\alpha}{3(\alpha+3)}$$

and

$$\int_0^1 v(1-v)(1-v^\alpha)dv = \frac{\alpha(\alpha+5)}{6(\alpha+2)(\alpha+3)}.$$

Remark 6. Following well-known results would be captured as special cases of our obtained result by varying different values of m and α:

1. If we choose $m=1$ in Theorem 8, then we get Theorem 5.
2. If we choose $\alpha=m=1$ in Theorem 8, then we get Theorem 8 of [18].

Corollary 3. If we choose $\alpha=1$ in Theorem 8, then we get the following Hermite-Hadamard-type inequality for twice differentiable $m-$convex function:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right| \leq \frac{(\varpi_2-\varpi_1)^2}{24(2)^{\frac{1}{q}}}$$

$$\times\left[\left(|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\right)^{\frac{1}{q}} + \left(m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + |\xi''(\varpi_2)|^q\right)^{\frac{1}{q}}\right].$$

2 Conclusion

The fractional Hermite-Hadamard inequality stands out as one of the most renowned within the realm of inequalities, boasting numerous generalizations across different classes of convex functions in existing literature. In this article, we present its extension for twice differentiable $m-$convex functions. Section 1 unveils three distinct findings concerning the estimated right bound of the fractional Hermite-Hadamard inequality in an absolute sense for twice differentiable $m-$convex functions. Here, we employ various methodologies, including Hölder's and Power mean integral inequalities. While some of these results are novel, others have been previously documented in the articles [16–18]. The final section is dedicated to providing remarks and offering future avenues of exploration for interested readers.

Now, we are going to summarize the results of Section 1 in Table 1.

Table 1

Result Summary of Section 1

S. No	m	α	Results	Found in
1	1	$-$	FHHTI for Ordinary Convex Functions	[16]
2	$-$	1	HHTI for $m-$Convex Functions	This Article
3	1	1	HHTI for Ordinary Convex Functions	[17, 18]

In the preceding table, the abbreviations FHHTI and HHTI refer to the Fractional Hermite-Hadamard type inequality and the Hermite-Hadamard type inequality, respectively, while the symbol "$-$" indicates validity for any value.

Now we are going to give some remarks and future ideas related to our stated results.

3 Remarks and Future Ideas

1. All the inequalities given in this article can be stated in the reverse direction for concave functions using the simple relation that ξ is concave if and only if ξ is convex.
2. One may also work on Fejér inequality by introducing weights in fractional Hermite-Hadamard inequality.
3. One may do similar work by using various distinct classes of convex functions.
4. One may try to state all the results given in this article for the discrete case.
5. One may also state all the results given in this article for Multi-dimensions.
6. One can extend this work to time scale domain or Quantum Calculus.
7. One can try to attain this work for Fuzzy theory.
8. One can try to work for finding refined bounds of all results.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Singularly perturbed problems with rapidly oscillating inhomogeneities in the case of discrete irreversibility of the limit operator

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We consider a linear singularly perturbed differential system, one of the points of the spectrum of the limiting operator of which goes to zero on some discrete subset of the segment of the independent variable. The problem belongs to the class of problems with unstable spectrum. Previously, S.A. Lomov's regularization method was used to construct asymptotic solutions of a similar system. However, it was applied in the case of absence of fast oscillations. The presence of the latter does not allow us to approximate the exact solution by a degenerate one, since the limit transition in the initial system when a small parameter tends to zero in a uniform metric is impossible. Therefore, when constructing the asymptotic solution, it is necessary to take into account the effects introduced into the asymptotics by fast oscillations. In developing the corresponding algorithm, one could use the ideas of the classical Lomov regularization method, but considering that its implementation requires numerous calculations (e.g., to construct the main term of the asymptotics in the simplest case of the second-order zero eigenvalue of the limit operator one has to solve three algebraic systems of order higher than the first), the authors considered it necessary to develop a more economical algorithm based on regularization by means of normal forms.

Keywords: singularly perturbed problem, normal form, discrete irreversibility of the operator, instability of the spectrum, regularized asymptotics, asymptotic solution, solvability of iterative problems, limit transition.

2020 Mathematics Subject Classification: 34E05, 34E15, 34E20.

1 Problem formulation and its regularization

Consider the singularly perturbed Cauchy's problem

$$\varepsilon\frac{dy}{dt} = A_0\left(t\right)y + h_0\left(t\right) + h_1\left(t\right)e^{i\frac{\beta(t)}{\varepsilon}},\quad y\left(0,\varepsilon\right) = y^0,\quad t\in\left[0,T\right] \tag{1_0}$$

where $y = \{y_1(t), ..., y_n(t)\}$ is an unknown vector function, $h_j = \{h_{1j}(t), ..., h_{nj}(t)\}$ are known vector functions, $y^0 = \left\{y_1^0, ..., y_n^0\right\}$ is the known constant vector, $\beta'(t) > 0$ is the frequency of rapidly oscillating inhomogeneity, $\varepsilon > 0$ is a small parameter. Let $\left\{\lambda_j(t),\ j = \overline{1,n}\right\}$ be the spectrum of the matrix $A_0(t)$. Assuming that the conditions:

1) $A_0(t) \in C^\infty\left([0,T], \mathbb{C}^{n\times n}\right),\ h_j(t) \in C^\infty\left([0,T],\ \mathbb{C}^n\right),\ j = 0,1,\ \beta(t) \in C^\infty\left([0,T], \mathbb{C}^1\right);$

2) there exists the subset $B \subset [0,T]$ such that

a) $\lambda_1(t) = l_1(t)\prod\limits_{j=1}^{r}(t-t_j)^{s_j},\ l_1(t) < 0,\ l_1(t) \in C^\infty[0,T],\ s_j = 2m_j \in \mathbb{Z}_+,\ t_j \in [0,T],$

$j = \overline{1,r},\ \ \lambda_k(t) \neq 0\ \ \forall t \in [0,T],\ \ k = \overline{2,n};$

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b) $\lambda_i(t) \neq \lambda_j(t)$, $\lambda_j(t) \neq \beta'(t)$, $i \neq j$, $i, j = \overline{1, n}$, $\forall t \in [0, T]$;

c) $\beta'(t) > 0$, $\mathrm{Re}\lambda_i(t) \leq 0$ $\forall t \in [0, T]$, $i = \overline{1, n}$

are satisfied, we develop an algorithm for constructing the asymptotic solution of the problem (1_0). The problem (1_0) belongs to the class of complex problems for the study of singularly perturbed systems with unstable spectrum [1]. In [2], a regularization method is developed for the case when the spectrum of the variable limit operator vanishes at individual points. In [3], the Cauchy problem, is studied in the presence of a "weak" turning point of the limit operator, and estimates are provided that characterize the behavior of singularities at $\varepsilon \to +0$. A generalization of the ideas of the regularization method for problems with a turning point at which the eigenvalues "stick together" at $t = 0$ and initializations are considered in works [4, 5]. An analytical method for solving a Burgers-type equation in a Banach space is investigated in [6]. Namely, after artificially introducing a small parameter into the equation, the existence of an analytical solution with respect to this parameter is proven. The concept of a pseudoanalytic (pseudoholomorphic) solution introduced by S.A. Lomov initiated the development of singularly perturbed analytic theory. In [7, 8], formally singularly perturbed equations are considered in topological algebras, which allows one to formulate the basic concepts of singularly perturbed analytic theory from the standpoint of maximum generality, and conditions for the existence of solutions holomorphic in the parameter are found in the case when the perturbing operator is bilinear. The study of finding conditions for the ordinary convergence of series in powers of a small parameter, representing solutions to perturbation theory problems, is considered in [9]. Their results were generalized to integro-differential equations in [10]. This paper is the first to apply the normal form method to study such problems. The purpose of this paper is to develop this algorithm to construct asymptotic solutions of the problem (1_0) in the presence of a rapidly oscillating inhomogeneity $h_0(t)\, e^{i\frac{\beta(t)}{\varepsilon}}$.

Since the function $e^{\frac{1}{\varepsilon}\beta(t)}$ satisfies the differential equation

$$\varepsilon \frac{dy_{n+1}(t,\varepsilon)}{dt} = i\beta'(t) y_{n+1}(t,\varepsilon), \quad y_{n+1}(0,\varepsilon) = e^{\frac{i}{\varepsilon}\beta(0)},$$

then from the system (1_0) of order n it will be necessary to pass to the system of order $(n+1)$:

$$\varepsilon \frac{d}{dt}\begin{pmatrix} y(t,\varepsilon) \\ y_{n+1}(t,\varepsilon)\end{pmatrix} = \begin{pmatrix} A_0(t) & h_1(t) \\ 0 & i\beta'(t)\end{pmatrix}\begin{pmatrix} y(t,\varepsilon) \\ y_{n+1}(t,\varepsilon)\end{pmatrix} + \begin{pmatrix} h_0(t) \\ 0\end{pmatrix}, \quad \begin{pmatrix} y(0,\varepsilon) \\ y_{n+1}(t,\varepsilon)\end{pmatrix} + \begin{pmatrix} y^0 \\ e^{\frac{i}{\varepsilon}\beta(0)}\end{pmatrix}$$

or

$$\varepsilon \frac{dz}{dt} = A(t)\, z + h(t), \quad z(0,\varepsilon) = z^0, \quad t \in [0, T], \tag{1}$$

where notations

$$z = \{y, y_{n+1}\}, \; z^0 = \left\{y^0, e^{\frac{i}{\varepsilon}\beta(0)}\right\}, \; A(t) = \begin{pmatrix} A_0(t) & h_1(t) \\ 0 & i\beta'(t)\end{pmatrix}, \; h(t) = \begin{pmatrix} h_0(t) \\ 0\end{pmatrix}$$

are introduced.

Let's denote by $e_i = \left\{0, ..., 0, \underset{(i)}{1}, 0, ..., 0\right\}$ the i-th ort in $\mathbb{C}^{n+1}$, $\bar{1} = \{1, ..., 1\} \in \mathbb{R}^{n+1}$ is the vector consisting solidly of units, $\lambda_{n+1}(t) = i\beta'(t)$, and through $\Lambda(t) = \mathrm{diag}\{\lambda_1(t), ..., \lambda_{n+1}(t)\}$ is the diagonal matrix with the spectrum of the matrix $A(t)$ on the diagonal. We regularize the problem (1) with the vector $u = \{u_1, ..., u_n, u_{n+1}\}$ of the regularizing variables satisfying the normal form*

$$\varepsilon \frac{du}{dt} = \Lambda(t)\, u + g_0(t)\, e_1 + \sum_{j=1}^{m} \varepsilon^j \sum_{i=1}^{r} g_j(t)\, e_1, \quad u(0,\varepsilon) = \bar{1}, \tag{2}$$

*On regularization by means of normal forms, see, for example, [10].

where the functions $g_j(t) \in C^\infty\left([0,T], \mathbb{C}^1\right)$ are calculated in the process of constructing the asymptotic solution of problem (1). The extended system corresponding to problem (1) will have the form

$$\varepsilon \frac{\partial \tilde{z}}{\partial t} + \frac{\partial \tilde{z}}{\partial u}\left[\Lambda(t)\,u + g_0(t)\,e_1 + \sum_{j=1}^{m} \varepsilon^j \sum_{i=1}^{r} g_j(t)\,e_1\right] - A(t)\,\tilde{z} = h(t), \quad \tilde{z}(t,u,\varepsilon)\,|_{t=0,u=\bar{1}} = y^0, \tag{3}$$

where the function $\tilde{z} = \tilde{z}(t,u,\varepsilon)$ is such that its contraction on the solution $u = u(t,\varepsilon)$ of the normal form (2) coincides with the exact solution $z(t,\varepsilon)$ of problem (1). Since problem (3) is regular in ε at $\varepsilon \to +0$, its solution can be sought in the form of series

$$\tilde{z}(t,u,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k z_k(t,u) \tag{4}$$

by non-negative powers of the parameter ε. Substituting series (4) into (3) and equating the coefficients at the same powers of ε, we obtain the following iterative problems:

$$Lz_0 \equiv \frac{\partial z_0}{\partial u}\Lambda(t)\,u - A(t)\,z_0 = h(t) - \frac{\partial z_0}{\partial u} g_0(t)\,e_1, \quad z_0(0,\bar{1}) = z^0; \tag{4_0}$$

$$Lz_1 = -\frac{\partial z_0}{\partial t} - \frac{\partial z_0}{\partial u} g_1(t)\,e_1 - \frac{\partial z_1}{\partial u} g_0(t)\,e_1, \quad y_1(0,\bar{1}) = 0; \tag{4_1}$$

$$Lz_{k+1} = -\frac{\partial z_k}{\partial t} - \frac{\partial z_0}{\partial u} g_{k+1}(t)\,e_1 - \frac{\partial z_{k+1}}{\partial u} g_0(t)\,e_1 - \sum_{j=1}^{k} \frac{\partial z_j}{\partial u} g_{k+1-j}(t)\,e_1, z_{k+1}(0,\bar{1}) = 0, \ k > 1. \tag{4_{k+1}}$$

Here $g_{kj}(t) \equiv 0$ at $k \geq m+1$.

2 *Solvability of the first iterative problem*

Under the described conditions on the spectrum of operator $A(t)$ there exists a matrix $C(t) \equiv (c_1(t), \ldots, c_{n+1}(t))$ with columns $c_j(t) \in C^\infty([0,T], \mathbb{C}^{n+1})$ such that for all $t \in [0,T]$ the identity

$$C^{-1}(t)A(t)C(t) \equiv \Lambda(t) = \operatorname{diag}(\lambda_1(t), \ldots, \lambda_{n+1}(t)) \tag{5}$$

holds. Let's denote by $d_j(t)$ the j-th column of the matrix $[C^{-1}(t)]^*$, $j = \overline{1, n+1}$. It is clear that for each $t \in [0,T]$ the following equality holds $A^*(t)d_j(t) = \overline{\lambda}_j(t)d_j(t)$ $(c_i(t), d_j(t)) \equiv \delta_{ij}$ $(i,j = \overline{1, n+1})$, where δ_{ij} is Kronecker's symbol (here and below $(*,*)$ denotes the scalar product in $\mathbb{C}^{n+1}$). Note that identity (5) excludes the rotation points in system (1).

The solution of each iterative problem (4_k) we will be defined in the space U of functions $z(t,u) = \{z_1, \ldots z_{n+1}\}$ of the form

$$z(t,u) = \sum_{j=1}^{n+1} z_j(t)\,u_j + z_0(t), \quad z_j(t) \in C^\infty([0,T], \mathbb{C}^{n+1}), \quad j = \overline{0, n+1} \tag{6}$$

in which the scalar product (at each $t \in [0,T]$)

$$< w, z > \equiv \left\langle \sum_{j=1}^{n+1} w_j(t)\,u_j + w_0(t), \sum_{j=1}^{n+1} z_j(t)\,u_j + z_0(t) \right\rangle \triangleq \sum_{j=0}^{n+1} (w_j(t), z_j(t)) \equiv \sum_{j=0}^{n+1} w_j^T(t)\,\bar{z}_j(t)$$

is introduced. Without developing the general theory of solvability of iterative problems (4_k), let us try to solve the problem (4_0). By defining its solution as an element of the space U given by (6):

$$z_0(t,u)=\sum_{j=1}^{n+1} z_j^{(0)}(t)\,u_j+z_0^{(0)}(t), \tag{7}$$

we obtain the following system of equations for the coefficients $z_j^{(0)}(t)$:

$$-A(t)\,z_0^{(0)}(t)=h(t)-g_0(t)\,z_1^{(0)}(t), \tag{8}$$

$$[\lambda_j(t)\,I-A(t)]\,z_j^{(0)}(t)=0,\quad j=\overline{1,n+1}. \tag{9}$$

Solutions of the systems (9) are defined in the form $z_j^{(0)}(t)=\alpha_j(t)\,c_j(t)$, where $\alpha_j(t)\in C^\infty\left([0,T],\mathbb{C}^1\right)$ are arbitrary scalar functions, $j=\overline{1,n+1}$. To compute these functions, we proceed to the iterative system (4_1). Defining its solution in the space U as a function

$$z_1(t,u)=\sum_{j=1}^{n+1} z_j^{(1)}(t)\,u_j+z_0^{(1)}(t),$$

we get similar systems

$$-A(t)\,z_0^{(1)}(t)=-\dot z_0^{(0)}(t)-z_1^{(0)}(t)\,g_1(t)-g_0(t)\,z_1^{(1)}(t),$$

$$[\lambda_j(t)\,I-A(t)]\,z_j^{(1)}(t)=-\dot\alpha_j(t)\,c_j(t)-\alpha_j(t)\,\dot c_j(t),\quad j=\overline{1,n+1}. \tag{10}$$

For the solvability of systems (10) in the class $C^\infty\left([0,T],\mathbb{C}^{n+1}\right)$ it is necessary and sufficient that

$$\left(-\dot\alpha_j(t)\,c_j(t)-\alpha_j(t)\,\dot c_j(t),d_j(t)\right)\equiv 0,\quad j=\overline{1,n+1}$$

from where we find the functions

$$\alpha_j(t)=\alpha_j(0)\exp\left\{-\int_0^t\left(\dot c_j(\theta),d_j(\theta)\right)d\theta\right\},\quad j=\overline{1,n+1}.$$

The initial values for these functions are found from the condition $z_0(0,\bar 1)=z^0$, which, taking into account (7), is written in the form

$$\sum_{j=1}^{n+1}\alpha_j(0)\,c_j(0)=z^0-z_0^{(0)}(0)\quad\Leftrightarrow\quad \alpha_j(0)=\left(z^0-z_0^{(0)}(0),d_j(0)\right),\quad j=\overline{1,n+1}. \tag{11}$$

However, no function has yet been found in (11) $z_0^{(0)}(t)$. Substituting $z_1^{(0)}(t)=\alpha_1(t)\,c_1(t)$ in (8) and making in the obtained system the transformation $z_0^{(0)}(t)=C(t)\,\xi\equiv\left(c_1(t),...,c_{n+1}(t)\right)\begin{pmatrix}\xi_1\\ ...\\ \xi_{n+1}\end{pmatrix}$, we obtain the following equations for the vector components ξ:

$$-\lambda_1(t)\,\xi_1=\left(h(t),d_1(t)\right)-g_0(t)\,\alpha_1(t),$$

$$-\lambda_j(t)\,\xi_j=\left(h(t),d_j(t)\right),\ j=\overline{2,n+1}.$$

Since $\lambda_j(t) \neq 0$ at $j = \overline{2, n+1}$, then the last equations of this system have unique solutions

$$\xi_j(t) = -\frac{(h(t), d_j(t))}{\lambda_j(t)}, \ j = \overline{2, n+1}.$$

In view of condition 2a), the first equation of the above system is solvable in the class $C^\infty\left([0, T], \mathbb{C}^1\right)$ then and only when

$$D^\nu(\alpha_1 g_0)(t_j) = D^\nu(h, d_1)(t_j), \quad j = \overline{1, r}, \quad \nu = \overline{0, s_j - 1}$$

(here and throughout the following, $D^\nu(f)(t_j)$ denotes the ν-th derivative of a function $f(t)$ at the point t_j).

It follows that the function $\alpha_1(t) g_0(t)$ is the Lagrange-Sylvester's polynomial of the function $(h(t), d_1(t))$, i.e.,

$$\alpha_1(t) g_0(t) = \sum_{j=1}^{r} \sum_{\nu=0}^{s_j-1} D^\nu(h, d_1)(t_j) K_{ji}(t), \tag{12}$$

where $\left\{K_{ji}(t), \ j = \overline{1, r}, \ i = \overline{0, s_j - 1}\right\}$ is the basis system of Lagrangian-Sylvester's polynomials constructed by the polynomial $\psi(t) = \prod\limits_{j=1}^{r} (t - t_j)^{s_j}$ [10; §9.2]. Suppose that the number $\alpha_1(0) = \left(z^0 - z_0^{(0)}(0), d_1(0)\right) \neq 0$. Then it follows from (12) that the function $g_0(t)$ is represented as

$$g_0(t) = \frac{1}{\alpha_1(0) p_1(t)} \sum_{j=1}^{r} \sum_{\nu=0}^{s_j-1} D^\nu(h, d_1)(t_j) K_{j\nu}(t), \tag{13}$$

where $p_1(t) = \exp\left\{-\int\limits_0^t (\dot{c}_1(\theta), d_1(\theta)) d\theta\right\}$. From (8), taking into account formula (12), we find the function $z_0^{(0)}(t)$:

$$\begin{gathered} z_0^{(0)}(t) = -A^{-1}(t)\left(h(t) - g_0(t) z_1^{(0)}(t)\right) = \\ = -C(t)\Lambda^{-1}(t) C^{-1}(t)(h(t) - g_0(t)\alpha_1(t) c_1(t)) = \\ = -\frac{(h(t), d_1(t)) - g_0(t)\alpha_1(t)}{\lambda_1(t)} c_1(t) - \sum_{j=2}^{n+1} \frac{(h(t), d_j(t))}{\lambda_j(t)} c_j(t) \equiv \\ \equiv -\frac{(h(t), d_1(t)) - \sum\limits_{j=1}^{r} \sum\limits_{\nu=0}^{s_j-1} D^\nu(h, d_1)(t_j) K_{j\nu}(t)}{\lambda_1(t)} c_1(t) - \sum_{j=2}^{n+1} \frac{(h(t), d_j(t))}{\lambda_j(t)} c_j(t). \end{gathered} \tag{14}$$

Hence, we can see that the function $z_0^{(0)}(t)$ does not depend on $\alpha_1(0)$. This allows us to find values $\alpha_j(0)$:

$$\begin{gathered} \alpha_1(0) = \left(z^0 - z_0^{(0)}(0), d_1(0)\right) = \left(z^0, d_1(0)\right) + \\ + \frac{(h(0), d_1(0)) - \sum\limits_{j=1}^{r} \sum\limits_{\nu=0}^{s_j-1} D^\nu(h, d_1)(t_j) K_{j\nu}(0)}{\lambda_1(t)}, \end{gathered} \tag{15}$$

$$\alpha_j(0) = \left(z^0, d_j(0)\right) + \sum_{j=2}^{n+1} \left(\tfrac{(h(0),d_j(0))}{\lambda_j(0)} c_j(0), d_j(0) \right) = \\ = \left(z^0, d_j(0)\right) + \tfrac{(h(0),d_j(0))}{\lambda_j(0)}, \quad j = \overline{2, n+1} \tag{16}$$

unambiguously and hence compute the solution (7) to problem (4_0) in the space U in a single-valued way. We come to the following result.

Theorem 1. Let conditions 1), 2a), 2b) be satisfied and the number $\alpha_1(0)$, defined by formula (15), is not equal to zero. Then whatever the functions $z_1(t,u) \in U$ and $g_1(t) \in C^\infty([0,T], \mathbb{C}^1)$, there exists a single function $g_0(t) \in C^\infty([0,T], \mathbb{C}^1)$, computed by formulas (13), such that the problem (4_0) under the additional condition

$$\left\langle -\frac{\partial z_0}{\partial t} - \frac{\partial z_0}{\partial u} g_1(t) e_1 - \frac{\partial z_1}{\partial u} g_0(t) e_1, d_j(t) u_j \right\rangle \equiv 0 \quad \forall t \in [0,T], \quad j = \overline{1, n+1}$$

has a single solution in the class U. This solution is given by formula (7), where the functions $\alpha_j(t)$ have the form $\alpha_j(t) = \alpha_j(0) \exp\left\{ -\int\limits_0^t (\dot{c}_j(\theta), d_j(\theta))\, d\theta \right\}$, $j = \overline{1, n+1}$, and the numbers $\alpha_j(0)$ calculated by the formulas (16).

Remark. If the right part $h(t)$ of system (1) is such that the following equations

$$D^\nu(h)(t_j) = 0 \Leftrightarrow D^\nu(h_1)(t_j) = 0, \quad j = \overline{1, r}, \quad \nu = \overline{0, s_j - 1} \tag{$*$}$$

are satisfied, then, as can be seen from formulas (13) and (14), the function $g_0(t) \equiv 0$, and the function $z_0^{(0)}(t)$ will have the form

$$z_0^{(0)}(t) = -\frac{(h(t), d_1(t))}{\lambda_1(t)} c_1(t) - \sum_{j=2}^{n+1} \frac{(h(t), d_j(t))}{\lambda_j(t)} c_j(t). \tag{$**$}$$

3 Algorithm for constructing solutions to iterative problems (4_k) at $k \geq 1$

Carrying out calculations similar to those used in constructing the solution of the first iterative problem (4_0), we obtain the following algorithm for the sequential solution of the problems (4_k), $k \geq 1$.

1) Each of the iterative systems (4_k), $k \geq 1$, is represented as

$$L\hat{z}_k \equiv \frac{\partial \hat{z}_k}{\partial u} \Lambda(t) u - A(t) \hat{z}_k = -\frac{\partial \hat{z}_{k-1}}{\partial t}, \tag{17}$$

$$-A(t) z_k^{(0)}(t) = -\frac{\partial z_{k-1}^{(0)}}{\partial t} - \frac{\partial \hat{z}_0}{\partial u} g_k(t) e_1 - \frac{\partial \hat{z}_k}{\partial u} g_0(t) e_1 - \sum_{j=1}^{k-1} \frac{\partial \hat{z}_k}{\partial u} g_{k-j}(t) e_1 \tag{18}$$

according to the representation of the solution $z(t,u) \in U$ as $z_k = \hat{z}_k(t,u) + z_k^{(0)}(t)$, where

$$\hat{z}_k(t,u) = \sum_{j=1}^{n+1} \hat{z}_j^{(k)}(t) u_j \in \hat{U}, \quad \hat{z}_k^{(0)}(t) \in U^{(0)} = C^\infty\left([0,T], \mathbb{C}^{n+1}\right).$$

2) We solve the system (17) in the space $\hat{U}$. For its solvability in this space it is necessary and sufficient that the identities $\left\langle -\frac{\partial \hat{z}_{k-1}}{\partial t}, d_j(t) u_j \right\rangle \equiv 0, \quad j = \overline{1, n+1}$ hold [10].

3) Writing the solution of the system (17) in the form $\hat{z}_k(t,u) = \sum_{j=1}^{n+1} \hat{z}_j^{(k)}(t)\,u_j$, substitute it into system (18) and find uniquely the function $g_k(t)$ (using Lagrange-Sylvester's polynomials) and the solution $\hat{z}_k^{(0)}(t) \in U^{(0)}$ of system (18) in the space $U^{(0)}$.

4) Let's compose the function $z_k(t,u) = \hat{z}_k(t,u) + z_k^{(0)}(t)$; it is a solution of the system (4_k), but is found ambiguously so far. To finally compute this function, we proceed to the following iterative problem (4_{k+1}).

The corresponding system $L\hat{z}_{k+1} = -\frac{\partial \hat{z}_k}{\partial t}$ will have a solution in $\hat{U}$ if and only if the following conditions hold

$$< -\frac{\partial \hat{z}_k}{\partial t}, d_j(t)\,u_j > \equiv 0, \quad j = \overline{1, n+1}.$$

These conditions and the initial condition $z_k(0,\bar{1}) = 0$ for the problem (4_k), $k \geq 1$ allow us to find the solution of $z_k(t,u) \in U$ in an unambiguous way.

4 Construction of the asymptotic solution of problem (1)

Let us proceed to the computation of the asymptotic solution of problem (1). Let the solutions $z_0(t,u), ...z_N(t,u) \in U$ of the problems $(4_0), ..., (4_N)$ respectively be constructed by the above algorithm. The functions $g_0(t), ..., g_N(t)$, participating in the formation of the normal form (2) (of order $m = N$) will be uniquely found. This form has the following solution:

$$\begin{aligned} &u_j(t,\varepsilon) = e^{\varepsilon^{-1}\int_0^t \lambda_j(s)ds}, \quad j = \overline{2, n+1}, \\ &u_1(t,\varepsilon) = e^{\varepsilon^{-1}\int_0^t \lambda_1(s)ds}\left[1 + \frac{1}{\varepsilon}\int_0^t e^{\varepsilon^{-1}\int_0^x \lambda_1(s)ds} g_0(x)dx\right] + \sum_{k=0}^{N-1} \varepsilon^k \int_0^t e^{\varepsilon^{-1}\int_0^x \lambda_1(s)ds} g_{k+1}(x)dx. \end{aligned} \tag{19}$$

Let us make a partial sum $S_N(t,u,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j z_j(t,u)$ of the series (4) and form a contraction of this sum on the solution (19) of the normal form (2). We denote the obtained function by $z_{\varepsilon N}(t)$. The following statement holds (which is proved in the same way as the analogous statement in [10; Chap. 3]).

Lemma 1. Let $\alpha_1(0)$, defined by formula (15), is not zero, and conditions 1), 2a) – 2c) are satisfied. Then the function $z_{\varepsilon N}(t)$ satisfies the problem

$$\varepsilon \frac{dz_{\varepsilon N}(t)}{dt} - A(t)z_{\varepsilon N}(t) = h(t) + \varepsilon^N R_N(t,\varepsilon), \quad z_{\varepsilon N}(0) = z^0,$$

where $\|R_N(t,\varepsilon)\|_{C[0,T]} \leq \bar{R}$, $\bar{R} > 0$ is a constant independent of $(t,\varepsilon) \in [0,T] \times (0,\varepsilon_0]$ ($\varepsilon_0 > 0$ is small enough).

Using this lemma, we prove the following result as in [10; Chap. 3, §3.5].

Theorem 2. Let all conditions of the lemma be satisfied. Then the following statements are true:

1. If the right-hand side $h(t)$ of problem (1) does not satisfy the requirement $(*)$, then there is an estimate

$$\|z(t,\varepsilon) - z_{\varepsilon N}(t)\|_{C[0,T]} \leq C_N \varepsilon^N, \tag{20}$$

where $z(t,\varepsilon)$ is the exact solution of problem (1), and $z_{\varepsilon N}(t)$ is the above constructed constriction of the N-th partial sum of the series sum of series (4) on the solution $u = u(t,\varepsilon)$ of the normal form (2) of order $m = N+1$, $C_N > 0$ is a constant independent of $(t,\varepsilon) \in [0,T] \times (0,\varepsilon_0]$, $\varepsilon_0 > 0$ is small enough.

2. If the right-hand side $h(t)$ of problem (1) satisfies the requirement $(**)$, then the estimate

$$\|z(t,\varepsilon)-z_{\varepsilon N}(t)\|_{C[0,T]}\leq C_{N+1}\varepsilon^{N+1},$$

where $z(t,\varepsilon)$ and $z_{\varepsilon N}(t)$ are the same functions as in (20) $C_{N+1}>0$ is a constant independent of $(t,\varepsilon)\in[0,T]\times(0,\varepsilon_0]$, $\varepsilon_0>0$ is quite small.

5 Example

Consider the differential equation

$$\varepsilon\dot{y}=-t^2 l_0(t)\,y+h_0(t)+h_1(t)\,e^{\frac{i}{\varepsilon}\beta(t)},\quad y(0,\varepsilon)=y^0,\quad t\in[0,T], \tag{21}$$

where $y=y(t,\varepsilon)$ is a scalar function, the coefficient $a(t)=-t^2l_0(t)$ goes to zero only at the point $t=0$ and $l_0(t)<0$, $\forall t\in[0,T]$, $l_0(t)$, $h_0(t)$, $h_1(t)\in C^\infty([0,T],\mathbb{R})$. For this equation we can write out the exact solution, but it will be very difficult to obtain the asymptotics at $\varepsilon\to+0$. Let's attempt to apply the algorithm developed above to extract the leading asymptotic term in this problem's solution. Denoting, as before,

$$z=\{y,y_2\},\quad z^0=\left\{y^0,e^{\frac{i}{\varepsilon}\beta(0)}\right\},\quad A(t)=\begin{pmatrix}a(t) & h_1(t)\\ 0 & i\beta'(t)\end{pmatrix},$$

$$h(t)=\begin{pmatrix}h_0(t)\\ 0\end{pmatrix},\quad \lambda_1(t)=a(t)=-t^2l_0(t),\quad \lambda_2(t)=i\beta'(t),$$

we obtain the system

$$\varepsilon\frac{dz}{dt}=A(t)\,z+h(t),\quad z(0,\varepsilon)=z^0,\quad t\in[0,T]. \tag{22}$$

Calculating the eigenvalues and eigenvectors of the matrices $A(t)$ and $A^*(t)$, we'll have:

$$c_1(t)=\begin{pmatrix}1\\0\end{pmatrix},\quad c_2(t)=\begin{pmatrix}\frac{h_0(t)}{-a(t)+\lambda_2(t)}\\ 1\end{pmatrix},$$

$$d_1(t)=\begin{pmatrix}1\\ \frac{h_0(t)}{a(t)-\bar{\lambda}_2(t)}\end{pmatrix},\quad d_2(t)=\begin{pmatrix}0\\1\end{pmatrix}.$$

By Theorem 2, in the case of $(h(t),d_1(t))\not\equiv 0\ \Leftrightarrow h_1(t)\not\equiv 0$ solution of the first-order normal form (2) $(m=1)$:

$$\begin{aligned}&\varepsilon\dot{u}_1=\lambda_1(t)u_1+(g_0(t)+\varepsilon g_1(t)),\quad u_1(0,1)=1,\\ &\varepsilon\dot{u}_2=\lambda_2(t)u_2,\quad u_2(0,1)=1\end{aligned} \tag{23}$$

contains a negative degree ε^{-1} since $g_0(t)=\alpha_1^{-1}(t)\,(h(t),d_1(t))\neq 0$. Thus the solution of problem (23) tends to infinity at $\varepsilon\to+0$. The physical content of the problem corresponds to bounded solutions, so we will consider problem (22) under the condition $(h(t),d_1(t))=h_1(t)\equiv 0$, $\forall t\in[0,T]$. Then $g_0(t)\equiv 0$, and the leading asymptotic term in the solution to problem (22) is given by (7)

$$z_{\varepsilon 0}(t)=\alpha_1(t)\,c_1(t)\,e^{\frac{1}{\varepsilon}\int_0^t\lambda_1(\theta)d\theta}\left[1+\int_0^t e^{-\frac{1}{\varepsilon}\int_0^s\lambda_1(\theta)d\theta}g_1(s)\,ds\right]+$$

$$+\alpha_2(t)\,c_2(t)\,e^{\frac{1}{\varepsilon}\int_0^t\lambda_2(\theta)d\theta}-\frac{(h(t),d_2(t))}{\lambda_2(t)}c_2(t),$$

where the functions $\alpha_1(t)$ and $\alpha_2(t)$ are calculated from the solvability condition of the problem (4_1) in the space U. Given our notations, we write the main term of the asymptotics of the solution of problem (21) in the following form

$$y_{\varepsilon 0}(t)=\left(y^0+\frac{h_0(0)\,e^{\frac{i}{\varepsilon}\beta(0)}}{a(0)-i\beta'(0)}\right)e^{\frac{1}{\varepsilon}\int\limits_0^t a(\theta)d\theta}\left[1+\int\limits_0^t e^{-\frac{1}{\varepsilon}\int\limits_0^s a(\theta)d\theta}g_1(s)\,ds\right]+\frac{h_0(t)\,e^{\frac{i}{\varepsilon}\beta(t)}}{-a(t)+i\beta'(t)}. \qquad (24)$$

Conclusion

From (24) we see that if $h_0(t)\neq 0$ on the segment $[0,T]$, the exact solution $y(t,\varepsilon)$ of problem (24) has no limit at $\varepsilon\to+0$ due to the oscillatory inhomogeneity $e^{\frac{i}{\varepsilon}\beta(t)}$ included in (24). If $h_0(t)=0$, $\forall t\in\in[0,T]$, then the main term of the asymptotics (24) takes the form of

$$y_{\varepsilon 0}(t)=y^0 e^{-\frac{1}{\varepsilon}\int\limits_0^t \theta^2 l_0(\theta)d\theta}\left[1+\int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_0^s \theta^2 l_0(\theta)d\theta}g_1(s)\,ds\right].$$

The zero of $t=0$ of the function $a(t)=-t^2 l_0(t)$ affects that the summand

$$y^0 e^{-\frac{1}{\varepsilon}\int\limits_0^t \theta^2 l_0(\theta)d\theta}\int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_0^s \theta^2 l_0(\theta)d\theta}g_1(s)\,ds$$

outside the boundary zone $[0,\delta(\varepsilon)]$ of length of order $\sqrt[3]{\varepsilon}$ "slows down" the tendency of the exact solution $y(t,\varepsilon)$ of problem (21) to the limit $\bar{\bar{y}}(t)\equiv 0$.

In the case of an exponential boundary layer occurring at $a(t)<0$, $\forall t\in[0,T]$, the exact solution $y(t,\varepsilon)$ differs from the limit outside the boundary layer by an order of magnitude of ε [11]. Thus, the effect of the slowed limit transition (as the small parameter approaches zero) in a singularly perturbed problem is associated with the point wise features of its spectrum.

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Conflict of Interest

The authors declare no conflict of interest.

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Numerical solution of singularly perturbed parabolic differential difference equations

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This study presents a computational method for the singularly perturbed parabolic differential difference equations with small negative shifts in convection and reaction terms. To handle the small negative shifts, the Taylor series expansion is applied. Then, the resulting asymptotically equivalent singularly perturbed parabolic convection-diffusion-reaction problem is discretized in the time variable using the implicit Euler technique on a uniform mesh, while the upwind method on a Shishkin mesh is used to discretize the space variable. Almost first-order convergence was achieved by establishing the stability and parameter-uniform convergence of the method. The Richardson extrapolation approach improved the rate of convergence to nearly second-order. Numerical experiments have been carried out in order to support the findings from the theory. The numerical results are presented in tables in terms of maximum absolute errors and graphs. The present results improve the existing methods in the literature. This finding highlights the efficiency of the method, paving the way for its application in other types of singularly perturbed parabolic problems. This method is capable of greatly improving computing performance in a variety of scenarios, which researchers can further explore.

Keywords: singular perturbation problem, differential difference equation, implicit Euler technique, upwind method, Shishkin mesh, parameter-uniform convergence, Richardson extrapolation.

2020 Mathematics Subject Classification: 65M06, 65M12, 65M15, 65M50.

Introduction

In singularly perturbed differential equations, the highest-order derivative term in the differential equation is multiplied by a small perturbation parameter ε $(0 < \varepsilon \ll 1)$. Various numerical solutions have been developed in the literature for a singularly perturbed parabolic problem with general shift arguments in the space variable in [1], retarded terms in [2–4], delay and advances in both reaction terms, differential-difference equations [5–7], functional-differential equations in [8–10]. Some numerical techniques have been devised in [11,12] to solve a singularly perturbed parabolic problem with delay in the reaction terms. Authors in [13–16] developed the numerical solutions for singularly perturbed parabolic differential equation with negative shifts in convection and reaction terms. Recently, authors in [17] considered and solved singularly perturbed partial functional-differential equation. Some numerical methods are devised in [18–20] to solve different types of singularly perturbed parabolic problems.

Therefore, the main purpose of this study is to construct an improved numerical method using implicit Euler method for time direction and upwind method on a Shishkin mesh for space direction together with Richardson extrapolation technique to solve the following singularly perturbed parabolic differential equation with negative shifts in convection and reaction terms

$$\mathcal{L}_{\varepsilon,\mu}u \equiv u_t - \varepsilon u_{xx} + r(x)u_x(x-\mu,t) + s(x)u(x-\mu,t) = f(x,t), \tag{1}$$

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with the initial data

$$u(x,0) = \phi_b(x), \qquad x \in \bar{\Omega}, \tag{2}$$

and the interval-boundary data

$$\begin{cases} u(x,t) = \phi_l(x,t), & (x,t) \in [-\mu, 0] \times \Omega_t, \\ u(1,t) = \phi_r(1,t), & t \in \Omega_t, \end{cases} \tag{3}$$

where $(x,t) \in D = \Omega_x \times \Omega_t = (0,1) \times (0,T]$ and $\bar{D} = \bar{\Omega} \times \bar{\Omega}_t = [0,1] \times [0,T]$ for some positive number $T > 0$. The parameter ε is a perturbation parameter such that $0 < \varepsilon \ll 1$, while the positive parameter μ is a small delay parameter (or negative shift) fulfilling $\mu < \varepsilon$. It is assumed that $r(x)$, $s(x)$, $\phi_b(x)$, $\phi_l(x,t)$, $\phi_r(1,t)$ and $f(x,t)$ are sufficiently smooth and bounded to ensure the possibility of a particular solution, and that $s(x)$ satisfies

$$s(x) \geq \beta > 0, \quad x \in \bar{\Omega},$$

for some constant β. Note that (1) contains negative shifts in the convection and reaction terms. When $\mu = 0$, (1) would reduce to the singularly perturbed parabolic differential equation. With a small ε, we observe layers that rely on the value of $r(x)$. We are interested in a related class of problems where both the convection and reaction terms have negative shifts, making it a two-parameter problem. A regular boundary layer appears in the region of the left boundary when $r(x) < 0$, and a boundary layer is located close to the right when $r(x) > 0$.

1 *The continuous problem*

It is reasonable to use the Taylor series approximation for terms involving delay in the case $\mu < \varepsilon$ [21]. Now, approximating $u(x-\mu,t)$ and $u_x(x-\mu,t)$ yields the following

$$\begin{aligned} u(x-\mu,t) &\approx u(x,t) - \mu u_x(x,t) + \frac{\mu^2}{2} u_{xx}(x,t) + O(\mu^3), \\ u_x(x-\mu,t) &\approx u_x(x,t) - \mu u_{xx}(x,t) + O(\mu^2). \end{aligned} \tag{4}$$

Plugging (4) into (1)–(3), we obtain an asymptotically equivalent time-dependent singularly perturbed convection-diffusion-reaction continuous problem of the following form

$$\mathcal{L}_{c_\varepsilon} u \equiv u_t - c_\varepsilon(x) u_{xx} + q(x) u_x + s(x) u(x,t) = f(x,t), \qquad (x,t) \in D, \tag{5}$$

with the initial condition

$$u(x,0) = \phi_b(x) \geq 0, \;\; x \in \bar{\Omega}_x, \tag{6}$$

and the boundary conditions

$$u(0,t) = \phi_l(t), \qquad u(1,t) = \phi_r(t) \geq 0, \qquad t \in \bar{\Omega}_t, \tag{7}$$

where $c_\varepsilon(x) = \varepsilon - \frac{\mu^2}{2} s(x) + \mu r(x)$ and $q(x) = r(x) - \mu s(x)$. With α and β being the lower limits for $r(x)$ and $s(x)$, respectively, we assume that $0 < c_\varepsilon(x) \leq \varepsilon - \frac{\mu^2}{2}\beta + \mu\alpha = c_\varepsilon$. We make the supposition that $q(x) = r(x) - \mu s(x) \geq \gamma > 0$, which suggests the presence of a boundary layer close to $x = 1$ with width $O(\varepsilon)$. The compatibility condition at the corner points, along with the smoothness of $\phi_l(t)$, $\phi_b(x)$, $\phi_r(t)$, can guarantee the existence and uniqueness of the solution for (1)–(3). We now offer bounds on the derivatives of the solution of (1)–(3). To get the bounds, one needs certain information about the solution.

Lemma 1. The solution $u(x,t)$ of (5)–(7) satisfies

$$|u(x,t)-\phi_b(x)| \le Ct,$$
$$|u(x,t)-\phi_l(t)| \le C(1-x), \qquad (x,t)\in \bar{D},$$

where C is a constant independent of c_ε.

Setting $c_\varepsilon = 0$ in (5)–(7) gives the reduced problem as

$$\begin{cases} \frac{\partial u^0}{\partial t} + q(x)\frac{\partial u^0}{\partial x} + s(x)u^0(x,t) = f(x,t), & (x,t)\in D,\\ u^0(0,t) = \phi_b(x), & x\in \bar{\Omega}_x,\\ u^0(0,t) = \phi_l(t), \qquad u^0(1,t) \ne \phi_r(t), & t\in \bar{\Omega}_t. \end{cases} \tag{8}$$

The solutions $u(x,t)$ of (5)–(7) and $u^0(x,t)$ of (8) are extremely similar for small values of c_ε. In order to show the bounds of the solution $u(x,t)$ of (5)–(7), we assume $\phi_b(x) = 0$ without compromising generality. Since $\phi_b(x)$ is sufficiently smooth, using the property of norm, we prove the following lemma:

Lemma 2. The bound of the solution $u(x,t)$ to (5)–(7) is given by

$$|u(x,t)| \le C, \quad (x,t)\in \bar{D}.$$

Proof. From Lemma 1, we have

$$|u(x,t)-\phi_b(x)| \le Ct.$$

From triangular inequality, we have

$$|u(x,t)| - |\phi_b(x)| \le |u(x,t)-\phi_b(x)| \le Ct.$$

This implies that

$$|u(x,t)| \le Ct + |\phi_b(x)|, \quad (x,t)\in \bar{D}.$$

Since $t\in[0,T]$ and $\phi_b(x)$ is bounded, we have

$$|u(x,t)| \le C,$$

which is the required result.

The problem (5)–(7) satisfies the following maximum principle.

Lemma 3. Let Θ be a sufficiently smooth function defined on D which satisfies $\Theta(x,t) \ge 0$, $\forall (x,t)\in \partial D$. Then, $\mathcal{L}_{c_\varepsilon}\Theta(x,t) \ge 0$, $(x,t)\in D$ implies that $\Theta(x,t)\ge 0$, $\forall (x,t)\in \bar{D}$.

Proof. See [16].

For the solution of (1), the above maximum principle immediately leads to the stability bound.

Lemma 4. The solution $u(x,t)$ of the continuous (5)–(7) is bounded as

$$|u(x,t)| \le \max\left\{|\phi_b(x)|, |\phi_l(t)|, |\phi_r(t)|\right\} + \frac{\|f\|}{\beta}.$$

Proof. We define two barrier-functions $\varpi^{\pm}$ as

$$\varpi^{\pm}(x,t) = \max\left\{|\phi_b(x)|, |\phi_l(t)|, |\phi_r(1,t)|\right\} + \frac{\|f\|}{\beta} \pm u(x,t).$$

Evaluating the barrier functions at the initial and boundary conditions, the required bound follows.

Theorem 1. [22] For $0 \leq l \leq 2,\ 0 \leq k \leq 3,\ 0 \leq l+k \leq 3$, the solution $u(x,t)$ of (5)–(7) is bounded by

$$\left|\frac{\partial^{l+k} u(x,t)}{\partial x^l \partial t^k}\right| \leq C\left(1 + c_\varepsilon^{-l} e^{-\gamma(1-x)/c_\varepsilon}\right).$$

Stronger bounds should be derived using Shishkin-type decomposition because the bounds on the solution's derivatives are not sufficiently sharp for the proof of uniform convergence. This can be achieved by decomposing the solution u as

$$u = v + w,$$

v is a regular component and w is a singular component. The solution of the non-homogeneous equation is the regular component v

$$\begin{cases} \mathcal{L}_{c_\varepsilon} v(x,t) = f(x,t), & x \in D, \\ v(0,t) = 0, & t \in \Omega_t, \quad v(x,0) = \phi_b(x), \quad x \in \bar{\Omega}_x, \end{cases}$$

and the singular component w represents the homogeneous equation's solution

$$\begin{cases} \mathcal{L}_{c_\varepsilon} w(x,t) = 0, & x \in D, \\ w(0,t) = 0, \quad w(1,t) = u(1,t) - v(1,t), & t \in \Omega_t, \\ w(x,0) = 0, & x \in \bar{\Omega}_x. \end{cases}$$

We can further decompose the regular component v as

$$v = v_0 + c_\varepsilon v_1 + c_\varepsilon^2 v_2,$$

where v_0 is the solution of the reduced problem and v_1 and v_2 are the solution of

$$\begin{cases} (v_1)_t + r(x)(v_1)_x + s(x)v_1 = (v_0)_{xx}, & (x,t) \in D, \\ v_1(x,0) = 0, \quad x \in \bar{\Omega}_x, \quad v_1(0,t) = 0, & t \in \bar{\Omega}_t, \end{cases}$$

and

$$\begin{cases} \mathcal{L}_{c_\varepsilon}(v_2)(x,t) = (v_1)_{xx}, & (x,t) \in D, \\ v_2(x,t) = 0, & (x,t) \in \partial D. \end{cases}$$

Now, we state the bounds for regular and singular components.

Theorem 2. Let v be a regular solution. Then v and its derivative satisfy the bound

$$\left|\frac{\partial^{i+j} v(x,t)}{\partial x^i \partial t^j}\right| \leq C(1 + c_\varepsilon^{2-k}), \qquad k = 0,1,2.$$

The derivative of regular solution generally bounded as

$$\left|\frac{\partial^{i+j} v(x,t)}{\partial x^i \partial t^j}\right| \leq C, \qquad k = 0,1,2,3.$$

Proof. See the proof in [22].

Theorem 3. Let w be the solution of (5)–(7). The bound of w is given by

$$|w(x,t)| \leq Ce^{-\gamma(1-x)/c_\varepsilon}, \qquad (x,t) \in D.$$

Proof. Considering the barrier functions $\Psi^{\pm}(x,t) = C\left(e^{-\gamma(1-x)/c_\varepsilon}\right)e^t \pm w(x,t),\ (x,t) \in \bar{D}$ and evaluating at the boundaries yields the required result.

Theorem 4. Solution of the singular component w and it derivatives satisfies the bound

$$\left|\frac{\partial^{i+j} w(x,t)}{\partial x^i \partial t^j}\right| \leq C c_\varepsilon^{-i} e^{-\gamma(1-x)/c_\varepsilon}, \qquad k = 0,1,2,3.$$

Proof. The proof follows from Theorem 3 and [22].

2 The discrete problem

We use a Shishkin mesh for the space direction and a uniform mesh for the time direction to discretize the problem. The space domain $[0,1]$ is divided into two sub-domains $[0,1-\sigma]$ and $(1-\sigma,1]$, to construct the Shishkin mesh. The transition parameter $1-\sigma$, which divides the coarse and fine regions of the mesh, is determined by taking

$$\sigma = \min\left\{\frac{1}{2}, \frac{\sigma_0 c_\varepsilon}{\gamma}\ln N\right\},$$

where σ_0 denotes a constant that represents the order of the method. We denote the space mesh points by

$$\Omega_x^N = \left\{0 = x_0, x_1, ...x_{N/2} = 1-\sigma, ..., x_N = 1\right\},$$

where

$$x_i = \begin{cases} iH, & i = 0, ...\frac{N}{2}, \\ 1-\sigma + \left(i - \frac{N}{2}\right)h, & i = \frac{N}{2}+1, ...N, \end{cases}$$

and let $N \geq 4$ be a positive even integer. Furthermore, we denote the space mesh size h_i as follows

$$h_i = \begin{cases} H = \frac{2(1-\sigma)}{N}, & i = 1, ..., \frac{N}{2}, \\ h = \frac{2\sigma}{N}, & i = \frac{N}{2}+1, ..., N. \end{cases}$$

To do the analysis, it was assumed that $\sigma = \frac{\sigma_0 c_\varepsilon}{\gamma}\ln N$; if not, N is exponentially larger than ε. It is clear from the above equation that $N^{-1} \leq H \leq 2N^{-1},\ \ h = \frac{2\sigma_0 c_\varepsilon}{\gamma}N^{-1}\ln N$, and the uniform mesh can be obtained by choosing $\sigma = 1/2$. A uniform mesh with a time step of Δt will be used for the time domain $[0,T]$ so that

$$\Omega_t^M = \left\{t_n = n\Delta t, \ \ n = 0, ...M, \ \ \Delta t = \frac{T}{M}\right\},$$

where M is the number of mesh intervals in the time variable over the interval $[0,T]$. We define the discretized domain $D^{N,\Delta t} = \Omega_x^N \times \Omega_t^M$. Before formulating the numerical method, we introduce the difference operators for a given mesh function $v(x_i,t_n) = v_i^n$ as follows

$$\delta_x^+ v_i^n = \frac{v_{i+1}^n - v_i^n}{h_{i+1}}, \qquad \delta_x^- v_i^n = \frac{v_i^n - v_{i-1}^n}{h_i},$$
$$\delta_x^2 v_i^n = \frac{2(\delta_x^+ v_i^n - \delta_x^- v_i^n)}{\tilde{h}_i} \ \text{ and } \ \delta_t^- v_i^n = \frac{v_i^n - v_i^{n-1}}{\Delta t},$$

where $\tilde{h}_i$ is defined by $\tilde{h}_i = h_i + h_{i+1},\ \ i = 1, ...N-1$. We now use the upwind method for the space derivative and the implicit Euler method for the time derivative to approximate (5)–(7). The discretisation of (5)–(7) thus assumes the following form:

$$\begin{cases} (\delta_t^- + \mathcal{L}_{c_\varepsilon}^{N,\Delta t})U_i^{n+1} = f_i^{n+1}, & i = 1, ...N-1, \quad n = 0, ..., M-1, \\ U_0^{n+1} = \phi_l(t_{n+1}), & U_N^{n+1} = \phi_r(t_{n+1}), \quad n = 0, ..., M-1, \\ U_i^0 = \phi_b(x_i), & i = 1, ...N-1, \end{cases} \tag{9}$$

where

$$\mathcal{L}_{c_\varepsilon}^{N,\Delta t} U_i^{n+1} = -c_\varepsilon \delta_x^2 U_i^{n+1} + r_i \delta_x^- U_i^{n+1} + s_i U_i^{n+1}.$$

The system of equations that follows is obtained by rearranging the terms in (9)

$$\begin{cases} r_i^- U_{i-1}^{n+1} + r_i^0 U_i^{n+1} + r_i^+ U_{i+1}^{n+1} = g_i^n, & i = 1, ...N-1, \quad n = 0, ...M-1, \\ U_0^{n+1} = \phi_l(t_{n+1}), & U_N^{n+1} = \phi_r(t_{n+1}), \\ U_i^0 = \phi_b(x_i), & i = 1, ...N-1, \end{cases}$$

where the coefficients are given by

$$\begin{cases} r_i^- = \Delta t \left(-\dfrac{2c_\varepsilon}{\tilde{h}_i h_i} - \dfrac{r_i}{h_i} \right), \quad r_i^+ = \Delta t \left(-\dfrac{2c_\varepsilon}{\tilde{h}_i h_{i+1}} \right), \quad r_i^0 = 1 + \Delta t s_i - r_i^- - r_i^+, \\ r_i = r(x_i), \quad s_i = s(x_i), \quad g_i^n = U_i^n + \Delta t f_i^{n+1}. \end{cases}$$

The coefficient matrix of the discrete scheme in (9) gives an $(N-1) \times (N-1)$ linear equation that can be solved uniquely using the Thomas algorithm for the unknowns $U_1, \cdots, U_{N-1}$.

3 Convergence analysis

It can be shown that the discrete maximum principle, which gives the difference operator $(\delta_t^- + \mathcal{L}_{c_\varepsilon}^{N,\Delta t})$ $\varepsilon-$uniform stability, is satisfied by the finite difference operator $(\delta_t^- + \mathcal{L}_{c_\varepsilon}^{N,\Delta t})$ defined in (9).

Lemma 5. Assume that the mesh function $\Psi(x_i, t_n)$ satisfies $\Psi(x_i, t_n) \geq 0$ on $(x_i, t_n) \in D^{N,\Delta t}$. Then, $(\delta_t^- + \mathcal{L}_{c_\varepsilon}^{N,\Delta t})\Psi(x_i, t_n) \geq 0$, $(x_i, t_n) \in D^{N,\Delta t}$ implies that $\Psi(x_i, t_n) \geq 0$ at each point of $(x_i, t_n) \in \bar{D}^{N,\Delta t}$.

The proposed method described in (9) converges ε-uniformly with first-order accuracy in both space and time variables as stated in the following theorem.

Theorem 5. Let U be the numerical solution in (9) and u be the continuous solution in (5)–(7). Therefore, the discrete solution's error $U^{N,\Delta t}$ fulfills the bound

$$|u(x_i, t_n) - U_i^n| \leq C(N^{-1} \ln N + \Delta t), \qquad 1 \leq i \leq N-1.$$

Proof. Readers who are interested may read the proof's details in [23].

The objective of this study was to obtain second-order ε-uniform convergence with respect to both the space and time directions by using the Richardson extrapolation technique to the discrete solution U_i^n of (9). Before introducing this technique, some lemmas are presented as follows:

Lemma 6. On $\bar{D}_x^N = \{x_i\}_0^N$, define the following mesh functions

$$S_i = \prod_{k=1}^{i} \left(1 + \frac{\alpha h_k}{c_\varepsilon}\right)^{-1}, \quad 1 \leq i \leq N,$$

with the usual convention that $S_0 = 1$ for $i = 0$. Then, there exists a positive constant C_1 such that for $i = 1, \cdots, N-1$, we have

$$(\delta_t^- + \mathcal{L}_{c_\varepsilon}^{N,M})S_i \geq \frac{C_1}{c_\varepsilon + \alpha h_i} S_i. \tag{10}$$

Moreover, for $N/2+1 \le i \le N-1$ and constant C_2, we have

$$(\delta_t^- + \mathcal{L}_{c_\varepsilon}^{N,M})S_i \ge C_2 c_\varepsilon^{-1} S_i. \tag{11}$$

Proof. Now, $S_i - S_{i-1} = \frac{\alpha h_i}{c_\varepsilon} S_{i-1}$. So, we have

$$\begin{aligned}(\delta_t^- + \mathcal{L}_{c_\varepsilon}^{N,M})S_i &= -\frac{2\alpha}{(h_i+h_{i+1})}(S_i - S_{i-1}) + \alpha\frac{\alpha}{c_\varepsilon}S_{i-1} + \beta S_i\\ &\ge \frac{\alpha}{c_\varepsilon}S_{i-1}\left[r_i - \frac{2\alpha h_i}{(h_i+h_{i+1})}\right]\\ &\ge \frac{C\alpha}{c_\varepsilon + \alpha h_i}S_i, \qquad 1 \le i \le N-1.\end{aligned}$$

As a result, since $h/c_\varepsilon < 4/\gamma$, (10) is proven, and (11) is a straightforward consequence of it.

Lemma 7. The following inequality is satisfied by the mesh function S_i

$$e^{-\gamma(1-x_i)/c_\varepsilon} \le \prod_{k=i+1}^{N}\left(1+\frac{\alpha h_k}{c_\varepsilon}\right)^{-1} = S_i, \qquad 0 \le i \le N, \tag{12}$$

and on Shishkin mesh, the mesh function $S_{c_\varepsilon,i}$ also satisfies the following inequality

$$\prod_{k=i+1}^{N}\left(1+\frac{\alpha h_k}{c_\varepsilon}\right)^{-1} \le CN^{-4(1-i/N)}, \qquad N/2 \le i \le N-1. \tag{13}$$

We solve the discrete scheme in (9) on the fine mesh $D^{2N,\Delta t/2} = \bar{\Omega}_x^{2N} \times \bar{\Omega}_t^{\Delta t/2}$ with $2N$ mesh intervals in the space direction and $2M$ mesh intervals in the time direction, where $\bar{\Omega}_x^{2N}$ is a piecewise uniform Shishkin mesh with the same transition point $1-\sigma$ as Ω_x^N. This improves the accuracy of the numerical solution $U^{N,\Delta t}$ using the Richardson extrapolation technique. Actually, by dividing each mesh interval of Ω_x^N in half, the discrete domain $\bar{\Omega}_x^{2N}$ may be produced. It is evident from this construction that $D^{N,\Delta t} = (x_i,t_n) \subset D^{2N,\Delta t/2} = \{(\tilde{x}_i,\tilde{t}_n)\}$. Thus, the suitable mesh widths in $D^{2N,\Delta t/2}$ may be obtained using

$$\tilde{x}_i - \tilde{x}_{i-1} = \begin{cases} H/2, & \text{for } \tilde{x}_i \in \bar{\Omega}_x^{2N} \cap [0,1-\sigma],\\ h/2, & \text{for } \tilde{x}_i \in \bar{\Omega}_x^{2N} \cap [1-\sigma,1],\end{cases}$$

and $\tilde{t}_n - \tilde{t}_{n-1} = \Delta t/2$, $\tilde{t}_n \in \bar{\Omega}_t^{\Delta t/2}$. On the mesh $D^{N,\Delta t}$, let $U(x_i,t_n)$ represent the numerical solution of the discrete scheme in (9). Thus, using Theorem 5, one may write on $(x_i,t_n) \in D^{N,\Delta t}$

$$\begin{aligned}U^{N,\Delta t}(x_i,t_n) - u(x_i,t_n) &= C(N^{-1}\ln N + \Delta t) + R_{N,\Delta t}(x_i,t_n)\\ &= C(N^{-1}(\gamma\sigma/\sigma_0 c_\varepsilon) + \Delta t) + R_{N,\Delta t}(x_i,t_n),\end{aligned} \tag{14}$$

where C is fixed constant and the remainder term $R_{N,\Delta t}(x_i,t_n)$ is $o(N^{-1}\ln N + \Delta t)$. Similarly, if $\tilde{U}^{2N,\Delta t/2}$ is the solution of the discrete (14) for $(\tilde{x}_i,\tilde{t}_n) \in D^{2N,\Delta t/2}$, then

$$\tilde{U}^{2N,\Delta t/2}(\tilde{x}_i,\tilde{t}_n) - \tilde{u}(\tilde{x}_i,\tilde{t}_n) = C\Big((2N)^{-1}(\sigma\gamma/\sigma_0 c_\varepsilon) + \Delta t/2\Big) + R_{2N,\Delta t/2}(x_i,t_n), \tag{15}$$

by considering the fact that $\tilde{U}(\tilde{x}_i,\tilde{t}_n)$ is obtained using the same transition point $1-\sigma$ and the remainder term $R_{2N,\Delta t/2}(x_i,t_n)$ is $o(N^{-1}\ln N + \Delta t)$. Now, eliminating the terms $O(N^{-1})$ and $O(\Delta t)$ from (14) and (15) leads to the following approximation

$$\begin{aligned}u_i^n - \Big(2\tilde{U}^{2N,\Delta t/2}(x_i,t_n) - U^{N,\Delta t}(x_i,t_n)\Big) &= -2R_{2N,\Delta t/2}(x_i,t_n) + R_{N,\Delta t}(x_i,t_n)\\ &= o(N^{-1}\ln N + \Delta t), \quad (x_i,t_n) \in \bar{D}^{N,\Delta t}.\end{aligned}$$

Therefore, we will utilize the following extrapolation formula:

$$U_{extp}^{N,\Delta t}(x_i,t_n) = 2\tilde{U}^{2N,\Delta t/2}(x_i,t_n) - U^{N,\Delta t}(x_i,t_n), \quad (x_i,t_n)\in \bar{D}^{N,M}, \tag{16}$$

to get a more accurate predicted numerical solution for $u(x,t)$. After extrapolating $U^{N,\Delta t}$, we obtain the estimate of the nodal error $|u(x_i,t_n) - U_{textp}^{N,\Delta t}(x_i,t_n)|$ by splitting the solution $U^{N,\Delta t}$ on the mesh $\bar{D}_{\sigma}^{N,M}$ into the sum

$$U^{N,\Delta t} = V^{N,\Delta t} + W^{N,\Delta t},$$

where the following discrete problems are solved by the regular component $V^{N,\Delta t}$ and the singular component $W^{N,\Delta t}$, respectively

$$\begin{cases} \mathcal{L}_{c_\varepsilon}^{N,\Delta t} V^{N,\Delta t} = f, \quad D^{N,\Delta t}, \quad V^{N,\Delta t} = v, \quad \partial D^{N,\Delta t}, \\ \mathcal{L}_{c_\varepsilon}^{N,\Delta t} W^{N,\Delta t} = 0, \quad D^{N,\Delta t}, \quad W^{N,\Delta t} = w, \quad \partial D^{N,\Delta t}. \end{cases} \tag{17}$$

Likewise, on the fine mesh $\bar{D}^{2N,\Delta t/2}$, we decomposed the solution $\tilde{U}^{2N,\Delta t/2}$ into the regular component $\tilde{V}^{2N,\Delta t/2}$ and the singular component $\tilde{W}^{2N,\Delta t/2}$ given by

$$\tilde{U}^{2N,\Delta t/2} = \tilde{V}^{2N,\Delta t/2} + \tilde{W}^{2N,\Delta t/2}.$$

The error can then be expressed using the form given below.

$$\begin{aligned} U^{N,\Delta t} - u &= \left(V^{N,\Delta t} - v\right) + \left(W^{N,\Delta t} - w\right), \\ \tilde{U}^{2N,\Delta t/2} - u &= \left(\tilde{V}^{2N,\Delta t/2} - v\right) + \left(\tilde{W}^{2N,\Delta t/2} - w\right). \end{aligned}$$

Lemma 8. Let $c_\varepsilon \le N^{-1}$. Then, the error associated with the smooth component $V^{N,\Delta t}$ after extrapolation fulfills the bound

$$|v(x_i,t_n) - V_{extp}^{N,\Delta t}(x_i,t_n)| \le C\left(N^{-2} + \Delta t^2\right), \qquad 1 \le i \le N-1.$$

Proof. It may be deduced from the extrapolation formula (15), Lemma 7, and (17) that

$$\begin{aligned} v(x_i,t_n) - V_{extp}^{N,\Delta t}(x_i,t_n) &= v(x_i,t_n) - \left(2\tilde{V}^{2N,\Delta t/2}(x_i,t_n) - V^{N,\Delta t}(x_i,t_n)\right) \\ &= -2\left(\tilde{V}^{2N,\Delta t/2} - v\right)(x_i,t_n) + \left(V^{N,\Delta t} - v\right)(x_i,t_n) \\ &= O\left(N^{-2} + \Delta t^2\right), \end{aligned}$$

from which the expected result is obtained.

Lemma 9. The extrapolated error for the layer component $W^{N,\Delta t}$ satisfies

$$|w(x_i,t_n) - W_{extp}^{N,\Delta t}(x_i,t_n)| \le C(N^{-2} + \Delta t^2), \qquad 1 \le i \le N/2.$$

Proof. Assume $1 \le i \le N/2$. This allows us to demonstrate, using (13) and the argument provided in [24] over $\bar{D}^{N,M}$, that

$$|W^{N,\Delta t}(x_i,t_n)| \le C \prod_{j=i+1}^{N} \left(1 + \frac{\alpha h_j}{c_\varepsilon}\right)^{-1} \le CN^{-2}.$$

We then derive $|w(x_i,t_n)| \le CN^{-2}$ from (12) and Theorem 3. So, we have

$$|W^{N,\Delta t} - w(x_i,t_n)| \le C(N^{-2} + \Delta t^2).$$

In the same way, $|\tilde{W}^{N,\Delta t} - w(x_i,t_n)| \le C(N^{-2} + \Delta t^2)$. The extrapolation formula (16) is used to acquire a required extrapolated error bound.

Lemma 10. Once the layer component $W^{N,\Delta t}$ has been extrapolated, the error associated with it satisfy

$$\left| w(x_i,t_n) - W^{N,\Delta t}_{extp}(x_i,t_n) \right| \leq C \left(N^{-2}\ln^2 N + \Delta t^2 \right), \qquad N/2 < i < N.$$

Proof. See [23].

The following theorem is the main finding of this study.

Theorem 6. Let $c_\varepsilon \leq N^{-1}$. Suppose u be the continuous problem solution and $U^{N,\Delta t}_{textp}$ be the solution that was obtained by solving the discrete problem using the Richardson extrapolation strategy. Consequently, the error connected to the solution $U^{N,\Delta t}_{textp}$ meets

$$\left| u(x_i,t_n) - U^{N,\Delta t}_{extp}(x_i,t_n) \right| \leq C \left(N^{-2}\ln^2 N + \Delta t^2 \right), \quad 1 \leq i \leq N-1. \tag{18}$$

Proof. For each $(x_i,t_n) \in \bar{D}^{N,\Delta t}$, we have

$$u(x_i,t_n) - U^{N,\Delta t}_{extp} = \left(v(x_i,t_n) - V^{N,\Delta t}_{extp} \right) + \left(w(x_i,t_n) - W^{N,\Delta t}_{extp} \right).$$

Thus, when Lemma 8 for the regular component and Lemmas 9 and 10 for the singular component are combined, the result (18) is obtained immediately.

4 Numerical computations and discussions

In order to verify the performance of the present method with the theoretical findings discussed in the preceding parts, we do numerical calculations in this section.

Example 1. Consider a singularly perturbed parabolic problem [16]:

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (2-x^2)\frac{\partial u}{\partial x} + (x^2+1+\cos(\pi x))u = 10t^2x(1-x)e^{-t}, & (x,t) \in [0,1]\times[0,1], \\ \begin{cases} u(x,0) = 0, & 0 \leq x \leq 1, \\ u(0,t) = 0, \qquad u(1,t) = 0, & 0 \leq t \leq 1. \end{cases} \end{cases}$$

Example 2. Consider a singularly perturbed parabolic problem [16]:

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (2-x^2)\frac{\partial u(x-\mu,t)}{\partial x} + (x^2+1+\cos(\pi x))u(x-\mu,t) = 10t^2(1-x)e^{-t}, (x,t) \in [0,1]\times[0,1], \\ \begin{cases} u(x,0) = 0, & 0 \leq x \leq 1, \\ u(x,t) = 0, \ \ \mu \leq x \leq 0, \ 0 \leq t \leq 1, \quad u(1,t) = 0, & 0 \leq t \leq 1. \end{cases} \end{cases}$$

Example 3. Consider a singularly perturbed parabolic problem [22]:

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + (1+x^2)u = 50(x(1-x))^3, & \\ & (x,t) \in [0,1]\times[0,2], \\ \begin{cases} u(x,0) = 0, & 0 \leq x \leq 1, \\ u(0,t) = 0, \qquad u(1,t) = 0, & 0 \leq t \leq 2. \end{cases} \end{cases}$$

Since there are no exact solutions for the examples, we estimate the maximum absolute errors for each (ε,μ) using the double mesh principle via the following formula:

$$e^{N,\Delta t}_{\varepsilon,\mu} = \max_{0\leq i\leq N; 0\leq j\leq M} \left| U^{N,\Delta t}(x_i,t_n) - U^{2N,\Delta t/2}(x_i,t_n) \right|,$$

before extrapolation and after extrapolation, we use the formula

$$(e_{\varepsilon,\mu}^{N,\Delta t})^{extr} = \max_{0\le i\le N; 0\le j\le M} \left|(U^{N,\Delta t})^{extr}(x_i,t_n) - (U^{2N,\Delta t/2})^{extr}(x_i,t_n)\right|,$$

where $U^{N,\Delta t}(x_i,t_n)$ is the numerical solution with $(N,\Delta t)$ mesh points and $U^{2N,\Delta t/2}(x_i,t_n)$ is the numerical solution at the finer mesh with $(2N,\Delta t/2)$ mesh points before extrapolation. The numerical solutions after extrapolation are $(U^{N,\Delta t})^{extr}(x_i,t_n)$ using the mesh points $(N,\Delta t)$ with mesh sizes h_i and Δt and $(U^{2N,\Delta t/2})^{extr}(x_i,t_n)$ using the mesh points $(2N,\Delta t/2)$ with mesh sizes $\frac{h_i}{2}$ and $\frac{\Delta t}{2}$. The (ε,μ)-maximum errors before and after extrapolations were calculated using the following formulas, respectively

$$e^{N,\Delta t} = \max_{\varepsilon,\mu} e_{\varepsilon,\mu}^{N,\Delta t} \quad \text{and} \quad (e^{N,\Delta t})^{extr} = \max_{\varepsilon,\mu} (e_{\varepsilon,\mu}^{N,\Delta t})^{extr}.$$

Furthermore, we compute the numerical rate of convergence before and after extrapolation with the following formulas, respectively

$$\rho_{\varepsilon,\mu}^{N,\Delta t} = \log_2\left(\frac{e_{\varepsilon,\mu}^{N,\Delta t}}{e_{\varepsilon,\mu}^{2N,\Delta t/2}}\right) \quad \text{and} \quad (\rho_{\varepsilon,\mu}^{N,\Delta t})^{extr} = \log_2\left(\frac{(e_{\varepsilon,\mu}^{N,\Delta t})^{extr}}{(e_{\varepsilon,\mu}^{2N,\Delta t/2})^{extr}}\right).$$

The (ε,μ)-maximum rates of convergence before and after extrapolations were calculated using the following formulas, respectively

$$\rho^{N,\Delta t} = \max_{\varepsilon,\mu} \rho_{\varepsilon,\mu}^{N,\Delta t} \quad \text{and} \quad \rho_{extr}^{N,\Delta t} = \max_{\varepsilon,\mu} (\rho_{\varepsilon,\mu}^{N,\Delta t})^{extr}.$$

Table 1

Computation of maximum point-wise errors and rate of convergence for $N = \frac{1}{\Delta t}, \mu = 0$, Example 1

$\varepsilon \downarrow$	Extrapolation	$N = 32$	64	128	256	512
	Before Extrapolation	8.5069e-03	5.1386e-03	2.8624e-03	1.5340e-03	8.0921e-04
	Rate	0.7273	0.8442	0.8999	0.9227	
10^{-6}	After Extrapolation	6.4391e-04	2.1982e-04	6.5090e-05	1.8622e-05	5.8390e-06
	Rate	1.5505	1.7558	1.8054	1.6732	
	Before Extrapolation	8.5068e-03	5.1384e-03	2.8621e-03	1.5337e-03	8.0896e-04
	Rate	0.7273	0.8442	0.9001	0.9229	
10^{-8}	After Extrapolation	6.4325e-04	2.1923e-04	6.4592e-05	1.8006e-05	5.2361e-06
	Rate	1.5529	1.7630	1.8429	1.7819	
	Before Extrapolation	8.5068e-03	5.1384e-03	2.8621e-03	1.5337e-03	8.0896e-04
	Rate	0.7273	0.8442	0.9001	0.9229	
10^{-10}	After Extrapolation	6.4325e-04	2.1923e-04	6.4584e-05	1.8001e-05	5.2298e-06
	Rate	1.5529	1.7632	1.8431	1.7832	
	Before Extrapolation	8.5068e-03	5.1384e-03	2.8621e-03	1.5337e-03	8.0896e-04
	Rate	0.7273	0.8442	0.9001	0.9229	
10^{-12}	After Extrapolation	6.4325e-04	2.1923e-04	6.4584e-05	1.8001e-05	5.2298e-06
	Rate	1.5529	1.7632	1.8431	1.7832	
$e^{N,\Delta t}$	Before Extrapolation	8.5069e-03	5.1386e-03	2.8624e-03	1.5340e-03	8.0921e-04
$\rho^{N,\Delta t}$	Rate	0.7273	0.8442	0.8999	0.9227	
$e^{N,\Delta t}_{extr}$	After Extrapolation	6.4391e-04	2.1982e-04	6.5090e-05	1.8622e-05	5.8390e-06
$\rho^{N,\Delta t}_{extr}$	Rate	1.5505	1.7558	1.8054	1.6732	

Table 2

Computation of maximum point-wise errors and rate of convergence for $N = \frac{1}{\Delta t}$, $\mu = 0.3\varepsilon$, Example 2

$\varepsilon \downarrow$	Extrapolation	$N = 32$	64	128	256	512
10^{-6}	Before Extrapolation	1.9445e-02	1.0633e-02	6.0314e-03	3.2579e-03	1.7222e-03
	Rate	0.8709	0.8180	0.8886	0.9197	
	After Extrapolation	1.6756e-03	6.1162e-04	1.9601e-04	7.6087e-05	2.8981e-05
	Rate	1.4540	1.6417	1.3652	1.3925	
10^{-8}	Before Extrapolation	1.9445e-02	1.0633e-02	6.0307e-03	3.2572e-03	1.7215e-03
	Rate	0.8709	0.8182	0.8887	0.9200	
	After Extrapolation	1.6737e-03	6.0967e-04	1.9661e-04	7.6346e-05	2.8879e-05
	Rate	1.4569	1.6327	1.3647	1.4025	
10^{-10}	Before Extrapolation	1.9445e-02	1.0633e-02	6.0307e-03	3.2572e-03	1.7215e-03
	Rate	0.8709	0.8182	0.8887	0.9200	
	After Extrapolation	1.6737e-03	6.0965e-04	1.9662e-04	7.6352e-05	2.8886e-05
	Rate	1.4570	1.6326	1.3647	1.4023	
10^{-12}	Before Extrapolation	1.9445e-02	1.0633e-02	6.0307e-03	3.2572e-03	1.7215e-03
	Rate	0.8709	0.8182	0.8887	0.9200	
	After Extrapolation	1.6737e-03	6.0965e-04	1.9662e-04	7.6352e-05	2.8886e-05
	Rate	1.4570	1.6326	1.3647	1.4023	
$e^{N,\Delta t}$	Before Extrapolation	1.9445e-02	1.0633e-02	6.0314e-03	3.2579e-03	1.7222e-03
$\rho^{N,\Delta t}$	Rate	0.8709	0.8180	0.8886	0.9197	
$e_{extr}^{N,\Delta t}$	After Extrapolation	1.6756e-03	6.1162e-04	1.9662e-04	7.6352e-05	2.8981e-05
$\rho_{extr}^{N,\Delta t}$	Rate	1.4540	1.6372	1.3647	1.3976	

Table 3

Comparison using $N = \frac{1}{\Delta t}$, $\mu = 0.3\varepsilon$ for Example 2

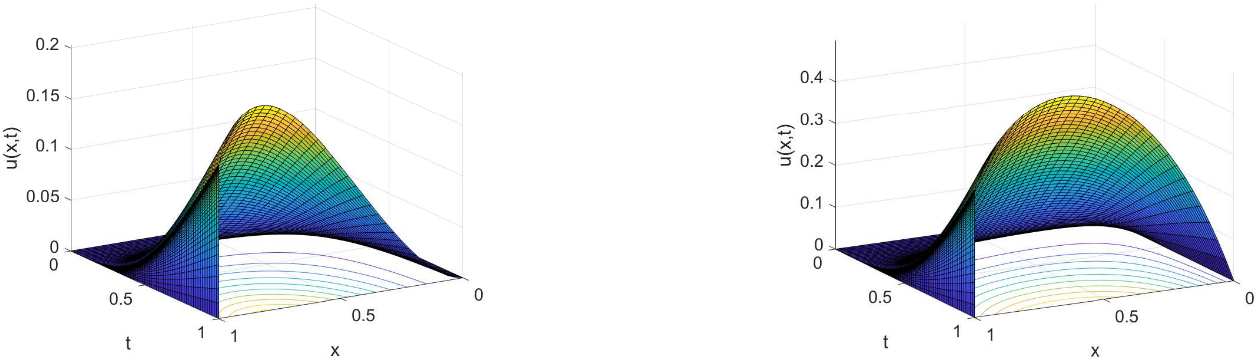
Extrapolation	$N = 16$	32	64	128
Present method				
Before Extrapolation	3.4791e-02	1.9445e-02	1.0633e-02	6.0314e-03
Rate	0.8393	0.8709	0.8180	
After Extrapolation	4.0244e-03	1.6756e-03	6.1162e-04	1.9662e-04
Rate	1.2641	1.4540	1.6372	
Result in [16]				
Before Extrapolation	1.3567e-02	7.7535e-03	4.1434e-03	2.5115e-03
Rate	0.8072	0.9040	0.7223	
After Extrapolation	7.5907e-03	2.3678e-03	8.2018e-04	2.5398e-04
Rate	1.6807	1.5295	1.6912	

Table 4

Computation of maximum point-wise errors and rate of convergenceat $\mu = 0$ for Example 3 with [22]

$\varepsilon \downarrow$	Extrapolation	$N = 32$ $\Delta t = 0.05$	64 $\frac{0.05}{2}$	128 $\frac{0.05}{2^2}$	256 $\frac{0.05}{2^3}$	512 $\frac{0.05}{2^4}$
2^{-6}	Before Extrapolation	1.2677e-2	7.4327e-3	4.0929e-3	2.1883e-3	1.1609e-3
	Rate	0.7703	0.8608	0.9033	0.9146	
	After Extrapolation	2.4529e-3	8.7923e-4	2.7423e-4	8.0145e-5	2.2686e-5
	Rate	1.4802	1.6809	1.7747	1.8208	
2^{-10}	Before Extrapolation	1.4598e-2	8.9967e-3	5.1615e-3	2.8253e-3	1.5371e-3
	Rate	0.6983	0.8016	0.8694	0.8782	
	After Extrapolation	3.8898e-3	1.6408e-3	5.8963e-4	1.8545e-4	5.3233e-5
	Rate	1.2453	1.4765	1.6688	1.8006	
2^{-14}	Before Extrapolation	1.5433e-2	9.6028e-3	5.5900e-3	3.0789e-3	1.6913e-3
	Rate	0.6845	0.7806	0.8604	0.8643	
	After Extrapolation	4.0459e-3	1.7118e-3	6.2066e-4	1.9732e-4	5.6955e-5
	Rate	1.2409	1.4636	1.6533	1.7926	
2^{-18}	Before Extrapolation	1.5485e-2	9.6442e-3	5.6179e-3	3.0960e-3	1.7018e-3
	Rate	0.6831	0.7796	0.8596	0.8633	
	After Extrapolation	4.0560e-3	1.7174e-3	6.2226e-4	1.9783e-4	5.7064e-5
	Rate	1.2398	1.4646	1.6533	1.7936	
2^{-20}	Before Extrapolation	1.5488e-2	9.6468e-3	5.6198e-3	3.0970e-3	1.7025e-3
	Rate	0.6830	0.7795	0.8597	0.8632	
	After Extrapolation	4.0565e-3	1.7177e-3	6.2234e-4	1.9785e-4	5.7068e-5
	Rate	1.2398	1.4647	1.6533	1.7937	
$e^{N,\Delta t}$	Before Extrapolation	1.5488e-2	9.6470e-3	5.6199e-3	3.0971e-3	1.7025e-3
$\rho^{N,\Delta t}$	Rate	0.6830	0.7795	0.8596	0.8633	
$e^{N,\Delta t}_{extr}$	After Extrapolation	4.0566e-3	1.7178e-3	6.2237e-4	1.9786e-4	5.7069e-5
$\rho^{N,\Delta t}_{extr}$	Rate	1.2398	1.4647	1.6533	1.7937	
Result in [22]						
$e^{N,\Delta t}$		1.021e-2	3.225e-3	1.066e-3	3.479e-4	1.111e-4
$\rho^{N,\Delta t}$		1.663	1.598	1.615	1.646	-

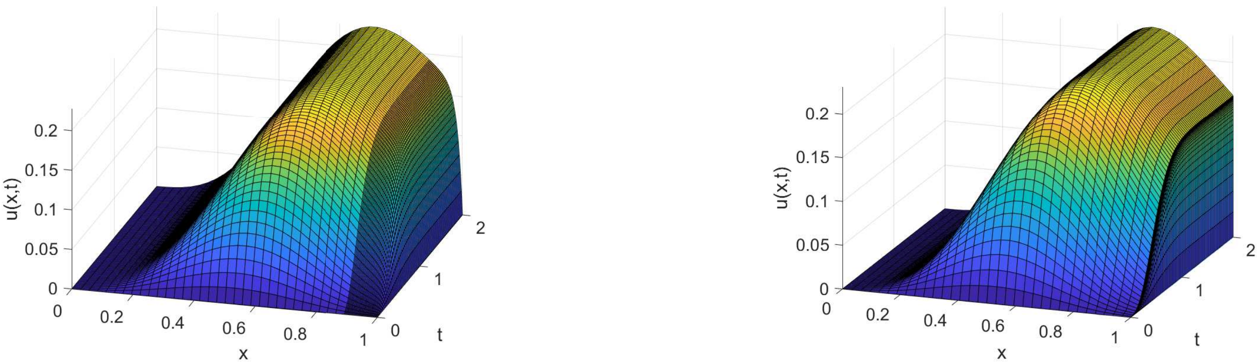
The computed maximum point-wise errors and the rate of convergence for Examples 1 and 2 are given in Tables 1 and 2, respectively. From these results, it is clear that the present method gives an ε-uniform convergence for Examples 1 and 2 before and after extrapolation. Comparison of Example 2 is given in Table 3. The computed maximum point-wise errors and the rate of convergence for Example 3 are given in Table 4 with its comparison. Numerical simulations for Examples 1 and 2 are plotted in Figure 1 and Example 3 in Figure 2. The maximum point-wise errors for Examples 1, 2, and 3 are plotted using log-log scale, as can be seen in Figures 3, 4, and 5, respectively. These figures clearly show that Richardson extrapolation increases the rate of convergence of the upwind scheme from $O(N^{-1}\ln N + \Delta t)$ to $O(N^{-2}\ln^2 N + \Delta t^2)$. Figures 6 and 7 show the effect of the perturbation parameter ε in terms of line graphs for Examples 1, 2, and 3. The effect of the singular perturbation parameter on the boundary layer of the solution for all Examples is shown in Figures 6 and 7. As observed in these Figures, as $\varepsilon \to 0$ strong boundary layer is formed near $x = 1$. The effect of the time level t in terms of line graphs for Examples 1, 2, and 3 is given in Figures 8 and 9. As observed from Figures 8 and 9, a strong boundary layer is formed near $x = 1$, and as the size of the time level increases, the thickness of the layer increases.



(a) Example 1 when $\mu = 0$

(b) Example 2 when $\mu = 0.3\varepsilon$

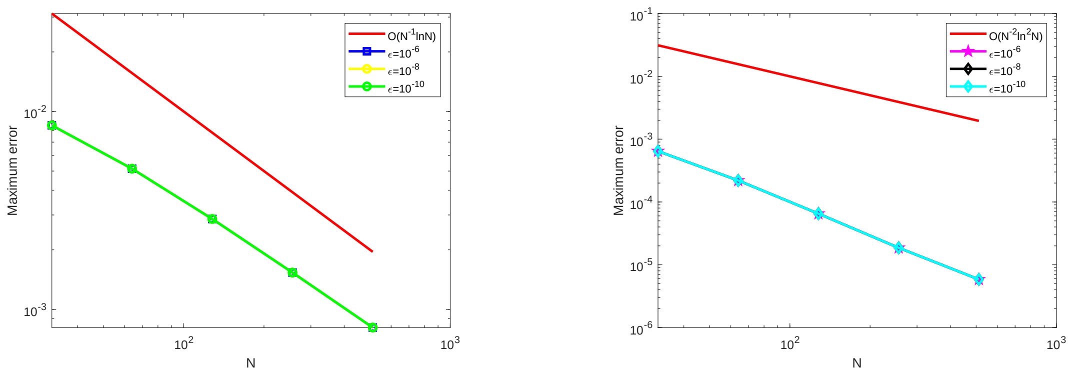
Figure 1. Surface plot of the numerical solution for $N = 64 = M$ and $\varepsilon = 10^{-6}$



(a) At $N = 64, M = 80$ and $\varepsilon = 2^{-6}$

(b) At $N = 64, M = 80$ and $\varepsilon = 2^{-16}$

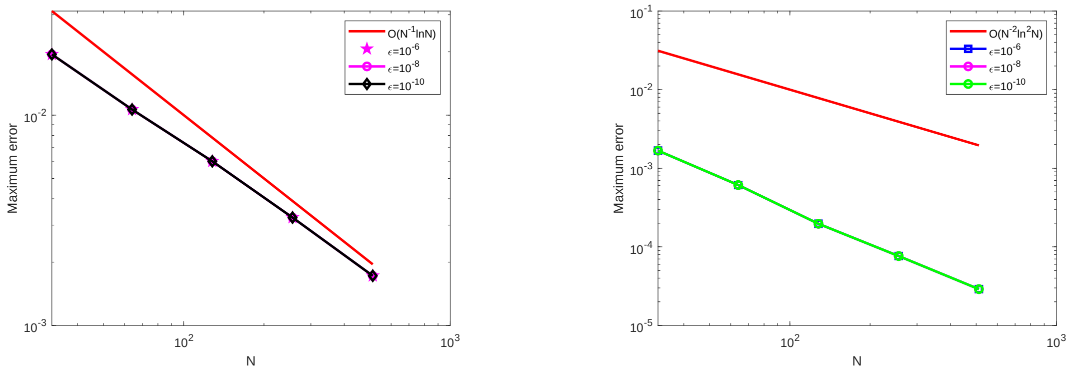
Figure 2. Surface plot of the numerical solution for Example 3 for $\mu = 0$



(a) Before extrapolation

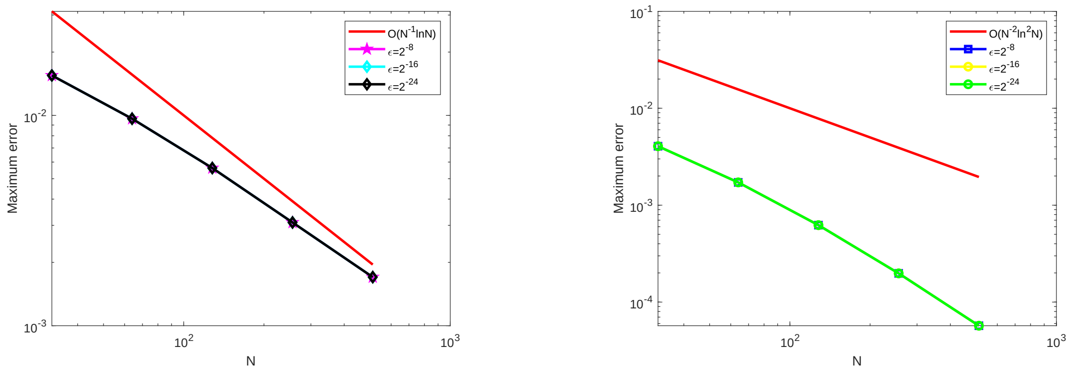
(b) After extrapolation

Figure 3. Log-log plot of the maximum point-wise errors at $\mu = 0$ for Example 1



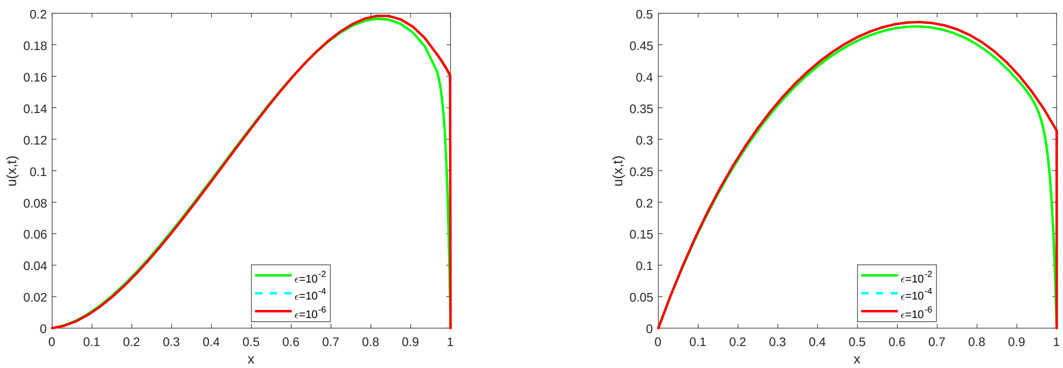
(a) Before extrapolation (b) After extrapolation

Figure 4. Log-log plot of the maximum point-wise errors at $\mu = 0.3\varepsilon$ for Example 2



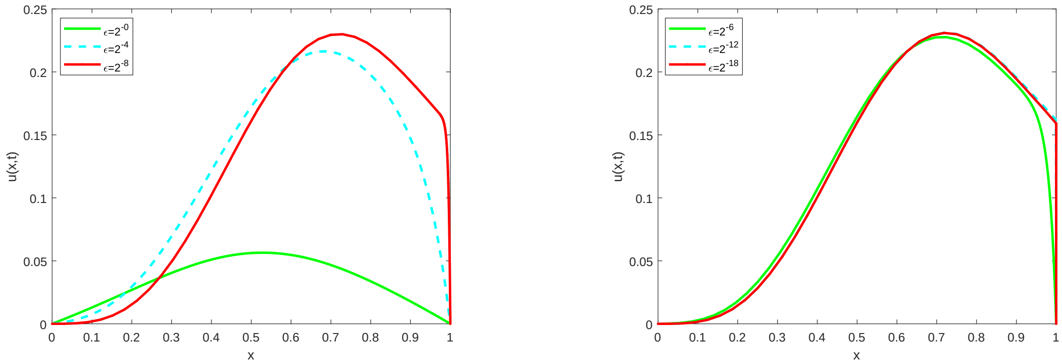
(a) Before extrapolation (b) After extrapolation

Figure 5. Log-log plot of the maximum point-wise errors at $\mu = 0$ for Example 3



(a) Example 1 at $N = 64 = M$, $\mu = 0$ (b) Example 2 at $N = 64 = M$, $\mu = 0.3\varepsilon$

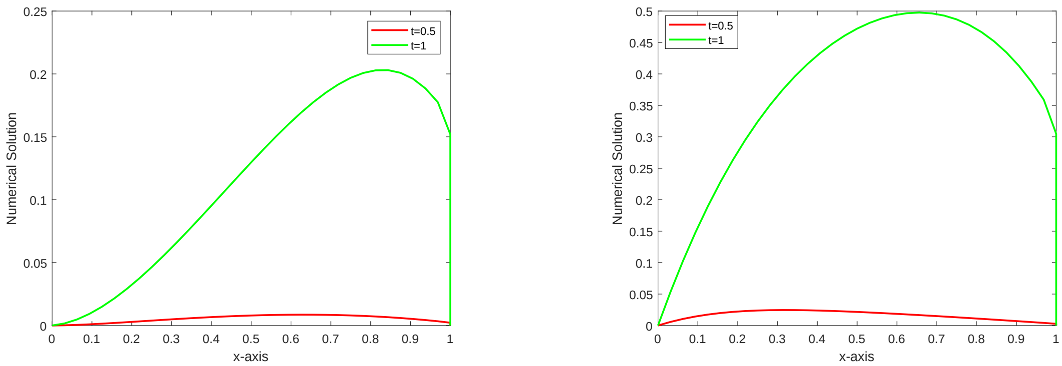
Figure 6. Effect of the perturbation parameter ε on the numerical solution



(a) At $N = 64, M = 80$, $\mu = 0$

(b) At $N = 64, M = 80$, $\mu = 0$

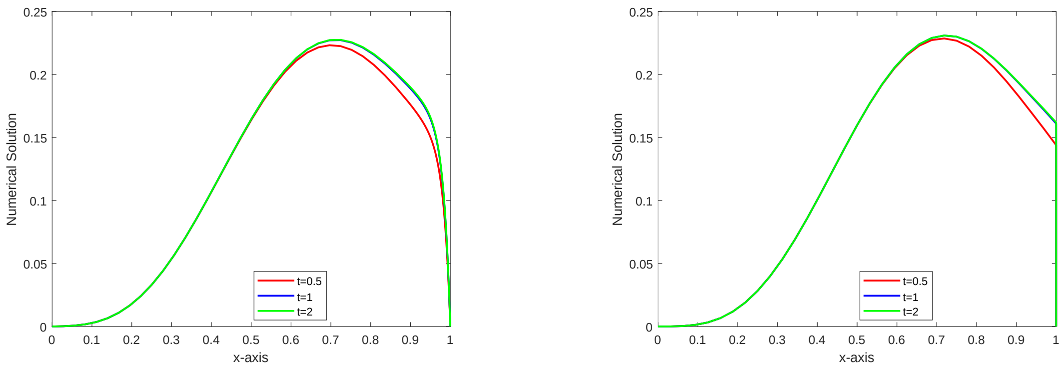
Figure 7. Effect of the parameter ε on the solution for Example 3



(a) Example 1 at $N = 64 = M$, $\varepsilon = 10^{-6}$, $\mu = 0$

(b) Example 2 at $N = 64 = M$, $\varepsilon = 10^{-6}$, $\mu = 0.3\varepsilon$

Figure 8. Effect of time t level on the solution



(a) At $N = 64, M = 80$, $\varepsilon = 2^{-6}$, $\mu = 0$

(b) At $N = 64, M = 80$, $\varepsilon = 2^{-16}$, $\mu = 0$

Figure 9. Effect of time t level on the solution interms of line graph for Example 3

Conclusion

This study presents a computational method that is almost second-order convergent for singularly perturbed parabolic differential difference equations with negative shifts. The Taylor series approximation is used to estimate the terms that involve delays. An implicit Euler technique for the time direction on a uniform mesh and an upwind difference method on a Shishkin mesh in the space direction are used to discretise the resulting singularly perturbed parabolic convection-diffusion-reaction equation. The stability and uniform convergence of the proposed method are established very well. The proposed method gives almost first-order convergence both in the time and space variables. The Richardson extrapolation technique is then applied to accelerate the order of convergence of the method in the time and space variables. Theoretically, we have proved that the extrapolation provides almost second-order $\varepsilon-$uniform convergence. To validate the applicability of the proposed method, some numerical examples are computed for different values of the perturbation parameter and delay parameter.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Generalizing Semi-n-Potent Rings

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The present article deals with the problem of characterizing a widely large class of associative and possibly non-commutative rings. So, we define and explore the class of rings R for which each element in R is a sum of a tripotent element from R and an element from the subring $\Delta(R)$ of R which commute with each other, calling them *strongly* Δ*-tripotent* rings, or shortly just *SDT* rings. Succeeding in obtaining a complete description of these rings R modulo their Jacobson radical $J(R)$ as the direct product of a Boolean ring and a Yaqub ring, our results somewhat generalize those established by Koşan-Yildirim-Zhou in Can. Math. Bull. (2019). Specifically, it is proved that if a ring R is SDT, then the factor ring $R/J(R)$ is always reduced and 6 lies in $J(R)$. Even something more, as already noticed before, it is shown that the quotient $R/J(R)$ is a tripotent ring, which means that each of its elements satisfies the cubic equation $x^3 = x$. Furthermore, examining triangular matrix rings $\mathrm{T}_n(R)$, we succeeded to classify its structure rather completely in the case where R is a local ring and $n \geq 3$ by establishing a satisfactory necessary and sufficient condition in terms of the ring R and its sections, resp., divisions.

Keywords: idempotent, tripotent, strongly nil-clean ring, Boolean ring, semi-tripotent ring, Δ subring, Jacobson radical, triangular matrix ring.

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Introduction and Motivation

Throughout this paper, all rings are assumed to be unital and associative. Almost all symbols, notation and concepts are standard being consistent with the classical book [1]. The Jacobson radical, the lower nil-radical, the set of nilpotent elements, the set of idempotent elements, and the set of units of R are denoted, respectively, by $J(R)$, $\mathrm{Nil}_*(R)$, $\mathrm{Nil}(R)$, $\mathrm{Id}(R)$, and $U(R)$. Additionally, we write $\mathrm{M}_n(R)$, $\mathrm{T}_n(R)$ and $R[x]$ for the $n \times n$ full matrix ring, the $n \times n$ upper triangular matrix ring, and the polynomial ring over R, respectively.

The core focus of this exploration is the set

$$\begin{aligned} J(R) \subseteq \Delta(R) &= \{x \in R : x + u \in U(R) \text{ for all } u \in U(R)\} \\ &= \{x \in R : 1 - xu \text{ is invertible for all } u \in U(R)\} \\ &= \{x \in R : 1 - ux \text{ is invertible for all } u \in U(R)\}, \end{aligned}$$

which was examined by Lam in [2; Exercise 4.24] and recently explored in detail by Leroy-Matczuk in [3]. It was indicated in [3; Theorems 3 and 6] that $\Delta(R)$ represents the (proper) largest Jacobson radical subring of R that remains closed under multiplication by all units (resp., quasi-invertible elements) of R, and it is an ideal of R exactly when $\Delta(R) = J(R)$.

In the contemporary ring theory, the class of strongly nil-clean rings possesses significant importance. A ring R is called *strongly nil-clean* if every element of R can be expressed as the sum of an

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idempotent in R and a nilpotent element in R that commute with each other (see [4–6]). Later on, Chen and Sheibani generalized in [7] this concept and introduced the so-called strongly 2-nil-clean rings: a ring R is defined as *strongly 2-nil-clean* if every element of R can be written as the sum of a tripotent element of R (i.e., an element $x \in R$ such that $x^3 = x$) and a nilpotent element of R that commute.

On the other hand, in a way of similarity, *strongly J-clean* rings are those rings in which every element can be written as the sum of an idempotent and an element from the Jacobson radical that commute [8, 9]. In this vein, Koşan et al. introduced in [10] the so-termed *semi-tripotent* rings R in which each element is the sum of a tripotent element from R and an element from $J(R)$.

Considering and analyzing these definitions, as well as the fact that $\Delta(R)$ is a (possibly proper) subset of $J(R)$, that is *not* necessarily an ideal, and which also does *not* have useful properties like the set $\mathrm{Nil}(R)$, a question naturally arises about the properties of those rings R for which each element is the sum of a tripotent element from R and an element from $\Delta(R)$ that commute with each other. The main objective of the current article is namely to investigate these types of rings and to conduct a comprehensive study of their structure.

Thereby, we come to the following key notion, motivated by the discussion alluded to above.

Definition 1. We say that R is a *strongly Δ-tripotent* ring, or just an *SDT ring* for short, if every element of R is the sum of a tripotent from R and an element from $\Delta(R)$ that commute with each other. Such a sum's presentation is also said to be an *SDT representation.*

Our further plan in the organization of our study is the following: In the next section, we obtain some crucial examples and principal properties of such rings establishing their connection with many standard properties – e.g., such as uniquely clean (see Corollary 2). In the subsequent section, we achieve the major result describing the algebraic structure of the SDT rings in an appropriate form showing that these rings modulo their Jacobson radical are the direct product of a Boolean ring and a Yaqub ring (see Theorem 1). Some other closely related statements are also proved such as Propositions 2 and 6. In the fourth section, we study the behavior of the given SDT concept under various ring extensions and, specifically, we characterize when $\mathrm{T}_n(R)$ is an SDT ring by finding a necessary and sufficient condition, provided that R is local and $n \geq 3$ (see Theorem 2). In the final fifth section, we conclude with some commentaries and two challenging open problems (see, e.g., Problems 1 and 2) which, hopefully, will stimulate a future intensive examination of the present subject.

1 Examples and Basic Properties

The following claim can easily be proven, so we omit the details leaving them to the interested reader for check.

Lemma 1. (1) Suppose $R = \prod_{i\in I} R_i$. Then, R is an SDT ring if, and only if, for each $i \in I$, R_i is an SDT ring.

(2) Suppose R is a ring and I is an ideal of R such that $I \subseteq J(R)$. Then, R/I is an SDT ring.

We proceed by proving the following three technical assertions.

Lemma 2. For every $e = e^3 \in R$ and $d \in \Delta(R)$, we have $(e \pm e^2)d, d(e \pm e^2), 2e^2d$, and $2ed \in \Delta(R)$.

Proof. For every $e = e^3 \in R$, we have

$$((1 - e^2) - e)((1 - e^2) - e) = 1 = ((1 - e^2) + e)((1 - e^2) + e).$$

Therefore, $(1 - e^2 \pm e) \in U(R)$, so it follows from [3; Lemma 1(2)] that, for every $d \in \Delta(R)$, both $((1 - e^2 \pm e)d$ and $d((1 - e^2 \pm e) \in \Delta(R)$. Since $\Delta(R)$ is a subring of R, we have $(e \pm e^2)d$ and $d(e \pm e^2) \in \Delta(R)$. This implies that $2ed$ and $2e^2d \in \Delta(R)$, as required. □

Lemma 3. Let R be an SDT ring, and $a \in R$. If $a^2 \in \Delta(R)$, then $a \in \Delta(R)$.

Proof. Assume that $a = e + d$ is an SDT representation. We have $a^2 = e^2 + 2ed + d^2$. By Lemma 2, it must be that

$$e^2 = a^2 - 2ed - d^2 \in \Delta(R) \cap \mathrm{Id}(R) = \{0\},$$

which implies $e = 0$. Thus, $a = d \in \Delta(R)$, as expected. □

Lemma 4. Let R be an SDT ring. Then, for every $a \in R$, $a - a^3 \in \Delta(R)$.

Proof. Assume $a = e + d$ is an SDT representation. We calculate that

$$a - a^3 = (d - d^3) - (2e^2d + 2ed) - (e^2d + ed^2).$$

Furthermore, according to Lemma 2, it suffices to show that $e^2d + ed^2 \in \Delta(R)$. But, Lemma 2 tells us that $e^2d + ed^2 \in \Delta(R)$ precisely when $ed + e^2d^2 \in \Delta(R)$. Consequently, we show that $ed + e^2d^2 \in \Delta(R)$.

To this target, assume $ed = f + b$ is an SDT representation. Then,

$$e^2d^2 = f^2 + 2fb + b^2.$$

Thus,

$$ed + e^2d^2 = (f + f^2) + (b + 2fb + b^2).$$

Now, multiplying by d and d^2 both sides of the previous relation, we have

$$ed^2 + e^2d^3 = (f + f^2)d + (b + 2fb + b^2)d \in \Delta(R),$$
$$ed^3 + e^2d^4 = (f + f^2)d^2 + (b + 2fb + b^2)d^2.$$

Owing to Lemma 2, we infer that $ed^2 + e^2d^3, ed^3 + e^2d^4 \in \Delta(R)$. Also,

$$ed^2 + e^2d^3 = ed^2 + e^2d^3 - ed^3 + ed^3 - e^2d^2 + e^2d^2 = e^2d^2 + ed^3 + (e^2 - e)d^3 + (e - e^2)d^2.$$

Thus, in virtue of Lemma 2, it follows that $e^2d^2 + ed^3 \in \Delta(R)$. Therefore, we get

$$\begin{cases} e^2d^2 + ed^3 \in \Delta(R), \\ ed^3 + e^2d^4 \in \Delta(R), \end{cases} \implies \quad e^2d^2 + e^2d^4 \in \Delta(R).$$

We now have that

$$(ed + e^2d^2)^2 = e^2d^2 + 2ed^3 + e^2d^4 \in \Delta(R).$$

So, Lemma 3 enables us that $ed + e^2d^2 \in \Delta(R)$, as pursued. □

We now arrive at the following concrete application of the last lemma.

Example 1. Let R be an arbitrary ring. Then, $R[x]$ is *not* an SDT ring.

Proof. Assume the contrary. Then, applying Lemma 4, we derive that $x - x^3 \in \Delta(R[x])$, and thus $1 - x + x^3 \in U(R[x])$, which is the wanted contradiction. □

With the previous example in mind, the ring $R[x]$ is surely not SDT. However, a logical question arises about the form of elements with an SDT representation in the polynomial ring $R[x]$. We will attempt to answer this question below.

Recall that a ring R is said to be *2-primal* if $\mathrm{Nil}_*(R) = \mathrm{Nil}(R)$. For instance, it is well known that any commutative ring and any reduced ring are definitely 2-primal.

Likewise, for an endomorphism σ of R, the ring R is called *σ-compatible* if, for every $a, b \in R$, the equality $ab = 0$ if, and only if, $a\sigma(b) = 0$ [11]. In this case, it is clear that σ is always injective.

We now manage to prove the following two pivotal statements.

Proposition 1. Let R be a 2-primal and α-compatible ring. Then,

$$\Delta(R[x,\alpha]) = \Delta(R) + Nil_*(R[x,\alpha])x.$$

Proof. Assuming $f = \sum_{i=0}^{n} a_i x^i \in \Delta(R[x,\alpha])$, then, for each $u \in U(R)$, we have that $1 - uf \in U(R[x,\alpha])$. Thus, taking into account [12; Corollary 2.14], $1 - ua_0 \in U(R)$ holds and, for every $1 \leq i \leq n$, it holds $ua_i \in \mathrm{Nil}_*(R)$. Since $\mathrm{Nil}_*(R)$ is an ideal, we deduce $a_0 \in \Delta(R)$ and hence, for each $1 \leq i \leq n$, we obtain $a_i \in Nil_*(R)$. Since R is a 2-primal ring, [12; Lemma 2.2] applies to get that $Nil_*(R)[x,\alpha] = Nil_*(R[x,\alpha])$, as desired.

Conversely, assume $f \in \Delta(R) + Nil_*(R[x,\alpha])x$ and $u \in U(R[x,\alpha])$. Then, employing [12; Corollary 2.14], we have $u \in U(R) + Nil_*(R[x,\alpha])x$. But, since R is a 2-primal ring, we receive $1 - uf \in U(R) + Nil_*(R[x,\alpha])x \subseteq U(R[x,\alpha])$, whence $f \in \Delta(R[x,\alpha])$, as promised. □

Proposition 2. Let R be a 2-primal and α-compatible ring, and let $e^3 = e = \sum_{i=0}^{n} e_i x^i \in R[x,\alpha]$. Then, $e_0^3 = e_0$ and, for every $1 \leq i \leq n$, the inclusion $e_i \in Nil(R)$ is true.

Proof. It is easy to see that $e_0^3 = e_0$, so it suffices to show that, for every $1 \leq i \leq n$, the relation $e_i \in \mathrm{Nil}(R)$ is valid. Since $e^3 = e$, we inspect that $e_n\alpha^n(e_n)\alpha^{2n}(e_n) = 0$. And because R is α-compatible, [13; Lemma 2.1] is applicable to get that $e_n^3 = 0$.

Now, set $g := f - e_n x^n$. Since $f^3 = f$ and $e_n \in \mathrm{Nil}_*(R)$, we have $g - g^3 \in \mathrm{Nil}_*(R)[x,\alpha]$, so $\bar{g} = \bar{g}^3 \in R/\mathrm{Nil}_*(R)[x,\alpha]$. Thus, one verifies that

$$e_{n-1}\alpha^{n-1}(e_{n-1})\alpha^{2n-2}(e_{n-1}) \in Nil_*(R).$$

But, since R is an α-compatible ring, [13; Lemma 2.1] works to obtain that $e_{n-1} \in \mathrm{Nil}(R)$. Continuing in this aspect, it can be shown that, for each $1 \leq i \leq n$, the condition $e_i \in Nil(R)$ is fulfilled, as asked for. □

To specify the elements with an SDT representation of the ring $R[x,\alpha]$, we need new notation. For convenience of the exposition, we just put the set of elements with an SDT representation in the ring R to be abbreviated as $SDT(R)$.

So, we have the validity of the following.

Lemma 5. Let R be a 2-primal and α-compatible ring. Then,

$$SDT(R[x,\alpha]) \subseteq SDT(R) + Nil_*(R)[x,\alpha]x.$$

Proof. Assume $f = \sum_{i=0}^{n} f_i x^i \in SDT(R[x,\alpha])$ and $f = \sum_{i=0}^{n} e_i x^i + \sum_{i=0}^{n} d_i x^i$ is an SDT representation. In accordance with Propositions 1 and 2, we have $e_0 = e_0^3$ and $d_0 \in \Delta(R)$, and hence clearly $e_0 d_0 = d_0 e_0$, so that $f_0 \in SDT(R)$.

Moreover, with the aid of Proposition 2, for every $1 \leq i \leq n$, it must be that $e_i, d_i \in \mathrm{Nil}_*(R)$, whence $f_i = e_i + d_i \in \mathrm{Nil}_*(R)$, as required. □

The next affirmation is crucial.

Lemma 6. Let R be an SDT ring. Then, $R/J(R)$ is reduced.

Proof. Assume $x^2 \in J(R) \subseteq \Delta(R)$. Thus, by Lemma 3, we have $x \in \Delta(R)$. Let $r \in R$. Since $1 - r^2x^2 \in U(R)$, we may set $u := 1 - rx^2r \in U(R)$. Therefore,

$$(1 - rx)(1 + rx) = 1 - rx + xr - rx^2r = xr - rx + u.$$

It suffices to show that $xr - rx \in \Delta(R)$. To this goal, assume $r = e + d$ is an SDT representation. Then,

$$xr - rx = x(e + d) - (e + d)x = xe - ex + (xd - dx),$$

and as $x, d \in \Delta(R)$, it is just sufficient to prove that $xe - ex \in \Delta(R)$.

Since
$$\left[e^2x(1-e^2)\right]^2 = 0 = \left[(1-e^2)xe^2\right]^2.$$

Lemma 3 assures that

$$\begin{cases} e^2x(1-e^2) \in \Delta(R) \Longrightarrow e^2x - e^2xe^2 \in \Delta(R), \\ (1-e^2)xe^2 \in \Delta(R) \Longrightarrow xe^2 - e^2xe^2 \in \Delta(R). \end{cases}$$

However, because $\Delta(R)$ is closed under addition, we arrive at $e^2x - xe^2 \in \Delta(R)$. Consequently,

$$xe - ex = xe + xe^2 - xe^2 - ex - e^2x + e^2x = e^2x - xe^2 + x(e+e^2) - (e+e^2)x \in \Delta(R).$$

Hence,
$$(1 - rx)(1 + xr) \in U(R).$$

But R was arbitrary, and so $x \in J(R)$, as needed. □

Given the truthfulness of Lemma 4, we have that, for every SDT ring R, $6 = 2^3 - 2 \in \Delta(R)$. This raises a logical question: if R is an SDT ring, is $6 \in J(R)$? We will answer this query in the following lemma.

Lemma 7. Let R be an SDT ring. Then, $6 \in J(R)$.

Proof. Invoking Lemma 4, we know that $6 \in \Delta(R)$, which implies $12 = 6 + 6 \in \Delta(R)$. Letting $r \in R$ be arbitrary, and letting $r = e + d$ be an SDT representation, Lemma 2 ensures that

$$1 - 12r = 1 - 12e - 12d = 1 - 2(6e) - 12d \in 1 + \Delta(R) \subseteq U(R).$$

Thus, $12 \in J(R)$.

Furthermore, since $6^2 = 36 = 3 \times 12 \in J(R)$, Lemma 6 helps us to conclude that $6 \in J(R)$, as stated. □

As a useful consequence, we deduce the following.

Corollary 1. Let R be an SDT ring. Then, the following two points hold:
(1) $2 \in U(R)$ if, and only if, $3 \in J(R)$.
(2) $3 \in U(R)$ if, and only if, $2 \in J(R)$.

Proof. The proof is pretty straightforward being based on Lemma 7, so we leave it voluntarily. □

The next two assertions are worthy of documentation.

Proposition 3. Let R be an SDT ring such that $2 \in U(R)$. Then, $\Delta(R)$ is an ideal. In particular, under these conditions, $\Delta(R) = J(R)$.

Proof. Since $\Delta(R)$ is closed under addition, it is sufficient to show that, for any $d \in \Delta(R)$ and $r \in R$, the relations $rd, dr \in \Delta(R)$ are valid. Assume, for this aim, that $rd = e+b$ and $r = f+b'$ are two SDT representations. Exploiting Lemma 2, we know $2fd \in \Delta(R)$. Since $2 \in U(R)$, [3; Lemma 1(2)] teaches us that $fd \in \Delta(R)$. So, we have

$$rd = e + b = fd + b'd \Longrightarrow e - fd = b'd - b \in \Delta(R).$$

But, since $fd \in \Delta(R)$, it follows that $e \in \Delta(R)$, so $e^2 \in \Delta(R) \cap Id(R) = \{0\}$, which forces $e = 0$. Therefore, $rd = b \in \Delta(R)$. Similarly, it can be shown that $dr \in \Delta(R)$, guaranteeing the claim. □

Proposition 4. Let R be an SDT ring with $3 \in U(R)$. Then, for any $a \in R$, we have $a = f + b$, where $f = f^2 \in R$, $b \in \Delta(R)$ and $fb = bf$.

Proof. Suppose $a = f + d$ is an SDT representation. Then,

$$a - a^2 = (f - f^2) + (d - 2fd - d^2).$$

Since $3 \in U(R)$ by Corollary 1, we get $2 \in J(R)$. Thus, $(f - f^2)^2 = -2(f - f^2) \in J(R)$ and, with Lemma 7 at hand, we observe that $f - f^2 \in J(R)$. This gives $a - a^2 \in \Delta(R)$.

On the other hand, since

$$a - f^2 = (a - a^2) + 2(a^2 - f^2 - fd) - d^2 \in \Delta(R),$$

by setting $e := f^2$, we finish the proof after all. □

A ring R is called an *SDI ring* if, for every $r \in R$, there exist $e = e^2 \in R$ and $b \in \Delta(R)$ such that $r = e + b$ and $eb = be$. Recall also that a ring is called a *ΔU ring*, provided $1 + \Delta(R) = U(R)$ [14].

The following closely related results are of some interest as well.

Lemma 8. Every SDI ring is a ΔU ring.

Proof. Suppose $u \in U(R)$ and $u = e + d$ is an SDI representation. Then, we have

$$e = u - d \in U(R) + \Delta(R) \subseteq U(R) \cap Id(R) = \{1\},$$

as required. □

Lemma 9. ([14; Proposition 2.3]) The ring R is a ΔU ring if and only if $U(R) + U(R) \subseteq \Delta(R)$; and then, $U(R) + U(R) = \Delta(R)$.

Recall that a ring R is said to be *uniquely clean*, provided that each element in R has a unique representation as the sum of an idempotent and a unit [15].

The next valuable consequence gives some transversal between the notions of SDI rings and unique cleanness.

Corollary 2. Let R be a ring. Then the following are equivalent:
(1) R is uniquely clean.
(2) R is SDI and all idempotents are central.

Proof. (1) $\Rightarrow$ (2). Assume R is a uniquely clean ring. Consulting with [15; Lemma 4], every idempotent in R is central. Besides, by virtue of [15; Theorem 20], for every $a \in R$, there exists a unique idempotent e such that $a - e \in J(R) \subseteq \Delta(R)$. Thus, there exists $d \in \Delta(R)$ such that $a = e + d$. Since all idempotents are central, we have $ed = de$.

(2) $\Rightarrow$ (1). Assume R is an SDI ring, and let $a \in R$ be arbitrary. Suppose $a + 1 = e + d$ is an SDI representation. Then, $a = e + (d - 1)$, which is a clean representation. Assume now that $e + u = f + v$ are two clean representations. So, Lemma 9 informs us that $e - f = v - u \in \Delta(R)$. Since all idempotents are central, we find $e - f = (e - f)^3$, and so $(e - f)^2 \in \Delta(R) \cap Id(R) = \{0\}$. Therefore, $e - f = (e - f)^3 = (e - f)(e - f)^2 = 0$. Hence, $e = f$, as it must be. □

2 *The Main Characterizations*

We start our considerations here with some relationships between certain classes of rings.

Proposition 5. Suppose R is an SDT ring and a domain. Then, R is a local ring.

Proof. Let $a \in R$. We want to show that either $a \in U(R)$ or $a \in \Delta(R)$. To that end, suppose $a = e + d$ is an SDT representation. If $e = 0$, then $a = d \in \Delta(R)$. If $e \neq 0$, then as $e^3 = e$ it must be $e(1 - e^2) = 0$. But, since R is a domain, $(1 - e)(1 + e) = 1 - e^2 = 0$, so either $e = 1$ or $e = -1$. Therefore, either $a = 1 + d \in U(R)$ or $a = -1 + d \in U(R)$. It can next easily be shown that R is a local ring if, and only if, $R = U(R) \cup \Delta(R)$, as required. □

As an immediate consequence, we yield:

Corollary 3. Suppose R is a strongly 2-nil clean and local ring. Then, R is an SDT ring.

Proof. It is pretty easy, because in a local ring the containment $\mathrm{Nil}(R) \subseteq J(R)$ always holds. □

The next assertion is of some importance by giving some close relevance between the notion of a semi-tripotent ring as stated in [10] and the new concept of an SDT ring given above.

Proposition 6. Suppose R is a semi-tripotent and local ring. Then, R is an SDT ring.

Proof. Since R is a local ring, either $2 \in J(R)$ or $2 \in U(R)$. If $2 \in J(R)$, then in virtue of [10; Theorem 3.5] the factor-ring $R/J(R)$ is Boolean. On the other hand, as R is local, it has to be that $R/J(R) \cong \mathbb{Z}_2$, and so $R = J(R) \cup (1 + J(R))$, yielding R is an SDT ring. If, however, $2 \in U(R)$, then again [10; Theorem 3.5] works to get that the quotient-ring $R/J(R)$ is a Yaqub ring. However, because R is local, it must be that $R/J(R) \cong \mathbb{Z}_3$, and thus $R = J(R) \cup (1 + J(R)) \cup (-1 + J(R))$ implying R is an SDT ring, as asserted. □

It is well known that a ring is Boolean if and only if it is a subdirect product of copies of $\mathbb{Z}_2$. Analogously, in [7], Chen and Sheibani called a non-zero ring R a *Yaqub ring* if it is a subdirect product of copies of $\mathbb{Z}_3$. They proved that R is a Yaqub ring if, and only if, 3 is nilpotent and R is a tripotent ring (that is, each of its element is tripotent).

We are now ready to attack the chief characterizing result, thereby completely describing the structure of the SDT rings.

Theorem 1. Assume R is an SDT ring. Then, $R/J(R)$ is a tripotent ring, i.e., $R/J(R) \cong R_1 \times R_2$, where R_1 is a Boolean ring and R_2 is a Yaqub ring.

Proof. Referring to Lemma 7, we have $6 \in J(R)$. Set $\bar{R} := R/J(R)$. Thanks to the famous Chinese Remainder Theorem, we write $\bar{R} \cong R_1 \times R_2$, where $R_1 := \bar{R}/2\bar{R}$ and $R_2 := \bar{R}/3\bar{R}$. Since R is an SDT ring, Lemma 1(2) guarantees that $\bar{R}$ is an SDT ring too. Therefore, again in view of Lemma 1(1), R_1 is an SDT ring. Since $2 = 0$ in R_1, we have $3 \in U(R_1)$. Thankfully, Proposition 4 yields R_1 is an SDI ring. Also, Lemma 6 implies that R_1 is reduced, and thus all idempotents in R_1 are central. Therefore, Corollary 2 shows that R_1 is a uniquely clean ring. Note that, as $J(R) = 0$, it must be that $J(R_1) = 0$. Using now [15; Theorem 19], we conclude that R_1 is a Boolean ring, as formulated.

On the other hand, since $3 = 0$ in $R_2 \neq \{0\}$, we have $2 \in U(R_2)$. Knowing Proposition 3, we obtain $J(R_2) = \Delta(R_2)$. This means, with the help of Lemma 4, that, for any $a \in R_2$, the relations $a - a^3 \in \Delta(R_2) = J(R_2) = 0$ are true. Thus, for any $a \in R_2$, we get that $a = a^3$. Furthermore, using [7; Lemma 4.4], we infer that R_2 is a Yaqub ring, as given. □

It is worthwhile noticing that the extra requirement on the first direct component R_1 and the second direct component R_2 to be not simultaneously $\{0\}$ can be freely ignored here, as opposed to what was shown in [16], where an analogous shortcoming was unambiguously detected for the main result of the paper [17].

Let R be a ring, and let $a \in R$. Suppose $ann_l a := \{r \in R : ra = 0\}$ and $ann_r a := \{r \in R : ar = 0\}$.

We continue by verifying the following two needed technicalities.

Lemma 10. Let R be a ring and $a = e + d$ an SDT representation in R. Then, $\mathrm{ann}_l(a) \subseteq \mathrm{ann}_l(e)$ and $\mathrm{ann}_r(a) \subseteq \mathrm{ann}_r(e)$.

Proof. Assume $ra = 0$. Now, Lemma 2 applies to ensure that there exists $d' \in \Delta(R)$ such that $a^2 = e^2 + d'$. Since $ra = 0$, we have $re^2 + rd' = 0$. Now, multiplying by e from the right, we get $re + red' = 0$, and so $re(1 + d') = 0$. Since $d' \in \Delta(R)$, it follows that $1 + d' \in U(R)$ which forces $re = 0$. Thus, $r \in \mathrm{ann}_l(e)$. Similarly, it can be shown that the inclusion $\mathrm{ann}_r(a) \subseteq \mathrm{ann}_r(e)$ is too valid, as required. □

Lemma 11. Let R be a ring and $e \in R$ an idempotent. If $a \in eRe$ is an SDT element in R, then a is an SDT element in the corresponding corner subring eRe.

Proof. Write $a = f + d$, where $f = f^3$, $d \in \Delta(R)$ and $fd = df$. Since $1 - e \in \mathrm{ann}_l(a) \cap \mathrm{ann}_r(a)$, Lemma 10 is a guarantor that $1 - e \in \mathrm{ann}_l(f) \cap \mathrm{ann}_r(f)$ implying $(1 - e)f = f(1 - e) = 0$. Thus, $f = ef = fe$. Likewise, since $a \in eRe$, we receive $a = ea = ae = eae$. But, subsequently multiplying $a = f + d$ by e from the left and right, we obtain that $a = efe + ede$. Note that, since $f = ef = fe$ and f is a tripotent, efe is also a tripotent. So, it suffices to show that $ede \in \Delta(eRe)$.

On the other hand, since $f = ef = fe = efe$ and $a = ea = ae = eae$, it is evident that

$$d = ed = de = ede \in \Delta(R) \cap eRe.$$

Now, we show that $eRe \cap \Delta(R) \subseteq \Delta(eRe)$ always holds. To this purpose, assume $r \in eRe \cap \Delta(R)$ and $u \in U(eRe)$. Then, $(u + (1-e))(u^{-1} + (1-e)) = 1$, so $u + (1-e) \in U(R)$. Since $r \in \Delta(R)$, there exists $v \in R$ such that $(1 - (u + (1-e))r)v = 1$. But $r \in eRe$, so that $(1 - ur)v = 1$. Furthermore, multiplying subsequently by e from the left and right, we extract that $(e - ur)eve = e$ forcing $r \in \Delta(eRe)$. Finally, $d \in \Delta(eRe)$, and we are done. □

As an automatic consequence, we yield the following.

Corollary 4. Let R be a ring, and let $e \in R$ be an idempotent. If R is an SDT ring, then so is the corresponding corner subring eRe.

Furthermore, in regard to the last corollary, a logically arising question is whether or not the converse in its formulation holds, that is, if both eRe and $(1 - e)R(1 - e)$ are SDT rings, is it true that so does R? However, the next construction, suggested to us by Dr. Omer Cantor to whom we express our sincere gratitude, illustrates that this question has a negative solution. In fact, let $R := \mathrm{M}_2(\mathbb{Z}_2)$ and set $e := E_{11}$. An easy check shows that both eRe and $(1 - e)R(1 - e)$ are isomorphic to $\mathbb{Z}_2$, so they are obviously SDT rings. However, it is readily to verify that $\Delta(R) = (0)$ by direct computation and, of course, some elements of R, such as E_{12}, are *not* tripotent or even *not* n-potent for any natural number $n \geq 3$. Therefore, R is *not* an SDT ring, as suspected.

3 Triangular Matrix Rings

As usual, a ring R is termed *local*, provided $R/J(R)$ is a division ring, that is, each element in $R \setminus J(R)$ is a unit, which set-theoretically means that $R = J(R) \cup U(R)$.

We begin here with the following technicality.

Lemma 12. Let R be a local ring with $2 \in U(R)$. Then, R has only trivial tripotent elements.

Proof. Suppose that $e = e^3 \in R$. If $e \in J(R)$, then $e(1-e^2) = 0$, whence $e = 0$. If now $e \in U(R)$, then $e^2 = 1$, and so $(1-e)(1+e) = 0$. Since $(1-e) + (1+e) = 2 \in U(R)$ and R is a local ring, we have either $1 - e \in U(R)$ or $1 + e \in U(R)$. This, in turn, means that either $e = 1$ or $e = -1$, as required. □

Based on the above claim, we now considerably extend the well-known Workhorse Lemma (see [18; Lemma 6]) as follows.

Lemma 13. (Generalized Workhorse Lemma) Let R be a local ring such that $2 \in U(R)$, $n \geq 2$ and $A, E \in \mathrm{T}_n(R)$. Suppose that, for all $(i,j) \neq (1,n)$, $(E^3)_{ij} = E_{ij}$ and $(AE - EA)_{ij} = 0$. Suppose also that

$$A = \begin{pmatrix} a & \alpha & c \\ & B & \beta \\ & & b \end{pmatrix} \text{ and } E = \begin{pmatrix} e & \gamma & z \\ & F & \delta \\ & & f \end{pmatrix},$$

where $B, F \in \mathrm{T}_{n-2}(R)$, $a, b, c, e, f, z \in R$, $\alpha, \gamma \in \mathrm{M}_{1,n-2}(R)$ and $\beta, \delta \in \mathrm{M}_{n-2,1}(R)$. Then, the following items are fulfilled:

(i) Given $e = f = 1$, then $E^3 = E$ if and only if $z = -1/2(\gamma F\delta + 2\gamma\delta)$, and in this case, $AE = EA$.

(ii) Given $e = f = -1$, then $E^3 = E$ if and only if $z = -1/2(\gamma F\delta - 2\gamma\delta)$, and in this case, $AE = EA$.

(iii) Given $e = f = 0$, then $E^3 = E$ if and only if $z = \gamma F\delta$, and in this case, $AE = EA$.

(iv) Given $e = 1$ and $f = -1$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta + 2c$.

(v) If $e = -1$ and $f = 1$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta - 2c$.

(vi) If $e = 1$ and $f = 0$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta + c$.

(vii) If $e = 0$ and $f = 1$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta - c$.

(viii) If $e = -1$ and $f = 0$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta - c$.

(w) If $e = 0$ and $f = -1$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta + c$.

Proof. (i) It is apparent that $E^3 = E$ if and only if $z = -1/2(\gamma F\delta + 2\gamma\delta)$. We show that $AE = EA$. Given the assumptions, we have

$$z = -1/2(\gamma F\delta + 2\gamma\delta), \tag{1}$$
$$\gamma F = -\gamma F^2, \tag{2}$$
$$F\delta = -F^2\delta, \tag{3}$$
$$\alpha + \gamma B = a\gamma + \alpha F, \tag{4}$$
$$F\beta + \delta b = B\delta + \beta. \tag{5}$$

In virtue of the above equations, we compute that

$$\begin{aligned}(EA)_{1n} &= c + \gamma\beta + zb \stackrel{(1)}{=} c + \gamma\beta - 1/2\gamma F\delta b - \gamma\delta b \\ &\stackrel{(5)}{=} c + \gamma\beta + \gamma(F\beta - B\delta - \beta) + 1/2\gamma F(F\beta - B\delta - \beta) \\ &= c + \gamma\beta + \gamma F\beta - \gamma B\delta - \gamma\beta + 1/2\gamma F^2\beta - 1/2\gamma FB\delta - 1/2\gamma F\beta \\ &\stackrel{(2)}{=} c - \gamma B\delta - 1/2\gamma FB\delta.\end{aligned}$$

$$\begin{aligned}(AE)_{1n} &= az+\alpha\delta+c \overset{(1)}{=} -a\gamma\delta-1/2a\gamma F\delta+\alpha\delta+c\\ &\overset{(4)}{=} (\alpha F-\alpha-\gamma B)\delta+1/2(\alpha F-\alpha-\gamma B)F\delta+\alpha\delta+c\\ &= \alpha F\delta-\alpha\delta-\gamma B\delta+1/2\alpha F^2\delta-1/2\alpha F\delta-1/2\gamma BF\delta+\alpha\delta+c\\ &\overset{(3)}{=} c-\gamma B\delta-1/2\gamma BF\delta = c-\gamma B\delta-1/2\gamma FB\delta.\end{aligned}$$

Note that, since $(AE-EA)_{ij}=0$, we establish $FB=BF$.

(ii) The proof is similar to part (i).

(iii) It is obvious that $E^3=E$ if and only if $z=\gamma F\delta$. We show that $AE=EA$. Given the assumptions, we have

$$z=\gamma F\delta, \tag{6}$$
$$\gamma=\gamma F^2, \tag{7}$$
$$\delta=F^2\delta, \tag{8}$$
$$\gamma B=a\gamma+\alpha F, \tag{9}$$
$$B\delta=F\beta+\delta b. \tag{10}$$

From the above equations, we calculate that

$$\begin{aligned}(EA)_{1n} &= \gamma\beta+zb \overset{(6)}{=} \gamma\beta+\gamma F\delta b\\ &\overset{(10)}{=} \gamma\beta+\gamma F(B\delta-F\beta)\\ &= \gamma\beta+\gamma FB\delta-\gamma F^2\beta\\ &\overset{(7)}{=} \gamma FB\delta.\end{aligned}$$

$$\begin{aligned}(AE)_{1n} &= az+\alpha\delta \overset{(6)}{=} a\gamma F\delta+\alpha\delta\\ &\overset{(9)}{=} (\gamma\beta-\alpha F)F\delta+\alpha\delta\\ &= \gamma BF\delta-\alpha F^2\delta+\alpha\delta\\ &\overset{(8)}{=} \gamma BF\delta=\gamma FB\delta.\end{aligned}$$

(iv) Assume $e=1$ and $f=-1$. Then, under the assumptions, we deduce that

$$\gamma F=-\gamma F^2,\quad F^2\delta=F\delta \implies \gamma F\delta=-\gamma F^2\delta=-\gamma F\delta \implies 2\,\gamma F\delta=0.$$

But, since $2\in U(R)$, we have $\gamma F\delta=0$. Thus, we get $(E^3)_{1n}=\gamma F\delta+z=z=E_{1n}$ and, therefore, $E^3=E$. Moreover, it is clear that $EA=AE$ if and only if

$$az+\alpha\delta-c=c+\gamma\beta+zb,$$

which is equivalent to

$$az-zb=\gamma\beta-\alpha\delta+2c.$$

(v) The proof is similar to part (iv).

(vi) Assume $e = 1$ and $f = 0$. So, under the given assumptions, we have

$$\gamma F = -\gamma F^2, \quad \delta = F^2\delta \implies \gamma F\delta = -\gamma F^2\delta = -\gamma\delta.$$

Consequently, we derive $(E^3)_{1n} = z + \gamma\delta + \gamma F\delta = z = E_{1n}$, and hence $E^3 = E$. It is also readily checked that $EA = AE$ if and only if

$$az - zb = \gamma\beta - \alpha\delta + c.$$

Finally, one sees that points (vii), (viii) and (w) possess proofs which are similar to that of (vi). □

The next preliminary facts are worthy of discussion: let $a \in R$. The mappings $l_a : R \to R$ and $r_a : R \to R$ represent the (additive) abelian group endomorphisms defined respectively by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Consequently, the expression $l_a - r_b$ defines an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$. According to [5], a local ring R is classified as *bleached* if, for any $a \in U(R)$ and $b \in J(R)$, both $l_a - r_b$ and $l_b - r_a$ are surjective. The category of bleached local rings includes many well-established examples, such as commutative local rings, local rings with nil Jacobson radicals, and local rings in which some power of each element of their Jacobson radicals is central [18; Example 13].

Now, we need the following.

Lemma 14. Let R be a local ring such that $2 \in U(R)$, and suppose that $A \in \mathrm{T}_n(R)$. Write A as (a_{ij}). Then, for any set $\{e_{ii}\}_{i=1}^n$ of tripotents in R such that $e_{ii} = e_{jj}$ whenever $l_{a_{ii}} - r_{a_{jj}}$ is not a surjective abelian group endomorphism of R, there exists a tripotent $E \in \mathrm{T}_n(R)$ such that $AE = EA$ and $E_{ii} = e_{ii}$ for every $i \in \{1, \dots, n\}$.

Proof. Leveraging Lemma 13, the proof process mirrors that of [18; Lemma 7]. To avoid redundancy, we omit the detailed proof. □

We are now in a position to attack the main result in this section, in which the proof we shall apply the established above Theorem 1.

Theorem 2. Let R be a local ring and $n > 2$. Then, the following conditions are equivalent:
(1) $\mathrm{T}_n(R)$ is an SDT ring;
(2) either
(2.1) R is a bleached ring and $R/J(R) \cong \mathbb{Z}_2$;
or
(2.2) R is a bleached ring, $R/J(R) \cong \mathbb{Z}_3$ and, if $a, b \in R$ such that $a - 1 \in \Delta(R)$ and $b + 1 \in \Delta(R)$, then $l_a - r_b : R \to R$ is surjective.

Proof. Since R is a local ring, we have either $2 \in J(R)$ or $2 \in U(R)$. We prove the theorem for both cases independently.

Case 1: If $2 \in J(R)$.

(1) ⇒ (2.1). Since 2 belongs to $J(R)$, Theorem 1 discovers that $R/J(R)$ is a Boolean ring. But, since R is local, we must have $R/J(R) \cong \mathbb{Z}_2$. Because $\mathrm{T}_n(R)$ is an SDT ring, Corollary 4 gives that $\mathrm{T}_2(R)$ is an SDT ring too. Moreover, Proposition 4 allows us to detect that $T_2(R)$ is an SDI ring.

Suppose now $a \in U(R)$ and $b \in J(R)$. We intend to show that $l_a - r_b : R \to R$ is surjective. Thereby, it suffices to prove that, for every $v \in R$, there exists $x \in R$ such that $ax - xb = v$. Put $r := \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$. Assume $r = g + j$ is an SDI representation, where $g = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix}$ and $j = \begin{pmatrix} d & y \\ 0 & d' \end{pmatrix}$. Since e is an idempotent and $a \in U(R)$, we deduce $e = 1$. However, since f is an idempotent and

$b \in J(R)$, we derive $f = 0$. Thus, $g = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$. Since $rg = gr$, we now have $ax - xb = v$. Therefore, $l_a - r_b : R \to R$ is surjective. Similarly, we can show that $l_b - r_a : R \to R$ is surjective, as desired.

(2.1) $\Rightarrow$ (1). Since $2 \in J(R)$, we only have the case $R/J(R) \cong \mathbb{Z}_2$. Thus, by [8; Theorem 4.4], there is nothing left to prove.

Case 2: If $2 \in U(R)$.

(1) $\Rightarrow$ (2.2). Since 2 belongs to $U(R)$, Theorem 1 demonstrates that $R/J(R)$ is a Yaqub ring. But, since R is local, we must have $R/J(R) \cong \mathbb{Z}_3$. Because $\mathrm{T}_n(R)$ is an SDT ring, Corollary 4 gives that $\mathrm{T}_2(R)$ is an SDT ring too.

Suppose now $a \in U(R)$ and $b \in J(R)$. We intend to show that $l_a - r_b : R \to R$ is surjective. Thereby, it suffices to establish that, for each $v \in R$, there is $x \in R$ such that $ax - xb = v$. Set $r := \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$. Assume $r = g + j$ is an SDT representation, where $g = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix}$ and $j = \begin{pmatrix} d & y \\ 0 & d' \end{pmatrix}$. Since $b \in J(R)$ and f is a tripotent, we detect $f = 0$. On the other hand, Lemma 12 allows us to conclude that R has no non-trivial tripotents. Hence, since $a \in U(R)$, e is simultaneously a unit and a tripotent element, and thus either $e = 1$ or $e = -1$. If $g = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$, then since $rg = gr$, we have $ax - xb = v$. If, however, $g = \begin{pmatrix} -1 & x \\ 0 & 0 \end{pmatrix}$, then again since $rg = gr$, we have $a(-x) - (-x)b = v$. Consequently, $l_a - r_b : R \to R$ is surjective. Similarly, we can establish that $l_b - r_a : R \to R$ is surjective.

We now show that under the given assumptions, the SDT representation of elements is unique. In this light, suppose $e + d = f + b$ are two SDT representations in R. Note that, Lemma 12 manifestly yields $e, f \in \{-1, 0, 1\}$, so that one easily sees that either $e = f$ or $e = -f$. If $e = -f$, then $2e = b - d \in \Delta(R)$. Since $2 \in U(R)$, we have $e \in \Delta(R)$. Thus, $e^2 \in \Delta(R) \cap \mathrm{Id}(R) = \{0\}$, which leads to $e = 0$. Therefore, $e = f = 0$.

Suppose now that $a = 1 + d$ and $b = -1 + d'$ are two SDT representations. Assume that

$$r = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$$

is an element of $\mathrm{T}_2(R)$. Also, suppose that $r = g + w$ is an SDT representation, where

$$g = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} d & y \\ 0 & d' \end{pmatrix}.$$

Bearing in mind the above note, we can assume without loss of generality that $e = 1$ and $f = -1$. Since $gw = wg$ and $2 \in U(R)$, we deduce $a(1/2)x - (1/2)xb = v$. This obviously implies that the map $l_a - r_b : R \to R$ is surjective.

(2.1) $\Rightarrow$ (1). Suppose $A \in \mathrm{T}_n(R)$. We show that A has an SDT representation such that $A = E + D$ in $\mathrm{T}_n(R)$. Since $R/J(R) \cong \mathbb{Z}_3$, we see with no any technical difficulty that $R = J(R) \cup (1 + J(R)) \cup (-1 + J(R))$. First, we construct the elements on the main diagonal E. Suppose

$$e_{ii} := \begin{cases} 0 & \text{if } a_{ii} \in J(R), \\ 1 & \text{if } a_{ii} \in 1 + J(R), \\ -1 & \text{if } a_{ii} \in -1 + J(R). \end{cases}$$

Therefore, one inspects that $a_{ii} - e_{ii} \in J(R)$ for each i. Notice that, since $2 \in U(R)$, it must be that $(1 + J(R)) \cap (-1 + J(R)) = \emptyset$. If $e_{ii} \neq e_{jj}$, then we come to

$$\begin{cases} (1)\ e_{ii} \in U(R) \text{ and } e_{jj} \in J(R), \\ (2)\ e_{ii} \in J(R) \text{ and } e_{jj} \in U(R), \\ (3)\ e_{ii} \text{ and } e_{jj} \in U(R). \end{cases}$$

We prove that, in all three cases, $l_{a_{ii}} - r_{a_{jj}} : R \to R$ is necessarily surjective.

In fact, for case (1), $a_{ii} \in U(R)$ and $a_{jj} \in J(R)$ and, because R is bleached, $l_{a_{ii}} - r_{a_{jj}} : R \to R$ is indeed surjective.

The case (2) is observed to be similar to case (1).

In case (3), with no harm of generality, assuming $e_{ii} = 1$ and $e_{jj} = -1$, we obtain that $a_{ii} - 1$, $a_{jj} + 1 \in \Delta(R)$. Therefore, by the requested assumption, $l_{a_{ii}} - r_{a_{jj}} : R \to R$ is surjective. Hence, with Lemma 14 in hand, there is a tripotent $E \in T_n(R)$ such that $AE = EA$ and $E_{ii} = e_{ii}$ for each $i \in \{1, \dots, n\}$. In addition,

$$A - E \in J(T_n(R)) \subseteq \Delta(T_n(R)),$$

thus completing the proof. □

The case when $n = 2$ can be considered separately in the following manner.

Example 2. Suppose R is an integral domain and an SDT ring. Then, $\mathrm{T}_2(R)$ is an SDT ring.

Proof. Utilizing Proposition 2, R is a local ring. In the other vein, since R is a domain, arguing as in the proof of Proposition 2, we can assume that R has no non-trivial tripotents.

Since R is local, we have either $2 \in U(R)$ or $2 \in J(R)$. First, we assume that $2 \in J(R)$, and let $A = \begin{pmatrix} a & \beta \\ 0 & b \end{pmatrix} \in \mathrm{T}_2(R)$. Note that an SDT ring with $2 \in J(R)$ is always an SDI ring. We show that $\mathrm{T}_2(R)$ is also SDI. Precisely, we consider the following four cases:

1. If $a, b \in J(R)$, then $A \in J(R)$, so $A = 0 + A$ is an SDI representation.

2. If $a, b \in U(R)$, then since R is both SDI and local, we have $a - 1 \in J(R)$ and $b - 1 \in J(R)$. Therefore,

$$A = I_2 + \begin{pmatrix} a-1 & \beta \\ 0 & b-1 \end{pmatrix}$$

is an SDI representation for A.

3. $a \in U(R), b \in J(R)$. Since R is an SDI ring, we obtain $a - 1 \in J(R)$. Thus,

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a-1 & \beta - \alpha \\ 0 & b \end{pmatrix}$$

is an SDT representation, where $\alpha = \beta((a-1) + (1-b))^{-1}$.

4. $b \in U(R), a \in J(R)$. Since R is an SDI ring, we receive $b - 1 \in J(R)$. So,

$$A = \begin{pmatrix} 0 & \alpha \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & \beta - \alpha \\ 0 & b-1 \end{pmatrix}$$

is an SDT representation, where $\alpha = \beta((b-1) + (1-a))^{-1}$.

Now, suppose $2 \in U(R)$.

1. If $a, b \in J(R)$, then $A \in J(R)$, so $A = 0 + A$ is an SDT representation.

2. Given $a, b \in U(R)$. If the SDT representations of a and b are of the form $a = 1 + (a-1)$ and $b = 1 + (b-1)$, then

$$A = I_2 + \begin{pmatrix} a-1 & \beta \\ 0 & b-1 \end{pmatrix}$$

is an SDT representation for A.

If the SDT representations of a and b are of the form $a = -1 + (a+1)$ and $b = -1 + (b+1)$, then

$$A = -I_2 + \begin{pmatrix} a+1 & \beta \\ 0 & b+1 \end{pmatrix}$$

is an SDT representation for A.

If the SDT representations of a and b are of the form $a = -1 + (a + 1)$ and $b = 1 + (b - 1)$, then

$$A = \begin{pmatrix} -1 & \alpha \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a+1 & \beta - \alpha \\ 0 & b-1 \end{pmatrix}$$

is an SDT representation, where $\alpha = 2\beta(2 + (b - 1) - (a + 1))^{-1}$.

If the SDT representations of a and b are of the form $a = 1 + (a - 1)$ and $b = -1 + (b + 1)$, then

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} a-1 & \beta - \alpha \\ 0 & b+1 \end{pmatrix}$$

is an SDT representation, where $\alpha = 2\beta(2 + (a - 1) - (b + 1))^{-1}$. Note that $2 \in U(R)$ is assumed.

3. Given $a \in U(R)$ and $b \in J(R)$. If the SDT representation of a is of the form $a = 1 + (a - 1)$, then

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a-1 & \beta - \alpha \\ 0 & b \end{pmatrix}$$

is an SDT representation for A, where $\alpha = \beta((1 - b) - (1 - a))^{-1}$.

If the SDT representation of a is of the form $a = -1 + (a + 1)$, then

$$A = \begin{pmatrix} -1 & \alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a+1 & \beta - \alpha \\ 0 & b \end{pmatrix}$$

is an SDT representation for A, where $\alpha = \beta((1 + b) - (1 + a))^{-1}$.

4. Given $a \in J(R)$ and $b \in U(R)$. If the SDT representation of b is of the form $b = 1 + (b - 1)$, then

$$A = \begin{pmatrix} 0 & \alpha \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & \beta - \alpha \\ 0 & b-1 \end{pmatrix}$$

is an SDT representation for A, where $\alpha = \beta((b - 1) + (1 - a))^{-1}$.

If the SDT representation of b is of the form $b = -1 + (b + 1)$, then

$$A = \begin{pmatrix} 0 & \alpha \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} a & \beta - \alpha \\ 0 & b+1 \end{pmatrix}$$

is an SDT representation for A, where $\alpha = \beta((1 + a) - (1 + b))^{-1}$, as claimed. □

Now, we manage to examine the above stated example in a more general situation like the following one.

Proposition 7. Let R be a ring that has no non-trivial tripotent elements. Then, the following conditions are equivalent:

(1) $T(R, V)$ is an SDT ring.

(2) Either $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$.

Proof. (1) $\Rightarrow$ (2). If $T(R, V)$ is an SDT ring, it is easily verified that R is also an SDT ring. Moreover, since R has no non-trivial tripotent elements, as shown in Proposition 2, we can prove that R is a local ring. Therefore, according to a combination of the locality of R and Theorem 1, we conclude $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$.

(2) $\Rightarrow$ (1). If $R/J(R) \cong \mathbb{Z}_2$, then from [15; Theorem 15] we deduce that $T(R, V)$ is a uniquely clean ring. Thus, it is an SDI ring and, consequently, an SDT ring.

If, however, $R/J(R) \cong \mathbb{Z}_3$, we so derive

$$R = J(R) \cup (1 + J(R)) \cup (-1 + J(R)).$$

Assume now that $\begin{pmatrix} a & v \\ 0 & a \end{pmatrix} \in T(R, V)$ is fulfilled. So, we have:

(a) If $a \in J(R)$, then $\begin{pmatrix} a & v \\ 0 & a \end{pmatrix} \in J(T(R, V))$.

(b) If $a \in 1 + J(R)$, then

$$\begin{pmatrix} a & v \\ 0 & a \end{pmatrix} = I_2 + \begin{pmatrix} a-1 & v \\ 0 & a-1 \end{pmatrix},$$

which is an SDT representation.

(c) If $a \in -1 + J(R)$, then

$$\begin{pmatrix} a & v \\ 0 & a \end{pmatrix} = -I_2 + \begin{pmatrix} a+1 & v \\ 0 & a+1 \end{pmatrix},$$

which is an SDT representation, as claimed. □

We finish our examinations with the following exhibitions which we leave to the interested reader for a direct check.

Example 3. Let R be a ring in which all tripotent elements are central. Then, the following issues hold:

(1) R is an SDT ring if and only if $R[[x]]$ is an SDT ring.
(2) R is an SDT ring if and only if $R[x]/(x^n)$ is an SDT ring.
(3) R is an SDT ring if and only if $T(R, R)$ is an SDT ring.

Concluding Discussion and Questions

As above noticed, in [10] the authors defined and investigated those rings R, calling them *semi-tripotent*, whose elements are a sum of a tripotent element from R and an element from the Jacobson radical of R which, generally, need *not* commute each other.

Now, regarding Proposition 6, one may ask whether the classes of semi-tripotent rings and SDT rings are independent of each other; that is, does there exist an SDT ring what is *not* semi-tripotent as well as a semi-tripotent ring that is *not* SDT? However, it was proved in [10; Theorem 3.5 (6)] that $R/J(R)$ has the same presentation as in our Theorem 1 plus the requirement that all idempotents of R lift modulo $J(R)$. That is why, it quite surprisingly follows that *every SDI ring whose idempotent lift modulo the Jacobson radical is always semi-tripotent.* However, as the opposite claim of Theorem 1 is not at all guaranteed in order to be a satisfactory criterion, we do *not* know yet if any semi-tripotent ring is SDT. Likewise, due to the lifting restriction of the idempotents, the reciprocal implication *cannot* happen in all generality or, in other words, there is an SDT ring that is *not* semi-tripotent.

Our first intriguing query is related to the study in-depth of a generalized version of the SDT rings like this, which presents a more general setting of the *semi-n-potent rings* as defined in [10].

Problem 1. Describe those rings R, naming them *strongly* Δ *n-potent*, whose elements are a sum of a n-potent element in R (i.e., an element $a \in R$ such that $a^n = a$ for some $n \in \mathbb{N}$) and an element from $\Delta(R)$ that commute with each other.

On the other side, in conjunction with [19], we close our work with the following interesting question.

Problem 2. Characterize those rings R, calling them $C\Delta$ *rings*, whose elements are a sum of an element from the center $\mathrm{Z}(R)$ and from $\Delta(R)$.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Properties of semigroups of elementary types of model classes

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The study of classes of first-order countable language models and their properties is an important direction in model theory. Of particular interest are axiomatizable classes of models (varieties, quasivarieties, finitely axiomatizable classes, Jonssonian classes, etc.). In this paper we present the results obtained on the properties of formula-definable classes of models and formula-definable semigroups of elementary types, namely, we study the properties of semigroups of elementary types of models in a first-order language. We consider products of elementary types which form a commutative semigroup with unit. A two-place relation of absorption of one elementary type by another is introduced, which allows us to distinguish formula-definable semigroups of elementary types and corresponding classes of models. On the basis of the axiomatizability property of formula-definite semigroups of elementary types, their connection with ultraproducts and infinite products is established. Examples of idempotently formula-definite and non-idempotently formula-definite semigroups are given, and their peculiarities are discussed. The paper demonstrates both the study of semigroups of elementary types and the study of properties of formula-definite classes of models.

Keywords: idempotent, axiomatizable class, formula-definable semigroups, properties of semigroups, model companion, formula-definable model, elementary types of model classes, non-formula-definable model classes, countable signature.

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Introduction

On the set of elementary types of a countable signature σ of the first-order language L, the product of elementary types is considered. This forms a commutative semigroup with an identity element. Certain properties of subsemigroups of the semigroup of elementary types are established. Within this semigroup, a binary relation of absorption of one elementary type by another is studied. This allows for the identification of formula-definable semigroups of elementary types and formula-definable classes of models. Several properties of formula-definable semigroups of elementary types and formula-definable classes of models are proven.

1 *Definitions and preliminary results*

Let L be a language of countable signature σ of first order. For any model A of the language L, let $Th(A)$ denote the set of all sentences (closed formulas) of the language L that are true in the model A. The theory $Th(A)$ is called the elementary type of the model A.

An arbitrary (abstract) class of all models of the counting signature σ of the first-order language L is divided into classes by the relation of elementary equivalence of models (classification of A. Tarski [1,2]). This results in a set of classes (elementary types).

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The cardinality of the set Th_L, consisting of all elementary types of the countable signature σ in L, does not exceed 2^ω. In what follows, T denotes the elementary type of a model.

Historically, within the class of all models of the language L, greatest research interest has been focused on axiomatizable subclasses of models defined by certain theories: varieties, quasivarieties, finitely axiomatizable classes, Jonsson classes, etc.

The symbol $*$ indicates a known result, with references provided.

Theorem 1. * [1] Filtered and direct products of models preserve elementary equivalence.

Definition 1. The *product* of the elementary types T_1 of model A and T_2 of model B is defined as $T_1 \cdot\ T_2 = Th(A \times B)$, where $A \times B$ is the Cartesian product of models A and B. Analogously, infinite products of theories $\prod_{i\in I} T_i$, ultrapowers $\prod_{i\in I} T_i/D$, ultraproducts T^I/D with ultrafilters D, and filtered products of theories are defined.

Proposition 1. Definition 1 is well-defined.

Proof. It follows from Theorem 1*.

The algebra $\langle Th_L, \cdot\rangle$ is a commutative semigroup with an identity element.

If K is some class of models of the language L, then the set of elementary types of all models in the class K is denoted by Th_K and is called the set of elementary types of the class K. If H is a set of elementary types of theories in L, then K_H is the class of all models of all elementary types in H.

Examples of subsemigroups of $\langle Th_L, \cdot\rangle$ with specific properties are studied in various articles, books, and monographs: semigroups of elementary types of models of Horn classes, varieties, and quasivarieties [1, 2]. J. Wierzejewski [3] proved that the set of stable (superstable, ω-stable) elementary types of models forms a semigroup of stable (superstable, ω-stable) types under the product. M.V. Shvidefski [4] explored the complexity of the lattice of subsemigroups of the semigroup of elementary types. D.E. Palchunov [5] studied the semigroup of elementary types of Boolean algebras.

Later in the text, in the class of all models of the language L, a binary relation of one model *absorbing* another is defined.

Definition 2. A model A *absorbs* a model B, if $Th(A \times B) = Th(A)$, where $A \times B$ is the direct product of models.

Then the definitions are given and the following statements are obtained, which immediately follow from the corresponding theorems with the sign *.

Definition 3. We say that an elementary type T_2 *absorbs* an elementary type T_1, denoted $T_1 \leq T_2$, if $T_1 \cdot T_2 = T_2$. An elementary type T is called *idempotent*, if $T \cdot T = T$. A model A is called an *idempotent model*, if $Th(A \times A) = Th(A)$.

Model B *absorbs* model A, if $Th(A) \leq Th(B)$.

The absorption relation on the set Th_L is antisymmetric and transitive.

Definition 4. A set H of elementary types of models in the language L is called *an axiomatizable set* of elementary types if the class K_H, consisting of all models of all elementary types in H, forms an axiomatizable class.

Not every set of elementary types is axiomatizable.

The problem of axiomatizability of model classes is one of the central questions in model theory [1, 2, 5].

Theorem 2. * (J. Keisler [1]) A class of models is axiomatizable, if and only if it is closed under ultraproducts and elementary equivalence.

Proposition 2. A set H of elementary types is axiomatizable, if and only if H is closed under ultraproducts.

Proof. Follows from Theorem 2*.

Theorem 3. * [1] For any two sets of models $\{A_i \mid i \in I\}, \{B_i \mid i \in I\}$, and any ultrafilter D on I, the following holds:

$$\prod_{i\in I}(A_i \times B_i)/D \cong \prod_{i\in I} A_i/D \times \prod_{i\in I} B_i/D.$$

Proposition 3. For any two sets of elementary types $\{T_i \mid i \in I\}, \{T_i' \mid i \in I\}$, and any ultrafilter D on I, the following holds:

$$\prod_{i\in I}(T_i \cdot T_i')/D = \prod_{i\in I} T_i/D \times \prod_{i\in I} T_i'/D.$$

Proof. Follows from Definition 1 of the ultraproduct of elementary types and Theorem 3*.

Theorem 4. [1] For any model A and any ultrafilter D, the following holds: $A \equiv A^I/D$.

Proposition 4. For any elementary type T and any ultrafilter D, the following holds: $T = T^I/D$.

Proof. Follows from Definition 1 of the ultrapower of elementary types and Theorem 4*.

Theorem 5. * Let A, B, and C be models of the language L. If $A \times B \times C \equiv A$, then $A \times B \equiv A$ ($\equiv$ denotes elementary equivalence of models).

Proposition 5. Let T_1 , T_2 , T_3 be elementary types. If $T_1 \cdot T_2 \cdot T_3 = T3$, then $T_1 \cdot T_3 = T_3$.

Proof. Follows from Definition 1 of the product of elementary types and Theorem 5*.

The main focus of the above results is the transition from studying the properties of model classes to examining the properties of sets of elementary types of these classes. This enables consideration of the semigroup $\langle Th_L, \cdot \rangle$ and the properties of its subsemigroups. That is, it allows us to discover new properties of model classes using the direct product operation for models.

Studies on axiomatizable classes of models closed with respect to direct products are available in textbooks and articles of many authors. However, the problem of characterizations of axiomatizable classes closed with respect to direct products is still open [6–8].

2 Formula-definable semigroups of elementary types

This section presents results related to formula-definable semigroups of elementary types and formula-definable model classes.

Definition 5. [8, 9] A set of elementary types H of a signature is called *a formula-definable set* of elementary types if there exists an elementary type T such that for any elementary type T_1, holds $T_1 \in H$ if and only if $T_1 \cdot T = T$. In this case, the elementary type T is called the *determinant* of the set H. If the determinant of H is idempotent, H is called an *idempotent formula-definable set* of elementary types.

Definition 6. A class of models K is called *a formula-definable model* class if Th_K is a formula-definable set of elementary types. If Th_K is an idempotent formula-definable set of elementary types, K is called *an idempotent formula-definable model class.* The model of the determinant of the set Th_K is called the determinant of the class K [10].

Examples:

1. The class of models with a single equivalence relation is formula-definable. The determinant of this class of models is a model with an infinite number of equivalence classes, each of which is infinite [11].

2. The set of all ω-stable, the set of all superstable and the set of all stable elementary types, these sets are not formula-definite sets of elementary types.

Example 1 is fairly self-explanatory.

Explanation of Example 2:

From an example provided in [3], there exists an unstable elementary type T such that $T \cdot T$ is ω-stable. If the set of all ω-stable types were formula-definable, i.e., defined by some elementary type T_1, it would follow that $T \cdot T \cdot T_1 = T_1$. Then, by Proposition 5, $T \cdot T_1 = T_1$. Hence, the set of all ω-stable elementary types is not formula-definable. The same reasoning applies to the sets of superstable and stable types.

By analogy, this is true for the set of all superstable and the set of all stable elementary types.

Theorem 6. If a set of elementary types H is closed under direct products, then there exists an idempotent T such that for any $T_1 \in H$, holds $T_1 \cdot T = T$.

Proof. Since H is closed under infinite products and the cardinality of elementary types is at most 2^ω, there exists an elementary type T in H such that the product of all types in H equals elementary type T. Applying Proposition 5, we conclude that for any $T_1 \in H$, holds $T_1 \cdot T = T$. The type T is idempotent.

However, the idempotent T obtained in Theorem 6 may not necessarily serve as the determinant of H.

Thus, a set of elementary types closed under infinite products may not be an idempotent formula-definable set, even if it is an axiomatizable set. Examples of such sets of elementary types can be found among quasivarieties. We will provide such an example later.

Theorem 7. * (J. Keisler [1]) By any proposition φ one can efficiently find a number n such that for any index set I and any models A_i , $i \in I$, there exists a subset J in I that contains at most n elements, and for any V, $J \subseteq V \subseteq I$, $\prod_{i \in V} A_i \models \varphi$ if and only if $\prod_{i \in I} A_i \models \varphi$.

Theorem 8. A formula-definable set of elementary types H is closed under ultraproducts, finite, and infinite direct products of elementary types. That is, H is an axiomatizable set of elementary types, forms a commutative semigroup with an identity, and the formula-definable class K_H of models is an axiomatizable class.

Proof. Let $T_1, ..., T_n \in H$. By definition, H is a formula-definable set, so there exists a type T such that $T_i \cdot T = T$, $i \leq n$. Since the operation $\cdot$ is commutative and associative, $T_1 \cdot ... \cdot T_n \cdot T = T$, which implies $T_1 \cdot ... \cdot T_n \in H$. Thus, H is closed under finite products.

Let $\{T_i \mid i \in I, T_i \in H\}$ The equality $\prod_{i \in I} T_i \cdot T = T$ follows from the closedness with respect to finite products and Theorem 7 *. Therefore, H is closed under infinite products.

Let $\prod_{i \in I} T_i/D$ be an ultraproduct of elementary types with ultrafilter D, where $T_i \in H$ for $i \in I$. Using Propositions 3 and 4,

$$\prod_{i \in I} T_i/D \cdot T = \prod_{i \in I} T_i/D \cdot T^I/D = \prod_{i \in I} (T_i \cdot T)/D = T.$$

Hence, H is closed under ultraproducts, meaning H is an axiomatizable set of elementary types. Consequently, the formula-definable class of models is axiomatizable class of models.

Not every axiomatizable class of models is a formula-definable class. For instance, the axiomatizable class of fields is not a formula-definable class. If it were, the product of fields would have to be a field, which is not generally true.

Therefore, the set of formula-definable sets of elementary types is a proper subset of the set of all axiomatizable sets of elementary types.

A formula-definable set of elementary types forms a commutative semigroup with an identity, referred to as *a formula-definable semigroup of elementary types* [9]. Each elementary type T defines a formula-definable set of elementary types $GT = \{T_1 \mid T_1 \cdot T = T, T_1 \in Th_L\}$. This set GT is axiomatizable, and the class of models H_{GT} is formula-definable class of models.

Definition 7. If the determinant of a formula-definable semigroup of elementary types is idempotent, then such a semigroup is called *an idempotent formula-definable semigroup* of elementary types. The class of all models of all elementary types in this semigroup is called *an idempotent formula-definable model class.*

Not every determinant of a formula-definable semigroup is idempotent. For example, the elementary theory of a dense order without endpoints defines a formula-definable semigroup of theories but is not itself idempotent.

Theorem 9. A formula-definable semigroup G of elementary types is an idempotent formula-definable semigroup, and the class H_G of models of this semigroup is an idempotent formula-definable model class.

Proof. Since G is a formula-definable semigroup, by Theorem 8 it is closed under infinite products. By Theorem 6, there exists an idempotent $T \in G$ such that for any $T_1 \in G$, holds $T_1 \cdot T = T$. We now show that the idempotent T is the determinant of G. Since G is formula-definable, there exists a determinant T^G such that for any elementary type $T_1 \in Th_L$ holds $T_1 \in G$ if and only if $T_1 \cdot T^G = T^G$. If for some of elementary type $T' \in Th_L$ holds $T' \cdot T = T$, then $T' \cdot T \cdot T^G = T^G$.

By Proposition 5, $T' \cdot T^G = T^G$. Therefore, $T' \in G$, meaning G is an idempotent formula-definable semigroup, and H_G is an idempotent formula-definable model class.

Examples of formula-definable and non-formula-definable model classes.

An example of minimal quasivarieties from A.I. Maltsev's work [2]:

"Consider the signature with two predicate symbols P and Q. The quasivariety K, defined by the formulas $x = y$ and $P(x) \to Q(x)$, consists of three single-element models U_1, U_2, U_3, having respective diagrams:

$$D(U_1) = \{P(a), Q(a)\},\ D(U_2) = \{\neg P(a), \neg Q(a)\},\ D(U_3) = \{\neg P(a), Q(a)\}.$$

The model U_1 is unitary, the model U_2 is absolutely free. The pair U_1, U_2 forms a minimal quasivariety defined by the formulas

$$x = y,\ P(x) \to Q(x),\ Q(x) \to P(x),$$

while the pair U_1, U_3 forms a minimal quasivariety defined by the formulas $x = y, Q(x)$, and the quasivariety K itself is not minima".

In this example, we can see that the subquasivariety $\{U_1, U_2\}$ is not an idempotently formula-definable class, but the subquasivariety $\{U_1, U_3\}$ is an idempotently formula-definable class like the quasivariety K itself.

That is, we have examples of idempotently formula-definable semigroups of elementary types and not idempotently formula-definable semigroups of elementary types.

Each idempotent defines a unique idempotent formula-definable semigroup of theories. And to each idempotent formula-definable semigroup of elementary types corresponds a unique idempotent determinant of this semigroup. This semigroup is an axiomatizable set of theories by Theorem 8.

By analogy, this can be said of idempotently formula-definable model classes.

From the previous considerations we see that idempotent formula-definable semigroups of elementary types differ from semigroups in the classical sense in that they consider infinite products and Proposition 5 and the idempotent determinant for each idempotent formula-definable semigroup of elementary types plays the role of a zero element.

Among quasivarieties there are quasivarieties V which are not idempotently formula-definable classes of models. But:

Theorem 10. If K is a variety of models, then Th_K is an idempotent formula-definable semigroup of elementary types of class K. In other words, any variety of models is an idempotent formula-definable class.

Proof. This follows from Theorem 6 and the fact that a variety is defined by identities that are stable under direct products of models.

For example, the set of all elementary types of Boolean algebras, under the product operation, forms an idempotent formula-definable semigroup. We give examples of formula-definable model classes that are quasivarieties but are not varieties.

Theorem 11. * [2] The class of semigroups embeddable in groups forms a quasivariety.

Let V be the class of semigroups embeddable in groups. Then, the corresponding set Th_V, consisting of all elementary types of semigroups in V, forms a semigroup under the product of elementary types. Moreover, it is closed under infinite products.

Question: Is the quasivariety of semigroups embeddable in groups an idempotent formula-definable class?

It is known [2] that for commutative semigroups, the validity of the quasidentity of contraction in the semigroup

$$xy = xz \to y = z \tag{*}$$

is sufficient for embedding the semigroup into a group.

Thus, the class of commutative semigroups satisfying the quasidentity forms a quasivariety of semigroups embeddable in groups. Consequently, the set of all elementary types of such semigroups forms a semigroup of elementary types.

Theorem 12. Let K be the class of commutative semigroups (with reduction) satisfying the quasidentity (*), and let Th_K denote the set of all elementary types of semigroups in K. Then Th_K is an idempotent formula-definable semigroup, and K is an idempotent formula-definable class of semigroups embeddable in groups.

Proof. Since K, the class of commutative semigroups embeddable in groups, is a quasivariety, Th_K is closed under infinite products. By Theorem 6, there exists an idempotent T such that for any $T_1 \in Th_K$ holds $T_1 \cdot T = T$. But, since K is defined by the quasidentity (*), this identity is present in T due to the multiplicative stability of quasidentities under products. Additionally, in any semigroup true $xx = xx \to x = x$.

If T' is the elementary type of a commutative semigroup that does not satisfy (*), then $T' \cdot T \neq T$. Thus, elementary type T serves as the determinant of Th_K, making Th_K an idempotent formula-definable semigroup.

It is clear that in this case, the class K is a quasivariety that is not a variety. Moreover, since in a semigroup embeddable in a group there can exist only one idempotent, which is the identity, the semigroup Th_K itself is not embeddable in a group.

The following theorem gives a sufficient condition when the formula-definite class will be an inductive class, that is, closed with respect to the union of chains.

Theorem 13. A formula-definite class K of models will be an inductive class, when the determinant T of the semigroup of elementary types of this class is a $\forall\exists$-elementary type.

Proof. Let $M_1 \subseteq M_2 \subseteq ...$ be a chain of models in K. By Theorem 8, K is an axiomatizable class. Take any model A of the elementary type T and consider the chain $M_1 \times A \subseteq M_2 \times A \subseteq ....$ Take the union of this chain. Since T is $\forall\exists$-elementary type, the union of this chain is the model of T. Since K is a formula-definable, axiomatizable class of models, the union $M_1 \subseteq M_2 \subseteq ...$ is a model in K.

Theorem 14. If G_1, G_2 are formula-definable semigroups of elementary types, their intersection $G1 \cap G_2$ is also a formula-definable semigroup of elementary types.

Proof. The intersection $G_1 \cap G_2 \neq \oslash$, as G_1 and G_2 both contain the identity element. $G_1 \cap G_2$ is closed under infinite direct products of theories. By Theorem 8, there exists an idempotent T such that for any $T' \in G_1 \cap G_2$ holds $T' \cdot T = T$. It remains to show that the elementary type T is the determinant of the semigroup $G_1 \cap G_2$. Let T_1 and T_2 be determinants of the semigroups G_1 and G_2, respectively. If for some elementary type $T^C \in Th_L$, $T^C \cdot T = T$, then $T_1 \cdot T \cdot T^C = T_1$ and $T_2 \cdot T \cdot T^C = T_2$. By Proposition 5, $T_1 \cdot T^C = T_1$ and $T_2 \cdot T^C = T_2$. Thus, $T^C \in G_1 \cap G_2$, that is, $T \cdot T^C = T$, meaning T is the determinant of the semigroup $G_1 \cap G_2$.

This theorem allows us to construct, for any set of elementary types M, a minimal formula-definable semigroup G such that $M \subseteq G$, where the model class K_G is the minimal formula-definable class satisfying $K_M \subseteq K_G$. The class K_G is an axiomatizable class of models.

Conclusion

In this paper we investigated properties of semigroups of elementary types of models in a first-order language. The formula-definite semigroups of elementary types, their relation to axiomatizable classes of models and the role of idempotent elements in their structure are considered. The presented results emphasize the importance of studying semigroups of elementary types for analyzing properties of classes of models and reveal new approaches to their classification.

The revealed properties of formula-definite and idempotently formula-definite semigroups demonstrate the potential of using these structures to solve open questions in model theory, such as the problem of axiomatizability of classes closed with respect to products. The examples given in the paper illustrate the variety and complexity of such structures.

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Author Contributions

All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Well-posed problems for the Laplace-Beltrami operator on a stratified set consisting of punctured circles and segments

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The Laplace-Beltrami operator is studied on a stratified set consisting of two punctured circles and an interval. A complete description of all well-posed boundary value problems for the Laplace-Beltrami operator on such a set is given. In the second part of the paper, a class of self-adjoint well-posed problems for the Laplace-Beltrami operator on the specified stratified set is identified. The obtained results can be considered as a generalization of known results on geometric graphs. In particular, the stratified set under consideration can be interpreted as graphs with loops. Studies on the spectral asymptotics of Sturm-Liouville operators on plane curves homotopic to a finite interval are also closely related to the present results paper. Since the punctured circle is diffeomorphic to a finite interval, the spectral methods applied to differential operators on a finite interval can be modified to study the spectral properties of differential operators on the punctured circle. The main results of this paper are obtained by modifications of methods that were previously used in the study of the asymptotic behavior of the eigenvalues of the Sturm-Liouville operator on a finite interval.

Keywords: graph, Laplace-Beltrami operator, unique solution, punctured circle, inhomogeneous system of equations, differential operators, eigenvalue, inhomogeneous equation, local coordinate.

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1 *Stratified set* Ω *and functions over* Ω

We consider the stratified set Ω formed by two punctured circles C_1, C_2 and interval $l = (0,1)$ as well as two points A and B. In this case, Ω is a connected set (Fig. 1), the role of one-dimensional strata is played by C_1, C_2, l, and the role of zero-dimensional strata is played by single-point sets $\{A\}$ and $\{B\}$.

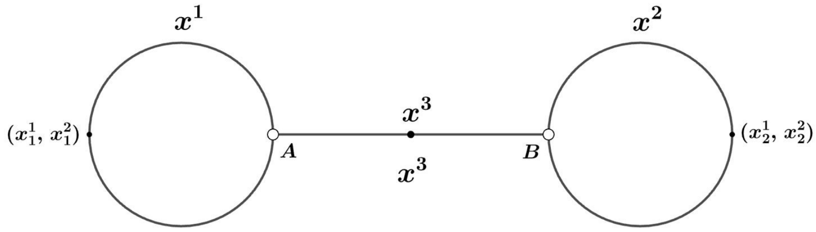

Figure 1. Stratified set Ω on the plane

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The facts given about the stratified set are sufficient for us; more general information about stratified sets can be found in the works [1, 2]. According to the work [3], a measure Ω is introduced on the set μ, as well as the corresponding function spaces. According to the specified work [4, 5] Ω is represented as a union of two non-intersecting parts: $\Omega_0 = C_1 \cup l \cup C_2$ and $\partial\Omega_0 = \{A, B\}$.

2 *Correctly solvable problems for the Laplace-Beltrami operator on a punctured circle C_1*

For convenience, we assume that the punctured circle C_1 is given by equation

$$C_1 = \{x_1 = (x_1^1, x_1^2) \in \mathbb{R}^{\nvDash} \backslash \{(0,0)\} : (x_1^2)^2 + (x_1^1 + 1)^2 = 1\}.$$

It is clear that the punctured circle C_1 can be defined using one card

$$\begin{cases} x_1^1 = \cos t_1 - 1, \\ x_1^2 = \sin t_1. \end{cases}$$

Moreover, the local coordinate t_1 runs through the interval $(0, 2\pi)$. In C_1, one can define classes of functions and the Laplace-Beltrami operator as was done in work [4]. In particular, the Laplace-Beltrami operator in this case represents the operator of twofold differentiation with respect to the variable t_1, if the function on C_1, is represented as a function on the interval $(0, 2\pi)$. If the function on C_1 is represented as a function of $x \in C_1$ then the value of the Laplace-Beltrami operator coincides with a two-fold tangent derivative. Since the Laplace-Beltrami operator is defined invariantly with respect to local coordinates, then when solving the corresponding equations, the equation can be solved in derivative local coordinates. Local coordinates can be chosen at one's discretion, and then the solution found in the chosen coordinates must be able to be written in other arbitrary local coordinates. From the above reasoning, it follows that the statement is true.

Theorem 1. For any numbers a, b and any function $f(x)$, defined on C_1 and belonging to $L_2(C_1)$ the inhomogeneous equation

$$(I - \Delta_{C_1})u(x) = f(x), \quad x \in C_1 \tag{1}$$

with conditions at the point $A(0, 0)$

$$U_0(u) = a_1, \quad U_1(u) = b_1 \tag{2}$$

has a unique solution $u(x) \in W_2^2(C_1)$.

Remark 1. If a point P on a circle precedes a point Q on the same circle, we briefly write $P \prec Q$. If points P and Q belong to the same oriented map, then the precedence of one point over another point of the same map is defined according to the orientation. Therefore, the notion of one-sided limit $\lim\limits_{\substack{P \to Q \\ P \prec Q}} f(P) = f(Q - 0)$ is correctly defined.

In Theorem 1, Δ_{C_1} denotes the Laplace-Beltrami operator on C_1. Here, in conditions (1), (2) there are linear functionals $U_0(\cdot)$, $U_1(\cdot)$, which are defined in the following way:

$$U_0(u) = \lim_{\substack{x \to A \\ x \succ A \\ x \in C_1}} u(x) - \lim_{\substack{x \to A \\ A \succ x \\ x \in C_1}} u(x),$$

$$U_1(u) = \lim_{\substack{x \to A \\ x \succ A \\ x \in C_1}} \frac{\partial u(x)}{\partial \tau} - \lim_{\substack{x \to A \\ A \succ x \\ x \in C_1}} \frac{\partial u(x)}{\partial \tau},$$

where $\frac{\partial u}{\partial \tau}$-means the derivative along the tangent to C_1 at point x. The proof of Theorem 1 can be found in the work of [4]. From Theorem 1 and from the results of M. Otelbaev [5–7] the following theorem follows.

Theorem 2. (i) For any function $f(x)$, defined on C_1 and belonging to $L_2(C_1)$ the inhomogeneous equation

$$(I-\Delta_{C_1})u(x)=f(x),\quad x\in C_1,$$

with conditions

$$U_0(u)=\int\limits_{C_1}(I-\Delta_{C_1})u(x)\overline{\sigma_0(x)}dl_x,\quad U_1(u)=\int\limits_{C_1}(I-\Delta_{C_1})u(x)\overline{\sigma_1(x)}dl_x, \tag{3}$$

has a unique solution $u(x)\in W_2^2(C_1)$, if $\sigma_0,\sigma_1\in L_2(C_1)$.

(ii) Let us assume that we add some conditions to the inhomogeneous operator equation (1) with conditions (2) so that equation (1) for all $f\in L_2(C_1)$ has a unique solution $u(x)\in W_2^2(C_1)$.

Then the added conditions are equivalent to conditions (3) for some $\sigma_0,\sigma_1\in L_2(C_1)$.

Proof. Proof of Theorem 2. The first part of Theorem 2 follows directly from Theorem 1 if

$$a_1=\int\limits_{C_1}f(x)\overline{\sigma_0(x)}dl_x,\quad b_1=\int\limits_{C_1}f(x)\overline{\sigma_1(x)}dl_x.$$

Now let us prove the second part of Theorem 2. By assumption, we add some conditions to equation (1) so that equation (1) for all $f\in L_2(C_1)$ has a unique solution $u(x)$, and

$$\|u(x)\|_{L_2(C_1)}\le M\|f(x)\|_{L_2(C_1)}, \tag{4}$$

where M does not depend on f.

So there is only one solution $u(x)$, subject to inequality (4). It follows from the embedding theorem that there exist values of linear functionals $U_0(u),U_1(u)$. It is easy to understand that linear functionals $U_0(\cdot),U_1(\cdot)$ according to inequality (4), are also functionals bounded in $L_2(C_1)$. Therefore, according to F. Riesz's theorem on the general form of a linear continuous functional in space $L_2(C_1)$ there exist functions $\sigma_0(x),\sigma_1(x)\in L_2(C_1)$ such that

$$U_0(u)=\int\limits_{C_1}f(x)\overline{\sigma_0(x)}dl_x,\quad U_1(u)=\int\limits_{C_1}f(x)\overline{\sigma_1(x)}dl_x.$$

Now it remains to replace $f(x)$ with $(I-\Delta_{C_1})u(x)$, from which the validity of the second part of Theorem 3 follows.

3 Well-solved problems for the Laplace-Beltrami operator on a stratified set Ω

In the previous paragraph we wrote out correctly solvable problems for the Laplace-Beltrami operator on a punctured circle C_1. In the same way, one can write out all possible correctly solvable linear problems for the Laplace-Beltrami operator on a punctured circle C_2. Note that correctly solvable linear problems for the operator of twofold differentiation on the interval $l=(0,1)$ are well known to [5–7]. Now, using the above results, we write out all possible correctly solvable linear problems for the Laplace-Beltrami operator on a stratified set Ω, consisting of C_1, C_2 and l. In this point, the punctured circle C_1 is defined as follows

$$C_1=\{x_1=(x_1^1,x_1^2)\in\mathbf{R^2}\backslash(\{0,0\}):(x_1^1+1)^2+(x_1^2)^2=1\},$$

where the role of local coordinates is played by the variable $t\in(0,2\pi)$:

$$\begin{cases}x_1^1=\cos t-1,\\ x_1^2=\sin t.\end{cases}$$

The punctured circle C_2 is defined as the following set

$$C_2 = \{x_2 = (x_2^1, x_2^2) \in \mathbf{R^2} \backslash \{(1,0)\} : (x_2^1 - 2)^2 + (x_2^2)^2 = 1\},$$

where the role of local coordinates is played by the variable τ:

$$x_2^1 = 2 + \cos\tau, \quad x_2^2 = \sin\tau, \quad \tau \in (\pi, 3\pi).$$

Interval l is defined as the horizontal open segment

$$l = \{x_3 = (x_3^1, x_3^2) \in \mathbf{R^2}: \ 0 < x_3^1 = S < 1, \ x_3^2 = 0\}.$$

Here the role of the local coordinate is played by the parameter S, which runs through the interval $(0,1)$. An analogue of Theorem 1 can be formulated for a punctured circle C_2 and interval l. As a result, we have the following statement.

Theorem 3. For any numbers $a_1, b_1, a_2, b_2, a_3, b_3$ and any functions $\vec{F} = \{f_1(x_1), f_2(x_2), f_3(s) \in L_2(\Omega)\}$ non-homogeneous system of equations

$$\begin{cases} (I - \Delta_{C_1})u_1(x_1) = f_1(x_1), \quad x_1 \in C_1, \\ (I - \Delta_{C_2})u_2(x_2) = f_2(x_2), \quad x_2 \in C_2, \\ u_3(s) - u_3''(s) = f_3(s), \quad s \in (0,1), \end{cases} \tag{5}$$

with conditions

$$\begin{gathered} U_0(u_1) = a_1, \quad U_1(u_1) = b_1, \\ V_0(u_2) = a_2, \quad V_1(u_2) = b_2, \\ u_3(0) = a_3, \quad u_3(1) = b_3 \end{gathered} \tag{6}$$

has a unique solution $u = (u_1, u_2, u_3) \in W_2^2(\Omega)$.

In Theorem 3 Δ_{C_2} denotes the Laplace-Beltrami operator on C_2. Also, linear forms determined by limiting ratios are designated by $V_0(\cdot)$ and $V_1(\cdot)$:

$$V_0(u_2) = \lim_{\substack{x \to B \\ x \succ B \\ x \in C_2}} u_2(x) - \lim_{\substack{x \to B \\ B \succ x \\ x \in C_2}} u_2(x),$$

$$V_1(u_2) = \lim_{\substack{x \to B \\ x \succ B \\ x \in C_2}} \frac{\partial u_2(x)}{\partial \tau} - \lim_{\substack{x \to B \\ B \succ x \\ x \in C_2}} \frac{\partial u_2(x)}{\partial \tau},$$

where $B = (1,0)$ and $\frac{\partial u}{\partial \tau}$-means the derivative along the tangent to C_2 at point x.

Similar results for graphs without loops were studied in [8]. This theorem can be interpreted as correctly solvable problems for the Laplace-Beltrami operator on graphs with loops. From Theorem 3 and the results [5–7] of the assertion follows.

Theorem 4. (i) For any function $\vec{F} = \{f_1, f_2, f_3\} \in L_2(\Omega)$ inhomogeneous system of equations (5) with conditions

$$\begin{cases} U_0(u_1) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1) \overline{\sigma_1(x_1)} dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2) \overline{\rho_1(x_2)} dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s)) \overline{\varphi_1(s)} ds, \\ U_1(u_1) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1) \overline{\sigma_2(x_1)} dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2) \overline{\rho_2(x_2)} dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s)) \overline{\varphi_2(s)} ds, \\ V_0(u_2) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1) \overline{\sigma_3(x_1)} dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2) \overline{\rho_3(x_2)} dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s)) \overline{\varphi_3(s)} ds, \\ V_1(u_2) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1) \overline{\sigma_4(x_1)} dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2) \overline{\rho_4(x_2)} dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s)) \overline{\varphi_4(s)} ds, \\ u_3(0) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1) \overline{\sigma_5(x_1)} dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2) \overline{\rho_5(x_2)} dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s)) \overline{\varphi_5(s)} ds, \\ u_3(1) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1) \overline{\sigma_6(x_1)} dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2) \overline{\rho_6(x_2)} dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s)) \overline{\varphi_6(s)} ds, \end{cases} \tag{7}$$

has a unique solution $u = \{u_1, u_2, u_3 \in W_2^2(\Omega)\}$, if

$$\sigma_j \in L_2(C_1), \quad \rho_j \in L_2(C_2), \quad \varphi_j \in L_2(0,1), \quad j = 1,2,3,4,5,6.$$

(ii) Let us assume that we add some conditions to the inhomogeneous system of equations (5) with conditions (6) so that equation (5) for all $\vec{F} = \{f_1, f_2, f_3\} \in L_2(\Omega)$ has a unique solution $u = (u_1, u_2, u_3) \in W_2^2(\Omega))$. Then the added conditions are equivalent to conditions of the form (7) for some

$$\sigma_j \in L_2(C_1), \quad \rho_j \in L_2(C_2), \quad \varphi_j \in L_2(0,1), \quad j = 1,2,3,4,5,6.$$

The proof of Theorem 4 repeats the proof of Theorem 2, only the theorem of F. Riesz is used, which concerns the Hilbert space $L_2(\Omega)$.

The formulation of correct boundary value problems for the Laplace operator in a punctured ball was discussed in the works [9–11]. A description of all possible well-defined problems for the Laplace-Beltrami operator on a punctured sphere can be found [12–14]. Everywhere correctly solvable problems for differential operators in punctured domains or in domains with cuts can be interpreted as singular perturbations of regular differential operators. From this point of view, singular differential operators are studied in the works [15–17], differential operators for the Dirichlet and Neumann problems are studied in the works [18, 19].

4 Examples of well-posed problems on a stratified set

In this section we will give specific examples that follow from the first part of Theorem 4. Let us recall Lemma 1 from work [4].

Lemma 1. [4] For any smooth 2π-periodic function $\hat{F}(t)$ the integral identity is valid

$$\int\limits_0^t \hat{F}(t) dt = \int\limits_{\gamma_x^1} F(\xi^1, \xi^2)(\xi^1 d\xi^2 - \xi^2 d\xi^1),$$

where γ_x^1 positively oriented arc of a punctured circle C_1^1, connecting the dots $(0,0)$ and $x = (x^1, x^2) \in C_1$.

Here the function $F(x)$ for $x \in C_1$ is generated by the function $\hat{F}(t)$ for $t \in (0, 2\pi)$ as follows: first, we expand $\hat{F}(t)$ into a trigonometric series

$$\hat{F}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt), \quad t \in (0, 2\pi)$$

and then according to the formulas $x^1+1 = \cos t$, $x^2 = \sin t$ we move from t to variables $(x^1, x^2) = x \in C_1$

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k T_k(x^1 + 1) + b_k x^2 U_{k-1}(x^1 + 1)),$$

where T_k and U_{k-1} Chebyshev polynomials of the first and second kind, respectively. Similarly, the integral $\int\limits_0^{\tau} \hat{\sigma}(\tau)d\tau$ at $\tau \in (\pi, 3\pi)$ we can rewrite it through the integral

$$\int_{\gamma_x^2} \sigma(\xi^1, \xi^2)(\xi^1 d\xi^2 - \xi^2 d\xi^1),$$

where γ_x^2 positively oriented arc pierced circle C_2, connecting points $(-1, 0)$ and $x = (x^1, x^2) \in C_2$. Here also $\sigma(x)$ for $x \in C_2$ is generated by the function $\hat{\sigma}(\tau)$ for $\tau \in (\pi, 3\pi)$ as follows:

First, we expand $\hat{\sigma}(\tau)$ for $\tau \in (\pi, 3\pi)$ into a trigonometric series

$$\hat{\sigma}(\tau) = \frac{c_0}{2} + \sum_{k=1}^{\infty}(c_k \cos k\tau + d_k \sin k\tau), \quad \tau \in (\pi, 3\pi),$$

and then according to the formulas $x^1 - 2 = \cos\tau$, $x^2 = \sin\tau$ we move from the parameter τ to the variables $(x^1, x^2) = x \in C_2$

$$\sigma(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty}(c_k T_k(x^1 - 2) + d_k x^2 U_{k-1}(x^1 - 2)).$$

In conditions (7) the integrals $\int\limits_{C_1} f_1(x_1)\overline{\sigma_1 x_1} dl_1$ and $\int\limits_{C_2} f_2(x_2)\overline{\sigma_2 x_2} dl_2$. These integrals can be rewritten in terms of local coordinates t and τ, respectively:

$$\int\limits_{C_1} f_1(x_1)\overline{\sigma_1(x_1)}dl_1 = \int\limits_0^{2\pi} f_1(\cos t - 1, \sin t)\overline{\sigma_1(\cos t - 1, \sin t)}dt = \int\limits_0^{2\pi} \hat{f}_1(t)\overline{\hat{\sigma}_1(t)}dt,$$

$$\int\limits_{C_2} f_2(x_2)\overline{\sigma_2(x_2)}dl_2 = \int\limits_{\pi}^{3\pi} f_2(2 + \cos\tau, \sin\tau)\overline{\sigma_2(2 + \cos\tau, \sin\tau)}d\tau = \int\limits_{\pi}^{3\pi} \hat{f}_2(\tau)\overline{\hat{\sigma}_2(\tau)}d\tau.$$

Now we are ready to rewrite the integral $\int\limits_{C_1} (I - \Delta_{C_1})u_1(x_1)\overline{\sigma_1(x_1)}dl_1$ in a form convenient for us

$$\int\limits_{C_1}(I - \Delta_{C_1})u_1(x_1)\overline{\sigma_1(x_1)}dl_1 = \int\limits_{C_1} f_1(x_1)\overline{\sigma_1(x_1)}dl_1 = \int\limits_0^{2\pi} \hat{f}_1(t)\overline{\hat{\sigma}_1(t)}dt = \int\limits_0^{2\pi} \Big(\hat{u}_1(t) - \hat{u}_1''(t)\Big)\overline{\hat{\sigma}_1(t)}dt.$$

We apply the integration by parts to the last integral, assuming that $\hat{\sigma}_1(t)$ is twice continuously differentiable function. As a result, we have

$$\int_0^{2\pi}\Big(\hat{u}_1(t)-\hat{u}_1''(t)\Big)\overline{\hat{\sigma}_1(t)}dt=\int_0^{2\pi}\hat{u}_1(t)\overline{\Big(\hat{\sigma}_1(t)-\hat{\sigma}_1''(t)\Big)}dt-\hat{u}_1'(t)\overline{\hat{\sigma}_1(t)}\Big|_{t=0}^{t=2\pi}+\hat{u}_1(t)\overline{\hat{\sigma}_1'(t)}\Big|_{t=0}^{t=2\pi}=$$

$$=\int_0^{2\pi}\hat{u}_1(t)\overline{\Big(\hat{\sigma}_1(t)-\hat{\sigma}_1''(t)\Big)}dt-\hat{u}'(2\pi-0)\overline{\hat{\sigma}_1(2\pi-0)}+\hat{u}_1(2\pi-0)\overline{\hat{\sigma}'(2\pi-0)}+\hat{u}_1'(+0)\overline{\hat{\sigma}_1(+0)}-\hat{u}_1(+0)\overline{\hat{\sigma}_1'(+0)}.$$

Now, as a result of the change of variables from the local coordinate t to the variables $(x_1^1,x_1^2)=x$, we have

$$\int_{C_1}(I-\Delta_{C_1})u_1(x_1)\overline{\sigma_1(x_1)}dl_1=\int_{C_1}u_1(x_1)\overline{(I-\Delta_{C_1})\sigma_1(x_1)}dl_1--\frac{\partial u_1}{\partial\tau}(\prec(0,0)\overline{\sigma_1(\prec(0,0))}+$$

$$+u_1(\prec(0,0))\overline{\frac{\partial}{\partial\tau}\sigma_1(\prec(0,0))}+\frac{\partial u_1(\prec(0,0))}{\partial\tau}\sigma_1(\prec(0,0))-u_1(\prec(0,0))\frac{\partial\sigma_1}{\partial\tau}(\prec(0,0)),\qquad(8)$$

where

$$g(\prec(0,0))=\lim_{\substack{x_1\to(0,0)\\x_1\prec(0,0)\\x_1\in C_1}}g(x_1),\quad g(\succ(0,0))=\lim_{\substack{x_1\to(0,0)\\x_1\succ(0,0)\\x_1\in C_1}}g(x_1),$$

$$\frac{\partial g(\prec(0,0))}{\partial\tau}=\lim_{\substack{x_1\to(0,0)\\x_1\prec(0,0)\\x_1\in C_1}}\frac{\partial g(x_1)}{\partial\tau},\quad\frac{\partial g(\succ(0,0))}{\partial\tau}=\lim_{\substack{x_1\to(0,0)\\x_1\succ(0,0)\\x_1\in C_1}}\frac{\partial g(x_1)}{\partial\tau},$$

where $\frac{\partial}{\partial\tau}$ is derivative along the tangent to C_1 at point x_1. In the same way, for any two sufficiently smooth C_2 functions on $u_2(x_2),\ \rho_2(x_2)$ the following identity holds

$$\int_{C_2}(I-\Delta_{C_2})u_2(x_2)\overline{\rho_2(x_2)}dl_2=\int_{C_2}u_2(x_2)\overline{(I-\Delta_{C_2})\rho_2(x_2)}dl_2-$$

$$-\frac{\partial u_2(\prec(-1,0))}{\partial\tau}\overline{\rho_2\prec(-1,0))}+u_2(\prec(-1,0))\overline{\frac{\partial\rho_2(\prec(-1,0))}{\partial\tau}}+\qquad(9)$$

$$+\frac{\partial u_2(\succ(-1,0))}{\partial\tau}\overline{\rho_2(\succ(-1,0))}-u_2(\succ(-1,0))\overline{\frac{\partial\rho_2(\succ(-1,0))}{\partial\tau}},$$

where $\frac{\partial}{\partial\tau}$ is derivative along the tangent to C_2 at the point x_2. The given auxiliary statements allow us to obtain consequences of Theorem 4. Now we will specify the choice of boundary functions $\sigma_j(x_1),\ \rho_j(x_2),\ \varphi_j(x_3)$ for $j=1,2,3,4,5,6$ from Theorem 4. Let for $j=1,2,3,4,5,6$ the functions $\sigma_j(x_1),\ \rho_j(x_2),\ \varphi_j(x_3)$ be chosen so that

$$(I-\Delta_{C_1})\sigma_j(x_1)=0,\quad x_1\in C_1,$$

$$(I-\Delta_{C_2})\rho_j(x_2)=0,\quad x_2\in C_2,$$

$$\varphi_j(s)-\varphi_j''(s)=0,\quad s\in(0,1).$$

Then, from relations (8) and (9) we have

$$\int\limits_{C_1}(I-\Delta_{C_1})u_1(x_1)\overline{\sigma_j(x_1)}dl_1=-\frac{\partial u_1(\prec(0,0))}{\partial\tau}\overline{\sigma_j(\prec(0,0))}+$$

$$+u_1\Big(\prec(0,0)\Big)\overline{\frac{\partial\sigma_j(\prec(0,0))}{\partial\tau}}+\frac{\partial u_1(\succ(0,0))}{\partial\tau}\overline{\sigma_j(\succ(0,0))}-u_1(\succ(0,0))\overline{\frac{\partial\sigma_j(\succ(0,0))}{\partial\tau}},$$

$$\int\limits_{C_2}(I-\Delta_{C_2})u_2(x_2)\overline{\rho_j(x_2)}dl_1=-\frac{\partial u_2(\prec(-1,0))}{\partial\tau}\overline{\rho_j(\prec(-1,0))}+$$

$$+u_2\Big(\prec(-1,0)\Big)\overline{\frac{\partial\rho_j(\prec(-1,0))}{\partial\tau}}+\frac{\partial u_2(\succ(-1,0))}{\partial\tau}\overline{\rho_j(\succ(-1,0))}-u_2(\succ(-1,0))\overline{\frac{\partial\rho_j(\succ(-1,0))}{\partial\tau}},$$

$$\int\limits_0^1(u_3(s)-u_3''(s))\overline{\varphi_j(s)}ds=-\frac{du_3(1-0)}{ds}\overline{\varphi_j(1-0)}+u_3(1-0)\frac{d\varphi_j(1-0)}{ds}+$$

$$\frac{du_3(+0)}{ds}\overline{\varphi_j(+0)}-u_3(+0)\frac{d\varphi_j(+0)}{ds}.$$

Thus, the boundary conditions (7) from Theorem 4 take the form for $j=1,2,3,4,5,6$

$$\begin{aligned}U_j=\overline{\hat\sigma_j(+0)}\big[\hat u_1'(+0)-\cosh 2\pi\,\hat u_1'(2\pi-0)+\sinh 2\pi\,\hat u_1'(2\pi-0)\big]+\\+\overline{\hat\sigma_j{}'(+0)}\big[\cosh 2\pi\hat u_1'(2\pi-0)-\sinh 2\pi\,\hat u_1'(2\pi-0)-\hat u_1'(+0)\big]+\\+\overline{\hat\rho_j(\pi+0)}\big[\hat u_2'(\pi+0)-\cosh 2\pi\,\hat u_2'(3\pi-0)+\sinh 2\pi\,\hat u_2'(3\pi-0)\big]+\\+\overline{\hat\rho_j{}'(\pi+0)}\big[\cosh 2\pi\hat u_2'(3\pi-0)-\sinh 2\pi\,\hat u_1'(3\pi-0)-\hat u_2'(\pi+0)\big]+\\+\overline{\varphi_j(+0)}\big[u_3'(+0)+\frac{\cosh 1}{\sinh 1}u_3(0)-\frac{1}{\sinh 1}u_3(1-0)\big]+\\+\overline{\varphi_j(1-0)}\big[\frac{\cosh 1}{\sinh 1}u_3(1-0)-u_3'(1-0)-\frac{1}{\sinh 1}u_3(+0)\big],\end{aligned}\tag{10}$$

where

$$U_1(u_1)=\hat u_1(+0)-\hat u_1(2\pi-0),\quad U_2(u_1)=\hat u_1'(+0)-\hat u_1'(2\pi-0),$$
$$U_3(u_2)=\hat u_2(\pi+0)-\hat u_2(3\pi-0),\quad U_4(u_2)=\hat u_2'(\pi+0)-\hat u_2'(3\pi-0),$$
$$U_5(u_3)=u_3(+0),\quad U_6(u_3)=u_3(1-0).$$

5 *Self-adjoint well-solved problems*

In the previous paragraph, examples of correctly solvable problems that are set using boundary conditions. Now we will select from them those problems that are self-adjoint. Correctly-solvable problems correspond to operators whose resolvent sets contain $\lambda=0$. At the same time, self-adjoint well-solvable problems correspond to operators whose eigenvalues provide nonzero real numbers. Thus, in this section, such well-solvable problems are distinguished whose spectrum is discrete and consists of nonzero real eigenvalues. Recall that for any two sufficiently smooth functions $u_1(x_1),\ u_2(x_2),\ u_3(s)$ and $\vartheta_1(x_1),\ \vartheta_2(x_2),\ \vartheta_3(s)$ the identity holds

$$\int\limits_{C_1}(I-\Delta_{C_1})u_1(x_1)\overline{\vartheta_1(x_1)}dl_1+\int\limits_{C_2}(I-\Delta_{C_2})u_2(x_2)\overline{\vartheta_2(x_2)}dl_2+\int\limits_0^1(u_3(s)-u_3''(s))\overline{\vartheta_3(s)}ds=$$

$$= \int\limits_{C_1} u_1(x_1)\overline{(I-\Delta_{C_1})\vartheta_1(x_1)}dl_1 + \int\limits_{C_2} u_2(x_2)\overline{(I-\Delta_{C_2})\vartheta_2(x_2)}dl_2 + \int\limits_0^1 u_3(s)\overline{(\vartheta_3(s)-\vartheta_3''(s))}ds+$$

$$\begin{aligned} &+(\hat{u}_1'(+0)-\hat{u}_1'(2\pi-0))\overline{\hat{\vartheta}_1(2\pi-0)}+\hat{u}_1'(+0)\overline{(\hat{\vartheta}_1(+0)-\hat{\vartheta}_1(2\pi-0))}-\\ &-(\hat{u}_1(+0)-\hat{u}_1(2\pi-0))\overline{\hat{\vartheta}_1'(2\pi-0)}+\hat{u}_1(+0)\overline{(\hat{\vartheta}_1'(2\pi-0)-\hat{\vartheta}_1'(+0))}+\\ &+(\hat{u}_2'(\pi+0)-\hat{u}_2'(3\pi-0))\overline{\hat{\vartheta}_2(3\pi-0)}+\hat{u}_2'(\pi+0)\overline{(\hat{\vartheta}_2(\pi+0)-\hat{\vartheta}_2(3\pi-0))}-\\ &-(\hat{u}_2(\pi+0)-\hat{u}_2(3\pi-0))\overline{\hat{\vartheta}_2'(3\pi-0)}+\hat{u}_2(\pi+0)\overline{(\hat{\vartheta}_2'(3\pi-0)-\hat{\vartheta}_2'(\pi+0))}-\\ &-u_3'(1-0)\overline{\vartheta_3(1-0)}+u_3'(+0)\overline{\vartheta_3(+0)}+u_3(1-0)\overline{\vartheta_3'(1-0)}-u_3(+0)\overline{\vartheta_3'(+0)}, \end{aligned} \tag{11}$$

where $\hat{u}_1(t)=u_1(\cos t-1,\sin t)$ for $t\in(0,2\pi)$, $\hat{u}_2(\tau)=u_2(2+\cos\tau,\sin\tau)$ for $\tau\in(\pi,3\pi)$.

Let D denote the set of functions $u_1(x_1),\ u_2(x_2),\ u_3(s)$ such that
(I) $u_1(x_1)\in W_2^2(C_1),\ u_2(x_2)\in W_2^2(C_2),\ u_3(s)\in W_2^2(0,1)$.

Let us also introduce the set D_0, consisting of functions $u_1(x_1),\ u_2(x_2),\ u_3(s)\in D$ such that
(II) $\hat{u}_1'(+0)=\hat{u}_1'(2\pi-0),\ \hat{u}_1'(+0)=0,\ \hat{u}_1(+0)=\hat{u}_1(2\pi-0), \hat{u}_1(+0)=0,$
$\hat{u}_2'(\pi+0)=\hat{u}_2'(3\pi-0),\ \hat{u}_2'(\pi+0)=0,\ \hat{u}_2(\pi+0)=\hat{u}_2(3\pi-0), \hat{u}_2(\pi+0)=0,$
$u_3(+0)=0,\ u_3'(+0)=0,\ u_3(1-0)=0,\ u_3'(1-0)=0.$

Let us introduce the operator L on D using the formula

$$L=(u_1(x_1),u_2(x_2),u_3(s))=((I-\Delta_{C_1})u_1(x_1),(I-\Delta_{C_2})u_2(x_2),(u_3(s)-u_s''(s))).$$

Let us denote by L_0 the restriction of the operator L on D_0. The operator L_0 is Hermitian and following the scheme from § 17 of the monograph [20] we write all possible self-adjoint extensions of the operator L_0. To do this, we need some properties of the operator L_0.

Lemma 2. Let $(f_1(x_1),f_2(x_2),f_3(s))\in L_2(\Omega)$. Equation

$$L_0=(u_1(x_1),u_2(x_2),u_3(s))=(f_1(x_1),f_2(x_2),f_3(s))$$

has a solution if and only if $(f_1(x_1),f_2(x_2),f_3(s))$ orthogonal to all solutions of the homogeneous system

$$(I-\Delta_{C_1})\omega_1(x_1)=0,\quad (I-\Delta_{C_2})\omega_2(x_2)=0,\quad \omega_3(s)-\omega_3''(s)=0. \tag{12}$$

Proof. Let us denote by $(u_1(x_1),u_2(x_2),u_3(s))$ the solution of the system

$$(I-\Delta_{C_1})u_1(x_1)=f_1(x_1),\ (I-\Delta_{C_2})u_2(x_2)=f_2(x_2),\ u_3(s)-u_3''(s)=f_3(s),$$

satisfying the condition

$$\hat{u}_1'(+0)=\hat{u}_1'(2\pi-0),\ \hat{u}_1(+0)=\hat{u}_1(2\pi-0),$$
$$\hat{u}_2'(\pi+0)=\hat{u}_2'(3\pi-0),\ \hat{u}_2(\pi+0)=\hat{u}_2(3\pi-0),\ u_3(+0)=0,u_3(1-0)=0.$$

From the results of the work [4] it follows that there is a unique solution $(u_1(x_1),u_2(x_2),u_3(s))$ to the indicated problem. In the work [4] the eigenvalues of the given problem are calculated and it is shown that there is no zero among the eigenvalues. For the found solution $(u_1(x_1),u_2(x_2),u_3(s))$ identity (11) will take the form

$$\int\limits_{C_1} f_1(x_1)\overline{\vartheta_1(x_1)}dl_1+\int\limits_{C_2} f_2(x_2)\overline{\vartheta_2(x_2)}dl_2+\int\limits_0^1 f_3(s)\overline{\vartheta_3(s)}ds=$$

$$= \int\limits_{C_1} u_1(x_1)\overline{(I-\Delta_{C_1})\vartheta_1(x_1)}dl_1 + \int\limits_{C_2} u_2(x_2)\overline{(I-\Delta_{C_2})\vartheta_2(x_2)}dl_2 + \int\limits_0^1 u_3(s)(\overline{\vartheta_3(s)-\vartheta_3''(s)})ds+$$

$$+\hat{u}_1'(+0)\overline{\hat{\vartheta}_1(+0)-\hat{\vartheta}_1(2\pi-0))} + \hat{u}_1(+0)\overline{(\hat{\vartheta}_1'(2\pi-0)-\hat{\vartheta}_1'(+0))}+ \tag{13}$$

$$+\hat{u}_2'(\pi+0)\overline{(\hat{\vartheta}_2(\pi+0)-\hat{\vartheta}_2(3\pi-0))} + +\hat{u}_2(\pi+0)\overline{(\hat{\vartheta}_2'(3\pi-0)-\hat{\vartheta}_2'(\pi-0))}-$$

$$-\hat{u}_3'(1-0)\overline{\vartheta_3(1-0)} + \hat{u}_3'(+0)\overline{(\vartheta_3)(+0)}.$$

Now let's choose $V_1 = (\vartheta_{11}(x_1), \vartheta_{12}(x_2), \vartheta_{13}(s))$ so that the homogeneous equations (12) and additional conditions are satisfied

$$\hat{\vartheta}_{11}(+0) - \hat{\vartheta}_{11}(2\pi-0) = 1, \;\; \hat{\vartheta}_{11}'(2\pi-0) - \hat{\vartheta}_{11}'(+0) = 0,$$

$$\hat{\vartheta}_{12}(\pi+0) - \hat{\vartheta}_{12}(3\pi-0) = 0, \;\; \hat{\vartheta}_{12}'(3\pi-0) - \hat{\vartheta}_{12}'(\pi+0) = 0,$$

$$\vartheta_{13}(1-0) = 0, \;\; \vartheta_{13}(+0) = 0.$$

In fact, $\vartheta_{13}(s) \equiv 0, \;\vartheta_{12}(x_2) \equiv 0, \;\hat{\vartheta}_{11}(t) = \frac{e^{2\pi-t}-e^t}{2(e^{2\pi}-1)}$. In this case, from relation (13) it follows

$$\int\limits_{C_1} f_1(x_1)\overline{\vartheta_{11}(x_1)}dl_1 = \hat{u}_1'(+0). \tag{14}$$

By choosing $V_2 = (V_{21}(x_1), V_{22}(x_2), V_{23}(s))$ in a reasonable way, we can obtain the relation

$$\int\limits_{C_1} f_1(x_1)\overline{\vartheta_{21}(x_1)}dl_1 = \hat{u}_1(+0). \tag{15}$$

Reasoning in the same way as in the monograph [20], we obtain the relations

$$\int\limits_{C_2} f_2(x_2)\overline{\vartheta_{32}(x_2)}dl_2 = \hat{u}_2'(+0), \tag{16}$$

$$\int\limits_{C_2} f_2(x_2)\overline{\vartheta_{42}(x_2)}dl_2 = \hat{u}_2(\pi+0), \tag{17}$$

$$\int\limits_0^1 f_3(s)\overline{\vartheta_{53}(s)}ds = -u_3'(1-0), \tag{18}$$

$$\int\limits_0^1 f_3(s)\overline{\vartheta_{63}(s)}ds = u_3'(+0). \tag{19}$$

From relations (14)–(19) the assertion of Lemma 1 follows.

We will also need the following assertion.

Lemma 3. Whatever the numbers

$$\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4,\, \alpha_5,\, \alpha_6,\, \beta_1,\, \beta_2,\, \beta_3,\, \beta_4,\, \beta_5,\, \beta_6$$

there exists a function $(u_1(x_1), u_2(x_2), u_3(s)) \in D$, satisfying the conditions

$$\hat{u}_1'(+0) - \hat{u}_1(2\pi - 0) = \alpha_1, \;\; \hat{u}_1'(+0) = \beta_1,$$

$$\hat{u}_1'(2\pi - 0) - \hat{u}_1'(+0) = \alpha_2, \;\; \hat{u}_1(+0) = \beta_2,$$

$$\hat{u}_2(\pi + 0) - \hat{u}_2(3\pi - 0) = \alpha_3, \;\; \hat{u}_2'(\pi + 0) = \beta_3,$$

$$\hat{u}_2'(3\pi - 0) - \hat{u}_2'(\pi + 0) = \alpha_4, \;\; \hat{u}_2(\pi + 0) = \beta_4,$$

$$u_3(1 - 0) = \alpha_5, \;\; -u_3'(1 - 0) = \beta_5$$

$$u_3(+0) = \alpha_6, \;\; u_3'(+0) = \beta_6.$$

Proof. The proof of Lemma 2 repeats the reasoning that was used in the proof of Lemma 2 § 17 of the monographs [20]. Now we can formulate the main result of this section, since the construction of § 17 of the monograph [20] in our case is carried out automatically.

Theorem 5. Every self-adjoint correctly solvable extension L_u of the operator L_0 is determined by boundary conditions of the form (10), and

$$\begin{gathered}
(\hat{\sigma}_j'(2\pi - 0) - \hat{\sigma}_j'(+0))\overline{(\hat{\sigma}_k(2\pi - 0) - \delta_{k2})} + (\hat{\sigma}_j'(+0) + \delta_{j1})\overline{(\hat{\sigma}_j(+0) - \hat{\sigma}_k(2\pi - 0))} - \\
-(\hat{\sigma}_j'(+0) - \hat{\sigma}_j'(2\pi - 0))\overline{(\hat{\sigma}_k(2\pi - 0) - \delta_{k1})} + (\hat{\sigma}_j'(+0) + \delta_{j2})\overline{(\hat{\sigma}_j'(2\pi - 0) - \hat{\sigma}_k'(+0))} - \\
-\,(\hat{\rho}_j'(3\pi - 0) - \hat{\rho}_j'(\pi + 0))\overline{(\hat{\rho}_k(3\pi - 0) - \delta_{k4})} + (\hat{\rho}_j'(+0) + \delta_{j3})\overline{(\hat{\rho}_j(\pi + 0) - \hat{\rho}_k(3\pi - 0))} - \\
-(\hat{\rho}_j'(\pi - 0) - \hat{\rho}_j'(3\pi + 0))(\hat{\rho}_k(3\pi - 0) - \delta_{k3}) + (\hat{\rho}_j'(\pi + 0) - \delta_{j4})(\hat{\rho}_j'(3\pi - 0) - \hat{\rho}_k'(\pi - 0)) - \\
-(\varphi_j'(1 - 0) - \delta_{j6})(\varphi_1(1 - 0)) + (\varphi_j'(+0) + \delta_{j5})(\varphi_k(+0)) + \\
+(\varphi_j(1 - 0))(\varphi_j'(1 - 0) - \delta_{k6}) - \varphi_j(+0)(\varphi_k'(+0) + \delta_{k5}) = 0.
\end{gathered} \tag{20}$$

Proof. Let us consider a well-posed problem defined by conditions (10). For convenience, we rewrite conditions (10) as

$$\begin{gathered}
-\overline{(\hat{\sigma}_j'(2\pi - 0) + \delta_{j1})}(\hat{u}_1(+0) - \hat{u}_1(2\pi - 0)) + \overline{(\hat{\sigma}_j(2\pi - 0) - \delta_{j2})}(\hat{u}_1'(+0) - \hat{u}_1'(2\pi - 0)) - \\
-\overline{(\hat{\rho}_j'(3\pi - 0) + \delta_{j3})}(\hat{u}_2(\pi + 0) - \hat{u}_2(3\pi - 0)) + \overline{(\hat{\rho}_j(3\pi - 0) - \delta_{j4})}(\hat{u}_2'(\pi + 0) - \hat{u}_2'(3\pi - 0)) - \\
-\overline{(\hat{\varphi}_j'(+0) + \delta_{j5})}u_3(+0) + \overline{(\hat{\varphi}_j'(1 - 0) - \delta_{j6})}u_3(1 - 0) + \\
+\overline{(\hat{\sigma}_j(+0) - \hat{\sigma}_j(2\pi - 0))}\hat{u}_1'(+0) + \overline{(\hat{\sigma}_j'(2\pi - 0) - \hat{\sigma}_j'(+0))}\hat{u}_1(+0) + \\
+\overline{(\hat{\rho}_j(\pi + 0) - \hat{\rho}_j(3\pi - 0))}\hat{u}_2'(\pi + 0) + \overline{(\hat{\rho}_j'(3\pi - 0) - \hat{\rho}_j'(\pi + 0))}\hat{u}_2(\pi + 0) + \\
+\overline{\varphi_j(+0)}u_3'(+0) - \overline{\varphi_j(1 - 0)}u_3'(1 - 0) = 0, \;\; j = 1, 2, 3, 4, 5, 6.
\end{gathered}$$

Let us introduce for $j = 1, 2, 3, 4, 5, 6$ a function $\vartheta_{j1}(x_1), \vartheta_{j2}(x_2), \vartheta_{j3}(s)$ such that

$$\hat{\vartheta}_{j1}(2\pi - 0) = \hat{\sigma}_j(2\pi - 0) - \delta_{j2}, \;\; \hat{\vartheta}_{j1}(+0) - \hat{\vartheta}_{j1}(2\pi - 0) = \hat{\sigma}_j(+0) - \hat{\sigma}_j(2\pi - 0),$$

$$\hat{\vartheta}_{j1}'(2\pi - 0) = \hat{\sigma}_j'(2\pi - 0) + \delta_{j1}, \;\; \hat{\vartheta}_{j1}'(2\pi - 0) - \hat{\vartheta}_{j1}'(+0) = \hat{\sigma}_j'(2\pi - 0) - \hat{\sigma}_j'(+0),$$

$$\hat{\vartheta}_{j2}(3\pi-0)=\hat{\rho}_j(3\pi-0)-\delta_{j4},\ \hat{\vartheta}_{j2}(\pi+0)-\hat{\vartheta}_{j2}(3\pi-0)=\hat{\rho}_j(\pi+0)-\hat{\rho}_j(3\pi-0),$$
$$\hat{\vartheta}'_{j2}(3\pi-0)=\hat{\rho}'_j(3\pi-0)+\delta_{j3},\ \hat{\vartheta}'_{j2}(3\pi-0)-\hat{\vartheta}'_{j2}(\pi+0)=\hat{\rho}'_j(3\pi-0)-\hat{\rho}'_j(\pi+0),$$
$$\hat{\vartheta}'_{j3}(+0)=\varphi'_j(+0)+\delta_{j5},\ \vartheta_{j3}(1-0)=\varphi_j(1-0),$$
$$\hat{\vartheta}'_{j3}(1-0)=\varphi'_j(1-0)-\delta_{j6},\ \vartheta_{j3}(+0)=\varphi_j(+0).$$

According to Lemma 2, such functions exist. In order for conditions (10) to be self-adjoint, according to theorem 4 from § 18 of the monographs [20], the following requirements must be met for any $j,k=1,2,3,4,5,6:$

$$\int\limits_{C_1}(I-\Delta_{C_1})\vartheta_{j1}(x_1)\overline{\vartheta_{k1}(x_1)}dl_1+\int\limits_{C_2}(I-\Delta_{C_2})\vartheta_{j2}(x_2)\overline{\vartheta_{k2}(x_2)}dl_2+\int\limits_0^1(\vartheta_{j3}(s)-\vartheta'_{j3}(s))\overline{\vartheta_{k3}(s)}ds=$$

$$=\int\limits_0^1\vartheta_{j3}(s)\overline{(\vartheta_{k3}(s)-\vartheta''_{k3}(s))}ds+\int\limits_{C_1}\vartheta_{j1}(x_1)\overline{(I-\Delta_{C_1})\vartheta_{k1}(x_1)}dl_1+\int\limits_{C_2}\vartheta_{j2}(x_2)\overline{(I-\Delta_{C_2})\vartheta_{k2}(x_2)}dx_2.$$

The above requirements can be written down using the Lagrange identity (11) in the form of the relation (20).

Conclusion

In this paper, the reasoning refers to a special stratified set Ω. The results presented can be extended to more complex stratified sets composed of one-dimensional and zero-dimensional manifolds. In this paper, an important tool is the transition from one-dimensional smooth manifolds defined by a single chart to intervals. In intervals, the theory of the Sturm-Liouville operator is quite advanced. Therefore, a reverse transition from the Sturm-Liouville operators on a system of intervals to the Laplace-Beltrami operators on stratified sets composed of one-dimensional smooth manifolds and zero-dimensional manifolds is possible.

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Author Contributions

B.E. Kanguzhin statement of the problem and proof of Theorems 1, 2, M.O. Mustafina proof of Lemma 1, 2, O.A. Kaiyrbek proof of Theorems 1, 2. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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Dirichlet type boundary value problem for an elliptic equation with three singular coefficients in the first octant

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The paper investigates a Dirichlet-type boundary value problem for a three-dimensional elliptic equation with three singular coefficients in the first octant. The uniqueness of the solution within the class of regular solutions is established using the energy integral method. To prove the existence of a solution, the Hankel integral transform method is employed. The use of the Hankel transform is particularly appropriate when the variables in the equation range from zero to infinity. This transform is an effective method for obtaining solutions to such problems. In three-dimensional space, to derive the image equation, the Hankel integral transform is applied to the original equation with respect to the variables x and y. As a result, a boundary value problem for an ordinary differential equation in the variable z arises. By solving this problem, a solution to the original boundary value problem is constructed in the form of a double improper integral involving Bessel functions of the first kind and Macdonald functions. To justify the uniform convergence of the improper integrals, asymptotic estimates of the Bessel functions of the first kind and Macdonald functions are utilized. Based on these estimates, bounds for the integrands are obtained, which ensure the convergence of the resulting double improper integral, that is, the solution to the original boundary value problem and its derivatives up to second order, inclusively, as well as the theorem of existence within the class of regular solutions.

Keywords: Hankel's integral transform, Bessel function, modified Bessel function, Macdonald function, singular coefficient, equation of elliptic type, Bessel operator, first octant.

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Introduction. Formulation of the problem

In recent years, interest in degenerate and singular equations has grown significantly, including equations containing the Bessel differential operator. These equations are often encountered in applications, for example, in problems with axial symmetry in continuum mechanics. Interest in problems related to the Bessel operator is also known from fundamental physics. This is due to its numerous applications in gas dynamics, shell theory, magnetohydrodynamics, and other fields of science and technology. A special place in the theory of degenerate and singular equations is occupied by equations containing the Bessel differential operator

$$B_q^z \equiv \frac{d^2}{dz^2} + \frac{2q+1}{z}\frac{d}{dz}, \quad q > -1/2.$$

According to the terminology by the Voronezh mathematician Ivan Aleksandrovich Kipriyanov, equations of three main classes containing the Bessel operator are called B-elliptic, B-hyperbolic, and B-parabolic, respectively. The monograph [1] studies boundary value problems for B-elliptic equations, in addition to this, the account of multi-dimension integral Fourier-Bessel-Hankel transformation theory is given in the monograph. The theory of boundary value problems for the equations with peculiarity

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has been reflected there, while the study of B-hyperbolic equations is presented in the monograph by A.K. Urinov, S.M. Sitnik and Sh.T. Karimov [2]. A wide range of questions for equations with Bessel operators was studied by I.A. Kipriyanov [1] and his students [3–5] and others.

In this paper, we study a Dirichlet-type problem for an elliptic equation with Bessel operators. The solution to the problem under consideration is solved by the Hankel transform method [6–8].

In the domain $\Omega = \{(x, y, z) : x \in (0, +\infty), y \in (0, +\infty), z = (0, +\infty)\}$, we consider the following three-dimensional equation with Bessel operators

$$Lu \equiv \left(B^x_{\alpha-1/2} + B^y_{\beta-1/2} + B^z_{\gamma-1/2}\right) u(x, y, z) = 0, \tag{1}$$

where $u(x, y, z)$ is an unknown function, and $0 < \alpha, \beta, \gamma < 1/2$.

In the domain Ω, equation (1) is of elliptic type. The planes $x = 0$, $y = 0$ and $z = 0$ are the planes of the singularity of the coefficients of the equation.

In the domain Ω, we consider the following problem for equation (1):

Problem D_∞. Find a solution to the equation (1) in the domain Ω, satisfying the conditions

$$u(x, y, z) \in C\left(\bar{\Omega}\right) \cap C^{2,2,2}_{x,y,z}(\Omega), \quad x^{2\alpha}u_x,\ y^{2\beta}u_y,\ z^{2\gamma}u_z \in C\left(\bar{\Omega}\right), \tag{2}$$

$$u(0, y, z) = 0, \ \lim_{x\to+\infty} u(x, y, z) = 0, \ y, z \in [0, +\infty), \tag{3}$$

$$u(x, 0, z) = 0, \ \lim_{y\to+\infty} u(x, y, z) = 0, \ x, z \in [0, +\infty), \tag{4}$$

$$u(x, y, 0) = f(x, y), \ \lim_{z\to+\infty} u(x, y, z) = 0, \ x, y \in [0, +\infty), \tag{5}$$

where $\bar{\Omega} = \{(x, y, z) : x \in [0, +\infty), \ y \in [0, +\infty), \ z = [0, +\infty)\}$, $f(x, y)$ is a given continuous function, such that $f(0, y) = 0$, $f(x, 0) = 0$, $\lim_{x\to+\infty} f(x, y) = 0$, $\lim_{y\to+\infty} f(x, y) = 0$.

In recent years, there has been a steady increase in interest in studying boundary value problems for elliptic equations that involve singularities. Examples of such studies can be found in works [9, 10], among others.

In this paper, we study the stated Problem D_∞ using the Hankel transform method. Many problems in physics, applied mathematics, and mathematical modeling reduce to solving differential, integral, and integro-differential equations. One of the effective methods for obtaining an analytical solution is the method of integral transforms. Among all Bessel-type transforms, the Hankel integral transform is the most thoroughly studied and widely used.

The integral Hankel transform of the order ν of a function is called the integral [6–8]

$$\bar{f}(p) = \int_0^{+\infty} f(t)\, t J_\nu(pt)\, dt, \ \ \nu \geq -1/2, \ \ 0 < p < +\infty,$$

where $J_\nu(z)$ is the Bessel function of the first kind of order ν [6].

The Hankel transform of a function $f(t)$ is true for any points on the interval $(0, +\infty)$ in which the function $f(t)$ is continuous or piecewise continuous with a finite number of discontinuity points of the first kind, and

$$\int_0^{+\infty} |f(t)|\, t^{1/2} dt < +\infty.$$

The inversion formula of the Hankel transform is determined by the integral

$$f(t) = \int_0^{+\infty} \bar{f}(p)\, p J_\nu(pt)\, dp, \ \ 0 < t < +\infty.$$

The function $\bar{f}(p)$ is often called the Fourier-Bessel-Hankel image [11], and the function $f(t)$ is the original.

The Hankel transform is advisable to apply, obviously, in the case when the variables in the equation change from 0 to $+\infty$.

1 Uniqueness of the solution to the problem D_∞

Theorem 1. If there exists solution to Problem D_∞, then it is unique.

Proof. Let Problem D_∞ have two solutions $u_1(x,y,z)$ and $u_2(x,y,z)$. Then $u(x,y,z)=u_1(x,y,z)-$ $-u_2(x,y,z)$ satisfies equation (1) and the homogeneous boundary conditions. We will prove that $u(x,y,z)\equiv 0$ in $\bar{\Omega}$. In the domain Ω the identity is valid

$$x^{2\alpha}y^{2\beta}z^{2\gamma}uLu=\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_x\right)_x+\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_y\right)_y+\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_z\right)_z-$$

$$-x^{2\alpha}y^{2\beta}z^{2\gamma}\left(u_x^2+u_y^2+u_z^2\right)=0.$$

Integrating this identity over the domain

$$\Omega_{\delta_1\delta_3\delta_5}^{\delta_2\delta_4\delta_6}=\left\{(x,y,z):\ \delta_1<x<\delta_2,\ \delta_3<y<\delta_4,\ \delta_5<z<\delta_6\right\},$$

where $\delta_j,\ j=\overline{1,6}$ are positive numbers, we have

$$\iiint\limits_{\Omega_{\delta_1\delta_3\delta_5}^{\delta_2\delta_4\delta_6}}\left[\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_x\right)_x+\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_y\right)_y+\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_z\right)_z\right]dxdydz=$$

$$=\iiint\limits_{\Omega_{\delta_1\delta_3\delta_5}^{\delta_2\delta_4\delta_6}}\left[x^{2\alpha}y^{2\beta}z^{2\gamma}\left(u_x^2+u_y^2+u_z^2\right)\right]dxdydz. \tag{6}$$

It is obvious that if $\delta_1,\ \delta_3,\ \delta_5\to 0,\ \ \delta_2,\delta_4,\delta_6\to+\infty$, then $\Omega_{\delta_1\delta_3\delta_5}^{\delta_2\delta_4\delta_6}\to\Omega$.

Applying the Gauss-Ostrogradsky formula [12] to the left side of equality (6), we have

$$\int\limits_{\delta_5}^{\delta_6}\int\limits_{\delta_3}^{\delta_4}y^{2\beta}z^{2\gamma}\left[\delta_2^{2\alpha}u(\delta_2,y,z)\,u_x(\delta_2,y,z)-\delta_1^{2\alpha}u(\delta_1,y,z)\,u_x(\delta_1,y,z)\right]dydz+$$

$$+\int\limits_{\delta_5}^{\delta_6}\int\limits_{\delta_1}^{\delta_2}x^{2\alpha}z^{2\gamma}\left[\delta_4^{2\beta}u(x,\delta_4,z)\,u_y(x,\delta_4,z)-\delta_3^{2\beta}u(x,\delta_3,z)\,u_y(x,\delta_3,z)\right]dxdz+$$

$$+\int\limits_{\delta_3}^{\delta_4}\int\limits_{\delta_1}^{\delta_2}x^{2\alpha}y^{2\beta}\left[{\delta_6}^{2\gamma}u(x,y,\delta_6)\,u_z(x,y,\delta_6)-\ \delta_5^{2\gamma}u(x,y,\delta_5)\,u_z(x,y,\delta_5)\right]dxdy=$$

$$=\iiint\limits_{\Omega_{\delta_1\delta_3\delta_5}^{\delta_2\delta_4\delta_6}}\left[x^{2\alpha}y^{2\beta}z^{2\gamma}\left(u_x^2+u_y^2+u_z^2\right)\right]dxdydz.$$

Hence, passing to the limit at $\delta_1,\ \delta_3,\ \delta_5 \to 0,\ \delta_2, \delta_4, \delta_6 \to +\infty$ and taking into account conditions (2), (3), (4) and (5) (for $f(x,y) \equiv 0$), from the last equality, we obtain

$$\iiint\limits_{\Omega} \left[x^{2\alpha} y^{2\beta} z^{2\gamma} \left(u_x^2 + u_y^2 + u_z^2\right)\right] dxdydz = 0.$$

From the last, we have

$$u_x(x,y,z) \equiv u_y(x,y,z) \equiv u_z(x,y,z) \equiv 0,\ \ (x,y,z) \in \Omega.$$

Then, $u(x,y,z) \equiv const$, $(x,y,z) \in \Omega$. Since $u \in C\left(\bar{\Omega}\right)$ and $u(0,y,z) \equiv 0$, then $u(x,y,z) \equiv 0$, $(x,y,z) \in \bar{\Omega}$. From this follows the statement of Theorem 1.

2 Existence of the solution to the problem D_∞

Let $\tilde{u}(\lambda,\mu,z)$ be the Hankel transformation of the unknown function $u(x,y,z)$ with respect to the variables x and y. Then, by the definition, we have

$$\tilde{u}(\lambda,\mu,z) = \int\limits_0^{+\infty}\int\limits_0^{+\infty} xy\left[x^{\alpha-1/2} y^{\beta-1/2} u(x,y,z)\right] J_{1/2-\alpha}(\lambda x)\, J_{1/2-\beta}(\mu y)\, dxdy. \tag{7}$$

Considering inverse thansform, we also have

$$u(x,y,z) = x^{1/2-\alpha} y^{1/2-\beta} \int\limits_0^{+\infty}\int\limits_0^{+\infty} \lambda\mu\tilde{u}(\lambda,\mu,z)\, J_{1/2-\alpha}(\lambda x)\, J_{1/2-\beta}(\mu y)\, d\lambda d\mu.$$

Based on (7), we introduce the functions

$$\tilde{u}_{\varepsilon_1\varepsilon_3}^{\varepsilon_2\varepsilon_4}(\lambda,\mu,z) = \int\limits_{\varepsilon_3}^{\varepsilon_4}\int\limits_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} y^{1/2+\beta} u(x,y,z)\, J_{1/2-\alpha}(\lambda x)\, J_{1/2-\beta}(\mu y)\, dxdy, \tag{8}$$

where ε_j, $j = \overline{1,4}$ are positive numbers.

It's obvious that $\lim\limits_{\substack{\varepsilon_1,\varepsilon_3\to 0\\ \varepsilon_2,\varepsilon_4\to+\infty}} \tilde{u}_{\varepsilon_1\varepsilon_3}^{\varepsilon_2\varepsilon_4}(\lambda,\mu,z) = \tilde{u}(\lambda,\mu,z)$.

Using the function (8) and the equation (1), we simplify the expression of $B_{\gamma-1/2}^z \tilde{u}_{\varepsilon_1\varepsilon_3}^{\varepsilon_2\varepsilon_4}(\lambda,\mu,z)$:

$$B_{\gamma-1/2}^z \tilde{u}_{\varepsilon_1\varepsilon_3}^{\varepsilon_2\varepsilon_4} = \int\limits_{\varepsilon_3}^{\varepsilon_4}\int\limits_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} y^{1/2+\beta} J_{1/2-\alpha}(\lambda x)\, J_{1/2-\beta}(\mu y)\, B_{\gamma-1/2}^z u(x,y,z)\, dxdy =$$

$$= -\int\limits_{\varepsilon_3}^{\varepsilon_4}\int\limits_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} y^{1/2+\beta} J_{1/2-\alpha}(\lambda x)\, J_{1/2-\beta}(\mu y)\left(B_{\alpha-1/2}^x + B_{\beta-1/2}^y\right) u(x,y,z)\, dxdy =$$

$$= -\int\limits_{\varepsilon_3}^{\varepsilon_4}\left[\int\limits_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x)\, B_{\alpha-1/2}^x u(x,y,z)\, dx\right] y^{1/2+\beta} J_{1/2-\beta}(\mu y)\, dy -$$

$$- \int\limits_{\varepsilon_1}^{\varepsilon_2}\left[\int\limits_{\varepsilon_3}^{\varepsilon_4} y^{1/2+\beta} J_{1/2-\beta}(\mu y)\, B_{\beta-1/2}^y u(x,y,z)\, dy\right] x^{1/2+\alpha} J_{1/2-\alpha}(\mu x)\, dx. \tag{9}$$

Applying the rule of integration by parts, from (9), we obtain

$$B^z_{\gamma-1/2}\tilde{u}^{\varepsilon_2\varepsilon_4}_{\varepsilon_1\varepsilon_3}(\lambda,\mu,z)=-\int\limits_{\varepsilon_3}^{\varepsilon_4}\left\{\left[J_{1/2-\alpha}(\lambda x)\,u_x-\lambda J_{-1/2-\alpha}(\lambda x)\,u\right]x^{1/2+\alpha}\Big|_{x=\varepsilon_1}^{x=\varepsilon_2}-\right.$$

$$\left.-\lambda^2\int\limits_{\varepsilon_1}^{\varepsilon_2}u(x,y,z)x^{1/2+\alpha}J_{1/2-\alpha}(\lambda x)\,dx\right\}y^{1/2+\beta}J_{1/2-\beta}(\mu y)\,dy-$$

$$-\int\limits_{\varepsilon_1}^{\varepsilon_2}\left\{\left[J_{1/2-\beta}(\mu y)\,u_y-\mu J_{-1/2-\beta}(\mu y)\,u\right]y^{1/2+\beta}\Big|_{y=\varepsilon_3}^{y=\varepsilon_4}-\right.$$

$$\left.-\mu^2\int\limits_{\varepsilon_3}^{\varepsilon_4}u(x,y,z)y^{1/2+\beta}J_{1/2-\beta}(\mu y)\,dy\right\}x^{1/2+\alpha}J_{1/2-\alpha}(\lambda x)\,dx. \tag{10}$$

By direct calculation, one can easily verify that the following limits for fixed $\lambda\in(0,+\infty)$ and $\mu\in(0,+\infty)$, exist and are finite:

$$\lim_{x\to 0}x^{1/2+\alpha}J_{-1/2-\alpha}(\lambda x)=2^{1/2+\alpha}\lambda^{-1/2-\alpha}/\Gamma(1/2-\alpha), \tag{11}$$

$$\lim_{y\to 0}y^{1/2+\beta}J_{-1/2-\beta}(\mu y)=2^{1/2+\beta}\mu^{-1/2-\beta}/\Gamma(1/2-\beta). \tag{12}$$

The behavior of the function $J_v(x)$ for sufficiently small and large values of x is described by the formulas given in [13], respectively:

$$J_\nu(x)\underset{x\to 0}{\approx}\frac{x^\nu}{2^\nu\Gamma(1+\nu)},\quad J_\nu(x)\underset{x\to+\infty}{\approx}\left(\frac{2}{\pi x}\right)^{1/2}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right). \tag{13}$$

From the equality (10), passing to the limit at $\varepsilon_1\to 0$, $\varepsilon_3\to 0$, $\varepsilon_2\to+\infty$, $\varepsilon_4\to+\infty$, and taking the conditions (2), (3), (4) and equalities (11), (12), (13) into account, as well as the notation (7), we obtain the following equation

$$\tilde{u}_{zz}(\lambda,\mu,z)+\frac{2\gamma}{z}\tilde{u}_z(\lambda,\mu,z)-\chi^2\tilde{u}(\lambda,\mu,z)=0,\;\;0<\lambda,\mu,z<+\infty, \tag{14}$$

where $\chi^2=\lambda^2+\mu^2$.

Moreover, due to the boundary conditions (5), from (7) it follows that the function $\tilde{u}(\lambda,\mu,z)$ satisfies the following boundary conditions:

$$\tilde{u}(\lambda,\mu,0)=f_{\lambda\mu},\quad \lim_{z\to+\infty}\tilde{u}(\lambda,\mu,z)=0, \tag{15}$$

where

$$f_{\lambda\mu}=\int\limits_0^{+\infty}\int\limits_0^{+\infty}x^{1/2+\alpha}y^{1/2+\beta}f(x,y)\,J_{1/2-\alpha}(\lambda x)\,J_{1/2-\beta}(\mu y)\,dxdy. \tag{16}$$

We solve the problem (14), (15). It knows that the general solution of the equation (14) has the form [9]

$$\tilde{u}(\lambda,\mu,z)=c_1z^{1/2-\gamma}I_{1/2-\gamma}(\chi z)+c_2z^{1/2-\gamma}K_{1/2-\gamma}(\chi z),\quad z\in[0,c], \tag{17}$$

where c_1 and c_2 are arbitrary constants, $I_l(x)$ and $K_l(x)$ are the Bessel function of the imaginary argument and the Macdonald function of order l [6], respectively.

From the equality (17), based on the asymptotic behavior of the functions $I_\nu(x)$ and $K_\nu(x)$ for sufficiently large x [13], we have

$$I_\nu(x) \approx \frac{e^x}{(2\pi x)^{1/2}}, \quad K_\nu(x) \approx \left(\frac{\pi}{2x}\right)^{1/2} e^{-x},$$

from which follows that the solution of equation (14) satisfying the second condition (15) is determined by the equality

$$\tilde{u}(\lambda,\mu,z) = c_2 z^{1/2-\gamma} K_{1/2-\gamma}(\chi z). \tag{18}$$

By the first condition of (15) from (17), we obtain the equality

$$\tilde{u}(\lambda,\mu,0) = c_2 2^{-1/2-\gamma}\chi^{-1/2+\gamma}\Gamma(1/2-\gamma) = f_{\lambda\mu},$$

from which we uniquely find c_2 as follows:

$$c_2 = 2^{1/2+\gamma}\chi^{1/2-\gamma} f_{\lambda\mu}/\Gamma(1/2-\gamma).$$

Substituting the value of c_2 into the equality (18), we uniquely find a solution to the problem (14), (15) in the form

$$\tilde{u}(\lambda,\mu,z) = \bar{K}_{1/2-\gamma}(\chi z) f_{\lambda\mu}, \tag{19}$$

where $\bar{K}_\nu(x) = 2^{1-\nu}x^\nu K_\nu(x)/\Gamma(\nu)$, $\nu > 0$.

The solution of the original problem will be obtained by using the inverse Hankel transform as follows:

$$u(x,y,z) = \int\limits_0^{+\infty}\int\limits_0^{+\infty} \lambda\mu X_\lambda(x) Q_\mu(y)\tilde{u}(\lambda,\mu,z)\,d\lambda d\mu, \tag{20}$$

where $X_\lambda(x) = x^{1/2-\alpha}J_{1/2-\alpha}(\lambda x), Q_\mu(y) = y^{1/2-\beta}J_{1/2-\beta}(\mu y)$, and $\tilde{u}(\lambda,\mu,z)$ is determined by the formula (19) and they are respectively solutions of the following equations:

$$B^x_{\alpha-1/2}X_\lambda(x) = -\lambda^2 X_\lambda(x), \ \ 0 < x < +\infty, \tag{21}$$

$$B^y_{\beta-1/2}Q_\mu(y) = -\mu^2 Q_\mu(y), \ \ 0 < y < +\infty, \tag{22}$$

$$B^z_{\gamma-1/2}\tilde{u}(\lambda,\mu,z) = \chi^2\tilde{u}(\lambda,\mu,z), \ \ \chi^2 = \lambda^2+\mu^2, \ \ 0 < \lambda,\mu,z < +\infty. \tag{23}$$

If differentiation under the integral sign is possible in (20), then the function $u(x,y,z)$ is a solution to equation (1). Indeed,

$$B^x_{\alpha-1/2}u(x,y,z) + B^y_{\beta-1/2}u(x,y,z) + B^z_{\gamma-1/2}u(x,y,z) =$$

$$= \int\limits_0^{+\infty}\int\limits_0^{+\infty} \lambda\mu\left[B^x_{\alpha-1/2}X_\lambda(x)\right] Q_\mu(y)\tilde{u}(\lambda,\mu,z)\,d\lambda d\mu +$$

$$+ \int\limits_0^{+\infty}\int\limits_0^{+\infty} \lambda\mu X_\lambda(x)\left[B^y_{\beta-1/2}Q_\mu(y)\right]\tilde{u}(\lambda,\mu,z)\,d\lambda d\mu +$$

$$+\int_0^{+\infty}\int_0^{+\infty}\lambda\mu X_\lambda(x)\,Q_\mu(y)\left[B^z_{\gamma-1/2}\tilde{u}(\lambda,\mu,z)\right]d\lambda d\mu.$$

Hence, by virtue of (21), (22) and (23), we have

$$B^x_{\alpha-1/2}u(x,y,z)+B^y_{\beta-1/2}u(x,y,z)+B^z_{\gamma-1/2}u(x,y,z)=0.$$

Let us demonstrate that the function (20) satisfies conditions (3) and (4). Using formulas (13), the functions $X_\lambda(x)$ and $Q_\mu(y)$ for small and large argument values, respectively, can be rewritten in the form [12]

$$X_\lambda(x)\approx\frac{\lambda^{1/2-\alpha}x^{1-2\alpha}}{2^{1/2-\alpha}\Gamma(3/2-\alpha)},\ \ 0<x,\lambda<1; \tag{24}$$

$$X_\lambda(x)\approx x^{-\alpha}\left(\frac{2}{\pi\lambda}\right)^{1/2}\sin\left(\lambda x+\frac{\alpha\pi}{2}\right),\ \ 1<x,\lambda<+\infty; \tag{25}$$

$$Q_\mu(y)\approx\frac{\mu^{1/2-\beta}y^{1-2\beta}}{2^{1/2-\beta}\Gamma(3/2-\beta)},\ \ 0<y,\mu<1;$$

$$Q_\mu(y)\approx y^{-\beta}\left(\frac{2}{\pi\mu}\right)^{1/2}\sin\left(\mu y+\frac{\beta\pi}{2}\right),\ \ 0<y,\mu<+\infty.$$

From these equalities, it follows that the function (20) satisfies the conditions (3) and (4).

Now, we prove several lemmas used in establishing the uniform convergence of the double integral (23).

Lemma 1. If $\alpha\in(0,1/2)$, then, with respect to the functions at $X_\lambda(x)=x^{1/2-\alpha}J_{1/2-\alpha}(\lambda x)$, as $x\in[0,+\infty)$, the following estimates hold:

$$|X_\lambda(x)|\leq\begin{cases}c_3x^{1-2\alpha}\lambda^{1/2-\alpha}, & 0<x,\lambda<1,\\ c_4x^{-\alpha}\lambda^{-1/2}, & 1<x,\lambda<+\infty,\end{cases} \tag{26}$$

$$\left|x^{2\alpha}X'_n(x)\right|\leq\begin{cases}c_5\lambda^{1/2-\alpha}, & 0<x,\ \lambda<1,\\ c_6\lambda^{1/2}x^{\alpha}, & 1<x,\lambda<+\infty,\end{cases} \tag{27}$$

$$\left|B^x_{\alpha-1/2}X_\lambda(x)\right|\leq\begin{cases}c_7x^{1-2\alpha}\lambda^{5/2-\alpha}, & 0<x,\lambda<1,\\ c_8x^{-\alpha}\lambda^{3/2}, & 1<x,\lambda<+\infty,\end{cases} \tag{28}$$

where $c_j,\ j=\overline{3,8}$ are positive constants.

Proof. From the equalities (24) and (25), we obtain estimate (26). Next, consider the functions $x^{2\alpha}X'_\lambda(x)=\lambda x^{1/2+\alpha}J_{-1/2-\alpha}(\lambda x)$ and (23). By virtue of the asymptotic formula (13), it is straightforward to show that these functions satisfy the estimates (27) and (28), respectively. Lemma 1 has been proved.

Similarly, the following lemma can be proved.

Lemma 2. If $\beta\in(0,1/2)$, then with respect to the functions $Q_\mu(y)=y^{1/2-\beta}J_{1/2-\beta}(\mu y)$, at $y\in[0,+\infty)$ the following estimates hold:

$$|Q_\mu(y)|\leq\begin{cases}c_9y^{1-2\beta}\mu^{1/2-\beta}, & 0<y,\mu<1,\\ c_{10}y^{-\beta}\mu^{-1/2}, & 1<y,\mu<+\infty,\end{cases} \tag{29}$$

$$\left|y^{2\beta}Q'_\mu(y)\right|\leq\begin{cases}c_{11}\mu^{1/2-\beta}, & 0<y,\mu<1,\\ c_{12}\mu^{1/2}y^{\beta}, & 1<y,\mu<+\infty,\end{cases}$$

$$\left|B^y_{\beta-1/2}Q_\mu\left(y\right)\right| \leq \begin{cases} c_{13}y^{1-2\beta}\mu^{5/2-\beta}, & 0<y,\mu<1, \\ c_{14}y^{-\beta}\mu^{3/2}, & 1<y,\mu<+\infty, \end{cases}$$

where $c_j,\ j=\overline{9,14}$ are positive constants.

Lemma 3. For any $\lambda,\mu,z\in(0,+\infty)$, the functions $\tilde{u}(\lambda,\mu,z)$, defined by equality (19) satisfy the estimates

$$|\tilde{u}(\lambda,\mu,z)| \leq |f_{\lambda\mu}|\,, \quad \left|B^z_{\gamma-1/2}\tilde{u}(\lambda,\mu,z)\right| \leq \chi^2\left|f_{\lambda\mu}\right|. \tag{30}$$

Proof. It is known [9] that if $\nu = const > 0$, then

$$\bar{K}_\nu\left(t\right)\leq 1, \quad \bar{K}_\nu\left(0\right)=1. \tag{31}$$

From equality (19), according to (31) the first estimate in (30) follows.

As demonstrated earlier, the function $\tilde{u}\left(\lambda,\mu,z\right)$ satisfies the equation (23). Therefore, by virtue of the first estimate in (30), the validity of the second estimate in (30) immediately follows. Lemma 3 has been proved.

Lemma 4. Let $\alpha,\beta,\gamma\in(0,1/2)$ and the function $f\left(x,y\right)$ satisfy the following conditions:

I. $f\left(x,y\right)\in C^{4,4}_{x,y}\left(\bar{\Pi}\right)$, where $\Pi=\left\{(x,y):\ 0<x<+\infty,\ 0<y<+\infty\right\};$

II. $\lim\limits_{x\to 0}\left(\partial^j/\partial x^j\right)f\left(x,y\right)=0,\ \ \lim\limits_{x\to+\infty}x^\alpha\left(\partial^j/\partial x^j\right)f\left(x,y\right)=0,\ \lim\limits_{y\to 0}\left(\partial^j/\partial y^j\right)f\left(x,y\right)=0,$ $\lim\limits_{y\to+\infty}y^\beta\left(\partial^j/\partial y^j\right)f\left(x,y\right)=0, \quad j=\overline{0,3}.$

Then, for the coefficients (16), the following estimate holds:

$$|f_{\lambda\mu}|\leq c_{15}(\lambda\mu)^{-4}, \tag{32}$$

where c_{15} is some positive constant.

Proof. The coefficients $f_{\lambda\mu}$, according to formula (16), can be rewritten as

$$f_{\lambda\mu}=\int\limits_0^{+\infty}y^{1/2+\beta}J_{1/2-\beta}\left(\mu y\right)F_{j\lambda}\left(y\right)dy, \tag{33}$$

where

$$F_\lambda\left(y\right)=\int\limits_0^{+\infty}x^{1/2+\alpha}J_{1/2-\alpha}\left(\lambda x\right)f\left(x,y\right)dx.$$

First, consider the function $F_\lambda\left(y\right)$. Using the equalities

$$x^{1/2+\alpha}J_{1/2-\alpha}\left(\lambda x\right)=-\frac{1}{\lambda}\frac{d}{dx}\left[x^{1/2+\alpha}J_{-1/2-\alpha}\left(\lambda x\right)\right],$$

the function $F_\lambda\left(y\right)$ can be represented as

$$F_\lambda\left(y\right)=-\frac{1}{\lambda}\int\limits_0^{+\infty}\frac{d}{dx}\left[x^{1/2+\alpha}J_{-1/2-\alpha}\left(\lambda x\right)\right]f\left(x,y\right)dx.$$

Applying integration by parts four times to the above integral, we obtain

$$F_\lambda\left(y\right)=-\frac{1}{\lambda}x^{1/2+\alpha}J_{-1/2-\alpha}\left(\lambda x\right)f\left(x,y\right)\Big|_{x=0}^{x=+\infty}+\frac{1}{\lambda^2}x^{1/2+\alpha}J_{1/2-\alpha}\left(\lambda x\right)f_x\left(x,y\right)\Big|_{x=0}^{x=+\infty}+$$

$$+\frac{1}{\lambda^3}x^{1/2+\alpha}J_{-1/2-\alpha}\left(\lambda x\right)B^x_{\alpha-1/2}f\left(x,y\right)\Big|_{x=0}^{x=+\infty}-\frac{1}{\lambda^4}x^{1/2+\alpha}J_{1/2-\alpha}\left(\lambda x\right)\frac{\partial}{\partial x}B^x_{\alpha-1/2}f\left(x,y\right)\Big|_{x=0}^{x=+\infty}+$$

$$+\frac{1}{\lambda^4}\int\limits_0^{+\infty} X_\lambda(x)\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\,dx. \tag{34}$$

By the conditions of Lemma 4, the boundary terms in (34) vanish. Consequently,

$$F_\lambda(y)=\frac{1}{\lambda^4}\int\limits_0^{+\infty} X_\lambda(x)\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\,dx. \tag{35}$$

Using the decomposition of the operator $B^x_{\alpha-1/2}$, it is easy to verify that the functions $\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)$, based on the conditions of Lemma 4, satisfy $\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\in C\left(\bar{\Pi}\right)$. Taking this into account and the fact that $X_n(x)\in C[0,+\infty)$, we conclude that the integral in (35) exists and that $F_\lambda(y)\in C[0,+\infty)$.

Now, consider the coefficient $f_{\lambda\mu}$, defined by equality (33). Similarly to the previous case, applying integration by parts four times to the integral in (33), we obtain

$$f_{\lambda\mu}=-\frac{1}{\mu}y^{1/2+\beta}J_{-1/2-\beta}(\mu y)F_\lambda(y)\Big|_{y=0}^{y=+\infty}+\frac{1}{\mu^2}y^{1/2+\beta}J_{1/2-\beta}(\mu y)F'_\lambda(y)\Big|_{y=0}^{y=+\infty}+$$
$$+\frac{1}{\mu^3}y^{1/2+\beta}J_{-1/2-\beta}(\mu y)B^y_{\beta-1/2}F_\lambda(y)\Big|_{y=0}^{y=+\infty}-\frac{1}{\mu^4}y^{1/2+\beta}J_{1/2-\beta}(\mu y)\frac{\partial}{\partial y}B^y_{\beta-1/2}F_\lambda(y)\Big|_{y=0}^{y=+\infty}+$$
$$+\frac{1}{\mu^4}\int\limits_0^{+\infty} Q_\mu(y)\frac{\partial}{\partial y}y^{2\beta}\frac{\partial}{\partial y}B^y_{\beta-1/2}F_\lambda(y)dy. \tag{36}$$

Since the integral in (36) converges uniformly with respect to y, all derivatives and operators with respect to y acting on the functions $F_\lambda(y)$ can be transferred to the functions $f(x,y)$. Then, by the conditions of Lemma 4, the boundary terms in (36) vanish, and therefore

$$f_{\lambda\mu}=\frac{1}{\mu^4}\int\limits_0^{+\infty} Q_\mu(y)\frac{\partial}{\partial y}y^{2\beta}\frac{\partial}{\partial y}B^y_{\beta-1/2}F_\lambda(y)\,dy.$$

Hence, taking (36) into account, we have

$$f_{\lambda\mu}=\frac{1}{\lambda^4\mu^4}\int\limits_0^{+\infty}\int\limits_0^{+\infty} X_\lambda(x)Q_\mu(y)\frac{\partial}{\partial y}y^{2\beta}\frac{\partial}{\partial y}B^y_{\beta-1/2}\left[\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\right]dxdy. \tag{37}$$

By virtue of the conditions of Lemma 4, the following hold:

$$\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\in C\left(\bar{\Pi}\right),\quad \frac{\partial}{\partial y}y^{2\beta}\frac{\partial}{\partial y}B^y_{\beta-1/2}f(x,y)\in C\left(\bar{\Pi}\right),$$

therefore

$$\frac{\partial}{\partial y}y^{2\beta}\frac{\partial}{\partial y}B^y_{\beta-1/2}\left[\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\right]\in C\left(\bar{\Pi}\right).$$

Taking this into account, along with $X_\lambda(x)Q_\mu(y)\in C\left(\bar{\Pi}\right)$, we conclude that the integrand is continuous on $\bar{\Pi}$, and the multiple integral in (37) exists. These considerations complete the proof of Lemma 4.

Based on (32), estimate (30) can be rewritten as

$$|\tilde{u}(\lambda,\mu,z)| \le c_{16}(\lambda\mu)^{-4}, \quad \left|B^{z}_{\gamma-1/2}\tilde{u}(\lambda,\mu,z)\right| \le c_{17}(\lambda\mu)^{-2}, \tag{38}$$

where c_{16} and c_{17} are positive constants.

Lemma 5. Let $\alpha,\beta,\gamma \in (0,1/2)$, and let $f(x,y)$ be a function such that for $\lambda,\mu \in (0,1)$, the following condition holds:

$$\int\limits_0^1\int\limits_0^1 xyf(x,y)dxdy < +\infty,$$

then the following estimate is valid:

$$|f_{\lambda\mu}| \le c_{18}\lambda^{1/2-\alpha}\mu^{1/2-\beta}, \;\; c_{18} = const > 0. \tag{39}$$

Proof. We estimate the coefficient $f_{\lambda\mu}$ defined by equality (16). Taking into account that $0 < x,y,\lambda,\mu < 1$, and using the asymptotic formulas for Bessel functions for small values of arguments (13), as well as the condition of Lemma 5, we obtain inequality (39).

Taking into account (39) and (31), the function in (19) is estimated in the following form

$$|\tilde{u}(\lambda,\mu,z)| \le c_{19}\lambda^{1/2-\alpha}\mu^{1/2-\beta}, \quad c_{19} = const > 0. \tag{40}$$

Now, let us analyze the function (20), i.e., we find an estimate for the function (20). By virtue of the estimates (26), (29), (38) and (40), the integral (23) is bounded, respectively, for $0 < x,y,z < 1$ and for $1 < x,y,z < +\infty$ by the following absolutely convergent improper double integrals:

$$|u(x,y,z)| \le \int\limits_0^{+\infty}\int\limits_0^{+\infty} |\lambda\mu X_\lambda(x)Q_\mu(y)\tilde{u}(\lambda,\mu,z)|\,d\lambda d\mu \le$$

$$\le c_{20}x^{1-2\beta}y^{1-2\beta}\int\limits_0^1\int\limits_0^1 \lambda^{1,5-\alpha}\mu^{1,5-\beta}d\lambda d\mu, \quad 0 < x,y < 1,$$

$$|u(x,y,z)| \le \int\limits_0^{+\infty}\int\limits_0^{+\infty} |\lambda\mu X_\lambda(x)Q_\mu(y)\tilde{u}(\lambda,\mu,z)|\,d\lambda d\mu \le$$

$$\le c_{21}x^{-\alpha}y^{-\beta}\int\limits_1^{+\infty}\int\limits_1^{+\infty} \lambda^{-3,5}\mu^{-3,5}d\lambda d\mu, \quad 1 < x,y < +\infty.$$

Similarly, it can be shown that the integrals $x^{2\alpha}u_x,\ y^{2\beta}u_y,\ z^{2\gamma}u_z,\ B^x_{\alpha-1/2}u,\ B^y_{\beta-1/2}u$ and $B^z_{\gamma-1/2}u$ are bounded by absolutely convergent improper double integrals.

According to Theorem 4 from [14; 233], the double integral in (20) converges uniformly.

Due to the uniform convergence of the double series (20), it can be integrated term by term, and for each term, the order of integration can be interchanged.

Consequently, the integrand in (20) is continuous, and the double integral in (20) converges uniformly for $0 < x,y,z < +\infty$. Therefore, by Theorem 1 from [14; 231], this integral represents a continuous function of x,y and z. Hence, $u(x,y,z)$ is a continuous function in its domain of definition.

Based on these statements, the following theorem holds:

Theorem 2. Let $\alpha,\beta,\gamma \in (0,1/2)$ and the function $f(x,y)$ satisfy the conditions of Lemma 4 and Lemma 5. Then the solution of Problem D_∞ exists and is given by formula (20).

Conclusion

In this work, a Dirichlet type boundary value problem for a three-dimensional elliptic equation with three singular coefficients is formulated and studied. The uniqueness of the solution to the problem has been proved by the method of energy integrals. The Hankel transform method was used to prove the existence of solutions. The solution of the original problem was obtained using the inverse Hankel transform in the form of a two-fold improper integral. Asymptotic methods were used to substantiate the uniform convergence of improper integrals. The obtained estimate made it possible to prove the convergence of these improper integrals and its derivatives up to and including the second order.

Author Contributions

M.R. Murodova collected and analyzed data, and led manuscript preparation, assisted in data collection and analysis. K.T. Karimov served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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Approximations of Theories of Unars

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Łoś's theorem states that a first-order formula holds in an ultraproduct of structures if and only if it holds in "almost all" factors, where "almost all" is understood in terms of a given ultrafilter. This fundamental result plays a key role in understanding the behavior of first-order properties under ultraproduct constructions. Pseudofinite structures – those that are elementarily equivalent to ultraproducts of finite models–serve as an important bridge between the finite and the infinite, allowing the transfer of finite combinatorial intuition to the study of infinite models. In the context of unary algebras (unars), a classification of unar theories provides a foundation for analyzing pseudofiniteness within this framework. Based on this classification, a characterization of pseudofinite unar theories is obtained, along with several necessary and sufficient conditions for a unar theory to be pseudofinite. Furthermore, various forms of approximation to unar theories are investigated. These include approximations not only for arbitrary unar theories but also for the strongly minimal unar theory. Different types of approximating sequences of finite structures are examined, shedding light on the model-theoretic and algebraic properties of unars and enhancing our understanding of their finite counterparts.

Keywords: pseudofinite theory, pseudofinite structure, strongly minimal unar, smoothly approximated structure, unar, Collatz Hypotesis, connected unar, bounded unar, ω-categorical unar.

2020 Mathematics Subject Classification: 03C13, 03C45, 03C52, 03C60.

Introduction

We are dealing with a structure called mono-unary algebra, or unar. Unars have often been studied in connection with various algebraic structures and branches of mathematics, such as universal algebra and model theory. Model-theoretic properties of theories formulated in the language of a single unary function have been studied in a number of works, including [1].

For additional properties, see [2–4]. Besides, unars can be applied in other fields such as computer science and sometimes engineer, physics and life sciences etc. [5–7]

The paper [8] considers surjective quadratic Jordan algebras, which has connections with problems of decomposition of algebraic structures, as in [9] which studies an algebraic approaches to binary formulas and compositions of theories. In both cases, the issues of decomposition and model construction are important.

Pseudofinite structures [10] are a fascinating area of mathematical logic that bridge the gap between finite and infinite structures. They allow for the study of infinite structures in ways that resemble finite structures, and they provide a connection to various other concepts in model theory. One of the most important examples of a pseudofinite structure is the ultraproduct of a sequence of finite structures. Given a sequence of finite structures $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, ...)$, their ultraproduct is an infinite structure that "approximates" each finite structure in the sequence. In fact, any first-order sentence that is true in almost all of the finite structures in the sequence (meaning all but a finite number) is true in the ultraproduct. This ultraproduct is a pseudofinite structure. Sergei Vladimirovich Sudoplatov raised

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a natural question [11, 12] about the types and powers of approximation of the theory. In paper [13], approximations of acyclic graphs are studied. It is proved that any theory of an acyclic graph (tree) of finite diameter is pseudofinite with respect to acyclic graphs (trees), that is, any such theory is approximated by theories of finite structures (acyclic graphs, trees). The works [14, 15] are devoted to the study of ranks, topologies and closures of families of theories, as well as algebras associated with definable families of theories.

The paper also investigates the smooth approximability of unars. Smoothly approximable structures were first studied in detail in the works [16, 17]. The model theory of smoothly approximable structures was significantly advanced by G. Cherlin and E. Hrushovskii. Automorphisms and their properties are an important aspect of the study of smoothly approximable unary algebras, as shown in [18, 19], which considers the features of automorphisms in more complex algebraic structures.

1 *Definitions and Basic Concepts*

As usual, we will use the standard terminology. A unar is a structure $\mathcal{U} = \langle U; f^{(1)}\rangle$, whose language consists of one single operation f. For any $u \in U$, let $f^0(u) = u$, $f^{n+1}(u) = f(f^n(u))$ for all $n \in \omega$, $f^{-1}(u) = \{w \in U | f(w) = u\}$. A unar $\mathcal{U}$ is called a *cycle* of length $n \in \mathbb{N}$, if there exists $u \in U$ such as $U = \{f^i(u)|0 \leq i < n\}$, $f^n(u) = u$, $f^i(u) \neq f^j(u)$ for all different $i, j \in \{0, \dots, n-1\}$. The set $\{u_i | i \in \omega\} \subseteq U$ is called a *semichain*, if $f(u_i) = u_{i+1}$ and $u_i \neq u_j$ for all distinct $i, j \in \omega$. The set $\{u_i | i \in \omega\} \subseteq U$ is called an infinite *antichain*, if $f(u_{i+1}) = u_i$ and $u_i \neq u_j$ for all distinct $i, j \in \omega$. If $|f^{-1}(u)| = k$, we say that u is a *k-branching point*, or *k-valence point*.

Definition 1. Let $X \subseteq U$ and $u, v \in U$. We say that u, v are *connected*, if there is $n, m \in \mathbb{N}$ such as $f^n(u) = f^m(v)$. The set $X \subseteq U$ is *connected* if any two elements of X are connected. A maximal connected set is called a *connected component* of U.

Definition 2. A theory T is said to be *bounded* if there exists a natural number N such that the following formula is true in T:

$$(\forall u)[\bigvee_{n,m=1}^{N} (f^n(u) = f^{n+m}(u))].$$

Fact. [20] The T is ω-categorical iff

i) T is bounded;

ii) if $\mathcal{U} \models T$, then there are only a finite number of non-isomorphic sets of the form $\bigcup_{n<\omega} f^{-n}(u)$ in $\mathcal{U}$ or equivalently, $\mathcal{U}$ realizes a finite number of 1-types.

The *root of depth* n of an element u is the set $K_n(u) = \{w \in \mathcal{U} | \exists i \leq n \text{ such that } f^i(w) = u\}$. The *root* of u is

$$K(u) = \bigcup_{i\in\omega} K_n(u).$$

A connected subset of the root $K_n(u)$ that contains u is called a subroot of depth n of the element u.

A set of N-neighborhood of $V \subseteq U$ is the set

$$\{u \in U : \exists v \in V \text{such that} \bigvee_{n,m}^{N} f^n(v) = f^m(u)\}.$$

The concept of pseudofiniteness was first introduced by J. Ax. A structure $\mathcal{M}$ in a fixed language L is called pseudofinite if it is infinite but satisfies the following property: for every sentence φ in L, if $\mathcal{M}$ satisfies φ, then there exists a finite structure $\mathcal{M}_0$ that also satisfies φ. The theory $T = Th(\mathcal{M})$ of a pseudofinite structure $\mathcal{M}$ is called a pseudofinite theory.

Ultraproducts have been a powerful tool in model theory since the 1950s and 1960s. They are also important in set theory because they are used to construct elementary embeddings, which are key to studying large cardinals. J. Ax linked the idea of pseudofiniteness to ultraproducts, showing how these constructions can help understand pseudofinite structures.

In classical logic, pseudofinite structures have an interesting property related to definable functions.

Proposition 1. Let $\mathcal{M}$ be a pseudofinite structure, and let $f : M^k \to M^k$ be a definable function. Then: f is injective (one-to-one) if and only if f is surjective (onto).

This property is a direct consequence of pseudofiniteness and highlights the "finite-like" behavior of pseudofinite structures, even though they are infinite.

Definition 3. [12] Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin \mathcal{T}$. The theory T is said to be *$\mathcal{T}$-approximated,* or *approximated by the family $\mathcal{T}$*, or a *pseudo-$\mathcal{T}$-theory,* if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ for which $\varphi \in T'$.

If the theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and theories $T' \in \mathcal{T}$ are said to be *approximations* for T.

Definition 4. [21] *A disjoint union* $\bigsqcup_{n\in\omega} \mathcal{M}_n$ of pairwise disjoint systems $\mathcal{M}_n$ of pairwise disjoint predicate signatures $\Sigma_n, n \in \omega$, is a system of signature $\bigcup_{n\in\omega} \Sigma_n \cup \{P_n^{(1)} | n \in \omega\}$ with support $\bigsqcup_{n\in\omega} M_n$, $P_n = M_n$, and interpretations of predicate symbols from Σ_n that coincide with their interpretations in systems $\mathcal{M}_n$, $n \in \omega$.

A disjoint union of theories T_n, pairwise disjoint predicate signatures Σ_n, respectively, $n \in \omega$, is the theory

$$\bigsqcup_{n\in\omega} T_n \rightleftharpoons Th(\bigsqcup_{n\in\omega} \mathcal{M}_n),$$

where $\mathcal{M}_n \models T_n,\ n \in \omega$.

Obviously, the $T_1 \sqcup T_2$ theory does not depend on the choice of the disjunctive union $\mathcal{M}_1 \sqcup \mathcal{M}_2$ of the models $\mathcal{M}_1 \models T_1$ and $\mathcal{M}_2 \models T_2$.

2 Smoothly Approximability of Unars

The study of countably infinite and countably categorical smoothly approximable structures is relevant in many areas of mathematics, including topology, analysis, and algebra.

A. Lachlan introduced the concept of smoothly approximable structures to shift the focus from analyzing finite structures to analyzing infinite ones. The idea is to classify large finite structures that behave as if they are "approximations" to an infinite limit structure. This approach provides a bridge between finite and infinite model theory.

Definition 5. [16] Let L be a countable signature and let $\mathcal{M}$ be a countable and ω-categorical L-structure. L-structure $\mathcal{M}$ (or $Th(\mathcal{M})$) is said to be *smoothly approximable* if there is an ascending chain of finite substructures $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \ldots \subseteq \mathcal{M}$ such that $\bigcup_{i\in\omega} \mathcal{M}_i = \mathcal{M}$ and for every i, and for every $\bar{a}, \bar{b} \in \mathcal{M}_i$ if $tp_{\mathcal{M}}(\bar{a}) = tp_{\mathcal{M}}(\bar{b})$, then there is an automorphism σ of M such that $\sigma(\bar{a}) = \bar{b}$ and $\sigma(\mathcal{M}_i) = \mathcal{M}_i$, or equivalently, if it is the union of an ω-chain of finite homogeneous substructures; or equivalently, if any sentence in $Th(\mathcal{M})$ is true of some finite homogeneous substructure of $\mathcal{M}$.

This means that $\mathcal{M}$ can be "approximated" by a sequence of finite substructures that are homogeneous in a certain sense.

It is important to note that a finitely homogeneous substructure does not necessarily mean that the substructure is homogeneous in the usual sense. Instead, it refers to a weaker property related to the existence of automorphisms preserving the substructure.

Theorem 1. Any infinite ω-categorical unar $\mathcal{U} = \langle U, f \rangle$ is smoothly approximable.

Proof. Let U be a countably categorical unar that does not have a ∞-branching point. Since by [20] in U the set of degrees of points is finite and U is bounded and realizes a finite number of 1-types, i.e., either all connected components are isomorphic, or U consists of a countable number of copies of non-isomorphic connected components. Each connected component can be considered as a finite homogeneous substructure. Thus, U can be represented as a union of finite homogeneous substructures, $U = \bigcup_{i\in\omega} U_i = \bigsqcup_{i\in\omega} U_i$, where U_i are finite homogeneous connected components.

Now U have the connected components with ∞-branching points. Let $U \setminus V$ is the connected components with ∞-branching points and V is the union of finite connected components. Then there are $W_0, W_1, \ldots, : |W_i| < \omega$, $W_0 \subset W_1 \subset \ldots$, and $U \setminus V = \bigsqcup_{i\in\omega} W_i$.

3 Pseudofiniteness of Unars

Theorem 2. A theory T of an infinite unar is pseudofinite if and only if any sentence $\varphi \in T$ is consistent with a theory of bounded unar.

Proof. Let T be the theory of pseudofinite unar, and let T' be the theory of bounded unar. By the definition of pseudofiniteness, any sentence $\varphi \in T$ has a finite model. Since $\varphi \cup T'$ is finitely consistent and, by the compactness theorem, T is consistent with T'.

To the opposite side. Since any sentence φ of a theory T of infinite unar is consistent with a theory T' of bounded unar, any sentence φ belongs to T'. Take $\varphi \in T \cap T'$. Again, by compactness, φ has a model that is either finite or infinite. Hence any sentence φ of the theory T has a finite model.

The following corollary is a direct consequence of Theorem 2 and summarizes Theorem 1.

Proposition 2. Any theory T of a bounded infinite unar is pseudofinite.

If in $\mathcal{U} = \langle U, f \rangle$ the unary function f is injective (surjective) then $\mathcal{U}$ is an *injective (surjective) unar.*

Proposition 3. Any surjective infinite unar is pseudofinite if and only if it is bijective.

Proof. Follows directly from Proposition 1.

Proposition 4. Any injective non-surjective infinite unar is not pseudofinite.

Proof. Let $\mathcal{U}$ be an infinite injective unar. The connected components in $\mathcal{U}$ can be classified to be either a copy of $\langle \mathbb{N}, succ \rangle$, $\langle \mathbb{Z}, succ \rangle$, or a cycle of period p, where $p \in \mathbb{N}^+$. We exclude the last two cases from consideration due to surjectivity. It remains to consider unary $\mathcal{U}$ components that isomorphic to $\langle \mathbb{N}, succ \rangle$. By Proposition 1, $\mathcal{U}$ is not pseudofinite, since there exists an element that does not have a preimage.

Remark 1. There are:

1) surjective pseudofinite and non-pseudofinite infinite unars, e.g., an infinite permutation or $\langle \mathbb{Z}, succ \rangle$ and, respectively, a function with at least two preimages for every element, or a cycle with its preimages out of this cycle;

2) injective non-surjective non-pseudofinite unars, e.g., a Peano successor function;

3) non-injective non-surjective pseudofinite and non-pseudofinite unars, e.g., a unar consisting of an element and its infinitely many preimages, and, respectively, this unar united with a connected component forming a Peano successor function.

These will be described in more detail in the following sections.

Remark 2. Consider the unary function

$$f(x) = \begin{cases} x/2, & \text{if } x \text{ is even,} \\ 3x+1, & \text{if } x \text{ is odd.} \end{cases}$$

Let's call the structure $\langle \mathbb{Z}^+, f \rangle$ as $3x+1$-*unar* or *Collatz unar*. It is easy to see that any point in this model is 1-branching or 2-branching. Therefore, the $3x+1$-unar is not strongly minimal and has an infinite number of antichains. Moreover, the $3x+1$-unar is a surjective unar. By Remark 1 is not pseudofinite.

3.1 Types of Approximations for Families of Theories of Unars

Definition 6. $\mathcal{T}$-approximated theory T is said to be *CYCLE-approximated,* if $\mathcal{T}$ is a family of theories of finite unars with cycles. Also, the $\mathcal{T}$-approximated theory T is said to be *FOREST-approximated,* if $\mathcal{T}$ is a family of theories of finite unars without cycles. In particular, if $\mathcal{T}$ is a family of the theory of connected unars, then T is said to be *TREE-approximated.*

Proposition 5. The theory T of unbounded unar is CYCLE-approximated if and only if each connected component contains a semichain and only one antichain.

Proof. Let $\mathcal{U}_n$ be a cycle of length $n < \omega$. Increasing the length of the cycle in the limit we obtain an acyclic unar $\mathcal{U} = \bigsqcup_{n \le i \le \omega} \mathcal{U}_i$, which is a copy of $\langle \mathbb{Z}, succ \rangle$ with a semichain and an antichain. The proof from right to left is similar to [13; Theorem 2].

Proposition 6. The theory T of unar is FOREST-approximated if and only if T is the theory of a non-injective and non-surjective bounded unar, each component containing an infinitely branching point.

Proof. By the definition of a FOREST-approximated theory T, the family $\mathcal{T}$=FOREST consists of finite acyclic unars. If all connected components are finite, then T is approximated by increasing the number of connected components. And if there is an ∞-branching point in the components, then T is approximated by increasing the valency of the root points. It is easy to see that T is a theory of neither injective nor surjective bounded unar. The proof from right to left is similar to [13; Theorem 4].

3.2 Approximations of Strongly Minimal Unars

The study of uncountable categoricity and ω-stability in certain types of structures is of principal importance.

Definition 7. A structure $\mathcal{M}$ is said to be *minimal*, if any subset definable in the structure using parameters is either finite or co-finite (a complement to a finite set). $\mathcal{M}$ is said to be *strongly minimal*, if any model of $Th(\mathcal{M})$ is minimal.

The notion of strong minimality is important in model theory because it provides a way to classify theories based on the complexity of their definable sets. Strongly minimal theories have many interesting and useful properties, including simplicity and stability, which make them amenable to study and applications in other areas of mathematics.

Proposition 7. The theory $T = Th(\mathfrak{A})$ of bounded strongly minimal unar $\mathfrak{A}$ is pseudofinite.

Proof. A bounded strongly minimal unar $\mathfrak{A}$ can have either one or no ∞-branching point. If bounded and has ∞-branching point, then $\mathfrak{A}$ is connected. By Proposition 2. $\mathfrak{A}$ is pseudofinite and by Proposition 6 the theory $Th(\mathfrak{A})$ is TREE-approximated.

If bounded and has no ∞-branching point, Then every connected component of $\mathfrak{A}$ is finite and all but finitely many connected components are cycles of the same length m. By the classical results of Zilber and Cherlin, Harrington, Lachlan [22, 23] say that strongly minimal (in fact ω-stable) ω-categorical theories are pseudofinite.

Proposition 8. There is a theory T of unbounded strongly minimal non-injective non-surjective pseudofinite unar.

Proposition 9. The theory $T = Th(\mathcal{U})$ of unbounded strongly minimal injective unar $\mathcal{U}$ is pseudofinite if and only if $\mathcal{U}$ is bijective.

Model-theoretic properties such as definable minimality of unars were studied in [2].

Proposition 10. Let T be the theory of a strongly minimal unar such that each vertex has n preimages for some natural n. Then the theory T is pseudofinite if and only if $n = 1$.

3.3 Connected Unars

Theorem 3. [24, 25] Let $\mathcal{U}$ be a connected unary without cycles, containing no infinite antichains, and there exist $m \in \omega$ and a semichain $S \subseteq \mathcal{U}$ such that

1) $|f^{-1}(u)| \leq m$ for all $u \in \mathcal{U}$;

2) for any $n \in \omega$ there are $u, v, u_0 \in S, v_0 \in \mathcal{U}$, satisfying the following conditions:

a) $a = f^n(u_0),\ b = f^n(v_0)$,

b) $b = f^{2n+k}(u)$ for some $k \in \omega$,

c) $f^{n-1}(v_0) \notin S$,

d) there is a finite partial isomorphism $\alpha : \mathcal{U} \to \mathcal{U}$ such that $dom\alpha = O^n(u)$, $rang\alpha \subseteq O^n(v)$ and $\alpha(u_0) = v_0$.

Then $\mathcal{U}$ is a pseudofinite unar.

In the work [26, 27] a study of pseudofinite polygons was started.

The following statements are easily derived from the above results.

Proposition 11. The theory T of connected unar is CYCLE-approximated if and only if it contains a semichain and only one antichain.

Proposition 12. The theory T of unar is TREE-approximated if and only if T is the theory of a non-injective and non-surjective bounded unar, containing an infinitely branching point.

Proposition 13. There is an pseudofinite unar that is not CYCLE-approximated and TREE-approximated.

4 Concluding remarks

On a base of classification of unar theories, a characterization of pseudofiniteness of unar theories is found, as well as some necessary and sufficient conditions of pseudofiniteness. Approximations of the theory of unars are shown, as well as for the strongly minimal theory of unars. Various types of approximation of the unar theory are considered. Unars are special cases of polygons. In the future, it is planned to study pseudofinite polygons and their approximations.

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Conflict of Interest

The author declare no conflict of interest

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The representation theorem of the Robinson hybrid

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This research lies within the domain of model theory, which investigates the properties of, broadly speaking, incomplete theories. The article introduces novel methods for classifying classes of structures whose associated theories are Jonssonian, forming a distinct subclass within the broader category of inductive theories. This subclass is characterized by satisfying the standard model-theoretic properties of joint embedding and amalgamation. The focus is placed specifically on the second kind of hybrids, those involving theories with different signatures. As a representative case of such hybrids among Jonsson theories, we examine the classical examples of the theory of unars and the theory of undirected graphs. The study proposes and formalizes several new notions, including the perfect Robinson hybrid, the center of a Robinson hybrid, the Kaiser class of a theory, and the concept of triple factorization. Within the framework of these definitions, we establish new results, among them a theorem confirming the existence of a unique countably categorical theory of S-acts, which is syntactically equivalent to the Robinson hybrid formed by the aforementioned classes.

Keywords: Jonsson theory, Robinson theory, hybrid, perfect Robinson hybrid, similarity, K_T-equivalence, ω-categorical, cosemanticness relation, S-act, triple factorization.

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Introduction

This work is part of the field of model theory, which examines the model-theoretic properties of, more generally, incomplete theories. It is widely recognized that modern model theory is a fast-evolving branch of mathematics with numerous significant topics. However, this framework is mainly developed for and tailored to the analysis of complete theories. The domain of incomplete theories is extensive, and within it, one can identify the subclass of inductive theories. This classification can be supported by at least the following reasoning. Specifically, a theory is considered inductive if every increasing chain of models remains a model of the theory itself. In other words, a theory is inductive when it is closed under chains of its models. On the other hand, it is a well-known result that such theories can be axiomatized by universal-existential sentences. It can also be observed that the main classical examples from algebra correspond to inductive theories. The most characteristic example of an inductive theory is group theory. Notably, this is also an example of an incomplete theory.

Within inductive theories, one can distinguish the well-studied subclass of Jonsson theories. For an introduction to this subclass, the reader may refer to the following literature: [1–3].

Among Jonsson theories, perfect Jonsson theories hold a particularly significant position. The study of this subclass has been the subject of several works, including [4–6].

The investigation of Jonsson theories is also valuable in the context of contemporary applications in information technology. This is not coincidental, as Jonsson theories, due to their general incompleteness, admit finite models. The identification and analysis of the relationship between infinite and finite

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models of Jonsson theories generates particular interest in this topic. This is because, unlike complete theories, which do not consider finite models, Jonsson theories examine the interplay of many classical concepts associated with complete theories within the framework of finite models. In particular, works such as [4, 7, 8] study such properties as categoricity, stability, various companions, axiomatizability, model completeness, atomic and prime models.

This paper explores two well-known examples of theories: the theory of all unars and the theory of undirected graphs. The study of elementary theories related to the structure of these signatures is widely recognized in the work of many researchers. These works contain many classical results describing various first-order properties related to the complete theories of these structures. Jonsson theories corresponding to these examples were studied in [4, 9, 10]. In the present work, we investigate hybrids of Jonsson theories, where the theories forming the hybrid are the theory of unars and the theory of undirected graphs. It should be noted that studies related to hybrids of Jonsson theories have been considered in [11, 12].

A notable development in the study of both Jonsson theories and inductive theories in general is the exploration of a distinguished subclass of models, referred to in this work as the Kaiser class. This class represents a natural extension of the class of existentially closed models associated with any inductive theory. Since it is well established that inductive theories possess a nonempty set of existentially closed models, the investigation of the Kaiser class introduces a novel and significant problem within the realms of classical model theory and universal algebra. When we refer to classical model theory, we mean problems related not only to incomplete theories but also to complete theories. Thus, in our view, the range of questions considered in this article is of particular interest in relation to topics that arise in classical model theory concerning the concept of hybrid of Jonsson theories.

1 *Essential concepts of Jonsson's model theory*

This section provides the foundational groundwork necessary for the further development of results concerning Jonsson theories and the corresponding classes of their models. The notions discussed here form the conceptual core of the model-theoretic framework within which the subsequent results are formulated and proved.

Jonsson model theory provides a natural semantic setting for analyzing algebraic structures such as unars and undirected graphs, which are known to satisfy the defining conditions of this class of theories. In particular, key properties such as universality and homogeneity serve as central invariants that characterize the semantic behavior of Jonsson theories and are tightly connected to the concept of saturation in models.

The notion of saturation, especially within universally homogeneous models, leads to the identification of a distinguished subclass of Jonsson theories, known as perfect Jonsson theories. These theories are of particular interest due to their stable semantic properties and the behavior of their existentially closed models.

An important feature of this subclass is that perfection is preserved under passage to the center of the theory. That is, if a Jonsson theory is perfect, then its center retains this property as well. This relationship reflects a deep structural symmetry within the semantic layers of Jonsson frameworks.

This section will focus primarily on universal Jonsson theories that describe two major classes of structures: unars with a single unary function symbol and undirected graphs formulated in a signature with one binary relation. To this end, the definition of universality is recalled, together with a formal introduction of the notion of κ-categoricity, which plays a central role in the classification of models in this context.

In what follows, we introduce the definitions and principal results required for the study of existentially closed models and the analysis of perfectness and categoricity within the Jonsson framework.

These notions play a crucial role in the formulation and proof of the main theorems presented in this paper.

Let's outline the key concepts and statements of model-theoretical constructs essential for understanding and working within the framework of Jonsson theories and their associated classes of models.

It has been established that many classical algebraic structures, such as unars and graphs, satisfy the conditions of Jonsson theories [4].

The notions of universality and homogeneity in a model emphasize the semantic invariant characteristic of any Jonsson theory, that is, its semantic model. Moreover, it has been demonstrated that whether this model is saturated or not has a profound impact on the structural features of both the Jonsson theory itself and its corresponding class of models.

The saturation of universally homogeneous models, in the sense defined by Jonsson, leads to the identification of a distinguished subclass of Jonsson theories, whose elements are termed perfect Jonsson theories.

It can be observed that if a Jonsson theory T is perfect, then its center T^*, i.e., the elementary theory of its semantic model $\mathfrak{C}_T$, is also a perfect Jonsson theory [4].

A characterization of perfect Jonsson theories was formulated in [4].

As the focus will be on universal Jonsson theories of all unars of the signature with one unary functional symbol and the theory of undirected graphs in a signature with one binary relation symbol, it is useful to recall the definition of universality. A theory T is called *universal* if it is equivalent to a set of universal sentences [1].

In order to establish the main results of this paper, it is necessary to introduce the framework of κ-categorical Jonsson theories, along with a characterization of existentially closed models within the theory T.

Definition 1. [4] A Jonsson theory T is said to be *κ-categorical* for some cardinal $\kappa \geq \omega$ if any two models of T with cardinality κ are isomorphic.

The following result, originally proven in [4], establishes the equivalence of ω-categoricity for a Jonsson theory and its center, provided that the theory is complete to $\forall\exists$-sentences.

Theorem 1. [4] Let T be $\forall\exists$-complete Jonsson theory. Then the following statements are equivalent:

1) T is ω-categorical.
2) The center T^* of T is ω-categorical.

The following theorem plays a central role in establishing one of the main results of this article. It provides a sufficient condition for a Jonsson theory to be perfect.

Theorem 2. [4] If a Jonsson theory T is ω-categorical, then T is perfect.

Definition 2. [1] A model A of theory T is said to be an *existentially closed model* of T, if for any extension $B \models T$ with $A \subseteq B$, and for any existential formula $\exists x\varphi(x, \bar{y})$, if $B \models \exists x\varphi(x, \bar{a})$ for some tuple $\bar{a} \in A$, then $A \models \exists x\varphi(x, \bar{a})$.

The class E_T, consisting of all existentially closed models of a Jonsson theory T, is guaranteed to be non-empty, due to the inductiveness of T. Clearly, $E_T \subseteq Mod(T)$, so E_T forms a natural subclass of the class of models of T.

Proposition 1. [1] Let T be an inductive theory. Then T has a model companion T' if and only if the class E_T of existentially closed models of T is elementary.

This criterion provides a useful tool for verifying the existence of model companions in the context of inductive theories.

In particular, if a Jonsson theory is perfect, then the class of its existentially closed models is known to be elementary.

The relationship between the two universal Jonsson theories, in terms of their centers and corresponding semantic models, is captured by the following proposition:

Proposition 2. [4] Let T_1 and T_2 be universal Jonsson theories. Then the following conditions are equivalent:

1) The theories T_1 and T_2 are equal; that is, they consist of exactly the same set of first-order sentences.
2) The semantic models $\mathfrak{C}_{T_1}$ and $\mathfrak{C}_{T_2}$ of the Jonsson theories T_1 and T_2, respectively, are isomorphic.
3) The centers of the theories, denoted T_1^* and T_2^*, are equal; that is, the elementary theories of their corresponding semantic models coincide.

2 Exploring the Robinson Spectrum in the Context of Jonsson Theories

The study of model-theoretic spectra associated with classes of first-order structures offers a rich framework for understanding the logical and semantic properties of these classes. Among such spectra, the Jonsson spectrum and its special case, the Robinson spectrum, serve as key tools in analyzing how certain theories interact with structural features of models.

Let L be a first-order language with signature σ, and let K denote a class of L-structures. In this context, we are interested in the collection of all Jonsson theories whose models include all elements of K. This leads naturally to the notion of the Jonsson spectrum of the class K, which captures the diversity of Jonsson axiomatizations that are valid across all structures in K.

Particularly notable is the subclass of Jonsson theories axiomatizable purely by universal sentences; these correspond precisely to the classical Robinson theories. Accordingly, the Robinson spectrum of K can be seen as a refined instance of the broader Jonsson spectrum, restricted to theories of a specific syntactic form. This interrelation allows for a layered approach: by first investigating the more general Jonsson setting, one can then derive meaningful insights into Robinson spectra and their applications.

An essential component in the structural analysis of these spectra is the concept of cosemanticness, which relates theories via their shared semantic core, or center. This equivalence relation partitions spectra into classes of semantically indistinguishable (though potentially syntactically distinct) theories, offering a deeper lens into the interplay between logic and model theory.

The present section introduces and develops the formal machinery underlying both Jonsson and Robinson spectra. We examine how these constructs are defined, how they behave under equivalence by cosemanticness, and how they manifest in concrete algebraic settings such as unars and undirected graphs. Through this analysis, we highlight fundamental differences between the two spectra, particularly in terms of the uniqueness of theories within equivalence classes, and trace the implications for broader concepts such as existential closure and categoricity.

This discussion culminates in a generalization of classical quasivarieties to what we term semantic Jonsson quasivarieties, which serve as a natural setting for interpreting Robinson spectra. These semantic structures, grounded in model-theoretic extensions of elementary theories, provide a fertile ground for exploring categorical properties and model completeness in enriched logical frameworks.

Let L be a first-order language with a signature σ, and let K be a class of L-structures. We consider a particular set of theories associated with K, known as the Jonsson spectrum of the class K. This concept is formally defined as follows:

Definition 3. [4] The *Jonsson spectrum* of the class K, denoted by $JSp(K)$, is the set of all Jonsson theories with signature σ such that every structure in K is a model of the theory. Formally,

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } \forall \mathfrak{A} \in K,\ \mathfrak{A} \models T\}.$$

A detailed treatment of the structure and characteristics of Jonsson spectra can be found in [4].

In the special case where a Jonsson theory is axiomatized solely by universal sentences, one recovers the classical notion of a Robinson theory. Thus, the Jonsson spectrum framework naturally extends to encompass the Robinson spectrum as a specific instance, providing a natural generalization of this concept.

Definition 4. The *Robinson spectrum* of the class K, denoted $RSp(K)$, consists of all Robinson theories with signature σ that are satisfied by every structure in K. Formally,

$$RSp(K) = \{T \mid T \text{ is a Robinson theory and } \forall \mathfrak{A} \in K,\, \mathfrak{A} \models T\}.$$

Within the framework of Jonsson theories, the notion of the cosemanticness relation plays a central role. Let T_1 and T_2 be Jonsson theories with centers T_1^* and T_2^*, respectively.

The following concept was originally formulated by Professor T.G. Mustafin:

Definition 5. [4] Two Jonsson theories T_1 and T_2 are said to be *cosemantic* (denoted $T_1 \bowtie T_2$) if their centers coincide, i.e., $T_1^* = T_2^*$.

It was established in [4] that this cosemanticness relation defines an equivalence relation on the class of Jonsson theories. Consequently, when this relation is applied to the Jonsson spectrum $JSp(K)$, the set is naturally partitioned into equivalence classes, referred to as cosemantic classes. The corresponding quotient set is denoted by $JSp(K)/_{\bowtie}$. This quotient set provides a useful framework for extending classical results and formulating broader generalizations within the theory. In an analogous manner, the quotient set $RSp(K)/_{\bowtie}$ can be introduced for the Robinson spectrum.

An essential result in the context of Robinsonian theories and the Robinson spectrum is the following proposition:

Proposition 3. [13] Let K be an arbitrary class of L-structures (possibly consisting of a single structure), and let $RSp(K)/_{\bowtie}$ be the quotient set of the Robinson spectrum of K with respect to cosemanticness. Then every cosemanticness class $[\Delta]$ contains exactly one theory. In other words, for any two Robinsonian L-theories T and T', the relation of cosemanticness is equivalent to the equality (logical equivalence) of theories; that is, $T \bowtie T' \Leftrightarrow T = T'$.

In the Robinson spectrum, when factorized by cosemanticness, each cosemanticness class is a singleton.

This proposition highlights a fundamental distinction between the Jonsson and Robinson spectra under the cosemanticness relation. In the case of the Robinson spectrum $RSp(K)$, factorization by cosemanticness yields a discrete partition: each equivalence class contains exactly one theory. This reflects the fact that for Robinsonian theories, semantic identity is equivalent to syntactic identity.

By contrast, for the Jonsson spectrum $JSp(K)$, the situation is more intricate. The equivalence relation of cosemanticness does not, in general, reduce to syntactic equality. That is, distinct Jonsson theories can share the same center and thus belong to the same cosemanticness class. Consequently, the quotient set $JSp(K)/_{\bowtie}$ may contain nontrivial equivalence classes, each consisting of multiple syntactically distinct yet semantically related theories.

This structural divergence between the two spectra is crucial for understanding the role of centers in classification problems and reflects deeper differences in the expressiveness and axiomatizability of Robinson versus Jonsson theories.

We now proceed to the formulation of the concept known as a semantic Jonsson quasivariety.

Let K be a class of quasivarieties of the first-order language L, as defined in [14], and let $L_0 \subset L$, where L_0 is the set of sentences of language L. Consider the elementary theory $Th(K)$ of this class K. By adding $\forall\exists$-sentences of language L, denoted by $\forall\exists(L_0)$, which are not contained in $Th(K)$, we can define the set of Jonsson theories $J(Th(K))$ as follows.

Denotation 1. $J(Th(K)) = \{\Delta \mid \Delta = Th(K) \cup \{\varphi^i\}\}$, where Δ is a Jonsson theory, φ^i denotes either a formula from $\forall\exists(L_0)$ or its negation, $i \in \{0, 1\}$, and $Th(K)$ is the elementary theory of the class of quasivarieties K.

Every theory $\Delta \in J(Th(K))$ is associated with a semantic model, denoted $\mathfrak{C}_\Delta$. We now define the set of all such models:

Denotation 2. $J\mathbb{C} = \{\mathfrak{C}_\Delta \mid \Delta \in J(Th(K)),\ \mathfrak{C}_\Delta \text{ is a semantic model of } \Delta\}$.

The set $J\mathbb{C}$ is referred to as a *semantic Jonsson quasivariety* associated with the class K, provided that its elementary theory $Th(J\mathbb{C})$ itself forms a Jonsson theory.

This construction generalizes the traditional notion of a quasivariety by integrating semantic properties tied specifically to Jonsson type extensions. Unlike standard quasivarieties, which are defined purely syntactically (e.g., by quasi-identities or Horn sentences), a semantic Jonsson quasivariety is formed by considering model-theoretic extensions of a given elementary theory $Th(K)$ via additional $\forall\exists$-sentences. These extensions do not necessarily follow from $Th(K)$ and may vary across different Jonsson theories $\Delta \in J(Th(K))$.

This concept differs substantially from the notion of a classical quasivariety. It is well known that if a quasivariety is countably categorical, then it is also uncountably categorical. However, this does not hold for a semantic Jonsson quasivariety. A counterexample is given by the theory of the semantic Jonsson quasivariety of abelian groups.

The Robinson spectra associated with universal unars and undirected graphs have been investigated within the framework of semantic Jonsson quasivarieties.

Let us consider an unar structure $\mathfrak{U}$, which is a model over the signature $\sigma_{\mathfrak{U}} = \langle f \rangle$, where f is a unary functional symbol. Define the sequence of iterated applications of f recursively as follows: $f_0(x) = x$, $f_{n+1}(x) = f(f_n(x))$, $n \in \omega$. Given elements $a, b \in U$ are called *U-connected* in X if there exist natural numbers m and n such that $f_m(a) = f_n(b)$ and $f_0(a) = f_m(a),\ f_0(b), \ldots, f_n(b) \in X$.

A subset $X \subseteq U$ is said to be *U-connected* if every pair of elements from X is U-connected. A subsystem $B \subseteq U$ whose universe forms is the maximal U-connected subset of carrier U is referred to as a *component* in the structure $\mathfrak{U}$. Furthermore, if B is a component, then the set $\{a \in B : \exists n \in \omega \text{ such that } \mathfrak{U} \models f_n(a) = a\}$ is called a *cycle* of the component.

Now consider a graph structure $\mathfrak{G}$, which is modeled as an algebraic system with signature $\sigma_{\mathfrak{G}} = \langle R \rangle$, where R is a binary symmetric relation. In this setting, elements of the universe are referred to as *vertices*, and a pair $\langle x, y \rangle$ forms an *edge* if $R(x, y)$ holds. A graph in which the relation R is empty, that is, contains no edges, is called a *totally disconnected graph*.

Based on the foundational results established in [4], it follows that the universal parts of the elementary theories of these structures denoted $Th_\forall(\mathfrak{U})$ and $Th_\forall(\mathfrak{G})$ for unars and undirected graphs, respectively, constitute their corresponding Robinson theories. Hence, these theories provide canonical examples of Robinson spectra for algebraic systems within the domain of semantic Jonsson quasivarieties.

Thus, we define the set

$$J\mathbb{C}_{\mathfrak{U}} = \{\mathfrak{C}_{\Delta_{\mathfrak{U}}} \mid \Delta_{\mathfrak{U}} \in J(Th(K_{\mathfrak{U}})),\ \mathfrak{C}_{\Delta_{\mathfrak{U}}} \models \Delta_{\mathfrak{U}}\}\ ,$$

where the signature $\sigma_{\mathfrak{U}} = \langle f \rangle$, and f is unary functional symbol. Here $\Delta_{\mathfrak{U}}$ denotes a Robinson theory of unars. The set $J\mathbb{C}_{\mathfrak{U}}$ is referred to as the *semantic Jonsson quasivariety of Robinson unars*, as introduced in [4].

Following [4], we define the Robinson spectrum of the set $J\mathbb{C}_{\mathfrak{U}}$ as follows:

Definition 6. Let $RSp(J\mathbb{C}_{\mathfrak{U}})$ denote the set of all Robinson theories $\Delta_{\mathfrak{U}}$ in the signature $\sigma_{\mathfrak{U}}$ such that every model $\mathfrak{C}_{\Delta_{\mathfrak{U}}} \in J\mathbb{C}_{\mathfrak{U}}$ satisfies the theory $\Delta_{\mathfrak{U}}$. That is,

$$RSp(J\mathbb{C}_{\mathfrak{U}}) = \{\Delta_{\mathfrak{U}} \mid \Delta_{\mathfrak{U}} \text{ is a Robinson theory of unars, and } \forall \mathfrak{C}_{\Delta_{\mathfrak{U}}} \in J\mathbb{C}_{\mathfrak{U}},\ \mathfrak{C}_{\Delta_{\mathfrak{U}}} \models \Delta_{\mathfrak{U}}\}.$$

This set is called the *Robinson spectrum* of the semantic Jonsson quasivariety $J\mathbb{C}_{\mathfrak{U}}$.

The quotient set of this spectrum is denoted by $RSp(J\mathbb{C}_{\mathfrak{U}})_{/\bowtie}$, which consists of equivalence classes $[\Delta_{\mathfrak{U}}]$ determined by the cosemanticness relation (that is, theories that share the same center).

Similarly, we can define a corresponding structure for undirected graphs. Consider the set

$$J\mathbb{C}_{\mathfrak{G}} = \{\mathfrak{C}_{\Delta_{\mathfrak{G}}} \mid \Delta_{\mathfrak{G}} \in J(Th(K_{\mathfrak{G}})),\ \mathfrak{C}_{\Delta_{\mathfrak{G}}} \models \Delta_{\mathfrak{G}}\},$$

where $\Delta_{\mathfrak{G}}$ is a Robinson theory formulated over the signature $\langle R\rangle$ of undirected graphs, R is a binary symmetric relation, i.e., the standard signature of undirected graphs. The set $J\mathbb{C}_{\mathfrak{G}}$ is thus interpreted as the *semantic Jonsson quasivariety of Robinson undirected graphs.*

Definition 7. Let $\sigma_{\mathfrak{G}}$ be the signature $\langle R\rangle$, where R is a binary symmetric relation. The set of all Robinson theories $\Delta_{\mathfrak{G}}$ such that every semantic model $\mathfrak{C}_{\Delta_{\mathfrak{G}}} \in J\mathbb{C}_{\mathfrak{G}}$ satisfies $\Delta_{\mathfrak{G}}$, that is,

$$RSp(J\mathbb{C}_{\mathfrak{G}}) = \{\Delta_{\mathfrak{G}} \mid \Delta_{\mathfrak{G}} \text{ is a Robinson theory of undirected graphs, and } \forall \mathfrak{C}_{\Delta_{\mathfrak{G}}} \in J\mathbb{C}_{\mathfrak{G}},\ \mathfrak{C}_{\Delta_{\mathfrak{G}}} \models \Delta_{\mathfrak{G}}\},$$

is called the *Robinson spectrum* of the semantic Jonsson quasivariety $J\mathbb{C}_{\mathfrak{G}}$ of Robinson undirected graphs.

As in previous constructions, one can define the corresponding *cosemantic quotient set*, denoted by $RSp(J\mathbb{C}_{\mathfrak{G}})/_{\bowtie}$, which consists of equivalence classes $[\Delta_{\mathfrak{G}}]$ under the cosemanticness relation, that is, theories whose centers coincide.

In the ω-categorical setting, a model-theoretic characterization of existentially closed models has been established for both unars and undirected graphs. The corresponding results are presented in the following theorems.

Theorem 3. Let $[\Delta_{\mathfrak{U}}]$ be a class of ω-categorical Robinson theories of unars. Then the following statements are equivalent:
1) $\mathfrak{A} \in E_{[\Delta_{\mathfrak{U}}]}$; that is, $\mathfrak{A}$ is an existentially closed model of the class $[\Delta_{\mathfrak{U}}]$;
2) $\mathfrak{A}$ is a disjoint union of components, each of which contains a cycle of the same length.

Theorem 4. Let $[\Delta_{\mathfrak{G}}]$ be a class of ω-categorical Robinson theories of undirected graphs, and let $E_{[\Delta_{\mathfrak{G}}]}$ denote the class of existentially closed models for this class. Then the following are equivalent:
1) $\mathfrak{B} \in E_{[\Delta_{\mathfrak{G}}]}$, i.e., $\mathfrak{B}$ is an existentially closed model of $[\Delta_{\mathfrak{G}}]$;
2) $\mathfrak{B}$ is an infinite totally disconnected graph.

Here, $E_{[\Delta_{\mathfrak{U}}]}$ and $E_{[\Delta_{\mathfrak{G}}]}$ denote the sets of existentially closed models corresponding to the cosemantic classes $[\Delta_{\mathfrak{U}}]$ and $[\Delta_{\mathfrak{G}}]$, respectively.

3 Jonsson theories similarity

The concept of similarity between first-order theories plays a central role in modern model theory, particularly in the classification and comparison of theories with respect to both syntactic and semantic characteristics. In this section, we focus on a specific class of theories – namely, Jonsson theories – and explore various notions of similarity that arise within this framework.

Our exposition begins with a foundation in generalized Jonsson theories, also known as α-Jonsson theories, which extend the classical definition by parameterizing inductiveness, amalgamation, and joint embedding properties via an ordinal index α. These properties ensure that models of the theory behave coherently when considered in chains, embeddings, or pushouts, and are crucial in establishing a robust structural framework for such theories.

To deepen the analysis of similarity, this section introduces two primary dimensions of comparison: syntactic similarity, based on mappings between formula algebras or existential lattices, and semantic similarity, defined via isomorphisms between so-called pure triples associated with models or semantic universes. These notions were initially developed for complete theories in the foundational work of

Professor T.G. Mustafin [15] and subsequently generalized to the Jonsson context by Professor A.R. Yeshkeyev.

The treatment of similarity culminates in precise criteria — such as bijective correspondences between existential lattices or structural isomorphism of model-theoretic automorphism groups — that allow us to relate two theories at a deep logical and algebraic level. Furthermore, the section highlights the critical insight that syntactic similarity always implies semantic similarity, whereas the converse does not necessarily hold.

The theoretical apparatus is complemented by illustrative examples and algebraic constructions, including S-acts (algebraic systems over a monoid), which serve as canonical models used to construct envelopes of arbitrary theories. These models provide a concrete setting for understanding how one theory can simulate or encapsulate the expressive power of another through inessential extensions.

Finally, this section culminates in the formalization of similarity at the level of Jonsson spectrum classes, offering an even broader perspective on how entire families of theories can be compared via their syntactic and semantic cores. The results obtained herein lay the groundwork for the subsequent sections, where the equivalence of centers, perfectness, and existential completeness play a decisive role in characterizing such similarities.

The following examples illustrate key concepts related to Γ-embeddings, Γ-chains, and model-theoretic properties of theories such as α-inductiveness, the α-joint embedding property (α-JEP), and the α-amalgamation property (α-AP). They help clarify how formulas from a given set Γ are preserved under various model-theoretic constructions and how theories behave with respect to chains and embeddings of varying levels of complexity [4].

Example 1 (On Γ-embeddings). Let Γ be the set of all quantifier-free formulas in the language $L = \{<\}$, and consider two structures $\mathcal{A} = (\mathbb{N}, <)$ and $\mathcal{B} = (\mathbb{Z}, <)$. Let $f\colon \mathbb{N} \to \mathbb{Z}$ be the inclusion map defined by $f(n) = n$. Since the order $<$ on $\mathbb{N}$ is preserved in $\mathbb{Z}$, and all quantifier-free formulas true in $\mathcal{A}$ remain true under f in $\mathcal{B}$, the map f is a Γ-embedding.

Example 2 (On Γ-chains). Consider a sequence of structures $\mathcal{A}_i = (\mathbb{Q}_i, <)$, where $\mathbb{Q}_i$ denotes the set of rational numbers with denominators at most 2^i. Then for each $i < j$, the inclusion $\mathcal{A}_i \subseteq_\Gamma \mathcal{A}_j$ holds with respect to $\Gamma = \{<\}$, since the order is preserved and extended. The sequence $\{\mathcal{A}_i\}_{i<\omega}$ thus forms a Γ-chain.

Example 3 (On α-inductiveness). Let T be the theory of linear orders. Consider a chain of countable models $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots$, where each $\mathcal{A}_i$ is a copy of $(\mathbb{N}, <)$ extended by adding isolated elements. The union of this Π_1-chain is again a model of T; hence, T is 1-inductive.

Example 4 (On α-joint embedding property). Let T be the theory of undirected graphs without additional properties. Any two graphs $\mathcal{A}$ and $\mathcal{B}$ can be jointly embedded into their disjoint union $\mathcal{M} = \mathcal{A} \sqcup \mathcal{B}$. The natural inclusion maps are Π_0-embeddings; thus, T satisfies 0-JEP.

Example 5 (On α-amalgamation property). Let T be the theory of vector spaces over a fixed field. Given three vector spaces $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2$ and linear embeddings $f_1\colon \mathcal{A} \to \mathcal{B}_1$, $f_2\colon \mathcal{A} \to \mathcal{B}_2$, the pushout (amalgam) exists and is also a vector space. Therefore, T satisfies 0-AP.

The concept of generalized Jonsson theories, also referred to as α-Jonsson theories, extends the classical notion of Jonsson theories by incorporating ordinal-indexed structural conditions. The following definition is based on the formulation presented in [2].

Consider the following definition, which introduces the notion of an α-Jonsson theory — a type of first-order theory characterized by specific model-theoretic properties.

Definition 8. [2] A theory T is called *α-Jonsson* (for ordinals $0 \leq \alpha \leq \omega$) if it has an infinite model and satisfies three key structural properties: closure under unions of Π_α chains (that is, α-inductiveness); the ability to jointly embed any two of its models into a common extension (α-JEP);

and the possibility of amalgamating models over a common substructure (α-AP). These conditions ensure that the theory possesses a well-behaved and robust class of models, suitable for advanced structural analysis.

By comparing this definition with that of a Jonsson theory, we observe a key difference: the latter is specialized to the case $\alpha = 0$, which yields the classical Jonsson theories. When $\alpha = \omega$ are referred to as complete Jonsson theories. In practice, the index $\alpha = 0$ is often omitted when referring to ordinary Jonsson theories. It is worth noting that, under this generalized framework, Jonsson theories are not necessarily complete.

As demonstrated in [2], Professor T.G. Mustafin established syntactic counterparts of the α-JEP and α-AP properties. These criteria provide an equivalent, formula-based perspective on the corresponding semantic conditions.

Proposition 4. [2] The following statements are equivalent:

1) The theory T satisfies the α-joint embedding property.
2) The α-JEP holds for all countable models of T.
3) For any disjoint tuples of variables $\overline{x}$ and $\overline{y}$, and any consistent sets of formulas $p(\overline{x})$ and $q(\overline{y})$ from $\Sigma_{\alpha+1}$, the union $T \cup p(\overline{x}) \cup q(\overline{y})$ is consistent, provided that both $T \cup p(\overline{x})$ and $T \cup q(\overline{y})$ are consistent separately.

Proposition 5. [2] The following conditions are equivalent:

1) The theory T satisfies the α-amalgamation property.
2) T satisfies the α-AP for countable structures.
3) For any two consistent sets of formulas $p(\overline{x})$ and $q(\overline{x})$ from $\Sigma_{\alpha+1}$ such that the following three sets are all consistent:

$$T \cup p(\overline{x}), \quad T \cup q(\overline{x}), \quad \text{and} \quad T \cup \{\neg\varphi(\overline{x}) \mid \varphi(\overline{x}) \in \Sigma_{\alpha+1},\, \varphi(\overline{x}) \notin p(\overline{x}) \cap q(\overline{x})\},$$

the union $T \cup p(\overline{x}) \cup q(\overline{x})$ is also consistent.
4) For every model $\mathcal{A} \models T$ and tuple $\overline{a} \in \mathcal{A}$, the set $\mathrm{Th}_{\Sigma_{\alpha+1}}(\mathcal{A}, \overline{a})$ can be extended to a unique maximal $\Sigma_{\alpha+1}$-type over T in the expanded language $L(\overline{a})$.

In the study of model theory, an important distinction is drawn between semantic and syntactic properties of theories. Semantic properties concern the behavior and structure of models, while syntactic properties are tied to the formal deductive system. The following propositions illustrate this distinction by clarifying the relationship between completeness and semantic similarity, and by enumerating key semantic notions that play a central role in classification theory.

Proposition 6. [15] If two theories T_1 and T_2 are complete, then they are necessarily semantically similar. However, the converse does not hold: semantically similar theories need not be syntactically similar.

Proposition 7. [15] The following concepts are classified as *semantic* in nature: type, forking, λ-stability, Lascar rank, strong type, Morley sequence, orthogonality, regularity of types, and $I(\aleph_\alpha, T)$ — the spectrum function.

We now turn our attention to a particular class of algebraic structures that will serve as the context for applying the main results established earlier. In the English-language model-theoretic literature, structures known as polygons over a monoid S are commonly referred to as S-acts [16]. Below, we provide a formal definition of this class.

Definition 9. [16] An S-act is a structure of the form $\langle A; f_\alpha : \alpha \in S\rangle$, where each f_α is a unary function on A, and the following axioms hold:

1) *Identity preservation:* $f_e(a) = a$ for all $a \in A$, where $e \in S$ is the identity element of the monoid.

2) *Compatibility with monoid operation:* $f_{\alpha\beta}(a) = f_\alpha(f_\beta(a))$ for all $\alpha, \beta \in S$ and for all $a \in A$.

The results that follow will demonstrate that for every complete theory, there exists another theory that is syntactically similar to it.

Theorem 5. [15] For every theory T_2 in a finite signature, there exists a theory T_1 of S-acts such that some inessential extension of T_1 is an almost envelope of T_2.

Theorem 6. [15] For every theory T_2 in an infinite signature, there exists a theory T_1 of S-acts such that some inessential extension of T_1 is an envelope of T_2.

This section presents a series of known results concerning syntactic and semantic similarities between Jonsson theories, as well as their extensions to classes of such theories. These notions generalize analogous concepts from the theory of complete first-order theories, as previously studied in works such as [13, 15], and have been systematically developed in [4].

In particular, the definitions of Jonsson syntactic similarity and Jonsson semantic similarity aim to capture structural equivalences between the existential fragments and semantic models of Jonsson theories. The formalization of these similarities relies on isomorphisms between lattices of existential formulas and between so-called semantic triples associated with the theories. The notion of the center of a Jonsson theory, denoted T^*, also plays a key role in transferring results from Jonsson theories to their complete analogues. Illustrative examples of Jonsson syntactic similarity between theories can be found in [4].

Analogously to the case of complete theories, Professor A.R. Yeshkeyev introduced the notion of Jonsson semantic similarity between two Jonsson theories [4]. The following result, which is similar to Proposition 6 but formulated in the context of Jonsson theories, was also established in [4].

Theorem 7. [4] Suppose that T_1 and T_2 are Jonsson theories that are syntactically similar in the Jonsson framework. Then they are also semantically similar within the same context.

By extending certain definitions from [15] and applying methods for working with Jonsson theories, it has been shown that, within the class of perfect existentially complete Jonsson theories, the introduced notions of syntactic and semantic similarity coincide with their counterparts in the class of complete theories, as defined in [13].

Theorem 8. [4] Let T_1 and T_2 be two existentially complete perfect Jonsson theories. Then the following statements are logically equivalent — that is, each holds if and only if the other does:

1) T_1 and T_2 are syntactically similar in the sense of Jonsson theories; that is, there exists a structure-preserving correspondence between their existential formulas that respects logical operations such as conjunction and existential quantification.

2) Their centers, T_1^* and T_2^*, are syntactically similar as complete theories; that is, the corresponding complete theories (obtained as the elementary theories of their respective semantic models) are related by a syntactic similarity that aligns their lattices of formulas.

To ensure precision in the subsequent exposition, we adopt the following designation. The syntactic and semantic similarities between two complete theories T_1 and T_2 will be denoted by $T_1 \overset{S}{\bowtie} T_2$ and $T_1 \underset{S}{\bowtie} T_2$, respectively. When dealing specifically with Jonsson theories, we will write $T_1 \overset{S}{\rtimes} T_2$ to indicate syntactic similarity in the Jonsson context, and $T_1 \underset{S}{\rtimes} T_2$ to denote their semantic similarity.

The following corollary for two Jonsson theories T_1 and T_2 in the language L was obtained in [4].

Corollary 1. [4] If the theories T_1 and T_2 are Jonsson syntactically similar ($T_1 \overset{S}{\rtimes} T_2$), then they are also Jonsson semantically similar ($T_1 \underset{S}{\rtimes} T_2$). Moreover, this is equivalent to the theories T_1 and T_2 being cosemantic, expressed as $T_1 \bowtie T_2$.

The notions of Jonsson semantic and syntactic similarity were further generalized to classes of Jonsson theories in [4]. As a result, a generalization of Theorem 7 was obtained for two classes from the Jonsson spectrum. This generalized result plays a crucial role in the proof of Theorem 11.

Lemma 1. [4] Let $\mathfrak{A} \in Mod(\sigma_1)$, $\mathfrak{B} \in Mod(\sigma_2)$, $[T_1] \in JSp(\mathfrak{A})/_{\bowtie}$, $[T_2] \in JSp(\mathfrak{B})/_{\bowtie}$ be perfect $\exists$-complete classes, then

$$[T_1] \overset{S}{\rtimes} [T_2] \Leftrightarrow [T_1^*] \overset{S}{\bowtie} [T_2^*].$$

4 Countable categoricity of Robinson hybrid and its similarity

In model theory, the notion of hybrid offers a constructive means of generating new theories by combining existing ones. Within the framework of Jonsson and Robinson theories, this operation enables the formation of syntactically or semantically enriched theories that retain key properties of their components. This section is devoted to the study of such hybrids, particularly their structure, categoricity, and the relations that govern their similarities.

The central object of analysis is the hybrid of Jonsson theories — a concept that allows two theories (with either identical or distinct signatures) to be combined via algebraic operations such as the Cartesian product, sum, or direct sum. These hybrid constructions fall into two main types, depending on whether the signatures of the input theories coincide. When extended to Robinson theories, those axiomatized by universal sentences, the same hybrid framework leads to the definition of Robinson hybrids, which inherit the logical rigor and syntactic simplicity characteristic of this subclass.

To support the analysis of such hybrids, we further examine the notions of perfectness, semantic models, and theoretical centers, particularly in the context of countable languages. A hybrid theory is said to be perfect if it coincides with the elementary theory of its saturated model; in such cases, its model-theoretic center plays a crucial role in determining categoricity and logical equivalence.

A key part of this section is the development of Kaiser equivalence, a newly introduced equivalence relation between Jonsson theories. This relation compares theories by their associated Kaiser classes, which capture the semantic behavior of existential fragments of models. Alongside this, we examine additional equivalence relations syntactic similarity and cosemanticness, that further refine the classification of theories within Robinson spectra.

The main results presented in this section show that, under certain conditions, the hybrid of two ω-categorical Robinson theories remains ω-categorical. Moreover, by applying triple factorization over Robinson spectra of semantic Jonsson quasivarieties (such as unars and undirected graphs), we construct a unique countably categorical theory of S-acts that is syntactically similar to a Robinson hybrid. This demonstrates not only the internal coherence of hybrid constructions but also the robustness of syntactic similarity in preserving key model-theoretic properties.

We begin by introducing the necessary definitions and preliminary results required to formulate the main theorems of this paper.

The concept of a hybrid of Jonsson theories was considered in [12]. By analogy, in the context of studying the Robinson spectra of semantic Jonsson quasivarieties for Robinson unars and undirected graphs, we introduce the notion of a Robinson hybrid corresponding to two Robinson theories.

Definition 10. 1) Let T_1 and T_2 be Robinson theories in a countable language L with the same signature σ, and let $\mathfrak{C}_{T_1}$ and $\mathfrak{C}_{T_2}$ be their semantic models, respectively. In the case where the Robinson theories T_1 and T_2 have a common signature, we define a hybrid of the first type of these Robinson theories as the theory $Th_{\forall}(\mathfrak{C}_1 \diamond \mathfrak{C}_2)$, provided that this theory is Robinson in the language of signature σ. We denote this hybrid as $HR(T_1, T_2)$, where the operation $\diamond \in \{\times, +, \oplus\}$ and $\mathfrak{C}_1 \diamond \mathfrak{C}_2 \in Mod\,\sigma$. Here, $\times$ represents the Cartesian product, $+$ denotes the sum, and $\oplus$ indicates the direct sum. Thus, the algebraic construction $\mathfrak{C}_1 \diamond \mathfrak{C}_2$ is referred to as the semantic hybrid of the theories T_1 and T_2.

2) If T_1 and T_2 are Robinson theories with different signatures σ_1 and σ_2, respectively, then the theory $HR(T_1, T_2) = Th_{\forall}(\mathfrak{C}_1 \diamond \mathfrak{C}_2)$ is called a hybrid of the second type, provided that this theory is Robinson in the language with the signature $\sigma = \sigma_1 \cup \sigma_2$ where $\mathfrak{C}_1 \diamond \mathfrak{C}_2 \in Mod\,\sigma$.

Clearly, 1) is a special case of 2).

Since Robinson theories are special cases of Jonsson theories, we can further use the notion of a perfect Robinson hybrid and also consider the concept of the center of a Robinson hybrid, which we denote by $HR^*(T_1, T_2)$, where $HR^*(T_1, T_2)$ is the center of the Robinson theory $Th_{\forall}(\mathfrak{C}_1 \diamond \mathfrak{C}_2)$.

Based on the definition of hybrids of Robinson theories, it is also possible to define hybrids corresponding to two classes of Robinson theories.

Definition 11. 1) Let K be an axiomatizable class of models in a countable language L with signature σ, and let $[T_1]$, $[T_2] \in RSp(K)/_{\bowtie}$. The hybrid of the first type $HR([T_1], [T_2])$ of the classes $[T_1]$ and $[T_2]$ is the theory $Th_{\forall}(\mathfrak{C}_1 \diamond \mathfrak{C}_2)$ provided that this theory is Robinson in the language with signature σ, where $\mathfrak{C}_i$ are semantic models of the classes $[T_i]$ for $i = 1, 2$, and $\diamond \in \{\times, +, \oplus\}$, where $\times$ denotes the Cartesian product, $+$ denotes the sum, and $\oplus$ denotes the direct sum of models.

2) Let K_1 and K_2 be axiomatizable classes of models of a countable language with different signatures σ_1 and σ_2, respectively, and let $[T_1] \in RSp(K_1)/_{\bowtie}$ and $[T_2] \in RSp(K_2)/_{\bowtie}$. Then the theory $HR([T_1], [T_2]) = Th_{\forall}(\mathfrak{C}_1 \diamond \mathfrak{C}_2)$ is called the hybrid of the second type of the classes $[T]_1$ and $[T_2]$, provided that this theory is Robinson in the language with signature $\sigma = \sigma_1 \cup \sigma_2$, where $\mathfrak{C}_1 \diamond \mathfrak{C}_2 \in Mod\sigma$.

To prove our result, we need a classical theorem on the characterization of countably categorical theories.

Theorem 9. [1] Let T be a complete theory. Then the following are equivalent:

a) T is ω-categorical;

b) for each $n < \omega$, T has only finitely many types in the variables $x_1, \ldots, x_n$.

In this article, we introduce a new concept, called Kaiser equivalence, between two Jonsson theories. As a starting point, we consider the definition of the Kaiser class of a theory.

Definition 12. A class $K_T = \{\mathfrak{A} \in Mod(T) : T^0(\mathfrak{A})$ is a Jonsson theory$\}$ is called a *Kaiser class* of the theory T, where $T^0(\mathfrak{A}) = Th_{\forall\exists}(\mathfrak{A})$.

Next, we consider a binary relation between the Kaiser classes of two Jonsson theories, T_1 and T_2.

Definition 13. Let T_1 and T_2 be Jonsson theories. We say that T_1 and T_2 are *K_T-equivalent* if $K_{T_1} = K_{T_2}$.

It is clear that the defined relation between two Jonsson theories is an equivalence relation.

Let $J\mathbb{C}\mathfrak{U}$ and $J\mathbb{C}\mathfrak{G}$ be the semantic Jonsson quasivarieties of Robinson unars and undirected graphs, respectively. Let $RSp(J\mathbb{C}\mathfrak{U})$ and $RSp(J\mathbb{C}\mathfrak{G})$ denote their corresponding Robinson spectra.

In addition, we define the following types of relations on these spectra:

1) syntactic similarity in the sense of Jonsson;

2) equivalence with respect to the class K_T;

3) the relation of cosemantic equivalence.

It is important to emphasize that, according to Proposition 3, each of these equivalence classes contains exactly one element.

It is straightforward to verify that each of the defined relations constitutes an equivalence relation. As a result, we can consider the corresponding quotient sets of the Robinson spectra of the classes $J\mathbb{C}_{\mathfrak{U}}$ and $J\mathbb{C}_{\mathfrak{G}}$ under these relations. This construction, which we refer to as triple factorization, is denoted by $RSp(J\mathbb{C}_{\mathfrak{U}})/_{\underset{K}{\overset{S}{\bowtie}}}$ and $RSp(J\mathbb{C}_{\mathfrak{G}})/_{\underset{K}{\overset{S}{\bowtie}}}$. Here, $[\dddot{\Delta}_{\mathfrak{U}}]$ denotes the equivalence class containing the theory $\Delta_{\mathfrak{U}}$ from $RSp(J\mathbb{C}_{\mathfrak{U}})/_{\underset{K}{\overset{S}{\bowtie}}}$, and similarly, $[\dddot{\Delta}_{\mathfrak{G}}]$ corresponds to the class of the theory $\Delta_{\mathfrak{G}}$ from $RSp(J\mathbb{C}_{\mathfrak{G}})/_{\underset{K}{\overset{S}{\bowtie}}}$. Each such equivalence class consists of a single theory of unars or undirected graphs.

We now proceed to the key findings of this article. It is important to note that in the results that follow, we consider only the Cartesian product as the operation $\diamond$.

Theorem 10. Let $[\dddot{\Delta}_{\mathfrak{U}}]$ and $[\dddot{\Delta}_{\mathfrak{G}}]$ denote the equivalence classes of ω-categorical Robinson theories corresponding to unars and undirected graphs, respectively. Then their Robinson hybrid $HR([\dddot{\Delta}_{\mathfrak{U}}],[\dddot{\Delta}_{\mathfrak{G}}])$ is also an ω-categorical Robinson theory.

Proof. Since, as stated in Proposition 3, these classes consist of a single element, we can further work only with theories. Also, by the definition of a Kaiser class of the theory, these theories are complete for universal (existential) sentences. Then, by Theorem 1, we obtain that the centers of these theories, denoted by $\dddot{\Delta}^*_{\mathfrak{U}}$ and $\dddot{\Delta}^*_{\mathfrak{G}}$, are also complete, countably categorical Robinson theories. Therefore, by Theorem 9, we have that for each $n<\omega$, $\dddot{\Delta}^*_{\mathfrak{U}}$ and $\dddot{\Delta}^*_{\mathfrak{G}}$ have only finitely many types in the variables $x_1,\ldots,x_n$.

As we know $\dddot{\Delta}^*_{\mathfrak{U}}$ and $\dddot{\Delta}^*_{\mathfrak{G}}$ are Robinson theories, then they have existentially closed semantic models $\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{U}}}$ and $\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{G}}}$, respectively, each of which realizes a finite number of types. Let us now consider a Cartesian product of their semantic models, $\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{U}}}\times\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{G}}}\in E_{HR(\dddot{\Delta}^*_{\mathfrak{U}},\dddot{\Delta}^*_{\mathfrak{G}})}$. By definition of the Cartesian product, this model also realizes a finite number of types. Therefore, the Robinson hybrid of second type of $\dddot{\Delta}^*_{\mathfrak{U}}$ and $\dddot{\Delta}^*_{\mathfrak{G}}$, denoted by $HR(\dddot{\Delta}^*_{\mathfrak{U}},\dddot{\Delta}^*_{\mathfrak{G}})=Th_{\forall}(\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{U}}}\times\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{G}}})$ is ω-categorical Robinson theory.

Note that, according to Theorem 2, $\dddot{\Delta}_{\mathfrak{U}}$ and $\dddot{\Delta}_{\mathfrak{G}}$ are perfect Robinson theories. Consequently, the classes of existentially closed models of $\dddot{\Delta}_{\mathfrak{U}}$ and $\dddot{\Delta}_{\mathfrak{G}}$ coincide with the classes of models of their centers. Since the Robinson hybrid of these theories is a universally (existentially) complete theory, it follows that this Robinson hybrid is countably categorical.

We can also extend one of the results from [12] by applying triple factorization to the Robinson spectra of the semantic Jonsson quasivarieties of unars and undirected graphs. As a result, we obtain a countably categorical theory of S-acts that is syntactically similar to the Robinson hybrid of these classes.

Theorem 11. Let $[\dddot{\Delta}_{\mathfrak{U}}]$ and $[\dddot{\Delta}_{\mathfrak{G}}]$ be the equivalence classes of ω-categorical Robinson theories of unars of the signature with one unary functional symbol and the theory of undirected graphs that is considered in the signature containing one binary relation symbol, respectively. Then there exists a ω-categorical class of Robinson theories of S-acts, that is Jonsson syntactically similar to the Robinson hybrid $HR([\dddot{\Delta}_{\mathfrak{U}}],[\dddot{\Delta}_{\mathfrak{G}}])$ of these classes, where each class is a single-element class.

Proof. Since, by Proposition 3, these classes are singletons, we can further work directly with the corresponding theories. By Theorem 2, the countably categorical hybrid $HR(\dddot{\Delta}_{\mathfrak{U}},\dddot{\Delta}_{\mathfrak{G}})$ is a perfect Robinson theory. Since its center, denoted by $HR^*(\dddot{\Delta}_{\mathfrak{U}},\dddot{\Delta}_{\mathfrak{G}})$, is complete, it follows from Theorem 5 that there exists a complete theory of the S-acts, denoted by $T_{S_{act}}$, such that $H^*(\dddot{\Delta}_{\mathfrak{U}},\dddot{\Delta}_{\mathfrak{G}})\overset{S}{\bowtie}T_{S_{act}}$. Then, by Proposition 6, we also have $HR^*(\dddot{\Delta}_{\mathfrak{U}},\dddot{\Delta}_{\mathfrak{G}})\underset{S}{\bowtie}T_{S_{act}}$.

Since the notion of a type is semantic according to Proposition 7, the notion of a formula is also semantic. Furthermore, since both JEP and AP are semantic concepts, the properties JEP and AP are equivalent to the consistency of certain formulas, which follows from Propositions 4 and 5.

As all axioms hold in the semantic model, $\forall$-axiomatizability is a semantic property. This, in turn, implies that the property of being a Robinson theory is also a semantic concept. Therefore, the theory $T_{S_{act}}$ qualifies as a Robinson theory as well.

Given that $HR^*(\dddot{\Delta}_{\mathfrak{U}},\dddot{\Delta}_{\mathfrak{G}})$ is a perfect hybrid, the semantic model $\mathfrak{C}_{HR(\dddot{\Delta}_{\mathfrak{U}},\dddot{\Delta}_{\mathfrak{G}})}$ of the hybrid $HR(\dddot{\Delta}_{\mathfrak{U}},\dddot{\Delta}_{\mathfrak{G}})$ is saturated. Moreover, since $HR^*(\dddot{\Delta}_{\mathfrak{U}},\dddot{\Delta}_{\mathfrak{G}})\underset{S}{\bowtie}T_{S_{act}}$ it follows from Definition 18 that the semantic triples of these theories are isomorphic. Hence, $\mathfrak{C}_{HR(\ddot{\Delta}_{\mathfrak{U}},\ddot{\Delta}_{\mathfrak{G}})} \cong \mathfrak{C}_{T_{S_{act}}}$. Therefore $\mathfrak{C}_{T_{S_{act}}}$ is also saturated, and thus $T_{S_{act}}$ is a perfect Robinson theory.

Consider $RSp(\mathfrak{C}_{T_{S_{act}}})$. Since the theory $T_{S_{act}}$ is perfect, we have that $|RSp(\mathfrak{C}_{T_{S_{act}}})_{/\underset{K}{\overset{S}{\bowtie}}}| = 1$. Let $\Delta \in RSp(\mathfrak{C}_{T_{S_{act}}})$, meaning Δ is Robinson theory and $\Delta^* = T_{S_{act}}$. We will show that Δ is a perfect $\exists$-complete Robinson theory.

Given that $HR^*(\dddot{\Delta}_{\mathfrak{U}}, \dddot{\Delta}_{\mathfrak{G}}) \underset{S}{\bowtie} \Delta^*$, it follows from the definition of semantic similarity for complete theories that Δ is a perfect Robinson theory. If, in addition, Δ is $\exists$-complete, then we may replace $T'_{S_{act}}$ with Δ. By Lemma 1, we then conclude that $HR(\dddot{\Delta}_{\mathfrak{U}}, \dddot{\Delta}_{\mathfrak{G}}) \overset{S}{\rtimes} \Delta = T'_{S_{act}}$. If Δ is not $\exists$-complete, we apply the following procedure to complete the theory. Since $\Delta \subset T_{S_{act}}$, for any existential sentence φ in the signature language of Δ such that $\Delta \not\vdash \varphi$ and $\Delta \not\vdash \neg\varphi$, but $\varphi \in T_{S_{act}}$, we define the theory $\Delta' = \Delta \cup \{\varphi\}$.

Since $\Delta \subset \Delta' \subset T_{S_{act}}$, and both Δ and $T_{S_{act}}$ are Robinson theories, it follows from Proposition 7 that Δ' is also a Robinson theory. If Δ' is not $\exists$-complete, we continue the process by successively adding existential sentences $\varphi \in T_{S_{act}}$ until Δ' becomes $\exists$-complete.

Let $\overline{\Delta} = \Delta \cup \{\varphi \mid \varphi \in \Sigma_1, \varphi \in T_{S_{act}}\}$ denote the result of the existential completion procedure applied to the theory Δ. In other words, $\overline{\Delta}$ is $\exists$-complete and is also a Robinson theory. We now show that $\overline{\Delta} \in RSp(\mathfrak{C}TS_{act})$, which implies that the theory $\overline{\Delta}$ is perfect.

Let us assume the opposite, that is, suppose $\overline{\Delta} \notin RSp(\mathfrak{C}_{T_{S_{act}}})$. This implies that $\mathfrak{C}_{T_{S_{act}}} \notin \mathrm{Mod}(\overline{\Delta})$. However, this cannot be the case because $\mathfrak{C}_{T_{S_{act}}} \models \Delta$, and for any sentence $\varphi \in \overline{\Delta} \setminus \Delta$, we have $\varphi \in T_{S_{act}}$. Therefore, $\mathfrak{C}_{T_{S_{act}}} \models \varphi$, which means that $\mathfrak{C}_{T_{S_{act}}} \in \mathrm{Mod}(\overline{\Delta})$. This leads to a contradiction, so we conclude that $\overline{\Delta} \in RSp(\mathfrak{C}_{T_{S_{act}}})$.

Since $\mathfrak{C}_{T_{S_{act}}}$ is saturated, it follows that $\overline{\Delta}$ is a perfect Robinson theory. Hence, by Lemma 1, we obtain the equivalence: $HR^*(\dddot{\Delta}_{\mathfrak{U}}, \dddot{\Delta}_{\mathfrak{G}}) \overset{S}{\bowtie} \overline{\Delta}^* \Leftrightarrow HR(\dddot{\Delta}_{\mathfrak{U}}, \dddot{\Delta}_{\mathfrak{G}}) \overset{S}{\rtimes} \overline{\Delta}$, where $\overline{\Delta} = T'_{S_{act}}$.

Conclusion

This study has explored the fundamental aspects of Jonsson theories and the associated Jonsson spectra of their model classes, with a particular focus on the Robinson spectrum and the relationship between syntactic and semantic similarity. By analyzing how these concepts interact within the framework of model-theoretic structures, we highlighted the relevance of definability, compactness, and saturation in understanding the classification and behavior of models determined by Jonsson theories.

A promising and relatively unexplored direction for future research involves extending these ideas to the setting of positive Jonsson theories. This includes formulating a precise definition of the positive Jonsson spectrum and investigating how the syntactic-semantic correspondence and model-theoretic equivalences, such as K_T-equivalence, manifest in this more restrictive yet expressive framework. Foundational concepts and definitions for developing positive model theory in the context of Jonssonness are already outlined in [4, 17], offering a solid starting point for such an investigation.

Altogether, the theoretical insights presented in this paper offer a clearer understanding of classical Jonsson structures and establish a meaningful foundation for advancing future research on their positive counterparts.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors have no conflict of interest to declare.

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On interval Riemann double integration on time scales

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This article explores the theory of Riemann double integration for functions whose values are intervals in the framework of time scale calculus. We define the Riemann double Δ-integral and Riemann double ∇-integral for interval valued functions, namely interval Riemann $\Delta\Delta$-integral and interval Riemann $\nabla\nabla$-integral. Some key theorems in the article discuss the uniqueness of the integral, the equality of the interval Riemann double integral to the Riemann double integral when function is degenerate, necessary and sufficient conditions for integrability, proving integrability of a function without knowing the actual value of the integral. Additionally the relationship between the interval Riemann double integral and Riemann double integral for two interval-valued functions is estableshed via Hausdorff-Pompeiu distance. Elementary properties of the integral such as linearity property, subset property and others are established. Using the concept of generalized Hukuhara difference, alternate definitions of the interval Riemann $\Delta\Delta$-integral and interval Riemann $\nabla\nabla$-integral are formulated and theorems proving the equivalence of the integrals defined in both approaches are established. Theorems proving the equivalence of interval Riemann Δ- and ∇-integrals previously defined in both approaches are also shown.

Keywords: interval valued functions, Hausdorff-Pompeiu distance, Riemann $\Delta\Delta$-integral, Riemann $\nabla\nabla$-integral, generalized Hukuhara difference, interval Riemann $\Delta\Delta$-integral, interval Riemann $\nabla\nabla$-integral, time scales.

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Introduction and Motivation

S. Hilger in 1988, as part of his Würzburg doctoral degree [1], introduced the theory of measure chain calculus (which came to be known as the time scale calculus); transcripts later published in 1990, [2]. Time scale calculus unifies and extends discrete and continuous calculus; the theory proves immensely useful when dealing with hybrid models [3]. As theoretical framework, Hilger formulated three axioms [2] (also view [4; 1997]); any set, say $\boldsymbol{T}$, that satisfied these axioms were called time scales. By nature any closed subset $\boldsymbol{T}$ of $\mathbb{R}$ is a time scale, an excerpt "*...any closed subset of* $\mathbb{R}$ *bears the structure of a measure chain in a natural manner.*" [2] concludes this.

Hilger introduced two operators [2]. The forward jump operator denoted by σ and the backward jump operator denoted by ρ. Mapping $\sigma : \boldsymbol{T} \to \boldsymbol{T}$ such that $\sigma(t) = \inf\{u \in \boldsymbol{T} : u > t\}$. Similarly, mapping $\rho : \boldsymbol{T} \to \boldsymbol{T}$ such that $\rho(t) = \sup\{u \in \boldsymbol{T} : u < t\}$.

Using the notion of forward jump operator, Hilger in [2] formulated the delta derivative (Δ-derivative). A decade later in 2000, C. D. Ahlbrandt et al. [5] introduced a notion of derivative, which they called the alpha derivative, consisting both the Δ-derivative and another derivative called the nabla derivative (∇-derivative) as special cases. This ∇-derivative was formulated using the notion of backward jump operator, officially named so in 2002 by F. M. Atici et al. [6].

Integrations of the Δ-derivative and ∇-derivative are extensively discussed in literature, including for the Riemann integration. The Riemann integral for real valued functions on time scales was formulated by S. Sailer [7], using the concept of Darboux sum definition of the integral; and by

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G.S. Guseinov et al., using the concept of Riemann sum definition of the integral [8, 9]. The Riemann double Δ-integral for real valued functions was defined by M. Bohner et al. [10, 11].

Below we give the definition of Riemann double Δ-integral (Riemann $\Delta\Delta$-integral) and Riemann double ∇-integral (Riemann $\nabla\nabla$-integral) for real valued functions as defined in [10].

Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ be two given time scales and put $\boldsymbol{T}_1 \times \boldsymbol{T}_2 = \{(\hat{t}, \check{t}) : \hat{t} \in \boldsymbol{T}_1, \check{t} \in \boldsymbol{T}_2\}$.

The intervals on which integrals are defined, i.e., intervals on time scale $\boldsymbol{T}$ are defined as assuming $v \leq w$ [11]:

$[v, w]_{\boldsymbol{T}} = \{t \in \boldsymbol{T} : v \leq t \leq w\}$; $(v, w)_{\boldsymbol{T}} = \{t \in \boldsymbol{T} : v < t < w\}$;
$[v, w)_{\boldsymbol{T}} = \{t \in \boldsymbol{T} : v \leq t < w\}$; $(v, w]_{\boldsymbol{T}} = \{t \in \boldsymbol{T} : v < t \leq w\}$.

For clarity E, F will represent partitions for the Δ-integral and G, H will represent partitions for the ∇-integral.

Let $[v, w]_{\boldsymbol{T}}$ and $[r, s]_{\boldsymbol{T}}$ be closed intervals on $\boldsymbol{T}$ such that $[v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}} = \{(\hat{t}, \check{t}) : \hat{t} \in [v, w]_{\boldsymbol{T}}, \check{t} \in [r, s]_{\boldsymbol{T}}\}$. We partition the intervals as $[v = \hat{t}_0 < \hat{t}_1 < \ldots < \hat{t}_p = w]$, $p \in \mathbb{N}$ and $[r = \check{t}_0 < \check{t}_1 < \ldots < \check{t}_q = s]$, $q \in \mathbb{N}$; $\mathcal{P}([v, w]_{\boldsymbol{T}})$ will denote the collection of all possible partitions of $[v, w]_{\boldsymbol{T}}$ and $\mathcal{P}([r, s]_{\boldsymbol{T}})$ will denote the collection of all possible partitions of $[r, s]_{\boldsymbol{T}}$.

Let $\mathrm{E} = \{v = \hat{t}_0 < \ldots < \hat{t}_p = w\} \in \mathcal{P}([v, w]_{\boldsymbol{T}})$ and $\mathrm{F} = \{r = \check{t}_0 < \ldots < \check{t}_q = s\} \in \mathcal{P}([r, s]_{\boldsymbol{T}})$. Subintervals are taken to be of the form $[\hat{t}_{e-1}, \hat{t}_e)_{\boldsymbol{T}}$ for $1 \leq e \leq p$ and $[\check{t}_{f-1}, \check{t}_f)_{\boldsymbol{T}}$ for $1 \leq f \leq q$, which we will call the $\Delta\Delta$-subintervals. From each of these $\Delta\Delta$-subintervals we choose $\hat{\vartheta}_e \in [\hat{t}_{e-1}, \hat{t}_e)_{\boldsymbol{T}}$ and $\check{\vartheta}_f \in [\check{t}_{f-1}, \check{t}_f)_{\boldsymbol{T}}$ arbitrarily and call it the $\Delta\Delta$-tags. We define the mesh of E as, mesh-(E) $= \max_{1 \leq e \leq p}(\hat{t}_e - \hat{t}_{e-1}) > 0$. For some $\delta > 0$, E_δ will represent a Δ-partition of $[v, w]_{\boldsymbol{T}}$ with mesh δ satisfying the property: for each $e = 1, 2, \ldots, p$ we have either- $(\hat{t}_e - \hat{t}_{e-1}) \leq \delta$ or $(\hat{t}_e - \hat{t}_{e-1}) > \delta \wedge \rho(\hat{t}_e) = \hat{t}_{e-1}$ (here $\wedge$ stands for "and"). Again, mesh-(F) $= \max_{1 \leq f \leq q}(\check{t}_f - \check{t}_{f-1}) > 0$. For some $\delta > 0$, F_δ will represent a Δ-partition of $[r, s]_{\boldsymbol{T}}$ with mesh δ satisfying the property: for each $f = 1, 2, \ldots, q$ we have either $(\check{t}_f - \check{t}_{f-1}) \leq \delta$ or $(\check{t}_f - \check{t}_{f-1}) > \delta \wedge \rho(\check{t}_f) = \check{t}_{f-1}$.

Riemann $\Delta\Delta$-sum, $\mathrm{R}_{\Delta\Delta}(g; \mathrm{E}_\delta; \mathrm{F}_\delta)$, of real valued function "g" evaluated at the $\Delta\Delta$-tags as follows,

$$\mathrm{R}_{\Delta\Delta}(g; \mathrm{E}_\delta; \mathrm{F}_\delta) := \sum_{e=1}^{p} \sum_{f=1}^{q} g(\hat{\vartheta}_e, \check{\vartheta}_f)(\hat{t}_e - \hat{t}_{e-1})(\check{t}_f - \check{t}_{f-1}).$$

Definition 1. [10] (Riemann $\Delta\Delta$-integral) Let function $g : [v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}} \to \mathbb{R}$ be a real valued function. Function g is said to be Riemann $\Delta\Delta$-integrable if there exists an $I_{\Delta\Delta} \in \mathbb{R}$ on $[v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-partitions E_δ and F_δ, we have $\left|\mathrm{R}_{\Delta\Delta}(g; \mathrm{E}_\delta; \mathrm{F}_\delta) - I_{\Delta\Delta}\right| < \varepsilon$. Here $I_{\Delta\Delta} = \mathrm{R}_{\Delta\Delta} \int_v^w \int_r^s g(\hat{t}, \check{t}) \Delta\hat{t} \Delta\check{t}$, where $\mathrm{R}_{\Delta\Delta} \int_v^w \int_r^s g(\hat{t}, \check{t}) \Delta\hat{t} \Delta\check{t}$ is called the Riemann $\Delta\Delta$-integral.

Let $\mathrm{G} = \{v = \hat{t}_0 < \ldots < \hat{t}_p = w\} \in \mathcal{P}([v, w]_{\boldsymbol{T}})$ and $\mathrm{H} = \{r = \check{t}_0 < \ldots < \check{t}_q = s\} \in \mathcal{P}([r, s]_{\boldsymbol{T}})$. Subintervals are taken to be of the form $(\hat{t}_{e-1}, \hat{t}_e]_{\boldsymbol{T}}$ for $1 \leq e \leq p$ and $(\check{t}_{f-1}, \check{t}_f]_{\boldsymbol{T}}$ for $1 \leq f \leq q$, which we will call the $\nabla\nabla$-subintervals. From each of these $\nabla\nabla$-subintervals we choose $\hat{\xi}_e \in (\hat{t}_{e-1}, \hat{t}_e]_{\boldsymbol{T}}$ and $\check{\xi}_f \in (\check{t}_{f-1}, \check{t}_f]_{\boldsymbol{T}}$ arbitrarily and call it the $\nabla\nabla$-tags. We define the mesh of G as, mesh-(G) $= \max_{1 \leq e \leq p}(\hat{t}_e - \hat{t}_{e-1}) > 0$. For some $\delta > 0$, G_δ will represent a partition of $[v, w]_{\boldsymbol{T}}$ with mesh δ satisfying the property: for each $e = 1, 2, \ldots, p$ we have either $(\hat{t}_e - \hat{t}_{e-1}) \leq \delta$ or $(\hat{t}_e - \hat{t}_{e-1}) > \delta \wedge \sigma(\hat{t}_{e-1}) = \hat{t}_e$. Again, mesh-(H) $= \max_{1 \leq f \leq q}(\check{t}_f - \check{t}_{f-1}) > 0$. For some $\delta > 0$, H_δ will represent a partition of $[r, s]_{\boldsymbol{T}}$ with mesh δ satisfying the property: for each $f = 1, 2, \ldots, q$ we have either $(\check{t}_f - \check{t}_{f-1}) \leq \delta$ or $(\check{t}_f - \check{t}_{f-1}) > \delta \wedge \sigma(\check{t}_{f-1}) = \check{t}_f$.

Riemann $\nabla\nabla$-sum, $\mathrm{R}_{\nabla\nabla}(g; \mathrm{G}_\delta; \mathrm{H}_\delta)$, of real valued function "g" evaluated at the $\nabla\nabla$-tags as follows,

$$\mathrm{R}_{\nabla\nabla}(g; \mathrm{G}_\delta; \mathrm{H}_\delta) := \sum_{e=1}^{p} \sum_{f=1}^{q} g(\hat{\xi}_e, \check{\xi}_f)(\hat{t}_e - \hat{t}_{e-1})(\check{t}_f - \check{t}_{f-1}).$$

Definition 2. (Riemann $\nabla\nabla$-integral) Let function $g : [v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}} \to \mathbb{R}$ be a real valued function. Function g is said to be Riemann $\nabla\nabla$-integrable if there exists an $I_{\nabla\nabla} \in \mathbb{R}$ on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-partitions G_δ and H_δ, we have $\left|\mathrm{R}_{\nabla\nabla}(g;\mathrm{G}_\delta;\mathrm{H}_\delta) - I_{\nabla\nabla}\right| < \varepsilon$. Here $I_{\nabla\nabla} = \mathrm{R}_{\nabla\nabla}\int_v^w\int_r^s g(\hat{t},\check{t})\nabla\hat{t}\nabla\check{t}$, where $\mathrm{R}_{\nabla\nabla}\int_v^w\int_r^s g(\hat{t},\check{t})\nabla\hat{t}\nabla\check{t}$ is called the Riemann $\nabla\nabla$-integral.

We take a quick look at the theory of interval analysis. R.E. Moore's monograph [12] and [13] played a vital role as a catalyst to the modern era of extensive research on interval analysis. This monograph was the outgrowth of his Stanford PhD thesis titled "Interval arithmetic and automatic error analysis in digital computing" [14]. Intuitively, interval analysis uses closed intervals of real numbers instead of just numbers for calculations. Following we present basic concepts on classical interval analysis, view [13] for more insight.

Let $\mathbb{R}_\mathrm{I}$ denote the class of all non-empty compact intervals of real numbers. $[\mathrm{P}] = [\mathrm{P}^-, \mathrm{P}^+] \in \mathbb{R}_\mathrm{I}$; P^- represents the left end point and P^+ represents the right end point of interval $[\mathrm{P}]$. If $\mathrm{P}^- = \mathrm{P}^+$ then $[\mathrm{P}]$ is said to be degenerate.

Given $[\mathrm{P}], [\mathrm{Q}] \in \mathbb{R}_\mathrm{I}$, some rules of ordinary interval arithmetic are

$$\text{Minkowski addition}: \quad [\mathrm{P}] \oplus [\mathrm{Q}] = \left[\mathrm{P}^- + \mathrm{Q}^-, \mathrm{P}^+ + \mathrm{Q}^+\right].$$

$$\text{Scalar Product}: \quad \text{for } r \in \mathbb{R}, \quad r[\mathrm{P}] = \big\{[r\mathrm{P}^-, r\mathrm{P}^+] \text{ if } r > 0; \quad [0] \text{ if } r = 0; \quad [r\mathrm{P}^+, r\mathrm{P}^-] \text{ if } r < 0\big\}.$$

$$\text{Order}: \quad [\mathrm{P}] < [\mathrm{Q}] \text{ implies } \mathrm{P}^+ < \mathrm{Q}^-.$$

$$\text{Subset}: \quad [\mathrm{P}] \subseteq [\mathrm{Q}] \text{ if and only if } \mathrm{Q}^- < \mathrm{P}^- \text{ and } \mathrm{P}^+ < \mathrm{Q}^+.$$

$$\text{Absolute value}: \quad \big|[\mathrm{P}]\big| = \max\big\{|\mathrm{P}^-|, |\mathrm{P}^+|\big\}.$$

Reader is referred to [13] and [15] for theory on ordinary interval analysis.

The Hausdorff-Pompeiu distance between intervals $[\mathrm{P}]$ and $[\mathrm{Q}]$ is defined as

$$\mathsf{s}\big([\mathrm{P}], [\mathrm{Q}]\big) = \max\big\{|\mathrm{P}^- - \mathrm{Q}^-|, |\mathrm{P}^+ - \mathrm{Q}^+|\big\}.$$

It is known that $\big(\mathbb{R}_\mathrm{I}, \mathsf{s}\big)$ is a complete metric space. Properties of "s" are

1. $\mathsf{s}\big([\mathrm{P}], [\mathrm{Q}]\big) = 0 \Leftrightarrow [\mathrm{P}] = [\mathrm{Q}]$;
2. $\mathsf{s}\big(\gamma[\mathrm{P}], \gamma[\mathrm{Q}]\big) = |\gamma|\mathsf{s}\big([\mathrm{P}], [\mathrm{Q}]\big)$ for all $\gamma \in \mathbb{R}$;
3. $\mathsf{s}\big([\mathrm{P}] \oplus [\mathrm{R}], [\mathrm{Q}] \oplus [\mathrm{R}]\big) = \mathsf{s}\big([\mathrm{P}], [\mathrm{Q}]\big)$;
4. $\mathsf{s}\big([\mathrm{P}] \oplus [\mathrm{R}], [\mathrm{Q}] \oplus [\mathrm{S}]\big) \leq \mathsf{s}\big([\mathrm{P}], [\mathrm{Q}]\big) + \mathsf{s}\big([\mathrm{R}], [\mathrm{S}]\big)$,

For details on "s" refer [16].

L. Stefanini in [16, 17] details the general limitation of substraction of sets. To partially overcome this situation, M. Hukuhara [18] introduced the H-difference (Hukuhara difference) which was further generalized by L. Stefanini [17], referring to it as the generalized Hukuhara difference. We will denote generalized Hukuhara difference by "$\ominus_{\mathrm{gH}}$" defined as

$$[\mathrm{P}^-, \mathrm{P}^+] \ominus_{\mathrm{gH}} [\mathrm{Q}^-, \mathrm{Q}^+] = [\mathrm{R}^-, \mathrm{R}^+] \Leftrightarrow \begin{cases} \mathrm{P}^- = \mathrm{Q}^- + \mathrm{R}^-, \mathrm{P}^+ = \mathrm{Q}^+ + \mathrm{R}^+, \\ \qquad\qquad \text{or} \\ \mathrm{Q}^- = \mathrm{P}^- - \mathrm{R}^-, \mathrm{Q}^+ = \mathrm{P}^+ - \mathrm{R}^+, \end{cases}$$

so that $[\mathrm{P}^-, \mathrm{P}^+] \ominus_{\mathrm{gH}} [\mathrm{Q}^-, \mathrm{Q}^+] = [\mathrm{R}^-, \mathrm{R}^+]$ is always defined by

$$\mathrm{R}^- = \min\{\mathrm{P}^- - \mathrm{Q}^-, \mathrm{P}^+ - \mathrm{Q}^+\}, \ \mathrm{R}^+ = \max\{\mathrm{P}^- - \mathrm{Q}^-, \mathrm{P}^+ - \mathrm{Q}^+\},$$

i.e., $[\mathrm{P}] \ominus_{\mathrm{gH}} [\mathrm{Q}] = \big[\min\{\mathrm{P}^- - \mathrm{Q}^-, \mathrm{P}^+ - \mathrm{Q}^+\}, \max\{\mathrm{P}^- - \mathrm{Q}^-, \mathrm{P}^+ - \mathrm{Q}^+\}\big]$.

Properties of "$\ominus_{\mathrm{gH}}$" are

1. $[\mathrm{P}] \ominus_{\mathrm{gH}} [\mathrm{P}] = \{0\}$;
2. $\big([\mathrm{P}] \oplus [\mathrm{Q}]\big) \ominus_{\mathrm{gH}} [\mathrm{Q}] = [\mathrm{P}]$; $\;[\mathrm{P}] \ominus_{\mathrm{gH}} \big([\mathrm{P}] \oplus [\mathrm{Q}]\big) = -[\mathrm{Q}]$;
3. $\mathtt{s}\big([\mathrm{P}], [\mathrm{Q}]\big) = \mathtt{s}\big([\mathrm{P}] \ominus_{\mathrm{gH}} [\mathrm{Q}], [0]\big)$; here $[0] = [0, 0]$;
4. $\mathtt{s}\big([\mathrm{P}], [\mathrm{Q}]\big) = 0 \Leftrightarrow [\mathrm{P}] \ominus_{\mathrm{gH}} [\mathrm{Q}] = \{0\}$.

For more details on properties of "$\ominus_{\mathrm{gH}}$" one may refer [16] and [17].

Let $[v, w]_{\boldsymbol{T}}$ be a closed interval on $\boldsymbol{T}$. Function h is said to be an interval valued function if it assigns a nonempty interval

$$[h(t)] = \big[h(t)^-, h(t)^+\big] = \big\{h : h(t)^- \le h \le h(t)^+\big\},$$

for each $t \in [v, w]_{\boldsymbol{T}}$, where $h^-, h^+ : [v, w]_{\boldsymbol{T}} \to \mathbb{R}$ are real valued functions.

$h : [v, w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ and $t \in [v, w]_{\boldsymbol{T}}$, $l \in \mathbb{R}_{\mathrm{I}}$ is said to be an interval limit of h as t tends to u, denoted by $\lim_{t\to u} h(t) = l$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathtt{s}(h(t), l) < \varepsilon$ for all $|t - u| < \delta$. Here,

$$\lim_{t\to u} h(t) = l \Leftrightarrow \lim_{t\to u} \big(h(t) \ominus_{\mathrm{gH}} l\big) = \{0\},$$

where the interval limits are in the metric "$\mathtt{s}$". For $h(t) = [h^-(t), h^+(t)]$, $\lim_{t\to u} h(t)$ exists if and only if $\lim_{t\to u} h^-(t)$ and $\lim_{t\to u} h^+(t)$ exists as finite numbers. Here,

$$\lim_{t\to u} h(t) = \Big[\lim_{t\to u} h^-(t), \lim_{t\to u} h^+(t)\Big].$$

$h : [v, w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ is said to be interval continuous at $u \in [v, w]_{\boldsymbol{T}}$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathtt{s}([h(t)], [h(u)]) < \varepsilon$ whenever $|t - u| < \delta$. Also, h is interval continuous at $u \in [v, w]_{\boldsymbol{T}}$ if and only if its end points h^- and h^+ are continuous functions at $u \in [v, w]_{\boldsymbol{T}}$. If h is interval continuous at every $t \in [v, w]_{\boldsymbol{T}}$, then we say that h is interval continuous. h is said to be interval bounded, if for $B > 0$, $|[h(t)]| < B$ for all $t \in [v, w]_{\boldsymbol{T}}$.

Reader is referred to [19] and [20].

Integration of functions whose values are intervals (interval valued functions) have garnered much attention in recent years for both continuous calculus and time scale calculus.

For interval valued functions in continuous calculus, the interval Riemann integral was defined by O. Caprani et al. in [15] (also view [13]); the interval Henstock integral was defined by C. Wu et al. in [21]; the interval Henstock-Stieltjes integral was defined by M. E. Hamid [22]; the interval AP-Henstock integral was defined by M. E. Hamid et al. [23]; the interval AP-Henstock-Stieltjes integral was defined by G. S. Eun et al. [24]; and the interval McShane and interval McShane-Stieltjes integrals are defined by C. K. Park [25].

In 2013, V. Lupulescu [19] introduced the notion of interval analysis to the concept of time scale calculus pioneering extensive research that followed soon. He formulated differentiability and integrability for interval valued functions on time scales using generalized Hukuhara difference.

For interval valued function in time scale calculus the interval Riemann integral was defined by D. Zhao et al. [26] (Δ-integral) and by M. Bohner et al. [20] (∇-integral and $\Diamond_\alpha$-integral), the interval Riemann integral defined using the notion of generalized Hukuhara difference was given by V. Lupulescu [19]; the interval Riemann-Stieltjes integral was defined in [27] (Δ-integral and ∇-integral) and interval Riemann-Stieltjes integral using the notion of generalized Hukuhara difference was also defined in the same [27] (Δ-integral and ∇-integral); the interval Henstock integral was defined by W.T. Oh et al. [28] (Δ-integral); the interval Henstock-Stieltjes integral was defined by J.H. Yoon [29] (Δ-integral); the interval McShane integral was defined by M.E. Hamid et al. [30] (Δ-integral); the interval McShane-Stieltjes integral was defined by M.E. Hamid [31] (Δ-integral); and the interval Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral was defined by D.A. Afariogun et al. [32, 33].

Given $\boldsymbol{T}_1 \times \boldsymbol{T}_2 = \{(\hat{t}, \check{t}) : \hat{t} \in \boldsymbol{T}_1, \check{t} \in \boldsymbol{T}_2\}$, and $[P] = [(P_1, P_2)]$, $[Q] = [(Q_1, Q_2)]$, "$\mathtt{s}$" forms a complete metric space defined as [32,33]

$$\begin{aligned}\mathtt{s}\big([P],[Q]\big) &= \mathtt{s}\big([(P_1,P_2)],[(Q_1,Q_2)]\big)\\ &= \max\left\{\sqrt{(Q_1^- - P_1^-)^2 + (Q_2^- - P_2^-)^2}, \sqrt{(Q_1^+ - P_1^+)^2 + (Q_2^+ - P_2^+)^2}\right\}.\end{aligned}$$

Below we give the definition of interval Riemann Δ-integral and interval Riemann ∇-integral according to D. Zhao et al. [26] and M. Bohner et al. [20], respectively.

We partition $[v, w]_{\boldsymbol{T}}$ as $\mathrm{E} = \big\{v = t_0 < \ldots < t_p = w\big\} \in \mathcal{P}([v, w]_{\boldsymbol{T}})$. Δ-subintervals are of the form $[t_{e-1}, t_e)_{\boldsymbol{T}}$; Δ-tags are $\vartheta_e \in [t_{e-1}, t_e)_{\boldsymbol{T}}$ taken arbitrarily. For some $\delta > 0$, E_δ will represent a Δ-partition of $[v, w]_{\boldsymbol{T}}$ with mesh δ.

Definition 3. [26](Interval Riemann Δ-integral) Let function $h : [v, w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann Δ-integrable if there exists an interval $[I_\Delta] \in \mathbb{R}_{\mathrm{I}}$ on $[v, w]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-partition E_δ, we have $\mathtt{s}\big(\mathrm{IR}_\Delta(h; \mathrm{E}_\delta), [I_\Delta]\big) < \varepsilon$. Here $[I_\Delta] = \mathrm{IR}_\Delta \int_v^w h(t)\Delta t$; $\mathrm{IR}_\Delta(h; \mathrm{E}_\delta) := \sum_{e=1}^{p} [h(\vartheta_e)](t_e - t_{e-1})$.

The set of all interval Riemann Δ-integrable functions on $[v, w]_{\boldsymbol{T}}$ will be denoted by $\big\{\mathrm{IR}_\Delta[v, w]_{\boldsymbol{T}}\big\}$.

The interval Riemann Δ-integral defined using the notion of generalized Hukuhara difference was given by V. Lupulescu [19] as

Definition 4. [19] Let function $h : [v, w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann Δ-integrable if there exists an interval $[I_\Delta] \in \mathbb{R}_{\mathrm{I}}$ on $[v, w]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-partition E_δ, we have $\mathtt{s}\big(\mathrm{IR}_\Delta(h; \mathrm{E}_\delta) \ominus_{gH} [I_\Delta], [0]\big) < \varepsilon$. Here $[I_\Delta] = \mathrm{IR}_\Delta \int_v^w h(t)\Delta t$; $\mathrm{IR}_\Delta(h; \mathrm{E}_\delta) := \sum_{e=1}^{p} [h(\vartheta_e)](t_e - t_{e-1})$.

We formulate a theorem (Theorem 1) which proves the equivalence of Definition 3 (as defined in [26]) and Definition 4 (as defined in [19]) below

Theorem 1. If $h \in \big\{\mathrm{IR}_\Delta[v, w]_{\boldsymbol{T}}\big\}$ then, h is interval Riemann Δ-integrable defined using the generalized Hukuhara difference and vice versa.

Proof. Suppose $h \in \big\{\mathrm{IR}_\Delta[v, w]_{\boldsymbol{T}}\big\}$ (Definition 3), then $\mathtt{s}\big(\mathrm{IR}_\Delta(h; \mathrm{E}_\delta), [I_\Delta]\big) < \varepsilon$. Hence,

$$\begin{aligned}&\mathtt{s}\Big(\Big[\min\Big\{\mathrm{IR}_\Delta(h^-; \mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+; \mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\}, \max\Big\{\mathrm{IR}_\Delta(h^-; \mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+; \mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\}\Big], [0]\Big)\\ &= \max\Big\{\Big|\min\Big\{\mathrm{IR}_\Delta(h^-; \mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+; \mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\} - 0^-\Big|, \Big|\max\Big\{\mathrm{IR}_\Delta(h^-; \mathrm{E}_\delta) - \mathrm{I}_\Delta^-,\\ &\mathrm{IR}_\Delta(h^+; \mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\} - 0^+\Big|\Big\} = \max\Big\{\Big|\min\Big\{\mathrm{IR}_\Delta(h^-; \mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+; \mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\}\Big|,\\ &\Big|\max\Big\{\mathrm{IR}_\Delta(h^-; \mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+; \mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\}\Big|\Big\} = \Big|\max\Big\{\mathrm{IR}_\Delta(h^-; \mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+; \mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\}\Big| < \varepsilon.\end{aligned}$$

Thus, if $h \in \big\{\mathrm{IR}_\Delta[v, w]_{\boldsymbol{T}}\big\}$ implies h is interval Riemann Δ-integral defined using the generalized Hukuhara difference. The converse is proved similarly.

For the ∇-integral, we partition $[v, w]_{\boldsymbol{T}}$ as $\mathrm{G} = \big\{v = t_0 < \ldots < t_p = w\big\} \in \mathcal{P}([v, w]_{\boldsymbol{T}})$. ∇-subintervals are of the form $(t_{e-1}, t_e]_{\boldsymbol{T}}$; ∇-tags are $\xi_e \in (t_{e-1}, t_e]_{\boldsymbol{T}}$ taken arbitrarily. For some $\delta > 0$, G_δ will represent a ∇-partition of $[v, w]_{\boldsymbol{T}}$ with mesh δ.

Definition 5. [20](Interval Riemann ∇-integral) Let function $h : [v, w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann ∇-integrable if there exists an interval $[I_\nabla] \in \mathbb{R}_{\mathrm{I}}$ on $[v, w]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-partition G_δ, we have $\mathtt{s}\big(\mathrm{IR}_\nabla(h; \mathrm{G}_\delta), I_\nabla]\big) < \varepsilon$. Here $[I_\nabla] = \mathrm{IR}_\nabla \int_v^w h(t)\nabla t$; $\mathrm{IR}_\nabla(h; \mathrm{G}_\delta) := \sum_{e=1}^{p} [h(\xi_e)](t_e - t_{e-1})$.

The set of all interval Riemann ∇-integrable functions on $[v,w]_{\boldsymbol{T}}$ will be denoted by $\{\mathrm{IR}_{\nabla}[v,w]_{\boldsymbol{T}}\}$.

The interval Riemann ∇-integral defined using the notion of generalized Hukuhara difference is given below

Definition 6. Let function $h : [v,w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann ∇-integrable if there exists an interval $[I_{\nabla}] \in \mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-partition G_{δ}, we have $\mathtt{s}\big(\mathrm{IR}_{\nabla}(h;\mathrm{G}_{\delta}) \ominus_{gH} [I_{\nabla}],[0]\big) < \varepsilon$. Here $[I_{\nabla}] = \mathrm{IR}_{\nabla}\int_v^w h(t)\nabla t$; $\mathrm{IR}_{\nabla}(h;\mathrm{G}_{\delta}) := \sum_{e=1}^{p}[h(\xi_e)](t_e - t_{e-1})$.

Theorem 2 states the equivalence of Definition 5 (as defined in [20]) and Definition 6; proof of the statement is omitted due to similarity with Theorem 1.

Theorem 2. If $h \in \{\mathrm{IR}_{\nabla}[v,w]_{\boldsymbol{T}}\}$, then h is interval Riemann ∇-integrable defined using the generalized Hukuhara difference and vice versa.

To the best of our knowledge, Riemann double integral for interval valued functions on time scales has not been discussed in literature. Hence, the primary objective of this paper is to define the interval Riemann $\Delta\Delta$- and $\nabla\nabla$-integrals and establish some fascinating results.

1 Interval Riemann double integration

Partitioning $[v,w]_{\boldsymbol{T}}$ as $\mathrm{E} = \{v = \hat{t}_0 < \ldots < \hat{t}_p = w\} \in \mathcal{P}([v,w]_{\boldsymbol{T}})$ and $[r,s]_{\boldsymbol{T}}$ as $\mathrm{F} = \{r = \breve{t}_0 < \ldots < \breve{t}_q = s\} \in \mathcal{P}([r,s]_{\boldsymbol{T}})$. $\Delta\Delta$-subintervals for $[v,w]_{\boldsymbol{T}}$ and $[r,s]_{\boldsymbol{T}}$ are of the form $[\hat{t}_{e-1},\hat{t}_e)_{\boldsymbol{T}}$ and $[\breve{t}_{f-1},\breve{t}_f)_{\boldsymbol{T}}$ respectively. $\Delta\Delta$-tags are $\hat{\vartheta}_e \in [\hat{t}_{e-1},\hat{t}_e)_{\boldsymbol{T}}$ and $\breve{\vartheta}_f \in [\breve{t}_{f-1},\breve{t}_f)_{\boldsymbol{T}}$ taken arbitrarily. For some $\delta > 0$, E_{δ} and F_{δ} will represent Δ-partitions of $[v,w]_{\boldsymbol{T}}$ and $[r,s]_{\boldsymbol{T}}$ respectively with mesh δ.

Interval Riemann $\Delta\Delta$-sum, $\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_{\delta};\mathrm{F}_{\delta})$, of interval valued function "h" evaluated at the $\Delta\Delta$-tags as follows,

$$\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_{\delta};\mathrm{F}_{\delta}) := \sum_{e=1}^{p}\sum_{f=1}^{q}\big[h(\hat{\vartheta}_e,\breve{\vartheta}_f)\big](\hat{t}_e - \hat{t}_{e-1})(\breve{t}_f - \breve{t}_{f-1}),$$

$$\text{i.e.,}\ \ \mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_{\delta};\mathrm{F}_{\delta}) = \big[h(\hat{\vartheta}_1,\breve{\vartheta}_1)^-\big(\hat{t}_1 - \hat{t}_0\big)\big(\breve{t}_1 - \breve{t}_0\big), h(\hat{\vartheta}_1,\breve{\vartheta}_1)^+\big(\hat{t}_1 - \hat{t}_0\big)\big(\breve{t}_1 - \breve{t}_0\big)\big] \oplus \ldots \oplus \\ \big[h(\hat{\vartheta}_p,\breve{\vartheta}_q)^-\big(\hat{t}_p - \hat{t}_{p-1}\big)\big(\breve{t}_q - \breve{t}_{q-1}\big), h(\hat{\vartheta}_p,\breve{\vartheta}_q)^+\big(\hat{t}_p - \hat{t}_{p-1}\big)\big(\breve{t}_q - \breve{t}_{q-1}\big)\big].$$

Here,

$$\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_{\delta};\mathrm{F}_{\delta}) := \sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^-(\hat{t}_e - \hat{t}_{e-1})(\breve{t}_f - \breve{t}_{f-1}),$$

$$\mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_{\delta};\mathrm{F}_{\delta}) := \sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^+(\hat{t}_e - \hat{t}_{e-1})(\breve{t}_f - \breve{t}_{f-1}).$$

Definition 7. (Interval Riemann $\Delta\Delta$-integral) Let function $h : [v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann $\Delta\Delta$-integrable if there exists an interval $[I_{\Delta\Delta}] \in \mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-partitions E_{δ} and F_{δ}, we have

$$\mathtt{s}\big(\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_{\delta};\mathrm{F}_{\delta}),[I_{\Delta\Delta}]\big) < \varepsilon.$$

Here $[I_{\Delta\Delta}] = \mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}$, where $\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}$ is called the interval Riemann $\Delta\Delta$-integral.

The set of all interval Riemann $\Delta\Delta$-integrable functions on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ will be denoted by $\{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$.

Example 1. 1. When $\boldsymbol{T} = \mathbb{R}$, the interval Riemann $\Delta\Delta$-integral coincides with the usual interval Riemann double integral in $\mathbb{R}$.

2. When $\boldsymbol{T} = a\mathbb{Z}$, here $a \in \mathbb{R}$ and $v,w,r,s \in a\mathbb{Z}$, if $h \in \{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$, then

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\check{t})\Delta\hat{t}\Delta\check{t} = a^2 \cdot \sum_{i=\frac{v}{a}}^{\frac{w}{a}-1}\sum_{j=\frac{r}{a}}^{\frac{s}{a}-1}\big[h(ai,aj)\big]$$

$$= a^2 \cdot \sum_{i=\frac{v}{a}}^{\frac{w}{a}-1}\sum_{j=\frac{r}{a}}^{\frac{s}{a}-1}\Big[h(ai,aj)^-, h(ai,aj)^+\Big].$$

If $a=1$, $\boldsymbol{T} = \mathbb{Z}$ and

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\check{t})\Delta\hat{t}\Delta\check{t} = \sum_{i=v}^{w-1}\sum_{j=r}^{s-1}\big[h(i,j)\big].$$

For the $\nabla\nabla$-integral, we partition $[v,w]_{\boldsymbol{T}}$ as $\mathrm{G} = \{v = \hat{t}_0 < \ldots < \hat{t}_p = w\} \in \mathcal{P}([v,w]_{\boldsymbol{T}})$ and $[r,s]_{\boldsymbol{T}}$ as $\mathrm{H} = \{r = \check{t}_0 < \ldots < \check{t}_q = s\} \in \mathcal{P}([r,s]_{\boldsymbol{T}})$. $\nabla\nabla$-subintervals for $[v,w]_{\boldsymbol{T}}$ and $[r,s]_{\boldsymbol{T}}$ are of the form $(\hat{t}_{e-1},\hat{t}_e]_{\boldsymbol{T}}$ and $(\check{t}_{f-1},\check{t}_f]_{\boldsymbol{T}}$, respectively. $\nabla\nabla$-tags are $\hat{\xi}_e \in (\hat{t}_{e-1},\hat{t}_e]_{\boldsymbol{T}}$ and $\check{\xi}_f \in (\check{t}_{f-1},\check{t}_f]_{\boldsymbol{T}}$ taken arbitrarily. For some $\delta > 0$, G_δ and H_δ will represent ∇-partitions of $[v,w]_{\boldsymbol{T}}$ and $[r,s]_{\boldsymbol{T}}$ respectively with mesh δ.

Interval Riemann $\nabla\nabla$-sum, $\mathrm{IR}_{\nabla\nabla}(h;\mathrm{G}_\delta;\mathrm{H}_\delta)$, of interval valued function h evaluated at the $\nabla\nabla$-tags as follows,

$$\mathrm{IR}_{\nabla\nabla}(h;\mathrm{G}_\delta;\mathrm{H}_\delta) := \sum_{e=1}^{p}\sum_{f=1}^{q}\big[h(\hat{\xi}_e,\check{\xi}_f)\big](\hat{t}_e - \hat{t}_{e-1})(\check{t}_f - \check{t}_{f-1}),$$

$$\text{i.e., } \mathrm{IR}_{\nabla\nabla}(h;\mathrm{G}_\delta;\mathrm{H}_\delta) = \big[h(\hat{\xi}_1,\check{\xi}_1)^-\big(\hat{t}_1-\hat{t}_0\big)\big(\check{t}_1-\check{t}_0\big), h(\hat{\xi}_1,\check{\xi}_1)^+\big(\hat{t}_1-\hat{t}_0\big)\big(\check{t}_1-\check{t}_0\big)\big] \oplus \ldots \oplus$$
$$\big[h(\hat{\xi}_p,\check{\xi}_q)^-\big(\hat{t}_p-\hat{t}_{p-1}\big)\big(\check{t}_q-\check{t}_{q-1}\big), h(\hat{\xi}_p,\check{\xi}_q)^+\big(\hat{t}_p-\hat{t}_{p-1}\big)\big(\check{t}_q-\check{t}_{q-1}\big)\big].$$

Here,

$$\mathrm{IR}_{\nabla\nabla}(h^-;\mathrm{G}_\delta;\mathrm{H}_\delta) := \sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\xi}_e,\check{\xi}_f)^-(\hat{t}_e - \hat{t}_{e-1})(\check{t}_f - \check{t}_{f-1}),$$

$$\mathrm{IR}_{\nabla\nabla}(h^+;\mathrm{G}_\delta;\mathrm{H}_\delta) := \sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\xi}_e,\check{\xi}_f)^+(\hat{t}_e - \hat{t}_{e-1})(\check{t}_f - \check{t}_{f-1}).$$

Definition 8. (Interval Riemann $\nabla\nabla$-integral) Let function $h : [v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann $\nabla\nabla$-integrable if there exists an interval $[I_{\nabla\nabla}] \in \mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-partitions G_δ and H_δ, we have

$$\mathsf{s}\big(\mathrm{IR}_{\nabla\nabla}(h;\mathrm{G}_\delta;\mathrm{H}_\delta), [I_{\nabla\nabla}]\big) < \varepsilon.$$

Here $[I_{\nabla\nabla}] = \mathrm{IR}_{\nabla\nabla}\int_v^w\int_r^s h(\hat{t},\check{t})\nabla\hat{t}\nabla\check{t}$, where $\mathrm{IR}_{\nabla\nabla}\int_v^w\int_r^s h(\hat{t},\check{t})\nabla\hat{t}\nabla\check{t}$ is called the interval Riemann $\nabla\nabla$-integral.

The set of all interval Riemann $\nabla\nabla$-integrable functions on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ will be denoted by $\{\mathrm{IR}_{\nabla\nabla}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$.

Example 2. 1. When $\boldsymbol{T} = \mathbb{R}$, the interval Riemann $\nabla\nabla$-integral coincides with the usual interval Riemann double integral in $\mathbb{R}$.

2. When $\boldsymbol{T} = a\mathbb{Z}$, here $a \in \mathbb{R}$ and $v, w, r, s \in a\mathbb{Z}$, if $h \in \{\mathrm{IR}_{\nabla\nabla}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$, then

$$\mathrm{IR}_{\nabla\nabla}\int_v^w \int_r^s h(\hat{t},\breve{t})\nabla\hat{t}\nabla\breve{t} = a^2 \cdot \sum_{i=\frac{v}{a}+1}^{\frac{w}{a}} \sum_{j=\frac{r}{a}+1}^{\frac{s}{a}} \big[h(ai,aj)\big]$$
$$= a^2 \cdot \sum_{i=\frac{v}{a}+1}^{\frac{w}{a}} \sum_{j=\frac{r}{a}+1}^{\frac{s}{a}} \Big[h(ai,aj)^-, h(ai,aj)^+\Big].$$

If $a = 1$, $\boldsymbol{T} = \mathbb{Z}$ and

$$\mathrm{IR}_{\nabla\nabla}\int_v^w \int_r^s h(\hat{t},\breve{t})\nabla\hat{t}\nabla\breve{t} = \sum_{i=v+1}^{w} \sum_{j=r+1}^{s} \big[h(i,j)\big].$$

Following statements and theorems will be given in regard to the $\Delta\Delta$-integral, $\nabla\nabla$-integral versions are omitted due to their similarity.

Remark 1. If $h \in \{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$, then the value of integral $[I_{\Delta\Delta}]$ is unique and well-defined.

If $h \in \{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$ and h is degenerate, then interval Riemann $\Delta\Delta$-integral (Definition 7) equals Riemann $\Delta\Delta$-integral (Definition 1).

Theorem 3. Let $h : [\hat{t}_0, \sigma(\hat{t}_0)]_{\boldsymbol{T}} \times [\breve{t}_0, \sigma(\breve{t}_0)]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$, then $h \in \{\mathrm{IR}_{\Delta\Delta}[\hat{t}_0, \sigma(\hat{t}_0)]_{\boldsymbol{T}} \times [\breve{t}_0, \sigma(\breve{t}_0)]_{\boldsymbol{T}}\}$ and

$$\mathrm{IR}_{\Delta\Delta}\int_{\hat{t}_0}^{\sigma(\hat{t}_0)} \int_{\breve{t}_0}^{\sigma(\breve{t}_0)} h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t} = \Big[h(\hat{t}_0,\breve{t}_0)^-\big(\sigma(\hat{t}_0)-\hat{t}_0\big)\big(\sigma(\breve{t}_0)-\breve{t}_0\big), h(\hat{t}_0,\breve{t}_0)^+\big(\sigma(\hat{t}_0)-\hat{t}_0\big)\big(\sigma(\breve{t}_0)-\breve{t}_0\big)\Big].$$

Theorem 4. Let $h : [\rho(\hat{t}_0), \hat{t}_0]_{\boldsymbol{T}} \times [\rho(\breve{t}_0), \breve{t}_0]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$, then $h \in \{\mathrm{IR}_{\Delta\Delta}[\rho(\hat{t}_0), \hat{t}_0]_{\boldsymbol{T}} \times [\rho(\breve{t}_0), \breve{t}_0]_{\boldsymbol{T}}\}$ and

$$\mathrm{IR}_{\Delta\Delta}\int_{\rho(\hat{t}_0)}^{\hat{t}_0} \int_{\rho(\breve{t}_0)}^{\breve{t}_0} h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t} = \Big[h\big(\rho(\hat{t}_0),\rho(\breve{t}_0)\big)^-\big(\hat{t}_0-\rho(\hat{t}_0)\big)\big(\breve{t}_0-\rho(\breve{t}_0)\big), h\big(\rho(\hat{t}_0),\rho(\breve{t}_0)\big)^+$$
$$\big(\hat{t}_0-\rho(\hat{t}_0)\big)\big(\breve{t}_0-\rho(\breve{t}_0)\big)\Big].$$

Theorem 5. An interval valued function $h : [v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ is interval Riemann $\Delta\Delta$-integrable on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ if and only if h^- and h^+ are Riemann $\Delta\Delta$-integrable on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ and

$$\mathrm{IR}_{\Delta\Delta}\int_v^w \int_r^s h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t} = \left[\mathrm{R}_{\Delta\Delta}\int_v^w \int_r^s h(\hat{t},\breve{t})^-\Delta\hat{t}\Delta\breve{t}, \mathrm{R}_{\Delta\Delta}\int_v^w \int_r^s h(\hat{t},\breve{t})^+\Delta\hat{t}\Delta\breve{t}\right].$$

Proof. If $h \in \{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$, then integral $[I_{\Delta\Delta}] = [I^-_{\Delta\Delta}, I^+_{\Delta\Delta}]$ such that for each $\varepsilon > 0$ there exists δ such that

$$\mathbf{s}\big(\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_\delta;\mathrm{F}_\delta), [I_{\Delta\Delta}]\big) = \max\left\{\Big|\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}\Big|, \Big|\mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big|\right\}$$
$$= \max\left\{\left|\sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^-(\hat{t}_e-\hat{t}_{e-1})(\breve{t}_f-\breve{t}_{f-1}) - I^-_{\Delta\Delta}\right|,\right.$$
$$\left.\left|\sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^+(\hat{t}_e-\hat{t}_{e-1})(\breve{t}_f-\breve{t}_{f-1}) - I^+_{\Delta\Delta}\right|\right\} < \varepsilon,$$

thus,

$$\left|\sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^-(\hat{t}_e-\hat{t}_{e-1})(\breve{t}_f-\breve{t}_{f-1})-I^-_{\Delta\Delta}\right|<\varepsilon \text{ and}$$
$$\left|\sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^+(\hat{t}_e-\hat{t}_{e-1})(\breve{t}_f-\breve{t}_{f-1})-I^+_{\Delta\Delta}\right|<\varepsilon,$$

hence we conclude.

Conversely, let h^-, h^+ be Riemann $\Delta\Delta$-integrable on $[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}$, then there exists $I_1, I_2\in\mathbb{R}$ such that for each $\varepsilon>0$, there exists δ such that

$$\left|\mathrm{R}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta)-I_1\right|<\varepsilon \text{ and } \left|\mathrm{R}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta)-I_2\right|<\varepsilon.$$

Letting $[I_{\Delta\Delta}]=[I_1,I_2]$, we have

$$\max\left\{\left|\mathrm{R}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta)-I_1\right|,\left|\mathrm{R}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta)-I_2\right|\right\}=\max\left\{\left|\sum_{e=1}^{p}\sum_{f=1}^{q}h(\hat{\vartheta}_e,\breve{\vartheta}_f)^-(\hat{t}_e-\hat{t}_{e-1})\right.\right.$$
$$\left.\left.(\breve{t}_f-\breve{t}_{f-1})-I^-_{\Delta\Delta}\right|,\left|\sum_{e=1}^{p}\sum_{f=1}^{q}h(\hat{\vartheta}_e,\breve{\vartheta}_f)^+(\hat{t}_e-\hat{t}_{e-1})(\breve{t}_f-\breve{t}_{f-1})-I^+_{\Delta\Delta}\right|\right\}<\varepsilon,$$

implies $\mathbf{s}\big(\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_\delta;\mathrm{F}_\delta),[I_{\Delta\Delta}]\big)<\varepsilon$ hence we conclude.

Without actually knowing the value of the integral, we can prove the integrability of a function via the criterion of integrability. It is stated as

Theorem 6. An interval valued function $h:[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}\to\mathbb{R}_{\mathrm{I}}$ is interval Riemann $\Delta\Delta$-integrable on $[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}$ if and only if for each $\varepsilon>0$ there exists δ such that any Δ-partitions $\mathrm{E}_{1_\delta},\mathrm{F}_{1_\delta}$ and $\mathrm{E}_{2_\delta},\mathrm{F}_{2_\delta}$ with mesh$<\delta$ implies

$$\mathbf{s}\big(\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_{1_\delta};\mathrm{F}_{1_\delta}),\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_{2_\delta};\mathrm{F}_{2_\delta})\big)<\varepsilon.$$

A function $h:[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}\to\mathbb{R}_{\mathrm{I}}$ is said to be interval continuous at $(\hat{t}_0,\breve{t}_0)\in[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}$ if for each $\varepsilon>0$ there exists $\delta>0$ such that $\mathbf{s}\big([h(\hat{t},\breve{t})],[h(\hat{t}_0,\breve{t}_0)]\big)<\varepsilon$, whenever $\sqrt{(\hat{t}_0-\hat{t})^2+(\breve{t}_0-\breve{t})^2}<\delta$.

Interval boundedness and interval continuity of a function are sufficient conditions for the existence of interval Riemann double integrability.

Theorem 7. Every bounded continuous interval valued function is interval Riemann $\Delta\Delta$-integrable, and

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}=\left[\mathrm{R}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})^-\Delta\hat{t}\Delta\breve{t},\mathrm{R}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})^+\Delta\hat{t}\Delta\breve{t}\right].$$

Below we establish a relation between interval Riemann $\Delta\Delta$-integral and Riemann $\Delta\Delta$-integral for two interval valued functions via Hausdorff-Pompeiu distance.

Theorem 8. Let $h_1,h_2\in\big\{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}\big\}$, if given $\mathbf{s}\big([h_1(\hat{t},\breve{t})],[h_2(\hat{t},\breve{t})]\big)$ is Riemann $\Delta\Delta$-integral then,

$$\mathbf{s}\Big(\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t},\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}\Big)\le\mathrm{R}_{\Delta\Delta}\int_v^w\int_r^s\mathbf{s}\big([h_1(\hat{t},\breve{t})],[h_2(\hat{t},\breve{t})]\big)\Delta\hat{t}\Delta\breve{t}.$$

Proof. By definition of distance we have,

$$\begin{aligned}
&\mathbf{s}\Big(\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}, \mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}\Big)\\
&= \max\Big\{\Big|\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat{t},\breve{t})^-\Delta\hat{t}\Delta\breve{t} - \mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat{t},\breve{t})^-\Delta\hat{t}\Delta\breve{t}\Big|,\\
&\quad \Big|\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat{t},\breve{t})^+\Delta\hat{t}\Delta\breve{t} - \mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat{t},\breve{t})^+\Delta\hat{t}\Delta\breve{t}\Big|\Big\}\\
&\le \max\Big\{\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s \big|h_1(\hat{t},\breve{t})^- - h_2(\hat{t},\breve{t})^-\big|\Delta\hat{t}\Delta\breve{t}, \mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s \big|h_1(\hat{t},\breve{t})^+ - h_2(\hat{t},\breve{t})^+\big|\Delta\hat{t}\Delta\breve{t}\Big\}\\
&\le \mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s \max\Big\{\big|h_1(\hat{t},\breve{t})^- - h_2(\hat{t},\breve{t})^-\big|, \big|h_1(\hat{t},\breve{t})^+ - h_2(\hat{t},\breve{t})^+\big|\Big\}\Delta\hat{t}\Delta\breve{t}\\
&= \mathrm{R}_{\Delta\Delta}\int_v^w\int_r^s \mathrm{s}\big([h_1(\hat{t},\breve{t})],[h_2(\hat{t},\breve{t})]\big)\Delta\hat{t}\Delta\breve{t}.
\end{aligned}$$

Theorem 9. Let $h_1, h_2 \in \{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$ and $\gamma \in \mathbb{R}$, then

1. $\gamma h_1 \in \{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$ and

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s \gamma h_1(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t} = \gamma\, \mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t},$$

2. $h_1 + h_2 \in \{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$ and

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s (h_1+h_2)(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t} = \mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t} +$$

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t},$$

3. $h_1(\hat{t},\breve{t}) \subseteq h_2(\hat{t},\breve{t})$

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t} \subseteq \mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}.$$

Definition 7 and Definition 8 can also be alternatively defined using the generalized Hukuhara difference as

Definition 9. Let function $h : [v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann $\Delta\Delta$-integrable if there exists an interval $[I_{\Delta\Delta}] \in \mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-partitions E_δ and F_δ, we have

$$\mathbf{s}\big(\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_\delta;\mathrm{F}_\delta) \ominus_{\mathrm{gH}} [I_{\Delta\Delta}], [0]\big) < \varepsilon.$$

Here $[I_{\Delta\Delta}] = \mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}$, where $\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}$ is called the interval Riemann $\Delta\Delta$-integral.

We establish a theorem which proves the equivalence of Definition 7 and Definition 9.

Theorem 10. If $h \in \{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$ then, h is interval Riemann $\Delta\Delta$-integrable defined using the generalized Hukuhara difference and vice versa.

Proof. Suppose $h \in \left\{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\right\}$ (Definition 7), then $\mathbf{s}\Big(\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_\delta;\mathrm{F}_\delta), \big[I_{\Delta\Delta}\big]\Big) < \varepsilon$. Hence,

$$\begin{aligned}
&\mathbf{s}\Big(\Big[\min\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\},\\
&\qquad \max\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\}\Big], [0]\Big)\\
&\qquad = \max\Big\{\Big|\min\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\} - 0^-\Big|,\\
&\qquad\quad \Big|\max\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\} - 0^+\Big|\Big\}\\
&\qquad = \max\Big\{\Big|\min\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\}\Big|,\\
&\qquad\quad \Big|\max\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\}\Big|\Big\}\\
&\qquad = \Big|\max\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\}\Big| < \varepsilon.
\end{aligned}$$

Thus, if $h \in \left\{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\right\}$ implies h is interval Riemann $\Delta\Delta$-integrable defined using the generalized Hukuhara difference. The converse is proved similarly.

Definition 8 is alternatively defined using the notion of generalized Hukuhara difference as

Definition 10. Let function $h : [v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann $\nabla\nabla$-integrable if there exists an interval $[I_{\nabla\nabla}] \in \mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-partitions G_δ and H_δ, we have

$$\mathbf{s}\big(\mathrm{IR}_{\nabla\nabla}(h;\mathrm{G}_\delta;\mathrm{H}_\delta) \ominus_{\mathrm{gH}} [I_{\nabla\nabla}], [0]\big) < \varepsilon.$$

Here $[I_{\nabla\nabla}] = \mathrm{IR}_{\nabla\nabla} \int_v^w \int_r^s h(\hat{t},\breve{t})\nabla\hat{t}\nabla\breve{t}$, where $\mathrm{IR}_{\nabla\nabla} \int_v^w \int_r^s h(\hat{t},\breve{t})\nabla\hat{t}\nabla\breve{t}$ is called the interval Riemann $\nabla\nabla$-integral.

We establish a theorem which proves the equivalence of Definition 8 and Definition 10; prove is omitted due to its similarity with Theorem 10.

Theorem 11. If $h \in \left\{\mathrm{IR}_{\nabla\nabla}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\right\}$, then h is interval Riemann $\nabla\nabla$-integrable defined using the generalized Hukuhara difference and vice versa.

Conclusion

This paper explores the theory of Riemann double integration for interval valued functions on time scales and discuss a few fascinating results.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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q-Analogues of Lyapunov-type inequalities involving Riemann–Liouville fractional derivatives

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In this article, new q-analogues of Lyapunov-type inequalities are presented for two-point fractional boundary value problems involving the Riemann–Liouville fractional q-derivative with well-posed q-boundary conditions. The study relies on the properties of the q-Green's function, which is constructed to solve such problems and allows for the analytical derivation of the inequalities. These inequalities find application in two directions: establishing precise lower bounds for the eigenvalues of corresponding q-fractional spectral problems and formulating criteria for the absence of real zeros in q-analogues of Mittag-Leffler functions. The obtained results generalize classical and fractional Lyapunov inequalities, offering new perspectives for the analysis of stability and spectral properties of q-fractional differential systems. The relevance of the work is driven by the growing interest in q-calculus in discrete models, such as viscoelastic systems or quantum circuits, where discrete dynamics play a key role. The convenience of closed-form analytical expressions makes the results practically applicable. The research lays the foundation for further generalizations, including Caputo derivatives or multidimensional q-systems, which may stimulate new discoveries in discrete fractional analysis.

Keywords: q-calculus, fractional q-derivative, Lyapunov-type inequality, Riemann–Liouville fractional derivative, Green's function, Mittag-Leffler function, eigenvalue problems, fractional integral.

2020 Mathematics Subject Classification: 26A33, 34A08, 39A13.

Introduction

Fractional calculus investigates integrals and derivatives of arbitrary (non-integer) order, has become an indispensable framework for modelling complex phenomena in physics, biology, engineering, and economics [1, 2]. Fractional differential equations (FDEs) naturally describe memory effects, nonlocal interactions, and anomalous diffusion; a representative example is C.F. Li et al.'s proof of positive solutions for nonlinear FDEs with boundary constraints [3].

A central analytical tool for boundary-value problems (BVPs) in the fractional setting is the Lyapunov-type inequality. R.A.C. Ferreira obtained the first variant for a Riemann–Liouville derivative with Dirichlet conditions [4]; M. Jleli and B. Samet extended the result to mixed boundary conditions [5]; and D. Basu et al. treated fractional boundary conditions, applying the inequality to spectral questions [6]. Subsequent refinements yielded sharper eigenvalue bounds and zero-free intervals for Mittag-Leffler functions [7].

Parallel to the continuous theory, q-fractional calculus blends quantum calculus with fractional analysis. Its origins trace back to Jackson's introduction of q-difference operators and integrals [8, 9] and R.D. Carmichael's work on q-difference equations [10]. Modern expositions by V. Kac and P. Cheung [11], T. Ernst [12, 13], and M.H. Annaby, Z.S. Mansour [14] have systematised the subject.

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Foundational notions of q-fractional integrals and derivatives, proposed by W.A. Al-Salam [15] and R.P. Agarwal [16], were rigorously formalised by P.M. Rajkovic et al. [17, 18].

Applications of q-fractional differential equations range from quantum mechanics to discrete dynamical systems. R.A.C. Ferreira analysed non-trivial and positive solutions for several classes of q-fractional BVPs [19, 20]; S. Shaimardan and collaborators established existence and uniqueness results for Cauchy-type problems with Riemann–Liouville derivatives [21]. The q-fractional framework has been connected with time–scale calculus through the work of F.M. Atici and P.W. Eloe [22]; with three-point and other non-local boundary conditions in the papers of S. Liang, J. Zhang, C. Yu, J. Wang, S. Wang et al. [23–25]; and further refined for related non-local problems by C. Zhai, J. Ren [26] and Y. Zhao, H. Chen, Q. Zhang [27]. Lyapunov-type inequalities for q-fractional equations were first obtained by M. Jleli and B. Samet [28].

In this work we derive two new Lyapunov-type inequalities for the q-fractional boundary-value problem

$$\begin{cases} D^{\alpha}_{q,a}u(t) + q(t)u(t) = 0, \quad a \le t \le b,\ 1 < \alpha \le 2,\ 0 \le \beta \le 1, \\ u(a) = 0, \quad D^{\beta}_{q,a}u(b) = 0,\ \ 0 < q < 1, \end{cases}$$

by exploiting properties of the associated q-Green function. The analysis combines topological fixed-point techniques [29], and existence principles in the Caratheodory framework [30]. Our results sharpen eigenvalue estimates, offer criteria for the real zeros of q-Mittag-Leffler functions, and advance the spectral theory of discrete fractional models.

1 Preliminaries

In this section, we introduce essential definitions and foundational concepts, including key aspects of q-calculus, which underpin the present study. For a comprehensive exploration of these topics, readers are referred to the monographs [11, 14].

For $\alpha \in \mathbb{R}$, the q-real number $[\alpha]_q$ is given by

$$[\alpha]_q = \frac{1-q^{\alpha}}{1-q}, \quad q \ne 1,$$

where $\lim\limits_{q\to 1} \frac{1-q^{\alpha}}{1-q} = \alpha$.

We introduce for $k \in \mathbb{N}$:

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{k=0}^{n-1}\left(1 - q^k a\right),\ (a;q)_\infty = \lim_{n\to\infty}(a,q)_n,\ (a;q)_\alpha = \frac{(a;q)_\infty}{(q^{\alpha}a;q)_\infty}.$$

The q-factorial $[n]_q!$, serving as the q-analogue of the binomial coefficient factorial, is defined as

$$[n]_q! = \begin{cases} 1, & \text{if } n = 0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q, & \text{if } n \in N. \end{cases}$$

The q-gamma function $\Gamma_q(x)$ is given by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \in R\backslash\{0,-1,-2,\ldots\}$$

and satisfies the functional relation $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.

Definition 1. [11] The q-analogue differential operator $D_q f(x)$ is

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1-q)},$$

and the q-derivatives $D_q^n(f(x))$ of higher order are defined inductively as follows:

$$D_q^0(f(x)) = f(x), \quad D_q^n(f(x)) = D_q\left(D_q^{n-1} f(x)\right) \quad (n = 1, 2, 3, \dots),$$

where $0 < q < 1$. Be aware that $\lim_{q\to 1} D_q f(x) = f'(x)$.

$$D_{q,x}(x-s)_q{}^{(\gamma)} = [\gamma]_q (x-s)_q{}^{(\gamma-1)}, \tag{1}$$
$$D_{q,s}(x-s)_q{}^{(\gamma)} = -[\gamma]_q (x-qs)_q{}^{(\gamma-1)}.$$

The q-integral (or Jackson integral) $\int_a^b f(x) d_q x$ is defined by

$$\int_0^a f(x) d_q x := (1-q)a \sum_{m=0}^{\infty} q^m f(aq^m),$$

for $a = 0$ and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

for $0 < a < b$. For further details, see [8, 9].

Definition 2. [21] For $\alpha > 0$, and a function f defined on $[a, b]$, the fractional q-integral of Riemann–Liouville type is characterized by $\left(I_{q,a}^0 f\right)(x) = f(x)$ and

$$\left(I_{q,a}^{\alpha} f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x-qt)_q^{(\alpha-1)} f(t) d_q t, \quad x \in [a, b].$$

Definition 3. [16]. Given $\alpha, \beta > 0$, the Riemann–Liouville fractional q-derivative is defined by setting $\left(D_{q,a}^0 f\right)(x) = f(x)$ and

$$\left(D_{q,a}^{\alpha} f\right)(x) = \left(D_{q,a}^{[\alpha]} I_{q,a}^{[\alpha]-\alpha} f\right)(x),$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

For $\lambda \in (-1, \infty)$, the following is valid [9]:

$$\left(D_{q,a}^{\alpha}(x-a)^{\lambda}\right)(x) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)}(x-a)^{\lambda-\alpha}. \tag{2}$$

The space $L_q^p = L_q^p[a, b]$ corresponding to $1 \le p < \infty$ is defined by

$$L_q^p[a, b] := \left\{ f : \left(\int_a^b |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\}.$$

Let $0 < a < b < \infty$ and $0 \le \lambda \le 1$. Then we introduce the space $C_{q,\lambda}[a,b]$ of functions f given on $[a,b]$, such that the functions with the norm

$$\|f\|_{C_{q,\lambda}[a,b]} := \max_{x\in[a,b]} \left|(x-qa)_q^{(\lambda)} f(x)\right| < \infty.$$

The collection of all q-absolutely continuous functions on $[a,b]$ is denoted $AC_q[a,b]$. For $n \in N := 1,2,3,\dots$ we denote by $AC_q^n[a,b]$ the space of real-valued functions $f(x)$ which have q-derivatives up to order $n-1$ on $[a,b]$ such that $D_q^{n-1} f(x) \in AC_q[a,b]$:

$$AC_q^n[a,b] := \left\{ f : [a,b] \to \mathbb{R}; D_q^{n-1} f(x) \in AC_q[a,b] \right\}.$$

Lemma 1. [18] Assume $\alpha > 0$, $\beta > 0$, and $1 \le p < \infty$. The semigroup property for the q-fractional integral holds as follows:
1. $(I_{q,a}^{\beta} I_{q,a}^{\alpha} f)(x) = (I_{q,a}^{\alpha+\beta} f)(x)$,
2. $(D_{q,a}^{\alpha} I_{q,a}^{\alpha} f)(x) = f(x)$,
3. $\left(D_{q,a}^{\beta} I_{q,a}^{\alpha} f\right)(x) = \left(I_{q,a}^{\alpha-\beta} f\right)(x)$,

where $f(x) \in L_q^p[a,b]$ for all $x \in [a,b]$.

Lemma 2. Suppose $\alpha > 0$, $p \in \mathbb{N}$, $q \in (0,1)$, and let $f \in AC_q^p[a,b]$ be a function with q-derivatives $D_{q,a}^k f$ defined at $x = a$ for $k = 0,1,\dots,p-1$. Following [19], the Riemann–Liouville q-fractional integral $I_{q,a}^{\alpha}$ and derivative $D_{q,a}^{\alpha}$ satisfy

$$\left(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f\right)(x) = \left(D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\right)(x) - \sum_{k=0}^{p-1} \frac{(x-a)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \left(D_{q,a}^{k} f\right)(a), \quad x \in [a,b].$$

Lemma 3. For $\gamma > -1$, $q \in (0,1)$, $a < b$, and $x \ge b$, the q-integral of the q-power function is given by

$$\int_a^b (x-qs)_q^{(\gamma)}\, d_q s = \frac{(x-a)^{\gamma+1}}{[\gamma+1]_q}, \tag{3}$$

where $(x-qs)_q^{(\gamma)} = (x-qs)^{\gamma}$ and $[\gamma+1]_q = \frac{1-q^{\gamma+1}}{1-q}$. See [9] for details.

2 *Main Results*

Theorem 1. Let $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 < \alpha - \beta < 1$, $q \in (0,1)$, and $h \in L_q^1[a,b]$. The q-fractional boundary value problem

$$D_{q,a}^{\alpha} u(t) + h(t) = 0, \quad t \in [a,b], \tag{4}$$

with boundary conditions

$$u(a) = 0, \quad D_{q,a}^{\beta} u(b) = 0, \tag{5}$$

has a unique solution given by

$$u(t) = \int_a^b G_q(t,s) h(s)\, d_q s,$$

where the q-Green's function $G_q(t,s)$ is defined as

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)}, & a \le t \le s \le b, \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)}, & a \le s \le t \le b. \end{cases} \tag{6}$$

Proof. By applying the operator $I_{q,a}^{\alpha}$ from definition 2 to both sides of (4) and employing Lemma 2 with $p = 2$, we obtain

$$u(t) = -I_{q,a}^{\alpha}h(t) + C_1(t-a)^{\alpha-1} + C_2(t-a)^{\alpha-2}, \tag{7}$$

for some $C_1, C_2 \in \mathbb{R}$. Applying the operator $D_{q,a}^{\beta}$ in condition (5) to both parts of the equation (7) and using the Lemma 1, we obtain

$$\begin{aligned} D_{q,a}^{\beta}u(t) = -D_{q,a}^{\beta}I_{q,a}^{\alpha}h(t) + C_1 D_{q,a}^{\beta}(t-a)^{\alpha-1} \\ + C_2 D_{q,a}^{\beta}(t-a)^{\alpha-2}, \end{aligned}$$

proceeding further, and using formula (2), we arrive at

$$\begin{aligned} D_{q,a}^{\beta}u(t) = -I_{q,a}^{\alpha-\beta}h(t) + C_1\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}(t-a)^{\alpha-\beta-1} \\ + C_2\frac{\Gamma_q(\alpha-1)}{\Gamma_q(\alpha-\beta-1)}(t-a)^{\alpha-\beta-2}. \end{aligned} \tag{8}$$

Using the boundary condition $u(a) = 0$ in equation (7) gives $C_2 = 0$. Applying the condition $D_{q,a}^{\beta}u(b) = 0$ to equation (8) then leads to

$$C_1 = \frac{1}{\Gamma_q(\alpha)(b-a)^{\alpha-\beta-1}}\int_a^b (b-qs)_q^{(\alpha-\beta-1)}h(s)\,d_qs.$$

Substituting the explicit expressions for C_1 and C_2 into equation (7), we obtain the unique solution of (4) as

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{(\alpha-1)}h(s)d_qs \\ &+ \frac{1}{\Gamma_q(\alpha)}\int_a^b \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)}h(s)d_qs \\ &= \frac{1}{\Gamma_q(\alpha)}\int_a^t \left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)}\right]h(s)d_qs \\ &+ \frac{1}{\Gamma_q(\alpha)}\int_t^b \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)}h(s)d_qs \\ &= \int_a^b G_q(t,s)h(s)d_qs. \end{aligned}$$

Hence, the result follows.

Corollary 1. Let $1 < \alpha \le 2$, $0 \le \beta \le 1$, $1 \le \alpha - \beta < 2$, $q \in (0,1)$, and $h \in L_q^1[a,b]$. The q-fractional boundary value problem

$$D_{q,a}^{\alpha}u(t) + h(t) = 0, \quad t \in [a,b],$$

with boundary conditions

$$u(a) = 0, \quad D_{q,a}^{\beta}u(b) = 0,$$

has a unique solution $u \in AC_q^{\alpha}[a,b]$ given by

$$u(t) = \int_a^b G_q(t,s)h(s)\,d_qs,$$

where the q-Green's function $G_q(t,s)$ is defined as

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)}, & a \le t \le s \le b, \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)}, & a \le s \le t \le b. \end{cases} \tag{9}$$

Proof. The result follows from Theorem 1 by identical arguments for the case $1 \le \alpha - \beta < 2$; the details are omitted.

We proceed to demonstrate the nonnegativity of the q-Green's functions and establish upper bounds for both the functions and their q-integrals.

Theorem 2. Let $1 < \alpha \le 2$, $0 \le \beta \le 1$, $0 < \alpha - \beta < 1$, $q \in (0,1)$, and let the q-Green's function $G_q(t,s)$ be defined as in Theorem 1. Then,

$$G_q(t,s) \ge 0 \quad \text{for all } (t,s) \in [a,b] \times [a,b].$$

Proof. We analyze the q-Green's function $G_q(t,s)$ defined in Theorem 1, considering its piecewise structure.

Case 1: $a \le t \le s \le b$. Here,

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} \cdot (b-qs)_q^{(\alpha-\beta-1)}.$$

Since $\Gamma_q(\alpha) > 0$, $(b-a)^{\alpha-\beta-1} > 0$, $(t-a)^{\alpha-1} \ge 0$ for $t \ge a$, and $(b-qs)_q^{(\alpha-\beta-1)} \ge 0$ for $s \le b$ (as $qs \le s$, $q \in (0,1)$, and $0 < \alpha - \beta < 1$), it follows that $G_q(t,s) \ge 0$.

Case 2: $a \le s \le t \le b$. In this case,

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)} \right].$$

Since $s \le t$, the q-power function is monotonic, so $t - qs \ge t - a$, and thus $(t-qs)_q^{(\alpha-1)} \le (t-a)^{\alpha-1}$. Additionally, as $qs \le s \le t \le b$, we have $b - qs \ge b - a$, implying $(b-qs)_q^{(\alpha-\beta-1)} \ge (b-a)^{\alpha-\beta-1}$. Therefore,

$$\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}(b-qs)_q^{(\alpha-\beta-1)} \ge (t-a)^{\alpha-1} \ge (t-qs)_q^{(\alpha-1)}.$$

Hence,

$$G_q(t,s) \ge \frac{1}{\Gamma_q(\alpha)} \left[(t-a)^{\alpha-1} - (t-qs)_q^{(\alpha-1)} \right] \ge 0.$$

Combining both cases, we conclude that $G_q(t,s) \ge 0$ for all $(t,s) \in [a,b] \times [a,b]$.

Remark 1. The nonnegativity of the q-Green's function $G_q(t,s)$, established in Theorem 2, is crucial for the qualitative analysis of the q-fractional boundary value problem in Theorem 1. Specifically, it ensures that the solution

$$u(t) = \int_a^b G_q(t,s)h(s)\,d_qs, \quad h \in L_q^1[a,b],$$

preserves the sign of the source term $h(s)$. For instance, if $h(s) \ge 0$ on $[a,b]$, then $u(t) \ge 0$; similarly, if $h(s) \le 0$, then $u(t) \le 0$, for all $t \in [a,b]$.

Corollary 2. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $1 \leq \alpha - \beta < 2$, $q \in (0,1)$, and let the q-Green's function $G_q(t,s)$ be defined as in Corollary 1 for $a < b$. Then,

$$G_q(t,s) \geq 0 \quad \text{for all } (t,s) \in [a,b] \times [a,b].$$

Proof. We analyze the piecewise definition of $G_q(t,s)$ from Corollary 1.

Case 1: $a \leq t \leq s \leq b$. Here,

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} \cdot (b-qs)_q^{(\alpha-\beta-1)}.$$

Since $\Gamma_q(\alpha) > 0$, $(t-a)^{\alpha-1} \geq 0$, $(b-a)^{\alpha-\beta-1} \geq 0$ (as $\alpha-\beta-1 \geq 0$), and $(b-qs)_q^{(\alpha-\beta-1)} \geq 0$ (as $qs \leq s \leq b$, $q \in (0,1)$), it follows that $G_q(t,s) \geq 0$.

Case 2: $a \leq s \leq t \leq b$. In this case,

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)} \right].$$

Since $a \leq qs \leq s \leq t \leq b$, we have $b - qs \geq b - a$, so $(b-qs)_q^{(\alpha-\beta-1)} \geq (b-a)^{\alpha-\beta-1}$. Also, $qs \geq a$, so $t - qs \leq t - a$, and the monotonicity of the q-power function [14] implies $(t-qs)_q^{(\alpha-1)} \leq (t-a)^{\alpha-1}$. Thus,

$$\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} \geq (t-a)^{\alpha-1} \geq (t-qs)_q^{(\alpha-1)}.$$

Hence,

$$G_q(t,s) \geq \frac{1}{\Gamma_q(\alpha)} \left[(t-a)^{\alpha-1} - (t-qs)_q^{(\alpha-1)} \right] \geq 0.$$

Thus, $G_q(t,s) \geq 0$ for all $(t,s) \in [a,b] \times [a,b]$.

Theorem 3. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $0 < \alpha - \beta < 1$, $q \in (0,1)$, $a < b$, and let the q-Green's function $G_q(t,s)$ be defined as in (6). Then, for $s \in [a,b]$,

$$\max_{t \in [a,b]} \frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}} = \frac{G_q(s,s)}{(b-qs)_q^{(\alpha-\beta-1)}},$$

and

$$\max_{s \in [a,b]} \frac{G_q(s,s)}{(b-qs)_q^{(\alpha-\beta-1)}} = \frac{(b-a)^{\beta}}{\Gamma_q(\alpha)}.$$

Proof. We analyze the ratio $\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}}$ for fixed $s \in [a,b]$. Since $qs \leq s \leq b$, $q \in (0,1)$, and $0 < \alpha - \beta < 1$, we have $\alpha - \beta - 1 \in (-1, 0)$, but $(b-qs)_q^{(\alpha-\beta-1)} \geq 0$ as per [14].

Case 1: $a \leq t \leq s \leq b$. From (6),

$$\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}} = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}.$$

Using the q-derivative (1),

$$D_{q,t}[(t-a)^{\alpha-1}] = [\alpha-1]_q (t-a)^{\alpha-2},$$

we obtain

$$D_{q,t}\left[\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}} \right] = \frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-\beta-1} \Gamma_q(\alpha-1)} \geq 0,$$

since $\alpha-2>-1$. At $t=a$, $(t-a)^{\alpha-2}$ may be singular ($\alpha-2\in(-1,0]$), but the q-derivative is defined for $t\in(a,s]$. Thus, the ratio is non-decreasing on $[a,s]$.

Case 2: $a\le s\le t\le b$. Here,

$$\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}}=\frac{1}{\Gamma_q(\alpha)}\left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}-\frac{(t-qs)_q^{(\alpha-1)}}{(b-qs)_q^{(\alpha-\beta-1)}}\right].$$

Computing the q-derivative,

$$D_{q,t}\left[\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}}\right]=\frac{1}{\Gamma_q(\alpha-1)}\left[\frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-\beta-1}}-\frac{(t-qs)_q^{(\alpha-2)}}{(b-qs)_q^{(\alpha-\beta-1)}}\right].$$

Since $qs\le s\le b$, we have $b-qs\ge b-a$, so $(b-qs)_q^{(\alpha-\beta-1)}\ge(b-a)^{\alpha-\beta-1}$. Also, $qs\ge a$, so $t-qs\le t-a$, and the monotonicity of the q-power function [14] implies $(t-qs)_q^{(\alpha-2)}\le(t-a)^{\alpha-2}$. Thus,

$$\frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-\beta-1}}\ge\frac{(t-qs)_q^{(\alpha-2)}}{(b-qs)_q^{(\alpha-\beta-1)}},$$

so

$$D_{q,t}\left[\frac{G_q(t,s)}{(b-qs)_q^{(\alpha-\beta-1)}}\right]\le 0.$$

Hence, the ratio is non-increasing on $[s,b]$. Combining both cases, the maximum occurs at $t=s$, where

$$\frac{G_q(s,s)}{(b-qs)_q^{(\alpha-\beta-1)}}=\frac{1}{\Gamma_q(\alpha)}\cdot\frac{(s-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}}.$$

For the second part, consider

$$\frac{G_q(s,s)}{(b-qs)_q^{(\alpha-\beta-1)}}=\frac{(s-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}\Gamma_q(\alpha)}.$$

Since $(s-a)^{\alpha-1}$ is increasing on $[a,b]$ ($\alpha-1>0$), the maximum occurs at $s=b$, yielding

$$\frac{(b-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}\Gamma_q(\alpha)}=\frac{(b-a)^{\beta}}{\Gamma_q(\alpha)}.$$

This completes the proof.

Corollary 3. Let $1<\alpha\le 2$, $0\le\beta\le 1$, $1\le\alpha-\beta<2$, $q\in(0,1)$, $a<b$, and let the q-Green's function $G_q(t,s)$ be defined as in (9). Then, for $s\in[a,b]$,

$$\max_{t\in[a,b]}G_q(t,s)=G_q(s,s),$$

and

$$\max_{s\in[a,b]}G_q(s,s)=\frac{(b-a)^{\beta}b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)}.$$

Proof. The statement follows from Theorem 3 by identical arguments applied to the range $1\le\alpha-\beta<2$; the details are omitted.

Corollary 4. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $1 \leq \alpha - \beta < 2$, $q \in (0,1)$, $a < b$, and let the q-Green's function $G_q(t,s)$ be defined as in (6) and (9). Then:

$$\max_{t\in[a,b]} \int_a^b G_q(t,s)\, d_q s = \frac{[\alpha-1]_q^{\alpha-1}}{\Gamma_q(\alpha+1)} \left(\frac{b-a}{[\alpha-\beta]_q}\right)^{\alpha}.$$

Proof. Consider the integral $I(t) = \int_a^b G_q(t,s)\, d_q s$, where $G_q(t,s)$ is defined in (6) and (9). Split the integral based on the definition of $G_q(t,s)$:

Case 1: $a \leq t \leq s \leq b$.

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)}.$$

Case 2: $a \leq s \leq t \leq b$.

$$G_q(t,s) = \frac{1}{\Gamma_q(\alpha)} \left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)}\right].$$

Thus,

$$I(t) = \int_a^t G_q(t,s)\, d_q s + \int_t^b G_q(t,s)\, d_q s.$$

Substitute the expression for $G_q(t,s)$:

$$\begin{aligned}
I(t) &= \int_a^t \frac{1}{\Gamma_q(\alpha)} \left[\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} - (t-qs)_q^{(\alpha-1)}\right] d_q s \\
&\quad + \int_t^b \frac{1}{\Gamma_q(\alpha)} \cdot \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} (b-qs)_q^{(\alpha-\beta-1)} d_q s \\
&= \frac{(t-a)^{\alpha-1}}{\Gamma_q(\alpha)(b-a)^{\alpha-\beta-1}} \int_a^b (b-qs)_q^{(\alpha-\beta-1)}\, d_q s - \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{(\alpha-1)}\, d_q s.
\end{aligned}$$

Using equation (3), under the conditions $x = b$ or $x = t \geq s$, we have

$$\int_a^b (b-qs)_q^{(\alpha-\beta-1)}\, d_q s = \frac{(b-a)^{\alpha-\beta}}{[\alpha-\beta]_q}, \quad \int_a^t (t-qs)_q^{(\alpha-1)}\, d_q s = \frac{(t-a)^{\alpha}}{[\alpha]_q},$$

we get

$$\begin{aligned}
I(t) &= \frac{(t-a)^{\alpha-1}(b-a)^{\alpha-\beta}}{\Gamma_q(\alpha)(b-a)^{\alpha-\beta-1}[\alpha-\beta]_q} - \frac{(t-a)^{\alpha}}{\Gamma_q(\alpha)[\alpha]_q} \\
&= \frac{(t-a)^{\alpha-1}(b-a)}{\Gamma_q(\alpha)[\alpha-\beta]_q} - \frac{(t-a)^{\alpha}}{\Gamma_q(\alpha)[\alpha]_q} \\
&= \frac{(t-a)^{\alpha-1}}{\Gamma_q(\alpha)} \left(\frac{b-a}{[\alpha-\beta]_q} - \frac{t-a}{[\alpha]_q}\right).
\end{aligned}$$

To find the maximum, compute the q-derivative:

$$D_{q,t}I(t) = \frac{1}{\Gamma_q(\alpha)}\left[[\alpha-1]_q(t-a)^{\alpha-2}\left(\frac{b-a}{[\alpha-\beta]_q}-\frac{t-a}{[\alpha]_q}\right)-(t-a)^{\alpha-1}\cdot\frac{1}{[\alpha]_q}\right]$$
$$= \frac{1}{\Gamma_q(\alpha)}\left[\frac{[\alpha-1]_q(t-a)^{\alpha-2}(b-a)}{[\alpha-\beta]_q}-\frac{(t-a)^{\alpha-1}([\alpha-1]_q+1)}{[\alpha]_q}\right]$$
$$= \frac{1}{\Gamma_q(\alpha)}\left[\frac{[\alpha-1]_q(t-a)^{\alpha-2}(b-a)}{[\alpha-\beta]_q}-(t-a)^{\alpha-1}\right],$$

where $[\alpha-1]_q+1=\frac{1-q^{\alpha-1}}{1-q}+1=\frac{1-q^{\alpha}}{1-q}=[\alpha]_q$.

Set $D_{q,t}I(t)=0$:

$$t^* = a+\frac{[\alpha-1]_q(b-a)}{[\alpha-\beta]_q}.$$

Substitute t^* into the expression for $I(t)$:

$$I(t^*) = \frac{\left(\frac{[\alpha-1]_q(b-a)}{[\alpha-\beta]_q}\right)^{\alpha-1}}{\Gamma_q(\alpha)}\left(\frac{b-a}{[\alpha-\beta]_q}-\frac{\frac{[\alpha-1]_q(b-a)}{[\alpha-\beta]_q}}{[\alpha]_q}\right)$$
$$= \frac{\left(\frac{[\alpha-1]_q(b-a)}{[\alpha-\beta]_q}\right)^{\alpha-1}}{\Gamma_q(\alpha)}\left(\frac{b-a}{[\alpha-\beta]_q}\left(1-\frac{[\alpha-1]_q}{[\alpha]_q}\right)\right)$$
$$= \frac{[\alpha-1]_q^{\alpha-1}(b-a)^{\alpha-1}}{\Gamma_q(\alpha)[\alpha-\beta]_q^{\alpha-1}}\cdot\frac{b-a}{[\alpha-\beta]_q}\cdot\frac{q^{\alpha-1}}{[\alpha]_q}$$
$$= \frac{[\alpha-1]_q^{\alpha-1}(b-a)^{\alpha}q^{\alpha-1}}{\Gamma_q(\alpha)[\alpha-\beta]_q^{\alpha}[\alpha]_q}.$$

The function $I(t)$ is increasing for $t<t^*$ ($D_{q,t}I(t)>0$) and decreasing for $t>t^*$ ($D_{q,t}I(t)<0$), confirming the maximum at t^*.

Theorem 4. Let $\mathfrak{B}_q=C_{q,\lambda}[a,b]$ denote the Banach space of functions continuous in the q-sense on the interval $[a,b]$, with norm

$$\|u\|_{C_{q,\lambda}}=\max_{t\in[a,b]}|u(t)|,$$

where $[a,b]=\{a,aq,aq^2,\ldots,aq^n=b\}$. Given $1<\alpha\le 2$, $0\le\beta\le 1$, $0<\alpha-\beta<1$, if the fractional q-difference boundary value problem

$$\begin{cases} D_{q,a}^{\alpha}u(t)+q(t)u(t)=0, & t\in[a,b],\\ u(a)=0, \quad D_{q,a}^{\beta}u(b)=0, \end{cases} \tag{10}$$

admits a nontrivial solution $u\in\mathfrak{B}_q$, then the following Lyapunov-type inequality holds:

$$\int_a^b (b-qs)_q^{(\alpha-\beta-1)}|q(s)|d_qs > \frac{\Gamma_q(\alpha)}{(b-a)^{\beta}}. \tag{11}$$

Proof. Any solution $u\in\mathfrak{B}_q$ of the boundary value problem (10) satisfies

$$u(t)=\int_a^b G_q(t,s)q(s)u(s)d_qs,$$

where $G_q(t,s)$ is the q-Green's function given by (6).

By applying the $C_{q,\lambda}$-norm, we obtain

$$\begin{aligned}\|u\|_{C_{q,\lambda}} &= \max_{t\in[a,b]}\left|\int_a^b G_q(t,s)q(s)u(s)d_qs\right| \\ &\le \max_{t\in[a,b]}\int_a^b |G_q(t,s)||q(s)||u(s)|d_qs \\ &\le \|u\|_{C_{q,\lambda}}\cdot\max_{t\in[a,b]}\int_a^b |G_q(t,s)||q(s)|d_qs.\end{aligned}$$

For a nontrivial solution ($\|u\|_{C_{q,\lambda}}\neq 0$), this implies

$$1\le \max_{t\in[a,b]}\int_a^b |G_q(t,s)||q(s)|d_qs.$$

By Theorem 3, the q-Green's function satisfies the bound

$$|G_q(t,s)|\le \frac{(b-a)^\beta(b-qs)_q^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)}.$$

Substituting this bound, we get

$$1<\max_{t\in[a,b]}\int_a^b |G_q(t,s)||q(s)|d_qs\le \frac{(b-a)^\beta}{\Gamma_q(\alpha)}\int_a^b (b-qs)_q^{(\alpha-\beta-1)}|q(s)|d_qs.$$

Therefore, dividing both sides by $\frac{(b-a)^\beta}{\Gamma_q(\alpha)}$, we obtain (11).

This completes the proof.

Corollary 5. Let $1<\alpha\le 2$, $0\le\beta\le 1$, and $1\le\alpha-\beta<2$. Suppose the fractional q-difference boundary-value problem

$$\begin{cases} D_{q,a}^\alpha u(t)+q(t)u(t)=0, & t\in[a,b],\\ u(a)=0, \quad D_{q,a}^\beta u(b)=0, \end{cases}$$

admits a nontrivial solution $u\in\mathfrak{B}_q=C_{q,\lambda}[a,b]$, where $C_{q,\lambda}[a,b]$ is the space of q-continuous functions on the q-interval $[a,b]$ with $0<q<1$. Then the following Lyapunov-type inequality holds:

$$\int_a^b |q(s)|\,d_qs>\frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}.$$

Proof. By Corollary 1, any solution $u\in C_{q,\lambda}[a,b]$ to the boundary-value problem satisfies:

$$u(t)=\int_a^b G_q(t,s)q(s)u(s)\,d_qs,$$

where $G_q(t,s)$ is the q-Green's function defined in (9).

Define the norm $\|u\|_{C_{q,\lambda}} = \sup\limits_{t\in[a,b]} |u(t)|$. From the solution representation:

$$|u(t)| \leq \int\limits_a^b |G_q(t,s)||q(s)||u(s)|\, d_q s \leq \|u\|_{C_{q,\lambda}} \int\limits_a^b |G_q(t,s)||q(s)|\, d_q s.$$

Taking the supremum over $t \in [a,b]$, we obtain

$$\|u\|_{C_{q,\lambda}} \leq \|u\|_{C_{q,\lambda}} \max_{t\in[a,b]} \int\limits_a^b |G_q(t,s)||q(s)|\, d_q s.$$

For a nontrivial solution ($\|u\|_{C_{q,\lambda}} > 0$), it follows that

$$1 \leq \max_{t\in[a,b]} \int\limits_a^b |G_q(t,s)||q(s)|\, d_q s.$$

By Corollary 2, $G_q(t,s)$ is non-negative, so $|G_q(t,s)| = G_q(t,s)$. By Corollary 3, the maximum of the Green's function is

$$\max_{t,s\in[a,b]} G_q(t,s) = \max_{s\in[a,b]} G_q(s,s) = \frac{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)}.$$

Thus, $G_q(t,s) \leq \max\limits_{s\in[a,b]} G_q(s,s)$, and

$$\int\limits_a^b G_q(t,s)|q(s)|\, d_q s \leq \frac{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)} \int\limits_a^b |q(s)|\, d_q s.$$

Combining with the previous inequality, we get

$$1 \leq \frac{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}{\Gamma_q(\alpha)} \int\limits_a^b |q(s)|\, d_q s.$$

Rearranging yields

$$\int\limits_a^b |q(s)|\, d_q s \geq \frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}.$$

To establish the strict inequality, suppose equality holds

$$\int\limits_a^b |q(s)|\, d_q s = \frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}.$$

This implies $G_q(t,s) = \max\limits_{s\in[a,b]} G_q(s,s)$ for all $t,s \in [a,b]$ where $q(s)u(s) \neq 0$. By Corollary 3, $G_q(t,s) = G_q(s,s)$ only when $t = s$, which has measure zero in the q-integral unless $u \equiv 0$. Since u is nontrivial, equality is impossible, so

$$\int\limits_a^b |q(s)|\, d_q s > \frac{\Gamma_q(\alpha)}{(b-a)^\beta b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}.$$

3 Applications

In this section, we investigate two applications of Theorem 4 and Corollary 5. First, we establish lower bounds for the eigenvalues of the Riemann–Liouville type fractional q-eigenvalue problems associated with (10). Second, we utilize these findings to identify intervals where the q-analogue of the two-parameter Mittag-Leffler function has no real zeros.

Theorem 5. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$ such that $0 < \alpha - \beta < 1$. Assume that y is a nontrivial solution of the Riemann–Liouville type fractional q-eigenvalue problem

$$\begin{cases} D^{\alpha}_{q,a}u(t) + \lambda u(t) = 0, & t \in [a,b], \\ u(a) = 0, \quad D^{\beta}_{q,a}u(b) = 0, \end{cases} \tag{12}$$

where $u(t) \neq 0$ for each $t \in (a,b)$. Then,

$$|\lambda| > \frac{[\alpha-\beta]_q \Gamma_q(\alpha)}{(b-a)^{\alpha}}.$$

Corollary 6. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$ such that $1 \leq \alpha - \beta < 2$. Assume that u is a nontrivial solution of the Riemann–Liouville type fractional q-eigenvalue problem (12), where $u(t) \neq 0$ for each $t \in (a,b)$. Then,

$$|\lambda| > \frac{\Gamma_q(\alpha)}{(b-a)^{\beta} b^{\alpha-\beta-1}(1-q)^{\alpha-\beta-1}}.$$

Consider the q-analogue of the two-parameter Mittag-Leffler function, defined as ([14]):

$$E_{q,\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(k\alpha+\beta)}, \quad z, \beta \in \mathbb{C}, \quad \Re(\alpha) > 0, \quad 0 < q < 1. \tag{13}$$

We use Theorem 5 and Corollary 6 to determine intervals where the function (13) has no real zeros.

Theorem 6. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $0 < \alpha - \beta < 1$, $q \in (0,1)$. The q-Mittag-Leffler function

$$E_{q,\alpha,\alpha-\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(k\alpha+\alpha-\beta)},$$

has no real zeros for

$$|z| \leq \frac{[\alpha-\beta]_q \Gamma_q(\alpha)}{(b-a)^{\alpha}}, \tag{14}$$

where $[\alpha-\beta]_q = \frac{1-q^{\alpha-\beta}}{1-q}$.

Proof. Consider the q-fractional eigenvalue problem

$$\begin{cases} D^{\alpha}_{q,a}u(t) + \lambda u(t) = 0, & t \in [a,b], \\ u(a) = 0, \quad D^{\beta}_{q,a}u(b) = 0, \end{cases}$$

where $D^{\alpha}_{q,a}$ is the Riemann–Liouville q-fractional derivative. The general solution is

$$u(t) = c_1(t-a)^{\alpha-1}E_{q,\alpha,\alpha}(-\lambda(t-a)^{\alpha}) + c_2(t-a)^{\alpha-2}E_{q,\alpha,\alpha-1}(-\lambda(t-a)^{\alpha}).$$

Let $g(t) = (t-a)^{\alpha-1}E_{q,\alpha,\alpha}(-\lambda(t-a)^{\alpha})$. Compute

$$D^{\alpha}_{q,a}g(t) = D^{\alpha}_{q,a}\left(\sum_{n=0}^{\infty} \frac{(-\lambda)^n (t-a)^{\alpha n+\alpha-1}}{\Gamma_q(\alpha n+\alpha)}\right) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma_q(\alpha n+\alpha)} D^{\alpha}_{q,a}(t-a)^{\alpha n+\alpha-1}.$$

Since $D^{\alpha}_{q,a}(t-a)^{\alpha n+\alpha-1} = \frac{\Gamma_q(\alpha n+\alpha)}{\Gamma_q(\alpha n)}(t-a)^{\alpha n-1}$, we get

$$D^{\alpha}_{q,a}g(t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma_q(\alpha n)}(t-a)^{\alpha n-1} = -\lambda g(t).$$

The condition $u(a) = 0$ implies $c_2 = 0$, since $(t-a)^{\alpha-2} \to \infty$ as $t \to a$. Thus,

$$u(t) = c_1(t-a)^{\alpha-1}E_{q,\alpha,\alpha}(-\lambda(t-a)^{\alpha}).$$

Compute

$$D^{\beta}_{q,a}u(t) = c_1 \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma_q(\alpha n+\alpha)} D^{\beta}_{q,a}(t-a)^{\alpha n+\alpha-1}.$$

Since $D^{\beta}_{q,a}(t-a)^{\alpha n+\alpha-1} = \frac{\Gamma_q(\alpha n+\alpha)}{\Gamma_q(\alpha n+\alpha-\beta)}(t-a)^{\alpha n+\alpha-\beta-1}$, we obtain

$$D^{\beta}_{q,a}u(t) = c_1(t-a)^{\alpha-\beta-1}E_{q,\alpha,\alpha-\beta}(-\lambda(t-a)^{\alpha}).$$

The condition $D^{\beta}_{q,a}u(b) = 0$ gives

$$c_1(b-a)^{\alpha-\beta-1}E_{q,\alpha,\alpha-\beta}(-\lambda(b-a)^{\alpha}) = 0 \implies E_{q,\alpha,\alpha-\beta}(-\lambda(b-a)^{\alpha}) = 0.$$

By Theorem 5, for a nontrivial solution $u \in \mathfrak{B}_q = C_{q,\lambda}[a,b]$,

$$|\lambda| > \frac{[\alpha-\beta]_q\Gamma_q(\alpha)}{(b-a)^{\alpha}}.$$

For $z = -\lambda(b-a)^{\alpha}$, we have

$$|z| = |\lambda|(b-a)^{\alpha} > [\alpha-\beta]_q\Gamma_q(\alpha).$$

Thus, $E_{q,\alpha,\alpha-\beta}(z) \neq 0$ for (14).

Corollary 7. Let $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$ such that $1 \leq \alpha - \beta < 2$. The q-Mittag-Leffler function $E_{q,\alpha,\beta}(z)$ has no real zeros for

$$|z| \leq \frac{\Gamma_q(\alpha)}{(b-a)^{\alpha}}.$$

Proof. Following the same reasoning as in Theorem 6, suppose $E_{q,\alpha,\beta}(\lambda) = 0$ for some real λ. The function $u(t) = E_{q,\alpha,\beta}(-\lambda(t-a)^{\alpha})$ satisfies the q-eigenvalue problem (12). By Corollary 6, any eigenvalue λ must satisfy:

$$|\lambda| > \frac{\Gamma_q(\alpha)}{(b-a)^{\alpha}}.$$

Hence, $E_{q,\alpha,\beta}(z) \neq 0$.

Conclusion

In this study, we derived two novel Lyapunov-type inequalities for boundary value problems involving the Riemann–Liouville fractional q-derivative within the regimes $0 < \alpha - \beta < 1$ and $1 \leq \alpha - \beta < 2$, thereby establishing precise estimates for eigenvalues and intervals free of zeros for q-Mittag-Leffler functions. By employing an analysis of the q-Green's function, we determined lower bounds for the eigenvalues of the problem $D^{\alpha}_{q,a}u + \lambda u = 0$ and identified regions devoid of real zeros for q-analogues of Mittag-Leffler functions, which holds significant importance for discrete systems with memory, such as viscoelastic lattices and quantum circuits. This work extends classical inequalities to the realm of q-calculus, thereby bridging continuous and discrete fractional analysis, and paves the way for further research on Caputo q-fractional derivatives and multidimensional q-lattices.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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