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MATHEMATICS

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Boundary Value Problems on a Star Thermal Graph and their Solutions

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In this study, heat conductivity boundary value problems on a star graph are considered, inspired by engineering applications, e.g., heat conduction phenomena in mesh-like structures. Based on the generalized function method, a unified technique for solving boundary value problems on such graphs is developed. Generalized solutions to transient and stationary boundary value problems are constructed for different conditions at the end edges, with the Kirchhoff conditions at the common node. Regular integral representations of solutions to boundary value problems are obtained using the properties and symmetry of the fundamental solution's Fourier transform. The derived results allow the action of various heat sources to be simulated, including concentrated ones by using singular generalized functions. The generalized function method enables a wide variety of boundary value problems to be tackled, including those with local boundary conditions at the ends of the graph, and various transmission conditions at the common node. Based on the research, the authors propose an analytical solution method under the action of various heat sources to solve various boundary value problems on a star thermal graph.

Keywords: star graph, temperature, heat flow, transmission conditions, generalized functions method, generalized solution, Fourier transform, boundary equations.

Mathematics Subject Classification: 35M10, 35K05, 35L05, 94C15.

Introduction

As a branch of applied mathematics, graph theory has wide applications in subjects such as economics, logistics, sociology, optimal control, and navigation. The properties of graphs are also actively used to solve boundary value problems (BVPs) on network-like structures, e.g., oil pipelines, gas pipelines, and electrical networks. The concept of graphs was first introduced by Leonhard Euler in 1736. In his work [1], he pioneered the approach of graph theory to solve the famous problem of the Königsberg bridges. At the beginning of the 20th century, the Hungarian mathematician Dénes König

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was the first to propose the term "graph" in reference to diagrams containing a discrete set of points connected by straight lines and began to study their general properties. In 1936, he published the first book on graph theory [2] that reviewed all related works from 1736 onwards (see also [3]).

Graph Theory (GT) has potential applications in the field of Computer Science (CS) for various purposes. Unique applications of GT in the field of CS, such as web document clustering, cryptography, and algorithm execution analysis, among others, are promising applications. Further, the concepts of GT can be used to simplify and analyze electronic circuits. Recently, graphs have been widely used in social networks (SN) for many purposes related to modeling and analyzing SN structures, modeling SN operation, analyzing SN users, and many other related aspects [4–9]. Determining a distributed parameter of conductance for a neuronal cable model on a tree graph was researched in [10, 11].

The development of analytical methods for the mathematical modeling of network-like structures is based on the solution and analysis of the corresponding direct and inverse problems for equation systems with parameters that are distributed on a graph. These methods primarily analyze the spectral completeness and basis properties of the corresponding BVPs' eigenfunctions in the space of square integrable functions, in addition to finding conditions for their unique solutions. The field of theory differential equations on networks is relatively new, with most articles on the subject focused on the direct problems of spectral theory.

The theory of ordinary differential equations on networks gave rise to research on the theory of inverse problems on geometric graphs, which can be traced to the articles of V. Yurko [12, 13]. The spectral problems for Sturm-Liouville operators on star graphs were also considered in [14–16]. A number of other contributions are devoted to different types of differential equations on graphs. In [17], discrete Yamabe equations on star graphs were considered, in [18, 19] the fractional boundary value problem on a graph was studied.

A new class of partial differential equations on graphs of different structures has only begun to be studied in the last decade. Boundary value problems for the d'Alembert wave equation, their spectral properties, and the issues of existence and uniqueness of solutions, as well as a number of inverse problems, were considered in the works [20–24].

The studies of boundary value problems for heat conduction equations on graphs appeared relatively recently. They are largely related to the solution of thermoelasticity problems on graphs for studying the strength properties of rod structures, widely used in mechanical engineering, robotics, and construction. Their solutions are based on the solution of systems of equations of spatially one-dimensional thermoelasticity problems [25–27].

Mathematical modeling of the process of heat propagation in a system of rods on a "tree-like" graph in the form of a bundle of linear differential operators was carried out by Yu. Martynova [28], who posed the inverse problem of finding the parameters of boundary conditions for given eigenvalues. Here we consider boundary value problems for the heat equation on an undirected star graph with an arbitrary number of links of different lengths, which may have different thermal characteristics. An analytical solution to the posed boundary value problems is constructed for given boundary conditions at the ends of the graph and a known total heat flow in its node. The constructed solution allows us to determine the temperature and heat flows in any link of the graph under periodic and non-stationary thermal effects. Other authors have not yet constructed such solutions to boundary value problems on thermal graphs.

We used the generalized function method to solve boundary value problems, leading to a differential equation solution with a singular right-hand side containing simple and double layers, the densities of which are determined by the boundary and initial conditions of the solution. The solution is constructed as the convolution of the Green's function of the equation with the appropriate right-hand side in every edge. To determine the unknown boundary values of the solution and its derivatives, connection equations are constructed boundary functions at the edge ends, employing the asymptotic properties of

Green's functions and their derivatives at zero. To construct a closed system of equations, the obtained algebraic equations for each edge of the graph are supplemented with transmission conditions at the node and linear boundary conditions at its ends. These conditions can be either locally or not locally connected. Thus, the proposed method is applicable to a wide range of BVPs, including those on mesh structures.

1 Statement of the boundary value problem on a heat star graph

We consider a heat star graph which contains N edges (A_0, A_j) of the length L_j $(j = 1, 2, ..., N)$ with a common node A_0 (Fig. 1). On each edge $S_j = \{x \in R^1 : 0 \le x \le L_j\}$, there is a coordinate system (x, t) whose origin can be found at the point $A_0 : x = 0$. At S_j, the temperature $\theta_j(x, t)$ satisfies the following heat conduction equation:

$$\frac{\partial \theta_j}{\partial t} - \kappa_j \frac{\partial^2 \theta_j}{\partial x^2} = F_j(x, t), \quad 0 \le x \le L_j, \quad t \ge 0. \tag{1}$$

Here, κ_j is the heat diffusivity coefficient, $F_j(x, t)$ describes the power of acting heat sources on the j-edge.

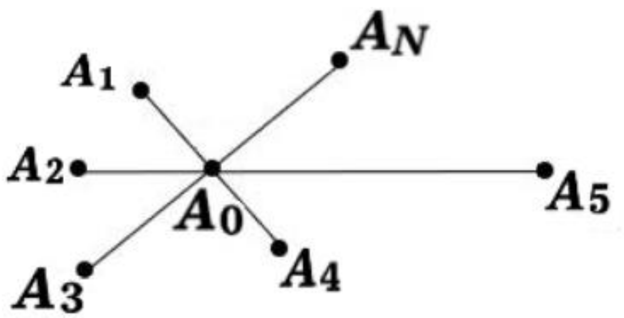

Figure 1. Star graph

The initial conditions at t = 0 for the temperature of the graph are as follows:
(Cauchy conditions)

$$\theta_j(x, 0) = \theta_0^j(x), \quad 0 \le x \le L_j, \quad \forall j, \tag{2}$$

where $\theta_0^j(x) \in C^2(R^1)$. Next, we consider the following boundary value problem (BVP).

Dirichlet problem. Temperature values are known at the ends of the graph:

$$\theta_j(L_j, 0) = \theta_2^j(t), \quad 0 \le x \le L_j, \quad t \ge 0. \tag{3}$$

The following continuity conditions and the Kirchhoff condition are specified in the common node A_0 of the graph:

$$\theta_1^1(t) = \theta_1^2(t) = \ldots = \theta_1^N(t), \quad x = 0, \quad t \ge 0, \tag{4}$$

$$\kappa_1 q_1^1(t) + \kappa_2 q_1^2(t) + \ldots + \kappa_N q_1^N(t) = 0. \tag{5}$$

Here $q_1^j(t) = \frac{\partial \theta_j}{\partial x}|_{x=0}$, $q_2^j(t) = \frac{\partial \theta_j}{\partial x}|_{x=L}$, the superscript indicates the number of the graph edge and $\theta_1^j(t) = \theta^j(0, t)$, $\theta_2^j(t) = \theta^j = (L_j, t)$ designate the temperatures at the ends of the j-th edge, $1 \le j \le N$.

We need to find the solution to the Dirichlet problem on this star graph.

2 Statement of the boundary value problem on the edge of a graph

To define the connection between the boundary values of temperature and heat flows on each element of a graph, at first we construct the solution to the boundary value problem (BVP) on the segment $[0, L]$.

Let's define $\theta(x,t)$, which is the solution of the heat equation:

$$\frac{\partial\theta}{\partial t} - \kappa\frac{\partial\theta}{\partial x^2} = F(x,t),\ 0 \leq x \leq L,\ t \geq 0. \tag{6}$$

The *initial conditions* are as follows, where the temperature is known at $t = 0$:

$$\theta(x,0) = \theta_0(x),\ \theta_0(x) \in C^2(R^1)\left\{0 \leq x \leq L\right\}. \tag{7}$$

We denote

$$\begin{aligned} \theta(0,t) = \theta_1(t),\ x = 0,\ t \geq 0, \\ \theta(L,t) = \theta_2(t),\ x = L,\ t \geq 0. \end{aligned} \tag{8}$$

The following matching conditions apply to the initial and boundary conditions:

$$\theta_1(t) = \theta_0(0), \quad \theta_2(t) = \theta_0(L).$$

We can construct the solutions of this BVP using the generalized function method [29].

Boundary conditions can be written for different BVP in a generalized form:

$$\begin{cases} (\alpha_1\theta_1 + \beta_1\Pi_1(t))|_{x=0} = G_1(t), \\ (\alpha_2\theta_2 + \beta_2\Pi_2(t))|_{x=L} = G_2(t), \end{cases} \tag{9}$$

where $\theta_j(t),\ \Pi_j(t) = -k\frac{\partial\theta}{\partial x}|_{x=x_j}$ $(j = 1,2)$ are the temperature and heat flows at the ends of segment, known functions $G_j(t) \in L_1(R_+),\ R_+ = [0,\infty)$; coefficients (α_j,β_j) are known arbitrary constants. By choosing them we can solve different BVPs. For example,

1. *Dirichlet problem:* $\alpha_1 = 1,\ \beta_1 = 0,\ \alpha_2 = 1,\ \beta_2 = 0$;
2. *Neumann problem:* $\alpha_1 = 0,\ \beta_1 = 1,\ \alpha_2 = 0,\ \beta_2 = 1$.

3 Generalized solution to boundary value problems on a graph segment using a generalized function method

To determine the solution, we consider a BVP in the space of generalized functions of slow growth $S'(R^2) = \left\{\hat{f}(x,t),\ (x,t) \in R^2\right\}$ [29,30]. To achieve this, we introduce a regular generalized function (marked with a cap):

$$\hat{\theta}(x,t) = \begin{cases} \theta(x,t),\ (x,t) \in D^-, \\ 0,\ x \notin D^-. \end{cases}$$

Here $D^- = [0,L] \times [0,\infty)$, $\theta(x,t)$ is the classical solution to the BVP in D^-. It can be represented as $\hat{\theta}(x,t) = \theta(x,t)H(L-x)H(x)H(t)$, where $H(x)$ is the Heaviside step function:

$$H(x) = \begin{cases} 1,\ x > 0, \\ 0,\ x < 0. \end{cases}$$

To construct the equation for $\hat{\theta}(x,t)$ in $S'(R^2)$, we find the generalized derivatives of $\hat{\theta}(x,t)$:

$$\frac{\partial\hat{\theta}}{\partial x} = \frac{\partial\theta}{\partial x}H_D^-(x,t) - \theta_2(t)\delta(L-x)H(t) + \theta_1(t)\delta(x)H(t),$$

$$\frac{\partial^2\hat{\theta}}{\partial x^2}=\frac{\partial^2\theta}{\partial x^2}H_D^-(x,t)-q_2(t)\delta(L-x)H(t)+q_1(t)\delta(x)H(t)+\theta_2(t)\delta'(L-x)H(t)+\theta_1(t)H(t)\delta'(x),$$

$$\frac{\partial\hat{\theta}}{\partial t}=\frac{\partial\theta}{\partial t}H_D^-(x,t)-\theta_0(t)H(L-x)\delta(t).$$

Here $H_D^-(x,t)=H_L^-(x)H(t)$ is characteristic function of the set D^-, $H_L^-(x)=H(L-x)H(x)$ is characteristic function of $[0,L]$, $\delta(x)$ is the singular generalized δ-function, $q_1(t)=\frac{\partial\theta}{\partial x}\big|_{x=0}$, $q_2(t)=\frac{\partial\theta}{\partial x}\big|_{x=L}$. Then Eq. (6) has the following form for $\hat{\theta}(x,t)$ in $S'(R^2)$:

$$\begin{gathered}\frac{\partial\hat{\theta}}{\partial t}-\kappa\frac{\partial^2\hat{\theta}}{\partial x^2}=\hat{F}(x,t)+\kappa q_2(t)\delta(L-x)H(x)H(t)-\kappa q_1(t)H(L-x)\delta(x)H(t)-\\-\kappa\theta_2(t)\delta'(L-x)H(x)H(t)-\kappa\theta_1(t)\delta'(x)H(L-x)H(t)+\theta_0(x)H_L^-(x)\delta(t),\\ \hat{F}(x,t)=F(x,t)H_D^-(x,t).\end{gathered}\tag{10}$$

Note that the right side of this equation includes all initial (7), (8) and boundary temperatures (9) and heat flows $\theta_j(t)\Pi_j(t)=\kappa q_j(t)$.

To shorten the formulas, we will use the following notation for partial derivatives: $u_{,x}=\frac{\partial u}{\partial x}$, $u_{,t}=\frac{\partial u}{\partial t}$.

According to the theory of generalized functions [30], the solution of Eq. (10) can be represented as a convolution of the fundamental solution to the heat equation (6) with the right-hand side of this equation:

$$\begin{gathered}\hat{\theta}(x,t)=\hat{F}_2(x,t)*U(x,t)+\kappa q_2(t)H(x)H(t)\underset{t}{*}U(L-x,t)-\\-\kappa q_1(t)H(L-x)H(t)\underset{t}{*}U(x,t)-\kappa\theta_2(t)H(t)H(x)\underset{t}{*}U_{,x}(L-x,t)-\\-\kappa\theta_1(t)H(L-x)H(t)\underset{t}{*}U_{,x}(x,t)+\theta_0(x)H(L-x)H(x)\underset{x}{*}U(x,t),\end{gathered}\tag{11}$$

where $U(x,t)$ is the Green's function of Eq. (6) — fundamental solution to the heat equation (6) by $F(x,t)=\delta(x)\delta(t)$ which decays at ∞ [30]:

$$U(x,t)=\frac{1}{\sqrt{2\pi\kappa t}}exp(-x^2/4\kappa t)H(t).\tag{12}$$

If $\hat{F}(x,t)$ is a regular function, then the formulae shown in (11) can be represented in the following integral form:

$$\begin{gathered}\hat{\theta}(x,t)=\theta(x,t)H(L-x)H(x)H(t)=\\=H(x)\int_0^t d\tau\int_{-\infty}^{+\infty}U(x-y,t-\tau)F(y,\tau)dy+\kappa H(x)H(t)\int_0^t q_2(t-\tau)U(L-x,\tau)d\tau-\\-\kappa H(L-x)H(t)\int_0^t U(x-y,t-\tau)q_1(\tau)d\tau-\kappa H(x)H(t)\int_0^t\theta_2(t-\tau)U_{,x}(L-x,\tau)d\tau-\\-\kappa H(L-x)H(t)\int_0^t U_{,x}(x,t-\tau)\theta_1(\tau)d\tau+\int_0^L U(x-y,t)\theta_0(y)H(L-y)H(y)dy.\end{gathered}\tag{13}$$

Comment. This formula determines the temperature inside a segment using known temperatures and heat flows at its ends. For this reason, it is very useful for engineering applications, as it allows one to study the influence of boundary temperature, heat flows, and initial conditions on the temperature of a segment. In practical problems, these can be measured, and this formula gives the heat state of various rod structures under arbitrary thermal conditions.

To determine unknown boundary functions (heat flows), the resolving boundary equations should be constructed using the boundary conditions at the ends of the segment.

4 Solution of BVP in Fourier transformation space in time. Resolving system of equations

To find four boundary functions in (13) and construct the resolving system of equations, we use the Fourier transform in time:

$$\begin{gathered}\bar{\theta}(x,\omega) = F\left[\hat{\theta}(x,t)\right] = H(x)H(L-x)\int_0^\infty \theta(x,t)e^{i\omega t}dt,\\ \hat{\theta}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\theta}(x,\omega)e^{-i\omega t}d\omega.\end{gathered} \tag{14}$$

To define the Fourier transform of the generalized solution (13), we use the property of the Fourier transform of convolution [30]:

$$\begin{aligned}\hat{\theta}(x,\omega)\bar{F}_2(x,\omega) \underset{x}{*} \bar{U}(x,\omega) + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)+\\ +\kappa\bar{q}_2(\omega)H(x)\bar{U}(L-x,\omega) - \kappa\bar{q}_1(\omega)H(L-x)\bar{U}(x,\omega)-\\ -\kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L-x,\omega) - \kappa\bar{\theta}_1 H(L-x)\bar{U}_{,x}(x,\omega).\end{aligned} \tag{15}$$

Here, a variable under the convolution sign $\left(\underset{x}{*}\right)$ shows convolution over the variable x. The integral representation of (15) has the following form:

$$\begin{aligned}\bar{\theta}(x,\omega)H(L-x)H(x)H(\omega) = H(x)\int_0^L \bar{U}(x-y,\omega)F_2(y,\omega)dy + \kappa H(x)\int_0^L \bar{U}(x-y,\omega)\theta_0(y)dy+\\ +\kappa\bar{q}_2(\omega)H(x)\bar{U}(L-x,\omega) - \kappa\bar{q}_1(x)H(L-x)\bar{U}(x,\omega)-\\ -\kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L-x,\omega) - \kappa\bar{\theta}_1 H(L-x)\bar{U}_{,x}(x,\omega),\end{aligned} \tag{16}$$

where the Fourier transform of the Green's function of the heat equation is equal to

$$\bar{U}(x,\omega) = -0,5\frac{sin(k|x|)}{\kappa(k+i0)}, \tag{17}$$

where $k = \sqrt{i\omega\kappa^{-1}} = e^{j\pi/4}\sqrt{\omega\kappa^{-1}} = (1+i)\sqrt{\frac{\omega}{2\kappa}}$. It satisfies the following equation:

$$\frac{d^2\bar{U}}{dx^2} + i\omega\kappa^{-1}\bar{U} = \delta(x).$$

Its derivative has a gap at the point $x = 0$, and is equal to

$$\frac{\partial\bar{U}(x,\omega)}{\partial x} = \bar{U}_{,x}(x,\omega) = -\frac{sgnx}{2\kappa}cos(\kappa|x|), \quad sgnx = \begin{cases} 1, & x > 0,\\ -1, & x < 0.\end{cases}$$

There are the next symmetry conditions:

$$\bar{U}(x,\omega) = \bar{U}(-x,\omega), \quad \bar{U}_{,x}(\pm 0,\omega) = \mp\frac{1}{2\kappa}. \tag{18}$$

We use these properties (17), (18) to construct the solving system of equations.

To find the unknown boundary functions, we pass in relation (16) to the limits at the ends of a segment:

$$\lim_{\varepsilon\to 0}\bar{\theta}(0+\varepsilon,\omega) = \bar{\theta}_1(x) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)|_{x=0} +$$
$$+\kappa\bar{q}_2(\omega)\bar{U}(L,\omega) - \kappa\bar{q}_1(\omega)\bar{U}(0,\omega) - \kappa\bar{\theta}_2(x)\bar{U}_x(L,\omega) - \kappa\bar{\theta}_1\bar{U}_{,x}(+0,\omega).$$

From (18) to follow: $-\kappa\bar{\theta}_1(\omega)\bar{U}_{,x}(+0,\omega) = 0,5\bar{\theta}_1(\omega)$. Next, we factor out the last term on the left side and obtain the following equation at the left end of the segment:

$$\frac{1}{2}\bar{\theta}_1(\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)|_{x=0} +$$
$$+\kappa\bar{q}_2(x)\bar{U}(L,\omega) - \kappa\bar{q}_1(\omega)\bar{U}(0,\omega) - \kappa\bar{\theta}_2(\omega)\bar{U}_{,x}(L,\omega).$$

Similarly, we consider the limit at $x = L-\varepsilon$, $\varepsilon \to +0$, and obtain the second equation on the right end of the segment. Let us formulate the obtained results.

Theorem 1. The Fourier time transformants of the boundary functions of the boundary value problems (6)–(9) satisfy the following system of linear algebraic equations:

$$\begin{bmatrix} 0,5 & 0 \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \begin{bmatrix} \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ 0,5 & 0 \end{bmatrix} \begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{Q}_1(0,\omega) \\ \bar{Q}_2(L,\omega) \end{bmatrix}, \tag{19}$$

where

$$\bar{Q}_1(0,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)|_{x=0},$$
$$\bar{Q}_2(L,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)|_{x=L} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)|_{x=L}.$$

The system (19) gives the possibility to determine only two boundary functions at the ends of the segment if two functions from $\bar{\theta}_1(\omega)\bar{q}_1(\omega)\bar{\theta}_2(\omega)\bar{q}_2(\omega)$ are known. If to add two boundary conditions (9) we have the full system of four linear algebraic equations to determine these boundary functions:

$$\mathbf{A}(\omega)\cdot B(\omega) = C(\omega), \tag{20}$$

where

$$\mathbf{A}(\omega) = \begin{pmatrix} 0,5 & 0 & \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) & 0,5 & 0 \\ \alpha_1 & \beta_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 \end{pmatrix},$$
$$B(\omega) = (\bar{\theta}_1(\omega), \bar{q}_1(\omega)\bar{\theta}_2(\omega), \bar{q}_2(\omega))^T,$$
$$C(\omega) = (\bar{Q}_1(0,\omega), \bar{Q}_2(L,\omega)\bar{b}_3(\omega), \bar{b}_4(\omega))^T.$$

The solution of resolving system (20) is

$$B(\omega) = \mathbf{A}^{-1}(\omega)\cdot C(\omega), \tag{21}$$

where $\mathbf{A}^{-1}$ is the inverse matrix of $\mathbf{A}(\omega)$.

Substituting (21) into (16) we obtain the temperature transformant at any point x. Performing the inverse Fourier transform (14), we obtain $\hat{\theta}(x,t)$. Thus, the temperature $\theta(x,t)$ on the interval $[0,L]$ has been determined at any time t. We have solved BVPs.

5 Resolving system of equations on a star thermal graph

Let's return to the consideration of Dirichlet problem for a star heat graph (Fig. 1) by using the system (12). On each edge L_j, $j = \overline{1,N}$, we have the following system of linear algebraic equations of connection of four boundary functions $(\bar{\theta}_1^j(\omega)\bar{q}_1^j(\omega), \bar{\theta}_2^j(\omega)\bar{q}_2^j(\omega))$:

$$\begin{pmatrix} 1 & 0 & -\cos(k_jL_j) & k_j^{-1}\sin(k_jL_j) \\ -\cos(k_jL_j) & -k_j^{-1}\sin(k_jL_j) & 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\theta}_1^j(\omega) \\ \bar{q}_1^j(\omega) \\ \bar{\theta}_2^j(\omega) \\ \bar{q}_2^j(\omega) \end{pmatrix} = \begin{pmatrix} \bar{F}_1^j(\omega) \\ \bar{F}_2^j(\omega) \end{pmatrix},$$

where j is the number of the corresponding graph edge, and $\bar{F}_1^j(\omega) = 2\bar{Q}_{1j}(0,\omega)$, $\bar{F}_2^j(\omega) = 2\bar{Q}_{2j}(L,\omega)$.

$$\bar{Q}_{1j}(0,\omega) = \bar{F}_j(x,\omega) \underset{x}{*} \bar{U}^j(x,\omega)|_{x=0} + \theta_0^j(x)H_L^-(x) \underset{x}{*} \bar{U}^j(x,\omega)|_{x=0},$$

$$\bar{Q}_{2j}(L,\omega) = \bar{F}_j(x,\omega) \underset{x}{*} \bar{U}^j(x,\omega)|_{x=L} + \theta_0^j(x)H_L^-(x) \underset{x}{*} \bar{U}^j(x,\omega)|_{x=L},\ \tilde{U}^j(x,\omega) = -0,5\frac{\sin(k_j|x|)}{k_j(k_j+i0)}.$$

Consequently, we have $2N$ equations for defining $4N$ boundary functions:

$$B(\omega) = (\bar{\theta}_1^1, \bar{q}_1^1, \bar{\theta}_2^1, \bar{q}_2^1, ..., \bar{\theta}_1^N, \bar{q}_1^N\bar{\theta}_2^N, \bar{q}_2^N)^T.$$

This graph has N edges with one boundary condition at the end of every edge. Consequently, we add N boundary conditions at the ends of this graph.

The next N equations contain the condition of continuity (4) and Kirchhoff condition (5) for N edges in common boundary points $x_1 = 0$ at the node A_0. Therefore, the complete resolving system of $4N$ equations has been written in the next form.

Theorem 2. The resolving system of boundary value problem equations (1)–(5) on a star heat graph with N different edges has the following form:

$$\mathbf{A}(\omega) \cdot B(\omega) = C(\omega), \tag{22}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \kappa_1\bar{U}^1_{,x}(L_1) & -\kappa_1\bar{U}^1(L_1) & .. & 0 & 0 & 0 & 0 \\ \kappa_1\bar{U}^1_{L_1} & \kappa_1\bar{U}^1(L_1) & 1 & 0 & .. & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & .. & ... & ... & ... & ... \\ ... & ... & ... & ... & .. & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & .. & 1 & 0 & \kappa_N\bar{U}^N_{,x}(L_N) & -\kappa_N\bar{U}^N(L_N) \\ 0 & 0 & 0 & 0 & .. & \kappa_N\bar{U}^N_{,x}(L_n,\omega) & \kappa_N\bar{U}^N(L_N) & 1 & 0 \\ 0 & 0 & 1 & 0 & .. & 0 & 0 & 0 & 0 \\ 0 & ... & ... & ... & .. & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & .. & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & .. & 0 & 0 & 0 & 0 \\ 1 & ... & ... & ... & .. & ... & ... & ... & ... \\ 1 & ... & ... & ... & .. & -1 & ... & 0 & 0 \\ 0 & \kappa_1 & 0 & 0 & .. & 0 & \kappa_n & 0 & 0 \end{bmatrix}$$

and dimension of the matrix $\mathbf{A}$ is $4N \times 4N$;

$$C(\omega) = \left(\bar{F}_1^1(0,\omega)\bar{F}_2^{-1}(L_1,\omega), ..., \bar{F}_1^N(L_N,\omega)\bar{F}_2^N(L_N,\omega); \bar{\theta}_1(L_1,\omega), ..., \bar{\theta}_N(L_N,\omega); \underbrace{0, ..., 0}_{N-1}, \bar{G}(\omega)\right)^T.$$

Its solution is equal to

$$B(\omega) = \mathbf{A}^{-1}(\omega) \cdot C(\omega),$$

$\mathbf{A}^{-1}(\omega)$ is inverse matrix to matrix $\mathbf{A}(\omega)$.

The first $2N$ rows of matrix A contain at "diagonal" the resulting system (22) for each edge of this graph. The other elements of these lines are null (see $A(\omega)$).

The next N lines contain the boundary conditions on the ends of graph (3). Here in the line $2N+j$ only the column $4j-1$ contains 1, others are null.

The line $(3N+j)$, $j=1,T\ldots,N-1$ in the first column stands 1, and -1 in column $(4j+1)$, $j=1,T\ldots,N-1$ (condition of temperature continuity (4)). In the last line (Kirchhoff condition (5)), the value κ_j stands in the column $2+4j$, $j=0,T\ldots,N-1$.

Substituting $(\bar{\theta}_1^j, \bar{q}_1^j, \bar{\theta}_2^j, \bar{q}_2^j)$ — the corresponding components of $B(\omega)$ for j-edge into (16), we obtain $\bar{\theta}_j(x,\omega)$ at any point x of this edge:

$$\bar{\theta}(x,\omega)H_D^-(x,t) = H(x)\int_0^{L^-} \bar{U}^j(x-y,\omega)\bar{F}^j(y,\omega)dy + \kappa_j H(x)\int_0^{L} \bar{U}^j(x-y,\omega)\theta_0^j(y)dy +$$
$$+\kappa_j\bar{q}_2^j(\omega)\bar{U}^j(L_j-x,\omega) - \kappa_j\bar{q}_1^j(\omega)H(L_j-x)\bar{U}^j(x,\omega) -$$
$$-\kappa_j\bar{\theta}_2^j(\omega)\bar{U}_{,x}^j(L_j-x,\omega) - \kappa_j\bar{\theta}_1^j(\omega)H(L_j-x)\bar{U}_{,x}^j(x,\omega), \quad 0 \le x \le L_j.$$

Performing the inverse Fourier transform (14), we obtain $\hat{\theta}(x,t) = \theta_j(x,t)H_D^-(x,t)$. Thus, the temperature on the star graph is determined at any time t at any edge.

So BVP for the heat star graph has been solved.

Conclusion

Using the generalized function method, the boundary value problems of thermal conductivity on a thermal star graph have been solved, which can be used to study various network-like structures under conditions of thermal heating (cooling). A unified technique has been developed for solving various boundary value problems typical for practical applications.

The action of heat sources can be modeled by both regular and singular generalized functions under various boundary conditions at the ends of the graph edge. The obtained regular integral representations of generalized solutions make it possible to determine the temperature and heat flows on each element and at any point of a graph for stationary oscillations with a constant frequency and in the case of periodic oscillations.

At first, a boundary value problem was solved on one edge of the graph. Using the generalized function method, a heat equation with a singular right-hand side was obtained. The solution to the Dirichlet problem was determined through the convolution of the fundamental solution with the singular right-hand side of the heat equation. Thus, the solution found on the edge was determined by the initial functions, boundary functions, and their derivatives (the unknown boundary functions). A resolving system of two linear algebraic equations in the space of the Fourier transform in time was constructed to determine the unknown boundary functions. After determining all the solutions on all graph edges and taking the continuity condition and Kirchhoff joint condition into account, we obtained the solution to the heat equation on the star graph.

The generalized function method presented here makes it possible to solve a wide range of boundary value problems at the ends of the graph edges and various transmission conditions at its common node and can be extended to network structures of very different types. This distinguishes this method from all others that are used to solve similar problems. This algorithm can be recommended and used for engineering calculations of heating networks and it will find wide application in the design and calculation of rod structures in mechanical engineering and construction.

The method of generalized functions presented here allows not only to solve a wide range of problems with different conditions at the ends of the edges of the graph and the conditions of transmission in its common node, but can also be extended to network structures of various types. This distinguishes this method from all others that are used to solve similar problems.

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Author contributions

L.A: Project administration, Conceptualization, Methodology, Formal Analysis, Investigation; D.P: Investigation; A.D: Investigation, Formal Analysis, Validation, Software; N.A: Investigation, Formal Analysis, Validation, Software.

Conflict of interest

The authors declare no conflict of interest.

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Particular solutions of the multidimensional singular ultrahyperbolic equation generalizing the telegraph and Helmholtz equations

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This article deals with the construction of particular solutions for a second-order multidimensional singular partial differential equation, which generalizes the famous telegraph and Helmholtz equations. The constructed particular solutions are expressed in terms of the multiple confluent hypergeometric function, which is analogous to the multiple Lauricella function and the famous Bessel function. A limit correlation theorem for the multiple confluent hypergeometric function is proved, and a system of partial differential equations associated with the confluent function is derived. Thanks to the proven properties of the multiple confluent hypergeometric function. The particular solutions of the multidimensional partial differential equation with the singular coefficients are written in explicit forms and it is determined that these solutions have a singularity at the vertex of a multidimensional cone.

Keywords: particular solution, Lauricella function, multiple confluent hypergeometric function, a limit correlation theorem, a system of the partial differential equations.

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Introduction

It is well known that particular solutions play an essential role in the study of partial differential equations. The set of particular solutions includes fundamental solutions that satisfy certain additional conditions. In case of the singular elliptic equations, the role of particular solutions is played by fundamental solutions. Formulation and solving of many local and non-local boundary value problems are based on these solutions. The explicit form of particular solutions gives a possibility to study the considered equation in detail.

In the case of PDE with singular coefficients, particular solutions, including fundamental ones, are expressed through hypergeometric functions, the number of variables of which is directly related to the number of singular coefficients. For instance, in the paper [1], particular solutions of the generalized Euler-Poisson-Darboux equation with three singular coefficients

$$u_{xx}+u_{yy}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y=u_{tt}+\frac{2\gamma}{t}u_t,\ x>0,\ y>0,\ t>0,\ \ 0<2\alpha,\,2\beta,\,2\gamma<1 \tag{1}$$

are written by a hypergeometric function $F_A^{(3)}$ in three variables introduced by Lauricella [2]. In addition, self-similar solutions of some model degenerate partial differential equations of the higher order are expressed by the higher order hypergeometric functions [3–6].

It is well known [7] that all linearly independent fundamental solutions at the origin of singular elliptic equation

$$\sum_{j=1}^{m}\frac{\partial^2 u}{\partial x_j^2}+\sum_{j=1}^{n}\frac{2\alpha_j}{x_j}\frac{\partial u}{\partial x_j}=0,\quad m\geq 2,\quad n\leq m \tag{2}$$

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in the first hyperoctant $x_1 > 0, \ldots, x_n > 0$ are expressed explicitly by the Lauricella function $F_A^{(n)}$ in n variables. Various applications of the fundamental solutions of equation (2) to the solution of boundary value problems for this equation can be found in the works [8–11].

In a recent work [12], particular solutions of the equation

$$\sum_{j=1}^{p} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^{p} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} = \sum_{j=p+1}^{n} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=p+1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j}, \;\; p = \overline{1, n} \tag{3}$$

are also expressed through the Lauricella function $F_A^{(n)}$, the variables of which differ from the variables of the Lauricella function included in the fundamental solutions of the equation (2) only by signs depending on the equation under consideration.

All fundamental solutions of the multidimensional Helmholtz equation with n singular coefficients

$$\sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} + \lambda u = 0, \;\; m \geq 2, \;\; n \leq m, \;\; -\infty < \lambda < +\infty \tag{4}$$

are presented by the confluent hypergeometric function in $n+1$ variables, the first n variables of which coincide with the variables of the fundamental solutions of equation (2). In this case, the last variable in the confluent hypergeometric function appears due to the presence of the parameter λ (for details, see [13]).

The following so-called multidimensional singular ultrahyperbolic equation

$$\sum_{j=1}^{p} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^{p} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} = \sum_{j=p+1}^{m} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=p+1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} + \lambda u, \;\; p \leq n \leq m \tag{5}$$

contains all four equations (1)–(4) considered above. Note that equation (5) generalizes also the well-known Helmholtz $u_{xx} + u_{yy} + cu = 0$ and telegraph $u_{xx} - u_{yy} + cu = 0$ equations.

In this paper we construct particular solutions of equation (5) in some multidimensional cone when $0 < p < n = m$ and prove that these solutions are simultaneously fundamental solutions of the considered equation near the origin. Note, if $p = 0$ or $p = n$, then the equation (5) becomes an equation of the singular elliptic type (4), particular (fundamental) solutions of which are found in [13].

The plan of this paper is as follows. In Section 1, we briefly give some preliminary information, which will be used later, and investigate new properties of the multiple confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$. In Section 2 we compose a system corresponding to the function $\mathrm{H}_A^{(n,1)}$ and find all particular solutions of this system. In Section 3 we study an ultrahyperbolic equation with singular coefficients, all particular solutions of which are written out explicitly through a multiple confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$. In Section 4, the properties of the constructed particular solutions are studied and the order of singularity of these solutions in the neighborhood of the origin is determined.

1 Hypergeometric functions of several variables

The great success of the theory of hypergeometric functions in one variable has stimulated the development of a corresponding theory in two and more variables. Horn [14] gave the general definition of the hypergeometric functions of two variables. He has investigated the convergence of hypergeometric functions in two variables and established the systems of partial differential equations which they satisfy (for details, see [15; Section 5.7]).

Following Horn we define a hypergeometric function of several variables.

Let a multiple power series be given

$$\sum_{|\mathbf{k}|=0}^{\infty} A(\mathbf{k}) \prod_{j=1}^{n} x_j^{k_j}, \tag{6}$$

where the summation is carried out over a multi-index $\mathbf{k} := (k_1, \ldots, k_n)$ with non-negative integer components $k_j \geq 0$, $j = 1, \ldots, n$, for which, as usual, $|\mathbf{k}| := k_1 + \ldots + k_n$.

A multiple power series (6) is a hypergeometric series if the following n relations

$$\frac{A(\mathbf{k}_j + \mathbf{e}_j)}{A(\mathbf{k})} = f_j(\mathbf{k}) \tag{7}$$

are rational functions of $\mathbf{k}$, where $\mathbf{e}_j := (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes a vector whose $j-$th component is equal to one, and the rest are equal to zero $(j = 1, \ldots, n)$.

Let's suppose

$$f_j(\mathbf{k}) = \frac{P_j(\mathbf{k})}{Q_j(\mathbf{k})}, \tag{8}$$

where P_j and Q_j are polynomials of $\mathbf{k}$ having degrees p_j and q_j respectively. It is assumed that Q_j has a multiplier of $k_j + 1$; P_j and Q_j have no common multipliers, with the possible exception of $k_j + 1$ $(j = 1, \ldots, n)$.

The largest of the numbers $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ is called *order* of the hypergeometric series (6).

The hypergeometric series (6) is called *complete*, if all the numbers $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ are the same, i.e. $p_1 = \ldots = p_n = q_1 = \ldots = q_n$, otherwise *confluent*.

A symbol $(\kappa)_\nu$ denotes the general Pochhammer symbol or the shifted factorial, since $(1)_l = l!$ $(l \in \mathbb{N} \bigcup \{0\}; \mathbb{N} := \{1, 2, \ldots\})$, which is defined (for $\kappa, \nu \in \mathbb{C}$), in terms of the familiar Gamma function, by

$$(\kappa)_\nu := \frac{\Gamma(\kappa + \nu)}{\Gamma(\kappa)} = \begin{cases} 1 & (\nu = 0;\ \kappa \in \mathbb{C} \setminus \{0\}), \\ \kappa(\kappa + 1) \ldots (\kappa + l - 1) & (\nu = l \in \mathbb{N};\ \kappa \in \mathbb{C}), \end{cases}$$

it is being understood conventionally that $(0)_0 := 1$ assumed tacitly that the $\Gamma-$quotient exists.

A Lauricella function $F_A^{(n)}$ in $n \in N$ real variables $\mathbf{x} := (x_1, \ldots, x_n)$ [2] (see, also [16])

$$F_A^{(n)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \ \mathbf{x} \right] = \sum_{|\mathbf{k}|=0}^{\infty} (a)_{|\mathbf{k}|} \prod_{j=1}^{n} \frac{(b_j)_{k_j}}{(c_j)_{k_j}} \frac{x_j^{k_j}}{k_j!}, \quad \sum_{j=1}^{n} |x_j| < 1 \tag{9}$$

is also a complete hypergeometric function of the order 2. Hereinafter $\mathbf{b} := (b_1, \ldots, b_n)$, $\mathbf{c} := (c_1, \ldots, c_n)$. In definition (9), as usual, the denominator parameters c_1,...,c_n are neither zero nor a negative integer.

Let a, b_k, c_k be real numbers, where $c_k \neq 0, -1, -2, \ldots$ and $a > |\mathbf{b}| > 0$ and $c_k > b_k$. Then for $n = 1, 2, \ldots$, the following limit correlation is true [17]

$$\lim_{\varepsilon \to 0} \left\{ \varepsilon^{-|\mathbf{b}|} F_A^{(n)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \ 1 - \frac{z_1(\varepsilon)}{\varepsilon}, \ldots, 1 - \frac{z_n(\varepsilon)}{\varepsilon} \right] \right\} = \frac{\Gamma(a - |\mathbf{b}|)}{\Gamma(a)} \prod_{k=1}^{n} \frac{|z_k(0)|^{-b_k} \Gamma(c_k)}{\Gamma(c_k - b_k)}, \tag{10}$$

where $|\mathbf{b}| := b_1 + \ldots + b_n$; $z_k(\varepsilon)$ are arbitrary functions, and $z_k(0) \neq 0$.

Note that the limit correlation formula (10) is applied in the theory of boundary value problems for the multidimensional singular elliptic equation (2), for instance, see [18].

Consider the following confluent hypergeometric function in $n + 1$ variables

$$\mathrm{H}_A^{(n,1)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \ \mathbf{x}, y \right] = \sum_{|\mathbf{k}|+l=0}^{\infty} (a)_{|\mathbf{k}|-l} \prod_{j=1}^{n} \frac{(b_j)_{k_j}}{(c_j)_{k_j}} \frac{x_j^{k_j}}{k_j!} \cdot \frac{y^l}{l!}, \quad \sum_{j=1}^{n} |x_j| < 1, \tag{11}$$

where $\mathbf{x}$ and y are real variables, and $l = 0, 1, 2, \ldots$

Note, this confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$ was first introduced and studied in a more general form in [13] and its particular cases ($n = 1, 2, 3$) were known in [15, 19, 20].

The confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$ has the following formula of derivation:

$$\frac{\partial^{|\mathbf{k}|+l}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n} \partial y^l} \mathrm{H}_A^{(n,1)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \; \mathbf{x}, y \right] = (a)_{|\mathbf{k}|-l} \prod_{j=1}^{n} \frac{(b_j)_{k_j}}{(c_j)_{k_j}} \cdot \mathrm{H}_A^{(n,1)} \left[\begin{matrix} a + |\mathbf{k}| - l, \mathbf{b} + \mathbf{k}; \\ \mathbf{c} + \mathbf{k}; \end{matrix} \; \mathbf{x}, y \right], \tag{12}$$

hereinafter, $\mathbf{k} := (k_1, \ldots, k_n)$ is an n-vector.

Using simple properties of the Pochhammer symbol

$$(a)_m (a + m)_k = (a)_{m+k}, \;\; (a)_k = \frac{(-1)^k}{(1 - a)_k},$$

we can represent the confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$ as

$$\mathrm{H}_A^{(n,1)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \; \mathbf{x}, y \right] = \sum_{k=1}^{\infty} \frac{(-1)^k}{(1 - a)_k} \frac{y^k}{k!} F_A^{(n)} \left[\begin{matrix} a - k, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \; \mathbf{x} \right], \tag{13}$$

where $F_A^{(n)}$ is the Lauricella function defined in (9).

It is obvious that

$$\mathrm{H}_A^{(n,1)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \; \mathbf{x}, 0 \right] = F_A^{(n)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \; \mathbf{x} \right]. \tag{14}$$

Theorem 1. Let a, b_k, c_k be real numbers, where $c_k \neq 0, -1, -2, \ldots$ and $a > |\mathbf{b}| > 0$ and $c_k > b_k$. Then for $n = 1, 2, \ldots$, the following limit correlation is true

$$\lim_{\varepsilon \to 0} \left\{ \varepsilon^{-|\mathbf{b}|} \mathrm{H}_A^{(n,1)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \; 1 - \frac{z_1(\varepsilon)}{\varepsilon}, \ldots, 1 - \frac{z_n(\varepsilon)}{\varepsilon}, \varepsilon y \right] \right\} = \frac{\Gamma\left(a - |\mathbf{b}|\right)}{\Gamma(a)} \prod_{k=1}^{n} \frac{|z_k(0)|^{-b_k}\, \Gamma\left(c_k\right)}{\Gamma\left(c_k - b_k\right)}, \tag{15}$$

where $|\mathbf{b}| := b_1 + \ldots + b_n$; $z_k(\varepsilon)$ are arbitrary functions, and $z_k(0) \neq 0$; y is a real variable.

Proof. The proof of Theorem 1 follows from expansion (13), obvious equality (14) and limit correlation formula (10).

2 *System of differential equations satisfied by the confluent function* $\mathrm{H}_A^{(n,1)}$

We represent the confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$, defined by the equality (11), in the form

$$\mathrm{H}_A^{(n,1)} \left[\begin{matrix} a, \mathbf{b}; \\ \mathbf{c}; \end{matrix} \; \mathbf{x}; y \right] = \sum_{|\mathbf{k}|+l=0}^{\infty} A\left(\mathbf{k}; l\right) \prod_{j=1}^{n} x_j^{k_j} \cdot y^l, \tag{16}$$

where

$$A\left(\mathbf{k}; l\right) = \frac{(a)_{|\mathbf{k}|-l} (b_1)_{k_1} \ldots (b_n)_{k_n}}{k_1! \ldots k_n! l! (c_1)_{k_1} \ldots (c_n)_{k_n}}.$$

By virtue of (7) and (8), we have

$$f_j\left(\mathbf{k}; l\right) = \frac{P_j\left(\mathbf{k}; l\right)}{Q_j\left(\mathbf{k}; l\right)}, \;\; j = \overline{1, n}; \;\;\; g\left(\mathbf{k}; l\right) = \frac{1}{G\left(\mathbf{k}; l\right)},$$

here

$$P_j(\mathbf{k};l) = (a+|\mathbf{k}|-l)(b_j+k_j),\ j=\overline{1,\,n};$$

$$Q_j(\mathbf{k};l) = (1+k_j)(c_j+k_j),\ j=\overline{1,\,n};$$

$$G = (1+l)(a-1+|\mathbf{k}|-l).$$

Series (16) satisfies a system of linear partial differential equations. Using differential operators

$$\delta_j \equiv x_j\frac{\partial}{\partial x_j},\ \ j=\overline{1,n};\ \ \delta' \equiv y\frac{\partial}{\partial y} \tag{17}$$

this system can be written in the form

$$\begin{cases} \left[Q_j\left(\delta_1,\ldots,\delta_n;\delta'\right)x_j^{-1} - P_j\left(\delta_1,\ldots,\delta_n;\delta'\right)\right]\omega = 0,\ j=\overline{1,\,n}, \\ \left[G\left(\delta_1,\ldots,\delta_n;\delta'\right)y^{-1}-1\right]\omega = 0. \end{cases} \tag{18}$$

Now, substituting differential operators (17) into (18), we get

$$\begin{cases} x_i(1-x_i)\,\omega_{x_ix_i} - x_i\sum\limits_{j=1,\,j\neq i}^{n} x_j\omega_{x_ix_j} + x_iy\omega_{x_iy} + \left[c_i-(a+1)\,x_i\right]\omega_{x_i} \\ \qquad - b_i\sum\limits_{j=1,\,j\neq i}^{n} x_j\omega_{x_j} + b_iy\omega_y - ab_i\omega = 0,\ \ i=\overline{1,n}, \\ y\omega_{yy} - \sum\limits_{j=1}^{n} x_j\omega_{x_jy} + (1-a)\,\omega_y + \omega = 0, \end{cases} \tag{19}$$

where $\omega(\mathbf{x};y) = \mathrm{H}_A^{(n,1)}\left[\begin{matrix} a,\mathbf{b}; \\ \mathbf{c}; \end{matrix}\ \mathbf{x},y\right]$.

Theorem 2. [13] System of differential equations (19) near the origin has 2^n linearly independent solutions:

$1:\{\,\mathrm{H}_A^{(n,1)}\left[\begin{matrix} a,b_1,\ldots,b_n; \\ c_1,\ldots,c_n; \end{matrix}\ \mathbf{x};y\right],$

$$C_n^1:\begin{cases} x_1^{1-c_1}\mathrm{H}_A^{(n,1)}\left[\begin{matrix} a+1-c_1,b_1+1-c_1,b_2,\ldots,b_n; \\ 2-c_1,c_2,\ldots,c_n; \end{matrix}\ \mathbf{x};y\right], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n^{1-c_n}\mathrm{H}_A^{(n,1)}\left[\begin{matrix} a+1-c_n,b_1,\ldots,b_{n-1},b_n+1-c_n; \\ c_1,\ldots,c_{n-1},2-c_n; \end{matrix}\ \mathbf{x};y\right], \end{cases}$$

$$C_n^2:\begin{cases} x_1^{1-c_1}x_2^{1-c_2}\mathrm{H}_A^{(n,1)}\left[\begin{matrix} a+2-c_1-c_2,b_1+1-c_1,b_2+1-c_2,b_3,\ldots,b_n; \\ 2-c_1,2-c_2,c_3,\ldots,c_n; \end{matrix}\ \mathbf{x};y\right], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_1^{1-c_1}x_n^{1-c_n}\mathrm{H}_A^{(n,1)}\left[\begin{matrix} a+2-c_1-c_n,b_1+1-c_1,b_2,...,b_{n-1},b_n+1-c_n; \\ 2-c_1,c_2,\ldots,c_{n-1},2-c_n; \end{matrix}\ \mathbf{x};y\right], \\ x_2^{1-c_2}x_3^{1-c_3}\mathrm{H}_A^{(n,1)}\left[\begin{matrix} a+2-c_2-c_3,b_1,b_2+1-c_2,b_3+1-c_3,b_4,\ldots,b_n; \\ c_1,2-c_2,2-c_3,c_4,\ldots,c_n; \end{matrix}\ \mathbf{x};y\right], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n-1}^{1-c_{n-1}}x_n^{1-c_n}\mathrm{H}_A^{(n,1)}\left[\begin{matrix} a+2-c_{n-1}-c_n,b_1,\ldots,b_{n-2},b_{n-1}+1-c_{n-1},b_n+1-c_n; \\ c_1,\ldots,c_{n-2},2-c_{n-1},2-c_n; \end{matrix}\ \mathbf{x};y\right], \end{cases}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$1:\left\{x_1^{1-c_1}\dots x_n^{1-c_n}\mathrm{H}_A^{(n,1)}\left[\begin{matrix} a+n-c_1-\dots-c_n, b_1+1-c_1,\dots,b_n+1-c_n;\\ 2-c_1,\dots,2-c_n;\end{matrix}\ \mathbf{x};y\right]\right.,$$

where $C_n^k=\dfrac{n!}{k!(n-k)!}$ are binomial coefficients.

When none of the numbers $c_1,\ c_2,\ \dots,\ c_n$ is equal to a negative integer, we obtain the general solution of system (19) by multiplying these 2^n partial solutions by arbitrary constants and then taking their sum.

It is easy to see that in the first group there is one solution $\left(C_n^0=1\right)$, in the second group there are $C_n^1=n$ solutions, the third group consists of $C_n^2=n(n-1)/2$ solutions, etc. So the system of hypergeometric equations (19) really has 2^n solutions.

However, within each group, the functions included in this group are symmetrical with respect to the numerical parameters. Therefore, for further purposes, it is enough to select one solution from each group, or more precisely, the solution that comes first in each group. So $n+1$ linearly independent solutions to the system of equations (19) will be identified by the formulas

$$\omega_0(\mathbf{x};y)=C_0\,\mathrm{H}_A^{(n,1)}\left[\begin{matrix} a,b_1,\dots,b_n;\\ c_1,\dots,c_n;\end{matrix}\ \mathbf{x};y\right], \tag{20}$$

$$\omega_i(\mathbf{x};y)=C_i\prod_{j=1}^{i}x_j^{1-c_j}\cdot\mathrm{H}_A^{(n,1)}\left[\begin{matrix} a+i-|\mathbf{c}_i|,b_1+1-c_1,\dots,b_i+1-c_i,b_{i+1},\dots,b_n;\\ 2-c_1,\ \dots,\ 2-c_i,\ c_{i+1},\dots,c_n;\end{matrix}\ \mathbf{x};y\right], \tag{21}$$

where $C_0,\dots,C_n$ are arbitrary constants; $|\mathbf{c}_i|:=c_1+\dots+c_i,\ i=\overline{1,n}$.

Using the derivation formula (12), it is easy to verify that the functions defined in (20) and (21) really satisfy to the system of partial differential equations (19).

3 Particular solutions

Consider the multidimensional ultrahyperbolic equation

$$L(u)\equiv\sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(\frac{\partial^2u}{\partial x_j^2}+\frac{2\alpha_j}{x_j}\frac{\partial u}{\partial x_j}\right)+\lambda u=0,\ \ p=\overline{1,n-1},\ \ n\ge 2 \tag{22}$$

in the n-dimensional cone

$$\Omega=\left\{(x_1,\dots,x_n):x_1^2+\dots+x_p^2>x_{p+1}^2+\dots+x_n^2,\ p=\overline{1,n-1};\ x_j>0,\ j=\overline{1,n}\right\},$$

where α_j are constants ($0<2\alpha_j<1$, $j=\overline{1,n}$); λ is a real number;

$$\mathrm{sgn}(z):=\begin{cases}\ \ 1, & \text{if } z\ge 0,\\ -1, & \text{if } z<0.\end{cases}$$

Let $x:=(x_1,\dots,x_n)$ be any point and $\xi:=(\xi_1,\dots,\xi_n)$ be any fixed point of Ω. We search for a solution of equation (22) as follows:

$$u\left(x;\xi\right)=P(r)\omega\left(\sigma_p,\eta_p\right),\quad p=\overline{1,n-1}, \tag{23}$$

where

$$P(r)=r_p^{-2\beta},\quad \beta=\frac{n-2}{2}+\sum_{j=1}^{n}\alpha_j; \tag{24}$$

ω is an unknown function, depending on $n+1$ variables

$$\sigma_p := (\sigma_{p1}, \sigma_{p2}, \ldots, \sigma_{pn}), \;\; \sigma_{pj} = -\mathrm{sgn}(p-j)\frac{4x_j\xi_j}{r_p^2}, \;\; j=\overline{1,n}, \tag{25}$$

$$\eta_p = \frac{1}{4}\lambda r_p^2, \quad r_p^2 = \sum_{k=1}^{n} \mathrm{sgn}(p-k)(x_k-\xi_k)^2, \;\; p=\overline{1,n-1}.$$

First, we calculate the derivatives of $u\,(x;\xi)$ with respect to the variables $x_1,\ldots,x_n$:

$$\frac{\partial u}{\partial x_j} = \frac{\partial P}{\partial x_j}\omega + P\left(\sum_{k=1}^{n}\frac{\partial \omega}{\partial \sigma_k}\frac{\partial \sigma_k}{\partial x_i} + \frac{\partial \omega}{\partial \eta}\frac{\partial \eta}{\partial x_j}\right),$$

$$\frac{\partial^2 u}{\partial x_j^2} = P\sum_{k=1}^{n}\frac{\partial^2\omega}{\partial x_k^2}\left(\frac{\partial \sigma_k}{\partial x_j}\right)^2 + 2P\sum_{k=1}^{n}\left(\sum_{l=k+1}^{n}\frac{\partial^2\omega}{\partial\sigma_k\partial\sigma_l}\frac{\partial\sigma_l}{\partial x_j} + \frac{\partial^2\omega}{\partial\sigma_k\partial\eta}\frac{\partial\eta}{\partial x_j}\right)\frac{\partial\sigma_k}{\partial x_j}$$
$$+\sum_{k=1}^{n}\left[\left(2\frac{\partial P}{\partial x_j}\frac{\partial\sigma_k}{\partial x_j} + P\frac{\partial^2\sigma_k}{\partial x_j^2}\right)\frac{\partial\omega}{\partial\sigma_k} + \left(2\frac{\partial P}{\partial x_j}\frac{\partial\eta}{\partial x_j} + P\frac{\partial^2\eta}{\partial x_j^2}\right)\frac{\partial\omega}{\partial\eta}\right] + \frac{\partial^2 P}{\partial x_j^2}\omega.$$

Now substituting product (23) into equation (22), we obtain

$$\begin{aligned}\sum_{k=1}^{n}A_k\frac{\partial^2\omega}{\partial\sigma_k^2} + A_{n+1}\frac{\partial^2\omega}{\partial\eta^2} + \sum_{k=1}^{n-1}\sum_{l=k+1}^{n}B_{k,l}\frac{\partial^2\omega}{\partial\sigma_k\partial\sigma_l} + \\ + \sum_{k=1}^{n}B_{k,n+1}\frac{\partial^2\omega}{\partial\sigma_k\partial\eta} + \sum_{k=1}^{n}D_k\frac{\partial\omega}{\partial\sigma_k} + D_{n+1}\frac{\partial\omega}{\partial\eta} + E\omega = 0,\end{aligned} \tag{26}$$

where

$$A_k = P\sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(\frac{\partial\sigma_k}{\partial x_j}\right)^2, \;\; A_{n+1} = P\sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(\frac{\partial\eta}{\partial x_j}\right)^2,$$

$$B_{k,l} = 2P\sum_{j=1}^{n}\mathrm{sgn}(p-j)\frac{\partial\sigma_k}{\partial x_j}\frac{\partial\sigma_l}{\partial x_j}, \;\; B_{k,n+1} = 2P\sum_{j=1}^{n}\mathrm{sgn}(p-j)\frac{\partial\sigma_k}{\partial x_j}\frac{\partial\eta}{\partial x_j},$$

$$D_k = \sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(P\frac{\partial^2\sigma_k}{\partial x_j^2} + 2\frac{\partial P}{\partial x_j}\frac{\partial\sigma_k}{\partial x_i} + 2P\frac{\alpha_j}{x_j}\frac{\partial\sigma_k}{\partial x_j}\right),$$

$$D_{n+1} = \sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(P\frac{\partial^2\eta}{\partial x_j^2} + 2\frac{\partial P}{\partial x_j}\frac{\partial\eta}{\partial x_j} + 2P\frac{\alpha_j}{x_j}\frac{\partial\eta}{\partial x_j}\right),$$

$$E = \sum_{j=1}^{n}\mathrm{sgn}(p-j)\left(\frac{\partial^2 P}{\partial x_j^2} + \frac{2\alpha_j}{x_j}\frac{\partial P}{\partial x_j}\right) + \lambda P.$$

Let us calculate the derivatives appearing in these coefficients:

$$\frac{\partial\sigma_k}{\partial x_k} = -\mathrm{sgn}(p-k)\left(\frac{4\xi_k}{r^2} + \frac{2\,(x_k-\xi_k)}{r^2}\sigma_k\right), \quad k=\overline{1,n}; \tag{27}$$

$$\frac{\partial\sigma_k}{\partial x_j} = -\mathrm{sgn}(p-j)\frac{2\,(x_j-\xi_j)}{r^2}\sigma_k, \quad j\neq k, \quad j,k=\overline{1,n}; \tag{28}$$

$$\frac{\partial^2\sigma_k}{\partial x_k^2} = \mathrm{sgn}(p-k)\left(\frac{4\xi_k}{x_k r^2}\sigma_k - \frac{6}{r^2}\sigma_k\right) + \frac{8\left(x_k-\xi_k\right)^2}{r^4}\sigma_k, \quad k=\overline{1,n}; \tag{29}$$

$$\frac{\partial^2\sigma_k}{\partial x_j^2} = -\mathrm{sgn}(p-j)\frac{2}{r^2}\sigma_k + \frac{8\left(x_j-\xi_j\right)^2}{r^4}\sigma_k, \quad j\neq k, \quad j,\, k=\overline{1,n}; \tag{30}$$

$$\frac{\partial\eta}{\partial x_j} = \frac{\lambda}{2}\mathrm{sgn}(p-j)\left(x_j-\xi_j\right), \ \frac{\partial^2\eta}{\partial x_j^2} = \frac{\lambda}{2}\mathrm{sgn}(p-j), \quad j=\overline{1,n}; \tag{31}$$

$$\frac{\partial P}{\partial x_j} = -2\beta r^{-2\beta-2}\mathrm{sgn}(p-j)\left(x_j-\xi_j\right), \quad j=\overline{1,n}; \tag{32}$$

$$\frac{\partial^2 P}{\partial x_j^2} = 4\beta r^{-2\beta-2}\left[(1+\beta)\frac{\left(x_j-\xi_j\right)^2}{r^2} - \frac{1}{2}\mathrm{sgn}(p-j)\right], \quad j=\overline{1,n}. \tag{33}$$

Taking into account (27)–(33), the coefficients of equation (26) take the form

$$A_k = -\frac{4P(r)}{r^2}\frac{\xi_k}{x_k}\sigma_k\left(1-\sigma_k\right), \ B_{k,n+1} = \frac{4P(r)}{r^2}\frac{\xi_k}{x_k}\sigma_k\eta + \frac{\lambda^2}{2}P(r)\sigma_k, \ k=\overline{1,n}; \tag{34}$$

$$B_{kl} = \frac{4P(r)}{r^2}\left(\frac{\xi_k}{x_k}+\frac{\xi_l}{x_l}\right)\sigma_k\sigma_l, \ k<l, \ \ k,l=\overline{1,n}; \ \ A_{n+1} = -\lambda P(r)\eta, \tag{35}$$

$$D_k = -\frac{4P(r)}{r^2}\left\{\left(2\alpha_k-\beta\sigma_k\right)\frac{\xi_k}{x_k} - \sigma_k\sum_{i=1}^{n}\frac{\xi_i}{x_i}\alpha_i\right\}, \ k=\overline{1,n}; \tag{36}$$

$$D_{n+1} = \frac{4P(r)}{r^2}\eta\sum_{i=1}^{n}\frac{\xi_i}{x_i}\alpha_i - \lambda P(r)\beta, \ E = \frac{4\beta P(r)}{r^2}\sum_{i=1}^{n}\frac{\xi_i}{x_i}\alpha_i + \lambda P(r). \tag{37}$$

Substituting coefficients (34)–(37) into equation (26) and grouping similar terms, we obtain

$$\begin{cases} \sigma_i\left(1-\sigma_i\right)\dfrac{\partial^2\omega}{\partial\sigma_i^2} - \sigma_i\displaystyle\sum_{j=1,j\neq i}^{n}\sigma_j\frac{\partial^2\omega}{\partial\sigma_i\partial\sigma_j} + \sigma_i\eta\frac{\partial^2\omega}{\partial\sigma_i\partial\eta} + \left[2\alpha_i-\left(\beta+\alpha_i+1\right)\sigma_i\right]\frac{\partial\omega}{\partial\sigma_i} \\ \qquad -\alpha_i\displaystyle\sum_{j=1,j\neq i}^{n}\sigma_j\frac{\partial\omega}{\partial\sigma_j} + \alpha_i\eta\frac{\partial\omega}{\partial\eta} - \beta\alpha_i\omega = 0, \ i=\overline{1,n}, \\ \eta\dfrac{\partial^2\omega}{\partial\eta^2} - \displaystyle\sum_{j=1}^{n}\sigma_j\frac{\partial^2\omega}{\partial\sigma_j\partial\eta} + (1-\beta)\frac{\partial\omega}{\partial\eta} + \omega = 0. \end{cases} \tag{38}$$

Thus, the multidimensional ultrahyperbolic equation (22) equivalently reduced to system (38).

Comparing system (38) with system (19) and, by virtue of (23), (20) and (21), we obtain particular solutions of equation (22):

$$q_{p0}\left(x;\xi\right) = C_{p0}\, r_p^{-2\beta}\mathrm{H}_A^{(n,1)}\left[\begin{array}{c} \beta,\alpha_1,\ldots,\alpha_n; \\ 2\alpha_1,\ldots,2\alpha_n; \end{array} \sigma_p,\eta_p\right], \tag{39}$$

$$q_{pj}\left(x;\xi\right) = C_{pj}\, r_p^{-2\beta-2j+4\alpha_1+\ldots+4\alpha_j}\prod_{k=1}^{j}\left(x_k\xi_k\right)^{1-2\alpha_k} \times$$

$$\times\, \mathrm{H}_A^{(n,1)}\left[\begin{array}{c} \beta+j-2\alpha_1-\ldots-2\alpha_j, 1-\alpha_1,\ldots,1-\alpha_j,\alpha_{j+1},\ldots,\alpha_n; \\ 2-2\alpha_1, \ \ldots, \ 2-2\alpha_j, \ \ 2\alpha_{j+1},\ldots,2\alpha_n; \end{array} \sigma_p,\eta_p\right], \tag{40}$$

where $C_{p0},\ldots,C_{pn}$ are arbitrary constants; β, σ_p and η_p are defined in (25); $j=\overline{1,n}$, $p=\overline{1,n-1}$.

4 Some properties of particular solutions

It can be shown directly that the particular solutions $q_{pi}\,(x;\xi)$ defined in (39) and (40) satisfy equation (22) with respect to the variables x, but these functions with respect to the same variables do not satisfy the adjoint equation

$$L^{*}(u) \equiv \sum_{j=1}^{n} \operatorname{sgn}(p-j)\left(\frac{\partial^2 u}{\partial x_j^2} - \frac{\partial}{\partial x_j}\left(\frac{2\alpha_j u}{x_j}\right)\right) + \lambda u = 0, \quad x \in \Omega. \tag{41}$$

Let's introduce some notations for brevity

$$x^{(2\alpha)} := \prod_{i=1}^{n} x_i^{2\alpha_i}, \;\; \tilde{x}_j^{(2\alpha)} := \prod_{i=1,\, i\neq j}^{n} x_i^{2\alpha_i}, \;\; j = \overline{1,n}.$$

Lemma 1. If $q_{pk}\,(x;\xi)$ are particular solutions to equation (22) with respect to the variables x, then the following functions

$$\tilde{q}_{pk}\,(x;\xi) = x^{(2\alpha)} q_{pk}\,(x;\xi) \tag{42}$$

are satisfied equation (22) with respect to the variables ξ and adjoint equation (41) with respect to the variables x, where $k = \overline{0,n}, \; p = \overline{1,n-1}$.

Proof. From the definition of variables ξ_p and η_p (see eq. (25)) it follows that each particular solution $q_{pk}\,(x;\xi)$ defined in (39) and (40) is symmetric with respect to the variables x and ξ. Therefore, the arbitrary solution of equation (22) with respect to the variables x is simultaneously the solution of the same equation with respect to the variables ξ and vice versa.

Now, assuming that the function $q_{pk}\,(x;\xi)$ satisfies equation $L\,(q_{pk}) = 0$, we substitute the function $\tilde{q}_{pk}\,(x;\xi)$ defined in (42) into the adjoint equation $L^{*}\,(\tilde{q}_{pk}) = 0$. First, we calculate the necessary partial derivatives

$$\frac{\partial \tilde{q}_{pk}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(x^{(2\alpha)} q_{pk}\right) = 2\alpha_j \tilde{x}_j^{(2\alpha)} x_j^{2\alpha_j - 1} q_{pk} + x^{(2\alpha)} \frac{\partial q_{pk}}{\partial x_j},$$

$$\frac{\partial^2 \tilde{q}_{pk}}{\partial x_j^2} = 2\alpha_j\,(2\alpha_j - 1)\,\tilde{x}_j^{(2\alpha)} x_j^{2\alpha_j-2} q_{pk} + 4\alpha_j \tilde{x}_j^{(2\alpha)} x_j^{2\alpha_j-1} \frac{\partial q_{pk}}{\partial x_j} + x^{(2\alpha)} \frac{\partial^2 q_{pk}}{\partial x_j^2},$$

$$\frac{\partial}{\partial x_j}\left(\frac{2\alpha_j \tilde{q}_{pk}}{x_j}\right) = 2\alpha_j\,(2\alpha_j - 1)\,\tilde{x}_j^{(2\alpha)} x_j^{2\alpha_j-2} q_{pk} + 2\alpha_j \tilde{x}_j^{(2\alpha)} x_j^{2\alpha_j-1} \frac{\partial q_{pk}}{\partial x_j}$$

and substitute them into adjoint equation (41):

$$L^{*}\,(\tilde{q}_{pk}) = x^{(2\alpha)} L\,(q_{pk}) = 0.$$

The last double relation completes the proof of Lemma 1.

Therefore, the following functions

$$q_{p0}\,(x;\xi) = C_{p0}\, r_p^{-2\beta} \prod_{j=1}^{n} x_j^{2\alpha_j} \cdot \mathrm{H}_A^{(n,1)} \left[\begin{array}{l} \beta, \alpha_1, \ldots, \alpha_n; \\ 2\alpha_1, \ldots, 2\alpha_n; \end{array} \xi_p; \eta_p\right], \tag{43}$$

$$q_{pk}\,(x;\xi) = C_{pk}\, r_p^{-2\beta - 2k + 4\alpha_1 + \ldots + 4\alpha_k} \prod_{j=1}^{n} x_j^{2\alpha_j} \cdot \prod_{j=1}^{k} (x_j \xi_j)^{1-2\alpha_j} \times$$

$$\times\, \mathrm{H}_A^{(n,1)} \left[\begin{array}{l} \beta + k - 2\alpha_1 - \ldots - 2\alpha_k, 1 - \alpha_1, \ldots, 1 - \alpha_k, \alpha_{k+1}, \ldots, \alpha_n; \\ 2 - 2\alpha_1, \;\; \ldots, \;\; 2 - 2\alpha_k, \;\; 2\alpha_{k+1}, \ldots, 2\alpha_n; \end{array} \xi_p; \eta_p\right], \tag{44}$$

are also partial solutions to equation (22).

Theorem 3. If $0 < 2\alpha_j < 1,$ then the particular solutions $q_{pk}\left(x;\xi\right)$ defined in (43) and (44) have a singularity of the order $\dfrac{1}{r_p^{n-2}}$ at $r_p \to 0$, where $k = \overline{0,n}, \quad j = \overline{1,n}, \quad p = \overline{1,n-1}.$

Proof. We consider the first particular solution $q_{p0}\left(x;\xi\right)$, defined in (43), the singularity of the remaining solutions is proved in a similar way.

By virtue of an equality $2\beta = n - 2 + 2\alpha$, where $\alpha := \alpha_1 + \ldots + \alpha_n$ (see eq. (24)), we can rewrite the particular solution $q_{p0}\left(x;\xi\right)$ in the form

$$q_0\left(x;\xi\right) = \frac{1}{r_p^{n-2}}\tilde{q}_0\left(x,\xi\right),$$

where

$$\tilde{q}_{p0} = C_{p0}\frac{x^{(2\alpha)}}{r_p^{2\alpha}}\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}\beta,\alpha_1,\ldots,\alpha_n;\\ 2\alpha_1,\ldots,2\alpha_n;\end{array} -\frac{4x_1\xi_1}{r_p^2},\ldots,-\frac{4x_p\xi_p}{r_p^2},\frac{4x_{p+1}\xi_{p+1}}{r_p^2},\ldots,\frac{4x_n\xi_n}{r_p^2},\frac{1}{4}\lambda r_p^2\right]. \quad (45)$$

Now we show that $\tilde{q}_{p0}\left(x,\xi\right)$ is bounded at $r_p \to 0$. On the right side (45) we make a replacement $x_j - \xi_j = \varepsilon t_j \; (j = \overline{1,n})$, where $t := (t_1,\ldots,t_n)$ are new variables and $\varepsilon \geq 0$, then

$$\tilde{q}_{p0}\left(x;\xi-\varepsilon t\right) = C_{p0}\frac{x^{(2\alpha)}\varepsilon^{-2\alpha}}{T_p^{2\alpha}}\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}\beta,\alpha_1,\ldots,\alpha_n;\\ 2\alpha_1,\ldots,2\alpha_n;\end{array} 1-\frac{z_1(\varepsilon)}{\varepsilon^2},\ldots,1-\frac{z_n(\varepsilon)}{\varepsilon^2},\frac{1}{4}\lambda\varepsilon^2T_p^2\right],$$

where

$$z_j(\varepsilon) = \frac{T_p^2\varepsilon^2 + \mathrm{sgn}(p-j)\cdot 4x_j\left(x_j-\varepsilon t_j\right)}{T_p^2}, \quad T_p^2 = \sum_{j=1}^{p}\mathrm{sgn}(p-j)t_j^2, \quad j = \overline{1,n}.$$

Using limit correlation (15), we have

$$\lim_{\varepsilon\to 0}\tilde{q}_{p0}\left(x,\xi-\varepsilon t\right) = C_{p0}\frac{\Gamma(\beta-\alpha)}{4^{2\alpha}\Gamma(\beta)}\prod_{j=1}^{n}\frac{\Gamma\left(2\alpha_j\right)}{\Gamma\left(\alpha_j\right)} < \infty.$$

Thus the function $\tilde{q}_{p0}\left(x;\xi\right)$ is bounded, hence the function $q_{p0}\left(x;\xi\right)$ has the singularity of the order $\dfrac{1}{r_p^{n-2}}$ at $r_p \to 0$.

Conclusion

In conclusion, we note that particular solutions satisfying the singular elliptic and ultrahyperbolic equations (2) and (3) (respectively, equations (4) and (22)) are always expressed in terms of the Lauricella function $F_A^{(n)}$ (respectively, the confluent hypergeometric function $\mathrm{H}_A^{(n,1)}$), the variables of which differ from each other only in signs, depending on the equation under consideration.

During the study, it became clear that solutions to second-order equations are expressed in terms of second-order hypergeometric functions, i.e. the order of the equation under consideration is equal to the order of the hypergeometric function through which particular (fundamental) solutions are expressed. This circumstance must be taken into account when constructing partial solutions of singular equations when their order exceeds two. For example, knowing that in [3] all 8 self-similar solutions to the equation

$$Lu = x^n y^m u_t - t^k y^m u_{xxx} - t^k x^n u_{yyy} = 0, \;\; m,n,k = const > 0$$

in the domain $D_1 = \{(x, y, t) : \ x > 0, \ y > 0, \ \ t > 0\}$ are written by third-order hypergeometric Kampé de Fériet function in two variables, we can guess that particular solutions of the equation

$$\prod_{j=1}^{n} x_j^{m_j} \cdot \frac{\partial u}{\partial t} - t^l \sum_{k=1}^{n} \left(\prod_{j=1, j\neq k}^{n} x_j^{m_j} \right) \frac{\partial^p u}{\partial x_k^p} = 0, \ \ l > 0, \ \ m_j > 0, \ j = \overline{1, n}$$

in the domain $D_2 = \{(\mathbf{x}, t) : x_1 > 0, \ldots, x_n > 0, \ t > 0\}$ are expressed through some confluent hypergeometric function of n variables with the order p.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Homogenization of Attractors to Reaction–Diffusion Equations in Domains with Rapidly Oscillating Boundary: Subcritical Case

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We consider the reaction–diffusion system of equations with rapidly oscillating terms in the equation and in boundary conditions in a domain with locally periodic oscillating boundary. In the subcritical case (the Fourier boundary condition is changed to the Neumann boundary condition in the limit) we proved that the trajectory attractors of this system converge in a weak sense to the trajectory attractors of the limit (homogenized) reaction–diffusion systems in domain independent of the small parameter, characterizing the oscillation rate. To obtain the results we use the approach of homogenization theory, asymptotic analysis and methods of the theory concerning trajectory attractors of evolution equations. Defining the appropriate functional and topological spaces with weak topology, we prove the existence of trajectory attractors and global attractors for these systems. Then we formulate the main Theorem and prove it with the help of auxiliary Lemmata. Applying the homogenization method and asymptotic analysis we derive the homogenized (limit) system of equations, prove the existence of trajectory attractors and global attractors and show the convergence of trajectory and global attractors.

Keywords: attractors, homogenization, reaction–diffusion equations, nonlinear equations, weak convergence, rapidly oscillating boundary.

2020 Mathematics Subject Classification: 35B27, 35B40, 35B41, 35D30.

Introduction

Recently, the attention of scientists has been attracted by various problems for evolution equations with dissipation, in which small parameters are present. To study such problems it is important to use asymptotic methods and homogenization theory. In the present paper we study homogenization problem for reaction–diffusion system of equations in domains with very rapidly oscillating boundary (see for detailed geometric settings [1]). We derive the homogenized (limit) system of equations in domain without oscillation of the boundary, then we prove the existence of trajectory and global attractors for the given and homogenized systems and also prove the convergence of attractors of the given system to the attractors of the homogenized system as the small parameter characterizing the oscillations, tends to zero, i.e. we prove the Hausdorff convergence of attractors as the small parameter

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tends to zero. In many pure mathematical papers, one can find the asymptotic analysis of problems in domains with rapidly oscillating boundaries (see, for example, [1–12]). We want to mention here the basic frameworks [13, 14], where one can find these approaches and methods as well as the detail bibliography.

The basic theory of attractors one can find, for instance, [15–17] and see also the references in these monographs (see also [18]). Homogenization of attractors were studied in [17,19–21] (see also [22–30]).

In this present paper we give the proofs of weak convergence of the trajectory attractor $\mathfrak{A}_\varepsilon$ of the reaction–diffusion systems in a domain with oscillating boundary, as $\varepsilon \to 0$, to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized systems in some natural functional space. We also proved the convergence of the global attractor $\mathcal{A}_\varepsilon$ to $\overline{\mathcal{A}}$ as $\varepsilon \to 0$. Here, the small parameter ε characterizes the period and the amplitude of the oscillations. The parameter ε is included also in Fourier condition on a part of the boundary, and we consider the case in which the Fourier condition transforms to the Neumann one (subcritical case) as the small parameter tends to zero.

The first section is devoted to basic settings, in the second section we describe the limiting (homogenized) reaction–diffusion system and its trajectory attractor. The third section contains auxiliary results and in the fourth section the proof of the main Theorem is presented.

1 Statement of the problem

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Ω lies in a half-space $x_d > 0$ and $\Gamma_1 \subset \{x : x_d = 0\}$. Given smooth nonpositive 1-periodic in the $\hat{\xi}$ function $F(\hat{x}, \hat{\xi})$, $\hat{x} = (x_1, ..., x_{d-1}), \hat{\xi} = (\xi_1, ..., \xi_{d-1})$, define the domain Ω_ε as follows: $\partial\Omega_\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2$, where we set $\Gamma_1^\varepsilon = \{x = (\hat{x}, x_d) : (\hat{x}, 0) \in \Gamma_1, x_d = \varepsilon^\alpha F(\hat{x}, \hat{x}/\varepsilon)\}$, $\alpha < 1$, i.e. we add thin oscillating layer $\Pi_\varepsilon = \{x = (\hat{x}, x_d) : (\hat{x}, 0) \in \Gamma_1, x_d \in [0, \varepsilon^\alpha F(\hat{x}, \hat{x}/\varepsilon))\}$ to the domain Ω. Usually, we assume $F(\hat{x}, \hat{\xi})$ to be compactly supported on Γ_1 uniformly in $\hat{\xi}$. Consider the following boundary-value problem:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \lambda \Delta u_\varepsilon - a\left(x, \frac{x}{\varepsilon}\right) f(u_\varepsilon) + h\left(x, \frac{x}{\varepsilon}\right), & x \in \Omega_\varepsilon, t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} + \varepsilon^\beta p(\hat{x}, \frac{\hat{x}}{\varepsilon}) u_\varepsilon = \varepsilon^{1-\alpha} g(\hat{x}, \frac{\hat{x}}{\varepsilon}), & x = (\hat{x}, x_d) \in \Gamma_1^\varepsilon, t > 0, \\ u_\varepsilon = 0, & x \in \Gamma_2, t > 0, \\ u_\varepsilon = U(x), & x \in \Omega_\varepsilon, t = 0, \end{cases} \tag{1}$$

where $u_\varepsilon = u_\varepsilon(x,t) = (u^1, \ldots, u^n)^\top$ is an unknown vector function, the nonlinear function $f = (f^1, \ldots, f^n)^\top$ is given, $h = (h^1, \ldots, h^n)^\top$ is the known right-hand side function, and λ is an $n \times n$-matrix with constant coefficients, having a positive symmetrical part: $\frac{1}{2}(\lambda + \lambda^\top) \geq \varpi I$ (where I is the unit matrix with dimension n). We assume that $p\left(\hat{x}, \hat{\xi}\right) = \operatorname{diag}\{p^1, \ldots, p^n\}$, $g\left(\hat{x}, \hat{\xi}\right) = (g^1, \ldots, g^n)^\top$ are continuous, 1-periodic in $\hat{\xi}$ and $p^i\left(\hat{x}, \hat{\xi}\right)$, $i = 1, \ldots n$, are positive. Here $\frac{\partial}{\partial \nu}$ is the normal derivative of the function multiplied by a matrix, i.e. $\frac{\partial}{\partial \nu} := \sum\limits_{j=1}^{n} \sum\limits_{k=1}^{d} \lambda_{ij} \frac{\partial}{\partial x_k} N_k$, $i = 1, \ldots, n$, and $N = (N_1, \ldots, N_d)$ is the unit outer normal to the boundary of the domain Ω_ε. Let us denote by $p_{\max}$ the maximum of p on Γ_1.

In this paper we investigate evolution equations and their trajectory attractors depending on a small parameter $\varepsilon > 0$ (see for details [26]). We consider the subcritical case, i.e. $\beta > 1 - \alpha$.

Function $a(x, \xi) \in C(\overline{\Omega}_\varepsilon \times \mathbb{R}^d)$ such that $0 < a_0 \leq a(x, \xi) \leq A_0$ with some coefficient a_0, A_0. Assuming that function $a_\varepsilon(x) = a\left(x, \frac{x}{\varepsilon}\right)$ has average $\overline{a}(x)$ when $\varepsilon \to 0+$ in space $L_{\infty,*w}(\Omega)$, that is

$$\int\limits_\Omega a\left(x, \frac{x}{\varepsilon}\right) \varphi(x) dx \to \int\limits_\Omega \overline{a}(x) \varphi(x) dx \quad (\varepsilon \to 0+) \tag{2}$$

for any function $\varphi \in L_1(\Omega)$.

Denote by V (respectively V_ε) the Sobolev space $H^1(\Omega,\Gamma_2)$ (respectively $H^1(\Omega_\varepsilon,\Gamma_2)$), i.e. the space of functions from the Sobolev space $H^1(\Omega)$ (respectively $H^1(\Omega_\varepsilon)$) with zero trace on Γ_2. We also denote by V' (respectively V'_ε) the dual space for V (respectively V_ε), i.e. the space of linear bounded functionals on V (respectively V_ε). Denote by Ω^+ such a domain that $\Omega_\varepsilon \subset \Omega^+$ for any ε. For the vector function $h\,(x,\xi)$, assume that for any $\varepsilon > 0$ the function $h^i_\varepsilon(x) = h^i\left(x, \frac{x}{\varepsilon}\right) \in L_2(\Omega^+)$ and has the average $\overline{h^i}(x)$ in the space $L_2(\Omega^+)$ for $\varepsilon \to 0+$, that is

$$h^i\left(x, \frac{x}{\varepsilon}\right) \rightharpoonup \overline{h^i}(x) \quad (\varepsilon \to 0+) \text{ weakly in } L_2(\Omega^+),$$

or

$$\int\limits_{\Omega^+} h^i\left(x, \frac{x}{\varepsilon}\right)\varphi(x)dx \to \int\limits_{\Omega^+} \overline{h^i}(x)\varphi(x)dx \quad (\varepsilon \to 0+) \tag{3}$$

for any function $\varphi \in L_2(\Omega^+)$ and for all $i = 1,\dots,n$.

From the condition (3), it follows that the norm of the function $h^i_\varepsilon(x)$ is bounded uniformly in ε, in the space $L_2(\Omega_\varepsilon)$, i.e.

$$\|h^i_\varepsilon(x)\|_{L_2(\Omega_\varepsilon)} \le M_0, \quad \forall\varepsilon \in (0,1]. \tag{4}$$

It is assumed that the vector function $f(v) \in C(\mathbb{R}^n;\mathbb{R}^n)$ satisfies the following inequalities

$$\sum_{i=1}^n |f^i(v)|^{p_i/(p_i-1)} \le C_0\left(\sum_{i=1}^n |v^i|^{p_i} + 1\right), \quad 2 \le p_1 \le \ldots \le p_{n-1} \le p_n, \tag{5}$$

$$\sum_{i=1}^n \gamma_i |v^i|^{p_i} - C \le \sum_{i=1}^n f^i(v)v^i, \quad \forall v \in \mathbb{R}^n, \tag{6}$$

for $\gamma_i > 0$ for any $i = 1,\dots,n$. The inequality (5) is due to the fact that in real reaction-diffusion systems, the functions $f^i(u)$ are polynomials with possibly different degrees. Inequality (6) calls *dissipativity condition* for the reaction-diffusion system (1). In a simple model case $p_i \equiv p$ for any $i = 1,\dots,n$, condition (5) and (6) reduce to the following inequalities

$$|f(v)| \le C_0\left(|v|^{p-1} + 1\right), \quad \gamma|v|^p - C \le f(v)v, \quad \forall v \in \mathbb{R}^n.$$

Note that the fulfillment of the Lipschitz condition for the function $f(v)$ with respect to the variable v is *not expected*.

Remark 1. Using the methods presented, it is also possible to study systems in which nonlinear terms have the form $\sum\limits_{j=1}^m a_j\left(x, \frac{x}{\varepsilon}\right) f_j(u)$, where a_j are matrices whose elements allow averaging and $f_j(u)$ polynomial vectors of u, which satisfy conditions of the form (5)–(6). For brevity, we study the case $m = 1$ and $a_1\left(x, \frac{x}{\varepsilon}\right) = a\left(x, \frac{x}{\varepsilon}\right) I$, where I is the identity matrix.

Denote

$$G(\hat{x}) = \int\limits_{[0,1)^{d-1}} \sqrt{|\nabla_{\hat{\xi}} F(\hat{x},\hat{\xi})|^2}\, g(\hat{x},\hat{\xi})\, d\hat{\xi}, \tag{7}$$

and we have the following convergence (see [1]):

$$\varepsilon^{1-\alpha}\int\limits_{\Gamma_1^\varepsilon} g^i\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) \upsilon\left(\hat{x}, \varepsilon^\alpha F\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right)\right) ds \to \int\limits_{\Gamma_1} G^i(\hat{x})\, \upsilon(x)\, ds$$

for any $\upsilon \in H^1(\Omega_\varepsilon)$ by $\varepsilon \to 0$. Here ds is the element of $(d-1)$-dimensional measure on the hypersurface.

Let us introduce the following notation for the spaces $\mathbf{H} := [L_2(\Omega)]^n$, $\mathbf{H}_\varepsilon := [L_2(\Omega_\varepsilon)]^n$, $\mathbf{V} := [H^1(\Omega, \Gamma_2)]^n$, $\mathbf{V}_\varepsilon := [H^1(\Omega_\varepsilon; \Gamma_2)]^n$. The norms in these spaces are determined as follows

$$\|v\|^2 := \int\limits_\Omega \sum_{i=1}^n |v^i(x)|^2 dx, \ \|v\|_\varepsilon^2 := \int\limits_{\Omega_\varepsilon} \sum_{i=1}^n |v^i(x)|^2 dx,$$

$$\|v\|_1^2 := \int\limits_\Omega \sum_{i=1}^n |\nabla v^i(x)|^2 dx, \ \|v\|_{1,\varepsilon}^2 := \int\limits_{\Omega_\varepsilon} \sum_{i=1}^n |\nabla v^i(x)|^2 dx.$$

We denote by $\mathbf{V}'$ the dual space to the space $\mathbf{V}$, and by $\mathbf{V}'_\varepsilon$ the dual space to the space $\mathbf{V}_\varepsilon$.

Let $q_i = p_i/(p_i - 1)$ for any $i = 1, \ldots, n$. We will use the following vector notation $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$, and also define spaces

$$\mathbf{L_p} := L_{p_1}(\Omega) \times \ldots \times L_{p_n}(\Omega), \ \mathbf{L}_{\mathbf{p},\varepsilon} := L_{p_1}(\Omega_\varepsilon) \times \ldots \times L_{p_n}(\Omega_\varepsilon),$$

$$\mathbf{L_p}(\mathbb{R}_+; \mathbf{L_p}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(\Omega)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(\Omega)),$$

$$\mathbf{L_p}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(\Omega_\varepsilon)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(\Omega_\varepsilon)).$$

As in [17, 31] we study weak solutions of the initial boundary value problem (1), that is, functions

$$u_\varepsilon(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}_\varepsilon) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon})$$

which satisfy the equation (1) in the distributional sense (the sense of generalized functions), that is, the integral identity holds

$$-\int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} u_\varepsilon \cdot \frac{\partial \psi}{\partial t} \, dxdt + \int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} \lambda \nabla u_\varepsilon \cdot \nabla \psi \, dxdt + \int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} a_\varepsilon(x) f(u_\varepsilon) \cdot \psi \, dxdt +$$

$$+\varepsilon^\beta \int\limits_{\Gamma_1^\varepsilon \times \mathbb{R}_+} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) u_\varepsilon \cdot \psi \, dsdt = \int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} h_\varepsilon(x) \cdot \psi \, dxdt + \varepsilon^{1-\alpha} \int\limits_{\Gamma_1^\varepsilon \times \mathbb{R}_+} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) \cdot \psi \, dsdt$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{V}_\varepsilon \cap \mathbf{L}_{\mathbf{p},\varepsilon})$. Here $y_1 \cdot y_2$ means scalar product of vectors $y_1, y_2 \in \mathbb{R}^n$.

If $u_\varepsilon(x,t) \in \mathbf{L_p}(0, M; \mathbf{L}_{\mathbf{p},\varepsilon})$, then from the condition (5) it follows that $f(u(x,t)) \in \mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon})$. At the same time, if $u_\varepsilon(x,t) \in \mathbf{L}_2(0, M; \mathbf{V}_\varepsilon)$, then $\lambda \Delta u_\varepsilon(x,t) + h_\varepsilon(x) \in \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon)$. Therefore, for an arbitrary weak solution $u_\varepsilon(x,s)$ to problem (1), satisfies

$$\frac{\partial u_\varepsilon(x,t)}{\partial t} \in \mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon).$$

From the Sobolev embedding theorem it follows that

$$\mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon) \subset \mathbf{L_q}\left(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}}\right),$$

where space $\mathbf{H}_\varepsilon^{-\mathbf{r}} := H^{-r_1}(\Omega_\varepsilon) \times \ldots \times H^{-r_n}(\Omega_\varepsilon)$, $\mathbf{r} = (r_1, \ldots, r_n)$ and indexes $r_i = \max\{1, d(1/q_i - 1/2)\}$ by $i = 1, \ldots, n$. Here $H^{-r}(\Omega_\varepsilon)$ denotes the space conjugate to the Sobolev space $\overset{\circ}{W}{}_2^r(\Omega_\varepsilon)$ with index $r > 0$ in the domain Ω_ε.

Therefore, for any weak solution $u_\varepsilon(x,t)$ to problem (1), its time derivative $\frac{\partial u_\varepsilon(x,t)}{\partial t}$ belongs to $\mathbf{L_q}(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}})$.

Remark 2. Existence of a weak solution $u(x,t)$ to problem (1) for any initial data $U \in \mathbf{H}_\varepsilon$ and fixed ε, can be proved in the standard way (see, for example, [16, 31]). This solution may not be unique, since the function $f(v)$ satisfies only the conditions (5), (6) and it is not assumed that the Lipschitz condition is satisfied with respect to v.

The following Lemma is proved in a similar way to the proposition XV. 3.1 from [17].

Lemma 1. Let $u_\varepsilon(x,t) \in \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon})$ be the weak solution of problem (1). Then

(i) $u_\varepsilon \in \mathbf{C}(\mathbb{R}_+;\mathbf{H}_\varepsilon)$;

(ii) function $\|u_\varepsilon(\cdot,t)\|^2$ is absolutely continuous on $\mathbb{R}_+$, and moreover

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(\cdot,t)\|^2 + \int\limits_{\Omega_\varepsilon} \lambda \nabla u_\varepsilon(x,t)\cdot\nabla u_\varepsilon(x,t)dx + \int\limits_{\Omega_\varepsilon} a_\varepsilon(x) f(u_\varepsilon(x,t))\cdot u_\varepsilon(x,t)dx + \tag{8}$$

$$+\varepsilon^\beta \int\limits_{\Gamma_1^\varepsilon} p\left(\hat{x},\frac{\hat{x}}{\varepsilon}\right) u_\varepsilon(x,t)\cdot u_\varepsilon(x,t)\,ds = \int\limits_{\Omega_\varepsilon} h_\varepsilon(x)\cdot u_\varepsilon(x,t)dx + \varepsilon^{1-\alpha}\int\limits_{\Gamma_1^\varepsilon} g\left(\hat{x},\frac{\hat{x}}{\varepsilon}\right)\cdot u_\varepsilon(x,t)\,ds,$$

for almost all $t \in \mathbb{R}_+$.

To define the trajectory space $\mathcal{K}_\varepsilon^+$ for (1), we use the general approaches of Section 2 from [26] and for every $[t_1,t_2] \in \mathbb{R}$, we have the Banach spaces

$$\mathcal{F}_{t_1,t_2} := \mathbf{L}_\mathbf{p}(t_1,t_2;\mathbf{L}_\mathbf{p}) \cap \mathbf{L}_2(t_1,t_2;\mathbf{V}) \cap \mathbf{L}_\infty(t_1,t_2;\mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}\left(t_1,t_2;\mathbf{H}^{-r}\right) \right\}$$

(sometimes we omit the parameter ε for brevity) with the following norm:

$$\|v\|_{\mathcal{F}_{t_1,t_2}} := \|v\|_{\mathbf{L}_\mathbf{p}(t_1,t_2;\mathbf{L}_\mathbf{p})} + \|v\|_{\mathbf{L}_2(t_1,t_2;\mathbf{V})} + \|v\|_{\mathbf{L}_\infty(0,M;\mathbf{H})} + \left\|\frac{\partial v}{\partial t}\right\|_{\mathbf{L}_\mathbf{q}(t_1,t_2;\mathbf{H}^{-r})}.$$

Setting $\mathcal{D}_{t_1,t_2} = \mathbf{L}_\mathbf{q}\left(t_1,t_2;\mathbf{H}^{-r}\right)$, we obtain $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$ and for $u(t) \in \mathcal{F}_{t_1,t_2}$, we have $A(u(t)) \in \mathcal{D}_{t_1,t_2}$. One considers now weak solutions to (1) as solutions of an equation in the general scheme of Section 2 from [26].

Consider the spaces

$$\mathcal{F}_+^{loc} = \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+;\mathbf{L}_\mathbf{p}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}^{loc}(\mathbb{R}_+;\mathbf{H}^{-r}) \right\},$$

$$\mathcal{F}_{\varepsilon,+}^{loc} = \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}^{loc}(\mathbb{R}_+;{\mathbf{H}_\varepsilon}^{-r}) \right\}.$$

We introduce the following notation. Let $\mathcal{K}_\varepsilon^+$ be the set of all weak solutions to (1). For any $U \in \mathbf{H}$, there exists at least one trajectory $u(\cdot) \in \mathcal{K}_\varepsilon^+$ such that $u(0) = U(x)$. Consequently, the space $\mathcal{K}_\varepsilon^+$ to (1) is not empty and is sufficiently large.

We define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ in the spaces $\mathcal{F}_{t_1,t_2}$ by means of the norms from $\mathbf{L}_2(t_1,t_2;\mathbf{H})$. We get

$$\rho_{t_1,t_2}(u,v) = \left(\int\limits_{t_1}^{t_2} \|u(t)-v(t)\|_\mathbf{H}^2 dt\right)^{1/2} \qquad \forall u(\cdot), v(\cdot) \in \mathcal{F}_{t_1,t_2}.$$

The topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$ is generated by these metrics. Let us recall that $\{v_k\} \subset \mathcal{F}_+^{loc}$ converges to $v \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\|v_k(\cdot) - v(\cdot)\|_{\mathbf{L}_2(t_1,t_2;\mathbf{H})} \to 0$ $(k \to \infty)$ for all $[t_1,t_2] \subset \mathbb{R}_+$. The topology Θ_+^{loc} is metrizable. We consider this topology in the trajectory space $\mathcal{K}_\varepsilon^+$ of (1). Similarly, we define the topology $\Theta_{\varepsilon,+}^{loc}$ in $\mathcal{F}_{\varepsilon,+}^{loc}$.

Denote by $S(\tau)$ the translation semigroup, i.e. $S(\tau)\,u(t) = u(t+\tau)$. The translation semigroup $S(\tau)$ acting on $\mathcal{K}_\varepsilon^+$, is continuous in the topology $\Theta_{\varepsilon,+}^{loc}$. It is easy to see that $\mathcal{K}_\varepsilon^+ \subset \mathcal{F}_{\varepsilon,+}^{loc}$ and the space $\mathcal{K}_\varepsilon^+$ is translation invariant, i.e. $S(\tau)\mathcal{K}_\varepsilon^+ \subseteq \mathcal{K}_\varepsilon^+$ for all $\tau \geq 0$.

Using the scheme of Section 3 from [18] and Section 2 from [26], one can define bounded sets in the space $\mathcal{K}_\varepsilon^+$ by means of the Banach space $\mathcal{F}_{\varepsilon,+}^b$. We naturally get

$$\mathcal{F}_{\varepsilon,+}^b = \mathbf{L}_\mathbf{p}^b(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}_2^b(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}^b(\mathbb{R}_+;\mathbf{H}_\varepsilon^{-r}) \right\}$$

and the space $\mathcal{F}_{\varepsilon,+}^b$ is a subspace of $\mathcal{F}_{\varepsilon,+}^{loc}$.

Definition 1. [17] A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called the *trajectory attractor* of the translation semigroup $\{S(\tau)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, if

(i) $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc};

(ii) the set $\mathfrak{A}$ is strictly invariant with respect to the semigroup: $S(\tau)\mathfrak{A} = \mathfrak{A}$ for all $\tau \geq 0$;

(iii) $\mathfrak{A}$ is an attracting set for $\{S(\tau)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, that is, for each $M > 0$, we have

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\mathfrak{A}) \to 0 \quad (\tau \to +\infty),$$

where $\mathrm{dist}_\mathcal{M}(X,Y) := \sup_{x\in X} \mathrm{dist}_\mathcal{M}(x,Y) = \sup_{x\in X}\inf_{y\in Y}\rho_\mathcal{M}(x,y)$ is the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$. We remember that the Hausdorff semidistance is not symmetric, for any $\mathcal{B} \subseteq \mathcal{K}^+$ bounded in $\mathcal{F}_+^b$ and for each $M > 0$.

Suppose that $\mathcal{K}_\varepsilon$ is the kernel to (1), that consists of all weak complete solutions $u(t), t \in \mathbb{R}$, to our system, bounded in

$$\mathcal{F}_\varepsilon^b = \mathbf{L}_\mathbf{p}^b(\mathbb{R};\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}_2^b(\mathbb{R};\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty(\mathbb{R};\mathbf{H}_\varepsilon) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}^b(\mathbb{R};\mathbf{H}_\varepsilon^{-r}) \right\}.$$

In analogous way we define the topology Θ_ε^{loc} in $\mathcal{F}_\varepsilon^b$.

Proposition 1. Problem (1) has the trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space $\Theta_{\varepsilon,+}^{loc}$. The set $\mathfrak{A}_\varepsilon$ is bounded in $\mathcal{F}_{\varepsilon,+}^b$ and compact in $\Theta_{\varepsilon,+}^{loc}$. Moreover, $\mathfrak{A}_\varepsilon = \Pi_+\mathcal{K}_\varepsilon$, the kernel $\mathcal{K}_\varepsilon$ is non-empty and bounded in $\mathcal{F}_\varepsilon^b$ and compact in Θ_ε^{loc}.

To prove this proposition we use the approach of the proof from [17]. To prove the existence of an absorbing set (bounded in $\mathcal{F}_{\varepsilon,+}^b$ and compact in $\Theta_{\varepsilon,+}^{loc}$) one can use Lemma 1 similar to [17].

It is easy to verify, that $\mathfrak{A}_\varepsilon \subset \mathcal{B}_0(R)$ for all $\varepsilon \in (0,1)$. Here $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_{\varepsilon,+}^b$ with a sufficiently large radius R. Due to the Aubin-Lions-Simon Lemma (see Lemma from [32]), we have

$$\mathcal{B}_0(R) \Subset \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon^{1-\delta}), \tag{9}$$

$$\mathcal{B}_0(R) \Subset \mathbf{C}^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon^{-\delta}), \quad 0 < \delta \leq 1. \tag{10}$$

Bearing in mind (9) and (10), the attraction to the constructed trajectory attractor can be strengthen.

Corollary 1. For any bounded in $\mathcal{F}_{\varepsilon,+}^b$ set $\mathcal{B} \subset \mathcal{K}_\varepsilon^+$, we get

$$\mathrm{dist}_{\mathbf{L}_2(0,M;\mathbf{H}_\varepsilon^{1-\delta})}(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\varepsilon) \to 0,$$
$$\mathrm{dist}_{\mathbf{C}([0,M];\mathbf{H}_\varepsilon^{-\delta})}(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\varepsilon) \to 0 \ (\tau \to \infty),$$

where M is a positive constant.

Recall that $\Omega \subset \Omega_\varepsilon$ and Ω lies in the positive half-space $\{x_d > 0\}$. Therefore, any function $u(x,t)$ with $x \in \Omega_\varepsilon$ that belongs to the space $\mathcal{F}_{\varepsilon,+}^b$ and is restricted to the domain Ω, belongs to the space $\mathcal{F}_+^b$ and, moreover,

$$\|u\|_{\mathcal{F}_+^b} \leq \|u\|_{\mathcal{F}_{\varepsilon,+}^b}.$$

Using this observation, we have

Corollary 2. The trajectory attractors $\mathfrak{A}_\varepsilon$ are uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$. The kernels $\mathcal{K}_\varepsilon$ are uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}^b$.

Definition 2. We say that the trajectory attractors $\mathfrak{A}_\varepsilon$ converge to the trajectory attractor $\overline{\mathfrak{A}}$ as $\varepsilon \to 0$ in the topological space Θ_+^{loc} if for any neighbourhood $\mathcal{O}(\overline{\mathfrak{A}})$ in Θ_+^{loc}, there is an $\varepsilon_1 \geq 0$ such that $\mathfrak{A}_\varepsilon \subseteq \mathcal{O}(\overline{\mathfrak{A}})$ for any $\varepsilon < \varepsilon_1$, that is, for each $M > 0$, we have

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0 \ (\varepsilon \to 0).$$

2 Homogenized reaction-diffusion system and its trajectory attractor (the case $\beta > 1 - \alpha$)

In the next sections, we study the behaviour of the problem (1) as $\varepsilon \to 0$ in the subcritical case $\beta > 1 - \alpha$. We have the following "formal" limit problem with inhomogeneous Fourier boundary condition

$$\begin{cases} \frac{\partial u_0}{\partial t} = \lambda \Delta u_0 - \overline{a}\,(x)\, f(u_0) + \overline{h}\,(x)\,, & x \in \Omega,\ t > 0, \\ \frac{\partial u_0}{\partial \nu} = G(\hat{x}), & x = (\hat{x}, 0) \in \Gamma_1,\ t > 0, \\ u_0 = 0, & x \in \Gamma_2,\ t > 0, \\ u_0 = U(x), & x \in \Omega,\ t = 0. \end{cases} \tag{11}$$

Here $\overline{a}(x)$ and $\overline{h}(x)$ are defined in (2) and (3), respectively, $G(\hat{x})$ was defined in (7).

As before, we consider weak solutions of the problem (11), that is, functions

$$u(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}_\mathbf{p}^{loc}\left(\mathbb{R}_+;\mathbf{L_p}\right),$$

which satisfy the following integral identity:

$$-\int\limits_{\Omega\times\mathbb{R}_+} u \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{\Omega\times\mathbb{R}_+} \lambda \nabla u \cdot \nabla \psi\, dxdt + \int\limits_{\Omega\times\mathbb{R}_+} \bar{a}(x) f(u) \cdot \psi\, dxdt = \int\limits_{\Omega\times\mathbb{R}_+} \bar{h}(x) \cdot \psi\, dxdt + \int\limits_{\Gamma_1\times\mathbb{R}_+} G\left(\hat{x}\right) \cdot \psi\, dsdt \tag{12}$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+;\mathbf{V} \cap \mathbf{L_p})$. For any weak solution $u(x,t)$ to problem (11), we have that $\frac{\partial u(x,t)}{\partial t} \in \mathbf{L_q}\left(0, M; \mathbf{H^{-r}}\right)$ (see Section 1). Recall, that the "limit" domain Ω in (11) and (12) is independent of ε and its boundary contains the plain part Γ_1.

Similar to (1), for any initial data $U \in \mathbf{H}$, the problem (11) has at least one weak solution (see Remark 2). Lemma 1 also holds true for the problem (11) with replacing the ε-depending coefficients a, h, p and g by the corresponding averaged coefficients $\overline{a}(x), \overline{h}(x), P(\hat{x})$, and $G(\hat{x})$.

As usual, let $\overline{\mathcal{K}}^+$ be the the trajectory space for (11) (the set of all weak solutions), that belong to the corresponding spaces $\mathcal{F}_+^{loc}$ and $\mathcal{F}_+^b$ (see Section 2 from [26]). Recall that $\overline{\mathcal{K}}^+ \subset \mathcal{F}_+^{loc}$ and the space $\overline{\mathcal{K}}^+$ is translation invariant with respect to translation semigroup $\{S(\tau)\}$, that is, $S(\tau)\overline{\mathcal{K}}^+ \subseteq \overline{\mathcal{K}}^+$ for all $\tau \geq 0$. We now construct the trajectory attractor in the topology Θ_+^{loc} for the problem (11) (see Section 1 and Section 2 from [26]).

Similar to Proposition 1, we have

Proposition 2. Problem (11) has the trajectory attractor $\overline{\mathfrak{A}}$ in the topological space Θ_+^{loc}. The set $\overline{\mathfrak{A}}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover,

$$\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}},$$

the kernel $\overline{\mathcal{K}}$ of the problem (11) is non-empty and bounded in $\mathcal{F}^b$.

We also have that $\overline{\mathfrak{A}} \subset \mathcal{B}_0(R)$, where $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_+^b$ with a sufficiently large radius R. Finally, the analog of Corollary 1 holds for the trajectory attractor $\overline{\mathfrak{A}}$.

Corollary 3. For any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subset \overline{\mathcal{K}}^+$, we have

$$\operatorname{dist}_{\mathbf{L}_2(0,M;\mathbf{H}^{1-\delta})}\left(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\overline{\mathcal{K}}\right) \to 0,$$
$$\operatorname{dist}_{\mathbf{C}([0,M];\mathbf{H}_\varepsilon^{-\delta})}\left(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\overline{\mathcal{K}}\right) \to 0 \ (\tau \to \infty), \ \forall M > 0.$$

3 Preliminary Lemmata (The case $\beta > 1 - \alpha$)

Next Lemmata are proved in [1].

Lemma 2. The convergence

$$v\left(\hat{x}, \varepsilon^\alpha F\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right)\right) \to v(\hat{x}, 0) \quad \text{as} \quad \varepsilon \to 0$$

strongly in $[L_2(\Gamma_1)]^n$ and the inequality

$$\|v\|_{[L_2(\Pi_\varepsilon)]^n} \leq C_1\sqrt{\varepsilon^\alpha}\|v\|_{\mathbf{V}_\varepsilon} \tag{13}$$

take place for any $v \in \mathbf{V}_\varepsilon$.

Lemma 3. Let (ds) be an element of the $(n-1)$-dimensional volume of Γ_1^ε. Then

$$ds = \left(\sqrt{1 + \varepsilon^{2-2\alpha}\left|\nabla_{\hat{\xi}}F\left(\hat{x}, \hat{\xi}\right)\right|^2}\,|_{\hat{\xi}=\frac{\hat{x}}{\varepsilon}}\right)d\hat{x}(1 + O(\varepsilon)) = \varepsilon^{\alpha-1}\left(\sqrt{\left|\nabla_{\hat{\xi}}F\left(\hat{x}, \hat{\xi}\right)\right|^2}\,|_{\hat{\xi}=\frac{\hat{x}}{\varepsilon}} + O(\varepsilon^{1-\alpha})\right)d\hat{x}.$$

Proposition 3. Uniformly in $u, v \in [H^{1/2}(\Gamma_1)]^n$

$$\left|\int_{\Omega_\varepsilon} u \cdot v\, d\hat{x}\right| \leq C_2\|u\|_{[H^{1/2}(\Gamma_1)]^n}\|v\|_{[H^{1/2}(\Gamma_1)]^n}.$$

Lemma 4. There exists such a positive constant C_3, independent of ε, that

$$\int_{\Omega_\varepsilon} |\nabla v|^2 dx + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) v \cdot v\, ds \geq C_3\|v\|_{\mathbf{V}_\varepsilon}$$

for any $v \in \mathbf{V}_\varepsilon$.

Let us consider auxiliary elliptic problems

$$\begin{cases} \lambda\Delta v_\varepsilon + h\left(x, \frac{x}{\varepsilon}\right) = 0, & x \in \Omega_\varepsilon, \\ \frac{\partial v_\varepsilon}{\partial \nu} + \varepsilon^\beta p(\hat{x}, \frac{\hat{x}}{\varepsilon})v_\varepsilon = \varepsilon^{1-\alpha}g(\hat{x}, \frac{\hat{x}}{\varepsilon}), & x = (\hat{x}, x_d) \in \Gamma_1^\varepsilon, \\ v_\varepsilon = 0, & x \in \Gamma_2, \end{cases} \tag{14}$$

and

$$\begin{cases} \lambda\Delta v_0 + \overline{h}(x) = 0, & x \in \Omega, \\ \frac{\partial v_0}{\partial \nu} = G(\hat{x}), & x = (\hat{x}, 0) \in \Gamma_1, \\ v_0 = 0, & x \in \Gamma_2, \end{cases}$$

and $\overline{h}(x)$ is defined in (3), $G(\hat{x})$ was defined in (7).

Lemma 5. Let $\beta > 1 - \alpha$. For all $v \in \mathbf{V}_\varepsilon$ the following convergences:

$$\left|\varepsilon^\beta \int_{\Gamma_1^\varepsilon} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) v\left(\hat{x}, \varepsilon^\alpha F\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right)\right) \cdot v\left(\hat{x}, \varepsilon^\alpha F\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right)\right) ds\right| \to 0,$$

$$\left|\varepsilon^{1-\alpha}\int\limits_{\Gamma_1^\varepsilon} g\left(\hat{x},\frac{\hat{x}}{\varepsilon}\right)\cdot v\left(\hat{x},\varepsilon^\alpha F\left(\hat{x},\frac{\hat{x}}{\varepsilon}\right)\right)ds-\int\limits_{\Gamma_1} G(\hat{x})\cdot v(x)\,ds\right|\to 0$$

are valid as $\varepsilon\to 0$.

Remark 3. Due to the smoothness of the boundary $\partial\Omega$, the solution v_0 belongs to $H^2(\Omega)$ [33], and, hence, can be continued on Π_ε to belong to $H^2(\Omega_\varepsilon)$ [34].

Lemma 6. Let $\beta>1-\alpha$ and $F(\hat{x},\hat{\xi})$, $g(\hat{x},\hat{\xi})$, $p(\hat{x},\hat{\xi})$ be periodic in ξ smooth functions. λ is a given matrix, $h(x,\frac{x}{\varepsilon})$ is right-hand function which satisfies conditions (3) and (4). Suppose that $F(\hat{x},\hat{\xi})$ compactly supported in $x\in\Gamma_1$ uniformly in ξ. Then, for all $\varepsilon>0$ the existence and uniqueness of solution to problem (14) follow, and the strong convergence

$$v_\varepsilon\to v_0 \tag{15}$$

in $\mathbf{V}$ as $\varepsilon\to 0$ is valid.

Proof. Due to Lemma 4 the existence and the uniqueness of solution to problem (1) can be obtained on the base of the Lax-Milgram Lemma ([35]). We extend the function v_0 to the oscillating layer keeping the norm. Then, after simple transformations, we find

$$\int_{\Omega_\varepsilon}\nabla(v_0-v_\varepsilon)\cdot\nabla w dx+\varepsilon^\beta\int_{\Gamma_1^\varepsilon}p(v_0-v_\varepsilon)\cdot w ds=\int_{\Omega_\varepsilon}\nabla v_0\cdot\nabla w dx-\int_{\Omega_\varepsilon}h\cdot w dx-\varepsilon^{1-\alpha}\int_{\Gamma_1^\varepsilon}g\cdot w ds+\varepsilon^\beta\int_{\Gamma_1^\varepsilon}pv_0\cdot w ds=$$

$$=\int_{\Omega}\nabla v_0\cdot\nabla w dx-\int_{\Omega_\varepsilon}h\cdot w dx-\varepsilon^{1-\alpha}\int_{\Gamma_1^\varepsilon}g\cdot w ds+\int_{\Pi_\varepsilon}\nabla v_0\nabla\cdot w dx+\varepsilon^\beta\int_{\Gamma_1^\varepsilon}pv_0\cdot w ds=$$

$$=\int_{\Pi_\varepsilon}\nabla v_0\cdot\nabla w dx-\varepsilon^{1-\alpha}\int_{\Gamma_1^\varepsilon}g\cdot w ds+\int_{\Gamma_1}G(\hat{x})\cdot w ds-\int_{\Pi_\varepsilon}h\cdot w dx+\varepsilon^\beta\int_{\Gamma_1^\varepsilon}pv_0\cdot w ds. \tag{16}$$

According to Lemma 3 and Proposition 3 the last integral in the right-hand side of (16) is estimated as follows

$$\varepsilon^\beta\left|\int_{\Gamma_1^\varepsilon}pv_0\cdot w ds\right|=\varepsilon^\beta\left|\int_{\Gamma_1^\varepsilon}pv_0\cdot w\left[\varepsilon^{\alpha-1}\left(\sqrt{\left|\nabla_{\hat{\xi}}F\left(\hat{x},\hat{\xi}\right)\right|^2}\Bigg|_{\hat{\xi}=\hat{x}/\varepsilon}+O(\varepsilon^{1-\alpha})\right)d\hat{x}\right]d\hat{x}\right|$$

$$\le\varepsilon^{\beta-1+\alpha}C_4\left|\int_{\Gamma_1}pv_0\cdot w d\hat{x}\right|\le\varepsilon^{\beta-1+\alpha}C_4\|w\|_{H^{1/2}(\Gamma_1)}\le\varepsilon^{\beta-1+\alpha}C_5\|w\|_{H^1(\Omega_\varepsilon)}.$$

Recall that in the subcritical case $\beta-1+\alpha>0$ and, therefore, this term is vanishes as $\varepsilon\to 0$.

By (13) considering the uniform boundedness of $\|v_0\|_{H^2(\Omega_\varepsilon)}$, we have

$$\left|\int_{\Pi_\varepsilon}\nabla v_0\cdot\nabla w dx\right|\le\|\nabla v_0\|_{L_2(\Pi_\varepsilon)}\|w\|_{H^1(\Omega_\varepsilon)}\le C_6\sqrt{\varepsilon^\alpha}\|v_0\|_{H^2(\Omega_\varepsilon)}\|w\|_{H^1(\Omega_\varepsilon)}$$

and

$$\left|\int_{\Pi_\varepsilon}h\cdot w dx\right|\le\|h\|_{L_2(\Pi_\varepsilon)}\|w\|_{L_2(\Pi_\varepsilon)}\le C_6\sqrt{\varepsilon^\alpha}\|h\|_{L_2(\Omega_\varepsilon)}\|w\|_{H^1(\Omega_\varepsilon)}.$$

Then, Lemma 5 implies

$$\left|\varepsilon^{1-\alpha}\int_{\Gamma_1^\varepsilon}g\cdot w ds-\int_{\Gamma_1}G(\hat{x})\cdot w ds\right|\to 0\ \text{ as }\varepsilon\to 0. \tag{17}$$

Combining these inequalities and convergence (17) with (16), we deduce

$$\left|\int_{\Omega_\varepsilon} \nabla(v_0 - v_\varepsilon)\cdot\nabla w dx\right| + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p(v_0 - v_\varepsilon)\cdot w ds \to 0 \ \text{ as } \ \varepsilon \to 0.$$

It remains to substitute $w = v_0 - v_\varepsilon$. Then, (15) follows from Lemma 4 and the Friedrichs type inequality (see, for example, [34], [36] and [37]). Lemma is proved.

Lemma 7. 1) All solutions $u_\varepsilon(t)$ to (1) satisfy

$$\|u_\varepsilon(t)\|_\varepsilon^2 \le \|u_\varepsilon(0)\|_\varepsilon^2 e^{-\varkappa_1 t} + R_1^2, \tag{18}$$

$$\varpi\int_t^{t+1}\|u_\varepsilon(s)\|_{\varepsilon,1}^2 ds + 2a_0\sum_{i=1}^n\gamma_i\int_t^{t+1}\|u_\varepsilon^i(s)\|_{L_{p_i}(\Omega_\varepsilon)}^{p_i} ds + 2p_{\max}\varepsilon^{1-\alpha}\int_t^{t+1}\|u_\varepsilon(s)\|_{\mathbf{L}_2(\Gamma_1^\varepsilon)}^2 ds \le \|u_\varepsilon(t)\|_\varepsilon^2 + R_2^2, \tag{19}$$

where $\varkappa_1 > 0$ is a constant independent of ε. Positive values R_1 and R_2 depend on M_0 (see (4)) and do not depend on $u_\varepsilon(0)$ and ε.

2) All solutions $u(t)$ to (11) satisfy the same inequalities (18) and (19) with the norms in the function spaces over the domain Ω instead Ω_ε.

Proof. We give a brief outline of the proof (see the details in [17]).

In the right hand side of (8) the integral over the part of the boundary Γ_1^ε is nonnegative, because of the positiveness of the matrix p. We integrate (8) with respect to t. Then, to estimate the terms

$$\varepsilon^{1-\alpha}\int_{\Gamma_1^\varepsilon} g\cdot w ds \qquad \text{and} \qquad \varepsilon^\beta\int_{\Gamma_1^\varepsilon} p u_\varepsilon\cdot w ds,$$

we use the Cauchy inequality and the compactness of embedding $\mathbf{L}_2(\Gamma_1^\varepsilon) \Subset \mathbf{V}_\varepsilon$. For other terms we use a standard procedure (see [17]). Lemma is proved.

4 Main assertion

Here we formulate the main result concerning the limit behaviour of the trajectory attractors $\mathfrak{A}_\varepsilon$ of the reaction-diffusion systems (1) as $\varepsilon \to 0$ in the subcritical case $\beta > 1 - \alpha$.

Theorem 1. The following limit holds in the topological space Θ_+^{loc}

$$\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}} \ \text{ as } \varepsilon \to 0+. \tag{20}$$

Moreover,

$$\mathcal{K}_\varepsilon \to \overline{\mathcal{K}} \text{ as } \varepsilon \to 0+ \ \text{ in } \ \Theta^{loc}. \tag{21}$$

Proof. It is easy to see that (21) implies (20). Hence, it is sufficient to prove (21), i.e., for every neighbourhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc}, there exists $\varepsilon_1 = \varepsilon_1(\mathcal{O}) > 0$, such that

$$\mathcal{K}_\varepsilon \subset \mathcal{O}(\overline{\mathcal{K}}) \quad \text{for} \quad \varepsilon < \varepsilon_1. \tag{22}$$

Assume that (22) is not true. Then there exists a neighbourhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\varepsilon_k \to 0+$ $(k \to \infty)$, and a sequence $u_{\varepsilon_k}(\cdot) = u_{\varepsilon_k}(t) \in \mathcal{K}_{\varepsilon_k}$, such that

$$u_{\varepsilon_k} \notin \mathcal{O}'(\overline{\mathcal{K}}) \quad \text{for all} \quad k \in \mathbb{N}.$$

The function $u_{\varepsilon_k}(x,t), t \in \mathbb{R}$ is a solution to

$$\begin{cases} \frac{\partial u_{\varepsilon_k}}{\partial t} = \lambda \Delta u_{\varepsilon_k} - a\left(x, \frac{x}{\varepsilon_k}\right) f(u_{\varepsilon_k}) + h\left(x, \frac{x}{\varepsilon_k}\right), & x \in \Omega_{\varepsilon_k}, \\ \frac{\partial u_{\varepsilon_k}}{\partial \nu} + \varepsilon_k^\beta p(\hat{x}, \frac{\hat{x}}{\varepsilon_k}) u_{\varepsilon_k} = \varepsilon_k^{1-\alpha} g(\hat{x}, \frac{\hat{x}}{\varepsilon_k}), & x \in \Gamma_1^{\varepsilon_k}, \\ u_{\varepsilon_k} = 0, & x \in \Gamma_2, \end{cases} \tag{23}$$

where $\beta > 1-\alpha$. To obtain a uniform estimate of the solution in ε, we use Lemma 7. By means of (18) and (19), we obtain that the sequence $\{u_{\varepsilon_k}(x,t)\}$ is bounded in $\mathcal{F}^b$, i.e.,

$$\|u_{\varepsilon_k}\|_{\mathcal{F}^b} = \sup_{t\in\mathbb{R}} \|u_{\varepsilon_k}(t)\| + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\varepsilon_k}(\vartheta)\|_1^2 \, d\vartheta \right)^{1/2} + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\varepsilon_k}(\vartheta)\|_{\mathbf{L_p}}^p \, d\vartheta \right)^{1/p} +$$
$$+ \varepsilon^\beta \sup_{t\in\mathbb{R}} \int_t^{t+1} \int_{\Gamma_1^\varepsilon} p\Big(\hat{x}, \frac{\hat{x}}{\varepsilon}\Big) u_\varepsilon(x,\vartheta) \cdot u_\varepsilon(x,\vartheta) \, ds \, d\vartheta + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \frac{\partial u_{\varepsilon_k}}{\partial t}(\vartheta) \right\|_{\mathbf{H}^{-r}}^q d\vartheta \right)^{1/q} \leq C \ \forall k \in \mathbb{N}. \tag{24}$$

Remind that here $\beta > 1-\alpha$. The constant C is independent of ε. Consequently, there exists a subsequence $\{u_{\varepsilon_k'}(x,t)\} \subset \{u_{\varepsilon_k}(x,t)\}$, such that $u_{\varepsilon_k'}(x,t) \to \overline{u}(x,t)$ as $k \to \infty$ in Θ^{loc}. Here $\overline{u}(x,t) \in \mathcal{F}^b$ and $\overline{u}(t)$ satisfy (24) with the same constant C. Because of (24), we get $u_{\varepsilon_k'}(x,t) \rightharpoonup \overline{u}(x,t)$ $(k \to \infty)$ weakly in $\mathbf{L}_2^{loc}(\mathbb{R};\mathbf{V})$, weakly in $\mathbf{L}_\mathbf{p}^{loc}(\mathbb{R};\mathbf{L_p})$, $*$-weakly in $\mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H})$ and $\frac{\partial u_{\varepsilon_k'}(x,t)}{\partial t} \rightharpoonup \frac{\partial \overline{u}(x,t)}{\partial t}$ $(k \to \infty)$ weakly in $\mathbf{L}_{\mathbf{q},w}^{loc}(\mathbb{R};\mathbf{H}^{-r})$. We claim that $\overline{u}(x,t) \in \overline{\mathcal{K}}$. We have $\|\overline{u}\|_{\mathcal{F}^b} \leq C$. Hence, we have to verify that $\overline{u}(x,t) = u_0(x,t)$, i.e. is a weak solution to (11).

Using (24) and (3), we find that

$$\frac{\partial u_{\varepsilon_k}}{\partial t} - \lambda \Delta u_{\varepsilon_k} - h_{\varepsilon_k}(x) \longrightarrow \frac{\partial \bar{u}}{\partial t} - \lambda \Delta \bar{u} - \overline{h}(x) \quad \text{as} \quad k \to \infty$$

in the space $D'(\mathbb{R};\mathbf{H}_\varepsilon^{-\mathbf{r}})$, since the derivative operators are continuous in the space of distributions.

Let us prove that

$$a\left(x, \frac{x}{\varepsilon_k}\right) f(u_{\varepsilon_k}) \rightharpoonup \bar{a}(x) f(\bar{u}) \ \text{ as } k \to \infty \tag{25}$$

weakly in $\mathbf{L}_{\mathbf{q},w}^{loc}(\mathbb{R};\mathbf{L_q})$. We fix an arbitrary number $M > 0$. The sequence $\{u_{\varepsilon_k}(x,t)\}$ is bounded in $\mathbf{L_p}(-M,M;\mathbf{L_p})$ (see (24)). Then, due to (5) the sequence $\{f(u_{\varepsilon_k}(t))\}$ is bounded in $\mathbf{L_q}(-M,M;\mathbf{L_q})$. Since $\{u_{\varepsilon_k}(x,t)\}$ is bounded in $\mathbf{L}_2(-M,M;\mathbf{V})$ and $\left\{\frac{\partial u_{\varepsilon_k}}{\partial t}(t)\right\}$ is bounded in $\mathbf{L_q}(-M,M;\mathbf{H^{-r}})$, we can assume that $u_{\varepsilon_k}(x,t) \to \bar{u}(x,t)$ as $k \to \infty$ strongly in $\mathbf{L}_2(-M,M;\mathbf{L}_2) = \mathbf{L}_2(\Omega\times]-M,M[)$ and, therefore,

$$u_{\varepsilon_k}(x,t) \to \bar{u}(x,t) \ \text{ as } k \to \infty \ \text{ for almost all } (x,t) \in \Omega\times]-M,M[.$$

Since the function $f(v)$ is continuous in $v \in \mathbb{R}$, we conclude that

$$f(u_{\varepsilon_k}(x,t)) \to f(\bar{u}(x,t)) \ \text{ as } k \to \infty \ \text{ for almost all } (x,t) \in \Omega\times]-M,M[. \tag{26}$$

We have

$$a\left(x, \frac{x}{\varepsilon_k}\right) f(u_{\varepsilon_k}) - \bar{a}(x) f(\bar{u}) = a\left(x, \frac{x}{\varepsilon_k}\right)(f(u_{\varepsilon_k}) - f(\bar{u})) + \left(a\left(x, \frac{x}{\varepsilon_k}\right) - \bar{a}(x)\right) f(\bar{u}). \tag{27}$$

Let us show that both terms in the right-hand side of (27) tend to zero as $k \to \infty$ weakly in $\mathbf{L_q}(-M,M;\mathbf{L_q}) = \mathbf{L_q}(\Omega \times]-M,M[)$. First, the sequence $a\left(x, \frac{x}{\varepsilon_k}\right)(f(u_{\varepsilon_k}) - f(\bar{u}))$ tends to zero as $k \to \infty$ for almost all $(x,t) \in \Omega \times]-M, M[$ (see (26)). Applying Lemma 1.3 from [38; Ch. 1, Sec. 1], we conclude that

$$a\left(x, \frac{x}{\varepsilon_k}\right)\left(f(u_{\varepsilon_k}) - f(\bar{u})\right) \rightharpoonup 0 \text{ as } k \to \infty$$

weakly in $\mathbf{L_q}(\Omega \times]-M, M[)$. Second, the sequence $\left(a\left(x, \frac{x}{\varepsilon_k}\right) - \bar{a}(x)\right) f(\bar{u})$ also tends to zero a $k \to \infty$ weakly in $\mathbf{L_q}(\Omega \times]-M, M[)$, since $a\left(x, \frac{x}{\varepsilon_k}\right) \rightharpoonup \bar{a}(x)$ as $k \to \infty$ $*$-weakly in $\mathbf{L}_{\infty,*w}(-M, M; \mathbf{L}_2)$ and $f(\bar{u}) \in \mathbf{L_q}(\Omega \times]-M, M[)$. Thus, (25) is proved.

The convergences of $\varepsilon_k^{\beta} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon_k}\right) u_{\varepsilon_k}$ to zero and $\varepsilon_k^{1-\alpha} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon_k}\right)$ to $G(\hat{x})$ are obvious due to Lemma 5.

Hence, for $\overline{u}(x,t) = u_0(x,t)$, we have

$$-\int_{-M}^{M}\int_{\Omega_{\varepsilon_k}} u_{\varepsilon_k} \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int_{-M}^{M}\int_{\Omega_{\varepsilon_k}} \lambda \nabla u_{\varepsilon_k} \cdot \nabla \psi\, dxdt + \int_{-M}^{M}\int_{\Omega_{\varepsilon_k}} a_{\varepsilon_k}(x) f(u_{\varepsilon_k}) \cdot \psi\, dxdt +$$

$$+{\varepsilon_k}^{\beta} \int_{-M}^{M}\int_{\Gamma_1^{\varepsilon_k}} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon_k}\right) u_{\varepsilon_k} \cdot \psi\, dsdt + {\varepsilon_k}^{1-\alpha} \int_{-M}^{M}\int_{\Gamma_1^{\varepsilon_k}} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon_k}\right) \cdot \psi\, dsdt \longrightarrow -\int_{-M}^{M}\int_{\Omega} u_0 \cdot \frac{\partial \psi}{\partial t}\, dxdt +$$

$$+\int_{-M}^{M}\int_{\Omega} \lambda \nabla u_0 \cdot \nabla \psi\, dxdt + \int_{-M}^{M}\int_{\Omega} \overline{a}(x) f(u_0) \cdot \psi\, dxdt + \int_{-M}^{M}\int_{\Gamma_1} G(\hat{x}) \cdot \psi\, dxdt$$

as $k \to \infty$.

Using (26), we pass to the limit in the equation (23) as $k \to \infty$ in the space $D'(\mathbb{R}; \mathbf{H}^{-r})$ and obtain that the function $u_0(x,t)$ satisfies the integral identity (12) and, therefore, it is a complete trajectory of the equation (11).

Consequently, $u_0 \in \overline{\mathcal{K}}$. We have proved above that $u_{\varepsilon_k} \to u_0$ as $k \to \infty$ in Θ^{loc}. Assumption $u_{\varepsilon_k} \notin \mathcal{O}'(\overline{\mathcal{K}})$ (see [39]) implies $u_0 \notin \mathcal{O}'(\overline{\mathcal{K}})$, and, hence, $u_0 \notin \overline{\mathcal{K}}$. We arrive to the contradiction that complete the proof of the theorem.

Using the compact inclusions (9) and (10), we can improve the convergence (20).

Corollary 4. For any $0 < \delta \le 1$ and for all $M > 0$

$$\operatorname{dist}_{\mathbf{L}_2([0,M];\mathbf{H}^{1-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0, \tag{28}$$

$$\operatorname{dist}_{\mathbf{C}([0,M];\mathbf{H}^{-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0 \quad (\varepsilon \to 0+). \tag{29}$$

To prove (28) and (29), we repeat the proof of Theorem 1 changing the topology Θ^{loc} on $\mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta})$ or $\mathbf{C}^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta})$.

Finally, we consider the reaction–diffusion systems for which the uniqueness theorem is true for the Cauchy problem. It suffices to assume that the nonlinear term $f(u)$ in (1) satisfies the condition

$$(f(v_1) - f(v_2), v_1 - v_2) \ge -C|v_1 - v_2|^2 \text{ for any } v_1, v_2 \in \mathbb{R}^n. \tag{30}$$

(see [17,31]). In [31] it was proved that if (30) is true, then (1) and (11) generate dynamical semigroups in $\mathbf{H}$, possessing global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ are bounded in $\mathbf{V}$ (see also [16], [15]). Moreover

$$\mathcal{A}_\varepsilon = \{u(0) \mid u \in \mathfrak{A}_\varepsilon\},\ \overline{\mathcal{A}} = \{u(0) \mid u \in \overline{\mathfrak{A}}\}.$$

The convergence (29) gives

Corollary 5. Under the assumption of Theorem 1 the limit formula takes place

$$\operatorname{dist}_{\mathbf{H}^{-\delta}}\left(\mathcal{A}_\varepsilon, \overline{\mathcal{A}}\right) \to 0 \ (\varepsilon \to 0+).$$

Conclusion

In the paper we study the reaction–diffusion systems of equations with rapidly oscillating terms in domains with locally periodic rapidly oscillating boundary depending on a small parameter. We construct the homogenized system of equations, define and proved the existence of the trajectory and global attractors to these systems and prove that they converge in a weak sense to the trajectory and global attractors of the limit (homogenized) reaction–diffusion systems in domain independent of the small parameter. In this paper we consider the subcritical case in which the Fourier type boundary condition transforms to the Neumann boundary condition under the limit passage.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Asymptotic estimates of the solution for a singularly perturbed Cauchy problem

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The article focuses on the initial problem for a third-order linear integro-differential equation with a small parameter at the higher derivatives, assuming that the roots of the additional characteristic equation have opposite signs. This paper presents a fundamental set of solutions and initial functions for a singularly perturbed homogeneous differential equation. The solution to the singularly perturbed initial integro-differential problem employs analytical formulas. A theorem concerning asymptotic estimates of the solution is established.

Keywords: singularly perturbed integro-differential equation, asymptotic estimates, Cauchy functions, fundamental solutions, small parameter.

2020 Mathematics Subject Classification: 34D15, 34E10, 34K26.

Introduction

Vasil'eva A.B. and Butuzov V.F. introduced the theory of singularly perturbed equations in their work [1]. Kassymov K.A. [2] investigated the most common cases of the Cauchy problem for singularly perturbed nonlinear systems of ordinary differential and integro-differential equations, as well as partial differential equations of hyperbolic type. Subsequently, singularly perturbed initial and boundary value problems with initial jumps were studied in [3,4]. Mirzakulova A.E. [5] extensively examines boundary value problems, particularly when the roots of the additional characteristic equation have opposing signs. While linear integro-differential equations are presented in [6], numerous papers have been dedicated to singularly perturbed integro-differential equations [7–12]. This article also provides an asymptotic solution for a singularly perturbed differential equation in a boundary value problem where the roots of the characteristic equation are opposite [13]. In addition, in recent years, significant work has been done on the numerical solution of integro-differential problems [14–17].

In this paper, we consider the initial problem for third-order linear integral-differential equations with a small parameter, where the roots of the corresponding characteristic equation have opposite signs. It is well known that there is no solution to a third-order linear differential equation with a small parameter (where the roots of the corresponding characteristic equation have opposite signs). However, we demonstrate that adding an integral term to the right-hand side yields an asymptotic formula for the solution. This article presents the findings of this research, which include an analytical formula and asymptotic estimates for solving a singularly perturbed integral-differential equation with a small parameter and initial conditions. Furthermore, a theorem on asymptotic estimation of these equations' solutions is proven.

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1 Statement of the problem

Consider the singularly perturbed integro-differential equation with a small parameter

$$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_2(t) y'' + A_1(t) y' + A_0(t) y = F(t) + \int\limits_0^1 \sum_{i=0}^{2} H_i(t,x) y^{(i)}(x,\varepsilon)\, dx, \tag{1}$$

in the initial conditions

$$y^{(i)}(0,\varepsilon) = \alpha_i, \ \ i = \overline{0,2}. \tag{2}$$

ε is a small parameter, while α_i are known constants.

Assume the following conditions hold:

I. Functions $A_i(t), \ i = \overline{0,2}$ and $F(t)$ are continuously differentiable on a segment $0 \le t \le 1$, and functions $H_i(t,x), \ i = \overline{0,2}$ are continuously differentiable on a domain $D = \{0 \le t \le 1, \ 0 \le x \le 1\}$.

II. Roots of "additional characteristic equation" $\mu^2 + A_2(t)\mu + A_1(t) = 0$ satisfy the following inequalities $\mu_1(t) \ne \mu_2(t)$ and $\mu_1(t) < -\gamma_1 < 0$, $\mu_2(t) > \gamma_2 > 0$.

To construct the solution, we first identify auxiliary functions. This article [13] serves as a reference. From the third formula to the seventh, we obtained the required results.

We look for the solution to equation (1) with conditions (2) in the form

$$y(t,\varepsilon) = \sum_{i=1}^{3} C_i y_i(t,\varepsilon) + \frac{1}{\varepsilon^2}\int\limits_0^t K_0(t,s,\varepsilon) z(s,\varepsilon) ds - \frac{1}{\varepsilon^2}\int\limits_t^1 K_1(t,s,\varepsilon) z(s,\varepsilon) ds; \tag{3}$$

here the fundamental solutions are $y_i(t,\varepsilon), \ i = \overline{1,3}$. $K_0(t,s,\varepsilon), \ K_1(t,s,\varepsilon)$ are auxiliary functions expressed by article [13], $C_i, \ i = \overline{1,3}$ are unknown constants.

We now denote the right side of equation (1) as follows

$$z(t,\varepsilon) = F(t) + \int\limits_0^1 \sum_{i=0}^{2} H_i(t,x) y^{(i)}(x,\varepsilon) dx.$$

Instead of $y(x,\varepsilon)$, we apply formula (3) to obtain the formula for $z(t,\varepsilon)$

$$z(t,\varepsilon) = f(t,\varepsilon) + \frac{1}{\varepsilon^2}\int\limits_0^1 \left[\int\limits_s^1 \Big(\sum_{i=0}^{2} H_i(t,x) K_0^{(i)}(x,s,\varepsilon) dx\right] z(s,\varepsilon) ds -$$

$$-\frac{1}{\varepsilon^2}\int\limits_0^1 \left[\int\limits_0^s \Big(\sum_{i=0}^{2} H_i(t,x) K_1^{(i)}(x,s,\varepsilon) dx\right] z(s,\varepsilon) ds,$$

here

$$f(t,\varepsilon) = F(t) + C_1 \int\limits_0^1 \Big(\sum_{i=0}^{2} H_i(t,x) y_1(x,\varepsilon)\Big) dx + C_2 \int\limits_0^1 \Big(\sum_{i=0}^{2} H_i(t,x) y_2(x,\varepsilon)\Big) dx +$$

$$+C_3 \int\limits_0^1 \Big(\sum_{i=0}^{2} H_i(t,x) y_3(x,\varepsilon)\Big) dx = F(t) + C_1\phi_1(t,\varepsilon) + C_2\phi_2(t,\varepsilon) + C_3\phi_3(t,\varepsilon),$$

$$\phi_i(t,\varepsilon)=\int\limits_0^1\Big(H_0(t,x)y_i(x,\varepsilon)+H_1(t,x)y_i'(x,\varepsilon)+H_2(t,x)y_i''(x,\varepsilon)\Big)dx,\quad i=\overline{1,3}. \tag{4}$$

Now, we provide the following notation

$$H(t,s,\varepsilon)=\frac{1}{\varepsilon^2}\int\limits_s^1\Big(\sum_{i=0}^{2}H_i(t,x)K_0^{(i)}(x,s,\varepsilon)\Big)dx-\frac{1}{\varepsilon^2}\int\limits_0^s\Big(\sum_{i=0}^{2}H_i(t,x)K_1^{(i)}(x,s,\varepsilon)\Big)dx.$$

III. The kernel $H(t,s,\varepsilon)$ does not have an eigenvalue with the value 1.

We get the Fredholm equation of the second type

$$z(t,\varepsilon)=f(t,\varepsilon)+\int\limits_0^1 H(t,s,\varepsilon)z(s,\varepsilon)ds. \tag{5}$$

If condition III is satisfied, the solution of integral equation (5) is the only one and it is written

$$z(t,\varepsilon)=f(t,\varepsilon)+\int\limits_0^1 R(t,s,\varepsilon)z(s,\varepsilon)ds, \tag{6}$$

where $R(t,s,\varepsilon)$ represents the resolvent of the kernel $H(t,s,\varepsilon)$.

Formula (6) is substituted for $z(s,\varepsilon)$ in formula (3), then formula (3) is written

$$y(t,\varepsilon)=\sum_{i=1}^{3}C_iy_i(t,\varepsilon)+\frac{1}{\varepsilon^2}\int\limits_0^t K_0(t,s,\varepsilon)\Big[F(s)+\sum_{i=1}^{3}C_i\phi_i(t,\varepsilon)+\int\limits_0^1 R(s,p,\varepsilon)\Big(F(p)+\sum_{i=1}^{3}C_i\phi_i(p,\varepsilon)\Big)dp\Big]ds-$$

$$-\frac{1}{\varepsilon^2}\int\limits_t^1 K_1(t,s,\varepsilon)\Big[F(s)+\sum_{i=1}^{3}C_i\phi_i(t,\varepsilon)+\int\limits_0^1 R(s,p,\varepsilon)\Big(F(p)+\sum_{i=1}^{3}C_i\phi_i(p,\varepsilon)\Big)dp\Big]ds.$$

Now, we provide the following notations:

$$Q_i(t,\varepsilon)=y_i(t,\varepsilon)+\frac{1}{\varepsilon^2}\int\limits_0^t K_0(t,s,\varepsilon)\overline{\phi_i}(s,\varepsilon)ds-\frac{1}{\varepsilon^2}\int\limits_t^1 K_1(t,s,\varepsilon)\overline{\phi_i}(s,\varepsilon)ds,\quad i=\overline{1,3}, \tag{7}$$

$$P(t,\varepsilon)=\frac{1}{\varepsilon^2}\int\limits_0^t K_0(t,s,\varepsilon)\overline{F}(s,\varepsilon)ds-\frac{1}{\varepsilon^2}\int\limits_t^1 K_1(t,s,\varepsilon)\overline{F}(s,\varepsilon)ds, \tag{8}$$

then the formulas for $y(t,\varepsilon)$ are abbreviated

$$y^{(j)}(t,\varepsilon)=C_1Q_1^{(j)}(t,\varepsilon)+C_2Q_2^{(j)}(t,\varepsilon)+C_3Q_3^{(j)}(t,\varepsilon)+P^{(j)}(t,\varepsilon),\quad j=\overline{0,2}. \tag{9}$$

Substituting equation (9) into condition (2) yields the algebraic system for calculating C_i, where $i=\overline{1,3}$:

$$\begin{cases}C_1Q_1(0,\varepsilon)+C_2Q_2(0,\varepsilon)+C_3Q_3(0,\varepsilon)+P(0,\varepsilon)=\alpha_0,\\ C_1Q_1'(0,\varepsilon)+C_2Q_2'(0,\varepsilon)+C_3Q_3'(0,\varepsilon)+P'(0,\varepsilon)=\alpha_1,\\ C_1Q''1(0,\varepsilon)+C_2Q_2''(0,\varepsilon)+C_3Q_3''(0,\varepsilon)+P''(0,\varepsilon)=\alpha_2.\end{cases} \tag{10}$$

As $\varepsilon \to 0$, the asymptotic behavior of $C_i,\ \ i=\overline{1,3}$, is obtained when the main determinant of system (10) is $\delta(\varepsilon) \neq 0$.

$$C_1(\varepsilon) = -\varepsilon\frac{\mu_2(0)(\alpha_0 y'_{30}(0)-\alpha_1)}{\mu_1(0)(\mu_2(0)-\mu_1(0))} + O(\varepsilon^2),$$

$$C_2(\varepsilon) = -\varepsilon\frac{\alpha_0}{\overline{\phi_2}(0)}\Big(\overline{\phi_3}(0) + y'_{30}(0)\mu_1(0)\mu_2(0) - \tag{11}$$

$$-y'_{30}(0)\frac{\mu_2(0)\overline{\phi_1}(0)}{\mu_1(0)(\mu_2(0)-\mu_1(0))}\Big) + \varepsilon\frac{\alpha_1}{\overline{\phi_2}(0)}\Big(\mu_1(0)\mu_2(0)-$$

$$-\frac{\mu_2(0)\overline{\phi_1}(0)}{\mu_1(0)(\mu_2(0)-\mu_1(0))}\Big) - \varepsilon\frac{\overline{F}(0)}{\overline{\phi_2}(0)} + O(\varepsilon^2),$$

$$C_3(\varepsilon) = \alpha_0 + O(\varepsilon).$$

By formula (4) $\phi_i(t,\varepsilon)$, $i=\overline{1,3}$ are found

$$\overline{\phi}_1(s,\varepsilon) = \frac{1}{\varepsilon}\Bigg(- \overline{H}_2(s,0)\mu_1(0)y_{10}(0) + O(\varepsilon)\Bigg) = \frac{1}{\varepsilon}\Big(\overline{\phi}_1(s) + O(\varepsilon)\Big),$$

$$\overline{\phi}_2(s,\varepsilon) = \frac{1}{\varepsilon}\Bigg(\overline{H}_2(s,1)\mu_2(1)y_{20}(1) + O(\varepsilon)\Bigg) = \frac{1}{\varepsilon}\Big(\overline{\phi}_2(s) + O(\varepsilon)\Big),$$

$$\overline{\phi}_3(s,\varepsilon) = \int\limits_0^1 \sum_{j=0}^{2} \overline{H}_j(s,x)y_{30}^{(j)}(x)dx + O(\varepsilon) = \overline{\phi}_3(s) + O(\varepsilon).$$

Given formulas (7) and (8), we obtain the asymptotic behaviors of $Q_i^{(j)}(t,\varepsilon)$ and $P^{(j)}(t,\varepsilon)$, $j=\overline{0,2}$, $i=\overline{1,3}$:

$$Q_1^{(j)}(t,\varepsilon) = \frac{1}{\varepsilon}\int\limits_0^t \frac{y_{30}^{(j)}(t)\overline{\phi}_1(s)}{\mu_1(s)\mu_2(s)y_{30}(s)}ds + \frac{1}{\varepsilon^j}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx}\Bigg(y_{10}(t)\mu_1^j(t) - \frac{\overline{\phi}_1(0)y_{10}(t)\mu_1^j(t)}{y_{10}(0)\mu_1^2(0)(\mu_2(0)-\mu_1(0))}\Bigg) +$$

$$+\frac{1}{\varepsilon^j}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx}\frac{\overline{\phi}_1(1)y_{20}(t)\mu_2^j(t)}{y_{20}(1)\mu_2^2(1)(\mu_2(1)-\mu_1(1))} + \frac{1}{\varepsilon^j}\frac{\overline{\phi}_1(t)}{(\mu_2(t)-\mu_1(t))}\Bigg(\frac{\mu_1^j(t)}{\mu_1^2(t)} - \frac{\mu_2^j(t)}{\mu_2^2(t)}\Bigg), \quad j=\overline{0,2}, \tag{12.a}$$

$$Q_2^{(j)}(t,\varepsilon) = \frac{1}{\varepsilon}\int\limits_0^t \frac{y_{30}^{(j)}(t)\overline{\phi}_2(s)}{\mu_1(s)\mu_2(s)y_{30}(s)}ds - \frac{1}{\varepsilon^j}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx}\frac{\overline{\phi}_2(0)y_{10}(t)\mu_1^j(t)}{y_{10}(0)\mu_1^2(0)(\mu_2(0)-\mu_1(0))} +$$

$$+\frac{1}{\varepsilon^j}e^{-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx}\Bigg(y_{20}(t)\mu_2^{(j)}(t) + \frac{\overline{\phi}_2(1)y_{20}(t)\mu_2^j(t)}{y_{20}(1)\mu_2^2(1)(\mu_2(1)-\mu_1(1))}\Bigg) +$$

$$+\frac{1}{\varepsilon^j}\frac{\overline{\phi}_2(t)}{(\mu_2(t)-\mu_1(t))}\Bigg(\frac{\mu_1^j(t)}{\mu_1^2(t)} - \frac{\mu_2^j(t)}{\mu_2^2(t)}\Bigg), \quad j=\overline{0,2}, \tag{12.b}$$

$$Q_3^{(j)}(t,\varepsilon) = y_{30}^{(j)}(t) + \int\limits_0^t \frac{y_{30}^{(j)}(t)\overline{\phi}_3(s)}{\mu_1(s)\mu_2(s)y_{30}(s)}ds - \frac{1}{\varepsilon^{j-1}}e^{\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx}\frac{\overline{\phi}_3(0)y_{10}(t)\mu_1^j(t)}{y_{10}(0)\mu_1^2(0)(\mu_2(0)-\mu_1(0))} +$$

$$+\frac{1}{\varepsilon^{j-1}}e^{-\frac{1}{\varepsilon}\int\limits_t^1 \mu_2(x)dx}\frac{\overline{\phi}_3(1)y_{20}(t)\mu_2^j(t)}{y_{20}(1)\mu_2^2(1)(\mu_2(1)-\mu_1(1))}+\frac{1}{\varepsilon^{j-1}}\frac{\overline{\phi}_3(t)}{(\mu_2(t)-\mu_1(t))}\left(\frac{\mu_1^j(t)}{\mu_1^2(t)}-\frac{\mu_2^j(t)}{\mu_2^2(t)}\right),\ j=\overline{0,2}, \tag{12.c}$$

$$P^{(j)}(t,\varepsilon)=\int\limits_0^t\frac{y_{30}^{(j)}(t)\overline{F}(s)}{\mu_1(s)\mu_2(s)y_{30}(s)}ds-\frac{1}{\varepsilon^{j-1}}e^{\frac{1}{\varepsilon}\int\limits_0^t \mu_1(x)dx}\frac{\overline{F}(0)y_{10}(t)\mu_1^j(t)}{y_{10}(0)\mu_1^2(0)(\mu_2(0)-\mu_1(0))}+$$

$$+\frac{1}{\varepsilon^{j-1}}e^{-\frac{1}{\varepsilon}\int\limits_t^1 \mu_2(x)dx}\frac{\overline{F}(s)y_{20}(t)\mu_2^j(t)}{y_{20}(1)\mu_2^2(1)(\mu_2(1)-\mu_1(1))}+\frac{1}{\varepsilon^{j-1}}\frac{\overline{F}(t)}{(\mu_2(t)-\mu_1(t))}\left(\frac{\mu_1^j(t)}{\mu_1^2(t)}-\frac{\mu_2^j(t)}{\mu_2^2(t)}\right),\ j=\overline{0,2}. \tag{12.d}$$

Theorem 1. If conditions I-III are valid, then the solution for integro-differential equation (1) and (2) holds the following asymptotic estimates as $\varepsilon\to 0$:

$$|y^{(j)}(t,\varepsilon)|\le C\Big(|\alpha_0|+\varepsilon|\alpha_1|+\varepsilon^2|\alpha_2|+\max_{0\le t\le 1}|F(t)|\Big)+$$

$$+\frac{C}{\varepsilon^{j-1}}e^{-\frac{t}{\varepsilon}\gamma_1}\Big(|\alpha_0|+|\alpha_1|+\varepsilon^2|\alpha_2|+\max_{0\le t\le 1}|F(t)|\Big)+ \tag{13}$$

$$+\frac{C}{\varepsilon^{j-1}}e^{-\frac{1-t}{\varepsilon}\gamma_2}\Big(|\alpha_0|+|\alpha_1|+\varepsilon^2|\alpha_2|+\max_{0\le t\le 1}|F(t)|\Big),\quad j=0,1,2,$$

where $C>0$ is a constant independent of ε.

Proof. Asymptotic estimates of $C_i,\ i=\overline{1,3}$ and $Q_i^{(j)}(t,\varepsilon),\ P^{(j)}(t,\varepsilon)$ are obtained by applying formulas (11)-(12.a-12.d):

$$|C_i|\le C\varepsilon\Big(|\alpha_0|+|\alpha_1|\Big),\quad i=1,2,$$

$$|C_3|\le C\Big(|\alpha_0|+\varepsilon^2|\alpha_2|\Big),$$

$$|Q_1^{(j)}(t,\varepsilon)|\le C\Big(1+\frac{1}{\varepsilon^j}e^{-\frac{t}{\varepsilon}\gamma_1}+\frac{1}{\varepsilon^j}e^{-\frac{1-t}{\varepsilon}\gamma_2}\Big),\quad j=\overline{0,2},$$

$$|Q_2^{(j)}(t,\varepsilon)|\le C\Big(1+\frac{1}{\varepsilon^j}e^{-\frac{t}{\varepsilon}\gamma_1}+\frac{1}{\varepsilon^j}e^{-\frac{1-t}{\varepsilon}\gamma_2}\Big),\quad j=\overline{0,2}, \tag{14}$$

$$|Q_3^{(j)}(t,\varepsilon)|\le C\Big(1+\frac{1}{\varepsilon^{j-1}}e^{-\frac{t}{\varepsilon}\gamma_1}+\frac{1}{\varepsilon^{j-1}}e^{-\frac{1-t}{\varepsilon}\gamma_2}\Big),\quad j=\overline{0,2},$$

$$|P^{(j)}(t,\varepsilon)\le C\max_{0\le t\le 1}|F(t)|\Big(1+\frac{1}{\varepsilon^{j-1}}e^{-\frac{t}{\varepsilon}\gamma_1}+\frac{1}{\varepsilon^{j-1}}e^{-\frac{1-t}{\varepsilon}\gamma_2}\Big),\quad j=\overline{0,2}.$$

We derive asymptotic estimations (13) from asymptotic behaviors (14). Theorem 1 has been proved.

Conclusion

This article examines the initial problem for a third-order linear integro-differential equation with a small parameter at the higher derivatives, assuming that the roots of the additional characteristic equation have opposite signs. This paper presents the construction of a fundamental system of solutions and a Cauchy function for a singularly perturbed homogeneous differential equation. The functions $Q_i(t,\varepsilon)$, $P(t,\varepsilon)$, $i=\overline{1,3}$, and constants C_i, $i=\overline{1,3}$ exhibit asymptotic behaviors and estimates. Furthermore, the article provides an analytical formula for solving this singularly perturbed initial problem. A theorem on asymptotic estimates of the solution is proven.

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Author Contributions

N.U. Bukanay used the fundamental solutions of the homogeneous equation to calculate the Cauchy function and derive the asymptotic formula. A.E. Mirzakulova checked all the calculations and wrote the asymptotic behavior of the solutions. A.T. Assanova conducted a comprehensive verification of the calculations. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Some estimates for the viscoelastic incompressible Kelvin-Voigt medium

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In this paper, we consider the application of the method of fictitious domains to a viscoelastic incompressible medium based on the Kelvin-Voigt model. Application of the method of fictitious domains allows solving the original problem in regions with complex geometric configuration. This makes it easier to automate the construction of a consistent difference mesh, and to solve the problem in areas of standard shape. Estimates for the proximity of the auxiliary problem's solution are obtained. The auxiliary problem is constructed by the method of fictitious domains. These estimates refer to the solution of the original problem. The original problem describes a viscoelastic incompressible medium. Convergence follows from the estimates of the proximity of the solutions of the original and auxiliary problems. Further, on the basis of the method of fictitious domains, two-sided estimates on a small parameter for the difference between the solution of the original problem and the solution of the auxiliary problem constructed by the method of fictitious domains are obtained. Moreover, the solution to the auxiliary problem is expanded as a series in powers of the small parameter. This is possible because that solution is represented as a functional series that converges absolutely in the original domain.

Keywords: Kelvin-Voigt medium, Kronecker symbol, approximate solution, fictitious area method, two-sided estimates, small parameter, stress, velocity, strain, displacement.

2020 Mathematics Subject Classification: 39A05.

Introduction and problem statement

This paper presents two-sided estimates for a small parameter of the solution of the initial problem. Since a priori estimates help to determine the interval or band in which the solution lies, this is relevant. In addition, since the initial problem does not have an analytical solution, two-sided estimates allow us to determine the initial approximation for finding an approximate solution to the problem, which is an important step in the process of finding a solution.

We consider the application of the method of fictitious domains for incompressible Kelvin-Voigt medium. Two-sided estimates of the convergence of the approximate solution to the exact solution by a small parameter α are obtained. We consider the formulation of a dynamic viscoelastic incompressible medium based on the Kelvin-Voigt model in a cylinder $Q = \{D \times [0 \le t \le t_1]\}$, where $D \subset R^3$ is a bounded singly connected region with a sufficiently smooth boundary γ. Let us introduce the notation $\gamma_t = \gamma \times [0, t_1]$, the strain and stress vector-functions $\overrightarrow{\varepsilon} = \{\varepsilon_{11},\ \varepsilon_{22},\ \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13},\ 2\varepsilon_{23}\}^T$, $\overrightarrow{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}\}^T$, here the symbol T is transpose, the displacement and velocity vector functions $\overrightarrow{U} = \{U_1, U_2, U_3\}^T$, $\overrightarrow{\vartheta} = \{\vartheta_1, \vartheta_2, \vartheta_3\}^T$. Following the work of [1] we consider the velocity-stress formulation. We find the solution satisfying the following relations:

$$\frac{\partial \overrightarrow{\vartheta}}{\partial t} + R\overrightarrow{\sigma} = \overrightarrow{f}, \quad (x,t) \in Q \tag{1}$$

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which is an equation of motion,

$$\frac{\partial \overrightarrow{\varepsilon}}{\partial t} - R\overrightarrow{\vartheta} = 0, \tag{2}$$

displacement-strain ratio,

$$B\overrightarrow{\sigma} = J\overrightarrow{\varepsilon} + D\frac{\partial \overrightarrow{\varepsilon}}{\partial t}. \tag{3}$$

The equation of state for the Kelvin-Voigt medium is

$$div\overrightarrow{u} = 0. \tag{4}$$

The condition of incompressibility of the medium, taking into account the stresses and the pressure function p, is given by the relationship

$$\sigma_{ik} = -\delta_{ik}p + 2\mu\varepsilon_{ik}, \quad i,k = 1,2,3. \tag{5}$$

Here δ_{ik} is the Kronecker symbol, $\overrightarrow{f}$ is the vector of mass forces, $B = B^T,\ C = C^T$ are symmetric positive-definite matrices depending on the Lamé constants and viscosity coefficient, J is an diagonal matrix, their form is given in [2].

R is a linear matrix-differential operator:

$$R = \begin{pmatrix} \nabla_1 & 0 & 0 & \nabla_2 & \nabla_3 & 0 \\ 0 & \nabla_2 & 0 & \nabla_1 & 0 & \nabla_3 \\ 0 & 0 & \nabla_3 & 0 & \nabla_1 & \nabla_2 \end{pmatrix}^T, \quad R^* = -R^T, \quad \nabla_i = \frac{\partial}{\partial x_i}, \quad i = 1,2,3.$$

The system of equations (1)–(5) transform to the following form, a vector function $\overrightarrow{\sigma}(x,t)$ that satisfies the following relations

$$B\frac{\partial^2 \overrightarrow{\sigma}}{\partial t^2} = A\overrightarrow{\sigma} + DA\frac{\partial \overrightarrow{\sigma}}{\partial t} + \overrightarrow{F}, \tag{6}$$

$A = -RR^*, \quad \overrightarrow{F} = R\overrightarrow{f}$, satisfies the initial conditions:

$$\overrightarrow{\sigma}(x,0) = \overrightarrow{q}(x), \quad \frac{\partial \overrightarrow{\sigma}}{\partial t}(x,0) = \overrightarrow{g}(x) \tag{7}$$

and boundary conditions

$$\sum_{k=1}^{3} \sigma_{ik}(x,t)\, n_k = 0, \quad (x,t) \in \gamma_t; \tag{8}$$

here $n_k = (n_1, n_2, n_3)^T$ is the vector of normal to γ, $\gamma_t = \gamma \times [0, t_1]$. Let us denote the problem (6)–(8) by the problem I.

Main provisions

In [2], we show the stationalization of the solution of the problem I to the solution of the static elastic problem.

$$R^*\overrightarrow{\sigma}^y(x) + \overrightarrow{F}(x) = 0, \quad \overrightarrow{\sigma}^y(x) = R\varepsilon^y(x),$$
$$\sum_{k=1}^{3} \sigma_{ik}^y(x)\, n_k = 0, \quad x \in \gamma. \tag{9}$$

In [2], the closeness estimate of the solution of problem I and problem (9) is obtained:

$$\left\|\overrightarrow{\sigma} - \overrightarrow{\sigma}^y\right\| \le e^{-\beta t} \cdot \left\|\overrightarrow{\sigma}(x,0) - \overrightarrow{\sigma}^y(x)\right\|,$$

where $\beta>0$ is a constant.

The following theorem is true for problem I:

Theorem 1. Let

$$\overrightarrow{\sigma}(x,t)\in W_2^{2,1}(Q), \quad \overrightarrow{g}(x)\in \dot{W}_2^1(D), \quad \overrightarrow{q}(x)\in L_2(D), \quad \overrightarrow{F}(x,t)\in L_2(Q),$$

then there exists a unique solution to the problem I and the following estimation is true

$$\|\overrightarrow{\sigma}(x,t)\|_{W_2^{2,1}(Q)} \le C_1(\left\|\overrightarrow{F}\right\|_{L_2(Q)} + \|\overrightarrow{q}\|_{L_2(D)} + \int_0^{t_1}\|\overrightarrow{g}\|_{\dot{W}_2^1(D)}d\tau).$$

The proof is similar to the proof of Theorem 1 in [3].

According to the method of fictitious domains [3–6], we augment the original region D with the region D_1 to a composite region $D_0 = D\cup D_1$, with boundary $\Gamma, \Gamma_t = \Gamma\times[0,t_1]$, $+Q_1 = D_1\times[0,t_1]$ and construct the auxiliary problem

$$L_\alpha\overrightarrow{\sigma}^\alpha = \overrightarrow{F},\ (x,t)\in Q,\quad L_\alpha\overrightarrow{\sigma}^\alpha = 0,\ \ (x,t)\in Q_1,$$

$$\sum_{k=1}^{3}(\overrightarrow{\sigma}^\alpha)_{ik}n_k = 0,\ \ (x,t)\in\gamma_t,\quad \overrightarrow{\sigma}^\alpha(x,0) = 0,\ \ x\in D_1,$$

$$\overrightarrow{\sigma}^\alpha(x,0) = \overrightarrow{q}(x),\ \ x\in D,\quad \frac{\partial\overrightarrow{\sigma}^\alpha}{\partial t}(x,0) = 0,\ \ x\in D_1$$

$$\left.\frac{\partial\overrightarrow{\sigma}^\alpha}{\partial t}\right|_t = \overrightarrow{g}(x),\ \ x\in D,\quad \sum_{k=1}^{3}\sigma_{ik}^\alpha n_k = 0,\ \ (x,t)\in\Gamma_t, \tag{10}$$

where $L_\alpha\overrightarrow{\sigma}^\alpha = B\frac{\partial^2\overrightarrow{\sigma}^\alpha}{\partial t^2} - a^\alpha A\overrightarrow{\sigma}^\alpha + JA\frac{\partial\overrightarrow{\sigma}^\alpha}{\partial t},\quad a^\alpha = \begin{cases} 1, & x\in D,\\ -\alpha^2, & x\in D_1,\end{cases}$ $\alpha>0$ is a small parameter.

On the coefficient gap curve γ_t, we set the following matching conditions

$$\left.\overrightarrow{\sigma}^\alpha\right|_{\gamma_t}^{+} = \left.\overrightarrow{\sigma}^\alpha\right|_{\gamma_t}^{-},\quad \left.\frac{\partial\overrightarrow{\sigma}^\alpha}{\partial N}\right|_{\gamma_t}^{+} = \frac{M}{\alpha}\left.\frac{\partial\overrightarrow{\sigma}^\alpha}{\partial n}\right|_{\gamma_t}^{-}.$$

Signs "+" or "–" mean convergence to the limit value of the function from inside or outside to the boundary γ_t. The parameter M takes the values –1 or +1 [6, 7].

Let us introduce the following series into consideration:

$$S_1 = \sum_{k=0}^{\infty}\alpha^k\overrightarrow{V_k},\ \text{on } Q,\quad S_2 = \sum_{k=1}^{\infty}\alpha^k\overrightarrow{W_k},\ \text{on } Q_1. \tag{11}$$

Putting (11) into (10), we obtain relations for determining $\overrightarrow{V_k}$ and $\overrightarrow{W_k}$:

$$L_\alpha\,\overrightarrow{V_0} = \ \overrightarrow{F},\ \ (x,t)\in QL_\alpha,\qquad \overrightarrow{W_1} = 0,\ \ (x,t)\in Q_1,$$

$$\sum_{k=1}^{3}(V_0)_{ik}n_k = 0,\ \ (x,t)\in\gamma_t,\quad \overrightarrow{W_1}(x,0) = 0,\quad \frac{\partial\overrightarrow{W_1}}{\partial t}(x,0) = 0,\quad x\in D_1.$$

$$\overrightarrow{V_0}(x,0) = \overrightarrow{q}(x),\ \frac{\partial\overrightarrow{V_0}}{\partial t}(x,0) = \overrightarrow{g}(x),\ x\in D,\quad \frac{\partial\overrightarrow{W_1}}{\partial n} = M\,\frac{\partial\overrightarrow{V_0}}{\partial N},\ (x,t)\in\gamma_t. \tag{12}$$

$$\sum_{k=1}^{3} (\overrightarrow{W_1})_{ik} n_k = 0, \qquad (x,t) \in \gamma_t.$$

And for k≥1

$$L_\alpha \overrightarrow{V_k} = \overrightarrow{F}, \quad (x,t) \in Q, \quad L_\alpha \overrightarrow{W}_{k+1} = 0, \quad (x,t) \in Q_1,$$

$$\sum_{k=1}^{3} (V_k)_{ik} n_k = 0, \quad (x,t) \in \gamma_t, \quad \sum_{k=1}^{3} (W_{k+1})_{ik} n_k = 0, \quad (x,t) \in \gamma_t,$$
$$\overrightarrow{V_k}(x,0) = 0, \quad x \in D, \quad \overrightarrow{W}_{k+1}(x,0) = 0, \quad x \in D_1,$$
$$\frac{\partial \overrightarrow{V_k}}{\partial t}(x,0) = 0, \quad x \in D, \quad \frac{\partial \overrightarrow{W}_{k+1}}{\partial t}(x,0) = 0, \quad x \in D_1,$$
$$\overrightarrow{V_k} = \overrightarrow{W_k}, \quad (x,t) \in \gamma_t.$$

Functions $\overrightarrow{V_k} \in W_2^{2,1}(Q)$, $k = 0,1,\ldots$, $\overrightarrow{W_k} \in W_2^{2,1}(Q_1)$, $k = 0,1,\ldots$

We obtain estimates of the convergence of the solution of the auxiliary problem to the solution of the original problem with respect to the small parameter α.

Theorem 2. If α_0 is such that $0 < \alpha < \alpha_0$, then the series S_1, S_2 absolutely converge to $W_2^{2,1}(Q)$ and $W_2^{2,1}(Q_1)$, and the following equalities are true

$$\overrightarrow{\sigma}^\alpha = S_1, \quad (x,t) \in Q, \quad \overrightarrow{\sigma}^\alpha = S_2, \quad (x,t) \in Q_1,$$

where $\overrightarrow{\sigma}^\alpha$ is the solution of problem (10).

Proof. We search obvious priori estimates [7, 8].

$$\left\|\overrightarrow{W_k}\right\|_{W_2^{2,1}(Q_1)} \leq C_1 \left\|\frac{\partial \overrightarrow{W_k}}{\partial n}\right\|_{W_2^{\frac{1}{2}}(\gamma_t)} \leq C_2 \left\|\frac{\partial \overrightarrow{V}_{k-1}}{\partial N}\right\|_{W_2^{\frac{1}{2},1}(\gamma_t)} \leq C_1 C_2 \left\|\overrightarrow{V}_{k-1}\right\|_{W_2^{\frac{1}{2}}(\gamma_t)}, \tag{13}$$

where C_1, C_2 are constants depending on the regions D, D_1 and not depending on α.

Now we show the convergence of the series S_1 to $W_2^{2,1}(Q_1)$ and S_2 to $W_2^{2,1}(Q_1)$, we have

$$\left\|\overrightarrow{V_k}\right\|_{W_2^{2,1}(Q)} \leq C_3 \left\|\overrightarrow{V_k}\right\|_{W_2^{\frac{3}{2},\ 1}(\gamma_t)} = C_3 \left\|\overrightarrow{W_k}\right\|_{W_2^{\frac{3}{2},1}(\gamma_t)} \leq C_3 C_4 \left\|\overrightarrow{W_k}\right\|_{W_2^{2,1}(Q_1)},$$

and using (8), (13), we obtain

$$\left\|\overrightarrow{V_k}\right\|_{W_2^{2,1}(Q)} \leq C_5 \left\|\overrightarrow{V}_{k-1}\right\|_{W_2^{\frac{3}{2},1}(Q)}, \quad k \geq 1,$$

then

$$\left\|\overrightarrow{V_0}\right\|_{W_2^{2,1}(Q)} \leq C_6 \left(\left\|\overrightarrow{F}\right\|_{L_2(Q)} + \|\overrightarrow{q}\|_{L_2(D)} + \int_0^{t_1} \|\overrightarrow{g}\|_{\dot{W}_2^1(D)} dt\right),$$

where $C_6 = C_1 C_2 C_3 C_4 C_5$.

Assuming $\alpha < \alpha_0 = C_6^{-1}$, we obtain the series S_1, is absolutely convergent to $W_2^{2,1}(Q)$ and correspondingly the series S_2 is absolutely convergent to $W_2^{2,1}(Q_1)$.

Multiplying (12) for $\overrightarrow{V_k}$ and $\overrightarrow{W_k}$ by α_k, and summing over k, we have

$$LS_1 = \overrightarrow{F}, \quad (x,t) \in Q, \quad L_\alpha S_2 = 0, \quad (x,t) \in Q_1,$$
$$S_1(x,0) = \overrightarrow{q}(x), \quad x \in DS_2(x,0) = 0, \quad x \in D_1, \tag{14}$$

$$\frac{\partial S_1}{\partial t}(x,0) = \overrightarrow{g}(x), \quad x \in D, \quad \frac{\partial S_2}{\partial t}(x,0) = 0, \quad x \in D_1.$$
$$S_2(x,t) = 0, \quad (x,t) \in \gamma_t,$$
$$S_1 = S_2, \quad (x,t) \in \Gamma_t, \quad \frac{\partial S_2}{\partial n} = \frac{M}{\alpha}\frac{\partial S_1}{\partial N}, \quad (x,t) \in \gamma_t,$$
$$L\overrightarrow{\sigma} = B\,\frac{\partial^2 \overrightarrow{\sigma}}{\partial t^2} - A\overrightarrow{\sigma} - JA\frac{\partial \overrightarrow{\sigma}}{\partial t}.$$

Hence, (14) we obtain that $\overrightarrow{\sigma}^{\alpha} = S_1$ in Q and $\overrightarrow{\sigma}^{\alpha} = S_2$ in Q_1, if the condition $0 < \alpha < \alpha_0$ is met. From the proof of Theorem 2 implies the following statement

$$\|\overrightarrow{\sigma} - \overrightarrow{\sigma}^{\alpha}_{+}\|_{W_2^{2,1}(Q)} \leq C_7\alpha, \quad \|\overrightarrow{\sigma} - \overrightarrow{\sigma}^{\alpha}_{-}\|_{W_2^{2,1}(Q)} \leq C_8\alpha, \tag{15}$$

here $\overrightarrow{\sigma}^{\alpha}_{+} = \overrightarrow{\sigma}^{\alpha}$, at $M = 1$, $\overrightarrow{\sigma}^{\alpha}_{-} = \overrightarrow{\sigma}^{\alpha}$, at $M = -1$. $\overrightarrow{\sigma}$ is the solution to the problem I. C_7, C_8 are constants depend on the areas D, D_1 and are independent of α.

Next, we can formulate a theorem giving two-sided estimates on α [9].

Theorem 3. If $0 < \alpha < \alpha_0$, $\overrightarrow{\sigma}$ is a solution of problem I, $\overrightarrow{\sigma}^{\alpha}_{+}$, $\overrightarrow{\sigma}^{\alpha}_{-}$ is a solution of problem (10) at $M = 1$ and $M = -1$, then the following estimation (16) is true

$$\left\|\overrightarrow{\sigma} - \frac{1}{2}\left(\overrightarrow{\sigma}^{\alpha}_{+} + \overrightarrow{\sigma}^{\alpha}_{-}\right)\right\|_{W_2^{2,1}(Q)} \leq C_9\alpha^2, \tag{16}$$

where

$$\overrightarrow{\sigma}^{\alpha} = S_1, \quad (x,t) \in Q, \quad \overrightarrow{\sigma}^{\alpha} = S_2, \quad (x,t) \in Q_1. \tag{17}$$

Proof. By virtue of Theorem 2, we have

$$\overrightarrow{\sigma}^{\alpha}_{+} = \sum_{k=0}^{\infty} \alpha^k \overrightarrow{V}^{+}_{k}, \quad (x,t) \in Q, \quad \overrightarrow{\sigma}^{\alpha}_{+} = \sum_{k=1}^{\infty} \alpha^k \overrightarrow{W}^{+}_{k}, \quad (x,t) \in Q_1, \tag{18}$$

here $\overrightarrow{V}^{+}_{k}$, $\overrightarrow{W}^{+}_{k}$ are solutions of (10) at $M = 1$, moreover

$$\overrightarrow{\sigma}^{\alpha}_{-} = \sum_{k=0}^{\infty} \alpha^k \overrightarrow{V}^{-}_{k}, \quad (x,t) \in Q, \quad \overrightarrow{\sigma}^{\alpha}_{-} = \sum_{k=1}^{\infty} \alpha^k \overrightarrow{W}^{-}_{k}, \quad (x,t) \in Q_1, \tag{19}$$

here $\overrightarrow{V}^{-}_{k}$, $\overrightarrow{W}^{-}_{k}$ are solutions of (10) at $M = -1$.

We obtain $\overrightarrow{V}^{+}_{0} \equiv \overrightarrow{V}^{-}_{0} \equiv \overrightarrow{\sigma}$, it is a solution of problem I.

We introduce the notation $\overrightarrow{W_1} = \overrightarrow{W}^{+}_{1} + \overrightarrow{W}^{-}_{1}$, the function $\overrightarrow{W_1}$ satisfies the following problem

$$L_\alpha \overrightarrow{W_1} = 0, \quad (x,t) \in Q_1, \quad \frac{\partial \overrightarrow{W_1}}{\partial n} = 0, \quad (x,t) \in \gamma_t,$$

$$\overrightarrow{W_1}(x,0) = 0, \quad \frac{\partial \overrightarrow{W_1}}{\partial t}(x,0) = 0, \quad x \in D_1, \quad \overrightarrow{W_1}(x,t) = 0, \quad (x,t) \in \Gamma_t,$$

hence, we obtain that $\overrightarrow{W_1} = 0$, or $\overrightarrow{W}^{+}_{1} = -\overrightarrow{W}^{-}_{1}$.

Further we introduce $\overrightarrow{V_1} = \overrightarrow{V}^{+}_{1} + \overrightarrow{V}^{-}_{1}$, the function $\overrightarrow{V_1}$ satisfies the problem

$$L_\alpha \overrightarrow{V_1} = 0, \quad (x,t) \in Q, \quad \frac{\partial \overrightarrow{V_1}}{\partial t}(x,0) = 0, \quad x \in D,$$

$$\overrightarrow{V_1}(x,0)=0, \quad (x,t)\in\Gamma_t, \quad \overrightarrow{V_1}(x,t)=0, \quad (x,t)\in\gamma_t,$$

from which we obtain $\overrightarrow{V_1}=0$, or $\overrightarrow{V}_1^+=-\overrightarrow{V}_1^-$. By sequentially introducing

$$\overrightarrow{W_2}=\overrightarrow{W}_2^+ + \overrightarrow{W}_2^-, \quad \overrightarrow{V_2}=\overrightarrow{V}_2^+ + \overrightarrow{V}_2^-,$$

we get

$$\overrightarrow{W_2}=\overrightarrow{W}_2^-, \quad \overrightarrow{V}_2^+=\overrightarrow{V}_2^-,$$

continuing this process, we get

$$\overrightarrow{V}_k^+=\overrightarrow{V}_k^-, \text{ if } k \text{ is even,}$$
$$\overrightarrow{V}_k^+=-\overrightarrow{V}_k^-, \text{ if } k \text{ is odd.}$$

Substituting (20) into (18), (19), we have

$$\overrightarrow{\sigma}_+^\alpha=\overrightarrow{\sigma}+\sigma\overrightarrow{V}_1^+ + \sigma^2\overrightarrow{V}_2^+ + \dots$$

$$\overrightarrow{\sigma}_-^\alpha=\overrightarrow{\sigma}-\sigma\overrightarrow{V}_1^+ + \sigma^2\overrightarrow{V}_2^+ + \dots$$

Applying decomposition (20), as well as estimation (17) at $0<\alpha<\alpha_0$, we obtain

$$\left\|\overrightarrow{\sigma}-\frac{1}{2}\left(\overrightarrow{\sigma}_+^\alpha+\overrightarrow{\sigma}_-^\alpha\right)\right\|_{W_2^{2,1}(Q)} \le \alpha^2\left\|\overrightarrow{V}_2^+ + \alpha^2\overrightarrow{V}_4^+ + \dots\right\|_{W_2^{2,1}(Q)} \le C_8\alpha^2\left\|\overrightarrow{V}_0^+\right\|_{W_2^{2,1}(Q)} \le C_9\alpha^2;$$

here $C_8=C_\alpha^2$, so for $x\in D$, $0<\alpha<\alpha_0$, we have a two-sided estimation

$$O\left(\alpha^2\right)+\min\left(\overrightarrow{\sigma}_+^\alpha\,,\ \overrightarrow{\sigma}_-^\alpha\right) \le \overrightarrow{\sigma} \le \max\left(\overrightarrow{\sigma}_+^\alpha\,,\ \overrightarrow{\sigma}_-^\alpha\right) + O(\alpha^2).$$

Thus, a two-sided estimate in terms of the small parameter of the solution to the original problem has been obtained through the solution of the auxiliary problem, where the parameter values $M=-1$, $M=1$ corresponds to $\overrightarrow{\sigma}_+^\alpha$.

Conclusion

The obtained estimate is essential for the application of the numerical solution of the auxiliary problem, and it is not considered in the works [10–16]. Continuation by the lowest coefficient in the fictitious region method leads to the same estimates. In works [10–16], the numerical implementation of the Kelvin-Voigt model in equivalent formulations is considered.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Approximation of fixed points for enriched $B_{\gamma,\mu}$ mapping using a new iterative algorithm in CAT(0) space

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In this paper, we define enriched $B_{\gamma,\mu}$ mapping in CAT(0) space and derive some fixed point results for such mapping with the help of averaged mapping. Our results extend some existing results. A new iterative algorithm is developed in CAT(0) space to approximate the fixed point of enriched $B_{\gamma,\mu}$ mapping. Δ-convergence and strong convergence of this iterative algorithm for enriched $B_{\gamma,\mu}$ mapping is proved.

Keywords: CAT(0) space, $B_{\gamma,\mu}$ mapping, averaged mapping, fixed point.

2020 Mathematics Subject Classification: 47H09, 47H10, 47J26.

Introduction

Fixed point theory is an active and vibrant area of research, which serves as a powerful tool in various fields of mathematics and other allied areas. Many implications across a diverse range of fields such as engineering, economics, dynamical system, differential equation, integral equation etc., can be formulated as a fixed point problem. In 1922, Banach proved a fundamental theorem [1] in metric fixed point theory known as Banach contraction theorem, and over the past century, this theorem has been generalized by various prominent authors considering different aspects.

In 2008, Suzuki [2] introduced a new class of mappings on a nonempty subset of a Banach space by proposing a condition (C). Later many authors generalized this class of mappings in different ways and derived different fixed point theorems and convergence results of different iteration schemes. These generalizations often involve relaxing the assumptions or considering more general settings, leading to broader applicability and deeper theoretical understanding.

In 2018, Patir et al. [3] introduced a new class of generalized nonexpansive mappings which is wider than the class of mappings satisfying Suzuki (C) condition. They proved some fixed point results as well as some properties of this class of mappings. In 2019, Berinde [4] introduced the class of enriched nonexpansive mappings in Hilbert space and approximated the fixed point of such mappings using Krasnoselskii iteration. Using the technique of enriching a mapping, many authors generalized and introduced several new classes of mappings with different aspects.

In 2024, Dashputre et al. [5] introduced SJR-iteration to approximate fixed point of generalized α-nonexpansive mapping in CAT(0) spaces and established strong and Δ-convergence theorems for such mapping. In the same year, Kim [6] introduced the concept of sequentially admissible mapping and sequentially admissible perturbation with the construction of a new iteration process corresponding to sequentially admissible mappings. They proved convergence results for the Mann type iterative method using uniformly L-Lipschitzian, sequentially admissible perturbation of asymptotically demicontractive mappings. Moreover, convergence result concerning Ishikawa type iterative method using uniformly

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L-Lipschitzian, sequentially admissible perturbation of asymptotically hemicontractive mappings to a fixed point in CAT(0) spaces was also established.

Although the fixed point theory in linear spaces (for example, Banach spaces and Hilbert spaces) have been developed extensively because of the linearity and convexity of the underlying spaces, but due to the unavailability of convex structure in metric space, it seemed impossible to extend the results of Banach space into metric space. Keeping in mind this situation, Reich et al. [7] introduced hyperbolic metric space using geodesic segment and Menger convexity [8]. This class of metric space includes all normed vector spaces, Hadamard manifolds, CAT(0) space, Hilbert balls, and the cartesian product of Hilbert balls, etc. CAT(0) space is a non-linear example of hyperbolic metric space.

Fixed point theory in CAT(0) space was first studied by Kirk [9] in the year 2003, where it was proved that every nonexpansive mapping defined on a bounded closed, convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single valued and multi-valued mappings in CAT(0) space have been rapidly developed with contributions from various prominent researchers (refer to [10–13]).

Motivated by this, in this paper, we define a new class of mappings called enriched $B_{\gamma,\mu}$ mappings in CAT(0) space. Some fixed point results are derived for such mappings. We also define a new iterative algorithm in CAT(0) space using enriched $B_{\gamma,\mu}$ mappings. The Δ-convergence and strong convergence results for this iterative algorithm are developed. This convergence is demonstrated graphically with the help of a numerical example.

1 Preliminaries

For a metric space (X, d), let $x, y \in X$ with $d(x, y) = m$. A geodesic path from x to y is a mapping $c : [0, m] \to X$ such that $c(0) = x$, $c(m) = y$, which is an isometry. A geodesic segment is the image of a geodesic path. A metric space (X, d) is termed a geodesic metric space if it satisfies the property that every pair of points in X can be connected by a geodesic segment. (X, d) is called a uniquely geodesic space if there exists exactly one geodesic segment connecting every two points.

In a geodesic metric space (X, d), a geodesic triangle $\Delta(x_1, x_2, x_3)$ is formed by three points (vertices) x_1, x_2, x_3 in X and a geodesic segment connecting each pair of these vertices. For a geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d), a comparison triangle is a triangle $\bar{\Delta}(x_1, x_2, x_3){:=}\Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in $\mathbb{R}^2$ that satisfies the property $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is referred to as a Cartan, Alexandrov, and Toponogov (0) space, in short, CAT(0) space with curvature bound 0, if it satisfies the CAT(0) inequality. That is, for each geodesic triangle $\Delta(x_1, x_2, x_3)$ in X and its corresponding comparison triangle $\bar{\Delta} := \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in $\mathbb{R}^2$, the inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(x', y')$$

holds for all $x, y \in \Delta$ and $x', y' \in \bar{\Delta}$.

Similarly, for any integer k, one can define CAT(k) space by comparing it with another space. CAT(0) space is uniquely geodesic space. For $k \in [0, 1]$, the notation $(1-k)x \oplus ky$ denotes the unique point z on the geodesic segment from x to y with $d(z, x) = kd(x, y)$ and $d(z, y) = (1-k)d(x, y)$. Suppose (X, d) is a CAT(0) space and $x, y, z \in (X, d)$ with $k \in [0, 1]$. Then

$$d((1-k)x \oplus ky, z) \leq (1-k)d(x, z) + d(y, z) \quad ([14]).$$

For detailed discussion on CAT(0) space, one may refer to [15, 16].

Now we recall some basic definitions and key results.

For a bounded sequence $\{x_n\}$ in a nonempty closed convex subset C of a CAT(0) space (X, d), define a functional $r(., \{x_n\}) : X \to \mathbb{R}^+$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, \{x_n\}),\ x \in X.$$

The asymptotic radius of the sequence $\{x_n\}$ with respect to C is defined by

$$r(\{x_n\}) = \inf_{x\in C} r(x,\{x_n\}).$$

The asymptotic center of the sequence $\{x_n\}$ with respect to C is defined by

$$A(\{x_n\}) = \{y \in C : r(y,\{x_n\}) = r(\{x_n\})\}.$$

Definition 1. [17] In a CAT(0) space (X,d), a sequence $\{x_n\}$ is said to be Δ-convergent to $x \in X$, if for every subsequence $\{z_n\}$ of $\{x_n\}$ x serves as the unique asymptotic center of $\{z_n\}$. It is denoted by $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and x is referred to as the Δ-limit of $\{x_n\}$.

A bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{z_n\})$ for every subsequence $\{z_n\}$ of $\{x_n\}$. In Banach space, every bounded sequence contains a regular subsequence [18].

Lemma 1. [19] Suppose $\{x_n\}$ is a sequence in a CAT(0) space (X,d) and $\{x_n\}$ is Δ-convergent to $x \in X$. Let $y \in X$ be such that $y \neq x$. Then

$$\limsup_{n\to\infty} d(x_n,x) < \limsup_{n\to\infty} d(x_n,y).$$

In a Banach space, the above condition is known as Opial property [20].

Definition 2. [21] For a closed convex subset C of a CAT(0) space (X,d), a bounded sequence $\{x_n\}$ in C converges weakly to $q \in C$ if and only if $\Phi(q) = \inf_{x\in C}\Phi(x)$, where $\Phi(x) = \limsup_{n\to\infty} d(x_n,x)$, $x \in C$.

Note that $\{x_n\}$ converges weakly to q if and only if $A(\{x_n\}) = \{q\}$ (refer to [17]).

Nanjaras and Panyanak [14] established the following relation between Δ-convergence and weak convergence in CAT(0) space.

Lemma 2. [14] For a bounded sequence $\{x_n\}$ in a CAT(0) space (X,d), let C be a closed convex subset of X which contains $\{x_n\}$. Then

(i) $\Delta\text{-}\lim_{n\to\infty} x_n = x$ implies $\{x_n\}$ converges weakly to x.

(ii) The converse of (i) is true if $\{x_n\}$ is regular.

Lemma 3. [22] For a closed convex subset C of a CAT(0) space (X,d) and a bounded sequence $\{x_n\}$ in C, the asymptotic center of $\{x_n\}$ is in C.

Lemma 4. [14] In a CAT(0) space, every bounded sequence has a Δ-convergent subsequence.

Lemma 5. [23] For a closed convex subset C of a CAT(0) space (X,d) and a bounded sequence $\{x_n\}$ in C, the asymptotic center $A(\{x_n\})$ contains exactly one point.

Lemma 6. [24] In a complete CAT(0) space (X,d) and $x \in X$, suppose $\{t_n\}$ is a sequence in $[p,q]$ for some $p,q \in (0,1)$ and $\{u_n\}$, $\{v_n\}$ are sequences in X satisfying

$$\limsup_{n\to\infty} d(u_n,x) \le r,$$

$$\limsup_{n\to\infty} d(v_n,x) \le r$$

and

$$\lim_{n\to\infty} d(t_n v_n \oplus (1-t_n)u_n, x) = r$$

for some $r \ge 0$. Then $\lim\limits_{n\to\infty} d(u_n,v_n) = 0$.

We recall that (refer to [25]) for a nonempty subset C of a metric space (X, d), a mapping T on C is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in C.$$

T is a quasi-nonexpansive mapping if

$$d(Tx, y) \leq d(x, y) \text{ for all } x \in C \text{ and for } z \in F(T) \neq \emptyset,$$

where $F(T)$ denotes the set of all fixed points of T.

T is a Suzuki nonexpansive mapping if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in C.$$

In 2020, Berinde et al. [26] defined enriched nonexpansive mapping in Banach space as follows:

Let X be a Banach space. A mapping $T : X \to X$ is said to be an enriched nonexpansive mapping if there exists $b \in [0, \infty)$ such that

$$||b(x - y) + Tx - Ty|| \leq (b + 1)||x - y||$$

for all $x, y \in X$.

Generalizing Suzuki nonexpansive mappings, Patir et al. [3] defined the following class of $B_{\gamma,\mu}$ mappings.

Definition 3. [3] For a nonempty subset C of a Banach space X, let $\gamma \in [0, 1]$ and $\mu \in [0, \frac{1}{2}]$ satisfying $2\mu \leq \gamma$. A mapping $T : C \to C$ is a $B_{\gamma,\mu}$ mapping if

$$\gamma||x - Tx|| \leq ||x - y|| + \mu||y - Ty||$$

implies $||Tx - Ty|| \leq (1 - \gamma)||x - y|| + \mu(||x - Ty|| + ||y - Tx||)$ for all $x, y \in C$.

In metric space setting the above definition will reduce to the following.

Definition 4. Let C be a nonempty subset of a metric space (X, d) and $\gamma \in$ [0,1], $\mu \in [0, \frac{1}{2}]$ so that $2\mu \leq \gamma$. A self-mapping $T : C \to C$ is a $B_{\gamma,\mu}$ mapping if

$$\gamma d(x, Tx) \leq d(x, y) + \mu d(y, Ty)$$

implies

$$d(Tx, Ty) \leq (1 - \gamma)d(x, y) + \mu(d(x, Ty) + d(y, Tx)) \text{ for all } x, y \in C.$$

Lemma 7. [3] Let C be a nonempty subset of a Banach space X and T be a $B_{\gamma,\mu}$ mapping on C. Then T is quasi-nonexpansive.

The concept of an averaged mapping appeared in the work of Krasnoselskii [27] in the context of Hilbert space, and the term averaged was given in [28].

Definition 5. [28] Given a mapping $T : X \to X$, where X is a Banach space, the averaged mapping $T_k : X \to X$ for $k \in (0, 1]$ is defined by

$$T_k(x) = (1 - k)x + kTx \text{ for all } x \in X.$$

Lemma 8. [29] For a self-mapping T on a convex subset C of a Banach space X and for any $k \in (0, 1]$, $F(T_k) = F(T)$.

2 Main results

In this section, we define an enriched class of mappings in CAT(0) space that generalizes the class of $B_{\gamma,\mu}$ mappings. We discuss some fixed point properties of this class. Next, we introduce an iterative algorithm in CAT(0) space involving such mappings with convergence properties.

Definition 6. Let (X, d) be a CAT(0) space and C be a nonempty subset of X. Let $\gamma \in [0,1]$, $\mu \in [0, \frac{1}{2}]$ be such that $2\mu \leq \gamma$. A mapping $T : C \to C$ is said to be an enriched $B_{\gamma,\mu}$ mapping if there exists $b \in [0, \infty)$ such that for $k = \frac{1}{b+1}$,

$$\gamma d(x, (1-k)x \oplus kTx) \leq d(x, y) + \mu d(y, (1-k)y \oplus kTy)$$

implies

$$\begin{aligned} d((1-k)x \oplus kTx, (1-k)y \oplus kTy) \leq (1-\gamma)d(x, y) + \mu(d(x, (1-k)y \oplus kTy) \\ + d(y, (1-k)x \oplus kTx)) \quad \text{for all } x, y \in C. \end{aligned}$$

It can be seen that every $B_{\gamma,\mu}$ mapping is an enriched $B_{\gamma,\mu}$ mapping with $b = 0$.

For $b = 0$, $\gamma = \mu = 0$, an enriched $B_{\gamma,\mu}$ mapping reduces to nonexpansive mapping. Again for $b = 0$, $\gamma = 1/2$, $\mu = 0$, it reduces to Suzuki nonexpansive mapping.

Example 1. Consider the CAT(0) space $(\mathbb{R}, d)$ with $d(x, y) = |x-y|$ for all $x, y \in \mathbb{R}$. Then $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 1 - 2x$ for $x \in \mathbb{R}$ is an enriched $B_{\gamma,\mu}$ mapping with $b = 2, \gamma = \frac{2}{3}, \mu = 0$. But it is not a $B_{\gamma,\mu}$ mapping.

Example 2. For the CAT(0) space $(\mathbb{R}^2, d)$ with

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \text{ for } x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$$

and $C = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, define $T : C \to C$ such that

$$T(x_1, x_2) = (1 - x_1, 1 - x_2) \text{ for all } (x_1, x_2) \in C.$$

Then T is an enriched $B_{\gamma,\mu}$ mapping with $b = 1$, $\gamma = \frac{1}{2}$, $\mu = 0$.

Lemma 9. Let T be a $B_{\gamma,\mu}$ mapping on a CAT(0) space (X, d) with $F(T) \neq \emptyset$. Then T is quasi-nonexpansive.

Proof. Let $z \in F(T)$. Then

$$\gamma d(z, Tz) = 0 \leq d(z, x).$$

Again, for $x(\neq z) \in X$, by $B_{\gamma,\mu}$ condition, we have

$$\begin{aligned} d(Tx, Tz) &\leq (1-\gamma)d(x, z) + \mu(d(x, Tz) + d(z, Tx)), \\ d(Tx, z) &\leq (1-\gamma)d(x, z) + \mu(d(x, z) + d(z, Tx)), \\ d(Tx, z) &\leq \frac{(1-\gamma+\mu)}{1-\mu} d(x, z) \leq d(x, z). \end{aligned}$$

So, T is quasi-nonexpansive.

Lemma 10. If T is an enriched $B_{\gamma,\mu}$ mapping on a CAT(0) space, then for $k = \frac{1}{b+1}$, the averaged mapping T_k is a $B_{\gamma,\mu}$ mapping.

Lemma 11. For a self-mapping T on a convex subset C of a CAT(0) space (X, d) and for any $k \in (0, 1]$, $F(T_k) = F(T)$, where T_k is the averaged mapping of T.

Proof. Clearly, if $x \in F(T)$, then $x \in F(T_k)$. So, $F(T) \subseteq F(T_k)$.
Let $x \in F(T_k)$. Then $T_k x = x$. Now,

$$\begin{aligned}
&d((1-k)x \oplus kTx, x) = kd(x, Tx),\\
&d(x, x) = kd(x, Tx),\\
&d(x, Tx) = 0,\\
&x \in F(T).
\end{aligned}$$

Lemma 12. For a nonempty subset C of a CAT(0) space (X, d), let T be an enriched $B_{\gamma,\mu}$ mapping on C. If $F(T) \neq \emptyset$, then $F(T)$ is closed.

Proof. Let $p \in \overline{F(T)}$ (the closure of $F(T)$). Then there exists a sequence $\{x_n\} \subseteq F(T)$ such that $x_n \to p$. Since T is an enriched $B_{\gamma,\mu}$ mapping, so, for $k = \frac{1}{b+1}$, using Lemma 10 and Lemma 9, T_k is quasi-nonexpansive.

Now,

$$0 = \lim_{n\to\infty} d(x_n, p) \geq \lim_{n\to\infty} d(x_n, T_k p) = d(T_k p, p).$$

So, $T_k p = p$. Therefore by Lemma 11, $p \in F(T)$. Hence $F(T)$ is closed.

Lemma 13. For a nonempty subset C of a CAT(0) space (X, d), let T be an enriched $B_{\gamma,\mu}$ mapping on C with the averaged mapping T_k, for $k = \frac{1}{b+1}$. Then for all $x, y \in C$, $c \in [0, 1]$

(i) $d(T_k x, T_k^2 x) \leq d(x, T_k x)$;

(ii) at least one of the following ((a) and (b)) holds:

(a) $\frac{c}{2} d(x, T_k x) \leq d(x, y)$;

(b) $\frac{c}{2} d(T_k x, T_k^2 x) \leq d(T_k x, y)$.

The condition (a) implies $d(T_k x, T_k y) \leq (1 - \frac{c}{2}) d(x, y) + \mu(d(x, T_k y) + d(y, T_k x))$, the condition (b) implies $d(T_k^2 x, T_k y) \leq (1 - \frac{c}{2}) d(T_k x, y) + \mu(d(T_k x, T_k y) + d(y, T_k^2 x))$;

(iii) $d(x, T_k y) \leq (3 - \frac{c}{2}) d(x, T_k x) + (1 - \frac{c}{2}) d(x, y) + \mu(2d(x, T_k x) + d(x, T_k y) + d(y, T_k x) + 2d(T_k x, T_k^2 x))$.

Proof. (i) For all $x \in C$

$$\gamma d(x, T_k x) \leq d(x, T_k x) + \mu d(T_k x, T_k^2 x).$$

So, by Lemma 10,

$$\begin{aligned}
d(T_k x, T_k^2 x) &\leq (1-\gamma) d(x, T_k x) + \mu d(x, T_k^2 x)\\
&\leq (1-\gamma) d(x, T_k x) + \mu d(x, T_k^2 x) + \mu d(T_k x, T_k^2 x),
\end{aligned}$$

that is,

$$d(T_k x, T_k^2 x) \leq \frac{1 - \gamma + \mu}{1 - \mu} d(x, T_k x) \leq d(x, T_k x).$$

(ii) If possible, let $\frac{c}{2} d(x, T_k x) > d(x, y)$ and

$$\frac{c}{2} d(T_k x, T_k^2 x) > d(T_k x, y) \quad \text{for some} \quad x, y \in C.$$

Now,

$$\begin{aligned} d(x, T_k x) \leq & d(x, y) + d(y, T_k x) \\ < & \frac{c}{2} d(x, T_k x) + \frac{c}{2} d(T_k x, T_k^2 x) \\ \leq & \frac{c}{2} d(x, T_k x) + \frac{c}{2} d(x, T_k x) \text{ (using (i))} \\ \leq & d(x, T_k x), \end{aligned}$$

that is, $d(x, T_k x) < d(x, T_k x)$, which is impossible. So, at least one of (a) and (b) holds.

If (a) holds, then, $\frac{c}{2} d(x, T_k x) \leq d(x, y)$. So,

$$\frac{c}{2} d(x, T_k x) \leq d(x, y) + \mu d(y, T_k y).$$

Therefore,

$$d(T_k x, T_k y) \leq \left(1 - \frac{c}{2}\right) + \mu(d(x, T_k y) + d(y, T_k x)).$$

If (b) holds, then, $\frac{c}{2} d(T_k x, T_k^2 x) \leq d(T_k x, y)$.

That is,

$$\frac{c}{2} d(T_k^2 x, T_k x) \leq d(T_k x, y).$$

So,

$$d(T_k^2 x, T_k y) \leq \left(1 - \frac{c}{2}\right) d(T_k x, y) + \mu(d(T_k x, T_k y) + d(y, T_k^2 x)).$$

(iii) We assume that (a) holds. Then

$$d(T_k x, T_k y) \leq \left(1 - \frac{c}{2}\right) d(x, y) + \mu(d(x, T_k y) + d(y, T_k x)).$$

Now,

$$\begin{aligned} d(x, T_k y) &\leq d(x, T_k x) + d(T_k x, T_k y) \\ &\leq d(x, T_k x) + \left(1 - \frac{c}{2}\right) d(x, y) + \mu(d(x, T_k y) + d(y, T_k x)) \\ &\leq \left(3 - \frac{c}{2}\right) d(x, T_k x) + \left(1 - \frac{c}{2}\right) d(x, y) + \mu(d(x, T_k y) + d(y, T_k x) \\ &+ 2d(x, T_k^2 x)), \text{ since } c \in [0, 1] \\ &\leq \left(3 - \frac{c}{2}\right) d(x, T_k x) + \left(1 - \frac{c}{2}\right) d(x, y) + \mu(d(x, T_k y) + d(y, T_k x) \\ &+ 2d(x, T_k x) + 2d(T_k x, T_k^2 x)). \end{aligned}$$

Now, suppose (b) holds. Then

$$d(T_k^2 x, T_k y) \leq \left(1 - \frac{c}{2}\right) d(T_k x, y) + \mu(d(T_k x, T_k y) + d(y, T_k^2 x)).$$

By triangle inequality,

$$
\begin{aligned}
d(x,T_ky) &\le d(x,T_kx)+d(T_k^2x,T_kx)+d(T_k^2x,T_ky)\\
&\le d(x,T_kx)+d(x,T_kx)+\left(1-\frac{c}{2}\right)d(T_kx,y)+\mu(d(T_kx,T_ky)+d(y,T_k^2x))\\
&\le 2d(x,T_kx)+\left(1-\frac{c}{2}\right)d(T_kx,x)+\left(1-\frac{c}{2}\right)d(x,y)+\mu(d(T_kx,T_ky)+d(y,T_k^2x))\\
&\le \left(3-\frac{c}{2}\right)d(x,T_kx)+\left(1-\frac{c}{2}\right)d(x,y)+\mu(d(x,T_k^2x)+d(T_kx,T_ky)+d(y,T_k^2x))\\
&\le \left(3-\frac{c}{2}\right)d(x,T_kx)+\left(1-\frac{c}{2}\right)d(x,y)+\mu(d(x,T_kx)+d(T_kx,T_k^2x)+d(T_kx,x)\\
&+d(x,T_ky)+d(y,T_kx)+d(T_kx,T_k^2x))\\
&=\left(3-\frac{c}{2}\right)d(x,T_kx)+\left(1-\frac{c}{2}\right)d(x,y)+\mu(2d(x,T_kx)+d(x,T_ky)\\
&+d(y,T_kx)+2d(T_kx,T_k^2x)).
\end{aligned}
$$

Next, we derive the following fixed point result using Δ-convergence.

Theorem 1. For a nonempty subset C of a CAT(0) space (X,d), let T be an enriched $B_{\gamma,\mu}$ mapping on C with the averaged mapping T_k, for $k=\frac{1}{b+1}$. Suppose $\{x_n\}$ is a sequence in C such that

(i) $\{x_n\}$ is Δ-convergent to z,
(ii) $\lim_{n\to\infty} d(T_kx_n,x_n)=0$, and
(iii) $d(z,T_kx_n)\le d(z,x_n)$.
Then $z\in F(T)$.

Proof. Since T is enriched $B_{\gamma,\mu}$ mapping, so, by Lemma 13 (iii), for $c\in[0,1]$,

$$
\begin{aligned}
d(x_n,T_kz) \le& \left(3-\frac{c}{2}\right)d(x_n,T_kx_n)+\left(1-\frac{c}{2}\right)d(x_n,z)+\mu(2d(x_n,T_kx_n)+d(x_n,T_kz)\\
&+d(z,T_kx_n)+2d(T_kx_n,T_k^2x_n)).
\end{aligned}
$$

Using condition (iii) and Lemma 13(i),

$$
\begin{aligned}
d(x_n,T_kz) \le& \left(3-\frac{c}{2}\right)d(x_n,T_kx_n)+\left(1-\frac{c}{2}\right)d(x_n,z)+\mu(2d(x_n,T_kx_n)+d(x_n,T_kz)\\
&+d(z,x_n)+2d(x_n,T_kx_n)).
\end{aligned}
\tag{1}
$$

Taking $n\to\infty$ on both sides of (1) and using conditions (ii) and (iii), we get

$$(1-\mu)\lim_{n\to\infty} d(x_n,T_kz)\le\left(1-\frac{c}{2}+\mu\right)\lim_{n\to\infty} d(x_n,z),$$

that is,

$$\lim_{n\to\infty} d(x_n,T_kz)\le\lim_{n\to\infty} d(x_n,z),$$
$$\limsup_{n\to\infty} d(x_n,T_kz)\le\limsup_{n\to\infty} d(x_n,z).$$

Since $\{x_n\}$ is Δ-convergent to z, so, if $T_kz\ne z$, then by Lemma 1,

$$\limsup_{n\to\infty} d(x_n,z)<\limsup_{n\to\infty} d(x_n,T_kz)\le\limsup_{n\to\infty} d(x_n,z),\quad \text{which is a contradiction.}$$

Hence $T_kz=z$.

Since $k\in(0,1]$, by Lemma 11, $z\in F(T)$.

Theorem 2. Let (X, d) be a CAT(0) space and C be a nonempty closed convex and bounded subset of X. Let T be an enriched $B_{\gamma,\mu}$ mapping on C with the averaged mapping T_k for $k = \frac{1}{b+1}$. Suppose $\{x_n\}$ is a bounded sequence in C that satisfies
(i) $\lim_{n\to\infty} d(x_n, T_k x_n) = 0$ and
(ii) $\lim_{n\to\infty} d(T_k x_n, T_k^2 x_n) = 0$.
Then $F(T) \neq \emptyset$.

Proof. Since $A(\{x_n\})$ contains exactly one point, let $z \in A(\{x_n\})$. Then by Lemma 3, $z \in C$.
By Lemma 13 (iii), for each $n \in \mathbb{N} \cup \{0\}$ and $c \in [0, 1]$, we have

$$\begin{aligned} d(x_n, T_k z) \leq \left(3 - \frac{c}{2}\right) d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) d(x_n, z) + \mu(2d(x_n, T_k x_n) + d(x_n, T_k z) \\ + d(z, T_k x_n) + 2d(T_k x_n, T_k^2 x_n)), \end{aligned}$$

that is,

$$d(x_n, T_k z) \leq \left(3 - \frac{c}{2}\right) d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) d(x_n, z) + \mu(2d(x_n, T_k x_n) + d(x_n, T_k z)+$$

$$d(z, x_n) + d(x_n T_k x_n) + 2d(T_k x_n, T_k^2 x_n)).$$

So,

$$(1 - \mu) \limsup_{n\to\infty} d(x_n, T_k z) \leq \left(1 - \frac{c}{2} + \mu\right) \limsup_{n\to\infty} d(x_n, z).$$

Therefore,

$$r(T_k z, \{x_n\}) \leq r(z, \{x_n\}).$$

Hence

$$T_k z \in A(\{x_n\}).$$

By uniqueness of asymptotic centers in CAT(0) space, we have $T_k z = z$. So, by Lemma 11, z is a fixed point of T.

3 *A new iterative algorithm in CAT(0) space*

In this section, we develop the following iterative scheme for approximating fixed points of enriched $B_{\gamma,\mu}$ mapping in CAT(0) space.

Let C be a nonempty subset of a CAT(0) space and $x_0 \in C$. Let T be an enriched $B_{\gamma,\mu}$ mapping on C with the averaged mapping T_k for $k = \frac{1}{b+1}$.

For $n \in \mathbb{N} \cup \{0\}$, we define

$$\begin{aligned} x_{n+1} &= (1 - \beta_n) y_n \oplus \beta_n T_k y_n, \\ y_n &= T_k z_n, \\ z_n &= (1 - \alpha_n) x_n \oplus \alpha_n T_k x_n, \end{aligned} \tag{2}$$

where $\alpha_n, \beta_n \in [0, 1]$.

Lemma 14. Let T be an enriched $B_{\gamma,\mu}$ mapping on a nonempty closed and convex subset C of a CAT(0) space (X, d). For $x_0 \in C$, let $\{x_n\}$ be a sequence defined by (2). If $F(T) \neq \emptyset$, then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.

Proof. For $p \in F(T)$ and $k = \frac{1}{b+1}$,

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\beta_n)y_n \oplus \beta_n T_k y_n, p) \\ &\leq (1-\beta_n)d(y_n, p) + \beta_n d(T_k y_n, p). \end{aligned}$$

Since T_k is quasi-nonexpansive,

$$\begin{aligned} d(x_{n+1}, p) &\leq (1-\beta_n)d(y_n, p) + \beta_n d(y_n, p) \\ &\leq d(y_n, p) \\ &= d(T_k z_n, p) \\ &\leq d(z_n, p) \\ &= d((1-\alpha_n)x_n \oplus \alpha_n T_k x_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(T_k x_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Thus, $\{d(x_n, p)\}$ is a monotonically decreasing sequence that is bounded below.

Hence, $\lim_{n\to\infty} d(z_n, p)$ exists for all $p \in F(T)$.

Theorem 3. Let C be a nonempty closed and convex subset of a complete CAT(0) space (X, d) and T be an enriched $B_{\gamma,\mu}$ mapping on C. For $x_0 \in C$, let $\{x_n\}$ be a sequence defined by (2). Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, T_k x_n) = 0$, where T_k is the averaged mapping with $k = \frac{1}{b+1}$.

Proof. Let $F(T) \neq \emptyset$ and $p \in F(T)$. So by Lemma 14, $\lim_{n\to\infty} d(x_n, p)$ exists and thus, $\{x_n\}$ is bounded.

Let $\lim_{n\to\infty} d(x_n, p) = a \geq 0$. For $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} d(x_n, T_k x_n) &\leq d(x_n, p) + d(T_k x_n, p) \\ &\leq 2d(x_n, p). \end{aligned}$$

So, $\lim_{n\to\infty} d(x_n, T_k x_n)$ exists.

Now,

$$d(x_{n+1}, p) \leq d(y_n, p) \leq d(z_n, p) \leq d(x_n, p),$$

that is,

$$a = \limsup_{n\to\infty} d(x_{n+1}, p) \leq \limsup_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(z_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = a.$$

Also,

$$a = \liminf_{n\to\infty} d(x_{n+1}, p) \leq \liminf_{n\to\infty} d(y_n, p) \leq \liminf_{n\to\infty} d(z_n, p) \leq \liminf_{n\to\infty} d(x_n, p) = a.$$

Hence

$$\limsup_{n\to\infty} d(z_n, p) = \liminf_{n\to\infty} d(z_n, p) = a.$$

So,

$$\lim_{n\to\infty} d(z_n, p) = a,$$

that is,

$$\lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T_k x_n, p) = a.$$

Also, we have

$$\limsup_{n\to\infty} d(x_n, p) \le a \quad \text{and} \quad \limsup_{n\to\infty} d(T_k x_n, p) \le a.$$

So, by Lemma 6, $\lim_{n\to\infty} d(x_n, T_k x_n) = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, T_k x_n) = 0$. Since $A(\{x_n\})$ contains exactly one point, let $p \in A(\{x_n\})$. By Lemma 13 (iii), for $c \in [0,1]$, we get

$$\begin{aligned} d(x_n, T_k p) &\le \left(3 - \frac{c}{2}\right) d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) d(x_n, p) + \mu(2d(x_n, T_k x_n) + d(x_n, T_k p) \\ &+ d(p, T_k x_n) + 2d(T_k x_n, T_k^2 x_n)) \\ &\le \left(3 - \frac{c}{2}\right) d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) d(x_n, p) + \mu(2d(x_n, T_k x_n) + d(x_n, T_k p) \\ &+ d(p, T_k x_n) + 2d(x_n, T_k x_n)). \end{aligned}$$

So,

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, T_k p) &\le \left(3 - \frac{c}{2}\right) \limsup_{n\to\infty} d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) \limsup_{n\to\infty} d(x_n, p) \\ &+ \mu(2 \limsup_{n\to\infty} d(x_n, T_k x_n) + \limsup_{n\to\infty} d(x_n, T_k p) + \limsup_{n\to\infty} d(p, T_k x_n) \\ &+ 2 \limsup_{n\to\infty} d(T_k x_n, T_k^2 x_n)) \\ &\le \left(3 - \frac{c}{2}\right) \limsup_{n\to\infty} d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) \limsup_{n\to\infty} d(x_n, p) \\ &+ \mu(2 \limsup_{n\to\infty} d(x_n, T_k x_n) + \limsup_{n\to\infty} d(x_n, T_k p) \\ &+ \limsup_{n\to\infty} d(p, T_k x_n) + 2 \limsup_{n\to\infty} d(x_n, T_k x_n)), \end{aligned}$$

that is,

$$(1-\mu) \limsup_{n\to\infty} d(x_n, T_k p) \le \left(1 - \frac{c}{2} + \mu\right) \limsup_{n\to\infty} d(x_n, p).$$

Since $2\mu \le \gamma$, taking $c = 2\gamma$, we get

$$\limsup_{n\to\infty} d(x_n, T_k p) \le \limsup_{n\to\infty} d(x_n, p).$$

Therefore,

$$r(T_k p, \{x_n\}) \le r(p, \{x_n\}).$$

Hence

$$T_k p \in A(\{x_n\}).$$

By Lemma 5, $T_k p = p$, that is, $p \in F(T_k)$ and by Lemma 11, $p \in F(T)$. Hence, $F(T) \ne \emptyset$.

In view of the above theorem, we can say that the sequence $\{x_n\}$ defined by (2), is Δ-convergent to a fixed point of T.

The next result deals with the strong convergence of the iterative algorithm (2) to a fixed point.

Theorem 4. Let C be a nonempty closed and convex subset of a complete CAT(0) space (X,d). Let T be an enriched $B_{\gamma,\mu}$ mapping on X and for $x_0 \in C$, $\{x_n\}$ be a sequence defined by (2). Let $F(T) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x,F(T)) = \inf\{d(x,p); p \in F(T)\}$.

Proof. Let $\{x_n\}$ be convergent to $p \in F(T)$. Then $\lim_{n\to\infty} d(x_n,p) = 0$. Now,

$$0 \leq d(x_n, F(T)) = inf\{d(x_n,p) : p \in F(T)\} \leq d(x_n,p).$$

Therefore, $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Hence

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Conversely, let

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

By Lemma 14, $\lim_{n\to\infty} d(x_n,p)$ exists for all $p \in F(T)$ and $\{d(x_n,p)\}$ is monotonically decreasing. Thus,

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, p_k) < \frac{1}{2^k} \text{ for all } k \in \mathbb{N} \text{ and for } \{p_k\} \subseteq F(T).$$

Since $\{d(x_n,p)\}$, $p \in F(T)$ is monotonically decreasing, so, for each k,

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

Now,

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

Hence $\{p_k\}$ is a Cauchy sequence in $F(T)$. Since $F(T)$ is closed, $\{p_k\}$ converges to some $p \in F(T)$.

Now,

$$d(x_{n_k}, p) \leq d(x_{n_k}, p_k) + d(p_k, p).$$

Taking $k \to \infty$, we get

$$\lim_{n\to\infty} d(x_{n_k}, p) = 0.$$

Since, $\lim_{n\to\infty} d(x_n,p)$ exists, so, $\lim_{n\to\infty} d(x_n,p) = 0$.

Hence $\{x_n\}$ converges to $p \in F(T)$.

We recall from [30] that a mapping T on a nonempty convex subset C of a CAT(0) space (X,d) with $F(T) \neq \emptyset$ satisfies the condition (I) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $f(d(x,F(T))) \leq d(x,Tx)$ for all $x \in C$. Here we use this condition (I) to prove the next strong convergence result.

Theorem 5. For a nonempty closed and convex subset C of a complete CAT(0) space (X, d), let T be an enriched $B_{\gamma,\mu}$ mapping on C with the averaged mapping T_k for $k = \frac{1}{b+1}$. Let $F(T) \neq \emptyset$ and for $x_0 \in C$, $\{x_n\}$ be a sequence defined by (2). If T_k satisfies condition (I) for a self-mapping f on $[0, \infty)$, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since $F(T) \neq \emptyset$, by Theorem 3, $\lim_{n\to\infty} d(x_n, T_k x_n) = 0$.
Also, by condition (I),

$$\lim_{n\to\infty} f(d(x_n, F(T_k))) \leq \lim_{n\to\infty} d(x_n, T_k x_n) = 0.$$

So,

$$\lim_{n\to\infty} f(d(x_n, F(T_k))) = 0.$$

Therefore,

$$\lim_{n\to\infty} (d(x_n, F(T_k))) = 0.$$

As $k \in (0, 1]$, so,

$$\lim_{n\to\infty} (d(x_n, F(T))) = 0.$$

Hence $\{x_n\}$ converges strongly to a fixed point of T.

We demonstrate the above theorem by the following example.

Example 3. Consider the CAT(0) space $X = [0, 1]$ with

$$d(x, y) = \begin{cases} x + y, & \text{if } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$$

Let $C = [0, \frac{1}{3}]$ and $T : C \to C$ be defined by $T(x) = 1 - 2x$, $x \in C$. Then T is an enriched $B_{\gamma,\mu}$ mapping with $b = 1$, $\gamma = \frac{2}{3}$, $\mu = 0$.

For $k = \frac{1}{2}$, the averaged mapping T_k is given by $T_k(x) = \frac{1-x}{2}$, $x \in C$.

Clearly, $F(T_k) = F(T) = \{\frac{1}{3}\} \neq \emptyset$.

We take f as the identity mapping on $[0, \infty)$. Then T_k satisfies condition (I) with respect to f.

So, by Theorem 5, for $x_0 \in C$, the sequence $\{x_n\}$ defined by (2) converges strongly to a fixed point of T_k.

Taking $x_0 = 0.05$, $x_0 = 0.1$, $x_0 = 0.2$ and $x_0 = 0.25$, we see the convergence of the iterative scheme as follows:

n	$x_0 = 0.05$	$x_0 = 0.1$	$x_0 = 0.2$	$x_0 = 0.25$
1	0.342188	0.340625	0.337500	0.335938
2	0.333057	0.333105	0.333203	0.333252
3	0.333342	0.333340	0.333337	0.333336
4	0.333333	0.333333	0.333333	0.333333
5	0.333333	0.333333	0.333333	0.333333
6	0.333333	0.333333	0.333333	0.333333

In Figure 1, the blue, purple, red, and green dotted lines represent the sequences defined by the iterative algorithm (2), when $x_0 = 0.05$, $x_0 = 0.1$, $x_0 = 0.2$ and $x_0 = 0.25$ respectively. It is seen that each sequence converges to the fixed point $\frac{1}{3}$.

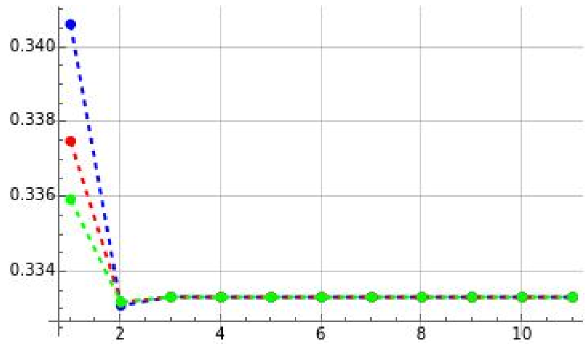

Figure 1. Convergence of the iteration scheme (2) with different initial points

Conclusion

We have established some fixed point results for enriched $B_{\gamma,\mu}$ mapping in CAT(0) space. Also, we introduced a new iteration scheme for such mappings in CAT(0) space and proved weak and strong convergence results of this iteration scheme. In 2022, Tufa et al. [31] constructed an iterative scheme to approximate the fixed point of a countable family of quasi-nonexpansive non-self-mapping in complete CAT(0) space. In this context, the investigation of common fixed points for a countable family of enriched $B_{\gamma,\mu}$ mappings is a scope of future study. Moreover, the comparison of the rate of convergence of our derived iteration scheme with some existing iteration schemes is another aspect for future discussion.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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A fractal-fractional gingerbread-man map generalized by $p-$fractal-fractional difference operator

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By using the generalization of the gamma function ($p-$gamma function: $\Gamma_p(.)$), we introduce a generalization of the fractal-fractional calculus which is called $p-$fractal-fractional calculus. Examples are illustrated including the basic power functions. As applications, we formulate the $p-$fractal-fractional difference operators. A class of maps, called gingerbread-man maps, is investigated. We present a new idea of a stability for continuous system, based on three parameters. Sufficient conditions are illustrated to obtain the stability of the system.

Keywords: fractional calculus, fractal calculus, fractional difference operator, fractal-fractional differential operator, fractal-fractional calculus, fractal-fractional discrete operator.

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Introduction

Combining the concepts of fractal and fractional operators in the notion of fractal-fractional operators [1], which are operators applied to functions defined on fractal sets and utilizing fractional calculus. Fractal-fractional operators are an interdisciplinary field spanning science, mathematics, and electronics [2, 3]. They have been applied in several domains, including as data processing, image evaluation, and the mathematical modeling of intricate systems including non-local and self-similar behavior [4, 5].

The normal gamma function is extended in the generalized gamma function (see [6]), which can be expressed for positive real numbers. It has several uses in the fields of mathematics, physics, engineering, and statistics (see [7–10]), when p is a positive integer. The fact that both the scale parameter and the form parameter are included makes its properties more complicated than those of the standard one. It is a key tool in many mathematical and scientific contexts, especially when dealing with circumstances with intricate and diverse data distributions [11, 12].

We offer a generalization of the fractal-fractional calculus utilizing the generalization of the gamma function called $p-$gamma function. The discrete $p-$fractal-fractional operators are also developed. We demonstrate that well-known examples are included in the generalized operators. The paper is organized as follows: Section 2 deals with the methods and observations (our main results). Section 3 provides the conclusion of this analysis with suggestions, as future works.

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1 *Presentation and Notices*

This section deals with all concepts that will be used in the sequel.

Definition 1. The $p-$gamma function can be generalized as follows [6] (see Table 1 for some values of $p-$gamma function)

$$\Gamma_p(\zeta) = \lim_{m\to\infty} \frac{m!p^m (m\,p)^{\frac{\zeta}{p}-1}}{(\zeta)_{m,p}}, \quad p > 0,$$

where $(\zeta)_{m,p} := \zeta(\zeta+p)(\zeta+2p)\dots(\zeta+(m-1)p)$ and $(\zeta)_{m,p} = \dfrac{\Gamma_p(\zeta+m\,p)}{\Gamma_p(\zeta)}$. Moreover, $\Gamma(\zeta) = \lim\limits_{p\to1} \Gamma_p(\zeta)$, $\Gamma_p(\zeta) = p^{\zeta/p-1}\Gamma(\zeta/p)$, $\Gamma_p(\zeta+p) = \zeta\Gamma_p(\zeta)$, and $\Gamma_p(p) = 1$.

Table 1

Exact and numerical solution of example 1

ζ	$\Gamma_1(\zeta)$	$\Gamma_2(\zeta)$	$\Gamma_3(\zeta)$	$\Gamma_4(\zeta)$	$\Gamma_5(\zeta)$
1	1.000	1.253	1.288	1.282	1.267
2	1.000	1.000	0.939	0.886	0.845
3	2.000	1.253	1.000	0.867	0.782
4	6.000	2.000	1.288	1.000	0.844
5	24.000	3.760	1.878	1.282	1.000

Definition 2. [1] Suppose that $\varphi(\chi)$ is a differentiable over $(0, b)$, then the Caputo fractal-fractional derivative is given as follows ($\mu, \nu \in (n-1, n]$):

$$^{\mathcal{C}}\Delta_\chi^{\mu,\nu}\varphi(\chi) := \frac{1}{\Gamma(n-\mu)} \int_0^\chi \frac{d\varphi(t)}{dt^\nu}(\chi - t)^{n-\mu-1}dt.$$

And for a continuous function $\varphi(\chi)$ and fractal differentiable over $(0, b)$, the Riemann-Liouville fractal-fractional differential operator is given by the formula ($\mu, \nu \in (n-1, n]$):

$$^{\mathcal{RL}}\Delta_\chi^{\mu,\nu}\varphi(\chi) := \frac{1}{\Gamma(n-\mu)} \frac{d}{d\chi^\nu} \int_0^\chi \varphi(t)(\chi - t)^{n-\mu-1}dt,$$

where

$$\frac{d\varphi(\chi)}{d\chi^\nu} = \lim_{\chi\to t} \frac{\varphi(\chi) - \varphi(t)}{\chi^\nu - t^\nu}.$$

More generalization is formulated for the above operators, as follows ($\mu, \nu, \gamma \in (n-1, n]$):

$$^{\mathcal{C}}\Delta_\chi^{\mu,\nu,\gamma}\varphi(\chi) := \frac{1}{\Gamma(n-\mu)} \int_0^\chi \frac{d^\gamma\varphi(t)}{dt^\nu}(\chi - t)^{n-\mu-1}dt.$$

And

$$^{\mathcal{RL}}\Delta_\chi^{\mu,\nu,\gamma}\varphi(\chi) := \frac{\frac{d^\gamma}{d\chi^\nu}}{\Gamma(n-\mu)} \int_0^\chi \varphi(t)(\chi - t)^{n-\mu-1}dt,$$

where

$$\frac{d^\gamma\varphi(\chi)}{d\chi^\nu} = \lim_{\chi\to t} \frac{\varphi^\gamma(\chi) - \varphi^\gamma(t)}{\chi^\nu - t^\nu}.$$

Correspondingly, the fractal-fractional integral operator of order $\mu, \nu > 0$ is formulated by the structure:

$$Y_\chi^{\mu,\nu}\varphi(\chi) := \frac{\nu}{\Gamma(\mu)} \int_0^\chi t^{\mu-1}\varphi(t)(\chi - \tau)^{\mu-1}\mathrm{d}t, \quad \mu, \nu > 0.$$

Combining the definition of $p-$gamma function and the fractal-fractional operators to get the generalized fractal-fractional operators, as follows:

Definition 3. Suppose that $\varphi(\chi)$ is a differentiable over the open interval $(0,b)$. Then the Caputo $p-$fractal-fractional derivative is given as follows ($\mu,\nu \in (n-1,n]$):

$$ {}_p^{\mathcal{C}}\Delta_\chi^{\mu,\nu}\varphi(\chi) := \frac{1}{p\Gamma_p(n-\mu)}\int_0^\chi \frac{d\varphi(t)}{dt^{\nu/p}}(\chi-t)^{n-\mu/p-1}dt. $$

And for a continuous function $\varphi(t)$ and fractal differentiable over $(0,b)$, the Riemann-Liouville $p-$fractal-fractional differential operator is given by the formula ($\mu,\nu \in (n-1,n]$):

$$ {}_p^{\mathcal{RL}}\Delta_\chi^{\mu,\nu}\varphi(\chi) := \frac{\frac{d}{d\chi^{\nu/p}}}{p\Gamma_p(n-\mu)}\int_0^\chi \varphi(t)(\chi-t)^{n-\mu/p-1}dt, $$

where

$$ \frac{d\varphi(\chi)}{d\chi^{\nu/p}} = \lim_{\chi\to t}\frac{\varphi(\chi)-\varphi(t)}{\chi^{\nu/p}-t^{\nu/p}}. $$

More generalization is considered for the above operators, as follows ($\mu,\nu \in (n-1,n]$):

$$ {}_p^{\mathcal{C}}\Delta_\chi^{\mu,\nu,\gamma}\varphi(\chi) := \frac{1}{p\Gamma_p(n-\mu)}\int_0^\chi \frac{d^{\gamma/p}\varphi(t)}{dt^{\nu/p}}(\chi-t)^{n-\mu/p-1}dt. $$

And

$$ {}_p^{\mathcal{RL}}\Delta_\chi^{\mu,\nu,\gamma}\varphi(\chi) := \frac{\frac{d^{\gamma/p}}{d\chi^{\nu/p}}}{p\Gamma_p(n-\mu)}\int_0^\chi \varphi(t)(\chi-t)^{n-\mu/p-1}dt, $$

where

$$ \frac{d^{\gamma/p}\varphi(\chi)}{d\chi^{\nu/p}} = \lim_{\chi\to t}\frac{\varphi^{\gamma/p}(\chi)-\varphi^{\gamma/p}(t)}{\chi^{\nu/p}-t^{\nu/p}}. $$

Correspondingly, the $p-$fractal-fractional integral operator of order $\mu,\nu > 0$ is formulated by the structure:

$$ {}_pY_\chi^{\mu,\nu}\varphi(\chi) := \frac{\nu}{p\Gamma_p(\mu)}\int_0^\chi t^{\mu/p-1}\varphi(t)(\chi-\tau)^{\mu/p-1}dt. $$

Example 1. Now for the generalized $p-$fractal-fractional operators, we have

$$ \frac{d\varphi(\chi)}{d\chi^\nu} = \lim_{\chi\to t}\frac{\chi^m-t^m}{\chi^{\nu/p}-t^{\nu/p}} = \frac{mpt^{m-\nu/p}}{\nu} $$

then ($\mu,\nu \in (0,1]$)

$$ \begin{aligned} {}_p^{\mathcal{C}}\Delta_\chi^{\mu,\nu}\chi^m &= \frac{mp}{\nu p\Gamma_p(1-\mu)}\int_0^\chi \left(t^{m-\nu/p}\right)(\chi-t)^{-\mu/p}dt \\ &= \frac{m}{\nu\Gamma_p(1-\mu)}\frac{\Gamma\left(1-\frac{\mu}{p}\right)\Gamma\left(m-\frac{\nu}{p}+1\right)\chi^{\frac{mp+p-\mu-\nu}{p}}}{\Gamma\left(m-\frac{-2p+\mu+\nu}{p}\right)}, \end{aligned} $$

$$ \begin{aligned} {}_p^{\mathcal{RL}}\Delta_\chi^{\mu,\nu}(\chi^m) &= \frac{1}{p\Gamma_p(1-\mu)}\frac{d}{d\chi^{\nu/p}}\int_0^\chi (t^m)(\chi-t)^{-\mu/p}dt \\ &= \frac{1}{p\Gamma_p(1-\mu)}\frac{d}{d\chi^{\nu/p}}\left(\frac{\Gamma(m+1)\Gamma\left(1-\frac{\mu}{p}\right)\chi^{m-\frac{\mu}{p}+1}}{\Gamma\left(m-\frac{\mu}{p}+2\right)}\right) \\ &= \frac{1}{p\Gamma_p(1-\mu)}\left(\frac{\Gamma(m+1)\Gamma(1-\mu/p)}{\Gamma(m-\mu/p+2)}\right)\left(\frac{d}{d\chi^{\nu/p}}\chi^{m-\mu/p+1}\right). \end{aligned} $$

But

$$\frac{d}{d\chi^{\nu/p}}\chi^{m-\mu/p+1} = \frac{(-\mu+mp+p)}{\nu}\chi^{(-\mu+mp+p-\nu)/p}$$

thus, we have

$${}_{p}^{\mathcal{RL}}\Delta_{\chi}^{\mu,\nu}(\chi^m) = \frac{1}{p\Gamma_p(1-\mu)}\left(\frac{\Gamma(m+1)\Gamma(1-\mu/p)}{\Gamma(m-\mu/p+2)}\right)\frac{(-\mu+mp+p)\chi^{(-\mu+mp+p-\nu)/p}}{\nu}.$$

The $p-$fractal-fractional integral implies that

$$\begin{aligned}
{}_{p}Y_{\chi}^{\mu,\nu}\chi^m &= \frac{\nu\int_0^\chi t^{\mu/p-1}t^m(\chi-t)^{\mu/p-1}dt}{p\Gamma_p(\mu)} \\
&= \frac{\nu}{p\Gamma_p(\mu)}\frac{\Gamma\left(\frac{\mu}{p}\right)\chi^{m+\frac{2\mu}{p}-1}\Gamma\left(m+\frac{\mu}{p}\right)}{\Gamma\left(m+\frac{2\mu}{p}\right)} \\
&= \frac{\nu\chi^{m+\frac{2\mu}{p}-1}\Gamma\left(m+\frac{\mu}{p}\right)}{p^{\frac{\mu}{p}-1}\Gamma\left(m+\frac{2\mu}{p}\right)}.
\end{aligned}$$

$$\frac{d^{\gamma/p}\varphi(\chi)}{d\chi^{\nu/p}} = \lim_{\chi\to t}\frac{\chi^{(\gamma/p)m}-t^{(\gamma/p)m}}{\chi^{\nu/p}-t^{\nu/p}} = \frac{\gamma m t^{(\gamma m-\nu)/p}}{\nu},$$

then we have

$$\begin{aligned}
{}_{p}^{\mathcal{C}}\Delta_{\chi}^{\mu,\nu,\gamma}(\chi^m) &= \frac{\gamma m\int_0^\chi \left(t^{(\gamma m-\nu)/p}\right)(\chi-t)^{-\mu/p}dt}{\nu p^{(1-\mu)/p}\Gamma\left(\dfrac{1-\mu}{p}\right)} \\
&= \frac{\gamma m\chi^{\frac{\gamma m-\mu+p-\nu}{p}}}{\nu\, p^{(1-\mu)/p}}\left(\frac{\Gamma(1-\dfrac{\mu}{p})\Gamma\left(\frac{\gamma m+p-\nu}{p}\right)}{\Gamma\left(\dfrac{1-\mu}{p}\right)\Gamma\left(-\frac{-\gamma m-2p+\nu+\mu}{p}\right)}\right).
\end{aligned}$$

$$\begin{aligned}
{}_{p}^{\mathcal{RL}}\Delta_{\chi}^{\mu,\nu,\gamma}(\chi^m) &= \frac{1}{p^{(1-\mu)/p}\Gamma\left(\dfrac{1-\mu}{p}\right)}\frac{d^{\gamma/p}}{d\chi^{\nu/p}}\int_0^\chi (t^m)(\chi-t)^{-\mu/p}dt \\
&= \frac{\gamma\,(p+mp-\mu)\,\chi^{-(p(\nu-(1+m)\gamma)+\gamma\mu)/p^2}}{\nu p^{(1-\mu)/p+1}}\left(\frac{\Gamma(m+1)\Gamma(1-\mu/p)}{\Gamma\left(\dfrac{1-\mu}{p}\right)\Gamma(m-\mu/p+2)}\right).
\end{aligned}$$

1.1 p−fractal-fractional differences operators

In light of the fact that the forward and backward difference operators are characterized as follows:

$$\curlywedge g(\chi) = g(\chi+1)-g(\chi), \quad \curlyvee g(\chi) = g(\chi)-g(\chi-1)$$

satisfying the iteration $\curlywedge^k = \curlywedge(\curlywedge^{k-1})$ and $\curlyvee^k = \curlyvee(\curlyvee^{k-1})$. And for the fractional order $\curlywedge^\mu = \curlywedge^n(\curlywedge^{-n+\mu})$ and $\curlyvee^\mu = (-1)^n \curlyvee^n (\curlyvee^{-n+\mu})$, where $n = [\mu]+1$. In [13], the Caputo fractional difference is defined as follows:

$$\begin{aligned} {}^C \curlywedge^\mu g(\chi) &= \curlywedge^{-(n-\mu)} \curlywedge^n g(\chi) \\ &= \frac{1}{\Gamma(n-\mu)} \sum_{k=0}^{\chi-(n-\mu)} (\chi-k-1)^{n-\mu-1} \curlywedge_k^n g(\chi), \end{aligned}$$

where the factor χ^μ is defined by

$$\chi^\mu = \frac{\Gamma(\chi+1)}{\Gamma(\chi+1-\mu)}, \quad \mu > 0;$$

and

$$\begin{aligned} {}^C \curlyvee^\mu g(\chi) &= \curlyvee^{-(n-\mu)} \curlyvee^n g(\chi) \\ &= \frac{1}{\Gamma(n-\mu)} \sum_{k=\chi+(n-\mu)}^{b} (k-1-\chi)^{n-\mu-1} \curlyvee_k^n g(\chi), \end{aligned}$$

correspondingly, the fractional integral difference operators are as follows:

$$\curlywedge^{-\mu} g(\chi) = \frac{1}{\Gamma(\mu)} \sum_{k=0}^{\chi-\mu} (\chi-k-1)^{\mu-1} \curlywedge_k^n g(\chi),$$

$$\curlyvee^{-\mu} g(\chi) = \frac{1}{\Gamma(\mu)} \sum_{k=\chi+\mu}^{b} (k-1-\chi)^{\mu-1} \curlyvee_k^n g(\chi).$$

We have the next generalized process of fractal-fractional difference formula.

Definition 4. The Caputo fractal-fractional difference operators are defined as follows:

$${}^C \curlywedge^{\mu,\nu} g(\chi) = \frac{1}{\Gamma(n-\mu)} \sum_{k=0}^{\chi-(n-\mu)} (\chi^\nu - (k+1)^\nu)^{n-\mu-1} \curlywedge_k^n g(\chi),$$

where the factor $(\chi^\nu)^\mu$ is defined by

$$(\chi^\nu)^\mu = \frac{\Gamma(\chi^\nu+1)}{\Gamma(\chi^\nu+1-\mu)}, \quad \mu,\nu > 0;$$

and

$${}^C \curlyvee^{\mu,\nu} g(\chi) = \frac{1}{\Gamma(n-\mu)} \sum_{k=\chi+(n-\mu)}^{b} ((k-1)^\nu - \chi^\nu)^{n-\mu-1} \curlyvee_k^n g(\chi),$$

correspondingly, the fractal-fractional integral difference operators are given by the formulas

$$\curlywedge^{-\mu,\nu} g(\chi) = \frac{1}{\Gamma(\mu)} \sum_{k=0}^{\chi-\mu} (\chi^\nu - (k+1)^\nu)^{\mu-1} \curlyvee_k^n g(\chi),$$

$$\curlyvee^{-\mu,\nu} g(\chi) = \frac{1}{\Gamma(\mu)} \sum_{k=\chi+\mu}^{b} ((k-1)^\nu - \chi^\nu)^{\mu-1} \curlyvee_k^n g(\chi).$$

More generalization is given by using $p-$fractal-fractional formula.

Definition 5. The Caputo $p-$fractal-fractional difference operators are defined as follows:

$$ {}^{C}_{p} \curlywedge^{\mu,\nu} g(\chi) = \frac{1}{p\Gamma_p(n-\mu)} \sum_{k=0}^{\chi-(n-\mu)} (\chi^\nu - (k+1)^\nu)^{n-\mu/p-1} \curlywedge_k^n g(\chi), $$

where the factor $\chi^{\mu/p}$ is defined by

$$ (\chi^\nu)^{\mu/p} = \frac{\Gamma_p(\chi^\nu+1)}{\Gamma_p(\chi^\nu+1-\mu/p)}, \quad \mu, \nu, p > 0; $$

$$ {}^{C}_{p} \curlyvee^{\mu,\nu} g(\chi) = \frac{1}{p\Gamma_p(n-\mu)} \sum_{k=\chi+(n-\mu)}^{b} ((k-1)^\nu - \chi^\nu)^{n-\mu/p-1} \curlyvee_k^n g(\chi), $$

correspondingly, the $p-$fractal-fractional integral difference operators are given by the formulas

$$ {}_{p} \curlywedge^{-\mu,\nu} g(\chi) = \frac{1}{p\Gamma_p(\mu)} \sum_{k=0}^{\chi-\mu} (\chi^\nu - (k+1)^\nu)^{\mu/p-1} \curlywedge_k^n g(\chi), $$

$$ {}_{p} \curlyvee^{-\mu,\nu} g(\chi) = \frac{1}{p\Gamma_p(\mu)} \sum_{k=\chi+\mu}^{b} ((k-1)^\nu - \chi^\nu)^{\mu/p-1} \curlyvee_k^n g(\chi). $$

Definition 6. The Caputo $p-$fractal-fractional difference operators are defined as follows:

$$ {}^{C}_{p} \curlywedge^{\mu,\nu} g(\chi) = \frac{1}{p^{(n-\mu)/p)}\Gamma(\frac{(n-\mu)}{p})} \sum_{k=0}^{\chi-(n-\mu)} (\chi^\nu - (k+1)^\nu)^{n-\mu/p-1} \curlywedge_k^n g(\chi), $$

where the factor $(\chi^\nu)^\mu$ is defined by

$$ (\chi^\nu)^{\mu/p} = \frac{p^{\frac{(\mu)}{p^2}} \Gamma\left(\frac{(\chi^\nu+1)}{p}\right)}{\Gamma\left(\frac{(\chi^\nu+1-\mu/p)}{p}\right)}, $$

$$ {}^{C}_{p} \curlyvee^{\mu,\nu} g(\chi) = \frac{1}{p^{(n-\mu)/p)}\Gamma(\frac{(n-\mu)}{p})} \sum_{k=\chi+(n-\mu)}^{b} ((k-1)^\nu - \chi^\nu)^{n-\mu/p-1} \curlyvee_k^n g(\chi), $$

correspondingly, the fractal-fractional integral difference operators are given by the formulas

$$ {}_{p} \curlywedge^{-\mu,\nu} g(\chi) = \frac{1}{p^{\mu/p}\Gamma(\frac{\mu}{p})} \sum_{k=0}^{\chi-\mu} (\chi^\nu - (k+1)^\nu)^{\mu/p-1} \curlywedge_k^n g(\chi), $$

$$ {}_{p} \curlyvee^{-\mu,\nu} g(\chi) = \frac{1}{p^{\mu/p}\Gamma(\frac{\mu}{p})} \sum_{k=\chi+\mu}^{b} ((k-1)^\nu - \chi^\nu)^{\mu/p-1} \curlyvee_k^n g(\chi). $$

Note that for a positive integer q and $\mu > 0$, we have

$$_{p}\,\text{⅄}^{-\mu,\nu}\left(\text{⅄}^{q} g(\chi)\right) = \text{⅄}^{q}\left({}_{p}\,\text{⅄}^{-\mu,\nu}\, g(\chi)\right) - \sum_{k=0}^{q-1} \frac{(\chi^{\nu}-a^{\nu})^{k+\mu/p-q}}{p^{\mu/p}\Gamma(\mu/p+k-q+1)}\,\text{⅄}^{k}\, g(a),$$

$$_{p}\,\Upsilon^{-\mu,\nu}\left(\Upsilon^{q} g(\chi)\right) = \Upsilon^{q}\left({}_{p}\,\Upsilon^{-\mu,\nu}\, g(\chi)\right) - \sum_{k=0}^{q-1} \frac{(b^{\nu}-\chi^{\nu})^{k+\mu/p-q}}{p^{\mu/p}\Gamma(\mu/p+k-q+1)}\,\Upsilon^{k}\, g(b).$$

As a consequence, when change $n-\mu/p$ instead of μ/p and n instead of q a calculation yields the following result:

Proposition 1. If $\nu, \mu, p > 0$ then

$$^{C}_{p}\,\text{⅄}^{\mu,\nu}\, g(\chi) = {}^{RL}_{p}\,\text{⅄}^{\mu,\nu}\, g(\chi) - \sum_{k=0}^{n-1} \frac{(\chi^{\nu}-a^{\nu})^{k-\mu/p}}{p^{\mu/p}\Gamma(k+1-\frac{\mu}{p})}\,\text{⅄}^{k}\, g(a),$$

and ($\chi \in [a,b], n = [\mu/p]+1$)

$$^{C}_{p}\,\Upsilon^{\mu,\nu}\, g(\chi) = {}^{RL}_{p}\,\Upsilon^{\mu,\nu}\, g(\chi) - \sum_{k=0}^{n-1} \frac{(b^{\nu}-\chi^{\nu})^{k-\mu/p}}{p^{\mu/p}\Gamma(k+1-\frac{\mu}{p})}\,\Upsilon^{k}\, g(b).$$

When $p=1$ and $\nu=1$, we obtain the result in [13] – Theorem 14.

1.2 Generalized gingerbread-man map (GGMM)

A gingerbread-man map is a two-dimensional chaotic map agreeing to the theory of dynamical systems. When specific initial circumstances and initial parameters are used, the map is chaotic. This map looks like a gingerbread man when the set of chaotic solutions is designed [14]

$$\begin{aligned} x_{k+1} &= 1 - y_k + |x_k|, \\ y_{k+1} &= x_k, \quad k \in \mathbb{N}\cup\{0\}. \end{aligned} \tag{1}$$

By using the generalized $p-$fractal-fractional operator $^{C}_{p}\text{⅄}^{\mu,\nu}$, the system turns into the following equations:

$$\begin{aligned} &^{C}_{p}\,\text{⅄}^{\mu,\nu}\, x(k) = 1 - y(k+\frac{\mu}{p}-1) + \left|x(k+\frac{\mu}{p}-1)\right| \\ &- x(k+\frac{\mu}{p}-1), \\ &^{C}_{p}\,\text{⅄}^{\mu,\nu}\, y(k) = x(k+\frac{\mu}{p}-1) - y(k+\frac{\mu}{p}-1), \end{aligned}$$

where $\mu, \nu \in (0,1],\ p \geq 1,\ k \in \mathbb{N}_{1+\frac{\mu}{p}}$. By using the generalized $p-$fractal-fraction integral form $^{C}_{p}\text{⅄}^{-\mu,\nu}$ with some preparations, we have

$$x(n) = x_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} (n^{\nu}-(k+1)^{\nu})^{\frac{\mu}{p}-1}\left(1 - y(k+\frac{\mu}{p}-1) + |x(k+\frac{\mu}{p}-1)| - x(k+\frac{\mu}{p}-1)\right),$$

$$y(n) = y_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} (n^{\nu}-(k+1)^{\nu})^{\frac{\mu}{p}-1}\left(x(k+\frac{\mu}{p}-1) - y(k+\frac{\mu}{p}-1)\right).$$

Now by using the factor

$$(\chi^\nu-\tau^\nu)^{\mu/p}=\frac{p^{\frac{(\mu)}{p^2}}\Gamma\left(\frac{(\chi^\nu-\tau^\nu+1)}{p}\right)}{\Gamma\left(\frac{(\chi^\nu-\tau^\nu+1-\mu/p)}{p}\right)},\quad \mu,\nu,p>0,$$

we get the system (see Figure 1):

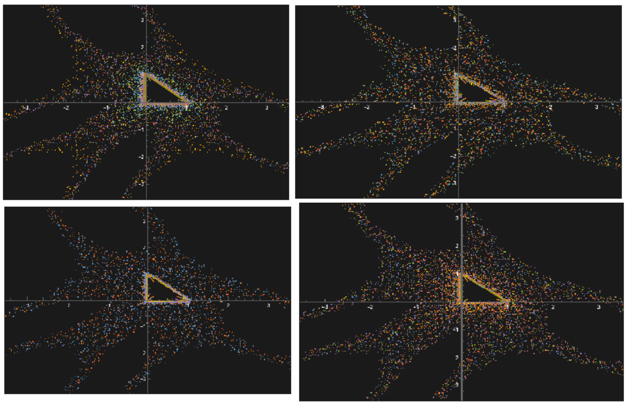

Figure 1. The plot of system (1) and system (2), when for different $p-$fractal-fractional value satisfying $\nu=\dfrac{\log\left(-\frac{9997p^{p/\mu}\Gamma(p/\mu)+10000}{9997p^{p/\mu}\Gamma(p/\mu)-10000}\right)}{\log(\mu)}$, $\nu=\dfrac{\log\left(-\frac{9998p^{p/\mu}\Gamma(p/\mu)+10000}{9998p^{p/\mu}\Gamma(p/\mu)-10000}\right)}{\log(\mu)}$, and $\nu=\dfrac{\log\left(-\frac{9999p^{p/\mu}\Gamma(p/\mu)+10000}{9999p^{p/\mu}\Gamma(p/\mu)-10000}\right)}{\log(\mu)}$ respectively. The iteration is selected for $n=1$ to 1000.

$$\begin{aligned}x(n)&=x_0+\frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})}\sum_{k=1}^{n-1-\mu}\frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\frac{\mu}{p}}{p}\right)}\left(n^\nu-(k+1)^\nu\right)^{\frac{\mu}{p}-1}\\&\times\left(1-y(k+\frac{\mu}{p}-1)+|x(k+\frac{\mu}{p}-1)|-x(k+\frac{\mu}{p}-1)\right),\\y(n)&=y_0+\frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})}\sum_{k=1}^{n-1-\mu}\frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\frac{\mu}{p}}{p}\right)}\left(x(k+\frac{\mu}{p}-1)-y(k+\frac{\mu}{p}-1)\right).\end{aligned}\tag{2}$$

System (2) can be recognized as follows:

$$\begin{aligned}x(n)&=x_0+\frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})}\sum_{k=1}^{n-1-\mu}\frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\mu/p}{p}\right)}\\&\times\left(n^\nu-(k+1)^\nu\right)^{\frac{\mu}{p}-1}\left(1-y(k+\frac{\mu}{p}-1)-2x(k+\frac{\mu}{p}-1)\right),\\y(n)&=y_0+\frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})}\sum_{k=1}^{n-1-\mu}\frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\frac{\mu}{p}}{p}\right)}\left(x(k+\frac{\mu}{p}-1)-y(k+\frac{\mu}{p}-1)\right).\end{aligned}$$

Thus, the characteristic polynomial is $P_1(\lambda) = \lambda^2 + 3\lambda + 3$, where the eigenvalues are $\lambda_{1,2} = 1/2(-3 \pm i\sqrt{3})$ corresponding to the eigenvectors $v_{1,2} = (1/2(-1 \pm i\sqrt{3}), 1)$. Hence, the system is in the steady behavior. Moreover, the equilibrium point is $(1/3, 1/3)$, while the fixed point is $2/7, 1/7$. In addition, we have

$$x(n) = x_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} \frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\mu/p}{p}\right)} \left(1 - y(k + \frac{\mu}{p} - 1)\right),$$

$$y(n) = y_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} \frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\mu/p}{p}\right)} \left(x(k + \frac{\mu}{p} - 1) - y(k + \frac{\mu}{p} - 1)\right).$$

Obviously, the characteristic polynomial is $P_1(\lambda) = \lambda^2 + \lambda + 1$, where the eigenvalues are $\lambda_{1,2} = 1/2(-1 \pm i\sqrt{3})$ corresponding to the eigenvectors $v_{1,2} = (1/2(-1 \pm i\sqrt{3}), 1)$. Hence, the system is in the steady behavior. Moreover, the equilibrium point is $(1, 1)$, while the fixed point is $2/3, 1/3$. The stability can be realized by the following generalized result.

Application 1. In this part, we introduce an application on control theory using the operator ${}^{\mathcal{C}}_{p}\Delta^{\mu,\nu,\gamma}_{\chi}\varphi(\chi)$. A p-fractal-fractional PID controller can be presented by

$$u(\chi) = K_a e(\chi) + K_i\, {}_pY^{\mu,\nu}_{\chi} e(\chi) + K_d\, {}^{\mathcal{C}}_{p}\Delta^{\mu,\nu,\gamma}_{\chi} e(\chi),$$

where

- $u(\chi)$ is the Control signal (output of the controller);
- K_a: Proportional gain (adjusts the control response to the current error);
- K_i: Integral gain (adjusts the control response based on the accumulated past error);
- K_d: Derivative gain (adjusts the control response based on the predicted future error);
- $e(\chi)$: Error signal;
- ${}_pY^{\mu,\nu}_{\chi} e(\chi)$: $p-$fractal-fractional integral operator, which is accounting for memory effects and fractal properties;
- ${}^{\mathcal{C}}_{p}\Delta^{\mu,\nu,\gamma}_{\chi} e(\chi)$: $p-$fractal-fractional differential operator, which is capturing anomalous diffusion and self-similar properties in the system.

Note that when $\mu = \nu = p = 1$, we obtain the integer case. By adding fractal and fractional dynamics, this controller goes beyond conventional PID control, improving performance in complicated, memory-dependent, and nonlinear systems. The stabilization of chaotic systems by the use of control mechanisms that lessen unpredictability and guarantee a desired steady-state or periodic behavior is known as chaos suppression. Feedback control, adaptive control, and sliding mode control are examples of traditional chaotic control techniques. However, by combining memory effects and multi-scale dynamics, fractal-fractional controllers provide special benefits.

By using the fractal-fractional weight w_k as follows

$$w_k := (n^\nu - k^\nu)^{\mu/p} = \frac{p^{\frac{(\mu)}{p^2}}\Gamma\left(\frac{(n^\nu-k^\nu+1)}{p}\right)}{\Gamma\left(\frac{(n^\nu-k^\nu+1-\mu/p)}{p}\right)}, \quad \mu, \nu, p > 0,$$

the controller of system (2) can be defined as follows:

$$u_n = -K_a e_n - K_i \sum_{k=0}^{n} w_k e_k - K_d(e_n - e_{n-1}),$$

where $e_n = x_n - x_d$ is the error with a desired fixed point x_d. The system exhibits a wide variety of x-values and stays chaotic for low K_a values. The system changes into a more steady, periodic behavior as K_a rises. The system completely stabilizes and displays a single fixed point after a particular threshold is reached. This demonstrates how more control gains reduce chaos and make a system more predictable.

The stability will be studied in the next section.

2 Stability

This section deals witht the sufficient conditions of the stability of the suggested system. If every zero of a polynomial with real coefficients has a negative real part, the polynomial is stable and/or a Hurwitz polynomial. A stable polynomial's coefficients share identical sign, as is widely recognized [15]. On the other hand, all of a matrix's eigenvalues, having negative real portions, indicates that the matrix is stable. In the domain of matrices, stability is frequently crucial to control theory and dynamic systems. We analyze the characteristic polynomial for λ in order to get the eigenvalues. The eigenvalues of matrix M are represented by the solutions (λ). The system is stable if the real components of all the eigenvalues are negative. Since stability describes how a system behaves in time, it is an essential topic in many domains, such as differential equations, control theory, and signal processing. We start with the linear system.

2.1 Linear system

We start with the next system.

Definition 7. Let $f(\chi) = f(\chi; \chi_0, f_0)$ be the solution of

$$ {}^{C}_{p}\,\curlywedge^{\mu,\nu} f(\chi) = F(\chi, f) $$

with the following details:
- $f(\chi)$ has a structure over $[\chi_0, \infty)$;
- the point $(\chi, f(\chi)) \in \mathbb{E}$, where

$$ \mathbb{E} := \{(\chi, \chi) : \chi \in (\chi_1, \infty),\ \|f\| < \chi_0, \chi > \chi_0\}. $$

Then f is called stable whenever a positive real number $\eta > 0$ exists for all solutions $f(\chi) = f(\chi; \chi_0, f_0) \in \mathbb{E}$ achieving the relation

$$ \|f_1 - f_0\| < \eta, $$

and for arbitrary numbers $\varepsilon > 0$ and $0 < \zeta \leq \eta$, the inequality

$$ \|f_1 - f_0\| < \zeta \Rightarrow \|f(\chi; \chi_0, f_0) - f(\chi; \chi_0, f_1)\| < \varepsilon, \quad \chi \in [\chi_0, \infty). $$

Additionally,

$$ \lim_{\chi \to \infty} \|f(\chi; \chi_0, f_0) - f(\chi; \chi_0, f_1)\| = 0 $$

then the solution f is asymptotically stable.

Theorem 2. Assume the linear system

$$ {}^{C}_{p}\,\curlywedge^{\mu,\nu} \begin{pmatrix} f(\chi) \\ g(\chi) \end{pmatrix} = \Upsilon_{2\times 2} \begin{pmatrix} f(\chi) \\ g(\chi) \end{pmatrix}. \tag{3} $$

Then the system is stable if and only if the solutions are bounded.

Moreover, if the characteristic polynomial of Υ is stable, the system is asymptotically stable.

Proof. Via creating a matrix-valued function with two variables, Υ, as follows:

$$\Upsilon(x,y)=\begin{bmatrix} \upsilon_{11}(x,y) & \upsilon_{12}(x,y) & \dots & \upsilon_{1n}(x,y) \\ \upsilon_{21}(x,y) & \upsilon_{22}(x,y) & \dots & \upsilon_{2n}(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \upsilon_{m1}(x,y) & \upsilon_{m2}(x,y) & \dots & \upsilon_{mn}(x,y) \end{bmatrix}.$$

Each element $\upsilon_{ij}(x,y)$ of the matrix is a scalar-valued function of x and y. Note that when $x=y$, then we obtain the matrix. In this case, $m=2$ and $n=2$, each element of the matrix is a linear combination of x and y.

Now, the boundedness of solutions of system (3) implies that there exists a fixed number $\varrho>0$ achieving the inequality $\|\Upsilon\|<\varrho$, where $\|.\|$ represents the max norm of the matrix $(\|\Upsilon\|=\max\limits_{1\le j\le n}\sum\limits_{i=1}^{m}|\upsilon_{ij}|)$. This leads to

$$\|f(\chi)-f_0(\chi)\|<\frac{\varepsilon}{2\varrho},\quad \|g(\chi)-g_0(\chi)\|<\frac{\varepsilon}{2\varrho},\quad \varepsilon>0.$$

As a result, we get

$$\|f(\chi;\chi_0,f_0)-f(\chi;\chi_0,\chi_1)\|=\|\Upsilon(\chi,\chi_0)(f_0-f_1)\|<\frac{\varrho\varepsilon}{2\varrho}=\frac{\varepsilon}{2}.$$

In a similar vein, there is

$$\|g(\chi;\chi_0,g_0)-g(\chi;\chi_0,g_1)\|=\|\Upsilon(\chi,\chi_0)(g_0-g_1)\|<\frac{\varrho\varepsilon}{2\varrho}=\frac{\varepsilon}{2}.$$

Let $R=(f,g)^t$, then

$$\begin{aligned}\|R(\chi)-R_0(\chi)\| &\le \|\Upsilon(\chi,\chi_0)\,(R(\chi)-R_0(\chi))\,\| \\ &\le \varrho\|R(\chi)-R_0(\chi)\| \\ &< \varrho(\frac{\varepsilon}{2\varrho}+\frac{\varepsilon}{2\varrho}) \\ &= \varepsilon.\end{aligned}$$

Based on the definition of stability, system (3) is stable.

On the other hand, the stability of the results, involving the zero-value solution, means that the inequality is satisfied by a constant with a positive value ω for a positive number $\varepsilon>0$, as follows:

$$\|R(\chi)\|<\omega\Rightarrow\|\Upsilon(\chi)R(\chi)\|<\varepsilon.$$

In particular,

$$\|f(\chi)\|=\|f(\chi;\chi_0,f_0)\|<\varepsilon/2$$

and

$$\|g(\chi)\|=\|g(\chi;\chi_0,g_0)\|<\varepsilon/2.$$

Hence, all stable solutions are bounded.

Now, the findings are asymptotically stable, provided the characteristic polynomial corresponding to Υ is stable (all its roots are negative).

$$\begin{aligned}\|f(\chi;\chi_0,f_0)-f(\chi;\chi_0,f_1)\| &\leq \varrho\exp\left(\frac{\Upsilon(\chi^\nu-\chi_0^\nu)^{\mu/p}}{\mu/p}\right)\|f_1-f_0\|\\ &\leq \varrho\exp(-\varepsilon_1\frac{\chi^{\mu\nu/p}}{\mu\nu/p}),\quad 0<\varepsilon_1<\varepsilon=0,\\ &\chi\to\infty,\quad \mu,\nu\in(0,1],\quad p\geq 1.\end{aligned}$$

In the same manner, we get

$$\begin{aligned}\|g(\chi;\chi_0,g_0)-g(\chi;\chi_0,g_1)\| &\leq c\exp\left(\frac{\Upsilon(\chi^\nu-\chi_0^\nu)^{\mu/p}}{\mu/p}\right)\|g_1-g_0\|\\ &\leq c\exp\left(-\varepsilon_1\frac{\chi^{\nu\mu/p}}{\mu\nu/p}\right),\quad 0<\varepsilon_1<\varepsilon=0,\\ &\chi\to\infty,\quad \mu,\nu\in(0,1],\quad p\geq 1,\end{aligned}$$

which implies the asymptotically stable outcomes.

Corollary 1. Suppose that all of the eigenvalues of the sup norm $\|\Upsilon\|<1$ fall inside the interval [0,1]. Then the system is stable, while if each solution of system (3) is bounded, then the system is asymptotically stable.

Proof. Assume the characteristic polynomial Υ such that $\|\Upsilon\|<1$ and all its eigenvalues are in the interval [0, 1]. Then it is an invertible positive contraction [15]. Then $\Upsilon_{2\times 2}^{-1}-\mathbb{I}_d$ is positive semi-definite, with positive determinant $|\Upsilon_{2\times 2}|>0$ (one can find the details in the proof of Proposition 3.5 in [16]). This leads to Υ characteristic polynomial is real stable. Hence, asymptotically stable is valid property, in light of Theorem 2.

2.2 Non-homogeneous system

We have the following result:

Theorem 3. Every solution for a non-homogeneous system that fits system (3)

$${}^{C}_{p}\curlywedge^{\mu,\nu}\begin{pmatrix}f(\chi)\\ g(\chi)\end{pmatrix}=\Upsilon_{2\times 2}\begin{pmatrix}f(\chi)\\ g(\chi)\end{pmatrix}+\begin{pmatrix}h_1(\chi)\\ h_2(\chi)\end{pmatrix}$$

is stable if and only if they are bounded and

$$\|H\|<\flat,\quad \flat\in(0,\infty),\quad H=(h_1,h_2)^t.$$

The system is asymptotically stable if the characteristic polynomial Υ is stable satisfying $\|\Upsilon\|<\varrho$ and

$$\varrho<\frac{e}{\flat},\quad \flat>0,\quad \|\Upsilon\|\leq\varrho.$$

Proof. Let $\|H(\chi)\|<\flat,\ \flat>0$. The condition of the theorem yields that

$$\begin{aligned}\|f(\chi)\| &\leq \varrho\exp\left(\varrho\flat\frac{(\chi^\nu-\chi_0^\nu)^{\mu/p}}{\mu/p}\right)\|f_0\|\\ &\leq \varrho\exp\left((\varrho\flat-e)\frac{\chi^{\nu\mu/p}}{\mu/p}\right)\|f_0\|\\ &=0,\quad \varrho\flat-e<0,\quad \nu,\mu\in(0,1],\quad \chi\to\infty.\end{aligned}$$

This proves the result.

Example 2. Consider the system

$$\begin{aligned}
{}_p^C \curlywedge^{\mu,\nu} x(\chi) &= 1 - y(\chi) + |x(\chi)| - x(\chi), \\
{}_p^C \curlywedge^{\mu,\nu} y(\chi) &= x(\chi) - y(\chi).
\end{aligned}$$

Then it can be divided into two cases, as follows:

$$\begin{aligned}
{}_p^C \curlywedge^{\mu,\nu} x(\chi) &= 1 - y(\chi), \\
{}_p^C \curlywedge^{\mu,\nu} y(\chi) &= x(\chi) - y(\chi).
\end{aligned} \tag{4}$$

And

$$\begin{aligned}
{}_p^C \curlywedge^{\mu,\nu} x(\chi) &= 1 - y(\chi) - 2x(\chi), \\
{}_p^C \curlywedge^{\mu,\nu} y(\chi) &= x(\chi) - y(\chi).
\end{aligned} \tag{5}$$

For system (4), the characteristic polynomial is $\Upsilon(\lambda) = \lambda^2 + \lambda + 1$, with two complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$ and $\|\Upsilon\| = \max(0+1, 1+1) = 2 < \varrho,\ \varrho > 2$. Moreover, $\|H\| = 1 < \flat,\ \flat > 1$. Thus, we have $2 < \varrho < e$ and $1 < \flat < e/\varrho$ yields $\varrho < e/\flat$. Thus, in view of Theorem 3, system (4) is asymptotically stable.

Now for system (5), we have the following data: $\Upsilon(\lambda) = \lambda^2 + 3\lambda + 3$, with two complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(-3 + i\sqrt{3}\right)$ and $\|\Upsilon\| = \max(2+1, 1+1) = 3 < \varrho,\ \varrho > 3$. Moreover, $\|H\| = 1 < \flat,\ \flat > 1$. As a consequence, the inequality $\varrho < e/\flat$ has no solution. Therefore, Theorem 3 is not applicable.

A mathematical method called perturbation analysis can be employed to examine how a system behaves in reactions to minor perturbations or adjustments. It is especially helpful in comprehending a system's sensitivity and stability. Experts can learn more about a system's overall functioning and make predictions about its eventual configurations by examining how it reacts to disturbances. When examining how dynamic systems behave when subjected to tiny perturbations, perturbation analysis is an invaluable resource. Its capacity to shed light on the system's general effectiveness, stability, and reactivate has made it a popular approach in a variety of scientific and technical fields.

The next example is a perturbation sample of the above system (see Figures 2, 3 and 4 for different values of ϵ_1 and ϵ_2).

Example 3. Consider the following system,

$$\begin{aligned}
{}_p^C \curlywedge^{\mu,\nu} x(\chi) &= 1 - y(\chi) + |x(\chi)| - x(\chi) + \epsilon_1, \\
{}_p^C \curlywedge^{\mu,\nu} y(\chi) &= x(\chi) - y(\chi) + \epsilon_2.
\end{aligned}$$

Then it can be divided into two cases, as follows:

$$\begin{aligned}
{}_p^C \curlywedge^{\mu,\nu} x(\chi) &= 1 - y(\chi) + \epsilon_1, \\
{}_p^C \curlywedge^{\mu,\nu} y(\chi) &= x(\chi) - y(\chi) + \epsilon_2.
\end{aligned} \tag{6}$$

And

$$\begin{aligned}
{}_p^C \curlywedge^{\mu,\nu} x(\chi) &= 1 - y(\chi) - 2x(\chi) + \epsilon_1, \\
{}_p^C \curlywedge^{\mu,\nu} y(\chi) &= x(\chi) - y(\chi) + \epsilon_2.
\end{aligned} \tag{7}$$

For system (6), the characteristic polynomial is $\Upsilon(\lambda) = \lambda^2 + \lambda + 1$, with two complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$ and $\|\Upsilon\| = \max(0+1, 1+1) = 2 < \varrho,\ \varrho > 2$. Moreover, $\|H\| = \max(1+\epsilon_1, \epsilon_2) = 1 + \epsilon_1 < \flat,\ \flat > 1 + \epsilon_1,\ \epsilon_2 \leq 1 + \epsilon_1$. Thus, we have $2 < \varrho < e$ and $0 < \epsilon_1 < \frac{(e-\varrho)}{\varrho}$ yields $\varrho < \frac{e}{\flat} < \frac{e}{1+\epsilon_1}$. Thus, in view of Theorem 3, system (6) (similarly, for system (7)) is asymptotically stable.

Now for system (7), we have the following data: $\Upsilon(\lambda) = \lambda^2 + 3\lambda + 3$, with two complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(-3 + i\sqrt{3}\right)$ and $\|\Upsilon\| = \max(2+1, 1+1) = 3 < \varrho,\ \varrho > 3$. Moreover, $\|H\| = \max(1+\epsilon_1, \epsilon_2) = 1 + \epsilon_1 < \flat,\ \flat > 1 + \epsilon_1,\ \epsilon_2 \leq \epsilon_1$. Then it follows that the inequality $\varrho < e/\flat$ has a solution, whenever $\varrho > 3$ and $-1 < \epsilon_1 < \frac{(e-\varrho)}{\varrho}$. Therefore, Theorem 3 indicates asymptotically solutions. A comparison of the system is shown in Figure 5 whenever $\mu = \nu = 0.999$ and $p = 1$, and vice versa when $\mu = \nu = 1$ and $p = 1$ for various values of ϵ_1 and ϵ_2.

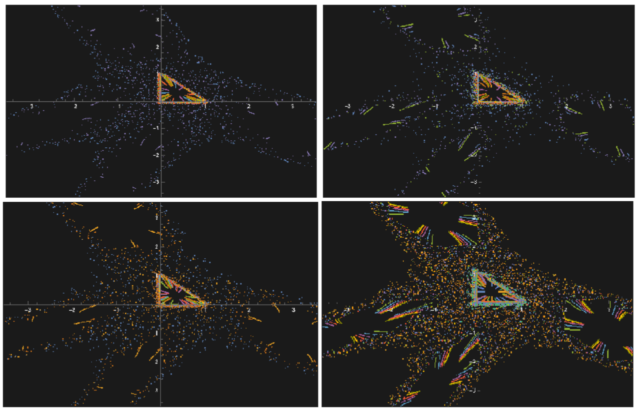

Figure 2. The plot of perterbation System when $\mu = \nu = 0.9992$ and $p = 1$ for $\epsilon_1 = \epsilon_2 = 0.0001, 0.001, 0.01$ and 0.1 respectively. The iteration is selected for $n = 1$ to 1000.

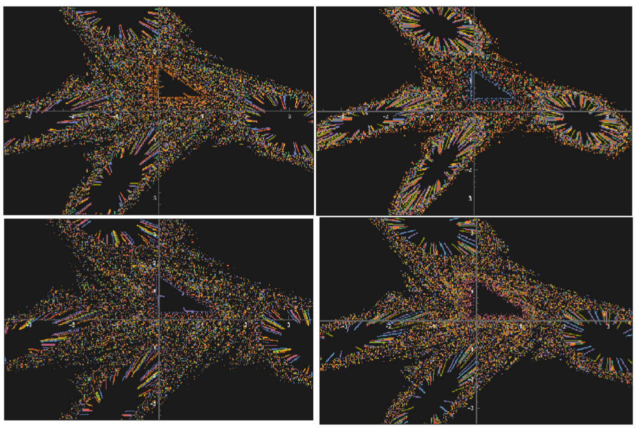

Figure 3. The plot of perterbation System when $\mu = \nu = 0.9992$ and $p = 1$ for $\epsilon_1 = \epsilon_2 = 0.5$, $\epsilon_1 = 0.3, \epsilon_2 = 0.5$, $\epsilon_1 = 0.5, \epsilon_2 = 0.3$ and $\epsilon_1 = 0.3, \epsilon_2 = 0.3$ respectively. The iteration is selected for $n = 1$ to 1000.

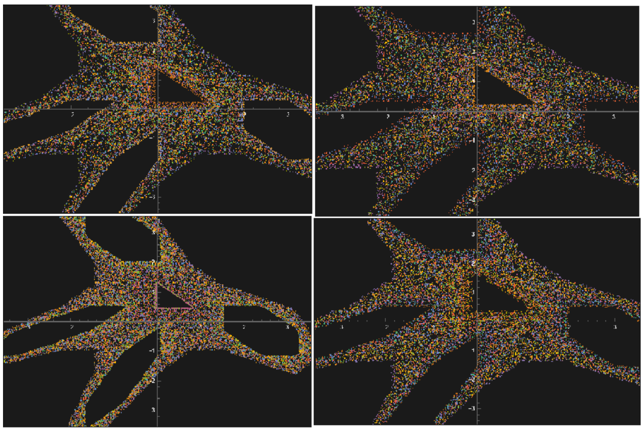

Figure 4. The plot of perterbation System when $\mu = \nu = 0.9999$ and $p = 1$ for $\epsilon_1 = \epsilon_2 = 0.3$, $\epsilon_1 = 0.5, \epsilon_2 = 0.3$, $\epsilon_1 = 0.3, \epsilon_2 = 0.5$ and $\epsilon_1 = 0.5, \epsilon_2 = 0.5$ respectively. The iteration is selected for $n = 1$ to 1000.

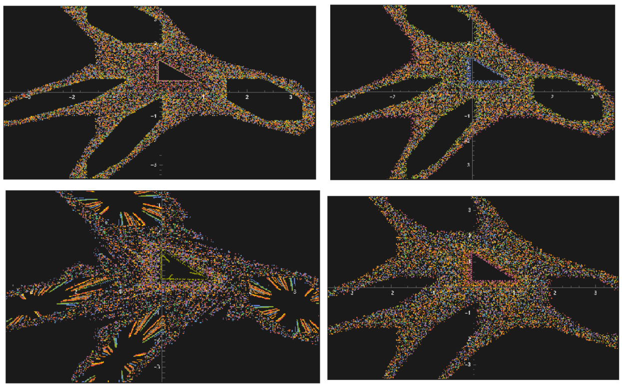

Figure 5. A comparison of the system, when $\mu = \nu = 0.999$ and $p = 1$ vice versa $\mu = \nu = 1$ and $p = 1$ for $\epsilon_1 = \epsilon_2 = 0.5$, (first line) and $\epsilon_1 = 0.3, \epsilon_2 = 0.3$ (second line) respectively. The iteration is selected for $n = 1$ to 1000.

Conclusions and suggestions

By utilizing the generalized gamma function (Γ_p), the fractal-fractional operators are modified. Moreover, the difference operators corresponding to the suggested $p-$fractal-fractional operators are introduced. Examples for the continuous types are illustrated. As an application, we suggested to study the generalized gingerbread-man map (GGMM). Some special cases are indicated for such a system. Stability of the linear and nonlinear systems are examined. We presented a set of conditions to obtain the asymptotic stability behavior of the proposed systems. A perturbation system is formulated with different graphics, based on the values of the perturbation factors. For future works, one can consider different types of stability of the generalized map.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Computational of the eigenvalues of the fractional Sturm-Liouville problem

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We study the asymptotic distribution for eigenvalues of fourth-order fractional Sturm-Liouville with Dirichlet boundary condition. In this work, we use the inverse Laplace transform method and the Asymptotic formula of the Mittag-Leffler function to get an analytical solution of the fractional Sturm-Liouville problems. When the fractional-order approaches 1, our results agree with the classical ones of fourth-order differential equations.

Keywords: Fractional Sturm-Liouville, Asymptotic formula, Laplace transform, Mittag-Leffler functions, Eigenvalues.

2020 Mathematics Subject Classification: Primary: 26A33; Secondary: 34A08, 33E12.

Introduction

The Sturm-Liouville Problems (SLPs), or eigenvalue problems, for ordinary differential equations play a very important role in theory and applications. These problems have been used to describe a large number of physical, biological and chemical phenomena. Among others, we can refer to the Sturm-Liouville analytical model of dirt transport in industrial washing of wool, which was developed by Caunce et al. [1], the one-dimensional heat and mass diffusion modeling software provided by Barouh and Mikhailov [2] as well as a set of boundary value models [3–6]. However, there are many phenomena in nature that cannot be characterized by classical derivative models [7, 8].

Fractional calculus is a theory that unifies and generalizes the notions of integer order differentiation and integration to any real or complex order. Various types of fractional derivative definitions were introduced in history, which the most popular definitions of fractional derivatives among them are Grünwald-Letnikov, Riemann-Liouville, Dzherbashyan-Caputo, Riesz-Fischer, two-scale fractal derivative [9] which is conformable with the traditional differential derivatives and a new fractional derivative with non-local and no-singular kernel is Atangana-Baleanu's fractional derivative [10] which is presented and applied to solve the fractional heat transfer model. Over the last decade, it has been demonstrated that many systems in science and engineering can be modeled more accurately by employing fractional order rather than integer order derivatives [11–16]. Along with developing the research area of fractional differential equations and applications, many studies have focused on the class of well-known fractional Sturm-Liouville problems (FSLPs). These types of FSLPs, due to their importance, have been a subject of numerous investigations, especially in various areas of science and in engineering fields, for example, chemistry, electricity, mechanics, biology, control theory, and economics [17, 18].

Since it is generally challenging to find analytical solutions for these problems and also FSLPs contain the composition of the left and right-sided derivative. Consequently, several numerical methods have been devoted to seeking approximate solutions, such as the Adomain decomposition method [19], Homotopy analysis method [20] and Fourier series [21].

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On the other hand, recently the FSLPs were studied by Kilmek and Agrawal in [22] and Rivero et al. in [23]. We refer the reader for the higher order of SLPs to [24–26]. Similarly, Dehghan and Mingarelli [27] obtained for the first time, asymptotic formulas of eigenvalues and eigenfunctions of 2α-order FSLP. Also Erdal Bas et al. investigated the conformable SLPs by spectral analysis in [28] and SLPs with a new generalized fractional derivative in [29], and Mortazaasl H. introduced two classes of conformable fractional Sturm-Liouville problem [30]. Significant research on fractional derivatives and numerical solutions of Sturm-Liouville problems can be mentioned by Babak Shiri [31, 32]. Moreover, Jafari et al. in [33], studied the SLPs with a generalized fractional derivative.

It should be noted that since finding analytical solutions for transcendental function is a challenging task, after studying, we realized that we can apply the asymptotic form of Mittag-Leffler's function to get the roots. The asymptotic behavior of Mittag-Leffler functions plays a very important role in the interpretation of the solution of various problems of physics connected with fractional reaction, fractional relaxation, fractional diffusion, and fractional reaction-diffusion, and so forth, in complex systems. The asymptotic expansion of $E_{\alpha,\beta}(z)$ is based on the integral representation of the Mittag-Leffler function in the form

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_C \frac{s^{\alpha-\beta}}{s^\alpha - z} e^s ds, \quad \Re(\alpha) > 0, \quad \Re(\beta) > 0, \quad s, \alpha, \beta \in \mathbb{C}, \tag{1}$$

where the path of integration C is a loop which starts and ends at $-\infty$ and encircles the circular disc $|s| \leq |z|^{\frac{1}{\alpha}}$ in the positive sense and $-\pi \leq \arg s \leq \pi$ on C (this curve is also the Hankel path). The integral representation (1) is used to obtain the asymptotic expansion of the Mittag-Leffler function at infinity [34].

In this work, the inverse Laplace transform method and the Asymptotic formula of the Mittag-Leffler function are applied to obtain analytical solutions of FSLPs. Using the introduced method, we obtained eigenvalues of the fractional Sturm-Liouville problems in three features. The results, show the simplicity and efficiency of this method. This paper's aim is to get an asymptotic formula for the eigenvalues of fourth-order FSLPs.

The paper is organized as follows: In Section 1, we have introduced some necessary definitions and preliminaries of fractional calculus theory. Three illustrative features are discussed in Section 2. The last section includes our conclusion.

1 Preliminaries

In this section, we recall some definitions and properties of fractional calculus theory used in this paper. The reader can refer for details to [35–37].

Definition 1. Let $\alpha \in \mathbb{R}$ with $\alpha \notin \mathbb{N}$ and $\alpha > 0$. The left and the right Riemann-Liouville fractional integrals $I^\alpha_{a^+}$ and $I^\alpha_{b^-}$ of order α are defined by

$$I^\alpha_{a^+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad x \in (a, b], \tag{2}$$

and

$$I^\alpha_{b^-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (\tau - x)^{\alpha-1} f(\tau) d\tau, \quad x \in [a, b).$$

$\Gamma(.)$ denotes the Euler Gamma function. The following property can be easily obtained.

Property 1. We have $I^\alpha_{a^+} C = \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} C$ and $I^\alpha_{b^-} C = \frac{(b-x)^\alpha}{\Gamma(\alpha+1)} C$. C is a constant.

Definition 2. The left and the right Caputo fractional derivatives ${}^cD^{\alpha}_{a^+}$ and ${}^cD^{\alpha}_{b^-}$ of order α are defined by

$$ {}^cD^{\alpha}_{a^+}f(x) := \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-\tau)^{-\alpha} f'(\tau)d\tau, \quad x > a, \tag{3}$$

and

$$ {}^cD^{\alpha}_{b^-}f(x) := \frac{-1}{\Gamma(1-\alpha)} \int_x^b (\tau - x)^{\alpha} f'(\tau)d\tau, \quad x < b,$$

where f is differentiable and $0 \le \alpha < 1$.

Definition 3. The left and the right Riemann-Liouville fractional derivatives $D^{\alpha}_{a^+}$ and $D^{\alpha}_{b^-}$ of order $0 \le \alpha < 1$ are defined by

$$ D^{\alpha}_{a^+}f(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-\tau)^{-\alpha} f(\tau)d\tau, \quad x > a, \tag{4}$$

and

$$ D^{\alpha}_{b^-}f(x) := \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (\tau - x)^{-\alpha} f(\tau)d\tau, \quad x < b. \tag{5}$$

Property 2. If $0 < \gamma < 1$ and $g \in AC[a,b]$ and $h \in L^q(a,b)(1 \le q < \infty)$. Then we have

$$ \int_a^b g(x) D^{\gamma}_{a^+}h(x)dx = \int_a^b h(x) {}^cD^{\gamma}_{a^+}g(x)dx + g(x) I^{1-\gamma}_{a^+}h(x) \mid_{x=a}^{x=b} .$$

Property 3. Let $0 < \alpha < \beta$, then the following identities hold:

$$\begin{aligned} I^{\alpha}_{a^+}(x-a)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}, \\ D^{\alpha}_{a^+}(x-a)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}, \\ I^{\alpha}_{b^-}(b-x)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1}, \\ D^{\alpha}_{b^-}(b-x)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-\alpha-1}. \end{aligned}$$

Property 4. For $a \le x < b$, $0 < \alpha < 1$, we have

$$\begin{aligned} I^{\alpha}_{a^+}(b-x)^{\alpha-1} &= \frac{(b-x)^{2\alpha-1}}{\Gamma(\alpha)} \int_{\frac{b-a}{b-x}}^1 (1-w)^{\alpha-1} w^{\alpha-1} dw, \\ &= -\frac{(b-x)^{2\alpha-1}}{\Gamma(\alpha)} \Big(B\Big(\frac{b-a}{b-x}; \alpha, \alpha\Big) - B(1;\alpha,\alpha)\Big), \end{aligned}$$

where $B(z;\alpha,\beta)$ is the "Incomplete Beta function" defined by

$$ B(z;\alpha,\beta) = \int_0^z w^{\alpha-1}(1-w)^{\beta-1}dw.$$

Property 5. If $\gamma > 0$ and $g \in L^q(a,b)$ $(1 \le q \le \infty)$, then the following equalities

$$\begin{aligned} D^{\gamma}_{a^+} \circ I^{\alpha}_{a^+} g(x) = g(x), \\ D^{\gamma}_{b^-} \circ I^{\gamma}_{b^-} g(x) = g(x), \end{aligned}$$

hold on $[a,b]$.

Property 6. If $0 < \alpha < 1$, $f \in L^1(a,b)$ and $I_{a^+}^{1-\alpha} f, I_{b^-}^{1-\alpha} f \in AC[a,b]$, then the following equalities

$$I_{a^+}^{\alpha} \circ D_{a^+}^{\alpha} f(x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} I_{a^+}^{1-\alpha} f(x) \mid_{x=a},$$
$$I_{b^-}^{\alpha} \circ D_{b^-}^{\alpha} f(x) = f(x) - \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} I_{b^-}^{1-\alpha} f(x) \mid_{x=b},$$

hold on $[a,b]$.

Property 7. Let $\Re(\alpha) > 0$ and $f(x) \in L^\infty(a,b)$ or $f(x) \in C[a,b]$. If $\Re(\alpha) \notin \mathbb{N}$ or $\alpha \in \mathbb{N}$, then

$$ {}^cD_{a^+}^{\alpha} \circ I_{a^+}^{\alpha} f(x) = f(x),$$
$$ {}^cD_{b^-}^{\alpha} \circ I_{b^-}^{\alpha} f(x) = f(x).$$

Property 8. Let $0 < \alpha \leq 1$. If $f \in AC[a,b]$, then

$$I_{a^+}^{\alpha} \circ {}^cD_{a^+}^{\alpha} f(x) = f(x) - f(a),$$
$$I_{b^-}^{\alpha} \circ {}^cD_{b^-}^{\alpha} f(x) = f(b) - f(x).$$

Next, we will review the Mittag-Leffler function. The function $E_\alpha(z)$ defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (z \in \mathbb{C},\, \Re(\alpha) > 0),$$

was introduced by Mittag-Leffler [36]. The generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C},$$

where $\beta \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Definition 4. The Laplace transform $\mathcal{L}$ of a function $f(x)$, is the function $F(s)$ which is defined by

$$F(s) = \mathcal{L}\{f(x)\} := \int_0^\infty e^{-sx} f(x) dx,$$

where $x \geq 0$ and s is the frequency parameter.

Definition 5. If $\mathcal{L}\{f(x)\} = F(s)$ then $f(x)$ is The inverse Laplace transform of $F(s)$ that is given by the complex integral

$$f(x) = \mathcal{L}^{-1}\{F(s)\} := \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{sx} F(s) ds.$$

Definition 6. Convolution of two functions $f(x)$ and $g(x)$ over on a finite rang $[0,x]$ is defined by

$$(f * g)(x) = \int_0^x f(s) g(x-s) ds, \qquad f, g : [0,\infty) \to \mathbb{R}.$$

Property 9. For $\Re(\alpha) > -1$, then

$$\begin{aligned} \mathcal{L}\{t^p\} &= \frac{\Gamma(p+1)}{s^{p+1}}, \\ \mathcal{L}^{-1}\{s^p\} &= \frac{1}{s^{p+1}\Gamma(p)}. \end{aligned} \tag{6}$$

Property 10. Suppose $f(t)$ is a differentiable function of exponential order, then

$$\mathcal{L}\{f'(t)\} = s\{f(t)\} - f(0).$$

Property 11. $\mathcal{L}\{f * g\}(x) = \mathcal{L}\{f(x)\} \cdot \mathcal{L}\{g(x)\}$.

Property 12. $\mathcal{L}^{-1}\{\frac{s^{\alpha-\beta}}{s^\alpha+\lambda}\} = x^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)$.

Property 13. According to the definition of the left fractional integral (2), we have

$$I_{0^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\left(f(x) * \frac{1}{x^{1-\alpha}}\right).$$

So, by (6) and Property 11, we get

$$\begin{aligned}\mathcal{L}\{I_{0^+}^\alpha f(x)\} &= \frac{1}{\Gamma(\alpha)} \cdot \mathcal{L}\{f(x)\}\mathcal{L}\{\frac{1}{x^{1-\alpha}}\} \\ &= \frac{1}{s^\alpha}\mathcal{L}\{f(x)\}.\end{aligned}$$

Property 14. According to the definition of the left Caputo fractional derivative (3) and Properties 13 and 10, we have, for $0 < \alpha < 1$,

$$\mathcal{L}\{{}^cD_{0^+}^\alpha f(x)\} = s^\alpha\mathcal{L}\{f(x)\} - s^{\alpha-1}f(0).$$

Property 15. According to the definition of the left Riemann-Liouville fractional derivative (4), for $0 < \alpha < 1$ by using Properties 13 and 10, we can write

$$\mathcal{L}\{D_{0^+}^\alpha f(x)\} = s^\alpha\mathcal{L}\{f(x)\} - I_{0^+}^{1-\alpha} f(x)\mid_{x=0} \mathcal{L}\{DI_{0^+}^{1-\alpha}f(x)\}.$$

2 *Eigenvalues of fourth order FSLP*

In this section, we consider three features of a differently defined fourth order fractional Sturm-Liouville operator. This operator is a composition of right Caputo fractional derivative with a left Riemann-Liouville fractional derivative as follows:

Feature 3.1.

$$ {}^cD_{b^-}^\alpha \circ D_{a^+}^\alpha \circ {}^cD_{b^-}^\alpha \circ D_{a^+}^\alpha y(x) = 0, \qquad 0 < \alpha < 1. \tag{7}$$

Applying the right fractional integral on (7), and using Property 8, we obtain

$$D_{a^+}^\alpha \circ {}^cD_{b^-}^\alpha \circ D_{a^+}^\alpha y(x) - D_{a^+}^\alpha \circ {}^cD_{b^-}^\alpha \circ D_{a^+}^\alpha y(t)|_{x=b} = 0.$$

Now, by taking the left fractional integral of the above equation and also by using the Properties 1 and 6, we get

$$\begin{aligned}{}^cD_{b^-}^\alpha \circ D_{a^+}^\alpha y(x) &- I_{a^+}^{1-\alpha c}D_{b^-}^\alpha \circ D_{a^+}^\alpha y(x)|_{x=a}\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \\ &- D_{a^+}^\alpha \circ {}^cD_{b^-}^\alpha \circ D_{a^+}^\alpha y(x)|_{x=b}\frac{(x-a)^\alpha}{\Gamma(\alpha+1)} = 0.\end{aligned}$$

Again by taking the right and left fractional integral and using Properties 1 and 6, we get it right away

$$\begin{aligned}y(x) &= I_{a^+}^{1-\alpha}y(t)|_{x=a}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} + D_{a^+}^\alpha y(x)|_{x=b}\frac{(x-a)^\alpha}{\Gamma(\alpha)} \\ &+ I_{a^+}^{1-\alpha c}D_{b^-}^\alpha y(x)|_{x=a}\phi_1(x;a,b,\alpha) + D_{a^+}^\alpha \circ {}^cD_{b^-}^\alpha \circ D_{a^+}^\alpha y(x)|_{x=b}\phi_2(x;a,b,\alpha),\end{aligned} \tag{8}$$

where

$$\phi_1(x;a,b,\alpha) = \frac{1}{\Gamma^3(\alpha)}\int_a^x \frac{(\tau-a)^{2\alpha-1}}{(x-\tau)^{1-\alpha}}\Big(B\Big(\frac{b-a}{\tau-a};\alpha,\alpha\Big) - B(1;\alpha,\alpha)\Big)d\tau,$$

and

$$\phi_2(x;a,b,\alpha) = \frac{1}{\Gamma^2(\alpha)\Gamma(\alpha+1)}\int_a^x \frac{(\tau-a)^{2\alpha-1}}{(x-\tau)^{1-\alpha}}\Big(B\Big(\frac{b-a}{\tau-a};\alpha+1,\alpha\Big) - B(1;\alpha+1,\alpha)\Big)d\tau.$$

It is worth noting that as α approach 1, (7) reduce $y^{(4)} = 0$, and (8) becomes

$$y(x) = y(a) + y'(b)(x-a) + y''(a)\,\phi_1(x;a,b,1) + y'''(b)\,\phi_2(x;a,b,1),$$

where $\phi_1(x;a,b,1)$ and $\phi_2(x;a,b,1)$ are polynomials of degrees 2 and 3, respectively, in terms of the variable x.

So, we can say fundamental solution is $\Big\{\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}, \frac{(x-a)^{\alpha}}{\Gamma(\alpha)}, \phi_1(x;a,b,\alpha), \phi_2(x;a,b,\alpha)\Big\}$, since all four of them satisfy the equation (7) separately and one can see that their Wronskian is not identically zero in $[a,b]$, having discontinuities at a and b. Associated to (7) is another similar but quite different composition.

Feature 3.2.

$$D^{\alpha}_{b^-} \circ {}^cD^{\alpha}_{a^+} \circ D^{\alpha}_{b^-} \circ {}^cD^{\alpha}_{a^+} y(x) = 0, \qquad 0 < \alpha < 1. \tag{9}$$

Applying the right fractional integral on (9) and using Property 6, we have

$${}^cD^{\alpha}_{a^+} \circ D^{\alpha}_{b^-} \circ {}^cD^{\alpha}_{a^+} y(x) - I^{1-\alpha c}_{b^-} D^{\alpha}_{a^+} \circ D^{\alpha}_{b^-} \circ {}^cD^{\alpha}_{a^+} y(x)|_{x=b} \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} = 0.$$

Now, by taking the left fractional on the above equation and using Properties 8 and 4, also introducing the function $\psi(t;a,b,\alpha)$ by

$$\psi(x;a,b,\alpha) = \frac{(b-x)^{2\alpha-1}}{\Gamma^2(\alpha)}\Big(B\Big(\frac{b-a}{b-x};\alpha,\alpha\Big) - B(1;\alpha,\alpha)\Big),$$

we obtain

$$D^{\alpha}_{b^-} \circ {}^cD^{\alpha}_{a^+} y(x) - D^{\alpha}_{b^-} \circ {}^cD^{\alpha}_{a^+} y(x)|_{x=a} - I^{1-\alpha c}_{b^-} D^{\alpha}_{a^+} \circ D^{\alpha}_{b^-} \circ {}^cD^{\alpha}_{a^+} y(x)|_{x=b} \cdot \psi(x;a,b,\alpha) = 0.$$

Again, by taking the right fractional on the above equation, we have

$$\begin{aligned} {}^cD^{\alpha}_{a^+} y(t) \quad &- \quad I^{1-\alpha c}_{b^-} D^{\alpha}_{b^-} y(x)|_{x=b} \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} - D^{\alpha}_{b^-} {}^cD^{\alpha}_{b^-} y(x)|_{x=a} \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \\ &- \quad I^{1-\alpha c}_{b^-} D^{\alpha}_{a^+} \circ D^{\alpha}_{b^-} \circ {}^cD^{\alpha}_{a^+} y(x)|_{x=b} \cdot \xi(x;a,b,\alpha) = 0, \end{aligned}$$

where

$$\xi(x;a,b,\alpha) = \frac{1}{\Gamma(\alpha)}\int_x^b (\tau-x)^{\alpha-1}\psi(\tau;a,b,\alpha)d\tau.$$

Finally

$$\begin{aligned} y(x) \quad &= \quad y(a) + I^{1-\alpha c}_{b^-} D^{\alpha}_{b^-} y(x)|_{x=b} \frac{(b-x)^{2\alpha-1}}{\Gamma^2(\alpha)} \cdot \Big(B\Big(\frac{b-a}{b-x};\alpha,\alpha\Big) - B(1;\alpha,\alpha)\Big) \\ &+ \quad D^{\alpha}_{b^-} \circ {}^cD^{\alpha}_{a^+} \circ y(x)|_{x=b} \frac{(b-x)^{2\alpha}}{\Gamma^2(\alpha)} \cdot \Big(B\Big(\frac{b-a}{b-x};\alpha+1,\alpha\Big) - B(1;\alpha+1,\alpha)\Big) \\ &+ \quad I^{1-\alpha c}_{b^-} D^{\alpha}_{a^+} \circ D^{\alpha}_{b^-} \circ {}^cD^{\alpha}_{a^+} y(x)|_{x=b} \cdot \int_a^x (x-\tau)^{\alpha-1}\xi(\tau;a,b,\alpha)d\tau. \end{aligned}$$

We see that $\psi(a;a,b,\alpha)$ and consequently $\xi(x;a,b,\alpha)=0$. Also if $\frac{1}{2}<\alpha<1$, we have

$$\lim_{x\to b}\psi(x;a,b,\alpha)=\frac{(b-a)^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}.$$

The functions $\psi(x;a,b,\alpha)$ and consequently $\xi(x;a,b,\alpha)$ have a discontinuity at $x=b$, when $0<\alpha\leq\frac{1}{2}$.

Feature 3.3. For $\frac{3}{4}<\alpha\leq 1$, consider the following fractional eigenvalue problem on $[0,1]$

$$-\,{}^cD^{\alpha}_{0^+}\circ D^{\alpha}_{0^+}\circ {}^cD^{\alpha}_{0^+}\circ D^{\alpha}_{0^+}y(x)=\lambda y(x), \tag{10}$$

with the boundary conditions

$$\begin{aligned}
&a)\quad I^{1-\alpha}_{0^+}y(x)|_{x=0}=0,\\
&b)\quad I^{1-\alpha}_{0^+}y(x)|_{x=1}=0,\\
&c)\quad I^{1-\alpha C}_{0^+}D^{\alpha}_{0^+}\circ D^{\alpha}_{0^+}y(x)|_{x=0}=0,\\
&d)\quad I^{1-\alpha C}_{0^+}D^{\alpha}_{0^+}\circ D^{\alpha}_{0^+}y(x)|_{x=1}=0.
\end{aligned} \tag{11}$$

By taking Laplace transformation on both side (10) and using Properties 7 and 8, we have

$$\begin{aligned}
\mathcal{L}(y(x)) &= \frac{s^{3\alpha}}{s^{4\alpha}+\lambda}I^{1-\alpha}_{0^+}y(x)|_{x=0}+\frac{s^{3\alpha-1}}{s^{4\alpha}+\lambda}D^{\alpha}_{0^+}y(x)|_{x=0}\\
&+ \frac{s^{\alpha}}{s^{4\alpha}+\lambda}I^{1-\alpha c}_{0^+}D^{\alpha}_{0^+}\circ D^{\alpha}_{0^+}y(x)|_{x=0}\\
&+ \frac{s^{\alpha-1}}{s^{4\alpha}+\lambda}D^{\alpha}_{0^+}\circ {}^cD^{\alpha}_{0^+}\circ D^{\alpha}_{0^+}y(x)|_{x=0}.
\end{aligned}$$

Now by taking inverse Laplace transformation in order to get $y(x)$, one can easily see that

$$\begin{aligned}
y(x)= \;& c_1x^{\alpha-1}E_{4\alpha,\alpha}(-\lambda x^{4\alpha})+c_2x^{\alpha}E_{4\alpha,\alpha+1}(-\lambda x^{4\alpha})+c_3x^{3\alpha-1}E_{4\alpha,3\alpha}(-\lambda x^{4\alpha})\\
&+c_4x^{3\alpha}E_{4\alpha,3\alpha+1}(-\lambda x^{4\alpha}).
\end{aligned} \tag{12}$$

Remark 1. When α approaches 1, equation (10) turns into $-y^{(4)}=\lambda y$ and its fundamental set of solution is

$$\left\{\cos\left(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}}\right)\cosh\left(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}}\right),\frac{\cos(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})\sinh(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})}{\sqrt{2}\sqrt[4]{\lambda}x}+\frac{\sin(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})\cosh(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})}{\sqrt{2}\sqrt[4]{\lambda}x},\frac{\sin(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})\sinh(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})}{\sqrt{\lambda}x^2},\frac{\sin(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}}-\frac{\pi}{4})\sinh(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})}{(\sqrt{\lambda}x)^3}\right\}.$$

Now by imposing the boundary conditions (11) on (12) and with the following formula

$$D^{\gamma}_{0^+}\left\{x^{\beta-1}E_{\alpha,\beta}(-\lambda x^{\alpha})\right\}=x^{\beta-\gamma-1}E_{\alpha,\beta-\gamma}(-\lambda x^{\alpha}),$$

finally we have

$$\begin{cases}
c_2xE_{4\alpha,2}(-\lambda x^{4\alpha})+c_4x^{-2\alpha+1}E_{4\alpha,2\alpha+2}(-\lambda x^{4\alpha})|_{x=1}=0,\\
c_2x^{-2\alpha+1}E_{4\alpha,-2\alpha+2}(-\lambda x^{4\alpha})+c_4xE_{4\alpha,2}(-\lambda x^{4\alpha}))|_{x=1}=0.
\end{cases}$$

Now in order to obtain eigenvalues of above system, the coefficient determinant of boundary conditions must be zero, i.e,

$$E^2_{4\alpha,2}(-\lambda)-E_{4\alpha,-2\alpha+2}(-\lambda)E_{4\alpha,2\alpha+2}(-\lambda)=0. \tag{13}$$

This is characteristic equations for eigenvalues, and we note that $E_{4\alpha,2}(-\lambda)$, $E_{4\alpha,-2\alpha+2}(-\lambda)$ and $E_{4\alpha,2\alpha+2}(-\lambda)$ are entire functions of order $\frac{1}{4\alpha}$. For solving (13), from the Mittag-Leffler integral representation (1), we have

$$E_{4\alpha,2}(-\lambda) = \frac{1}{2\pi i}\int_C \frac{s^{4\alpha-2}}{s^{4\alpha}+\lambda}e^s ds.$$

For solving this integral, we use Cauchy's residue theorem

$$s^{4\alpha}+\lambda = 0 \implies s_k = \lambda^{\frac{1}{4\alpha}} e^{i\left(\frac{2k\pi+\pi}{4\alpha}\right)}.$$

On the other hand

$$\frac{3}{4} < \alpha \leq 1 \implies \frac{(2k+1)\pi}{4} < \frac{(2k+1)\pi}{4\alpha} \leq \frac{(2k+1)\pi}{3}.$$

Acceptable poles are

$$s_{-2} = \lambda^{\frac{1}{4\alpha}} e^{-i\left(\frac{3\pi}{4\alpha}\right)} \ , \ s_{-1} = \lambda^{\frac{1}{4\alpha}} e^{-i\left(\frac{\pi}{4\alpha}\right)} \ , \ s_0 = \lambda^{\frac{1}{4\alpha}} e^{i\left(\frac{\pi}{4\alpha}\right)} \ , \ s_1 = \lambda^{\frac{1}{4\alpha}} e^{i\left(\frac{3\pi}{4\alpha}\right)}.$$

Thus

$$E_{4\alpha,2}(-\lambda) = \frac{1}{2\pi i}\int_C \frac{s^{4\alpha-2}}{s^{4\alpha}+\lambda}e^s ds = \frac{1}{2\pi i}\left\{2\pi i \sum_{i=-2}^{1} \frac{e^{s_i}}{4\alpha s_i}\right\} = \frac{1}{4\alpha}\sum_{i=-2}^{1}\frac{e^{s_i}}{s_i}$$

$$= \frac{1}{4\alpha}\left\{\frac{e^{\lambda^{\frac{1}{4\alpha}} e^{i\frac{-3\pi}{4\alpha}}}}{\lambda^{\frac{1}{4\alpha}} e^{i\frac{-3\pi}{4\alpha}}} + \frac{e^{\lambda^{\frac{1}{4\alpha}} e^{i\frac{-\pi}{4\alpha}}}}{\lambda^{\frac{1}{4\alpha}} e^{i\frac{-\pi}{4\alpha}}} + \frac{e^{\lambda^{\frac{1}{4\alpha}} e^{i\frac{\pi}{4\alpha}}}}{\lambda^{\frac{1}{4\alpha}} e^{i\frac{\pi}{4\alpha}}} + \frac{e^{\lambda^{\frac{1}{4\alpha}} e^{i\frac{3\pi}{4\alpha}}}}{\lambda^{\frac{1}{4\alpha}} e^{i\frac{3\pi}{4\alpha}}}\right\}.$$

Finally

$$E_{4\alpha,2}(-\lambda) = \left\{ \frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{3\pi}{4\alpha}\right))}}{2\alpha\lambda^{\frac{1}{4\alpha}}}\cos\left(\lambda^{\frac{1}{4\alpha}}\sin\frac{3\pi}{4\alpha} - \frac{3\pi}{4\alpha}\right) + \frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{\pi}{4\alpha}\right)}}{2\alpha\lambda^{\frac{1}{4\alpha}}}\cos\left(\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha} - \frac{\pi}{4\alpha}\right)\right\}. \quad (14)$$

In similarly on $E_{4\alpha,2\alpha+2}(-\lambda)$ and $E_{4\alpha,-2\alpha+2}(-\lambda)$, we have

$$E_{4\alpha,2\alpha+2}(-\lambda) = \frac{\lambda^{\frac{-2\alpha-1}{4\alpha}}}{2\alpha}\left\{ e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{3\pi}{4\alpha}\right)}\sin\left(\lambda^{\frac{1}{4\alpha}}\sin\frac{3\pi}{4\alpha} - \frac{3\pi}{4\alpha}\right) - e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{\pi}{4\alpha}\right)}\sin\left(\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha} - \frac{\pi}{4\alpha}\right)\right\}, \quad (15)$$

and

$$E_{4\alpha,-2\alpha+2}(-\lambda) = \frac{\lambda^{\frac{2\alpha-1}{4\alpha}}}{2\alpha}\left\{ e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{3\pi}{4\alpha}\right)}\sin\left(\lambda^{\frac{1}{4\alpha}}\sin\frac{3\pi}{4\alpha} - \frac{3\pi}{4\alpha}\right) - e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{\pi}{4\alpha}\right)}\sin\left(\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha} - \frac{\pi}{4\alpha}\right)\right\}. \quad (16)$$

With substitution (14), (15) and (16) in (13), we get

$$\left\{\frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{3\pi}{4\alpha}\right)}}{2\alpha\lambda^{\frac{1}{4\alpha}}}\cos\left(\lambda^{\frac{1}{4\alpha}}\sin\frac{3\pi}{4\alpha} - \frac{3\pi}{4\alpha}\right) + \frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{\pi}{4\alpha}\right)}}{2\alpha\lambda^{\frac{1}{4\alpha}}}\cos\left(\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha} - \frac{\pi}{4\alpha}\right)\right\}^2$$
$$-\left\{\frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{3\pi}{4\alpha}\right)}}{2\alpha\lambda^{\frac{1}{4\alpha}}}\sin\left(\lambda^{\frac{1}{4\alpha}}\sin\frac{3\pi}{4\alpha} - \frac{3\pi}{4\alpha}\right) - \frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{\pi}{4\alpha}\right)}}{2\alpha\lambda^{\frac{1}{4\alpha}}}\sin\left(\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha} - \frac{\pi}{4\alpha}\right)\right\}^2 = 0,$$

which implies

$$\frac{e^{\lambda^{\frac{1}{4\alpha}}\cos(\frac{3\pi}{4\alpha})}}{2\alpha}\left[\sin\left(\frac{\pi}{4}-\lambda^{\frac{1}{4\alpha}}\sin\frac{3\pi}{4\alpha}+\frac{3\pi}{4\alpha}\right)\right]+\frac{e^{\lambda^{\frac{1}{4\alpha}}\cos(\frac{3\pi}{4\alpha})}}{2\alpha}\left[\sin\left(\frac{\pi}{4}+\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha}-\frac{\pi}{4\alpha}\right)\right]=0,$$

we denote above transcendental equation with $h_\alpha(-\lambda)=f_\alpha(-\lambda)+g_\alpha(-\lambda)$, with

$$f_\alpha(-\lambda)=\frac{e^{\lambda^{\frac{1}{4\alpha}}\cos(\frac{3\pi}{4\alpha})})}{2\alpha}\left[\sin\left(\frac{\pi}{4}-\lambda^{\frac{1}{4\alpha}}\sin\frac{3\pi}{4\alpha}+\frac{3\pi}{4\alpha}\right)\right],$$

and

$$g_\alpha(-\lambda)=\frac{e^{\lambda^{\frac{1}{4\alpha}}\cos(\frac{3\pi}{4\alpha})})}{2\alpha}\left[\sin\left(\frac{\pi}{4}+\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha}-\frac{\pi}{4\alpha}\right)\right]. \tag{17}$$

It is obvious that

$$g_n=\left(\frac{n\pi+\frac{\pi}{4\alpha}-\frac{\pi}{4}}{\sin\frac{\pi}{4\alpha}}\right)^{4\alpha},\quad n=-1,0,1,\ldots,$$

are positive zeros of $g_\alpha(-\lambda)$ and $g_\alpha(0)=\frac{1}{2\alpha}\sin(\frac{\pi}{4}-\frac{\pi}{4\alpha})$, and $f_\alpha(0)=\frac{1}{2\alpha}\sin(\frac{\pi}{4}-\frac{\pi}{4\alpha})$.

In order to get positive eigenvalues, since $\cos(\frac{3\pi}{4\alpha})$ is negative as long as $\frac{3}{4}<\alpha\le 1$, then for all sufficiently large λ, $f_\alpha(-\lambda)\to o$, thus

$$h_\alpha(-\lambda)\simeq g_\alpha(-\lambda).$$

Now from $h_\alpha(-\lambda)=0$, it can be concluded that

$$\sin\left(\frac{\pi}{4}+\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha}-\frac{\pi}{4\alpha}\right)=0.$$

Finally from there we obtain the following asymptotic formula

$$\lambda_n\sim\left(\frac{n\pi-\frac{\pi}{4\alpha}+\frac{\pi}{4}}{\sin\frac{\pi}{4\alpha}}\right)^{4\alpha},\quad n\to\infty. \tag{18}$$

A glance at (17) shows that the set of all λ such that

$$\pi(1+2n)<\sin\left(\frac{\pi}{4}+\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha}-\frac{\pi}{4\alpha}\right)<2\pi(1+n),$$

implies

$$\left(\frac{2n\pi+\frac{3\pi}{4}+\frac{\pi}{4\alpha}}{\sin\frac{\pi}{4\alpha}}\right)^{4\alpha}<\lambda_n(\alpha)<\left(\frac{2n\pi+\frac{7\pi}{4}+\frac{\pi}{4\alpha}}{\sin\frac{\pi}{4\alpha}}\right)^{4\alpha},\qquad n=0,1,2,\ldots$$

Remark 2. As the last discussion, the asymptotic formula (18) is the generalization of classic one i.e. as $\alpha\to 1^-$, this corresponds exactly with the well known classical asymptotic estimate $\lambda_n\sim(\frac{2n\pi}{\sqrt{2}})^4$ as $n\to\infty$.

Remark 3. The uniqueness of the answer is obtained by the fixed point theorem. For this case, as well as the convergence of 10 with boundary conditions (11), we refer the reader to ([30], 3.2).

For the equation (10), Table 1 and Figures 1, 2 show the eigenvalues and eigenfunctions (EFs) for different $\alpha=0.75$, $\alpha=0.86$ and $\alpha=1$, respectively.

Table 1

The eigenvalues λ_n

α=0.76	α= 0.86	α=1.0
λ_1=13334.95	λ_1=83.59	λ_1=97.41
λ_2=192878.31	λ_2=654.87	λ_2= 1558.54
λ_3= 979175.49	λ_3=3098.92	λ_3=7890.14
	λ_4=6255.26	λ_4= 24936.72
		λ_5=60880.68
		$\vdots$

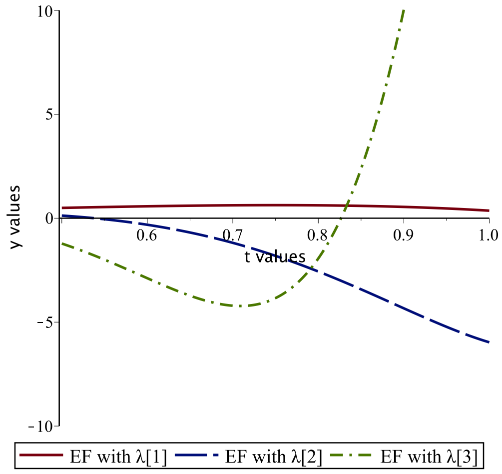

Figure 1.

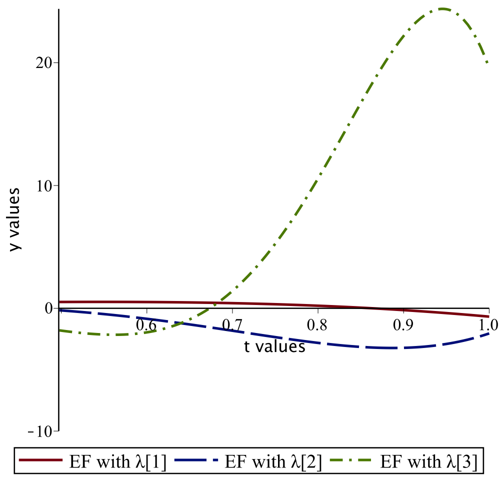

Figure 2.

Here, for the purpose of comparison, we present an example from reference [38].

Example 1. Consider the following fourth order fractional eigenvalue problem

$$D^{\alpha}y(x) = \lambda y(x), \quad x \in (0,1), \tag{19}$$

with the boundary conditions:

$$y(0) = y'(0) = 0, \; y(1) = y'(1) = 0, \tag{20}$$

where $3 < \alpha = \frac{p}{q} \leq 4$.

By the Laplace transform of the Caputo derivative, we have:

$$y(x) = AE_{\alpha,1}(\lambda x^{\alpha}) + BxE_{\alpha,2}(\lambda x^{\alpha}) + Cx^2E_{\alpha,3}(\lambda x^{\alpha}) + Dx^3E_{\alpha,4}(\lambda x^{\alpha}).$$

Notice that the first derivative of y is given by

$$y'(x) = \frac{A}{\alpha}E_{\alpha,0}(\lambda x^{\alpha}) + BE_{\alpha,1}(\lambda x^{\alpha}) + CxE_{\alpha,2}(\lambda x^{\alpha}) + Dx^2E_{\alpha,3}(\lambda x^{\alpha}).$$

By applying the boundary conditions, to obtain the non-trivial eigenvalues, we set the determinant of the coefficient matrix equal to zero, thus we get

$$(E_{\alpha,3}(\lambda))^2 - E_{\alpha,2}(\lambda)E_{\alpha,4}(\lambda) = 0,$$

a result that was obtained in [38] using a different method, while we applied the asymptotic method introduced in this paper in order to determine the eigenvalues.

Conclusions

In this article, the eigenvalues and the eigenfunctions were derived by studying three features of fractional Sturm-Liouville equations of mixed Riemann-Liouville and Caputo fractional derivatives type. The focus of the paper is on the asymptotic distribution of the eigenvalues obtained from transcendental equation under the assumption the potential function is considered to be zero.

On the other hand in order to get eigenvalues, we studied transcendental equation based on asymptotic behavior of the Mittag-Leffler function rather than numerical method. Also our results showed that it is consistent with the classical one as $\alpha \to 1^{-}$.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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On two four-dimensional curl operators and their applications

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Academician O.A. Ladyzhenskaya emphasized the importance of constructing a fundamental system in the space of solenoidal functions for simple domains such as squares, cubes, and similar regions. This article examines the problem of constructing such fundamental systems for a four-dimensional parallelepiped and cube. As is well known, applying the stream functions known from the two- and three-dimensional cases, the spectral problem for the Stokes operator reduces to the so-called clamped plate problem, which, in turn, has no solution in domains such as the square, cube, or parallelepiped. Thus, in higher-dimensional cases, the necessity of an analogous stream function becomes evident. In this work, the authors propose two curl operators that satisfy the above-mentioned requirements. Using the introduced curl operators, the spectral problem for the biharmonic operator in a four-dimensional parallelepiped and cube is formulated. Alternative approaches to constructing a fundamental system are presented, given the unsolvability of the spectral problem. Furthermore, the growth orders of the obtained eigenvalues are established.

Keywords: spectral problem, fundamental system, curl operator.

2020 Mathematical Subject Classification: 35P05, 35P10, 47A75, 53C65.

Introduction

As is well known, the theoretical foundation of classical electromagnetic field (EMF) theory is based on Maxwell's equations, which generalize the experimental results obtained by the end of the 18th century. The development of classical EMF theory led to its description as an antisymmetric second-rank tensor, from which Maxwell's equations follow. These equations played a key role in the development of theoretical physics and had a profound influence on the creation of the special theory of relativity and other theories. By the early 20th century, classical electrodynamics was considered a completed science, and the EMF theory received its further development in the form of quantum electrodynamics.

In this work, we consider two four-dimensional curl operators. While the first curl operator is closely related to electromagnetic field theory and Maxwell's equations, the second curl operator is introduced artificially. The first curl operator is introduced (theoretically well-founded) using an antisymmetric second-rank tensor [1; 146, 149]. In fact, the four-dimensional curl operator is introduced on a six-dimensional vector field. In contrast to the first, the artificially chosen four-dimensional curl operator is introduced on a four-dimensional vector field. These operators are used by us to construct fundamental systems in the space of solenoidal functions. These systems are not only important theoretically but also computationally efficient for the approximate solution of boundary value problems for the Stokes and Navier-Stokes equation systems.

It should be noted that spectral problems for the Stokes operator (with periodicity conditions) in a cubic domain were also considered in the works [2–4]. In the work [2], the spectra of the curl and Stokes operators in a cube for functions satisfying the periodicity condition are studied. The Cauchy

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problem for the 3-D Navier-Stokes equations with periodic boundary conditions in the spatial variable was studied in [4].

First of all, let us formulate the spectral problem for the Stokes operator. Let $x = (x_1, ..., x_d) \in \Omega \subset \mathbb{R}^d,\ d \geq 2$, be an open bounded (simply connected) domain with boundary $\partial\Omega$. We seek nontrivial solutions $\{\vec{w}_k(x),\ p_k(x),\ x \in \Omega,\ \ k \in \mathbb{N}\}$ and corresponding values of the parameter $\{\mu_k^2,\ k \in \mathbb{N}\}$ for the following boundary value problem [5; 38]:

$$\begin{cases} -\Delta\vec{w}(x) + \nabla p(x) & = & \mu^2 w(x), & x \in \Omega, \\ \operatorname{div}\{\vec{w}(x)\} & = & 0, & x \in \Omega, \\ \vec{w}(x) & = & 0, & x \in \partial\Omega. \end{cases} \qquad (A)$$

In the terminology of inverse problem theory for differential equations [6], problem (A) can be interpreted as a coefficient inverse problem, where the condition of overdetermination is represented by the requirement of the nontriviality of the solution $\{\vec{w}(x), \nabla p(x)\}$, corresponding to the sought coefficient μ^2.

Let us introduce the main spaces that will be used. Let $x = (x_1, ..., x_d) \in \Omega \subset \mathbb{R}^d,\ d \geq 2$, be an open bounded (simply connected) domain with a sufficiently smooth boundary $\partial\Omega$, and $m \geq 0$ be an integer,

$$W_2^m(\Omega) = \left\{ v |\ \partial_x^{|\alpha|} v \in L^2(\Omega),\ |\alpha| \leq m \right\}, \quad \text{where} \ \ \partial_x^{|\alpha|} = \partial_{x_1}^{\alpha_1} ... \partial_{x_d}^{\alpha_d},\ \ |\alpha| = \sum_{j=1}^{d} \alpha_j,\ \ \partial_{x_j} = \frac{\partial}{\partial_{x_j}},$$

$$\overset{\circ}{W}{}_2^m(\Omega) = \left\{ v |\ v \in W_2^m(\Omega),\ \partial_{\vec{n}}^j v = 0,\ j = 0, 1, 2..., m-1,\ \vec{n} \text{ is the outward normal to } \partial\Omega \right\}.$$

Otherwise, in the notation of the spaces, we will follow the monograph [7].

1 *The first four-dimensional curl operator*

Let us consider the four-dimensional case of the spectral problem (A). We start from the case of the four-dimensional rectangular parallelepiped.

Let $\Omega_4 = \{x_0 < x < x_1,\ y_0 < y < y_1,\ z_0 < z < z_1,\ \zeta_0 < \zeta < \zeta_1\}$ be a rectangular parallelepiped, where $x_0, x_1, y_0, y_1, z_0, z_1, \zeta_0, \zeta_1$ are given.

Problem 1.1. Find the vector function $\vec{U}(x, y, z, \zeta)$ for the given solenoidal vector function $\vec{w}(x, y, z, \zeta)$, i.e.

$$\operatorname{curl} \vec{U}(x, y, z, \zeta) = \vec{w}(x, y, z, \zeta),\ \ \operatorname{div} \vec{w}(x, y, z, \zeta) = 0,\ \ (x, y, z, \zeta) \in \Omega_4, \qquad (1)$$

where $\vec{U} = \{U_1, U_2, U_3, U_4, U_5, U_6\},\ \vec{w} = \{w_1, w_2, w_3, w_4\}$,

$$U_k \in W_2^2(\Omega_4),\ \ k = 1, 2, 3, 4, 5, 6;\ \ \ w_j \in W_2^1(\Omega_4),\ \ j = 1, 2, 3, 4. \qquad (2)$$

We introduce the first four-dimensional curl operator in the following way

$$\vec{w} = \operatorname{curl} \vec{U} = \begin{pmatrix} -\partial_y U_1 - \partial_z U_2 - \partial_\zeta U_3 \\ \partial_x U_1 + \partial_\zeta U_5 - \partial_z U_6 \\ \partial_x U_2 + \partial_y U_6 - \partial_\zeta U_4 \\ \partial_x U_3 + \partial_z U_4 - \partial_y U_5 \end{pmatrix},\ \ \operatorname{div}\operatorname{curl} \vec{U} = 0. \qquad (3)$$

Remark 1. The curl operator in equation (3) acts on a six-dimensional vector $\vec{U}$, which, in particular, corresponds to the following vector composed of the intensity vectors of the electric field $\vec{E}$ and the magnetic field $\vec{H}$: $\vec{E} = \{E^1, E^2, E^3\}$, $\vec{H} = \{H^1, H^2, H^3\}$ [1; 149, 274] namely, $\vec{U} = \{E^1, E^2, E^3, H^1, H^2, H^3\}$.

We introduce the following notations

$$\begin{gathered} y = x_1, \quad z = x_2, \quad \zeta = x_3, \quad x = x_4, \\ U_k = E^k = ic\varepsilon_0 E_k, \quad U_{k+3} = H^k = \tfrac{1}{\mu_0} B_k, \quad k = 1,2,3, \\ w_4 = \tfrac{\varrho}{\varepsilon_0}, \quad w_k = j_k, \quad k = 1,2,3, \end{gathered} \tag{4}$$

where $E_k,\ k = 1,2,3$ are the components of the electric field intensity vector, $B_k,\ k = 1,2,3$ are the components of the magnetic field intensity vector, c is the speed of light in a vacuum, ε_0 is the dielectric constant in a vacuum, μ_0 is the magnetic permeability in a vacuum, ϱ is the charge density, $\vec{j} = \{j_1, j_2, j_3\}$ is the electric current density vector, and $i = \sqrt{-1}$.

Proposition 1. According to (3)–(4) and [1; 149] we will have Maxwell's equations for the electromagnetic field in a vacuum:

$$\begin{cases} -ic\varepsilon_0 \partial_{x_1} E_1 + \frac{1}{\mu_0}\partial_{x_3} B_3 - \frac{1}{\mu_0}\partial_{x_4} B_2 = j_1, \\ -ic\varepsilon_0 \partial_{x_1} E_2 + \frac{1}{\mu_0}\partial_{x_4} B_1 - \frac{1}{\mu_0}\partial_{x_2} B_3 = j_2, \\ -ic\varepsilon_0 \partial_{x_1} E_3 + \frac{1}{\mu_0}\partial_{x_2} B_2 - \frac{1}{\mu_0}\partial_{x_3} B_1 = j_3, \\ ic\varepsilon_0 \partial_{x_1} E_1 + ic\varepsilon_0 \partial_{x_2} E_2 + ic\varepsilon_0 \partial_{x_3} E_3 = \frac{\varrho}{\varepsilon_0}, \end{cases} \tag{5}$$

$$\begin{cases} \partial_{x_2} B_1 + \partial_{x_3} B_2 + \partial_{x_4} B_3 = 0, \\ \partial_{x_4} B_1 + \frac{i}{c}\partial_{x_3} E_2 - \frac{i}{c}\partial_{x_2} E_3 = 0, \\ \partial_{x_4} B_2 + \frac{i}{c}\partial_{x_1} E_3 - \frac{i}{c}\partial_{x_3} E_1 = 0, \\ \partial_{x_4} B_3 + \frac{i}{c}\partial_{x_2} E_1 - \frac{i}{c}\partial_{x_1} E_2 = 0. \end{cases} \tag{6}$$

Proof of Proposition 1. Indeed, if according to (3)–(4) and [1; 149] we introduce new independent variables instead of the spacetime coordinates (x, y, z, t) as

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict, \tag{7}$$

where c is the speed of light in a vacuum, one can observe a remarkable symmetry in Maxwell's equations describing the electromagnetic field.

With the notation (7), the equations satisfied by the electrodynamic potentials (V, A_1, A_2, A_3) can be written as:

$$\begin{cases} \partial^2_{x_1} A_k + \partial^2_{x_2} A_k + \partial^2_{x_3} A_k + \partial^2_{x_4} A_k & = & -\mu_0 j_k, \\ \partial^2_{x_1} V + \partial^2_{x_2} V + \partial^2_{x_3} V + \partial^2_{x_4} V & = & -\frac{\varrho}{\varepsilon_0}, \end{cases}$$

where $k = 1,2,3$, and the Lorentz condition can be written as:

$$\partial_{x_1} A_1 + \partial_{x_2} A_2 + \partial_{x_3} A_3 + \frac{i}{c}\partial_{x_4} V = 0.$$

Introducing the notation

$$\Phi_k = A_k, \quad k = 1,2,3; \quad \Phi_4 = \frac{i}{c} V, \tag{8}$$

the relations between the electromagnetic field vectors $\vec{E} = \{E_1, E_2, E_3\}$, $\vec{B} = \{B_1, B_2, B_3\}$, and the potentials V, $\vec{A} = \{A_1, A_2, A_3\}$:

$$\vec{E} = -\nabla V - \vec{A}, \ \ \vec{B} = \nabla \times \vec{A},$$

can be written as follows:
$$\begin{cases} E_k = -\partial_{x_k} V + \frac{c}{i}\partial_{x_4} A_k, & k = 1,2,3, \\ B_j = \partial_{x_k} A_l - \partial_{x_l} A_k, & j,k,l = 1,2,3, \end{cases}$$
or, taking into account (8), we will have
$$\begin{cases} -\frac{i}{c}E_k &= \partial_{x_k}\Phi_4 - \partial_{x_4}\Phi_k, \quad k = 1,2,3, \\ B_j &= \partial_{x_k}\Phi_l - \partial_{x_l}\Phi_k, \quad j,k,l = 1,2,3. \end{cases} \tag{9}$$

Considering the right-hand side of the relations (9), we will define the elements of the matrix $F_{\mu\nu}$ using Table 1.

T a b l e 1

Matrix $F_{\mu\nu}$

$\mu \parallel \nu$	1	2	3	4
1	0	B_3	$-B_2$	$-(i/c)E_1$
2	$-B_3$	0	B_1	$-(i/c)E_2$
3	B_2	$-B_1$	0	$-(i/c)E_2$
4	$(i/c)E_1$	$(i/c)E_2$	$(i/c)E_3$	0

Therefore, we have
$$\begin{array}{lll} F_{12} = B_3, & F_{13} = -B_2, & F_{23} = B_1, \\ F_{14} = -\frac{i}{c}E_1, & F_{24} = -\frac{i}{c}E_2, & F_{34} = -\frac{i}{c}E_3, \end{array}$$
and we confirm the validity of the conditions
$$F_{\mu\nu} = -F_{\nu\mu}.$$
In these notations, the relations (9) will be written as:
$$F_{\mu\nu} = \partial_{x_\mu}\Phi_\nu - \partial_{x_\nu}\Phi_\mu, \quad \mu,\nu = 1,2,3,4. \tag{10}$$
From here, according to (3)–(4), we obtain the equations (5).

It remains to establish the equations (6). From (10), we obtain
$$\partial_{x_\lambda}F_{\mu\nu} + \partial_{x_\nu}F_{\lambda\mu} + \partial_{x_\mu}F_{\nu\lambda} = 0, \quad \lambda,\mu,\nu = 1,2,3,4. \tag{11}$$
It can be verified that, according to Table 1, the relations (11) are equivalent to the relations (6).

On the other hand, taking into account the notations (4) we can now write the spectral problem for the Stokes operator corresponding to the equations (5). We have
$$\begin{cases} -\Delta\vec{w}(x) + \nabla p(x) &= \mu^2 w(x), & x \in \Omega_4, \\ \operatorname{div}\{\vec{w}(x)\} &= 0, & x \in \Omega_4, \\ \vec{w}(x) &= 0, & x \in \partial\Omega_4, \end{cases} \tag{12}$$
where, when returning from the notation of independent variables (x_1, x_2, x_3, x_4) to the notation (x, y, z, ζ):
$$x = x_1, \quad y = x_2, \quad z = x_3, \quad \zeta = x_4,$$

we get

$$w_4(x,y,z,\zeta)=\frac{\varrho(x,y,z,\zeta)}{\varepsilon_0},\quad w_k(x,y,z,\zeta)=j_k(x,y,z,\zeta),\quad k=1,2,3,$$

$p(x,y,z,\zeta)$ is the scalar function of the artificial pressure.

Let us introduce the notation for the spaces

$$\mathbf{W}_2^2(\Omega_4)=\left(W_2^2(\Omega_4)\right)^6,\ \ \mathbf{W}_2^1(\Omega_4)=\left(W_2^1(\Omega_4)\right)^4. \tag{13}$$

Proposition 2. In the notation (4), the following equality of the sets holds [5; 470]:

$$\operatorname{curl}\left\{\mathbf{W}_2^2(\Omega_4)\right\}=\mathbf{W}(\Omega_4)=\left\{\vec{w}\in\mathbf{W}_2^1(\Omega_4),\ \operatorname{div}\vec{w}=0\right\}. \tag{14}$$

Proposition 3. If $U_1=U_2=U_3=U_4=U_5=U_6=U(x,y,z,\zeta)\in W_2^2(\Omega_4)$, then instead of (14) we obtain:

$$\vec{w}\in\mathbf{W}_0(\Omega_4)=\operatorname{curl}\{\vec{U}\}_{|U_1=U_2=U_3=U_4=U_5=U_6=U\in W_2^2(\Omega_4)}\subset\mathbf{W}(\Omega_4).$$

The following statement has been proven.

Proposition 4. For each four-dimensional vector function $\vec{w}(x,y,z,\zeta)\in\mathbf{W}(\Omega_4)$ (14), there exists a six-dimensional vector function $\vec{U}(x,y,z,\zeta)\in\mathbf{W}_2^2(\Omega_4)$ (13) that satisfies the relations (1)–(3). The converse statement is also true: for each six-dimensional vector function $\vec{U}(x,y,z,\zeta)\in\mathbf{W}_2^2(\Omega_4)$ (13), there exists a four-dimensional vector function $\vec{w}(x,y,z,\zeta)\in\mathbf{W}(\Omega_4)$ (14) that satisfies the relations (1)–(3).

Now we turn to the case of the four-dimensional cube $\Omega=\{0<x,y,z,\zeta<l\}$.

Let $U_1=U_2=U_3=U_4=U_5=U_6=U$,

$$\vec{U}=\{U,U,U,U,U,U\},\ \ (x,y,z,\zeta)\in\Omega, \tag{15}$$

then we have:

$$\vec{U}(x,y,z,\zeta)=\{U,U,U,U,U,U\},$$

$$\vec{w}(x,y,z,\zeta)=\{w_1,w_2,w_3,w_4\}.$$

We introduce the curl operator (3) for the "four-dimensional cube" Ω under the condition (15) in the following way

$$\vec{w}=\operatorname{curl}\vec{U}=\begin{pmatrix}(-\partial_y-\partial_z-\partial_\zeta)U\\(\partial_x+\partial_\zeta-\partial_z)U\\(\partial_x+\partial_y-\partial_\zeta)U\\(\partial_x+\partial_z-\partial_y)U\end{pmatrix},\ \ \operatorname{div}\operatorname{curl}\vec{U}=0.$$

In this case, the spectral problem for the Stokes operator (12) takes the form:

$$(-\Delta)^2U=\lambda^2(-\Delta)U,\quad (x,y,z,\zeta)\in\Omega,$$

$$U=\partial_{\vec{n}}U=0,\quad (x,y,z,\zeta)\in\partial\Omega,$$

where $\boxed{\lambda^2=3\mu^2}$ and $\vec{n}$ is the outward unit normal to $\partial\Omega$.

The spectral problem (A1), along with its extensions to polyharmonic operators, has been the subject of extensive research [8–11]. It has been shown that an explicit solution for a square domain is unattainable due to the failure of the separation of variables method in this case. The only known exceptions are circular and spherical domains [12–14]. In [15], lower bounds for the first eigenvalue of problem (A1) were derived for various manifolds. Numerical approximations of problem (A1) are provided in [16, 17]. Various issues of biharmonic operators were also studied in [18–20].

Let us replace the biharmonic operator in the problem (12) with a fourth-order differential operator.

Problem 1.1.

$$(\partial_x^4+\partial_y^4+\partial_z^4+\partial_\zeta^4)U=\lambda^2(-\Delta)U,\ \ (x,y,z,\zeta)\in\Omega, \tag{16}$$

$$U_{|\partial\Omega}=\partial_{\vec n}U_{|\partial\Omega}=0. \tag{17}$$

The spaces $V_1(\Omega)$ and $V_2(\Omega)$, with dim $=4$. Let $V_1(\Omega)$ and $V_2(\Omega)$ denote the spaces equipped with the scalar products: $(\nabla u,\nabla v)_{L^2(\Omega)}\ \forall\, u,v\in\overset{\circ}{W}{}_2^1(\Omega)$ and $((u,v))\overset{\text{def}}{=}\left(\partial_x^2u,\partial_x^2v\right)_{L^2(\Omega)}+\left(\partial_y^2u,\partial_y^2v\right)_{L^2(\Omega)}+$ $+\left(\partial_z^2u,\partial_z^2v\right)_{L^2(\Omega)}+\left(\partial_\zeta^2u,\partial_\zeta^2v\right)_{L^2(\Omega)}\ \forall\, u,v\in\overset{\circ}{W}{}_2^2(\Omega)$.

We will show that the set of "generalized eigenfunctions" of the inverse operator T^{-1} to the operator from (20), belonging to the space $V_2(\Omega)$, forms an orthonormal basis in the space $V_1(\Omega)$.

For this purpose, we will consider the following auxiliary boundary value problem.

$$\left(\partial_x^4+\partial_y^4+\partial_z^4+\partial_\zeta^4\right)u(x,y,z,\zeta)=(-\Delta)\,h(x,y,z,\zeta)\quad\text{in}\quad\Omega, \tag{18}$$

$$u(x,y,z,\zeta)=\partial_{\vec n}u(x,y,z,\zeta)=0\quad\text{on}\quad\partial\Omega, \tag{19}$$

which, in operator form, is expressed as:

$$Tu=B_1h, \tag{20}$$

$$T\in\mathscr{L}(\overset{\circ}{W}{}_2^2(\Omega);W_2^{-2}(\Omega)),\ \ B_1\in\mathscr{L}(L^2(\Omega);W_2^{-2}(\Omega)).$$

Let $h\in L^2(\Omega)$, i.e. $B_1h\in W_2^{-2}(\Omega)$. Then the boundary value problem (18)–(19) (or (20)) can be written as the following (highlighted) identity:

$$\langle Tu,v\rangle\overset{\text{def}}{=}\underline{((u,v))}=\langle B_1h,v\rangle\overset{\text{def}}{=}\underline{(h,-\Delta v)}_{L^2(\Omega)}\ \ \forall\ v(x,y,z,\zeta)\in\ V_2(\Omega), \tag{21}$$

where

$$((u,v))\overset{\text{def}}{=}\left(\partial_x^2u,\partial_x^2v\right)_{L^2(\Omega)}+\left(\partial_y^2u,\partial_y^2v\right)_{L^2(\Omega)}+\left(\partial_z^2u,\partial_z^2v\right)_{L^2(\Omega)}+\left(\partial_\zeta^2u,\partial_\zeta^2v\right)_{L^2(\Omega)}, \tag{22}$$

$$((u,u))\geq\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)}. \tag{23}$$

Remark 2. According to (24) from [7; 117, 125] for convex bounded domains $x=(x_1,...,x_d)\in\Omega\subset\mathbb{R}^d$, $d\geq 2$, with a piecewise smooth boundary $\partial\Omega$, the following inequality holds:

$$\|u_{xx}\|_{L^2(\Omega)}\leq C\|\Delta u\|_{L^2(\Omega)},\ \ \forall\, u\in W^2_{2,0}(\Omega)\equiv W_2^2(\Omega)\cap\overset{\circ}{W}{}_2^1(\Omega); \tag{24}$$

$$(u,v)^{(2)}_{2,\Omega}=\int\limits_{\Omega}(uv+u_xv_x+u_{xx}v_{xx})\,dx,\ \ u_{xx}=\left(u_{x_jx_k}\right),\ j,k=1,...,d,$$

therefore, due to the inequalities (24) and (23), the equivalent norm (22) is defined in the space $W_2^2(\Omega)\cap\overset{\circ}{W}{}_2^1(\Omega)$. This norm induces an equivalent norm on the subspace $\overset{\circ}{W}{}_2^2(\Omega)\subset W^2_{2,0}(\Omega)$.

Theorem 1. Let $h\in L^2(\Omega))$, i.e. $B_1h\in W_2^{-2}(\Omega)$. Then the boundary value problem (16)–(17) has a unique solution $u\in\ V_2(\Omega)$.

Remark 3. The statement of Theorem 1 remains valid if $h\in W^2_{2,0}(\Omega)$, i.e., $Bh\in L^2(\Omega)$, where $B\in\mathscr{L}(W^2_{2,0}(\Omega);L^2(\Omega))$.

Remark 4. According to (23) and (24) for the continuous bilinear functional $((u,v))$ (22), the condition of positive definiteness holds:

$$((u,u)) \geq K\|u\|^2_{V_2(\Omega)} \;\; \forall\, u \in V_2(\Omega). \tag{25}$$

Theorem 2. Let the operator B in the spectral problem (16)–(17) be defined as in Remark 3: $B \in \mathscr{L}(W^2_{2,0}(\Omega); L^2(\Omega))$. Then the operator T^{-1}:

$$T^{-1}: \; L^2(\Omega) \to V_2(\Omega) \subset L^2(\Omega)$$

is self-adjoint and compact, as it acts in the space $L^2(\Omega)$, i.e. the set of "generalized eigenfunctions" of the operator T^{-1}, belonging to the space $V_2(\Omega)$, forms an orthonormal basis in the space $V_1(\Omega)$.

Proof of Theorem 1. The statement of Theorem 1 follows from (21), (23), (25), and the continuity of the bilinear functional (22) in the space $\overset{\circ}{W}{}^2_2(\Omega)$ [21; 629, 653]. Definition of condition (E) [22; 169]: positive definiteness of the principal self-adjoint part of the system — the ellipticity condition for one equation.

Proof of Theorem 2. To prove Theorem 2, let us consider the mapping $T^{-1}: Bh \to u$, defined from the statement of Remark 3 to Theorem 1. It is linear and acts continuously from $L^2(\Omega)$ to $V_2(\Omega)$. Due to the compact embedding $V_2(\Omega) \hookrightarrow L^2(\Omega)$, the linear operator T^{-1}, considered as a linear operator on $L^2(\Omega)$, is compact. This operator is also self-adjoint ("relative to the operator B") because

$$(T^{-1}Bh_1, Bh_2)_{L^2(\Omega)} = ((u_1,u_2)) = \big(Bh_1, T^{-1}Bh_2\big)_{L^2(\Omega)},$$

where

$$T^{-1}Bh_i = u_i, \;\; Bh_i \in L^2(\Omega), \;\; T^{-1}Bh_i \in V_2(\Omega), \;\; i = 1,2.$$

Consequently, the operator T^{-1} possesses a complete orthonormal sequence of "generalized eigenfunction" (see the formulas below in (27)): $v_j \in V_2(\Omega)$,

$$T^{-1}Bv_j = \lambda_j^{-2} v_j, \quad j \geq 1, \quad \lambda_j^{-2} > 0, \quad \lambda_j^{-2} \to +0, \quad j \to +\infty. \tag{26}$$

By multiplying equatio (26) scalarly by Tv, we obtain

$$v_j \in V_2(\Omega), \;\; ((v_j, v)) = \lambda_j^2 (\nabla v_j, \nabla v)_{L^2(\Omega)}, \;\; \forall\, v \in V_2(\Omega), \tag{27}$$

where $j \geq 1$, $\lambda_j^2 > 0$, $\lambda_j^2 \to +\infty$, $j \to +\infty$, we indeed have

$$(v_j, Tv)_{L^2(\Omega)} = \underline{((v_j, v))} = \lambda_j^2 (Bv_j, v)_{L^2(\Omega)} = \underline{\lambda_j^2 (\nabla v_j, \nabla v)_{L^2(\Omega)}}.$$

From the underlined identity (which coincides with (27)), we obtain, as usual:

$$(\nabla v_j, \nabla v_k)_{L^2(\Omega)} = \delta_{jk}, \quad ((v_j, v_k)) = \lambda_j^2 \delta_{jk}, \quad \forall\, j, k.$$

This completes the proof of Theorem 2.

Theorem 3. The spectral problem (16)–(17) has the following solution:

$$U_n(x,y,z,\zeta) = X_n(x) Y_n(y) Z_n(z) \Upsilon_n(\zeta), \quad \lambda_n^2, \quad n \in \mathbb{N}, \tag{28}$$

where $X_n(x) = \Phi_n(\sigma)_{|\sigma=x}$, $Y_n(y) = \Phi_n(\sigma)_{|\sigma=y}$, $Z_n(z) = \Phi_n(\sigma)_{|\sigma=z}$, $\Upsilon_n(\zeta) = \Phi_n(\sigma)_{|\sigma=\zeta}$ are defined as follows

$$\begin{cases} \Phi_{2n-1}(\zeta) = \sin^2 \frac{\lambda_{2n-1}\zeta}{2}, \; \lambda^2_{2n-1} = \left(\frac{2(2n-1)\pi}{l}\right)^2, \;\; n \in \mathbb{N}, \\ \Phi_{2n}(\zeta) = \left[\lambda_{2n} l - \sin \lambda_{2n} l\right] \sin^2 \frac{\lambda_{2n}\zeta}{2} - \sin^2 \frac{\lambda_{2n} l}{2} \left[\lambda_{2n}\zeta - \sin \lambda_{2n}\zeta\right], \\ \qquad \lambda^2_{2n} = \left(\frac{2\nu_n}{l}\right)^2, \;\; n \in \mathbb{N}, \end{cases} \tag{29}$$

and $\{\nu_n, \; n \in \mathbb{N}\}$ are the positive roots of the equation $\tan \nu = \nu$, and $\mu_n^2 = \lambda_n^2/3$, $n \in \mathbb{N}$.

The arrangement of eigenvalues on the positive half-axis is shown in Figure 1 (here $l = 2$).

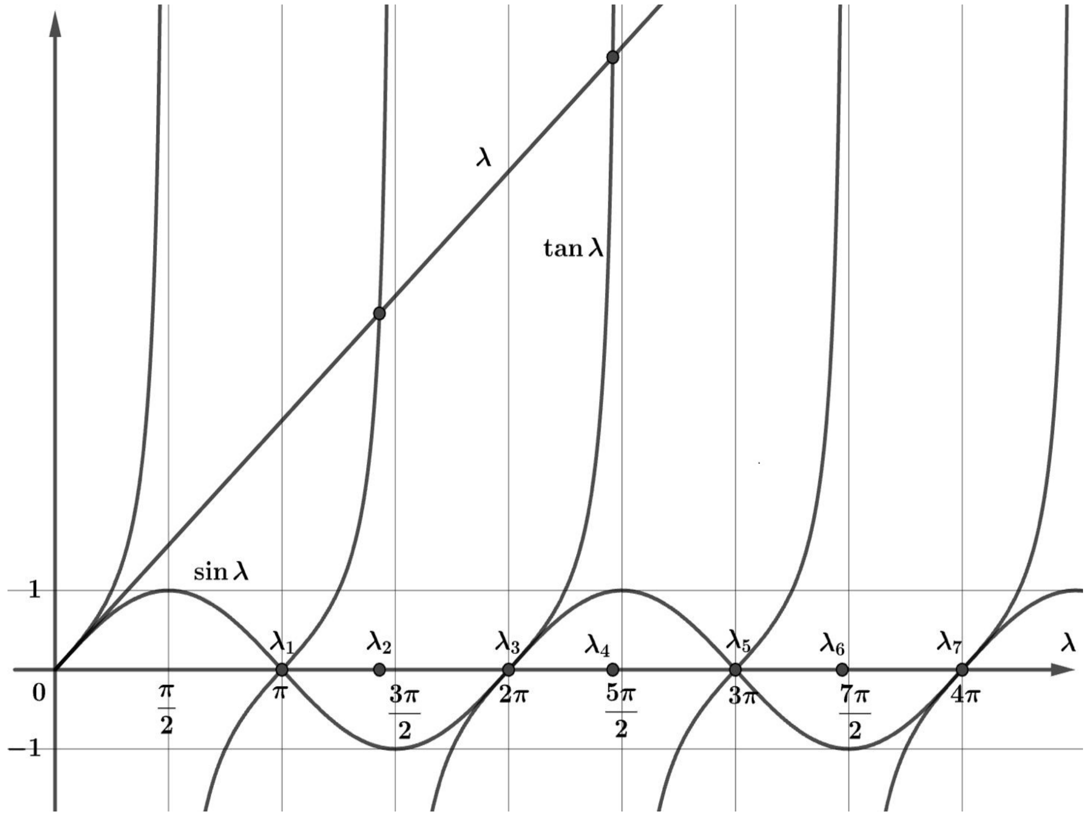

Figure 1. The positive roots of the equations (for $l = 2$):
$\tan\nu_n = \nu_n,\ \ \nu_n = \frac{\lambda_n l}{2} = \lambda_n;\ \ \sin\lambda_n = 0,\ \ n \in \mathbb{N}.$

From Figure 1 we get:

$$0 < \lambda_1 = \pi < \lambda_2 = \frac{3\pi}{2} - \varepsilon_1 < \lambda_3 = 2\pi < \lambda_4 = \frac{5\pi}{2} - \varepsilon_2 <$$

$$< \lambda_5 = 3\pi < \lambda_6 = \frac{7\pi}{2} - \varepsilon_3 < \lambda_7 = 4\pi < ...$$

Next, from Theorem 1, we obtain:

Corollary 1. The eigenvalues $\{\lambda_{2n},\ n \in \mathbb{N}\}$ are ordered as follows:

$$0 < \lambda_{2n} = \frac{2\nu_n}{l} < \frac{(2n+1)\pi}{2},\ \ \forall\, n \in \mathbb{N},$$

$$\lambda_{2n} = \frac{2\nu_n}{l} \to \frac{(2n+1)\pi}{2},\ \ n \to \infty,$$

where $\{\nu_n,\ n \in \mathbb{N}\}$ are the positive roots of the equation $\tan\nu = \nu$.

Theorem 4. The system of functions $\{U_n(x, y, z, \zeta)\}_{n=1}^{\infty} \subset V_2(\Omega)$, defined by the relations (28)–(29), forms a complete orthogonal sequence of “generalized eigenfunctions” in the space $V_1(\Omega)$.

Let us introduce the notations:

$$\overset{\circ}{\mathbf{W}}{}_2^2(\Omega) = (\overset{\circ}{W}{}_2^2(\Omega))^6\,, \quad \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) = (\overset{\circ}{W}{}_2^1(\Omega))^4\,, \tag{30}$$

$$\overset{\circ}{\mathbf{W}}(\Omega) = \{\vec{w} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega),\ \operatorname{div}\vec{w} = 0\}\,. \tag{31}$$

Proposition 5. For each four-dimensional vector-function $\vec{w}(x,y,z,\zeta) \in (V_1(\Omega))^4$, satisfying the condition $\operatorname{div}\vec{w}(x,y,z,\zeta) = 0$, there exists a unique six-dimensional vector-function $\vec{U}(x,y,z,\zeta) \in (V_2(\Omega))^6$. The converse statement is also true.

The validity of Proposition 5 directly follows from the proof of Proposition 4.

Proposition 6. The equality of sets holds [5; 470]:

$$\operatorname{curl}\{\overset{\circ}{\mathbf{W}}{}_2^2(\Omega)\} = \overset{\circ}{\mathbf{W}}(\Omega).$$

Let $\mathbf{H}(\Omega)$ denote the space of solenoidal functions, defined as follows:

$$\mathbf{H}(\Omega) = \{\vec{w}|\, \vec{w} \in \mathbf{L}^2(\Omega),\ \operatorname{div}\vec{w} = 0,\ \vec{w}\cdot\vec{n}|_{\partial\Omega} = 0\},\ \ \mathbf{L}^2(\Omega) = (L^2(\Omega))^4, \tag{32}$$

where $\vec{w}\cdot\vec{n}$ is the normal component of the vector $\vec{w}$.

Proposition 7. If $U_1 = U_2 = U_3 = U_4 = U_5 = U_6 = U(x,y,z,\zeta) \in \overset{\circ}{W}{}_2^2(\Omega)$, then we get

$$\vec{w} \in \overset{\circ}{\mathbf{W}}_0(\Omega) = \operatorname{curl}\{\vec{U}\}_{|U_1=U_2=U_3=U_4=U_5=U_6=U\in \overset{\circ}{W}{}_2^2(\Omega)} \subset \overset{\circ}{\mathbf{W}}(\Omega).$$

Next, we introduce the extended system of functions $\left\{U_m^1(x,y,z,\zeta)\right\}_{m=0}^{\infty}$, where

$$U_0^1(x,y,z,\zeta) \equiv 0,\ \ U_m^1(x,y,z,\zeta) = U_m(x,y,z,\zeta),\ \ m \in \mathbb{N},$$

and construct vector-functions

$$\vec{U}^1_{mjkqrs}(x,y,z,\zeta) = \left\{U_m^1, U_j^1, U_k^1, U_q^1, U_r^1, U_s^1\right\} \in \overset{\circ}{\mathbf{W}}{}_2^2(\Omega), \tag{33}$$

where $m,j,k,q,r,s \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, which, as is evident, will form a fundamental system in the space $\mathbf{V}_1(\Omega) = (V_1(\Omega))^6$.

Theorem 5. By applying the curl operator from (3) to the extended system of vector-functions (33), we obtain the desired fundamental system $(m,j,k,q \in \mathbb{N}_0)$:

$$\vec{w}_{mjkqrs}(x,y,z,\zeta) = \{w_{1,mjk}, w_{2,mrs}, w_{3,jsq}, w_{4,kqr}\} \in \overset{\circ}{\mathbf{W}}(\Omega) \tag{34}$$

in the space of solenoidal functions $\mathbf{H}(\Omega)$ (32), where

$$w_{1,mrs}(x,y,z,\zeta) = \partial_x U_m^1 + \partial_\zeta U_r^1 - \partial_z U_s^1,\ \ (x,y,z,\zeta)\in\Omega,\ \ m,r,s \in \mathbb{N}_0,$$

$$w_{2,jsq}(x,y,z,\zeta) = \partial_x U_j^1 + \partial_y U_s^1 - \partial_\zeta U_q^1,\ \ (x,y,z,\zeta)\in\Omega,\ \ j,s,q \in \mathbb{N}_0,$$

$$w_{3,kqr}(x,y,z,\zeta) = \partial_x U_k^1 + \partial_z U_q^1 - \partial_y U_r^1,\ \ (x,y,z,\zeta)\in\Omega,\ \ k,q,r \in \mathbb{N}_0,$$

$$w_{4,mjk}(x,y,z,\zeta) = -\partial_y U_m^1 - \partial_z U_j^1 - \partial_\zeta U_k^1,\ \ (x,y,z,\zeta)\in\Omega,\ \ m,j,k \in \mathbb{N}_0,$$

$$\operatorname{div}\vec{w}_{mjkqrs}(x,y,z,\zeta) = 0,\ \ (x,y,z,\zeta)\in\Omega,\ \ m,j,k,q,r,s \in \mathbb{N}_0, \tag{35}$$

$$\vec{w}_{mjkqrs}(x,y,z,\zeta) = 0,\ \ (x,y,z,\zeta)\in\partial\Omega,\ \ m,j,k,q,r,s \in \mathbb{N}_0. \tag{36}$$

Proof of Theorem 5. From the formulas defining the vector-function (34), we sequentially obtain:

$$|w_1 - w_1^\varepsilon|^2 \le 3\left[|\partial_x(U_1 - U_1^\varepsilon)|^2 + |\partial_\zeta(U_5 - U_5^\varepsilon)|^2 + |\partial_z(U_6 - U_6^\varepsilon)|^2\right],$$

$$|w_2 - w_2^\varepsilon|^2 \le 3\left[|\partial_x(U_2 - U_2^\varepsilon)|^2 + |\partial_y(U_6 - U_6^\varepsilon)|^2 + |\partial_\zeta(U_4 - U_4^\varepsilon)|^2\right],$$

$$|w_3 - w_3^\varepsilon|^2 \le 3\left[|\partial_x(U_3 - U_3^\varepsilon)|^2 + |\partial_z(U_4 - U_4^\varepsilon)|^2 + |\partial_y(U_5 - U_5^\varepsilon)|^2\right],$$

$$|w_4 - w_4^\varepsilon|^2 \le 3\left[|\partial_y(U_1 - U_1^\varepsilon)|^2 + |\partial_z(U_2 - U_2^\varepsilon)|^2 + |\partial_\zeta(U_3 - U_3^\varepsilon)|^2\right],$$

i.e.,

$$\|\vec{w} - \vec{w}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \le 2\sqrt{3}\|\vec{U} - \vec{U}^\varepsilon\|_{\mathbf{V}_1(\Omega)} \le 2\sqrt{3}\,\varepsilon,$$

where, first, according to Proposition 4, for each vector-function $\vec{w}(z, y, z, \zeta) \in \overset{\circ}{\mathbf{W}}(\Omega)$ from (31), there corresponds a unique vector-function $\vec{U}(z, y, z, \zeta) \in \overset{\circ}{\mathbf{W}}{}_2^2(\Omega)$ (30).

Secondly, the finite sum

$$\vec{U}^\varepsilon(x, y, z, \zeta) = \sum_{n=0}^{N_\varepsilon} a_n \vec{U}_n^1(x, y, z, \zeta), \quad N_\varepsilon < \infty,$$

ensuring the fulfillment of inequality

$$\|\vec{U}(x, y, z, \zeta) - \vec{U}^\varepsilon(x, y, z, \zeta)\|_{\mathbf{V}_1(\Omega)} \le \varepsilon$$

corresponds to the sum defined by the formulas:

$$\vec{w}^\varepsilon(x, y, z, \zeta) = \sum_{n=0}^{N_\varepsilon} a_n \vec{w}_n(x, y, z, \zeta), \quad N_\varepsilon < \infty,$$

and satisfying inequality:

$$\frac{1}{2\sqrt{3}}\|\vec{w}(x, y, z, \zeta) - \vec{w}^\varepsilon(x, y, z, \zeta)\|_{\mathbf{H}(\Omega)} \le \|\vec{U}(x, y, z, \zeta) - \vec{U}^\varepsilon(x, y, z, \zeta)\|_{(V_1(\Omega))^6} \le \varepsilon,$$

due to the fact that equality

$$\|\vec{w}(x, y, z, \zeta) - \vec{w}^\varepsilon(x, y, z, \zeta)\|_{\mathbf{L}^2(\Omega)} = \|\vec{w}(x, y, z, \zeta) - \vec{w}^\varepsilon(x, y, z, \zeta)\|_{\mathbf{H}(\Omega)}$$

and identity

$$\|\vec{w} - \vec{w}^\varepsilon\|_{\mathbf{H}^\perp(\Omega)} = 0, \;\; \mathbf{L}^2(\Omega) = \mathbf{H}(\Omega) \oplus \mathbf{H}^\perp(\Omega)$$

hold. The relations (28)–(31) show that the system of four-dimensional vector-functions

$$\{\vec{w}_n(x, y, z, \zeta)\}_{n=0}^\infty \subset (V_1(\Omega))^4,$$

forms a fundamental system in the space of solenoidal vector fields $\mathbf{H}(\Omega)$, satisfying the condition

$$\operatorname{div} \vec{w}_n(x, y, z, \zeta) = 0.$$

This concludes the proof of Theorem 5.

Proposition 8. Any vector-functions of the form

$$\vec{w}(x, y, z, \zeta) = \{0, 0, 0, w_4(x, y, z, \zeta)\},$$

$$\vec{w}(x, y, z, \zeta) = \{0, 0, w_3(x, y, z, \zeta), 0\},$$

$$\vec{w}(x, y, z, \zeta) = \{0, w_2(x, y, z, \zeta), 0, 0\},$$

$$\vec{w}(x, y, z, \zeta) = \{w_1(x, y, z, \zeta), 0, 0, 0\},$$

from the space $(\overset{\circ}{W}{}_2^1(\Omega))^4$ (where the functions $w_1(x, y, z, \zeta)$, $w_2(x, y, z, \zeta)$, $w_3(x, y, z, \zeta)$ and $w_4(x, y, z)$ are identically nonzero) cannot be solenoidal, i.e., they will not satisfy both the equation (incompressibility condition of the incompressible fluid) $\operatorname{div} \vec{w}(x, y, z, \zeta) = 0$ (35) and the boundary condition (36).

Thus, we have constructed a fundamental system in the space of solenoidal functions for a four-dimensional "cubic" domain.

2 *The second four-dimensional* curl *operator for a 4-D domain*

Let us formulate Problem 1.1 for the second four-dimensional curl operator. We start from the case of the four-dimensional rectangular parallelepiped, $\dim \Omega_4 = 4$.

Problem 2.1. Find the vector-function $\vec{U}(x, y, z, \zeta)$ for a given solenoidal vector-function $\vec{w}(x, y, z, \zeta)$, i.e.

$$\operatorname{curl} \vec{U}(x, y, z, \zeta) = \vec{w}(x, y, z, \zeta), \;\; \operatorname{div} \vec{w}(x, y, z, \zeta) = 0, \;\; (x, y, z, \zeta) \in \Omega_4, \tag{37}$$

where $\vec{U} = \{U_1, U_2, U_3, U_4\}, \;\; \vec{w} = \{w_1, w_2, w_3, w_4\}$,

$$U_j \in W_2^2(\Omega_4), \;\; w_j \in W_2^1(\Omega_4), \;\; j = 1, 2, 3, 4. \tag{38}$$

We introduce curl operator [5; 141] in the following way

$$\vec{w} = \operatorname{curl} \vec{U} = \begin{pmatrix} \partial_y U_3 - \partial_z U_2 - \partial_\zeta U_2 \\ \partial_z U_4 - \partial_\zeta U_3 - \partial_x U_3 \\ \partial_\zeta U_1 + \partial_x U_2 - \partial_y U_4 \\ \partial_x U_2 + \partial_y U_3 - \partial_z U_1 \end{pmatrix}, \;\; \operatorname{div} \operatorname{curl} \vec{U} = 0. \tag{39}$$

We recall the notation for the spaces

$$\mathbf{W}_2^2(\Omega_4) = \left(W_2^2(\Omega_4)\right)^4, \;\; \mathbf{W}_2^1(\Omega_4) = \left(W_2^1(\Omega_4)\right)^4. \tag{40}$$

Proposition 9. If $U_1 = U_2 = U_3 = U_4 = U(x, y, z, \zeta) \in W_2^2(\Omega_4)$, then instead of (14), we obtain:

$$\vec{w} \in \mathbf{W}_0(\Omega_4) = \operatorname{curl}\{\vec{U}\}_{|U_1=U_2=U_3=U_4=U \in W_2^2(\Omega_4)} \subset \mathbf{W}(\Omega_4).$$

The following statement is true.

Proposition 10. For each four-dimensional vector-function $\vec{w}(x, y, z, \zeta) \in \mathbf{W}(\Omega_4)$ (14), there exists a four-dimensional vector-function $\vec{U}(x, y, z, \zeta) \in \mathbf{W}_2^2(\Omega_4)$ (40) that satisfies the relations (37)–(39). The converse statement is also true: for each four-dimensional vector-function $\vec{U}(x, y, z, \zeta) \in \mathbf{W}_2^2(\Omega_4)$ (40), there exists a four-dimensional vector-function $\vec{w}(x, y, z, \zeta) \in \mathbf{W}(\Omega_4)$ (14) that satisfies the relations (37)–(39).

Now we turn to the case of the four-dimensional cube $\Omega = \{0 < x, y, z, \zeta < l\}$.

Let

$$U_1 = U_2 = U_3 = U_4 = U, \;\; \vec{U} = \{U, U, U, U\}, \;\; (x, y, z, \zeta) \in \Omega, \tag{41}$$

then we have

$$\vec{U}(x, y, z, \zeta) = \{U, U, U, U\},$$
$$\vec{w}(x, y, z, \zeta) = \{w_1, w_2, w_3, w_4\}.$$

The curl operator (39) for the “four-dimensional cube” Ω under the condition (41).

$$\vec{w} = \operatorname{curl} \vec{U} = \begin{pmatrix} (\partial_y - \partial_z - \partial_\zeta)U \\ (\partial_z - \partial_\zeta - \partial_x)U \\ (\partial_\zeta + \partial_x - \partial_y)U \\ (\partial_x + \partial_y - \partial_z)U \end{pmatrix}, \;\; \operatorname{div} \operatorname{curl} \vec{U} = 0.$$

In this case, the spectral problem for the Stokes operator (A) takes the form:

$$\Delta(\Delta - S)U = \mu^2(-\Delta + S)U, \;\; (x, y, z, \zeta) \in \Omega, \tag{42}$$

$$U = \partial_{\vec{n}} U = 0, \ \ (x,y,z,\zeta) \in \partial\Omega, \tag{43}$$

$$S = \frac{4}{3}\left(\partial^2_{xz} + \partial^2_{yz} + \partial^2_{y\zeta} - \partial^2_{x\zeta}\right), \tag{44}$$

where $\vec{n}$ is the outward unit normal to $\partial\Omega$.

In equation (42), let the operator $S \equiv 0$, and then replace the resulting biharmonic operator $(-\Delta)^2$ with a fourth-order differential operator. As a result, instead of the spectral problem (42)–(44), we obtain the spectral problem (16)–(17) with $\lambda^2 = \mu^2$, for which Theorems 3 and 4 remain valid.

Let us introduce the notations:

$$\overset{\circ}{\mathbf{W}}{}^2_2(\Omega) = (\overset{\circ}{W}{}^2_2(\Omega))^4\,, \quad \overset{\circ}{\mathbf{W}}{}^1_2(\Omega) = (\overset{\circ}{W}{}^1_2(\Omega))^4\,, \tag{45}$$

$$\overset{\circ}{\mathbf{W}}(\Omega) = \{\vec{w} \in \overset{\circ}{\mathbf{W}}{}^1_2(\Omega), \ \operatorname{div}\vec{w} = 0\}\,. \tag{46}$$

Proposition 11. For each four-dimensional vector-function $\vec{w}(x,y,z,\zeta) \in (V_1(\Omega))^4$, satisfying the condition $\operatorname{div}\vec{w}(x,y,z,\zeta) = 0$, there exists a unique four-dimensional vector-function $\vec{U}(x,y,z,\zeta) \in (V_2(\Omega))^6$. The converse statement is also true.

The validity of Proposition 11 directly follows from the proof of Proposition 10.

Proposition 12. If $U_1 = U_2 = U_3 = U_4 = U(x,y,z,\zeta) \in \overset{\circ}{W}{}^2_2(\Omega)$, then we get:

$$\vec{w} \in \overset{\circ}{\mathbf{W}}_0(\Omega) = \operatorname{curl}\{\vec{U}\}_{|U_1=U_2=U_3=U_4=U\in \overset{\circ}{W}{}^2_2(\Omega)} \subset \overset{\circ}{\mathbf{W}}(\Omega).$$

Next, we introduce the extended system of functions $\left\{U^1_m(x,y,z,\zeta)\right\}_{m=0}^{\infty}$, where

$$U^1_0(x,y,z,\zeta) \equiv 0, \ \ U^1_m(x,y,z,\zeta) = U_m(x,y,z,\zeta), \ \ m \in \mathbb{N},$$

and construct vector-functions

$$\vec{U}^1_{mjkq}(x,y,z,\zeta) = \left\{U^1_m, U^1_j, U^1_k, U^1_q\right\} \in \overset{\circ}{\mathbf{W}}{}^2_2(\Omega), \tag{47}$$

$m,j,k,q \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, which, as evident, will form a fundamental system in the space $\mathbf{V}_1(\Omega) = (V_1(\Omega))^4$.

Theorem 6. By applying the curl operator from (39) to the extended system of vector-functions (47), we obtain the desired fundamental system ($m,j,k,q \in \mathbb{N}_0$):

$$\vec{w}_{mjkq}(x,y,z,\zeta) = \{w_{1,kjj}, w_{2,qkk}, w_{3,mjq}, w_{4,jkm}\} \in \overset{\circ}{\mathbf{W}}(\Omega) \tag{48}$$

in the space of solenoidal functions $\mathbf{H}(\Omega)$ (32), where

$$w_{1,kjj}(x,y,z,\zeta) = \partial_y U^1_k - \partial_z U^1_j - \partial_\zeta U^1_j, \ \ (x,y,z,\zeta) \in \Omega, \quad j,k \in \mathbb{N}_0,$$

$$w_{2,qkk}(x,y,z,\zeta) = \partial_z U^1_q - \partial_\zeta U^1_k - \partial_\zeta U^1_k, \ \ (x,y,z,\zeta) \in \Omega, \quad k,q \in \mathbb{N}_0,$$

$$w_{3,mjq}(x,y,z,\zeta) = \partial_\zeta U^1_m + \partial_x U^1_j - \partial_y U^1_q, \ \ (x,y,z,\zeta) \in \Omega, \quad m,j,q \in \mathbb{N}_0,$$

$$w_{4,jkm}(x,y,z,\zeta) = \partial_x U^1_j + \partial_y U^1_k - \partial_z U^1_m, \ \ (x,y,z,\zeta) \in \Omega, \quad m,j,k \in \mathbb{N}_0,$$

$$\operatorname{div}\vec{w}_{mjkq}(x,y,z,\zeta) = 0, \ \ (x,y,z,\zeta) \in \Omega, \ \ m,j,k,q \in \mathbb{N}_0, \tag{49}$$

$$\vec{w}_{mjkq}(x,y,z,\zeta) = 0, \ \ (x,y,z,\zeta) \in \partial\Omega, \ \ m,j,k,q \in \mathbb{N}_0. \tag{50}$$

Proof of Theorem 6. From the formulas defining the vector-function (48), we sequentially obtain

$$|w_1 - w_1^\varepsilon|^2 \le 3\left[|\partial_y(U_3 - U_3^\varepsilon)|^2 + |\partial_z(U_2 - U_2^\varepsilon)|^2 + |\partial_\zeta(U_2 - U_2^\varepsilon)|^2\right],$$

$$|w_2 - w_2^\varepsilon|^2 \le 3\left[|\partial_x(U_3 - U_3^\varepsilon)|^2 + |\partial_z(U_4 - U_4^\varepsilon)|^2 + |\partial_\zeta(U_3 - U_3^\varepsilon)|^2\right],$$

$$|w_3 - w_3^\varepsilon|^2 \le 3\left[|\partial_x(U_2 - U_2^\varepsilon)|^2 + |\partial_y(U_4 - U_4^\varepsilon)|^2 + |\partial_\zeta(U_1 - U_1^\varepsilon)|^2\right],$$

$$|w_4 - w_4^\varepsilon|^2 \le 3\left[|\partial_x(U_2 - U_2^\varepsilon)|^2 + |\partial_y(U_3 - U_3^\varepsilon)|^2 + |\partial_z(U_1 - U_1^\varepsilon)|^2\right],$$

i.e.,

$$\|\vec{w} - \vec{w}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \le 2\sqrt{3}\|\vec{U} - \vec{U}^\varepsilon\|_{\mathbf{V}_1(\Omega)} \le 2\sqrt{3}\,\varepsilon,$$

where, first, according to Proposition 11, for each vector-function $\vec{w}(z,y,z,\zeta) \in \overset{\circ}{\mathbf{W}}(\Omega)$ from (46), there corresponds a unique vector-function $\vec{U}(z,y,z,\zeta) \in \overset{\circ}{\mathbf{W}}{}_2^2(\Omega)$ (45).

Secondly, the finite sum

$$\vec{U}^{\,1\varepsilon}(x,y,z,\zeta) = \sum_{n=0}^{N_\varepsilon} a_n \vec{U}_n^1(x,y,z,\zeta), \quad N_\varepsilon < \infty, \tag{51}$$

ensuring the fulfillment of inequality

$$\|\vec{U}^1(x,y,z,\zeta) - \vec{U}^{\,1\varepsilon}(x,y,z,\zeta)\|_{\mathbf{V}_1(\Omega)} \le \varepsilon, \tag{52}$$

corresponds to the sum defined by the formulas:

$$\vec{w}^{\,\varepsilon}(x,y,z,\zeta) = \sum_{n=0}^{N_\varepsilon} a_n \vec{w}_n(x,y,z,\zeta), \quad N_\varepsilon < \infty, \tag{53}$$

and satisfying inequality:

$$\frac{1}{2\sqrt{3}}\|\vec{w}(x,y,z,\zeta) - \vec{w}^{\,\varepsilon}(x,y,z,\zeta)\|_{\mathbf{H}(\Omega)} \le \|\vec{U}^1(x,y,z,\zeta) - \vec{U}^{\,1\varepsilon}(x,y,z,\zeta)\|_{(V_1(\Omega))^6} \le \varepsilon, \tag{54}$$

due to the fact that equality

$$\|\vec{w}(x,y,z,\zeta) - \vec{w}^{\,\varepsilon}(x,y,z,\zeta)\|_{\mathbf{L}^2(\Omega)} = \|\vec{w}(x,y,z,\zeta) - \vec{w}^{\,\varepsilon}(x,y,z,\zeta)\|_{\mathbf{H}(\Omega)},$$

and identity

$$\|\vec{w} - \vec{w}^{\,\varepsilon}\|_{\mathbf{H}^\perp(\Omega)} = 0, \;\; \mathbf{L}^2(\Omega) = \mathbf{H}(\Omega) \oplus \mathbf{H}^\perp(\Omega)$$

hold.

The relations (51)–(54) show that the system of four-dimensional vector-functions

$$\{\vec{w}_n(x,y,z,\zeta)\}_{n=0}^\infty \subset (V_1(\Omega))^4,$$

forms a fundamental system in the space of solenoidal vector fields $\mathbf{H}(\Omega)$, satisfying the condition

$$\operatorname{div} \vec{w}_n(x,y,z,\zeta) = 0.$$

This concludes the proof of Theorem 6.

Proposition 13. Any vector-functions of the form

$$\vec{w}(x,y,z,\zeta) = \{0,0,0,w_4(x,y,z,\zeta)\}\,,$$
$$\vec{w}(x,y,z,\zeta) = \{0,0,w_3(x,y,z,\zeta),0\}\,,$$
$$\vec{w}(x,y,z,\zeta) = \{0,w_2(x,y,z,\zeta),0,0\}\,,$$
$$\vec{w}(x,y,z,\zeta) = \{w_1(x,y,z,\zeta),0,0,0\}\,,$$

from the space $(\overset{\circ}{W}{}_2^1(\Omega))^4$ (where the functions $w_1(x,y,z,\zeta)$, $w_2(x,y,z,\zeta)$, $w_3(x,y,z,\zeta)$ and $w_4(x,y,z)$ are identically nonzero) cannot be solenoidal, i.e. they will not satisfy both the equation (incompressibility condition of the incompressible fluid) $\operatorname{div}\vec{w}(x,y,z,\zeta)=0$ (49), and the boundary condition (50).

Thus, we have constructed a fundamental system in the space of solenoidal functions for a "four-dimensional cubic" domain.

Conclusion

The paper considers two four-dimensional curl operators. The first is the classical one, which is used in the description of Maxwell's equations for electromagnetic fields. The second curl operator is new and has not been known before. Based on these operators, an explicit construction of a fundamental system in the space of solenoidal functions for the "four-dimensional cube" is obtained. This fundamental system of functions can be used for the approximate solution of boundary value problems for stationary and evolutionary Stokes and Navier-Stokes equations. It is worth noting that in the works [23] and [24], the solution to the spectral problem (A1) for the biharmonic operator in the domain Ω, represented by a 3-D sphere, was found.

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Author Contributions

M.T. Jenaliyev served as the principal investigator of the research grant and supervised the research process. A.S. Kassymbekova assisted in data collection and analysis. M.G. Yergaliyev collected and analyzed data, and led manuscript preparation. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2025M2/122-136 *Research article*

Spectral analysis of second order quantum difference operator over the sequence space l_p $(1 < p < \infty)$

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In this article, we study the spectrum, fine spectrum and boundedness property of second order quantum difference operator Δ_q^2 $(0 < q < 1)$ over the class of sequence l_p $(1 < p < \infty)$, the p^{th} summable sequence space. The second order quantum difference operator Δ_q^2 is a lower triangular triple band matrix $\Delta_q^2(1, -(1+q), q)$. We also determine the approximate point spectrum, defect spectrum, compression spectrum, and Goldberg classification of the operator on the class of sequence. We obtained the results by solving an infinite system of linear equations and computing the inverse of a lower triangular infinite matrix. We also provide appropriate examples along with graphical representations where necessary.

Keywords: spectrum, difference operator, infinite matrices, triple-band matrix.

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Introduction

Spectral theory of bounded linear operators on Banach or Hilbert spaces holds a significant place in different branches of Mathematics due to its many applications. The fundamental principle of the modern spectral theorem is that certain linear operators on infinite dimensional spaces can be represented in a "diagonal" matrix form. From this diagonal form, we can determine the spectrum of the operator. The spectrum of an operator can be classified into three parts: the point spectrum, the continuous spectrum, and the residual spectrum. These three disjoint parts together are referred to as the "fine spectrum".

In operator theory, one of the most important linear operators is the difference operator. The spectrum of this operator and its different forms on various sequence spaces have been studied by many researchers. In recent times, researchers have started analyzing the spectrum of the quantum version of some well-known operators, one of which is the difference operator. In our study, we analyze the spectrum of a second order quantum difference operator. The $q-$analog of the second order difference operator is defined as $(\Delta_q^2 u)_k = u_k - (1+q)u_{k-1} + qu_{k-2}$, for all $k \in \mathbb{N}$ and any term with negative indices are zero. The matrix representation of this operator is given below

$$\Delta_q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ -(1+q) & 1 & 0 & 0 & \dots \\ q & -(1+q) & 1 & 0 & \dots \\ 0 & q & -(1+q) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The n^{th} order $q-$difference operator is defined as $(\Delta_q^n u)_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i}_q q^{\binom{i}{2}} u_{n+k-i}$, which was introduced by Bustoz and Gordillo [1].

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The initial study of the spectrum of the difference operator Δ, $(\Delta y_k = y_k - y_{k+1})$ was conducted by Altay and Basar [2], Kayaduman and Furkan [3], and Akhmedov and Basar [4, 5] in the spaces c_o, c, $; l_1$, bv and l_p, bv_p respectively. The spectrum of the second-order difference operator Δ^2 was studied by Dutta and Baliarsingh [6] over the space c_0. After this the operator Δ was generalized to $B(r,s)$, $B(r,s)(y_k) = (ry_k + sy_{k-1})$. The spectrum of this operator was studied by Altay and Basar [7], Kayaduman et al. [8], Bilgic and Furkan [9], Dutta and Tripathy [10] over the spaces c_0, c; l_1, bv; l_p, bv_p $(1 \leq p < \infty)$ and cs respectively. $B(r,s)$ was further generalized to $B(r,s,t)$, $B(r,s,t)(y_k) = (ry_k + sy_{k-1} + ty_{k-2})$. The spectrum of this operator was studied by Bilgic and Furkan [11]; Furkan et al. [12, 13] over the spaces l_1, bv, c_0, c and l_p, bv_p respectively. Srivastava and Kumar [14, 15]; Akhmedov and El-Shabrawy [16] studied the spectrum of the generalised difference operator Δ_v, where $\Delta_v(y_k) = (v_k y_k - v_{k-1} y_{k-1})$ over the sequence spaces c_0, l_1 and c, l_p respectively. Akhmedov and El-Shabrawy [17, 18]; Dutta and Baliarsingh [19] obtained the spectrum of the operator Δ_{ab}, where $\Delta_{ab}(y_k) = (a_k y_k + b_{k-1} y_{k-1})$ over the sequence spaces c_0, c and l_p, bv_p respectively. Panigrahi and Srivastava [20, 21] also analyzed the spectrum of Δ^2_{uv}, where $\Delta^2_{uv}(y_k) = (u_k y_k - v_{k-1} y_{k-1} + u_{k-2} y_{k-2})$ and Δ^2_{uvw}, where $\Delta^2_{uvw}(y_k) = (u_k y_k + v_{k-1} y_{k-1} + w_{k-2} y_{k-2})$ over the sequence spaces c_0 and l_1 respectively. Then, Altundağ and Abay [22] studied on the fine spectrum of generalized upper triangular triple-band matrices $(\Delta^2_{uvw})^t$ where the transpose of matrix operator Δ^2_{uvw} over the sequence space l_1. Patra and Srivastava [23] considered a new generalized difference operator $A(p_1, p_2;\ q_1, q_2;\ r_1, r_2)$ and determined its spectrum over the sequence space l_p $(1 \leq p < \infty)$. The operators mentioned above can be expressed using a lower triangular band matrix. Spectral analysis of the quantum versions of some well-known operators has been conducted in recent years. The spectrum of $q-$Cesàro matrix was studied by Yildirim [24], Durna and Turkay [25] over the sequence space c_0 and c respectively. Yaying et al. [26, 27] studied the spectrum of second order $q-$difference operator over the sequence space c_0, l_1 respectively. Spectrum of weighted $q-$difference operator was studied by Yaying et al. [28] over the sequence space c_0.

$q-$Analog: A $q-$analog of a number, a theorem, an identity or an expression is a generalization that involves a new parameter q and it reduced to the original number, theorem, identity as the limit $q \longrightarrow 1^-$. In the 19^{th} century, the basic hypergeometric series became the first $q-$analog to be extensively studied. In recent research of many areas of Mathematics like combinatorics, approximation theory, difference and integral equations, etc., $q-$calculus have been used extensively.

The $q-$analog $[m]_q$ of m for $q \in (0,1)$ can be determined as

$$[m]_q = \begin{cases} \sum_{k=0}^{m-1} q^k, & m = 1, 2, 3, \ldots, \\ 0, & m = 0. \end{cases}$$

One might observe that $[m]_q = m$ whenever $q \longrightarrow 1^-$. The $q-$analog $\binom{m}{k}_q$ of binomial coefficient $\binom{m}{k}$ can be determined as

$$\binom{m}{q}_q = \begin{cases} \dfrac{[m]_q!}{[m-k]_q! [k]_q!}, & m \geq k, \\ 0, & k > m, \end{cases}$$

where the $q-$analog of the factorial, i.e., $q-$factorial, is defined as

$$[m]_q! = \begin{cases} \prod_{k=1}^{m} [k]_q, & m = 1, 2, 3, \ldots, \\ 1, & m = 0. \end{cases}$$

The $q-$analog of some specific binomials such as $\binom{0}{0} = \binom{m}{0} = \binom{m}{m} = 1$, also $\binom{m}{m-k}_q = \binom{m}{k}_q$. For an in-depth study of quantum calculus, we refer to the book [29].

1 Some Definitions and Preliminaries

Consider $M : U \longrightarrow V$ be a bounded linear operator, in which U and V are Banach spaces, the following collections

$$\begin{aligned} R(M) &= \{v \in V : v = Mu,\ u \in U\} \\ \text{and}\ \ B(U,V) &= \{M : U \longrightarrow V : M \text{ is continuous and linear}\} \end{aligned}$$

are termed as the range of the operator M and the set of all bounded linear operators from U to V respectively. The adjoint operator M^* of M is defined from V^* to U^*, where V^* and U^* represent the dual space of V and U respectively. Again, it is defined as $(M^*f)(u) = f(Mu)$, for all $f \in V^*$ and $u \in U$.

Let $M : D(M) \longrightarrow U$, where $D(M)$ denotes the domain of M. From M we can get an operator,

$$M_\mu = M - \mu I,$$

where $\mu \in \mathbb{C}$ and I is the identity operator. A regular value $\mu \in \mathbb{C}$ of M is such that M_μ is invertible, and its inverse (M_μ^{-1}) is bounded and defined on a set A and call it the resolvent operator of M, where A is dense in U. The collections of such μ is called the resolvent set and is denoted by $\rho(M,U)$. In the complex plane $\mathbb{C}$, the compliment of $\rho(M,U)$ is denoted by $\sigma(M,U)$, and is called the spectrum of M.

Further, $\sigma(M,U)$ is classified into three disjoint subsets, namely, the point spectrum $\sigma_p(M,U)$, the continuous spectrum $\sigma_c(M,U)$, and the residual spectrum $\sigma_r(M,U)$. In point spectrum, M_μ^{-1} does not exist for any $\mu \in \sigma_p(M,U)$, while in continuous spectrum, M_μ^{-1} exist but unbounded for every $\mu \in \sigma_c(M,U)$, and also defined on a set that is dense in U. On the other hand, in the residual spectrum, M_μ^{-1} exists but may or may not be bounded for $\mu \in \sigma_r(M,U)$ and is not dense in U.

There are more subdivisions of the spectrum of a bounded operator such as approximate point spectrum $\sigma_{ap}(M,U)$, defect spectrum $\sigma_\delta(M,U)$ and compression spectrum $\sigma_{co}(M,U)$, which are defined as follows:

- $\sigma_{ap}(M,U) = \{\mu \in \mathbb{C} : (M - \mu I) \text{ is not bounded below}\}$;
- $\sigma_\delta(M,U) = \{\mu \in \mathbb{C} : (M - \mu I) \text{ is not surjective}\}$;
- $\sigma_{co}(M,U) = \{\mu \in \mathbb{C} : \overline{R(M - \mu I)} \neq U\}$.

2 Goldberg's classification of spectrum

A detailed classification of the spectrum of an operator was given by Goldberg [30]. This classification is based on the nature of the set $R(M_\mu)$ and the inverse M_μ^{-1}.

If $M \in B(U,U)$, then there are three possibilities for $R(M_\mu)$:

(*P*) $R(M_\mu) = U$,
(*Q*) $\overline{R(M_\mu)} = U$, but $R(M_\mu) \neq U$,
(*R*) $\overline{R(M_\mu)} \neq U$

and three possibilities for M_μ^{-1}:

(*1*) Exist and continuous,
(*2*) Exist but discontinuous,
(*3*) Does not exist.

Combination of the possibilities $P,\ Q,\ R$ and $1,\ 2,\ 3$ leads to nine different states. They are identified as $P_1,\ P_2,\ P_3,\ Q_1,\ Q_2,\ Q_3,\ R_1,\ R_2$, and R_3.

If $M_\mu \in P_1$ or $M_\mu \in Q_1$, then $\mu \in \rho(M,X)$. If $M_\mu \in R_2$, then M_μ^{-1} exists and is unbounded, and $\overline{R(M_\mu)} \neq X$ and we can write $\mu \in R_2\sigma(M,X)$. We can summarize this classification in the following Table 1.

Table 1

Goldberg's classification of spectrum of linear operator

		1	2	3
		M_μ^{-1} exist and bounded	M_μ^{-1} exist and unbounded	M_μ^{-1} does not exist
P	$R(M_\mu)=U$	$\mu\in\rho(M,U)$	-	$\mu\in\sigma_p(M,U)$ $\mu\in\sigma_{ap}(M,U)$
Q	$\overline{R(M_\mu)}=U$	$\mu\in\rho(M,U)$	$\mu\in\sigma_c(M,U)$ $\mu\in\sigma_\delta(M,U)$ $\mu\in\sigma_{ap}(M,U)$	$\mu\in\sigma_p(M,U)$ $\mu\in\sigma_\delta(M,U)$ $\mu\in\sigma_{ap}(M,U)$
R	$\overline{R(M_\mu)}\neq U$	$\mu\in\sigma_r(M,U)$ $\mu\in\sigma_\delta(M,U)$ $\mu\in\sigma_{co}(M,U)$	$\mu\in\sigma_r(M,U)$ $\mu\in\sigma_\delta(M,U)$ $\mu\in\sigma_{co}(M,U)$ $\mu\in\sigma_{ap}(M,U)$	$\mu\in\sigma_p(M,U)$ $\mu\in\sigma_\delta(M,U)$ $\mu\in\sigma_{co}(M,U)$ $\mu\in\sigma_{ap}(M,U)$

Proposition 1. ([31], p. 28) Spectral and sub-spectral relationships of an operator M and its adjoint operator M^* are provided below.

(a) $\sigma(M^*,Y^*)=\sigma(M,Y)$,
(b) $\sigma_{ap}(M^*,Y^*)=\sigma_\delta(M,Y)$,
(c) $\sigma_\delta(M^*,Y^*)=\sigma_{ap}(M,Y)$,
(d) $\sigma_p(M^*,Y^*)=\sigma_{co}(M,Y)$,
(e) $\sigma(M,Y)=\sigma_{ap}(M,Y)\bigcup\sigma_p(M^*,Y^*)=\sigma_p(M,Y)\bigcup\sigma_{ap}(M^*,Y^*)$.

Lemma 1. ([30], p. 60) The adjoint operator M^* of M is onto if and only if M has a bounded inverse.

Lemma 2. ([30], p. 59) The bounded linear operator $M:U\longrightarrow V$ has dense range if and only if M^* is one to one.

Throughout this work, the aforementioned spaces c_0, c, l_1, l_p, bv, bv_p, cs and l_∞ represent the spaces of all null, convergent, absolutely summable, $p-$absolutely summable, bounded variation, $p-$ bounded variation, convergent series, and bounded sequences, respectively.

Before going to the main results we state a remark, "if z is a complex number, then $\sqrt{z}$ means the square root of z with non-negative real part. If $Re\sqrt{z}=0$, then $\sqrt{z}$ means the square root of z with $Im(z)\geq 0$".

3 Spectrum of $\Delta_q^2(1,-(1+q),q)$ on l_p

Theorem 1. $\Delta_q^2\in B(l_p)$ with $(1+(1+q)^p+q^p)^{1/p}\leq ||\Delta_q^2||_{l_p}\leq 2(q+1)$ for $0<q<1$, where $1<p<\infty$.

Proof. The linearity of Δ_q^2 is straightforward to prove, so it is omitted. Now, consider $e^{(0)}=(1,0,0,\ldots)$ in l_p. Then, $(\Delta_q^2)e^{(0)}=(1,-(1+q),q,0,0,\ldots)$ and it is obtained that

$$\frac{||(\Delta_q^2)e^{(0)}||_{l_p}}{||e^{(0)}||_{l_p}}=(1+(1+q)^P+q^p)^{1/p}.$$

From this we get, $(1+(1+q)^P+q^p)^{1/p}\leq ||(\Delta_q^2)e^{(0)}||$. Again for any $u=(u_k)\in l_p$ and using the

Minkowaski inequality, we get

$$\begin{aligned}
||\Delta_q^2 u||_{l_p} &= \left(\sum_k |qu_{k-1} + (-(1+q))u_k + u_{k+1}|^p\right)^{1/p} \\
&\leq \left(\sum_k |qu_{k-1}|^p\right)^{1/p} + \left(\sum_k |(1+q)u_k|^p\right)^{1/p} + \left(\sum_k |u_{k+1}|^p\right)^{1/p} \\
&= \left(q^p \sum_k |u_{k-1}|^p\right)^{1/p} + \left((1+q)^p \sum_k |u_k|^p\right)^{1/p} + \left(\sum_k |u_{k+1}|^p\right)^{1/p} \\
&= (q + (1+q) + 1)\, ||u||_{l_p} \\
&= 2(1+q)||u||_{l_p}.
\end{aligned}$$

As a result, we get $(1 + (1+q)^p + q^p)^{1/p} \leq ||\Delta_q^2||_{l_p} \leq 2(q+1)$.

Theorem 2. The point spectrum $\sigma_p(\Delta_q^2, l_p) = \phi$ (the empty set).

Proof. We prove this theorem by the method of contradiction. Consider $\sigma_p(C_1(q), l_p) \neq \phi$. Then for any $0 \neq u \in l_p$ with $(\Delta_q^2)u = \lambda u$, we get the following equalities:

$$\begin{aligned}
u_0 &= \lambda u_0, \\
-(1+q)u_0 + u_1 &= \lambda u_1, \\
qu_0 - (1+q)u_1 + u_2 &= \lambda u_2, \\
qu_1 - (1+q)u_2 + u_3 &= \lambda u_3, \\
&\vdots \\
qu_{m-2} - (1+q)u_{m-1} + u_m &= \lambda u_m, \\
&\vdots
\end{aligned}$$

If u_m is the first non zero entry of the sequence $u = (u_m)$, then from the above equations we get $\lambda = 1$. Putting the value of $\lambda = 1$ in the proceeding equation, we get $u_m = 0$, which contradicts our assumption. Thus, $\sigma_p(\Delta_q^2, l_p) = \phi$.

Lemma 3. ([32], p. 126) The matrix $A = (a_{nk})$ defines a bounded linear operator $T \in B(l_1)$, mapping l_1 to itself, if and only if the supremum of l_1 norms of the columns of A is bounded.

Lemma 4. ([32], p. 126) The matrix $A = (a_{nk})$ defines a bounded linear operator $T \in B(l_\infty)$, mapping l_∞ to itself, if and only if the supremum of l_1 norms of the rows of A is bounded.

Lemma 5. ([33], p. 174, Theorem 9) If $1 < p < \infty$ and $A \in (l_1, l_1) \cap (l_\infty, l_\infty)$. Then $A \in (l_p, l_p)$.

Theorem 3. $\sigma(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$.

Proof. We consider

$$S = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$$

and let $\mu \notin S$. So, we get $\mu \neq 1$, and this implies that $(\Delta_q^2 - \mu I)$ has an inverse. Now,

$$(\Delta_q^2 - \mu I) = \begin{bmatrix} 1-\mu & 0 & 0 & 0 & 0 & \dots \\ -(1+q) & 1-\mu & 0 & 0 & 0 & \dots \\ q & -(1+q) & 1-\mu & 0 & 0 & \dots \\ 0 & q & -(1+q) & 1-\mu & 0 & \dots \\ 0 & 0 & q & -(1+q) & 1-\mu & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $(\Delta_q^2 - \mu I)$ is a lower triangular matrix, its inverse can be obtained easily and has been given below:

$$(\Delta_q^2 - \mu I)^{-1} = \begin{bmatrix} m_1 & 0 & 0 & 0 & \dots \\ m_2 & m_1 & 0 & 0 & \dots \\ m_3 & m_2 & m_1 & 0 & \dots \\ m_4 & m_3 & m_2 & m_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$m_1 = \frac{1}{1-\mu},$$
$$m_2 = \frac{q+1}{(1-\mu)^2},$$
$$m_3 = \frac{(q+1)^2 - q(1-\mu)}{(1-\mu)^3},$$
$$\vdots$$

here the sequence (m_k) satisfies the following recurrence relation

$$m_k = \frac{(q+1)m_{k-1} - qm_{k-2}}{1-\mu}, \text{ for } k \geq 3.$$

From this recurrence relation, we get the characteristic equation as

$$(1-\mu)w^2 - (1+q)w + q = 0,$$

whose solutions are:

$$w_1 = \frac{(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}}{2(1-\mu)},$$
$$w_2 = \frac{(1+q) - \sqrt{(1+q)^2 - 4(1-\mu)q}}{2(1-\mu)}.$$

From elementary calculations on recurrence sequence, we get

$$m_k = \frac{w_1^k - w_2^k}{\sqrt{(1+q)^2 - 4(1-\mu)q}}. \tag{1}$$

Now, we can proceed to the proof in two cases.

Case 1: If $(1+q)^2 = 4q(1-\mu)$, then we get

$$m_k = \left(\frac{2k}{1+q}\right)\left[\frac{(1+q)}{2(1-\mu)}\right]^k.$$

It can be easily proved that $(m_k) \in l_p$ if $\left|\dfrac{(1+q)}{2(1-\mu)}\right| < 1$. So, $\mu \notin S$ implies that $(m_k) \in l_p$.

Case 2: If $(1+q)^2 \neq 4q(1-\mu)$. Since $\mu \notin S$, we have $|w_1| < 1$. Again, using the inequality $|1-\sqrt{z}| \leq |1+\sqrt{z}|$ for any $z \in \mathbb{C}$, we get

$$\left|\frac{(1+q)}{2(1-\mu)} - \frac{\sqrt{(1+q)^2 - 4(1-\mu)q}}{2(1-\mu)}\right| \leq \left|\frac{(1+q)}{2(1-\mu)} + \frac{\sqrt{(1+q)^2 - 4(1-\mu)q}}{2(1-\mu)}\right|$$
$$\implies |w_2| \leq |w_1| < 1.$$

Using this in equation (1), it is obtained that $(m_k) \longrightarrow 0$ as $k \longrightarrow \infty$. Now

$$\begin{aligned}||(\Delta_q^2 - \mu I)^{-1}||_{(l_1:l_1)} &= \sup_{k\in\mathbb{N}} \sum_{i=k}^{\infty} |m_i| = \sum_{i=1}^{\infty} |m_i| \\ &\leq \frac{1}{|\sqrt{(1+q)^2 - 4(1-\mu)q}|}\left(\sum_{i=1}^{\infty} |w_1|^i + \sum_{i=1}^{\infty} |w_2|^i\right) < \infty.\end{aligned}$$

Since $|w_1| < 1$ and $|w_2| < 1$, it follows that $(\Delta_q^2 - \mu I)^{-1} \in (l_1, l_1)$. Again, since $(m_k) \in l_1$, the supremum of l_1 norms of the rows of $(\Delta_q^2 - \mu I)^{-1}$ is finite. This results in $(\Delta_q^2 - \mu I)^{-1} \in (l_\infty, l_\infty)$. Now, using Lemma 5, we get $(\Delta_q^2 - \mu I)^{-1} \in (l_1, l_1) \cap (l_\infty, l_\infty) \implies (\Delta_q^2 - \mu I)^{-1} \in (l_p, l_p)$. It proves that $\sigma(\Delta_q^2, l_p) \subseteq S$.

Now consider $\mu \in S$. If $\mu = 1$, then we get

$$(\Delta_q^2 - \mu I) = (\Delta_q^2 - I) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ -(1+q) & 0 & 0 & 0 & 0 & \dots \\ q & -(1+q) & 0 & 0 & 0 & \dots \\ 0 & q & -(1+q) & 0 & 0 & \dots \\ 0 & 0 & q & -(1+q) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now $(\Delta_q^2 - I)u = \theta \implies u = \theta$. So, we get $(\Delta_q^2 - I) : l_p \longrightarrow l_p$ is one-one but not onto. This implies $(\Delta_q^2 - I)$ is not invertible.

If we take μ from S other than 1, then it is obtained from *Case 1* that $\left|\dfrac{(1+q)}{2(1-\mu)}\right| \geq 1$. It implies $(\Delta_q^2 - \mu I)^{-1} \notin B(l_p)$. Again, from *Case 2*, it is obtained that $|w_1| \geq 1$ and the inequality $|w_1| > |w_2|$, which directly implies $(w_k) \nrightarrow 0$, and so $\sum_{i=1}^{\infty} |w_i|^p$ diverges. Consider $v = (1, 0, 0, \dots) \in l_p$. Then $(\Delta_q^2 - \mu I)^{-1} v = (m_1, m_2, m_3, \dots)$, which doesn't belong to l_p. Therefore, $(\Delta_q^2 - \mu I)^{-1} \notin B(l_p)$ and this proves that $S \subseteq \sigma(\Delta_q^2, l_p)$. Finally, it is obtained $\sigma(\Delta_q^2, l_p) = S$.

Lemma 6. ([34], p. 215) For any $A \in B(l_p)$ $(1 < p < \infty)$, the adjoint operator $A^* \in B(l_q)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and can be represented by the transpose of A matrix.

Theorem 4. $\sigma_p((\Delta_q^2)^*, l_p^* \cong l_q) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| < \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$.

Proof. Consider the set $S_1 = \left\{\mu \in \mathbb{C} : |2(1-\mu)| < \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$, and let $\mu \in S_1$. Now, solving $(\Delta_q^2)^* u = \mu u$, for $\theta \neq u \in l_p^* \cong l_q$, i.e.,

$$\begin{bmatrix} 1 & -(1+q) & q & 0 & 0 & 0 & \dots \\ 0 & 1 & -(1+q) & q & 0 & 0 & \dots \\ 0 & 0 & 1 & -(1+q) & q & 0 & \dots \\ 0 & 0 & 0 & 1 & -(1+q) & q & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix} = \mu \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix},$$

we get the following linear equations

$$\begin{aligned} & u_0 - (1+q)u_1 + qu_2 = \mu u_0, \\ & u_1 - (1+q)u_2 + qu_3 = \mu u_1, \\ & u_2 - (1+q)u_3 + qu_4 = \mu u_2, \\ & \vdots \\ & u_k - (1+q)u_{k+1} + qu_{k+2} = \mu u_k, \\ & \vdots \end{aligned}$$

Now, if we take $\mu = 1$, then we get an eigenvector $(1, 0, 0, \dots)$ corresponding to $\mu = 1$. We consider $\mu \in S_1$ other than 1. The above linear equations can also be expressed in terms of u_1 and u_0 as

$$\begin{aligned} u_2 &= \frac{(1+q)}{q} u_1 - \frac{1-\alpha}{q} u_0, \\ u_3 &= \frac{(1+q)^2 - q(1-\mu)}{q^2} u_1 - \frac{(1-\mu)(1+q)}{q^2}, \\ &\vdots \\ u_k &= \frac{m_k(1-\mu)^k}{q^{k-1}} u_1 - \frac{m_{k-1}(1-\mu)^k}{q^{k-1}} u_0, \quad k \geq 2, \end{aligned} \tag{2}$$

in which (m_k) comes from equation (1). Now, we can find the eigenvector (u_k), for $\mu \neq 1$. Here we can make a choice for u_0 and u_1. Let $u_0 = 1$ and $u_1 = \dfrac{2(1-\mu)}{(1+q) + \sqrt{(1+q)^2 - 4q(1-\mu)}}$.

Already, we have obtained that w_1 and w_2 are roots of the characteristic equation $(1-\mu)w^2 - (1+ q)w + q = 0$. So, we get $w_1 \cdot w_2 = \dfrac{q}{1-\mu}$, and $w_1 - w_2 = \dfrac{\sqrt{(1+q)^2 - 4q(1-\mu)}}{1-\mu}$.

It can also be seen that $u_1 = \frac{1}{w_1}$. Using these facts in the relation of the sequence (u_k), we get

$$\begin{aligned}
u_k &= \frac{m_k(1-\mu)^k}{q^{k-1}}u_1 - \frac{m_{k-1}(1-\mu)^k}{q^{k-1}}u_0 \\
&= \left(\frac{1-\mu}{q}\right)^{k-1}(1-\mu)(-m_{k-1}u_0 + m_k u_1) \\
&= \frac{1}{(w_1w_2)^{k-1}}(1-\mu)(-m_{k-1} + m_k\frac{1}{w_1}) \\
&= \frac{1}{(w_1w_2)^{k-1}}(1-\mu)\left(\frac{-w_1^{k-1}+w_2^{k-1}}{\sqrt{(1+q)^2-4q(1-\mu)}} + \frac{w_1^k - w_2^k}{\sqrt{(1+q)^2-4q(1-\mu)}}w_1^{-1}\right) \\
&= \frac{1}{w_1^{k-1}w_2^{k-1}}\frac{1-\mu}{\sqrt{(1+q)^2-4q(1-\mu)}}(-w_1^{k-1}+w_2^{k-1}+w_1^{k-1}-w_2^k w_1^{-1}) \\
&= \frac{1}{w_1^{k-1}w_2^{k-1}}\left(\frac{1}{w_1-w_2}\right)w_2^{k-1}\left(1-\frac{w_2}{w_1}\right) \\
&= \frac{1}{w_1^{k-1}}\frac{1}{w_1} \\
&= \frac{1}{w_1^k} \\
&= u_1^k.
\end{aligned}$$

Finally, we obtained that the sequence $(u_k) = u_1^k$, for all $k \geq 2$. If we consider $w_1 = w_2$, i.e., for the case $(1+q)^2 = 4q(1-\mu)$, we get the same result. Thus, it is clearly seen that the sequence $u = (u_k) \in l_p^*$, since $|u_1| < 1$. As a result, $S_1 \subseteq \sigma_p((\Delta_q^2)^*, l_q)$.

Now, we consider $\mu \notin S_1$, i.e.,

$$\begin{aligned}
|2(1-\mu)| &\geq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right| \\
\implies &\left|\frac{1}{w_1}\right| \geq 1 \\
\implies &|w_1| \leq 1.
\end{aligned}$$

Here, we have to show that $\mu \notin \sigma_p((\Delta_q^2)^*, l_q)$. From equation (2), it is obtained that

$$\frac{u_{k+1}}{u_k} = \left(\frac{1-\mu}{q}\right)\left(\frac{m_k}{m_{k-1}}\right)\left(\frac{-u_0 + \dfrac{m_{k+1}}{m_k}u_1}{-u_0 + \dfrac{m_k}{m_{k-1}}u_1}\right).$$

Based on the roots w_1 and w_2, we will consider three cases.

Case 1: $|w_2| < |w_1| \leq 1$.

For this case, we get $(1+q)^2 \neq 4q(1-\mu)$ and

$$\begin{aligned}\lim_{k\to\infty}\frac{m_k}{m_{k-1}} = \lim_{k\to\infty}\frac{m_{k+1}}{m_k} &= \lim_{k\to\infty}\frac{w_1^{k+1}-w_2^{k+1}}{w_1^k-w_2^k}\\ &= \lim_{k\to\infty}\frac{w_1^{k+1}\left[1-\left(\frac{w_2}{w_1}\right)^{k+1}\right]}{w_1^k\left[1-\left(\frac{w_2}{w_1}\right)^{k}\right]}\\ &= w_1.\end{aligned}$$

Again, from equation (2), we get

$$u_k = \left(\frac{1-\mu}{q}\right)^{k-1}(1-\mu)(-m_{k-1}u_0+m_ku_1). \tag{3}$$

If $-u_0+w_1u_1=0$, then from equation (3), we find $(u_k)=\left(\frac{u_0}{w_1^k}\right)$, which doesn't belong to l_q as $|w_1|\leq 1$. Otherwise,

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|^q = \frac{1}{|w_1|^q|w_2|^q}|w_1|^q = \frac{1}{|w_2|^q} > 1.$$

Case 2: $|w_2|=|w_1|<1$.

For this case, we get $(1+q)^2 = 4q(1-\mu)$ and, using the formula

$$m_k = \left(\frac{2k}{1+q}\right)\left[\frac{1+q}{2(1-\mu)}\right]^k, \text{ for all } k\geq 1,$$

we get that

$$\lim_{k\to\infty}\left|\frac{m_k}{m_{k-1}}\right|^q = \left|\frac{1+q}{2(1-\mu)}\right|^q = |w_1|^q$$

which leads to

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|^q = \frac{1}{|w_1|^q|w_2|^q}|w_1|^q = \frac{1}{|w_2|^q} > 1.$$

This implies that (u_k) doesn't belong to l_q.

Case 3: $|w_1|=|w_2|=1$.

For this case, we get $(1+q)^2 = 4q(1-\mu)$ and $\left|\dfrac{1+q}{2q}\right| = 1$. Assume that $\mu\in\sigma_p((\Delta_q^2)^*, l_q)$, then there exists $\theta \neq u \in l_q$. Now, rewriting equation (2), we get

$$\begin{aligned} u_k &= \frac{m_k(1-\mu)^k}{q^{k-1}}u_1 - \frac{m_{k-1}(1-\mu)^k}{q^{k-1}}u_0 \\ &= \frac{\left(\frac{2k}{1+q}\right)\left(\frac{1+q}{2(1-\mu)}\right)^k(1-\mu)^k}{q^{k-1}}u_1 - \frac{\left(\frac{2(k-1)}{1+q}\right)\left(\frac{1+q}{2(1-\mu)}\right)^{k-1}(1-\mu)^k}{q^{k-1}}u_0 \\ &= \frac{k(1+q)^{k-1}}{(2q)^{k-1}}u_1 - \frac{(k-1)\left(\frac{1+q}{2}\right)^{k-2}(1-\mu)}{q^{k-1}}u_0 \\ &= \frac{k(1+q)^{k-1}}{(2q)^{k-1}}u_1 - \frac{(k-1)\left(\frac{1+q}{2}\right)^{k-2}\frac{(1+q)^2}{4q}}{q^{k-1}}u_0 \\ &= \frac{k(1+q)^{k-1}}{(2q)^{k-1}}u_1 - \frac{(k-1)(1+q)^k}{(2q)^k}u_0 \\ &= \left(\frac{1+q}{2q}\right)^{k-1}\left[ku_1 - (k-1)\frac{1+q}{2q}u_0\right]. \end{aligned}$$

Since $\lim_{k\to\infty} u_k = 0 \implies \lim_{k\to\infty}\left[ku_1 - (k-1)\frac{1+q}{2q}u_0\right] = 0$, and we must have $u_0 = u_1 = 0$. Consequently, it implies $u = \theta$, a contradiction. So, we get $\mu \notin \sigma_p((\Delta_q^2)^*, l_q)$. Thus, $\sigma_p((\Delta_q^2)^*, l_q) \subseteq S_1$, and hence $\sigma_p((\Delta_q^2)^*, l_q) = S_1$.

Theorem 5. The residual spectrum:

$$\sigma_r(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| < \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}.$$

Proof. From Lemma 2, we get $\sigma_r(\Delta_q^2, l_p) = \sigma_p((\Delta_q^2)^*, l_q)\backslash\sigma_p(\Delta_q^2, l_p)$. Now, applying the Theorems 2 and 4, we get the required result.

Theorem 6. The continuous spectrum:

$$\sigma_c(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| = \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}.$$

Proof. We have $\sigma(\Delta_q^2, l_p) = \sigma_p(\Delta_q^2, l_p) \cup \sigma_r(\Delta_q^2, l_p) \cup \sigma_c(\Delta_q^2, l_p)$ and the corresponding sets are pairwise disjoint. Now, applying Theorems 3, 4 and 5, we get the required result.

Theorem 7. $P_3\sigma(\Delta_q^2, l_p) = Q_3\sigma(\Delta_q^2, l_p) = R_3\sigma(\Delta_q^2, l_p) = \phi$.

Proof. From Table 1, we get $\sigma_p(\Delta_q^2, l_p) = P_3\sigma(\Delta_q^2, l_p) \cup Q_3\sigma(\Delta_q^2, l_p) \cup R_3\sigma(\Delta_q^2, l_p)$. Again, from Theorem 2, we get $\sigma_p(\Delta_q^2, l_p) = \phi$, and consequently we get the required result.

Theorem 8. The operator Δ_q^2 satisfies the following relations:

(a) $Q_2\sigma(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| = \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$,

(b) $R_2\sigma(\Delta_q^2, l_p) \supseteq \left\{\mu \in \mathbb{C} : |2(1-\mu)| < \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}\backslash\{1\}$,

(c) $R_1\sigma(\Delta_q^2, l_p) \subseteq \{1\}$.

Proof. We have from Table 1 that $\sigma_c(\Delta_q^2, l_p) = Q_2\sigma(\Delta_q^2, l_p)$. Now, using Theorem 6, we get the result of (a).

Again, from Theorem 3, if for any $\mu \in \sigma_r(\Delta_q^2, l_p)\backslash\{1\}$ then the operator $(\Delta_q^2 - \mu I)^{-1} \notin \Delta_q^2(l_p)$. From this, we get $\sigma_r(\Delta_q^2, l_p)\backslash\{1\} \subseteq R_2\sigma(\Delta_q^2, c_o)$ and $R_1\sigma(\Delta_q^2, l_p) \subseteq \{1\}$.

Theorem 9. For the operator Δ_q^2 the following results hold.

(a) $\sigma_{ap}(\Delta_q^2, l_p) \supseteq \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\} \backslash \{1\}$,

(b) $\sigma_{ap}((\Delta_q^2)^*, l_p^*) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$,

(c) $\sigma_\delta(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$,

(d) $\delta_{co}(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| < \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$.

Proof. (a) From Table 1, we get

$$\sigma_{ap}(\Delta_q^2, l_p) = \sigma(\Delta_q^2, l_p) \backslash R_1\sigma(\Delta_q^2, l_p).$$

Now, applying the Theorems 3 and 8, we get the required result.

(b) The result in (b) is obtained from the relation (e) in Proposition 1.

(c) The result in (c) is obtained from the relation (b) in Proposition 1.

(d) The result in (d) is obtained from the relation (d) in Proposition 1.

4 *Example*

Taking particular values for $q \in (0, 1)$, we construct some examples of spectrum of Δ_q^2 that are given below.

(i) If $q = \frac{1}{2}$, then the spectrum of $\Delta_{1/2}^2$ is given by

$$\sigma(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|\frac{3}{2} + \sqrt{\frac{9}{4} - 2(1-\mu)}\right|\right\}.$$

(ii) If $q = \frac{1}{4}$, then the spectrum of $\Delta_{1/4}^2$ is given by

$$\sigma(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|\frac{5}{4} + \sqrt{\frac{25}{16} - (1-\mu)}\right|\right\}.$$

The graphical representation of the spectra for these examples is presented in Figures 1 and 2.

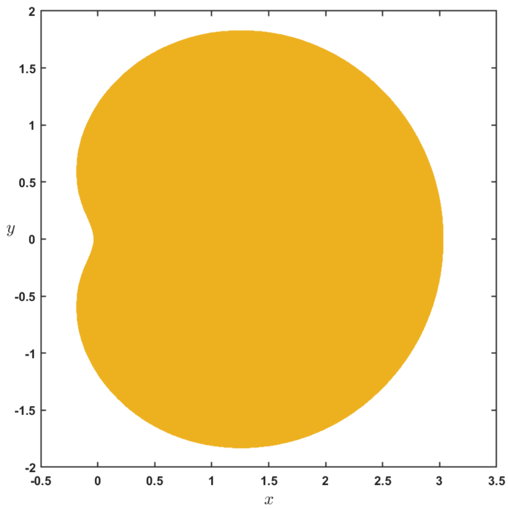

Figure 1. Spectrum of $\Delta_{1/2}^2$

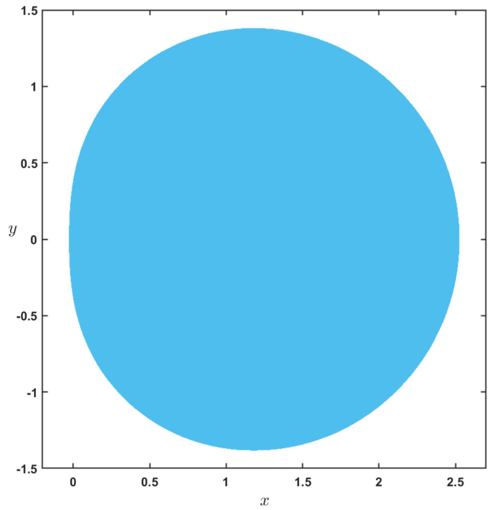

Figure 2. Spectrum of $\Delta^2_{1/4}$

Conclusion

In our study, we have determined the spectrum and fine spectrum of $q-$analog of second order difference operator. This operator reduces to second order difference operator when $q \to 1$. Like a generalized difference operator, we have a generalized quantum difference operator. Spectral analysis of generalized quantum difference operator can also be done in different sequence spaces.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Liftings from Lorentzian α-Sasakian manifolds to tangent bundles

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The goal of the current study is to investigate the complete lift of Lorentzian α-Sasakian manifolds to the tangent bundle TM. We also examine the complete lift of the different four types of Lorentzian α-Sasakian manifolds and find that (TM, g^C) is an η-Einstein manifold in each instance. In order to show that a Lorentzian α-Sasakian manifold exists on TM, a non-trivial example by means of partial differential equations is built in the final section.

Keywords: complete lift, tangent bundle, mathematical operators, Weyl conformal curvature tensor, conharmoic curvature tensor, projective curvature tensor, concircular curvature tensor, trans-Sasakian manifolds, Lorentzian α-Sasakian manifolds, partial differential equations.

2020 Mathematics Subject Classification: 53D10, 53C25, 53C15, 58A30.

Introduction

The geometry of trans-Sasakian manifolds received significant contributions from Blair and Oubina [1]. An almost contact metric manifold $(\mathsf{M}, \phi, \xi, \eta, g)$ with dim $= (2n+1)$ was defined due to Blair [2]. The geometry of the almost Hermitian manifold $(\bar{\mathsf{M}}, J, G)$ helps to determine the geometry of the almost contact metric manifold $(\mathsf{M}, \phi, \xi, \eta, g)$, that offers various structures on M (Sasakian, quasi-Sasakian, Kenmotsu, etc.) [1, 3–6], where $\bar{\mathsf{M}} = \mathsf{M} \times R$, J represents the almost complex structure and G stands for the Hermitian metric. A structure $(\phi, \xi, \eta, g, \alpha, \beta)$ on M is referred to as a trans-Sasakian structure where α, β are smooth functions using the structure in the class W_4 on $(\bar{\mathsf{M}}, J, G)$. On the nearly Hermitian manifold $(\bar{\mathsf{M}}, J, G)$, sixteen different types of structures are known to exist [7].

It is noted that cosymplectic [2], β-Kenmotsu [6] and α-Sasakian [6] are trans-Sasakian structures of type $(0,0), (0,\beta)$ and $(\alpha, 0)$ respectively. De and Tripathi [8] have explored trans-Sasakian manifolds and achieved outstanding findings. Lorentzian α-Sasakian manifolds were the subject of Yildiz and Murathan's research [9]. Yoldas developed a few classes of generalized recurrent α-cosymplectic manifolds [10–12]. We cite [13, 14] for additional research on the aforementioned subject.

Assuming that $(\mathsf{M}, g), n = \dim \mathsf{M} > 3$ is connected semi Riemannian manifold of class C^∞ and represent by ∇ its Levi-Civita connection. We write the Riemannian-Christoffel curvature tensor R, the Weyl conformal curvature tensor $\mathcal{C}$, the conharmonic curvature tensor K [13], the projective curvature tensor P and the concircular curvature tensor $\tilde{\mathcal{C}}$ of (M, g) by

$$R(s_1, s_2)s_3 = \nabla_{s_1}\nabla_{s_2}s_3 - \nabla_{s_2}\nabla_{s_1}s_3 - \nabla_{[s_1,s_2]}s_3,$$

$$\begin{aligned}\mathcal{C}(s_1, s_2)s_3 &= R(s_1, s_2)s_3 \\ &+ \frac{1}{n-2}[S(s_1, s_3)s_2 - S(s_2, s_3)s_1 + g(s_1, s_3)Qs_2 - g(s_2, s_3)Qs_1] \\ &- \frac{\tau}{(n-1)(n-2)}[g(s_1, s_3)s_2 - g(s_2, s_3)s_1],\end{aligned}$$

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$$\begin{aligned} K(s_1,s_2)s_3 &= R(s_1,s_2)s_3 \\ &- \frac{1}{n-2}[S(s_2,s_3)s_1 - S(s_1,s_3)s_2 + g(s_2,s_3)Qs_1 - g(s_1,s_3)Qs_2], \end{aligned}$$

$$P(s_1,s_2)s_3 = R(s_1,s_2)s_3 - \frac{1}{n-2}[g(s_2,s_3)Qs_1 - g(s_1,s_3)Qs_2],$$

$$\tilde{\mathcal{C}}(s_1,s_2)s_3 = R(s_1,s_2)s_3 - \frac{\tau}{n(n-2)}[g(s_1,s_3)s_2 - g(s_2,s_3)s_1],$$

respectively. In this scenario, Q is the Ricci operator as defined by $S(s_1,s_2) = g(Qs_1,s_2)$, $s_1,s_2,s_3 \in \Im_0^1(\mathsf{M})$, S is the Ricci tensor and $\tau = tr(S)$ is the scalar curvature.

The fundamental characteristics of curvature tensors and the idea of the liftings of tensor fields and connections to their tangent bundle was developed in [15]. In their study, Dida and Hathout [16] looked into Ricci soliton structures on tangent bundles of Riemannian manifolds. Numerous scholars have examined several connections and geometric structures on the tangent bundle and established their fundamental geometric features [17–22].

These works serve as our inspiration as we investigate the complete lift of Lorentzian α-Sasakian manifolds to tangent bundle TM. Additionally, we investigate the complete lift of ϕ-conformally flat, ϕ-conharmonically flat, ϕ-projectively flat and ϕ-concircularly flat Lorentzian α-Sasakian manifolds and derive (TM, g^C) as an η^C-Einstein manifold in each instance where g^C is the Lorentzian metric.

Notations. The notations below appear in several places throughout the text: Both $\Im_a^b(\mathsf{M})$ and $\Im_a^b(\mathsf{TM})$ stand for the set of all tensor fields of type (a,b) [23, 24].

1 *Preliminaries*

If there is a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η, and a Lorentzian metric g that satisfy $\forall s_1, s_2 \in \Im_0^1(\mathsf{M})$ [1, 2, 8, 25]

$$\begin{aligned} \eta(\xi) &= -1, & (1)\\ \phi^2 &= I + \eta \otimes \xi, & (2)\\ g(\phi s_1, \phi s_2) &= g(s_1,s_2) + \eta(s_1)\eta(s_2), & (3)\\ g(s_1,\xi) &= \eta(s_1), & (4)\\ \phi\xi &= 0, \quad \eta(\phi s_1) = 0, \end{aligned}$$

then the differentiable manifold M is said to be a Lorentzian α-Sasakian manifold.

The following relationships are also true in a Lorentzian α-Sasakian manifold M [26–30]

$$\begin{aligned} \nabla_{s_1}\xi &= -\alpha\phi s_1, & (5)\\ (\nabla_{s_1}\eta)s_2 &= -\alpha g(\phi s_1, s_2), & (6) \end{aligned}$$

and M becomes η-Einstein if its Ricci tensor S is given by

$$S(s_1,s_2) = ag(s_1,s_2) + b\eta(s_1)\eta(s_2), \;\; \forall s_1,s_2 \in \Im_0^1(\mathsf{M}), \tag{7}$$

in above equations ∇ represents the covariant differentiation operator w.r.t. g and a,b are functions on M.

For curvature tensor R, we have [8]

$$
\begin{aligned}
R(\xi, s_1)s_2 &= \alpha^2(g(s_1,s_2)\xi + \eta(s_2)s_1), && (8)\\
R(s_1, s_2)\xi &= \alpha^2(\eta(s_2)s_1 + \eta(s_1)s_2), && (9)\\
R(\xi, s_1)\xi &= \alpha^2(\eta(s_1)\xi + s_1), && (10)\\
S(s_1, \xi) &= (n-1)\alpha^2\eta(s_1), && (11)\\
Q\xi &= (n-1)\alpha^2\xi, && (12)\\
S(\xi, \xi) &= -(n-1)\alpha^2, && (13)\\
S(\phi s_1, \phi s_2) &= S(s_1, s_2) + (n-1)\alpha^2\eta(s_1)\eta(s_2), && (14)
\end{aligned}
$$

where Q stands for the Ricci operator with $S(s_1, s_2) = g(Qs_1, s_2)$.

2 The complete lift from a Lorentzian α-Sasakian manifold to its tangent bundle

Let us consider a local coordinate system $(x^i), i = 1, ..., n$ on differentiable manifold M and let (x^i, y^i), $i = 1, ..., n$ be an induced local coordinate system on tangent bundle TM. If $s_1 = {s_1}^i \frac{\partial}{\partial x^i}$ is a local vector field on M, then its vertical and complete lifts in the term of partial differential equations are

$$
\begin{aligned}
{s_1}^V &= {s_1}^i \frac{\partial}{\partial y^i},\\
{s_1}^C &= {s_1}^i \frac{\partial}{\partial x^i} + \frac{\partial {s_1}^i}{\partial x^j} y^j \frac{\partial}{\partial y^i}.
\end{aligned}
$$

Let η, s_1 and ϕ, respectively, represent 1-form, vector field, and a tensor field of type (1,1). Denote the complete and vertical lifts of η, s_1 and ϕ by $\eta^C, {s_1}^C, \phi^C$ and $\eta^V, {s_1}^V, \phi^V$, respectively. Then by using mathematical operators on η, s_1 and ϕ, we have [15, 31]

$$
\begin{aligned}
\eta^V({s_1}^C) = \eta^C({s_1}^V) = \eta(s_1)^V,\ \eta^C({s_1}^C) = \eta(s_1)^C,\\
\phi^V {s_1}^C = (\phi s_1)^V,\ \phi^C {s_1}^C = (\phi s_1)^C,\\
[s_1, s_2]^V = [{s_1}^C, {s_2}^V] = [{s_1}^V, {s_2}^C],\ [s_1, s_2]^C = [{s_1}^C, {s_2}^C],\\
\nabla^C_{{s_1}^C} {s_2}^C = (\nabla_{s_1} s_2)^C, \qquad \nabla^C_{{s_1}^C} {s_2}^V = (\nabla_{s_1} s_2)^V,
\end{aligned}
$$

∇^C is used for the complete lift of ∇ on TM.

Using the complete lift on (1)–(7), we conclude

$$
\begin{aligned}
\eta^C(\xi^C) &= \eta^V(\xi^V) = 0,\ \ \eta^C(\xi^V) = \eta^V(\xi^C) = -1, && (15)\\
(\phi^2)^C &= I + \eta^C \otimes \xi^V + \eta^V \otimes \xi^C, && (16)\\
g^C((\phi s_1)^C, (\phi s_2)^C) &= g^C({s_1}^C, {s_2}^C) + \eta^C({s_1}^C)\eta^V({s_2}^C)\\
&+ \eta^V({s_1}^C)\eta^C({s_2}^C),\\
g^C({s_1}^C, \xi^C) &= \eta^C({s_1}^C),\\
\phi^C\xi^C &= \phi^V\xi^V = \phi^C\xi^V = \phi^V\xi^C = 0, && (17)\\
\eta^C(\phi s_1)^C &= \eta^V(\phi s_1)^V = \eta^C(\phi s_1)^V = \eta^V(\phi s_1)^C = 0, && (18)
\end{aligned}
$$

$\forall {s_1}^C, {s_2}^C \in \Im^1_0(\mathsf{TM})$, and

$$
\begin{aligned}
\nabla^C_{{s_1}^C}\xi^C &= -\alpha(\phi s_1)^C,\\
(\nabla^C_{{s_1}^C}\eta^C){s_2}^C &= -\alpha g^C((\phi s_1)^C, {s_2}^C).
\end{aligned}
$$

When the complete lift of Ricci tensor S holds for

$$S^C({s_1}^C, {s_2}^C) = ag^C({s_1}^C, {s_2}^C) + b\{\eta^C({s_1}^C)\eta^V({s_2}^C) + \eta^V({s_1}^C)\eta^C({s_2}^C)\},$$

then the Lorentzian α-Sasakian manifold M on TM is thought to be η^C-Einstein.

Taking the complete lift on (8)–(14), we infer

$$\begin{aligned}
R^C(\xi^C, {s_1}^C){s_2}^C &= \alpha^2\{g^C({s_1}^C, {s_2}^C)\xi^V + g^C({s_1}^V, {s_2}^C)\xi^V \\
&- \eta^C({s_2}^C){s_1}^V + \eta^V({s_2}^C){s_1}^C\}, \\
R^C({s_1}^C, {s_2}^C)\xi^C &= \alpha^2\{\eta^C({s_2}^C){s_1}^V + \eta^V({s_2}^C){s_1}^C \\
&- \eta^C({s_1}^C){s_2}^V + \eta^V({s_1}^C){s_2}^C\}, \\
R^C(\xi^C, {s_1}^C)\xi^C &= \alpha^2\{\eta^C({s_1}^C)\xi^V + \eta^V({s_1}^C)\xi^C + {s_1}^C\}, \\
S^C({s_1}^C, \xi^C) &= (n-1)\alpha^2\eta^C({s_1}^C), \\
(Q\xi)^C &= (n-1)\alpha^2\xi^C, \\
S^C(\xi^C, \xi^C) &= -(n-1)\alpha^2, \\
S^C((\phi s_1)^C, (\phi s_2)^C) &= S^C({s_1}^C, {s_2}^C) + (n-1)\alpha^2\{\eta^C({s_1}^C)\eta^V({s_2}^C) \\
&+ \eta^V({s_1}^C)\eta^C({s_2}^C)\},
\end{aligned}$$

where $S^C({s_1}^C, {s_2}^C) = g^C((Qs_1)^C, {s_2}^C)$ and $S^({s_1}^V, {s_2}^C) = g^C((Qs_1)^V, {s_2}^C)$.

3 Main Results

Definition 1. Consider a differentiable manifold (M^n, g) with $n > 3$. Then M is said to be

- ϕ-conformally flat if
$$\phi^2\mathcal{C}(\phi s_1, \phi s_2)\phi s_3 = 0, \tag{19}$$
- ϕ-conharmonically flat if
$$\phi^2 K(\phi s_1, \phi s_2)\phi s_3 = 0,$$
- ϕ-projectively flat provided
$$\phi^2 P(\phi s_1, \phi s_2)\phi s_3 = 0,$$
- ϕ-concircularly flat if
$$\phi^2\tilde{\mathcal{C}}(\phi s_1, \phi s_2)\phi s_3 = 0.$$

Theorem 1. Let M^n be ϕ-conformally flat Lorentzian α-Sasakian manifold and denote by TM its tangent bundle. Then (TM, g^C) is an η^C-Einstein manifold, g^C being the Lorentzian metric of TM.

Proof. For the given assumptions, we see that

$$\phi^2\mathcal{C}(\phi s_1, \phi s_2)\phi s_3 = 0 \Longleftrightarrow g(\mathcal{C}(\phi s_1, \phi s_2)\phi s_3, \phi s_4) = 0, \tag{20}$$

$\forall s_1, s_2, s_3, s_4 \in \Im^1_0(\mathsf{M})$.

Complete lifts on (1) produce

$$\begin{aligned}
\mathcal{C}^C({s_1}^C, {s_2}^C){s_3}^C &= R^C({s_1}^C, {s_2}^C){s_3}^C \\
&+ \frac{1}{n-2}[S^C({s_1}^C, {s_3}^C){s_2}^V + S^C({s_1}^V, {s_3}^C){s_2}^C \\
&- S^C({s_2}^C, {s_3}^C){s_1}^V - S^C({s_2}^V, {s_3}^C){s_1}^C \\
&+ g^C({s_1}^C, {s_3}^C)(Qs_2)^V + g^C({s_1}^V, {s_3}^C)(Qs_2)^C \\
&- g^C({s_2}^C, {s_3}^C)(Qs_1)^V - g^C({s_2}^V, {s_3}^C)(Qs_1)^C] \\
&- \frac{\tau}{(n-1)(n-2)}[g^C({s_1}^C, {s_3}^C){s_2}^V + g^C({s_1}^V, {s_3}^C){s_2}^C \\
&- g^C({s_2}^C, {s_3}^C){s_1}^V - g^C({s_2}^V, {s_3}^C){s_1}^C].
\end{aligned}$$

In view of (20) and (21) ϕ-conformally flat means

$$
\begin{aligned}
& (g(R(\phi s_1, \phi s_2)\phi s_3, \phi s_4))^C \\
= \ & \frac{1}{n-2}[g^C((\phi s_2)^C, (\phi s_3)^C)S^C((\phi s_1)^V, (\phi s_4)^C) \\
+ \ & g^C((\phi s_2)^V, (\phi s_3)^C)S^C((\phi s_1)^C, (\phi s_4)^C) \\
- \ & g^C((\phi s_1)^C, (\phi s_3)^C)S^C((\phi s_2)^V, (\phi s_4)^C) \\
- \ & g^C((\phi s_1)^V, (\phi s_3)^C)S^C((\phi s_2)^C, (\phi s_4)^C) \\
+ \ & S^C((\phi s_2)^C, (\phi s_3)^C)g^C((\phi s_1)^V, (\phi s_4)^C) \\
+ \ & S^C((\phi s_2)^V, (\phi s_3)^C)g^C((\phi s_1)^C, (\phi s_4)^C) \\
- \ & S^C((\phi s_1)^C, (\phi s_3)^C)g^C((\phi s_2)^V, (\phi s_4)^C) \\
- \ & S^C((\phi s_1)^V, (\phi s_3)^C)g^C((\phi s_2)^C, (\phi s_4)^C)] \\
- \ & \frac{\tau}{(n-1)(n-2)}[g^C((\phi s_2)^C, (\phi s_3)^C)g^C((\phi s_1)^V, (\phi s_4)^C) \\
+ \ & g^C((\phi s_2)^V, (\phi s_3)^C)g^C((\phi s_1)^C, (\phi s_4)^C) \\
- \ & g^C((\phi s_1)^C, (\phi s_3)^C)g^C((\phi s_2)^V, (\phi s_4)^C) \\
- \ & g^C((\phi s_1)^V, (\phi s_3)^C)g^C((\phi s_2)^C, (\phi s_4)^C)]. \qquad (21)
\end{aligned}
$$

Let $\{\varepsilon_i : i = 1, ..., n-1, \xi\}$ represent a local orthonormal basis. Then $\{(\phi\varepsilon_i)^C : i = 1, ..., n-1, (\phi\xi)^C\}$ is in TM.

Setting $s_1 = s_4 = \varepsilon_i$ in (21) produces

$$
\begin{aligned}
& \sum_{i=1}^{n-1} g(R(\phi\varepsilon_i, \phi s_2)\phi s_3, \phi\varepsilon_i) \\
= \ & \frac{1}{n-2}\sum_{i=1}^{n-1}[g^C((\phi s_2)^C, (\phi s_3)^C)S^C((\phi\varepsilon_i)^V, (\phi\varepsilon_i)^C) \\
+ \ & g^C((\phi s_2)^V, (\phi s_3)^C)S^C((\phi\varepsilon_i)^C, (\phi\varepsilon_i)^C) \\
- \ & g^C((\phi\varepsilon_i)^C, (\phi s_3)^C)S^C((\phi s_2)^V, (\phi\varepsilon_i)^C) \\
- \ & g^C((\phi\varepsilon_i)^V, (\phi s_3)^C)S^C((\phi s_2)^C, (\phi\varepsilon_i)^C) \\
+ \ & S^C((\phi s_2)^C, (\phi s_3)^C)g^C((\phi\varepsilon_i)^V, (\phi\varepsilon_i)^C) \\
+ \ & S^C((\phi s_2)^V, (\phi s_3)^C)g^C((\phi\varepsilon_i)^C, (\phi\varepsilon_i)^C) \\
- \ & S^C((\phi\varepsilon_i)^C, (\phi s_3)^C)g^C((\phi s_2)^V, (\phi\varepsilon_i)^C) \\
- \ & S^C((\phi\varepsilon_i)^V, (\phi s_3)^C)g^C((\phi s_2)^C, (\phi\varepsilon_i)^C)] \\
- \ & \frac{\tau}{(n-1)(n-2)}\sum_{i=1}^{n-1}[g^C((\phi s_2)^C, (\phi s_3)^C)g^C((\phi\varepsilon_i)^V, (\phi\varepsilon_i)^C) \\
+ \ & g^C((\phi s_2)^V, (\phi s_3)^C)g^C((\phi\varepsilon_i)^C, (\phi\varepsilon_i)^C) \\
- \ & g^C((\phi\varepsilon_i)^C, (\phi s_3)^C)g^C((\phi s_2)^V, (\phi\varepsilon_i)^C) \\
- \ & g^C((\phi\varepsilon_i)^V, (\phi s_3)^C)g^C((\phi s_2)^C, (\phi\varepsilon_i)^C)]. \qquad (22)
\end{aligned}
$$

In view of (15), (16), (17), (18) and (19), we infer

$$\begin{aligned}\sum_{i=1}^{n-1}(g(R(\phi\varepsilon_i,\phi s_2)\phi s_3,\phi\varepsilon_i))^C &= S^C((\phi s_2)^C,(\phi s_3)^C)\\ &+ g^C((\phi s_2)^C,(\phi s_3)^C), \qquad (23)\\ \sum_{i=1}^{n-1}S^C((\phi\varepsilon_i)^C,(\phi\varepsilon_i))^C &= \tau-(n-1)\alpha^2, \qquad (24)\\ \sum_{i=1}^{n-1}[g^C((\phi\varepsilon_i)^C,(\phi s_3)^C)S^C((\phi s_2)^V,(\phi\varepsilon_i)^C) &+ g^C((\phi\varepsilon_i)^V,(\phi s_3)^C)S^C((\phi s_2)^C,(\phi\varepsilon_i)^C)\\ &= S^C((\phi s_2)^C,(\phi s_3)^C) \qquad (25)\\ \sum_{i=1}^{n-1}g^C((\phi\varepsilon_i)^C,(\phi\varepsilon_i))^C &= n-1, \qquad (26)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n-1}[g^C((\phi\varepsilon_i)^C,(\phi s_3)^C)g^C((\phi s_2)^V,(\phi\varepsilon_i)^C) &+ g^C((\phi\varepsilon_i)^V,(\phi s_3)^C)g^C((\phi s_2)^C,(\phi\varepsilon_i)^C)\\ &= g^C((\phi s_2)^C,(\phi s_3)^C). \qquad (27)\end{aligned}$$

Making use of (23)–(27) the equation (22) can be expressed as

$$S^C((\phi s_2)^C,(\phi s_3)^C)=L_1 g^C((\phi s_2)^C,(\phi s_3)^C), \qquad (28)$$

where $L_1=[\frac{\tau}{n-1}-(n-1)\alpha^2-(n-2)]$. Using (17) and (19), the equation (28) becomes

$$S^C({s_2}^C,{s_3}^C)=L_1 g^C({s_2}^C,{s_3}^C)+L_1\{\eta^C({s_2}^C)\eta^V({s_3}^C)+\eta^V({s_2}^C)\eta^C({s_3}^C)\}.$$

Thus (TM,g^C) is an η^C-Einstein manifold.

On the similar devices of Theorem 4.1 and using definition 1, we have

Theorem 2. Let M^n, $(n>3)$ be ϕ-conharmonically flat Lorentzian α-Sasakian manifold and TM be its tangent bundle. Then (TM,g^C) is an η^C-Einstein manifold.

Theorem 3. For any ϕ-projectively flat Lorentzian α-Sasakian manifold M^n $(n>3)$, (TM,g^C) is an η^C-Einstein manifold.

Theorem 4. For any ϕ-concircularly flat Lorentzian α-Sasakian manifold M^n $(n>3)$, (TM,g^C) will be an η^C-Einstein manifold.

4 Example

Assume a differentiable manifold $\mathsf{M}=\{(u,v,w):u,v,w\in\Re^3,w>0\}$ and denote the L.I. vector fields on M by $\varsigma_1,\varsigma_2,\varsigma_3$ given by [32]

$$\varsigma_1=\varsigma^{-w}\frac{\partial}{\partial v},\varsigma_2=\varsigma^{-w}(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}),\varsigma_3=\alpha\frac{\partial}{\partial w}=\xi.$$

Further for 1-form η on M, one can write

$$g(\varsigma_1,\varsigma_2)=g(\varsigma_1,\varsigma_3)=g(\varsigma_2,\varsigma_3)=0,\quad g(\varsigma_1,\varsigma_1)=g(\varsigma_2,\varsigma_2)=1,\quad g(\varsigma_3,\varsigma_3)=-1$$

and

$$\eta(s_3) = g(s_3, \varsigma_3), \quad s_3 \in \Im_0^1(\mathsf{M}),$$

where g is the Lorentzian metric.

Suppose ϕ stands for $(1,1)$-tensor field satisfying

$$\phi\varsigma_1 = \varsigma_1, \quad \phi\varsigma_2 = \varsigma_2, \quad \phi\varsigma_3 = 0.$$

With the linearity of ϕ, one concludes $\eta(\varsigma_3) = -1$, $\phi^2\zeta_3 = s_3 + \eta(s_3)\varsigma_3$ and $g(\phi s_1, \phi s_2) = g(s_1, s_2) + + \eta(s_1)\eta(s_2)$.

Thus, (for $\varsigma_3 = \xi$) M becomes Lorentzian almost paracontact metric manifold with Lorentzian almost paracontact metric structure (ϕ, ξ, η, g) on M.

Also,

$$[\varsigma_1, \varsigma_2] = 0, \quad [\varsigma_1, \varsigma_3] = \alpha\varsigma_1, \quad [\varsigma_2, \varsigma_3] = \alpha\varsigma_2.$$

The Koszul's formula is written as

$$\begin{aligned} 2g(\nabla_{\varsigma_1}\varsigma_2, s_3) &= Xg(s_2, s_3) + s_2 g(s_3, s_1) - \varsigma_3 g(s_1, s_2) \\ &- g(s_1, [s_2, s_3]) + g(s_2, [s_3, s_1]) + g(s_3, [s_1, s_2]), \end{aligned}$$

and we have [32]

$$\nabla_{\varsigma_1}\varsigma_1 = \alpha\varsigma_3, \quad \nabla_{\varsigma_1}\varsigma_3 = \alpha\varsigma_1, \quad \nabla_{\varsigma_2}\varsigma_2 = \alpha\varsigma_3, \quad \nabla_{\varsigma_2}\varsigma_3 = \alpha\varsigma_2, \tag{29}$$

$$\nabla_{\varsigma_1}\varsigma_2 = \nabla_{\varsigma_3}\varsigma_1 = \nabla_{\varsigma_3}\varsigma_2 = \nabla_{\varsigma_2}\varsigma_1 = \nabla_{\varsigma_3}\varsigma_3 = 0. \tag{30}$$

We can easily verify that

$$\begin{aligned} \nabla_{s_1}\xi &= -\alpha\phi s_1, \\ (\nabla_{s_1}\eta)s_2 &= -\alpha g(\phi s_1, s_2). \end{aligned}$$

Hence, M is a "Lorentzian α-Sasakian manifold".

Let us denote the complete and vertical lifts of $\varsigma_1, \varsigma_2, \varsigma_3$ on $T\mathsf{M}$ by $\varsigma_1^C, \varsigma_2^C, \varsigma_3^C$ and $\varsigma_1^V, \varsigma_2^V, \varsigma_3^V$. Further, suppose that g^C be the complete lift of a Riemannian metric g on TM holding

$$\begin{aligned} g^C({s_1}^V, \varsigma_3^C) &= (g^C(s_1, \varsigma_3))^V = (\eta(s_1))^V, \\ g^C({s_1}^C, \varsigma_3^C) &= (g^C(s_1, \varsigma_3))^C = (\eta(s_1))^C, \\ g^C(\varsigma_3^C, \varsigma_3^C) &= -1, \quad g^V({s_1}^V, \varsigma_3^C) = 0, \quad g^V(\varsigma_3^V, \varsigma_3^V) = 0 \end{aligned} \tag{31}$$

and so on.

Consider the $(1,1)$-tensor field ϕ and its complete and vertical lifts ϕ^C and ϕ^V by

$$\phi^V(\varsigma_1^V) = \varsigma_1^V, \quad \phi^C(\varsigma_1^C) = \varsigma_1^C.$$

$$\phi^V(\varsigma_2^V) = \varsigma_2^V, \quad \phi^C(\varsigma_2^C) = \varsigma_2^C,$$

$$\phi^V(\varsigma_3^V) = \phi^C(\varsigma_3^C) = 0.$$

Above equation produces

$$(\phi^2 X)^C = {s_1}^C + \eta^V(s_1)\varsigma_3^C + \eta^C(s_1)\varsigma_3^V, \tag{32}$$

$$\begin{aligned} g^C((\phi\varsigma_1)^C, (\phi\varsigma_2)^C) &= g^C(\varsigma_1^C, \varsigma_2^C) + (\eta(\varsigma_1))^C(\eta(\varsigma_2))^V \\ &+ (\eta(\varsigma_1))^V(\eta(\varsigma_2))^C. \end{aligned}$$

Thus, for $\varsigma_3 = \xi$ in (31)–(32), the structure $(\phi^C, \xi^C, \eta^C, g^C)$ is a Lorentzian almost paracontact metric structure on TM.

The Koszul's formula for ∇^C can be viewed as

$$\begin{aligned} 2g^C(\nabla^C_{\varsigma_1^C}\varsigma_2^C, s_3^C) &= X^C g^C(s_2^C, s_3^C) + s_2^C g^C(s_3^C, s_1^C) - \varsigma_3^C g^C(s_1^C, s_2^C) \\ &- g^C(s_1^C, [s_2^C, s_3^C]) + g^C(s_2^C, [s_3^C, s_1^C]) + g^C(s_3^C, [s_1^C, s_2^C]). \end{aligned}$$

Taking the complete lift on (29) and (30), we conclude

$$\begin{aligned} \nabla^C_{\varsigma_1^C}\varsigma_1^C &= \alpha\varsigma_3^C, \;\; \nabla^C_{\varsigma_1^C}\varsigma_3^C = \alpha\varsigma_1^C, \;\; \nabla^C_{\varsigma_2^C}\varsigma_2^C = \alpha\varsigma_3^C, \;\; \nabla^C_{\varsigma_2^C}\varsigma_3^C = \alpha\varsigma_2^C, \\ \nabla^C_{\varsigma_1^C}\varsigma_2^C &= \nabla^C_{\varsigma_3^C}\varsigma_1^C = \nabla^C_{\varsigma_3^C}\varsigma_2^C = \nabla^C_{\varsigma_2^C}\varsigma_1^C = \nabla^C_{\varsigma_3^C}\varsigma_3^C = 0. \end{aligned}$$

We can easily verify that

$$\begin{aligned} \nabla^C_{s_1^C}\xi^C &= -\alpha\phi^C s_1^C, \\ (\nabla^C_{s_1^C}\eta^C)s_2^C &= -\alpha g^C((\phi s_1)^C, s_2^C). \end{aligned}$$

Hence, $(\phi^C, \xi^C, \eta^C, g^C, \mathsf{TM})$ is a Lorentzian α-Sasakian manifold.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

All authors contributed equally to this work.

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https://doi.org/10.31489/2025M2/147-162 *Research article*

On g_{MT}- and g_β-convexity and the Ostrowski type inequalities

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In this study, motivated by recent results on Ostrowski-type inequalities, we introduce a new identity that serves as a basis for establishing fractional Ostrowski inequalities. Specifically, we focus on functions whose modulus of the first derivatives are g_{MT}-convex and g_β-convex. This approach allows us to extend classical results to more general settings. Several special cases are discussed, recovering known inequalities while highlighting the versatility of our method.

Keywords: g_{MT}- convexity, g_β-convexity, power mean inequality, generalized Riemann-Liouville fractional integrals.

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Introduction

The Ostrowski inequality (see [1]), can be stated as follows:

Theorem 1. Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$, be a differentiable mapping with bounded first derivatives, then

$$\left|\varepsilon(x) - \frac{1}{k-r}\int\limits_r^k \varepsilon(\mathfrak{u})d\mathfrak{u}\right| \le M(k-r)\left[\frac{1}{4} + \frac{(x-\frac{r+k}{2})^2}{(k-r)^2}\right] \tag{1}$$

holds, where $r, k \in \mathfrak{I}$ with $r < k$.

In recent years, such inequalities were studied extensively by many researchers. Regarding some papers with closed relationship with inequality (1) we refer readers to [2–16], and references cited therein.

In [17], Liu used the so-called MT-convex function defined by Tunç [18,19] and derived the following fractional Ostrowski type inequalities:

Definition 1. [18, 19] Let the function $\varepsilon : \mathfrak{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$, if for all η, $\mu \in \mathfrak{I}$ and $t \in [0, 1]$

$$\varepsilon(t\eta + (1-t)\mu) \le \frac{\sqrt{t}}{2\sqrt{1-t}}\varepsilon(\eta) + \frac{\sqrt{1-t}}{2\sqrt{t}}\varepsilon(\mu) \tag{2}$$

holds, then ε is said an MT-convex. If (2) holds in the opposite sense, then ε is said MT-concave.

Theorem 2. [17] Let the differentiable function $\varepsilon : [r, k] \to \mathbb{R}$ with $\varepsilon' \in L^1[r, k]$. If $|\varepsilon'|$ is MT-convex on $[r, k]$, where $\alpha > 0$, $0 \le r < k$ and for $x \in [r, k] : |\varepsilon'(x)| \le \mathfrak{M}$, then we have

$$\begin{aligned}&\left|\frac{(x-r)^\alpha + (k-x)^\alpha}{k-r}\varepsilon(x) - \frac{\Gamma(\alpha+1)}{k-r}\left[J^\alpha_{x^-}\varepsilon(r) + J^\alpha_{x^+}\varepsilon(k)\right]\right| \\ \le &\mathfrak{M}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\frac{(x-r)^{\alpha+1} + (k-x)^{\alpha+1}}{k-r}.\end{aligned}$$

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Theorem 3. [17] Let the differentiable function $\varepsilon : [r,k] \to \mathbb{R}$ with $\varepsilon' \in L^1[r,k]$. If $|\varepsilon'|^q$ is MT-convex on $[r,k]$, where $\alpha > 0$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $0 \le r < k$ and for $x \in [r,k] : |\varepsilon'(x)| \le \mathfrak{M}$, then we have

$$\begin{aligned}&\left|\tfrac{(x-r)^\alpha+(k-x)^\alpha}{k-r}\varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[J^\alpha_{x-}\varepsilon(r) + J^\alpha_{x+}\varepsilon(k)\right]\right| \\ &\le \tfrac{\mathfrak{M}}{(1+p\alpha)^{\frac{1}{p}}}\left(\tfrac{\pi}{2}\right)^{\frac{1}{q}}\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}.\end{aligned}$$

Theorem 4. [17] Let the differentiable function $\varepsilon : [r,k] \to \mathbb{R}$ with $\varepsilon' \in L^1[r,k]$. If $|\varepsilon'|^q$ is MT-convex on $[r,k]$, where $\alpha > 0$, $q > 1$, $0 \le r < k$ and for $x \in [r,k] : |\varepsilon'(x)| \le \mathfrak{M}$, then we have

$$\begin{aligned}&\left|\tfrac{(x-r)^\alpha+(k-x)^\alpha}{k-r}\varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[J^\alpha_{x-}\varepsilon(r) + J^\alpha_{x+}\varepsilon(k)\right]\right| \\ &\le \tfrac{\mathfrak{M}}{(1+\alpha)^{1-\frac{1}{q}}}\left(\tfrac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}\right)^{\frac{1}{q}}\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}.\end{aligned}$$

The concept of convex functions and sets are extended to the new class of g-convex function and g-convex sets (see [20]). This class is more general and plays an important role in nonlinear programming problems and optimization theory in which the constraint set the objective function are g-convex.

The objective of this research is to use extended Riemann-Liouville fractional integrals to construct new Ostrowski inequalities for functions whose absolute value of first derivatives is g_{MT}- and g_β-convex. These results generalize those of [8] and give fresh estimates of this kind of disparity.

1 Preliminaries

The following section is devoted to some definitions and remarks.

Definition 2. [21] Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$, if for all η, $\mu \in \mathfrak{I}$ and $\varpi \in [0,1]$

$$\varepsilon(\varpi\eta + (1-\varpi)\mu) \le \varpi\varepsilon(\eta) + (1-\varpi)\varepsilon(\mu)$$

holds, then ε is called a convex function.

Definition 3. [22] Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$, if for all η, $\mu \in \mathfrak{I}$ and $\varpi \in [0,1]$

$$\varepsilon(\varpi\eta + (1-\varpi)\mu) \le \varepsilon(\eta) + \varepsilon(\mu)$$

holds, then ε is an P-convex function.

Definition 4. [23] Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$ be a nonnegative function, if for all η, $\mu \in \mathfrak{I}$, some fixed $s \in (0,1]$ and $\varpi \in [0,1]$

$$\varepsilon(\varpi\eta + (1-\varpi)\mu) \le \varpi^s\varepsilon(\eta) + (1-\varpi)^s\varepsilon(\mu)$$

holds, then ε is an s-convex function.

Definition 5. [24] Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$ be a nonnegative function, if for all η, $\mu \in \mathfrak{I}$ and $\varpi \in (0,1)$

$$\varepsilon(\varpi\eta + (1-\varpi)\mu) \le \varpi(1-\varpi)[\varepsilon(\eta) + \varepsilon(\mu)]$$

holds, then ε is a tgs-convex function.

Definition 6. [25] Let $\varepsilon : \mathfrak{I} \to \mathbb{R}$, if for all η, $\mu \in \mathfrak{I}$, p, $q > -1$ and $\varpi \in (0,1)$

$$\varepsilon(\varpi\eta + (1-\varpi)\mu) \le \varpi^p(1-\varpi)^q\varepsilon(\eta) + \varpi^q(1-\varpi)^p\varepsilon(\mu)$$

holds, then ε is an β-convex function.

Remark 1. For $(p,q) \in \{(0,0),(s,0),(1,1),(1,0)\}$, Definition 6, recapture the P-convexity, s-convexity, tgs-convexity and classical convexity, respectively.

Definition 7. [20] We say that a set $K_g \subseteq \mathbb{R}^n$ is g-convex, if there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ and

$$\varpi g(\eta) + (1-\varpi)g(\mu) \in K_g$$

holds $\forall\ \eta,\ \mu \in \mathbb{R}^n : g(\eta),\ g(\mu) \in K_g$ and $\varpi \in [0,1]$.

Definition 8. [20] Let $\varepsilon : \mathbb{R}^n \to \mathbb{R}$, if there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ and for all $\eta,\ \mu \in \mathbb{R}^n : g(\eta)$, $g(\mu) \in K_g$ and $\varpi \in [0,1]$

$$\varepsilon(\varpi g(\eta) + (1-\varpi)g(y)) \leq \varpi\varepsilon(g(\eta)) + (1-\varpi)\varepsilon(g(\mu))$$

holds, then ε is an g-convex on K_g.

Remark 2. Every convex function ε on a convex set K_g is a g-convex function, where g is the identity map. However, the converse is not true.

Example 1. Let $K_g \subset \mathbb{R}^2$ be given as

$$K_g = \left\{(\eta,\mu) \in \mathbb{R}^2 : (\eta,\mu) = \alpha_1(0,\ 0) + \alpha_2(0,\ 3) + \alpha_3(2,\ 1)\right\}$$

with $\alpha_i > 0$, $\sum\limits_{i=1}^{3} \alpha_i = 1$, and define a mapping $g : \mathbb{R}^2 \to \mathbb{R}^2$ as $g(\eta,\mu) = (0,\ \mu)$.

The function $\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\varepsilon(\eta,\ \mu) = \begin{cases} \eta^3, & \text{if } \ \mu < 1, \\ \eta\mu^3, & \text{if } \ \mu \geq 1 \end{cases}$$

is g-convex on K_g but is not convex.

In [26], Sarikaya defined the so-called g_h-convex functions which are a generalization of the aforementioned convex functions.

Definition 9. [26] Let the functions $h : (0;1) \to (0;1)$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $\varepsilon : \mathfrak{I} \subset \mathbb{R} \to [0,+\infty)$. If for all $\eta,\mu \in \mathbb{R}^n :\ g(\eta), g(\mu) \in K_g$ and $\varpi \in [0,1]$

$$\varepsilon(\varpi g(\eta) + (1-\varpi)g(\mu)) \leq h\left(\varpi\right)\varepsilon(g(\eta)) + h(1-\varpi)\varepsilon(g(\mu))$$

holds, then ε is a g_h-convex function.

Among the subclasses of Definition 9 we mention the classes of g_{MT}- and g_β-convex functions as follows:

Definition 10. Let $\varepsilon : K_g \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative and $g : \mathbb{R} \to \mathbb{R}$, if for all $\eta,\mu \in \mathbb{R}$: $g(\eta), g(\mu) \in K_g$, and $\varpi \in (0,1)$

$$\varepsilon(\varpi g(\eta) + (1-\varpi)g(\mu)) \leq \tfrac{\sqrt{\varpi}}{2\sqrt{1-\varpi}}\varepsilon(g(\eta)) + \tfrac{\sqrt{1-\varpi}}{2\sqrt{\varpi}}\varepsilon(g(\mu))$$

holds, then ε is a g_{MT}-convex function.

Definition 11. Let $\varepsilon : K_g \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative and $g : \mathbb{R} \to \mathbb{R}$, if for all $\eta,\mu \in \mathbb{R}$: $g(\eta), g(\mu) \in K_g$, $p,\ q > -1$ and $\varpi \in (0,1)$

$$\varepsilon(\varpi g(\eta) + (1-\varpi)g(\mu)) \leq \varpi^p(1-\varpi)^q\varepsilon(g(\eta)) + \varpi^q(1-\varpi)^p\varepsilon(g(\mu))$$

holds, then ε is a g_β-convex function.

Definition 12. [27–29] The Riemann-Liouville integrals $I^{\alpha}_{\eta^+}\varepsilon$ and $I^{\alpha}_{\mu^-}\varepsilon$ of order $\alpha > 0$ with $\eta \geq 0$ where $\varepsilon \in L^1[\eta,\mu]$ are defined by

$$I^{\alpha}_{\eta^+}\varepsilon(x) = \frac{1}{\Gamma(\alpha)}\int\limits_{\eta}^{x}(x-\varpi)^{\alpha-1}\varepsilon(\varpi)d\varpi, \quad x > \eta, \tag{3}$$

and

$$I^{\alpha}_{\mu^-}\varepsilon(x) = \frac{1}{\Gamma(\alpha)}\int\limits_{x}^{\mu}(\varpi-x)^{\alpha-1}\varepsilon(\varpi)d\varpi, \quad x < \mu, \tag{4}$$

respectively, where $\Gamma(\alpha) = \int\limits_{0}^{\infty} e^{-\varpi}\varpi^{\alpha-1}d\varpi$, $\alpha > 0$ is the gamma function. Here $I^{0}_{\eta^+}\varepsilon(x) = I^{0}_{\mu^-}\varepsilon(x) = \varepsilon(x)$. For $\alpha = 1$, (3) and (4) recapture the classical integral.

Definition 13. [30] The left- and right-sided generalized Riemann-Liouville fractional integrals of order $\alpha > 0$, where $\varepsilon \in L^1[g(\eta), g(\mu)]$, with $g(\eta) < g(\mu)$, are given by

$$I^{\alpha}_{g(\eta)^+}\varepsilon(x) = \frac{1}{\Gamma(\alpha)}\int\limits_{g(\eta)}^{x}(x-\varpi)^{\alpha-1}\varepsilon(\varpi)d\varpi, \quad 0 \leq g(\eta) < x < g(\mu), \tag{5}$$

and

$$I^{\alpha}_{g(\mu)^-}\varepsilon(x) = \frac{1}{\Gamma(\alpha)}\int\limits_{x}^{g(\mu)}(\varpi-x)^{\alpha-1}\varepsilon(\varpi)d\varpi, \quad 0 \leq g(\eta) < x < g(\mu). \tag{6}$$

It is clear from (5) and (6) that $I^{\alpha}_{g(\eta)^+}\varepsilon(g(\eta)) = 0$ and $I^{\alpha}_{g(\mu)^-}\varepsilon(g(\mu)) = 0$.

2 *Main results*

Throughout this paper $K_g = [g(r), g(k)]$, $g(r) < g(k)$.

2.1 *Ostrowski type fractional integral inequalities for g_{MT}-convex functions*

Lemma 1. Let $\varepsilon : [g(r), g(k)] \to \mathbb{R}$ be a differentiable function on $(g(r), g(k))$ with $g(r) < g(k)$, where $g : \mathbb{R} \to \mathbb{R}$ is a mapping satisfying for all $x \in (r,k) : g(r) < g(x) < g(k)$. If $\varepsilon' \in L^1[g(r), g(k)]$, then for all $x \in [r,k]$ such that $g(x) \in [g(r), g(k)]$, and $\alpha > 0$, then

$$\begin{aligned}
&\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^-}\varepsilon(g(r)) + I^{\alpha}_{g(x)^+}\varepsilon(g(k))\right] \\
&= \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int\limits_{0}^{1}\varpi^{\alpha}\varepsilon'\left(\varpi g(x) + (1-\varpi)g(r)\right)d\varpi \\
&\quad - \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int\limits_{0}^{1}\varpi^{\alpha}\varepsilon'\left(\varpi g(x) + (1-\varpi)g(k)\right)d\varpi.
\end{aligned} \tag{7}$$

Proof. Integrating by parts, it yields

$$I_1 = \int_0^1 \varpi^\alpha \varepsilon' \left(\varpi g(x) + (1-\varpi)g(r)\right) d\varpi$$
$$= \frac{\varepsilon(g(x))}{g(x)-g(r)} - \frac{\alpha}{g(x)-g(r)} \int_0^1 \varpi^{\alpha-1} \varepsilon \left(\varpi g(x) + (1-\varpi)g(r)\right) d\varpi,$$

with the change of variable $u = \varpi g(x) + (1-\varpi)g(r)$, it follows that

$$I_1 = \frac{\varepsilon(g(x))}{g(x)-g(r)} - \frac{\Gamma(\alpha+1)}{(g(x)-g(r))^{\alpha+1}} \frac{1}{\Gamma(\alpha)} \int_{g(r)}^{g(x)} (u-g(r))^{\alpha-1} \varepsilon(u) du$$
$$= \frac{\varepsilon(g(x))}{g(x)-g(r)} - \frac{\Gamma(\alpha+1)}{(g(x)-g(r))^{\alpha+1}} I^\alpha_{g(x)^-} \varepsilon(g(r)).$$

Similarly, we obtain

$$I_2 = \int_0^1 \varpi^\alpha \varepsilon' \left(\varpi g(x) + (1-\varpi)g(k)\right) d\varpi = \frac{\varepsilon(g(x))}{g(x)-g(k)} + \frac{\Gamma(\alpha+1)}{(g(k)-g(x))^{\alpha+1}} I^\alpha_{g(x)^+} \varepsilon(g(k)).$$

Multiplying I_1 by $\frac{(g(x)-g(r))^{\alpha+1}}{k-r}$, and I_2 by $\frac{(g(k)-g(x))^{\alpha+1}}{k-r}$, we have

$$\frac{(g(x)-g(r))^{\alpha+1}}{k-r} \int_0^1 \varpi^\alpha \varepsilon' \left(\varpi g(x) + (1-\varpi)g(r)\right) d\varpi$$
$$= \frac{(g(x)-g(r))^{\alpha}}{k-r} \varepsilon(g(x)) - \frac{\Gamma(\alpha+1)}{k-r} I^\alpha_{g(x)^-} \varepsilon(g(r)) \tag{8}$$

and

$$\frac{(g(k)-g(x))^{\alpha+1}}{k-r} \int_0^1 \varpi^\alpha \varepsilon' \left(\varpi g(x) + (1-\varpi)g(k)\right) dt$$
$$= -\frac{(g(k)-g(x))^{\alpha}}{k-r} \varepsilon(g(x)) + \frac{\Gamma(\alpha+1)}{k-r} I^\alpha_{g(x)^+} \varepsilon(g(k)). \tag{9}$$

Subtracting (9) from (8), we get (7).

Remark 3. Lemma 1 gives Lemma 1 from [4], for $g(x) = x$.

Theorem 5. Let $\varepsilon : K_g \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on K_g° such that $\varepsilon' \in L^1[g(r), g(k)]$. If $|\varepsilon'|$ is g_{MT}-convex function with respect to g where $g : \mathbb{R} \to \mathbb{R}$ is a mapping satisfying $g(r) < g(x) < g(k)$ for all $x \in (r,k)$ and $|\varepsilon'(z)| \leq M$, $z \in K_g$, then

$$\left| \frac{(g(x)-g(r))^\alpha + (g(k)-g(x))^\alpha}{k-r} \varepsilon(g(x)) - \frac{\Gamma(\alpha+1)}{k-r} \left[I^\alpha_{g(x)^-} \varepsilon(g(r)) + I^\alpha_{g(x)^+} \varepsilon(g(k)) \right] \right|$$
$$\leq \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{2\Gamma(\alpha+1)} \frac{\left((g(x)-g(r))^{\alpha+1} + (g(k)-g(x))^{\alpha+1}\right)\sqrt{\pi}}{k-r} M$$

holds for all $x \in [r,k]$ with $g(x) \in K_g$ and $\alpha > 0$ and Γ is the gamma function.

Proof. By Lemma 1 and modulus, we have

$$\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^{-}}\varepsilon(g(r))+I^{\alpha}_{g(x)^{+}}\varepsilon(g(k))\right]\right|\\
\leq&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int\limits_0^1\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|d\varpi\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int\limits_0^1\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|d\varpi.
\end{aligned}$$

Since $|\varepsilon'|$ is g_{MT}-convex with respect to the function g, and taking into account that $|\varepsilon'(x)|\leq M$, $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, we obtain

$$\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^{-}}\varepsilon(g(r))+I^{\alpha}_{g(x)^{+}}\varepsilon(g(k))\right]\right|\\
\leq&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int\limits_0^1\varpi^{\alpha}\left(\tfrac{\sqrt{\varpi}}{2\sqrt{1-\varpi}}\left|\varepsilon'(g(x))\right|+\tfrac{\sqrt{1-\varpi}}{2\sqrt{\varpi}}\left|\varepsilon'(g(r))\right|\right)d\varpi\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int\limits_0^1\varpi^{\alpha}\left(\tfrac{\sqrt{\varpi}}{2\sqrt{1-\varpi}}\left|\varepsilon'(g(x))\right|+\tfrac{\sqrt{1-\varpi}}{2\sqrt{\varpi}}\left|\varepsilon'(g(k))\right|\right)d\varpi\\
\leq&\tfrac{M(g(x)-g(r))^{\alpha+1}}{2(k-r)}\int\limits_0^1\left(\varpi^{\alpha+\frac{1}{2}}(1-\varpi)^{\frac{-1}{2}}+\varpi^{\alpha-\frac{1}{2}}(1-\varpi)^{\frac{1}{2}}\right)d\varpi\\
&+\tfrac{M(g(k)-g(x))^{\alpha+1}}{2(k-r)}\int\limits_0^1\left(\varpi^{\alpha+\frac{1}{2}}(1-\varpi)^{\frac{-1}{2}}+\varpi^{\alpha-\frac{1}{2}}(1-\varpi)^{\frac{1}{2}}\right)d\varpi\\
=&\tfrac{\left((g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}\right)}{2(k-r)}\left(\beta(\alpha+\tfrac{3}{2},\tfrac{1}{2})+\beta(\alpha+\tfrac{1}{2},\tfrac{3}{2})\right)M\\
=&\tfrac{\left((g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}\right)\sqrt{\pi}}{2(k-r)}\tfrac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)}M,
\end{aligned}$$

where β is the beta function, defined by: $\beta(x,y)=\int\limits_0^1\varpi^{x-1}(1-\varpi)^{y-1}d\varpi=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ \ x>0,\ \ y>0.$

Remark 4. For $g(x)=x$, Theorem 5 becomes Theorem 2.

Theorem 6. Let the differentiable mapping $\varepsilon:K_g\subset\mathbb{R}_+\to\mathbb{R}$ with $\varepsilon'\in L^1\left[g(r),g(k)\right]$. If $|\varepsilon'|^q$ is g_{MT}-convex function with respect to g, where $g:\mathbb{R}\to\mathbb{R}$ is a mapping satisfying $g(r)<g(x)<g(k)$ and $|\varepsilon'(z)|\leq M$, $z\in K_g$, then

$$\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^{-}}\varepsilon(g(r))+I^{\alpha}_{g(x)^{+}}\varepsilon(g(k))\right]\right|\\
\leq&\left(\tfrac{1}{1+\alpha p}\right)^{\frac{1}{p}}\left(\tfrac{\pi}{2}\right)^{\frac{1}{q}}\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{k-r}M
\end{aligned}$$

holds for all $x\in[r,k]$, p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha>0$.

Proof. By Lemma 1 and Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r))+I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|\\
\le&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|d\varpi\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|d\varpi\\
\le&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\int_0^1\varpi^{\alpha p}d\varpi\right)^{\frac1p}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^q d\varpi\right)^{\frac1q}\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\int_0^1\varpi^{\alpha p}d\varpi\right)^{\frac1p}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^q d\varpi\right)^{\frac1q}\\
=&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\tfrac{1}{1+\alpha p}\right)^{\frac1p}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^q d\varpi\right)^{\frac1q}\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\tfrac{1}{1+\alpha p}\right)^{\frac1p}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^q d\varpi\right)^{\frac1q}.
\end{aligned}\tag{10}$$

Since $|\varepsilon'|^q$ is g_{MT}-convex with respect to g and $|\varepsilon'(x)|\le M$, we have

$$\begin{aligned}
\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^q d\varpi\le&\int_0^1\left(\tfrac{\sqrt{\varpi}}{2\sqrt{1-\varpi}}\left|\varepsilon'(g(x))\right|^q+\tfrac{\sqrt{1-\varpi}}{2\sqrt{\varpi}}\left|\varepsilon'(g(r))\right|^q\right)d\varpi\\
\le&M^q\int_0^1\left(\tfrac{\sqrt{\varpi}}{2\sqrt{1-\varpi}}+\tfrac{\sqrt{1-\varpi}}{2\sqrt{\varpi}}\right)d\varpi=\tfrac{\pi}{2}M^q.
\end{aligned}\tag{11}$$

From (10) and (11), we get the result.

Remark 5. For $g(x)=x$, Theorem 6 will be reduced to Theorem 3.

Theorem 7. Let the differentiable mapping $\varepsilon: K_g\subset\mathbb{R}_+\to\mathbb{R}$ with $\varepsilon'\in L^1[g(r),g(k)]$. If $|\varepsilon'|^q$ is g_{MT}-convex function with respect to g, where $g:\mathbb{R}\to\mathbb{R}$ is a mapping satisfying $g(r)<g(x)<g(k)$ for all $x\in(r,k)$, $q\ge1$, and $|\varepsilon'(z)|\le M$, $z\in K_g$, then

$$\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r))+I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|\\
\le&\left(\tfrac{(1+\alpha)\Gamma(\alpha+\frac12)}{2\Gamma(\alpha+1)}\sqrt{\pi}\right)^{\frac1q}\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{(1+\alpha)(k-r)}M
\end{aligned}$$

holds for all $x\in[r,k]$ with $g(x)\in K_g$ and $\alpha>0$, where Γ is the gamma function.

Proof. By the identity of Lemma 1, modulus and the so-called power mean inequality, it yields

$$
\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{g(x)^-}\varepsilon(g(r))+I^{\alpha}_{g(x)^+}\varepsilon(g(k))\right]\right|\\
\leq&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int\limits_0^1\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|d\varpi\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int\limits_0^1\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|d\varpi\\
\leq&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\int\limits_0^1\varpi^{\alpha}d\varpi\right)^{1-\frac{1}{q}}\left(\int\limits_0^1\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\int\limits_0^1\varpi^{\alpha}d\varpi\right)^{1-\frac{1}{q}}\left(\int\limits_0^1\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^q dt\right)^{\frac{1}{.q}}\\
=&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\tfrac{1}{(1+\alpha)^{1-\frac{1}{q}}}\left(\int\limits_0^1\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^q d\varpi\right)^{\frac{1}{q}}\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\tfrac{1}{(1+\alpha)^{1-\frac{1}{q}}}\left(\int\limits_0^1\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^q d\varpi\right)^{\frac{1}{.q}}.
\end{aligned}
\tag{12}
$$

Since $|\varepsilon'|^q$ is g_{MT}-convex with respect to a function g, and $|\varepsilon'(x)|\leq M$, we get

$$
\begin{aligned}
&\int\limits_0^1\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^q d\varpi\\
\leq&\int\limits_0^1\left(\tfrac{\varpi^{\alpha}\sqrt{\varpi}}{2\sqrt{1-\varpi}}\left|\varepsilon'(g(x))\right|^q+\tfrac{\varpi^{\alpha}\sqrt{1-\varpi}}{2\sqrt{\varpi}}\left|\varepsilon'(g(r))\right|^q\right)d\varpi\\
\leq&M^q\int\limits_0^1\left(\tfrac{\varpi^{\alpha}\sqrt{\varpi}}{2\sqrt{1-\varpi}}+\tfrac{\varpi^{\alpha}\sqrt{1-\varpi}}{2\sqrt{\varpi}}\right)d\varpi\\
=&\tfrac{1}{2}M^q\int\limits_0^1\left(\varpi^{\alpha+\frac{1}{2}}(1-\varpi)^{-\frac{1}{2}}+\varpi^{\alpha-\frac{1}{2}}(1-\varpi)^{\frac{1}{2}}\right)d\varpi\\
=&M^q\left(\tfrac{\Gamma(\alpha+\frac{3}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+2)}+\tfrac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{3}{2})}{2\Gamma(\alpha+2)}\right)\\
=&M^q\left(\left(\alpha+\tfrac{1}{2}\right)+\tfrac{3}{2}\right)\tfrac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+2)}=\tfrac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}M^q
\end{aligned}
\tag{13}
$$

and

$$
\int\limits_0^1\varpi^{\alpha}\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)k\right)\right|^q d\varpi\leq M^q\tfrac{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\alpha+1)}. \tag{14}
$$

From (12)–(14), we get the result.

Remark 6. For $g(x)=x$, Theorem 7 will be reduced to Theorem 4.

2.2 Fractional Ostrowski's inequalities for g_β-convexity

Theorem 8. Let the differentiable mapping $\varepsilon : K_g \subset \mathbb{R}_+ \to \mathbb{R}$ with $\varepsilon' \in L^1[g(r), g(k)]$. If $|\varepsilon'|$ is g_β-convex function with respect to g where $g:\mathbb{R}\to\mathbb{R}$ is a mapping satisfying for all $x \in (r,k)$: $g(r) < g(x) < g(k)$ and $|\varepsilon'(z)| \le M$, $z \in K_g$, then

$$\begin{aligned}&\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r)) + I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|\\ &\le \tfrac{\Gamma(\alpha+p+1)\Gamma(q+1)+\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\left(\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{k-r}\right)M \end{aligned} \tag{15}$$

holds for all $x \in [r,k]$ with $g(x) \in K_g$ and $\alpha > 0$, p, $q > -1$ and Γ is the gamma function.

Proof. Using Lemma 1 and modulus, we get

$$\begin{aligned}&\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r)) + I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|\\ &\le \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int_0^1 \varpi^\alpha \left|\varepsilon'\left(\varpi g(x) + (1-\varpi)g(r)\right)\right| d\varpi\\ &\quad + \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int_0^1 \varpi^\alpha \left|\varepsilon'\left(\varpi g(x) + (1-\varpi)g(k)\right)\right| d\varpi.\end{aligned}$$

The fact that $|\varepsilon'|$ is g_β-convex with respect to g and $|\varepsilon'(x)| \le M$, gives

$$\begin{aligned}&\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r)) + I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|\\ &\le \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int_0^1 \varpi^\alpha \left[\varpi^p(1-\varpi)^q\left|\varepsilon'(g(x))\right| + \varpi^q(1-\varpi)^p\left|\varepsilon'(g(r))\right|\right] d\varpi\\ &\quad + \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int_0^1 \varpi^\alpha \left[\varpi^p(1-\varpi)^q\left|\varepsilon'(g(x))\right| + \varpi^q(1-\varpi)^p\left|\varepsilon'(g(k))\right|\right] d\varpi\\ &\le \tfrac{M(g(x)-g(r))^{\alpha+1}}{k-r}\int_0^1 \left[\varpi^{\alpha+p}(1-\varpi)^q + \varpi^{\alpha+q}(1-\varpi)^p\right] d\varpi\\ &\quad + \tfrac{M(g(k)-g(x))^{\alpha+1}}{k-r}\int_0^1 \left[\varpi^{\alpha+p}(1-\varpi)^q + \varpi^{\alpha+q}(1-\varpi)^p\right] d\varpi\\ &= \left(\beta(\alpha+p+1,q+1) + \beta(\alpha+q+1,p+1)\right)\left(\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{k-r}\right)M\\ &= \tfrac{\Gamma(\alpha+p+1)\Gamma(q+1)+\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\left(\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{k-r}\right)M.\end{aligned}$$

The proof is completed.

Corollary 1. In (15), if we choose $g(x)=x$, i.e., $|\varepsilon'|$ is β-convex, we get

$$\begin{aligned}&\left|\tfrac{(x-r)^\alpha+(k-x)^\alpha}{k-r}\varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{x^-}\varepsilon(r) + I^\alpha_{x^+}\varepsilon(k)\right]\right|\\ &\le \tfrac{\Gamma(\alpha+p+1)\Gamma(q+1)+\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}\right)M.\end{aligned}$$

We now give some special cases which can be derived from the preceding corollary.

Corollary 2. In Corollary 1, taking $p = q = 0$, we get

$$\begin{aligned}&\left|\tfrac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x)-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I_{x^-}^{\alpha}\varepsilon(r)+I_{x^+}^{\alpha}\varepsilon(k)\right]\right|\\ \leq&\tfrac{2\left((x-r)^{\alpha+1}+(k-x)^{\alpha+1}\right)}{(\alpha+1)(k-r)}M.\end{aligned}$$

Corollary 3. In Corollary 1, taking $p = s$, $q = 0$, we get

$$\begin{aligned}&\left|\tfrac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x)-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I_{x^-}^{\alpha}\varepsilon(r)+I_{x^+}^{\alpha}\varepsilon(k)\right]\right|\\ \leq&\tfrac{\Gamma(\alpha+s+1)+\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}\right)M.\end{aligned}$$

Corollary 4. In Corollary 1, taking $p = q = 1$, we get

$$\begin{aligned}&\left|\tfrac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x)-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I_{x^-}^{\alpha}\varepsilon(r)+I_{x^+}^{\alpha}\varepsilon(k)\right]\right|\\ \leq&\tfrac{2\left((x-r)^{\alpha+1}+(k-x)^{\alpha+1}\right)}{(\alpha+3)(\alpha+2)(k-r)}M.\end{aligned}$$

Corollary 5. For $x = \frac{r+k}{2}$ Corollary 1 gives the following midpoint inequality:

$$\begin{aligned}&\left|\varepsilon(\tfrac{r+k}{2})-\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-r)^{\alpha}}\left[I_{(\frac{r+k}{2})^-}^{\alpha}\varepsilon(r)+I_{(\frac{r+k}{2})^+}^{\alpha}\varepsilon(k)\right]\right|\\ \leq&\left(\tfrac{\Gamma(\alpha+p+1)\Gamma(q+1)+\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\right)\tfrac{(k-r)M}{2}.\end{aligned}$$

Theorem 9. Let the differentiable mapping $\varepsilon : K_g \subset \mathbb{R}_+ \to \mathbb{R}$ with $\varepsilon^{'} \in L^1[g(r), g(k)]$. If $|\varepsilon'|^{\mu}$ is g_{β}-convex function with respect to g, where $g : \mathbb{R} \to \mathbb{R}$ is a mapping satisfying for all $x \in (r, k)$: $g(r) < g(x) < g(k)$ and $|\varepsilon'(z)| \leq M$, $z \in K_g$, then

$$\begin{aligned}&\left|\tfrac{(g(x)-g(r))^{\alpha}+(g(k)-g(x))^{\alpha}}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I_{g(x)^-}^{\alpha}\varepsilon(g(r))+I_{g(x)^+}^{\alpha}\varepsilon(g(k))\right]\right|\\ \leq&\left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\left(\tfrac{2\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\right)^{\frac{1}{\mu}}\left(\tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{k-r}\right)M\end{aligned}\tag{16}$$

holds for all $x \in [r, k]$ with $g(x) \in K_g$ and $\alpha > 0$, p, $q > -1$, λ, $\mu > 1$ with $\frac{1}{\lambda} + \frac{1}{\mu} = 1$, where Γ is the gamma function.

Proof. By Lemma 1, modulus and Hölder's inequality, we have

$$
\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x))-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r))+I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right|\\
\leq&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|d\varpi\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int_0^1\varpi^\alpha\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|d\varpi\\
\leq&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\int_0^1\varpi^{\alpha\lambda}d\varpi\right)^{\frac{1}{\lambda}}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}}\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\int_0^1\varpi^{\alpha\lambda}d\varpi\right)^{\frac{1}{\lambda}}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}}\\
=&\tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}}\\
&+\tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\left(\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}}. \qquad (17)
\end{aligned}
$$

Since $|\varepsilon'|^\mu$ is g_β-convex with respect to a function g, and $|\varepsilon'(x)|\leq M$, we get

$$
\begin{aligned}
&\int_0^1\left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi\\
\leq&\int_0^1\left[\varpi^p(1-\varpi)^q\left|\varepsilon'(g(x))\right|^\mu+\varpi^q(1-\varpi)^p\left|\varepsilon'(g(r))\right|^\mu\right]d\varpi\\
\leq&M^\mu\int_0^1\left[\varpi^p(1-\varpi)^q+\varpi^q(1-\varpi)^p\right]d\varpi\\
=&2M^\mu\beta(p+1,q+1)=2M^\mu\tfrac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}. \qquad (18)
\end{aligned}
$$

From (17) and (18), we get the result.

Corollary 6. In (16), if we choose $g(x)=x$, i.e. $|\varepsilon'|^\mu$ is β-convex, we have

$$
\begin{aligned}
&\left|\tfrac{(x-r)^\alpha+(k-x)^\alpha}{k-r}\varepsilon(x)-\tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{x^-}\varepsilon(r)+I^\alpha_{x^+}\varepsilon(k)\right]\right|\\
\leq&\left(\tfrac{2\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}\right)^{\frac{1}{\mu}}\left(\tfrac{1}{\alpha\lambda+1}\right)^{\frac{1}{\lambda}}\left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r}\right)M.
\end{aligned}
$$

Some particular situations that may be derived from the earlier corollary are given below.

Corollary 7. In Corollary 6, taking $p = q = 0$, we get

$$\begin{aligned} &\left| \tfrac{(x-r)^\alpha+(k-x)^\alpha}{k-r} \varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r} \left[I^\alpha_{x^-} \varepsilon(r) + I^\alpha_{x^+} \varepsilon(k) \right] \right| \\ &\leq 2^{\frac{1}{\mu}} \left(\tfrac{1}{\alpha\lambda+1} \right)^{\frac{1}{\lambda}} \left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r} \right) M. \end{aligned}$$

Corollary 8. In Corollary 6, taking $p = s$, $q = 0$, we get

$$\begin{aligned} &\left| \tfrac{(x-r)^\alpha+(k-x)^\alpha}{k-r} \varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r} \left[I^\alpha_{x^-} \varepsilon(r) + I^\alpha_{x^+} \varepsilon(k) \right] \right| \\ &\leq \left(\tfrac{2}{s+1} \right)^{\frac{1}{\mu}} \left(\tfrac{1}{\alpha\lambda+1} \right)^{\frac{1}{\lambda}} \left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r} \right) M. \end{aligned}$$

Corollary 9. In Corollary 6, taking $p = q = 1$, we get

$$\begin{aligned} &\left| \tfrac{(x-r)^\alpha+(k-x)^\alpha}{k-r} \varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r} \left[I^\alpha_{x^-} \varepsilon(r) + I^\alpha_{x^+} \varepsilon(k) \right] \right| \\ &\leq \left(\tfrac{1}{3} \right)^{\frac{1}{\mu}} \left(\tfrac{1}{\alpha\lambda+1} \right)^{\frac{1}{\lambda}} \left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{k-r} \right) M. \end{aligned}$$

Corollary 10. For $x = \frac{r+k}{2}$, Corollary 6 gives the following midpoint inequality:

$$\begin{aligned} &\left| \varepsilon\left(\tfrac{r+k}{2}\right) - \tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-r)^\alpha} \left[I^\alpha_{\left(\frac{r+k}{2}\right)^-} \varepsilon(r) + I^\alpha_{\left(\frac{r+k}{2}\right)^+} \varepsilon(k) \right] \right| \\ &\leq \left(\tfrac{2\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} \right)^{\frac{1}{\mu}} \left(\tfrac{1}{\alpha\lambda+1} \right)^{\frac{1}{\lambda}} \tfrac{(k-r)M}{2}. \end{aligned}$$

Theorem 10. Let the differentiable mapping $\varepsilon : K_g \subset \mathbb{R}_+ \to \mathbb{R}$ with $\varepsilon' \in L^1[g(r), g(k)]$. If $|\varepsilon'|^\mu$ is g_β-convex function with respect to g, where $g : \mathbb{R} \to \mathbb{R}$ is a mapping satisfying for all $x \in (r, k)$: $g(r) < g(x) < g(k)$ and $|\varepsilon'(z)| \leq M$, $z \in K_g$, then

$$\begin{aligned} &\left| \tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r} \varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r} \left[I^\alpha_{g(x)^-} \varepsilon(g(r)) + I^\alpha_{g(x)^+} \varepsilon(g(k)) \right] \right| \\ &\leq \left(\tfrac{(\alpha+1)\Gamma(\alpha+p+1)\Gamma(q+1)+(\alpha+1)\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)} \right)^{\frac{1}{\mu}} \tfrac{(g(x)-g(r))^{\alpha+1}+(g(k)-g(x))^{\alpha+1}}{(\alpha+1)(k-r)} M \end{aligned} \tag{19}$$

holds for all $x \in [r, k]$ with $g(x) \in K_g$ and $\alpha > 0$ and $p, q > -1$, $\mu > 1$, where Γ is the gamma function.

Proof. By Lemma 1, modulus and power mean inequality, we get

$$\begin{aligned}
&\left|\tfrac{(g(x)-g(r))^\alpha+(g(k)-g(x))^\alpha}{k-r}\varepsilon(g(x)) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{g(x)^-}\varepsilon(g(r)) + I^\alpha_{g(x)^+}\varepsilon(g(k))\right]\right| \\
\leq& \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\int\limits_0^1 \varpi^\alpha \left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right| d\varpi \\
&+ \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\int\limits_0^1 \varpi^\alpha \left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right| d\varpi \\
\leq& \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\left(\int\limits_0^1 \varpi^\alpha d\varpi\right)^{1-\frac{1}{\mu}}\left(\int\limits_0^1 \varpi^\alpha \left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}} \\
&+ \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\left(\int\limits_0^1 \varpi^\alpha d\varpi\right)^{1-\frac{1}{\mu}}\left(\int\limits_0^1 \varpi^\alpha \left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^\mu d\varpi\right)^{\frac{1}{\cdot\mu}} \\
=& \tfrac{(g(x)-g(r))^{\alpha+1}}{k-r}\tfrac{1}{(\alpha+1)^{1-\frac{1}{\mu}}}\left(\int\limits_0^1 \varpi^\alpha \left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi\right)^{\frac{1}{\mu}} \\
&+ \tfrac{(g(k)-g(x))^{\alpha+1}}{k-r}\tfrac{1}{(\alpha+1)^{1-\frac{1}{\mu}}}\left(\int\limits_0^1 \varpi^\alpha \left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(k)\right)\right|^\mu d\varpi\right)^{\frac{1}{\cdot\mu}}. \qquad (20)
\end{aligned}$$

Since $|\varepsilon'|^\mu$ is g_β-convex with respect to g, and $|\varepsilon'(x)| \leq M$, we get

$$\begin{aligned}
&\int\limits_0^1 \varpi^\alpha \left|\varepsilon'\left(\varpi g(x)+(1-\varpi)g(r)\right)\right|^\mu d\varpi \\
\leq& \int\limits_0^1 \varpi^\alpha \left[\varpi^p(1-\varpi)^q\left|\varepsilon'(g(x))\right|^\mu + \varpi^q(1-\varpi)^p\left|\varepsilon'(g(r))\right|^\mu\right] d\varpi \\
\leq& M^\mu \int\limits_0^1 \left[\varpi^{\alpha+p}(1-\varpi)^q + \varpi^{\alpha+q}(1-\varpi)^p\right] d\varpi \\
=& \left(\beta(\alpha+p+1,q+1)+\beta(\alpha+q+1,p+1)\right)M^\mu \\
=& \tfrac{\Gamma(\alpha+p+1)\Gamma(q+1)+\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}M^\mu. \qquad (21)
\end{aligned}$$

From (20) and (21), we get the result.

Corollary 11. In (19), if we choose $g(x) = x$, i.e. $|\varepsilon'|^\mu$ is β-convex, we have

$$\begin{aligned}
&\left|\tfrac{(x-r)^\alpha+(k-x)^\alpha}{k-r}\varepsilon(x) - \tfrac{\Gamma(\alpha+1)}{k-r}\left[I^\alpha_{x^-}\varepsilon(r) + I^\alpha_{x^+}\varepsilon(k)\right]\right| \\
\leq& \left(\tfrac{(\alpha+1)\Gamma(\alpha+p+1)\Gamma(q+1)+(\alpha+1)\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\right)^{\frac{1}{\mu}}\left(\tfrac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{(\alpha+1)(k-r)}\right)M.
\end{aligned}$$

We will now show some special cases that can be extracted from the previous result.

Corollary 12. In Corollary 11, taking $p = q = 0$, we get

$$\left|\frac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x) - \frac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r) + I^{\alpha}_{x^+}\varepsilon(k)\right]\right| \leq 2^{\frac{1}{\mu}}\left(\frac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{(\alpha+1)(k-r)}\right)M.$$

Corollary 13. In Corollary 11, taking $p = s$, $q = 0$, we get

$$\left|\frac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x) - \frac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r) + I^{\alpha}_{x^+}\varepsilon(k)\right]\right|$$
$$\leq \left(\frac{(\alpha+1)\Gamma(\alpha+s+1)+\Gamma(\alpha+2)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right)^{\frac{1}{\mu}}\left(\frac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{(\alpha+1)(k-r)}\right)M.$$

Corollary 14. In Corollary 11, taking $p = q = 1$, we get

$$\left|\frac{(x-r)^{\alpha}+(k-x)^{\alpha}}{k-r}\varepsilon(x) - \frac{\Gamma(\alpha+1)}{k-r}\left[I^{\alpha}_{x^-}\varepsilon(r) + I^{\alpha}_{x^+}\varepsilon(k)\right]\right|$$
$$\leq \left(\frac{2(\alpha+1)}{(\alpha+3)(\alpha+2)}\right)^{\frac{1}{\mu}}\left(\frac{(x-r)^{\alpha+1}+(k-x)^{\alpha+1}}{(\alpha+1)(k-r)}\right)M.$$

Corollary 15. For $x = \frac{r+k}{2}$ Corollary 11 gives the following midpoint inequality:

$$\left|\varepsilon\left(\frac{r+k}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(k-r)^{\alpha}}\left[I^{\alpha}_{\left(\frac{r+k}{2}\right)^-}\varepsilon(r) + I^{\alpha}_{\left(\frac{r+k}{2}\right)^+}\varepsilon(k)\right]\right|$$
$$\leq \left(\frac{(\alpha+1)\Gamma(\alpha+p+1)\Gamma(q+1)+(\alpha+1)\Gamma(\alpha+q+1)\Gamma(p+1)}{\Gamma(\alpha+p+q+2)}\right)^{\frac{1}{\mu}}\frac{(k-r)M}{2(\alpha+1)}.$$

Conclusion

In this study, we have explored fractional Ostrowski inequalities for functions whose modulus of the first derivatives exhibit g_{MT}-convexity and g_{β}-convexity. Several new results have been established, contributing to the advancement of fractional integral inequalities. Additionally, by considering specific cases, we have successfully recovered some well-known results, demonstrating the broad applicability and generality of our approach. This work extends classical Ostrowski inequalities and provides deeper insights into their behavior under generalized convexity assumptions. Future research could further investigate the potential applications of these inequalities in fields such as numerical analysis, optimization, and approximation theory.

Author Contributions

All authors participated in the conception and preparation of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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A combined problem with local and nonlocal conditions for a class of mixed-type equations

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This paper investigates the issues of existence and uniqueness of a solution to a combined boundary value problem with local and nonlocal conditions for a specific class of mixed elliptic-hyperbolic-type equations with singular coefficients. A distinctive feature of the considered problem is that on one part of the boundary characteristic, the values of the desired function are specified, while on the other part, nonlocal conditions are imposed. These conditions establish pointwise connections between the values of the sought function on different parts of the boundary characteristics using the Riemann-Liouville fractional differentiation operator. At the same time, a portion of hyperbolic domain's boundary remains free from boundary conditions. The proof of the solution's uniqueness is based on the application of an analogue of A.V. Bitsadze's extremum principle for mixed-type equations with singular coefficients. The existence of the solution is reduced to the analysis of a Tricomi singular integral equations' system with a shift, containing a non-Fredholm operator with isolated first-order singularity in kernel. By applying the Carleman-Vekua regularization method, these equations are reduced to a Wiener-Hopf integral equation, for which it is proved that the index is equal to zero. This, in turn, reduces the problem to a Fredholm integral equation of the second kind, the uniqueness of whose solution ensures the well-posedness of the given problem.

Keywords: mixed-type equation with singular coefficients, nonlocal condition, regularization, systems of singular integral equations, Wiener-Hopf equation, index.

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Introduction

The theory of local and nonlocal boundary value problems for mixed-type equations plays an important role in engineering and nature, particularly in gas dynamics, state processes, the development of oil reservoirs, groundwater filtration, heat and mass transfer in objects with complex structures, electrical oscillations in conductors, fluid flow in a channel surrounded by a porous medium, aerodynamics, and other phenomena.

The development of the theory of degenerate equations of mixed type originates from the fundamental works of G. Darboux, F. Tricomi, E. Holmgren, and S. Gellerstedt, published in 1894, 1923, 1927, and 1938, respectively. The problem for the model equation of mixed type was first formulated and solved by F. Tricomi, and it is now known as the Tricomi problem. After this work, the theory of local and nonlocal problems for mixed-type equations was developed in fundamental studies of E. Holmgren, S. Gellerstedt, A.V. Bitsadze, A.A. Samarskii, V.I. Zhegalov, A.M. Nakhushev, I. Frankl, S.G. Mikhlin, K.I. Babenko, M.M. Smirnov, M. Protter, M.M. Meredov, Sh.A. Alimov, E.I. Moiseev, A.P. Soldatov, M.S. Salakhitdinov, and others.

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Among the works devoted to boundary value problems for mixed-type equations, special attention should be given to the work of A.V. Bitsadze [1], in which a number of important problems in both two-dimensional and spatial cases were studied. These studies have stimulated further research in this direction and have attracted numerous mathematicians to this field.

By the late 1970s, many issues in the theory of boundary value problems for degenerate partial differential equations, including mixed-type equations with smooth coefficients in the considered domain, had acquired a mathematically complete form. Further progress in this field was largely determined by qualitatively new problems for equations with non-smooth coefficients, particularly for mixed-type equations with singular coefficients. Despite the large number of studies on hyperbolic equations and equations of mixed type, problems with combined local and nonlocal conditions for a degenerate mixed-type equation with singular coefficients remain poorly studied.

A nonlocal problem for a mixed-type equation with a singular coefficient in an unbounded domain was studied in the work of M. Ruziev and M. Reissig [2], while the works of Z.G. Feng [3], Z. Feng, and J. Kuang [4] are devoted to the study of boundary value problems for nonlinear mixed-type equations.

This work is devoted to the study of the solvability of a combined problem with local and nonlocal conditions for a certain class of mixed-type equations with singular coefficients. The degenerate equations with singular coefficients considered in this article differ from the well-known classical problems in that the correctness of the known Cauchy problem (in the hyperbolic region) and Holmgren's problem (in the elliptic domain) does not always hold. In the considered domains, these problems in their standard formulation may turn out to be unsolvable if the mixed-type equation degenerates along a line that is simultaneously a characteristic (an envelope of a family of characteristics) or if coefficients of the equation at lower-order terms are singular. Therefore, in these cases, it is natural to consider modified Cauchy and Holmgren problems, where the condition on the degeneration line is given with weight functions. Therefore, in the problem formulation of the study, along with local and nonlocal boundary conditions on the degeneration line of the equation, discontinuous matching conditions for the normal derivatives of the sought function with weight functions are specified. The problem conditions are given in a combined form: locally on the boundary of the ellipticity of the equation and on one part of the boundary characteristic, while on the other part, nonlocal conditions are imposed, establishing pointwise relationships between the values of the sought function at different sections of the characteristic boundary using the fractional differentiation operator in the sense of Riemann-Liouville. At the same time, part of the boundary of the hyperbolic domain remains free from boundary conditions.

By modifying A.V. Bitsadze's extremum principle for a mixed-type equation with singular coefficients, the uniqueness of the combined problem has been proven.

The solvability of the problem is reduced to the study of non-standard singular Tricomi integral equations with a numerical parameter in the non-singular part of the kernel and a non-Fredholm operator on the right-hand side of the equation.

The obtained singular integral equation is characterized by the following properties:

- generalizes the singular integral equation of F. Tricomi. In a particular case, this equation is reduced to the equation studied by F. Tricomi;
- the "nonsingular" part of the kernel has non-Carleman shifts;
- the non-characteristic part of the singular integral equation contains non-Fredholm integral operators; more precisely, the kernels of these operators have isolated singularities of the first order.

An algorithm has been developed for solving such non-standard integral equations: first, temporarily assuming the non-characteristic part of the equation as a known quantity, a singular integral equation of Tricomi with a shift is obtained; then, by regularizing it using the Carleman method developed by S.G. Mikhlin, the Wiener-Hopf equation is derived, which, through the Fourier transform, is reduced to a Riemann boundary value problem in the theory of functions of a complex variable.

Furthermore, it is proved that the index of the Riemann problem is equal to zero, which ensures the unique regularization of the Wiener-Hopf equation into a Fredholm integral equation of the second kind, whose unique solvability follows from the uniqueness of the solution to the formulated problems.

1 *Statement of the problem A*

In a finite simply connected domain D of the plane of independent variables x, y, bounded for $y>0$ by a regular curve

$$\sigma_0: \quad x^2+4(m+2)^{-2}y^{m+2}=1$$

with endpoints $A=A(-1,0)$ and $B=B(1,0)$, and for $y<0$ with characteristics AC and BC of a mixed-type equation with singular coefficients of the following form

$$(signy)|y|^m u_{xx}+u_{yy}+\frac{\beta_0}{y}u_y=0 \tag{1}$$

is considered, where $m>0$, $-\frac{m}{2}<\beta_0<1$.

Let D^+ and D^- denote the parts of the domain D, located, respectively, in the half-planes $y>0$ and $y<0$, and let C_0 and C_1, respectively represent the intersection points of the characteristics AC and BC with the characteristic originating from the point $E(c,\, 0)$, where $c\in I=(-1,\, 1)$ is the interval of the $y=0$ axis.

Problem A. Find the generalized solution $u(x,y)$ of equation (1), that satisfies the following conditions:

$$u\left(x,y\right)|_{\sigma_0}=\varphi\left(x\right), \quad -1\le x\le 1, \tag{2}$$

$$u|_{AC_0}=\psi\left(x\right), \quad -1\le x\le \frac{c-1}{2}, \tag{3}$$

$$\begin{gathered} a_0(x)(1+x)^{\beta}D^{1-\beta}_{c,x}u\left[\theta_0\left(x\right)\right]+b_0(x)(1-x)^{\beta}D^{1-\beta}_{x,1}u\left[\theta_1(x)\right]=c_0(x)u(x,0)+\\ +d_0(x)\lim_{y\to -0}(-y)^{\beta_0}\frac{\partial u}{\partial y}+f_0(x), \quad c<x<1, \end{gathered} \tag{4}$$

where $D^{1-\beta}_{c,x}$, $D^{1-\beta}_{x,1}$ are fractional differentiation operators of an order $1-\beta$; $\beta=\frac{m+2\beta_0}{2(m+2)}$; $\theta_0(x)$ and $\theta_1(x)$ represent the corresponding affixes of the intersection points of characteristics AC and BC with a characteristic originating from a point $M\left(x_0,0\right)$, where $x_0\in[c,\, 1]$:

$$\theta_0\left(x_0\right)=\frac{x_0-1}{2}-i\left(\frac{m+2}{4}\left(1+x_0\right)\right)^{\frac{2}{m+2}},$$

$$\theta_1\left(x_0\right)=\frac{x_0+1}{2}-i\left(\frac{m+2}{4}\left(1-x_0\right)\right)^{\frac{2}{m+2}}.$$

Given functions $\psi\left(x\right)$, $a_0\left(x\right)$, $b_0\left(x\right)$, $c_0(x)$, $d_0(x)$, $f_0(x)$ are continuously differentiable on the closure of their definition's domain, with the conditions:

$$a_0^2(x)+b_0^2(x)\neq 0, \quad c_0(x)\ge 0, \quad d(x)=a_0(x)+b_0(x)-d_0(x)>0, \quad x\in(c,1),$$

$$d(c)+\lambda\pi\,\mathrm{ctg}3\alpha\pi\left(a_0(c)-b_0(c)\neq 0, \quad \lambda=\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)}, \quad \alpha=\frac{1}{4}\left(1-2\beta\right)\right.$$

and the function $\varphi(x)$ is represented in the form

$$\varphi(x)=(1-x^2)^{2(1-\beta_0)}\tilde{\varphi}(x), \tag{5}$$

where $\tilde{\varphi}(x) \in C^1(\bar{I})$, $\psi(-1) = 0$.

Under the generalized solution of Problem A in the domain D, we refer to a function $u(x,y) \in C\left(\overline{D}\right) \cap C^2\left(D^+\right)$, which satisfies equation (1) in the domain D^+, while in the domain D^-, it is a generalized solution of a class R_1 [5, 6] and in the degeneration interval I it satisfies the following conjugation condition

$$\lim_{y\to -0}(-y)^{\beta_0}\frac{\partial u}{\partial y} = \lim_{y\to +0} y^{\beta_0}\frac{\partial u}{\partial y}\ , \quad x \in I\backslash\left\{c\right\},$$

and these limits at $x = \pm 1,\ x = c$ may have singularities of an order no higher than $1 - 2\beta$, while satisfying conditions (2)–(4).

Note that Problem A is a generalization of the problem by F. Tricomi [7] and, for the limiting value $c = -1$ from the Problem A it reduces to the problem of A.M. Nakhushev [8], and for the limiting value $c = 1$, with the additional conditions $c_0(x) = 0,\ d_0(x) = 0$,

$$a_0(x)(1+x)^{\beta} D_{c,x}^{1-\beta}\ \psi(0)|_{\substack{x=1\\ c=1}} + b_0(x)(1-x)^{\beta} D_{x,1}^{1-\beta}\,\varphi(1)|_{x=1} = c_0(1),$$

it leads to the problem of F. Tricomi [7]. Problem A in a particular case, was studied in [9].

For degeneration on the boundary of the hyperbolic domain with singular coefficients, the generalized Tricomi problem with Goursat conditions was studied in [10]. Solvability issues and spectral properties of local and nonlocal problems for model mixed-type equations were investigated in [1, 5–8, 11–14].

2 The Extremum Principle and Uniqueness of the Solution to Problem A

Before proceeding to the proof of solution uniqueness for Problem A, we present, without proof, the extremum principle and the local properties of the solution to equation (1) in the domain D^+.

Let us consider the Gellerstedt equation with a singular coefficient

$$E(u) = y^m u_{xx} + u_{yy} + \frac{\beta_0}{y}\ u_y = 0,\ y > 0, \tag{6}$$

where $m > 0,\ -\frac{m}{2} < \beta_0 < 1$, in a finite simply connected domain Ω of the complex plane $z = x + iy$ which is limited by a simple Jordan arc Γ with endpoints $A(-1,0),\ B(1,0)$ lying on the half plane $y > 0$ and a segment AB of the axis $y = 0$.

Lemma 1. (The Extremum Principle) [1,14] Any regular solution $u(x,y)$ of equation (6), continuous in Ω, does not achieve its positive maximum or negative minimum at the interior points of the domain Ω.

Let a regular solution $u(x,y)$ of equation (6) achieve its positive maximum in the domain $\bar{\Omega}$ at the point $(b,\,0)$ along the axis $y = 0$.

We derive the inequality in the neighborhood of the point $(b,\,0)$ for the function

$$\nu(x) = \lim_{y\to +0} y^{\beta_0}\frac{\partial u}{\partial y},\ \ x \in (-1,1). \tag{7}$$

Let limit (7) exists at the point $(b,\,0)$ [13, 14].

Lemma 2. (Analog of the Zaremba-Giraud Principle) [1, 14] Let 1) the function $u(x,y) \in C(\bar{\Omega}) \cap C^2(\Omega)$ continuous in $\bar{\Omega}$ satisfy the inequality $E\left(u\right) \geq 0$ (≤ 0) and take its maximum positive value (minimum negative value) at some point $(b,\,0)$, $b \in (-1,1)$;

2) the value $u\left(x,y\right)$ on the curve Γ is less (greater), than at the point $(b,\,0)$. Then

$$\lim_{y\to +0} y^{\beta_0}\frac{\partial u}{\partial y} < 0\ \ (> 0)\,,$$

provided that this limit exists.

Suppose that the limit (7) does not exist at the point $(b,\,0)$ and in the neighborhood of this point, the partial derivatives of the solution $u(x,y)$ to equation (6) are allowed to have the following order of singularities

$$\left|y^{\beta_0}u_y(x,y)\right| \le O\left(\rho^{2\beta-1+\varepsilon_0}\right), \qquad |u_x(x,y)| \le O\left(\rho^{\varepsilon_0-1}\right), \tag{8}$$

where ε_0 is a sufficiently small positive constant, $\rho=(x-b)^2+\frac{4}{(m+2)^2}y^{m+2}$.

Lemma 3. [6] If the solution of equation (6) in the domain Ω achieves its positive maximum (negative minimum) at the point $(b,\,0)$ on the axis $y=0$ and the estimates (8) are valid at this point, then there exists a neighborhood $(b-r_1,\ b+r_1)$ of the point $(b,\,0)$ for which $\int\limits_{b-r_1}^{b+r_1}\nu(x)dx<0\ \ (>0)$.

By virtue of Darboux's formula [9]

$$\begin{gathered}u(x,y)=\gamma_1\int\limits_{-1}^{1}\tau\left[x+\tfrac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right](1-t)^{\beta-1}(1+t)^{\beta-1}dt+\\ +\gamma_2(-y)^{1-\beta_0}\int\limits_{-1}^{1}\nu\left[x+\tfrac{2t}{m+2}(-y)^{\frac{m+2}{2}}\right](1-t)^{-\beta}(1+t)^{-\beta}dt,\end{gathered}$$

where

$$\beta=\frac{m+2\beta_0}{2(m+2)},\ \gamma_1=\frac{\Gamma(2\beta)}{\Gamma^2(\beta)}2^{1-2\beta},\ \gamma_2=-\frac{\Gamma(2-2\beta)}{(1-\beta_0)\,\Gamma^2(1-\beta)}2^{2\beta-1},$$

giving a solution to the modified Cauchy problem

$$u(x,0)=\tau(x),\ \ x\in I,\ \lim_{y\to-0}(-y)^{\beta_0}\frac{\partial u}{\partial y}=v(x)\,,\quad x\in I,$$

from the boundary conditions (3) and (4), respectively, we have

$$\nu\left(x\right)=\gamma D_{-1,x}^{1-2\beta}\tau\left(x\right)+\Psi\left(x\right),\ \ x\in(-1,\ c)\,, \tag{9}$$

$$d(x)\nu(x)=\ \gamma a_0(x)\ D_{-1,x}^{1-2\beta}\,\tau(x)+\gamma b_0(x)\,D_{x,1}^{1-2\beta}\,\tau(x)\ +c_0(x)\tau(x)+f(x),\ x\in(c,\ 1), \tag{10}$$

where

$$\gamma=\frac{2\Gamma\left(2\beta\right)\Gamma\left(1-\beta\right)}{\Gamma\left(\beta\right)\Gamma\left(1-2\beta\right)}\left(\frac{m+2}{4}\right)^{2\beta},\ \Psi\left(x\right)=\frac{-\gamma\Gamma\left(\beta\right)}{\Gamma\left(2\beta\right)}D_{-1,x}^{1-\beta}\psi\left(\frac{x-1}{2}\right),\quad x\in(-1,\ c),$$

$$f(x)=\frac{2^{1-2\beta}}{\gamma_2\Gamma(1-\beta)(m+2)^{1-2\beta}}\left[f_0\left(x\right)+\frac{a_0\left(x\right)}{\Gamma(\beta)}\frac{d}{dx}\int\limits_{-1}^{c}\frac{\psi((t-1)/2)\,dt}{(x-t)^{1-\beta}}\right],\quad x\in(c,\ 1),$$

here $\Psi(x)\in C[-1,\ c]\cap C^1(-1,\ c),\ f(x)\in C[c,1]\cap C^1(c,\ 1)$.

Theorem 1. (Analogue of A.V. Bitsadze's Extremum Principle) [1] The solution $u(x,y)$ to Problem A under the conditions: $\psi\left(x\right)\equiv 0$, $f_0(x)\equiv 0$,

$$a_0(x)\ge 0,\quad b_0(x)\ge 0,\quad c_0(x)\ge 0,\quad a_0(x)+b_0(x)\ge d_0(x), \tag{11}$$

achieves its positive maximum or negative minimum in the closed domain D^+ only at points on the arc σ_0.

Proof. Let $u(x,y)$ be a solution to Problem A, satisfying the conditions of Theorem 1. Clearly, by the extremum principle, the solution $u(x,y)$ in the domain D^+ cannot achieve its extreme. Assume the function $u(x,y)$ achieves its positive maximum in the closed domain $\bar{D}^+$ at the point $P(x_0,0)$, $x_0 \in I\backslash\{c\}$, i.e. $\max\limits_{(x,y)\in\bar{D}^+} u(x,y) = u(x_0,0) = \tau(x_0) > 0$. Using the fact that the fractional derivatives $D_{c,x}^{1-2\beta}\tau(x)$, $D_{x,1}^{1-2\beta}\tau(x)$ at the point of the positive maximum of the function $\tau(x)$ are strictly positive from (9) and (10) based on (11), we have $\nu(x_0) > 0$, it contradicts the known analogue of the Zaremba-Giraud principle, stating that at the point of positive maximum $\nu(x_0) < 0$ (Lemma 2) and hence it follows that $x_0 \notin I\backslash\{c\}$.

Now suppose that the solution $u(x,y)$ achieves its positive maximum (negative minimum) at the point $E(c,0)$. Then, by Theorem 2 there exists $r_1 > 0$, such that for the interval the following holds

$$\int\limits_{c-r_1}^{c+r_1} \nu(x)dx < 0 \quad (>0). \tag{12}$$

On the other hand, using (9) and (10), for the specified r_1, we have

$$\int\limits_{c-r_1}^{c+r_1} \nu(x)dx = \int\limits_{c-r_1}^{c} \nu(x)dx + \int\limits_{c}^{c+r_1} \nu(x)dx > 0 \quad (<0). \tag{13}$$

Inequality (13) contradicts inequality (12), i.e., the function $u(x,y)$ does not achieve its positive maximum (negative minimum) at the point $E(c,0)$.

Therefore, the function $u(x,y)$ achieves its positive maximum in the domain $\bar{D}^+$ at points on the curve σ_0.

It can also be shown that the function $u(x,y)$, which satisfies the conditions of Theorem 1, attains its negative minimum within the domain $\bar{D}^+$, including at points on the curve σ_0. Theorem 1 is proved.

From Theorem 1 follows

Corollary. Problem A under condition (11) has no more than one solution.

3 Existence of a solution to problem A

Theorem 2. Let the following conditions be hold:

$$\frac{\sin\alpha\pi\cos\beta\pi\,[b_0(c)-a_0(c)]\,[(2a_0(c)-d_0(c))\sin\beta\pi + 2b_0(c)-d_0(c)]}{[ch2\pi y + \cos 2\alpha\pi]\left\{[d(c)(1+\sin\beta\pi)]^2 + [(b_0(c)-a_0(c))\cos\beta\pi]^2\right\}} < 1, \tag{14}$$

where $\alpha = \frac{1-2\beta}{4}$, $y = \frac{1+x}{1+t}$. Then there exists a solution to Problem A.

The proof of Theorem 2 will be carried out in several stages. From the known solution to the modified problem N (6), (2) and (7) [14], we obtain a functional relationship between the unknown functions $\tau(x)$ and $v(x)$ carried over to I from the domain D^+

$$\tau(x) = -k_1 \int\limits_{-1}^{1} \left[|x-t|^{-2\beta} - (1-xt)^{-2\beta}\right] \nu(t)dt + \Phi(x), \qquad x \in \bar{I}, \tag{15}$$

where $k_1 = \left(\frac{4}{m+2}\right)^{2\beta} \frac{\Gamma^2(\beta)}{4\pi\Gamma(2\beta)}$,

$$\Phi(x) = 2k_1\left(\frac{m+2}{2}\right)^{2\beta}(1-x^2)\int\limits_{-1}^{1}(1-t^2)^{\beta-1/2}(1-2xt+x^2)^{-1-\beta}\varphi(t)dt.$$

From relationships (9), (10) and (14), excluding the function $\tau(x)$, and taking into account that the resulting equations have singularities at $x \in (-1,\ c)$ and $x \in (c,1)$, we obtain

$$\begin{gathered}\nu\left(x\right)=-\lambda\int\limits_{-1}^{c}\left(\frac{1+s}{1+x}\right)^{1-2\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)\nu\left(s\right)ds-\\-\lambda\int\limits_{c}^{1}\left(\frac{1+s}{1+x}\right)^{1-2\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)\nu\left(s\right)ds+\tilde{F}_0\left(x\right),\quad x\in\left(-1,c\right),\end{gathered}\tag{16}$$

$$\begin{gathered}d(x)v(x)=-\lambda a_0(x)\int\limits_{-1}^{c}\left(\frac{1+s}{1+x}\right)^{1-2\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)v(s)ds-\\-\lambda a_0(x)\int\limits_{c}^{1}\left(\frac{1+s}{1+x}\right)^{1-2\beta}\left(\frac{1}{s-x}-\frac{1}{1-xs}\right)v(s)ds+\lambda b_0(x)\int\limits_{-1}^{c}\left(\frac{1-s}{1-x}\right)^{1-2\beta}\left(\frac{1}{s-x}+\frac{1}{1-xs}\right)v(s)ds+\\+\lambda b_0(x)\int\limits_{c}^{1}\left(\frac{1-s}{1-x}\right)^{1-2\beta}\left(\frac{1}{s-x}+\frac{1}{1-xs}\right)v(s)ds-\frac{k_1c_0(x)}{1+\sin\beta\pi}\int\limits_{-1}^{c}\left[|x-s|^{-2\beta}-(1-xs)^{-2\beta}\right]\nu(s)ds-\\-\frac{k_1c_0(x)}{1+\sin\beta\pi}\int\limits_{c}^{1}\left[|x-s|^{-2\beta}-(1-xs)^{-2\beta}\right]\nu(s)ds+\tilde{F}_1(x),\quad x\in(c,1),\end{gathered}\tag{17}$$

where

$$\tilde{F}_0(x)=\frac{\gamma D_{-1,x}^{1-2\beta}\Phi\left(x\right)+\Psi\left(x\right)}{1+\sin\beta\pi},\quad \tilde{F}_1(x)=\frac{\gamma a_0(x)D_{-1,x}^{1-2\beta}\Phi\left(x\right)+\gamma b_0(x)D_{x,1}^{1-2\beta}\Phi\left(x\right)+f(x)+c_0(x)\Phi\left(x\right)}{1+\sin\beta\pi}.$$

Proceeding similarly to the approaches in works [6, 14], due to the conditions imposed on the given functions of the problem, particularly condition (5), in the case of a normal curve, it can be shown that

$$\begin{gathered}\tilde{F}_1\left(x\right)\in C\left(c,\ 1\right]\cap C^{(0,2\beta)}\left(c,\ 1\right),\\ \tilde{F}_1\left(x\right)\in C\left(c,\ 1\right]\cap C^{(0,2\beta)}\left(c,\ 1\right),\ \ \tilde{F}_1\left(x\right)=O\left(\left(x-c\right)^{2\beta-1}\right).\end{gathered}$$

Note that relations (16) and (17) hold for $x \in (-1,\ c)$ and $x \in (c,1)$ respectively. To consider them in a single interval $I = (-1,1)$ in (15) replace x with $ax-b$, and in (17), replace x with $bx+a$, where $a = (1+c)/2$, $b = (1-c)/2$,$a+b = 1$, $a-b = c$ and then perform the substitution of variables $s = at-b$ for integrals over the interval $(-1,c)$ and $s = bt+a$ for integrals over the interval $(c,1)$, where $t \in [-1,1]$. By isolating the characteristic part in the integrals with singular properties and performing some transformations, we have

$$\begin{gathered}\nu_0\left(t\right)+\lambda\int\limits_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{a}{(ax-b)(at-b)}\right)\nu_0\left(t\right)dt=\\=\lambda\int\limits_{-1}^{1}\frac{b\,v_1(t)dt}{bt-ax+1}+\mathrm{T}_1[\nu_1]+F_0(x),\ x\in(-1,1),\end{gathered}\tag{18}$$

$$\begin{gathered}D(x)\nu_1(x)+K(x)\int\limits_{-1}^{1}\left(\frac{1+t}{1+x}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{b}{1-(bx+a)(bt+a)}\right)\nu_1\left(t\right)dt=\\=\mu\int\limits_{-1}^{1}\frac{av_0(t)dt}{at-bx-1}+H_0[v_0]+H_1[v_1]+F_1(x),\ x\in(-1,1),\end{gathered}\tag{19}$$

where

$$\nu_0(x)=\nu(ax-b),\ \nu_1(x)=\nu(bx+a),\ A\left(x\right)=\lambda a_0\left(bx+a\right),\ B\left(x\right)=\lambda b_0\left(bx+a\right),$$

$$D\left(x\right)=d(bx+a),\ F_0\left(x\right)=\tilde{F}_0\left(ax-b\right),\ F_1\left(x\right)=\tilde{F}_1\left(bx+a\right),\ \lambda=\frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)},$$

$$\mathrm{T}_1[\nu_1] = \lambda \int_{-1}^{1} \left[\left(\frac{1+a+bt}{a\,(1+x)} \right)^{1-2\beta} - 1 \right] \frac{b\, v_1(t)dt}{bt-ax+1} + \lambda \int_{-1}^{1} \left(\frac{1+a+bt}{a\,(1+x)} \right)^{1-2\beta} \frac{b\, v_1(t)dt}{1-(ax-b)(bt+a)},$$

$$K(x,t) = A(x) - B(x) \left(\frac{1-x}{1-t} \right)^{2\beta} \left(\frac{1+x}{1+t} \right)^{1-2\beta} \frac{1+bt+a}{1+bx+a},$$

$$K(x) = K(x,x) = A(x) - B(x), \;\; \mu = -K(-1),$$

$$H_1[v_1] = A\,(x) \int_{-1}^{1} \left\{ \left[\left(\tfrac{1+a+bt}{1+a+bx} \right)^{1-2\beta} - \left(\tfrac{1+t}{1+x} \right)^{1-2\beta} \right] \left(\tfrac{1}{t-x} - \tfrac{b}{1-(bx+a)(bt+a)} \right) \right\} \nu_1\,(t)\,dt -$$
$$- \tfrac{bk_1 c_0 (bx+a)}{1+\sin\beta\pi} \int_{c}^{1} \left[(|b(x-t))|^{-2\beta} - (1-(bx+a)(bt+a))^{-2\beta} \right] v_1(t)dt -$$
$$- \int_{-1}^{1} \left(\tfrac{1+t}{1+x} \right)^{1-2\beta} \left(\tfrac{1}{t-x} - \tfrac{b}{1-(bx+a)(bt+a)} \right) (K(x,t) - K(x,x)) \nu_1(t)dt$$

are regular operators.

3.1 Regularization of the System of Singular Integral Equations of the Tricomi Problem with a Shift

Let us proceed to the regularization of the system of singular integral equations (18), (19). In equation (19) we will perform the following step-by-step:

- Temporarily consider the right-hand side of equation (19) as a known function from the class $L_\rho\,(-1,1)$, $p > 1$ satisfying Hölder's condition. By regularizing equation (19) using the Carleman-Vekua method [15–17] in the class of functions H, where $(1+x)^{1-2\beta}\nu_1(x)$ is bounded on the left end and may be unbounded on the right end of the interval $\bar{I}$, we transform equation (19) with respect to the function $(1+x)^{1-2\beta}\nu_1(x)$.
- In the obtained equation, considering assumptions introduced above for the right-hand side of (19) and after some transformations, isolating the characteristic part of the equation, it is easy to establish that the kernel of the obtained integral equation when the condition $M(-1) \neq 0$ is fulfilled, where

$$M(x) = \mu\left(D^*(x) + \pi ctg 3\alpha\pi K^*(x)\right),$$

$$M(-1) = \mu\;\left(D^*(-1) + \pi ctg 3\alpha\pi\; K^*(-1)\right) =$$
$$= \tfrac{\mu(D(-1)+\pi ctg 3\alpha\, K(-1))}{D^2(-1)+\pi^2 K^2(-1)} = \tfrac{\mu[(a_0(c)+b_0(c)-d_0(c)-\lambda\mu\pi ctg 3\alpha\pi\,)]}{(a_0(c)+b_0(c)-d_0(c))^2+\lambda\pi^2\mu^2},$$

$$\mu = \lambda\left[b_0(c) - a_0(c)\right],\;D^*(x) = \frac{D(x)}{D^2(x)+\pi^2 K^2(x)},\;\;K^*(x) = \frac{K(x)}{D^2(x)+\pi^2 K^2(x)}$$

for $t = 1,\;x = -1$ has a first-order singularity [17, 18], therefore, this operator is non-Fredholm. In the obtained equations, after some simple transformations, we have

$$\nu_1(x) = -\int_{-1}^{1} n\left(\frac{1+x}{1+t} \right) \frac{\tilde{\nu}_0(t)}{1+t} dt + \bar{N}_0[\tilde{\nu}_0] + N_1[\nu_1] + F_2(x), \tag{20}$$

where $\tilde{\nu}_0(x) = \nu_0(-x)$

$$n(y) = \frac{M(-1)}{1+by/a}, \tag{21}$$

$$N_0[\nu_0]=(M(x)-M(-1))\int\limits_{-1}^{1}\tfrac{a\nu_0(t)dt}{at-bx-1}+\int\limits_{-1}^{1}K(x,t)\nu_0(t)dt+$$
$$+D^*(x)\tilde{H}_0[\nu_0]-K^*(x)\int\limits_{-1}^{1}\left(\tfrac{1+t}{1+x}\right)^{3\alpha}\left(\tfrac{1-t}{1-x}\right)^{2\alpha}\left(\tfrac{1-(bx+a)c}{1-(bt+a)c}\right)^{\alpha}\times$$
$$\times\tfrac{\omega(x)(D(x)+i\pi K(x))}{\omega(t)(D(t)+i\pi K(t))}\left(\tfrac{1}{t-x}-\tfrac{b}{1-(bx+a)(bt+a)}\right)H_0[\nu_0]dt-$$
$$-\mu\, aK^*(x)\int\limits_{-1}^{1}\nu_0(s)ds\int\limits_{-1}^{1}\left(\tfrac{1+t}{1+x}\right)^{3\alpha}\left(\tfrac{1-t}{1-x}\right)^{2\alpha}\left[\left(\tfrac{1-(bx+a)c}{1-(bt+a)c}\right)^{\alpha}\tfrac{\omega(x)(D(x)+i\pi K(x))}{\omega(t)(D(t)+i\pi K(t))}-1\right]\times$$
$$\times\left(\tfrac{1}{t-x}-\tfrac{b}{1-(bx+a)(bt+a)}\right)\tfrac{dt}{as-bt-1}.$$

The operator $\bar{N}_0[\tilde{\nu}_0]$ is obtained from the operator N_0 by substituting $\nu_0(-x)$ with $\tilde{\nu}_0(x)$.

$$N_1[\nu_1]=D^*(x)H_1[\nu_1]-K^*(x)\int\limits_{-1}^{1}\left(\tfrac{1+t}{1+x}\right)^{3\alpha}\left(\tfrac{1-t}{1-x}\right)^{2\alpha}\left(\tfrac{1-(bx+a)c}{1-(bt+a)c}\right)^{\alpha}\times$$
$$\times\tfrac{\omega(x)(D(x)+i\pi K(x))}{\omega(t)(D(t)+i\pi K(t))}\left(\tfrac{1}{t-x}-\tfrac{b}{1-(bx+a)(bt+a)}\right)H_1[\nu_1]dt,$$

$N_1[\nu_1]$ is a regular operator,

$$F_2(x)=D^*(x)F_1(x)-K^*(x)\int\limits_{-1}^{1}\left(\tfrac{1+t}{1+x}\right)^{3\alpha}\left(\tfrac{1-t}{1-x}\right)^{2\alpha}\left(\tfrac{1-(bx+a)c}{1-(bt+a)c}\right)^{\alpha}\times$$
$$\times\tfrac{\omega(x)(D(x)+i\pi K(x))}{\omega(t)(D(t)+i\pi K(t))}\left(\tfrac{1}{t-x}-\tfrac{b}{1-(bx+a)(bt+a)}\right)F_1(t)dt,\ \ \omega(x)=\tfrac{1+a-ax}{bx+a}.$$

Now consider equation (18). Equation (18) will be considered as a singular integral equation with a shift in the non-summable part of the kernel relative to an unknown function. Here, proceeding similarly to the case of equation (19) and performing analogous calculations, we obtain

$$\tilde{\nu}_0\left(x\right)=-\int\limits_{-1}^{1}m\left(\frac{1+x}{1+s}\right)\frac{v_1\left(s\right)ds}{1+s}+T_2^*\left[v_1\right]+F_3\left(-x\right),\tag{22}$$

where

$$m\left(y\right)=A\frac{b^{1+\alpha}}{a^{\alpha}(b+ay)y^{\alpha}},\quad A=\sin\alpha\pi/\pi,\tag{23}$$

$$T_2[v_1]=\lambda\tfrac{\cos\beta\pi}{2\pi}\int\limits_{-1}^{1}b\,\nu_1(s)ds\int\limits_{-1}^{1}\left(\tfrac{1+t}{1+x}\right)^{2a}\left(\tfrac{1-t}{1-x}\right)^{a}\left(\tfrac{1+a+bs}{a(1+t)}\right)^{4a}\left[\left(\tfrac{1-c(at-b)}{1-c(ax-b)}\right)^{a}-1\right]\times$$
$$\times\left(\tfrac{1}{t-x}-\tfrac{a}{1-(ax-b)(at-b)}\right)dt+\left(\tfrac{\sin\alpha\pi}{\pi}\right)^2T_1[v_1]+\tfrac{\sin\alpha\pi}{\pi}\int\limits_{-1}^{1}\left(\tfrac{b(1+s)}{a(1+x)}\right)^{\alpha}\tfrac{b\nu_1(s)ds}{1+(ax+b)(bs+a)},$$

$$T_1[v_1]=-\tfrac{\pi}{\sin\alpha\pi}\int\limits_{-1}^{1}\left(\tfrac{b(1+s)}{a(1-x)}\right)^{\alpha}\left[\left(\tfrac{1+a+bs)}{a(1+x)}\right)^{2\alpha}-1\right]\left(\tfrac{1}{bs-ax+1}-\tfrac{1}{1-(ax-b)(bs+a)}\right)b\nu_1(s)ds+$$
$$+\int\limits_{-1}^{1}\left(\tfrac{1+a+bs}{a}\right)^{4\alpha}\tfrac{1}{(1+x)^{2\alpha}(1-x)^{\alpha}}\times$$
$$\times\left\{-\frac{2^{\alpha}B\left(1-2\alpha,\alpha\right)}{\left(1+x\right)^{2\alpha}}\left[F\left(1-2\alpha,-\alpha,1-\alpha;\frac{1-x}{2}\right)-\left(\frac{a(1+x)}{1+a+bs}\right)^{2\alpha}F\left(1-2\alpha,-\alpha,1-\alpha;-\frac{b}{a}\frac{1-s}{2}\right)\right]+\right.$$
$$\left.+\,a(bs+a)M_1\left(s\right)-a(ax-b)M_2\left(x\right)\right\}b_1v(s)ds,$$

are regular operators,

$$M_1(s) = \int_{-1}^{1} \frac{(1-t)^{\alpha}}{(1+t)^{2\alpha}} \frac{dt}{1-(bs+a)(at-b)} = \frac{2^{1-\alpha} B(1-2\alpha, \alpha)}{1-c(bs+a)} F\left(1+\alpha, 1, 2-\alpha; -\frac{2a(bs+a)}{1-c(bs+a)}\right),$$

$$M_2(x) = \int_{-1}^{1} \frac{(1-t)^{\alpha}}{(1+t)^{2\alpha}} \frac{dt}{1-(ax-b)(at-b)} = \frac{2^{1-\alpha} B(1+\alpha, 1-2\alpha)}{1-c(ax-b)} F\left(1-2\alpha, 1, 2-\alpha; -\frac{2a(ax-b)}{1-c(ax-b)}\right).$$

Here $B(\alpha, \beta)$ and $F(a, b, c, z)$ are beta and hypergeometric Gauss functions respectively.

$$F_3(x) = \frac{1+\sin\beta\pi}{2} F_0(x) - \frac{\cos\beta\pi}{2\pi} \int_{-1}^{1} \left(\frac{1+t}{1+x}\right)^{2a} \left(\frac{1-t}{1-x}\right)^{a} \left(\frac{1-c(at-b)}{1-c(ax-b)}\right)^{a} \times$$
$$\times \left(\frac{1}{t-x} - \frac{a}{1-(ax-b)(at-b)}\right) F_0(t) dt,$$

where $\tilde{v}_0(x) = v_0(-x)$, and the operator $T_2^*[v_1]$ is obtained from operator $T_2[v_1]$ by substituting x with $-x$.

3.2 Derivation and Analysis of the Wiener-Hopf Integral Equation

Thus, equation (22) together with equation (20) form a system of integral equations with respect to unknown functions $\tilde{v}_0(x)$ and $v_1(x)$ with a singular feature in the kernel [19]. From equations (20) and (22), excluding the function $\tilde{v}_0(x)$ with respect to the function $v_1(x)$, we obtain the equation

$$v_1(x) = \int_{-1}^{1} \frac{\Omega(x,t) v_1(t) dt}{1+t} + L[v_1] + F_4(x), \tag{24}$$

where $L[v_1]$ is a regular operator and

$$\Omega(x,t) = \int_{-1}^{1} n\left(\frac{1+x}{1+s}\right) m\left(\frac{1+s}{1+t}\right) \frac{ds}{1+s}, \tag{25}$$

$$L[v_1] = -\int_{-1}^{1} n\left(\frac{1+x}{1+t}\right) \frac{T_1^*[v_1]}{1+t} dt + \bar{N}_0 \left[-\int_{-1}^{1} m\left(\frac{1+x}{1+s}\right) \frac{v_1(s)\, ds}{1+s} + T_2^*[v_1]\right] + N_1[v_1],$$

$$F_4(x) = -\int_{-1}^{1} n\left(\frac{1+x}{1+t}\right) \frac{F_3(-t)}{1+t} dt + \bar{N}_0[F_3(-x)] + F_2(x).$$

We evaluate the kernel $\Omega(x,t)$, for this in (25) making a substitution $\frac{1+s}{1+t} = r$, we have

$$\Omega(x,t) = \int_{0}^{2/(1+t)} n\left(\frac{1+x}{r(1+t)}\right) m(r) \frac{dr}{r} = \Omega_1(x,t) - \Omega_2(x,t), \tag{26}$$

where

$$\Omega_1(x,t) = \int_{0}^{\infty} n\left(\frac{1+x}{r(1+t)}\right) m(r) \frac{dr}{r}, \quad \Omega_2(x,t) = \int_{2/(1+t)}^{\infty} n\left(\frac{1+x}{r(1+t)}\right) m(r) \frac{dr}{r}.$$

Based on (21) and (23) it is easy to establish the estimate

$$|\Omega_2(x,t)| = \int\limits_{2/(1+t)}^{\infty} \left| n\left(\frac{1+x}{r(1+t)}\right)\right| |m(r)| \frac{dr}{r} \leq M(-1)A\left(\frac{b}{a}\right)^{1+\alpha} \left(\frac{1+t}{r}\right)^{\alpha}.$$

From this, it follows that $\frac{\Omega_2(x,t)}{1+t}$ is a regular kernel. Now, by direct computation, we have

$$\Omega_1(x,t) = \int\limits_0^{\infty} n\left(\frac{1+x}{r(1+t)}\right) m(r) \frac{dr}{r} = M(-1)\frac{y^{\alpha}-1}{y^{\alpha}(y-1)}, \tag{27}$$

where $y = (1+x)/(1+t)$. Now, based on (26) and (27), we rewrite equation (24) in the form

$$v_1(x) = M(-1)\int\limits_{-1}^{1} \left(\frac{1+t}{1+x}\right)^{\alpha} \frac{\left(\frac{1+x}{1+t}\right)^{\alpha}-1}{\frac{1+x}{1+t}-1} \frac{v_1(t)\,dt}{1+t} + L_1[v_1] + F_4(x), \tag{28}$$

where $L_1[v_1] = L[v_1] - \int\limits_{-1}^{1} \frac{\Omega_2(x,t)v_1(t)dt}{1+t}$ is a regular operator.

Now, introducing the notation in (28) $\rho(z) = e^{(\alpha-1)z/2}\nu_1(2e^{-z}-1)$, after some transformations, we obtain

$$\rho(z) = M(-1)\int\limits_0^{\infty} \frac{sh(\alpha(z-s)/2)\rho(s)ds}{sh((z-s)/2)} + L_3[\rho] + F_5(z), \tag{29}$$

where $L_3[\rho] = L_2\left[e^{(1-\alpha)z/2}\rho(z)\right]$, $F_5(z) = e^{(\alpha-1)z/2}F_4(2e^{-z}-1)$. Equation (29) is an integral Wiener-Hopf equation [20]. Under the condition

$$\frac{M(-1)\pi\sin\alpha\pi}{ch2y\pi + \cos 2\alpha\pi} < 1 \tag{30}$$

it is easy to calculate that the index of equation (29) is equal to 0. Therefore, equation (29) is uniquely reduced to a Fredholm integral equation of the second kind, and the unique solvability of this equation follows from the uniqueness of the solution to Problem A. From condition (30) due to the entered notations, by straight calculations we obtain (14).

Thus, Theorem 2 is proved.

Conclusion

The paper investigates the existence and uniqueness of a solution to a combined problem with local and nonlocal conditions for one class of mixed elliptic-hyperbolic type equations with singular coefficients.

The studied problem differs from known problems in that the values of the sought function are specified on one part of the characteristic boundary, while, nonlocal conditions are imposed on the other part. These nonlocal conditions relate the values of the sought function on one part of the characteristic boundary pointwise to the values on the characteristic boundary of another family using the fractional differentiation operator in the sense of Riemann-Liouville. At the same time, a part of the boundary of the hyperbolic region of the domain is freed from boundary conditions.

The uniqueness of the solution to formulated problem is proved using an analogue of extremum principle by A.V. Bitsadze for a mixed type equation with singular coefficients.

The proof of the existence of a solution is reduced to solving a system of singular Tricomi integral equations with a shift in the non-summable part of the kernel and a non-Fredholm operator with an isolated first-order singularity in the kernel of the operator. By the Carleman-Vekua regularization method, the obtained singular integral equations with a non-zero operator on the right are reduced to the Wiener-Hopf integral equation. It is proved that the index of the Wiener-Hopf integral equation is zero. Consequently, the Wiener-Hopf equation is uniquely reduced to the Fredholm integral equation of the second kind, the unambiguous solvability of which follows from the uniqueness of the solution of the problem A.

An algorithm has been developed for solving non-standard singular integral Tricomi equations with a shift in the non-summable part of the kernel and a non-Fredholm operator with an isolated first-order singularity in the operator's kernel.

Thus, the issues of the unique solvability of a combined problem with local and nonlocal conditions for a certain class of mixed-type equations with singular coefficients have been formulated and studied. It has been established that the well-posedness of the combined problem, defined by local and nonlocal conditions on a single characteristic boundary, significantly depends on the ratio of the coefficients of the nonlocal conditions at the junction point of the local and nonlocal conditions, which lies on the degeneration line of the equation.

In conclusion, we note that the developed methods for studying non-standard singular integral equations can be applied to a broader class of partial differential equations with singular coefficients, including for other values of the parameter β_0 in equation (1).

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Solutions of boundary value problems for loaded hyperbolic type equations

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This paper investigates a class of second-order partial differential equations describing wave processes with nonlocal effects, including cases involving fractional derivatives. Such equations often arise in the theory of elasticity, aerodynamics, acoustics, and electrodynamics. The presented equations include both integral and differential terms, evaluated either at a fixed point $x = x_0$ or $x = \alpha(t)$. An equation with a fractional derivative of order $0 \leq \beta < 1$ is considered, making it possible to model memory effects and other nonlocal properties. For each equation, supplemented by initial conditions, either a closed-form analytical solution is obtained or the main steps of its derivation are outlined. The article employs the Laplace transform to solve the resulting integral equation, enabling the solution to be presented in an explicit form.

Keywords: differential equations, partial derivatives, loaded equations, boundary value problem, Laplace transform, convolution, wave equations, fractional derivative, integral equations, Cauchy problem.

2020 Mathematics Subject Classification: 35L05, 35L10, 35R11, 44A10.

Introduction

The study of optimal control problems and long-term forecasting has led to the emergence of a new class of mathematical equations known as “loaded equations”.

Initially, these equations were studied by N.N. Nazarov and N.N. Kochin, though the term “loaded equation” was first introduced by A.M. Nakhushev [1–4], who developed a general classification and applications for this type of equations. Loaded equations can be differential, integral, integro-differential, or functional equations, in which the differentiation, integration, or functional transformation operator is applied not to the entire function, but only to its value at a specific point or over a certain set.

Loaded second-order partial differential equations are of particular interest and have been examined in a number of studies [5–9]. These equations find applications in various fields of science and technology, for instance, in modeling heat propagation, wave processes, population dynamics, and other phenomena, and they remain highly relevant today [10–15].

It should be noted that the authors of the present article have experience in solving boundary value problems for hyperbolic equations [16–18]. This expertise has enabled them to analyze loaded equations, taking into account the specifics of methods developed for classical partial differential equations [19–21].

Thus, this study contributes to the theory of loaded equations by providing new methods and results that can be used to model and analyze a variety of physical, biological, and technical processes.

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1 *Solution of the boundary value problem for a loaded wave equation, where the load depends on the value of the function $u(x,t)$ at a fixed point $x = x_0$*

Let us consider the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \mu u(x,t)\Big|_{x=x_0} + f(x,t), \quad -\infty < x < \infty, \quad t > 0 \tag{1}$$

with initial conditions

$$u(x,t)\Big|_{t=0} = g_1(x), \tag{2}$$

$$\frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = g_2(x), \tag{3}$$

where a is const, the function $f(x,t)$ is continuous on $-\infty < x < \infty,\ t > 0$, functions $g_1(x), g_2(x)$ are continuous on $-\infty < x < \infty$, μ is parameter (load coefficient). The parameter μ equation (1) can take values depending on the physical context of the problem, with $\mu \in \mathbb{R}$ (an arbitrary real number). For example:

$\mu > 0$: models an increase in the load (an additional force);

$\mu = 0$: in this case, the equation reduces to the classical wave equation;

$\mu < 0$: may describe dissipative effects.

Here, the load in equation (1) signifies that the system's state at the point x_0 directly impacts wave propagation throughout the entire space.

The solution to problem (1)–(3) can be expressed as follows

$$u(x,t) = \frac{\mu}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} u(x,\tau)\Big|_{x=x_0} d\xi d\tau + \frac{1}{2} u_1(x,t) =$$

$$= \mu \int_0^t (t-\tau) u(x,\tau)\Big|_{x=x_0} d\tau + \frac{1}{2} u_1(x,t), \tag{4}$$

where

$$u_1(x,t) = [g_1(x-at) + g_1(x+at)] + \frac{1}{a} \int_{x-at}^{x+at} g_2(\xi) d\xi + \frac{1}{a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau) d\xi d\tau. \tag{5}$$

Now, from (4) we obtain

$$u(x,t)\Big|_{x=x_0} = \mu \int_0^t (t-\tau) u(x,\tau)\Big|_{x=x_0} d\tau + \frac{1}{2} u_1(x,t)\Big|_{x=x_0}. \tag{6}$$

We introduce the notation $\psi(t) = u(x,t)\Big|_{x=x_0}$. Then equation (6) takes the form

$$\psi(t) = \mu \int_0^t (t-\tau)\psi(\tau) d\tau + \frac{1}{2} u_1(x,t)\Big|_{x=x_0}. \tag{7}$$

To solve equation (7), we use the Laplace transform. Let $\Psi(p) = L[\psi(t)]$ be the transform of $\psi(t)$. Let $U(p) = L\left[\frac{1}{2}u_1(x,t)\Big|_{x=x_0}\right]$ be the transform of $\frac{1}{2}u_1(x,t)\Big|_{x=x_0}$. Since $L[t] = \frac{1}{p^2}$, $\int\limits_0^t (t-\tau)\psi(\tau)d\tau = t * \psi(t)$ is convolution. Applying the convolution theorem for the Laplace transform:

$$L[f(t) * g(t)] = L[f(t)]L[g(t)], \tag{8}$$

where $f(t) * g(t) = \int\limits_0^t f(\tau)g(t-\tau)d\tau$, we obtain from (7):

$$\Psi(p) = \mu L[t * \psi(t)] + U(p).$$

Taking (8) into account, we get

$$\Psi(p) = \mu L[t]L[\psi(t)] + U(p).$$

Since $L[t] = \frac{1}{p^2}$, $\Psi(p) = L[\psi(t)]$, we have:

$$\Psi(p) = \frac{\mu}{p^2}\Psi(p) + U(p).$$

Hence,

$$\Psi(p) = \frac{p^2 U(p)}{p^2 - \mu}.$$

Thus, we have found the transform $\Psi(p)$ of the function $\psi(t)$ in the image space. Applying the inverse Laplace transform to $\Psi(p)$:

$$\psi(t) = L^{-1}\left[\frac{p^2 U(p)}{p^2-\mu}\right] = L^{-1}\left[U(p) + \frac{\mu U(p)}{p^2-\mu}\right],$$

$$\psi(t) = L^{-1}[U(p)] + L^{-1}\left[\frac{\mu U(p)}{p^2-\mu}\right].$$

For the second term, we use a Laplace transform table:

$$L^{-1}\left[\frac{\mu}{p^2-\mu}\right] = \sqrt{\mu}\sinh(\sqrt{\mu}t).$$

Therefore, the second term can be written as:

$$L^{-1}\left[\frac{\mu U(p)}{p^2-\mu}\right] = \sqrt{\mu}\sinh(\sqrt{\mu}t) * \frac{1}{2}u_1(x,t)\Big|_{x=x_0}.$$

Here "*" denotes convolution. Consequently,

$$\psi(t) = \frac{1}{2}u_1(x,t)\Big|_{x=x_0} + \sqrt{\mu}\sinh(\sqrt{\mu}t) * \frac{1}{2}u_1(x,t)\Big|_{x=x_0}.$$

This is the solution to the original integral equation (7). In our particular case, we need the convolution of $\sqrt{\mu}\sinh(\sqrt{\mu}t)$ и $\frac{1}{2}u_1(x_0,t)$. Thus,

$$\psi(t) = \frac{1}{2}u_1(x,\tau)\Big|_{x=x_0} + \frac{\sqrt{\mu}}{2}\int\limits_0^t u_1(x_0,\tau)\sinh(\sqrt{\mu}(t-\tau))d\tau.$$

Hence, the solution to problem (1)–(3) is given by

$$u(x,t)=\frac{\mu}{2}\int\limits_0^t(t-\tau)\left(u_1(x,\tau)\Big|_{x=x_0}+\sqrt{\mu}\int\limits_0^\tau u_1(x_0,\tau_1)\sinh(\sqrt{\mu}(\tau-\tau_1))d\tau_1\right)d\tau+\frac{1}{2}u_1(x,t).$$

Influence of the parameter μ on the class of solutions. For $\mu\neq 0$ the solution (6) includes integral terms with a load. In this case, we require smoothness: $u(x,t)\in C^2$.

For $\mu>0$, the solution contains hyperbolic functions. For $\mu<0$, trigonometric functions appear. In the case $\mu=0$, the equation becomes the classical wave equation with the d'Alembert solution. In this case, we require smoothness $g_1(x)\in C^2$ and $g_2(x)\in C^1$.

2 *Solution of the boundary value problem for a loaded wave equation, where the load depends on the derivative of the function $u(x,t)$ at the fixed point $x=x_0$*

Consider the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}=a^2\frac{\partial^2 u(x,t)}{\partial x^2}+\mu\frac{\partial u(x,t)}{\partial x}\Big|_{x=x_0}+f(x,t), \tag{9}$$

subject to the initial conditions (2) and (3).

The nonlocal term in equation (9) indicates that the rate of change of the function at the point x_0 affects wave propagation.

The solution to problem (8), (2), (3) can be written in the form

$$u(x,t)=\frac{\mu}{2a}\int\limits_0^t\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)}\frac{\partial u(x,\tau)}{\partial x}\Big|_{x=x_0}d\xi d\tau+\frac{1}{2}u_1(x,t)=$$

$$=\mu\int\limits_0^t(t-\tau)\frac{\partial u(x,\tau)}{\partial x}\Big|_{x=x_0}d\tau+\frac{1}{2}u_1(x,t).$$

Introducing the notation

$$\psi(t)=\frac{\partial u(x,\tau)}{\partial x}\Big|_{x=x_0},$$

we obtain:

$$u(x,t)=\frac{\mu}{2a}\int\limits_0^t\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)}\psi(\tau)d\xi d\tau+\frac{1}{2}u_1(x,t)=\mu\int\limits_0^t(t-\tau)\psi(\tau)d\tau+\frac{1}{2}u_1(x,t). \tag{10}$$

From (10), take the derivative with respect to x and then substitute $x=x_0$. This yields an explicit representation for the unknown function $\psi(t)$:

$$\psi(t)=\frac{1}{2}\frac{\partial u_1(x,t)}{\partial x}\Big|_{x=x_0}.$$

Hence, the solution to the boundary value problem (9), (2), (3) is

$$u(x,t)=\frac{\mu}{2}\int\limits_0^t(t-\tau)\frac{\partial u_1(x,\tau)}{\partial x}\Big|_{x=x_0}d\tau+\frac{1}{2}u_1(x,t). \tag{11}$$

Thus, the following theorem holds:

Theorem 1. Let the following conditions be satisfied:
a) the functions $g_1(x) \in C^2(\mathbb{R})$, $g_2(x) \in C^1(\mathbb{R})$;
b) the function $f(x,t) \in C(\mathbb{R} \times [0,T])$;
c) the parameters $a > 0$, $\mu \in \mathbb{R}$, $x_0 \in \mathbb{R}$ are fixed.

Then, for equation (9) with a nonlocal term, under the initial conditions (2) and (3), there exists a unique solution $u(x,t) \in C^2(\mathbb{R} \times [0,T])$, representable in the form (11), where $u_1(x,t)$ is the solution of the corresponding problem without the nonlocal term ($\mu = 0$), computed by formula (5).

Next, consider the boundary value problem for a loaded equation in which the loaded term involves the time derivative:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \mu \left.\frac{\partial u(x,t)}{\partial t}\right|_{x=x_0} + f(x,t). \tag{12}$$

A solution to problem (12), (2), (3) can be written as

$$u(x,t) = \frac{\mu}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} \left.\frac{\partial u(x,\tau)}{\partial \tau}\right|_{x=x_0} d\xi d\tau + \frac{1}{2} u_1(x,t) =$$

$$= \mu \int_0^t (t-\tau) \left.\frac{\partial u(x,\tau)}{\partial \tau}\right|_{x=x_0} d\tau + \frac{1}{2} u_1(x,t). \tag{13}$$

Since

$$\frac{d}{dt}\left[\mu \int_0^t (t-\tau) \left.\frac{\partial u(x,\tau)}{\partial \tau}\right|_{x=x_0} d\tau\right] = \mu \int_0^t \frac{\partial u(x_0,\tau)}{\partial \tau} d\tau,$$

it follows that

$$\frac{\partial u(x,t)}{\partial t} = \mu \int_0^t \left.\frac{\partial u(x,\tau)}{\partial \tau}\right|_{x=x_0} d\tau + \frac{1}{2} \frac{\partial u_1(x,t)}{\partial t}.$$

Substituting $x = x_0$ gives

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{x=x_0} = \mu \int_0^t \left.\frac{\partial u(x,\tau)}{\partial t}\right|_{x=x_0} d\tau + \frac{1}{2} \left.\frac{\partial u_1(x,t)}{\partial t}\right|_{x=x_0}.$$

Introduce $\psi(t) = \left.\frac{\partial u(x,t)}{\partial t}\right|_{x=x_0}$, then

$$\psi(t) = \mu \int_0^t \psi(\tau) d\tau + \frac{1}{2} \left.\frac{\partial u_1(x,t)}{\partial t}\right|_{x=x_0}.$$

Taking the derivative with respect to t:

$$\psi'(t) - \mu \psi(t) = \frac{1}{2} \left.\frac{\partial^2 u_1(x,t)}{\partial t^2}\right|_{x=x_0}. \tag{14}$$

From condition (3), it follows that

$$\psi(t)|_{t=0} = g_2(x_0). \tag{15}$$

Let us solve the problem (14), (15). We introduce the notation $\widetilde{f}(t) = \frac{1}{2}\frac{\partial^2 u_1(x_0,t)}{\partial t^2}$ and find the integrating factor:

$$M(t) = e^{-\mu t}.$$

Multiply both sides of the equation by $M(t)$:

$$e^{-\mu t}\psi'(t) - \mu e^{-\mu t}\psi(t) = e^{-\mu t}\widetilde{f}(t).$$

The left-hand side is the derivative of a product:

$$\frac{d}{dt}\left(e^{-\mu t}\psi(t)\right) = e^{-\mu t}\widetilde{f}(t).$$

Integrate both sides with respect to t from 0 to t:

$$e^{-\mu t}\psi(t) - \psi(0) = \int_0^t e^{-\mu\tau}\widetilde{f}(\tau)d\tau.$$

Taking into account the initial condition $\psi(0) = g_2(x_0)$, we obtain

$$e^{-\mu t}\psi(t) = g_2(x_0) + \int_0^t e^{-\mu\tau}\widetilde{f}(\tau)d\tau,$$

hence,

$$\psi(t) = e^{\mu t}g_2(x_0) + e^{\mu t}\int_0^t e^{-\mu\tau}\widetilde{f}(\tau)d\tau.$$

Substituting $\widetilde{f}(t)$, we get the solution of problem (14), (15):

$$\psi(t) = e^{\mu t}g_2(x_0) + \frac{1}{2}e^{\mu t}\int_0^t e^{-\mu\tau}\frac{\partial^2 u_1(x_0,\tau)}{\partial\tau^2}d\tau.$$

Let us analyze the effect of the parameter μ on the solution to problem (13), (14). When $\mu = 0$, the solution simplifies to:

$$\psi(t) = g_2(x_0) + \frac{1}{2}\int_0^t \frac{\partial^2 u_1(x_0,\tau)}{\partial\tau^2}d\tau = g_2(x_0) + \frac{1}{2}\left[\frac{\partial u_1(x_0,t)}{\partial t} - \frac{\partial u_1(x_0,0)}{\partial t}\right].$$

For $\mu > 0$ the solution contains a growing exponential term, which requires additional conditions on $\widetilde{f}(t)$ to ensure boundedness.

For $\mu < 0$ the solution decays exponentially as $t \to \infty$.

Substituting $\psi(t)$ into (13) yields the solution of problem (12), (2), (3).

Theorem 2. Let the following conditions be satisfied:

a) the initial functions satisfy the smoothness requirements: $g_1(x) \in C^2(\mathbb{R})$, $g_2(x) \in C^1(\mathbb{R})$;

b) the inhomogeneous term $f(x,t) \in C(\mathbb{R} \times [0,T])$;

c) the problem parameters satisfy $a > 0$, $\mu \in \mathbb{R}$, $x_0 \in \mathbb{R}$.

Then, for the equation with a nonlocal term in time (12) under the initial conditions (2), (3), the following statements hold:

if $\mu < 0$, there exists a unique solution $u(x,t) \in C^2(\mathbb{R} \times [0,T])$;

if $\mu = 0$, a solution exists and is unique in the class $C^2(\mathbb{R} \times [0,T])$;

if $\mu > 0$, a solution may exist only locally on some interval $[0, t_0)$.

Finally, consider the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \mu \frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0} + f(x,t), \tag{16}$$

in which the nonlocal term depends on the second derivative of $u(x,t)$ with respect to t at the fixed point $x = x_0$. This implies that the acceleration of the wave process at x_0 influences the wave propagation.

Then the solution to problem (16) under conditions (2) and (3) can be represented as

$$u(x,t) = \frac{\mu}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} \frac{\partial^2 u(x,\tau)}{\partial \tau^2}\bigg|_{x=x_0} d\xi d\tau + \frac{1}{2} u_1(x,t) =$$

$$= \mu \int_0^t (t-\tau) \frac{\partial^2 u(x,\tau)}{\partial \tau^2}\bigg|_{x=x_0} d\tau + \frac{1}{2} u_1(x,t). \tag{17}$$

Since

$$\frac{d}{dt}\left[\mu \int_0^t (t-\tau) \frac{\partial^2 u(x,\tau)}{\partial \tau^2}\bigg|_{x=x_0} d\tau\right] = \mu \frac{\partial u(x_0,t)}{\partial t}\bigg|_0^t = \mu \frac{\partial u(x,t)}{\partial t}\bigg|_{x=x_0} - \mu \frac{\partial u(x,0)}{\partial t}\bigg|_{x=x_0},$$

$$\frac{d}{dt}\left[\mu \frac{\partial u(x,t)}{\partial t}\bigg|_{x=x_0} - \mu \frac{\partial u(x,0)}{\partial t}\bigg|_{x=x_0}\right] = \mu \frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0},$$

then by setting $x = x_0$ in (17), we obtain

$$\frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0} = \mu \frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0} + \frac{1}{2} \frac{\partial^2 u_1(x,t)}{\partial t^2}\bigg|_{x=x_0}$$

or

$$\frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0} = \frac{1}{2(1-\mu)} \frac{\partial^2 u_1(x,t)}{\partial t^2}\bigg|_{x=x_0}.$$

From this, we arrive at the following theorem.

Theorem 3. Suppose the following conditions hold:

1) $f(x,t) \in C((-\infty, +\infty) \times [0, +\infty))$, $g_1(x), g_2(x) \in C(-\infty, +\infty)$, a is a const,

2) $\mu \neq 1$, μ is a parameter,

then the solution to problem (16), (2), (3) exists, is unique, and can be expressed in terms of the solution of the classical wave problem together with an integral representation:

$$u(x,t) = \mu \int_0^t (t-\tau) \frac{\partial^2 u(x,\tau)}{\partial \tau^2}\bigg|_{x=x_0} d\tau + \frac{1}{2} u_1(x,t),$$

where

$$\frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0} = \frac{1}{2(1-\mu)} \frac{\partial^2 u_1(x,t)}{\partial t^2}\bigg|_{x=x_0},$$

and the function $u_1(x,t)$ is given by formula (5).

3 Solution of the boundary value problem for a hyperbolic equation with a fractional derivative

Now let us consider the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \mu \left[{}_0 D_t^{\beta} u(x,t) \right]_{x=\alpha(t)} + f(x,t), \tag{18}$$

subject to conditions (2), (3), where $0 \leq \beta < 1$. Here, the loading term is the fractional derivative with respect to time of order β of $u(x,t)$, evaluated at the point $x = \alpha(t)$, which moves through space as a function of time. The presence of a fractional time derivative in the nonlocal term implies that the state of the system at any given time depends on all of its past states, affecting how the wave propagates.

Assume the functions

$$f(x,t) \in C((-\infty, +\infty) \times [0, +\infty)), \quad g_1(x), g_2(x) \in C(-\infty, +\infty), \tag{19}$$

$$\left[{}_0 D_t^{\beta} u(x,t) \right] \Big|_{x=\alpha(t)} \in C(-\infty, +\infty). \tag{20}$$

Then the solution of the problem takes the form

$$u(x,t) = \frac{\mu}{2a} \int\limits_0^t \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} \left[{}_0 D_\tau^{\beta} u(x,\tau) \right] \Big|_{x=\alpha(\tau)} d\xi d\tau + \frac{1}{2} u_1(x,t) =$$

$$= \mu \int\limits_0^t (t-\tau) \left[{}_0 D_\tau^{\beta} u(x,\tau) \right] \Big|_{x=\alpha(\tau)} d\tau + \frac{1}{2} u_1(x,t). \tag{21}$$

Here,

$$\left[{}_0 D_t^{\beta} u(x,t) \right] \Big|_{x=\alpha(t)} = {}_0 D_t^{\beta} \left\{ \mu \int\limits_0^t (t-\tau) \left[{}_0 D_\tau^{\beta} u(x,\tau) \right] d\tau \right\} \Big|_{x=\alpha(\tau)} + \widehat{u}_1(x,t) \Big|_{x=\alpha(t)}, \tag{22}$$

where

$$\widehat{u}_1(x,t) \Big|_{x=\alpha(t)} = {}_0 D_t^{\beta} \left\{ \frac{1}{2} u_1(x,t) \right\} \Big|_{x=\alpha(t)} = \frac{1}{2\Gamma(1-\beta)} \int\limits_0^t \frac{\frac{\partial u_1(x,\tau)}{\partial \tau}}{(t-\tau)^{\beta}} d\tau \Big|_{x=\alpha(t)}. \tag{23}$$

Introduce the notation

$$\psi(t) = {}_0 D_t^{\beta} u(x,t) \Big|_{x=\alpha(t)}. \tag{24}$$

Then equation (22) becomes:

$$\psi(t) = \mu {}_0 D_t^{\beta} \left\{ \int\limits_0^t (t-\tau) \psi(\tau) d\tau \right\} + \widehat{u}_1(x,t) \Big|_{x=\alpha(t)} =$$

$$= \frac{\mu}{\Gamma(1-\beta)} \int\limits_0^t \frac{\left[\int\limits_0^{\tau} (\tau - \tau_1) \psi(\tau_1) d\tau_1 \right]'_{\tau}}{(t-\tau)^{\beta}} d\tau + \widehat{u}_1(x,t) \Big|_{x=\alpha(t)} =$$

$$= \frac{\mu}{\Gamma(1-\beta)} \int\limits_0^t \frac{\int\limits_0^\tau \psi(\tau_1) d\tau_1}{(t-\tau)^\beta} d\tau + \widehat{u}_1(x,t)\Big|_{x=\alpha(t)}. \tag{25}$$

Let us make a change of variables for the outer integral: $t - \tau = \eta$. Then

$$\psi(t) = \frac{\mu}{\Gamma(1-\beta)} \int\limits_0^t \frac{\int\limits_0^{t-\eta} \psi(\tau_1) d\tau_1}{\eta^\beta} d\eta + \widehat{u}_1(x,t)\Big|_{x=\alpha(t)}.$$

Hence,

$$\psi'(t) = \frac{\mu}{\Gamma(1-\beta)} \int\limits_0^t \frac{\psi(t-\eta)}{\eta^\beta} d\eta + \frac{\partial \widehat{u}_1(x,t)}{\partial t}\Big|_{x=\alpha(t)}.$$

Reversing the substitution, we get

$$\psi'(t) = \frac{\mu}{\Gamma(1-\beta)} \int\limits_0^t \frac{\psi(\tau)}{(t-\tau)^\beta} d\tau + \frac{\partial \widehat{u}_1(x,t)}{\partial t}\Big|_{x=\alpha(t)}$$

or

$$\psi'(t) = \frac{\mu}{\Gamma(1-\beta)} \psi(t) * t^{-\beta} + \frac{\partial \widehat{u}_1(x,t)}{\partial t}\Big|_{x=\alpha(t)}. \tag{26}$$

From conditions (2) and (25) it follows that $\psi(0) = 0$. We shall solve this equation using the Laplace transform. Let $\Psi(p) = \mathcal{L}[\psi(t)]$ denote the Laplace transform of $\psi(t)$ and let $U_1(p) = \mathcal{L}\left[\frac{\partial \widehat{u}_1(x,t)}{\partial t}\Big|_{x=\alpha(t)}\right]$ be the transform of $\frac{\partial \widehat{u}_1(x,t)}{\partial t}\Big|_{x=\alpha(t)}$, then $\mathcal{L}[\psi'(t)] = p\Psi(p) - \psi(0) = p\Psi(p)$, $\mathcal{L}[t^{-\beta}] = \frac{\Gamma(1-\beta)}{p^{1-\beta}}$. Substituting into (26), we obtain

$$p\Psi(p) = \frac{\mu}{\Gamma(1-\beta)} \mathcal{L}[\psi(t) * t^{-\beta}] + U_1(p).$$

By the convolution theorem for the Laplace transform,

$$p\Psi(p) = \frac{\mu}{\Gamma(1-\beta)} \Psi(p) \frac{\Gamma(1-\beta)}{p^{1-\beta}} + U_1(p).$$

From this, it follows that

$$\Psi(p) = \frac{p^{1-\beta}}{p^{2-\beta} - \mu} U_1(p). \tag{27}$$

From formula (1.80) [8, p. 21] with $k = 0$ we have

$$\mathcal{L}[t^{\widetilde{\beta}-1} E_{\widetilde{\alpha}\widetilde{\beta}}(\pm s t^{\widetilde{\alpha}})] = \frac{p^{\widetilde{\alpha}-\widetilde{\beta}}}{p^{\widetilde{\alpha}} \mp s},$$

where $s = \mu$, $\widetilde{\alpha} = 2 - \beta$, $\widetilde{\alpha} - \widetilde{\beta} = 1 - \beta$, and which implies $\widetilde{\beta} = 1$. Therefore,

$$\mathcal{L}^{-1}\left[\frac{p^{1-\beta}}{p^{2-\beta} - \mu}\right] = E_{2-\beta;1}(\mu t^{2-\beta}),$$

where $E_{2-\beta;1}(\mu t^{2-\beta})$ is the Mittag-Leffler function [8, p. 16-17] . Applying the inverse Laplace transform to (27) yields:

$$\psi(t) = E_{2-\beta;1}(\mu t^{2-\beta}) * \left.\frac{\partial \widehat{u}_1(x,t)}{\partial t}\right|_{x=\alpha(t)} \tag{28}$$

or

$$\psi(t) = \int_0^t E_{2-\beta,1}(\mu(t-s)^{2-\beta}) \cdot \left[\left.\frac{\partial \hat{u}_1(x,s)}{\partial s}\right|_{x=\alpha(s)} \right] ds.$$

Thus, the theorem is proved.

Theorem 4. The integro-differential equation (26) with the homogeneous initial condition $\psi(0) = 0$ has a solution in the class $C([0,T])$ for any continuous right-hand side $\left[{}_0D_t^{\beta} u(x,t)\right]_{x=\alpha(t)}$, where $0 \le \beta < 1$, and its solution is given by formula (28).

By substituting (28) into (21) and taking into account (24), we obtain the solution of problem (18), (2), (3).

Theorem 5. Let the functions $f(x,t)$, $g_1(x)$, $g_2(x)$ satisfy conditions (19). Then the boundary-value problem (18), (2), (3) has, in the class (20), a unique solution given by

$$u(x,t) = \mu \int\limits_0^t (t-\tau)\psi(\tau)d\tau + \frac{1}{2}u_1(x,t),$$

where the functions $\psi(t)$, $u_1(x,t)$ и $\left.\widehat{u}_1(x,t)\right|_{x=\alpha(t)}$ are determined by formulas (28), (23) and (5).

Conclusion

Thus, in this work, various forms of the wave equation with nonlocal terms and a fractional derivative were investigated. Solutions were presented for equations whose nonlocal terms depend on the values of the function and its derivatives at fixed points, and the case of an equation with a fractional derivative was examined, which makes it possible to model memory effects. The solution method, based on applying the Laplace transform, enabled us to obtain analytical or semi-analytical solutions. These solutions illustrate the influence of nonlocal effects and hereditary properties on the dynamics of wave processes. The results obtained are of great significance for understanding the behavior of complex systems, where interaction is not restricted to local interactions and where the system's past states affect its current behavior.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Exponentially fitted finite difference methods for a singularly perturbed nonlinear differential-difference equation with a small negative shift

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This study presents exponentially fitted finite difference methods for solving a singularly perturbed nonlinear differential-difference equation that consists of a small negative shift. The quasilinearization technique is applied to the nonlinear problem and a sequence of linear problems is obtained. The resulting linear problems are treated with exponentially fitted finite difference methods of higher order. The methods developed in this paper are studied for stability and convergence. Numerical results using the proposed methods are presented for two test problems and hence the efficiency of the methods is demonstrated.

Keywords: singular perturbation theory, nonlinear, delay differential equations, differential-difference equations, boundary value problems, finite difference scheme.

2020 Mathematics Subject Classification: 65L10, 65L11.

Introduction

Singularly perturbed nonlinear differential equations pose significant challenges in various fields of engineering and applied science [1]. As a result of the nonlinearity, the perturbation parameter, and the delay parameter, these problems become more challenging to solve. Lange and Miura [2] initiated the analysis of singularly perturbed nonlinear differential-difference equations and explored the existence and uniqueness of solutions to these boundary value problems.

A B-spline collocation method was proposed by Kadalbajoo and Kumar in [3] to solve a singularly perturbed nonlinear differential-difference equation with a negative shift. In an effort to improve the accuracy of the exact solution in the boundary layer region, they developed a piecewise uniform mesh. Ravi Kanth and Murali [4] developed an exponentially fitted spline method to solve nonlinear singularly perturbed delay differential equations. The same authors in [5] presented a numerical technique for solving a certain type of nonlinear singularly perturbed delay differential equation, which utilizes parametric cubic splines and a special mesh structure.

Lalu and Phaneendra [6] proposed a computational method for nonlinear delay differential equations, which consists of the reduction of the nonlinear problem into a sequence of linear problems using the quasilinearization technique and trigonometric spline approach suggested to solve the sequence of singularly perturbed linear delay differential equations. Chandru et al. [7] utilized a hybrid difference scheme on a Shishkin mesh to solve a singularly perturbed reaction-diffusion problem with a discontinuous source term.

The authors Sirisha et al, in [8] introduced a mixed finite difference approach for solving singularly perturbed differential-difference equations with mixed shifts by applying domain decomposition. Subburayan and Ramanujam [9] provided uniformly convergent finite difference methods with piecewise

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linear interpolation on Shishkin meshes to solve singularly perturbed boundary value problems for second order ordinary delay differential equations of convection-diffusion type.

Woldaregay and Duressa [10] proposed the numerical treatment of singularly perturbed differential difference equations involving mixed small shifts in the reaction terms. Woldaregay and Duressa [11] devised a fitted operator mid-point upwind finite difference technique for solving a singularly perturbed differential equation with a small delay in convection and reaction terms. Ranjan and Gowrisankar [12] developed numerical methods for linear convection-diffusion singular perturbation problems on Shishkin mesh utilizing the non-symmetric discontinuous Galerkin finite element method with interior penalty methods.

Prathap and Rao [13] presented a uniformly convergent finite difference technique on a uniform mesh, for solving singularly perturbed boundary value problems of second order ordinary differential-difference equation of convection-diffusion type. The same authors in [14], developed numerical techniques using non-polynomial splines for singularly perturbed boundary value problems with mixed shifts on a Shishkin mesh. Chakravarthy and Kumar [15] developed a numerical technique utilizing an adaptive grid to solve a time-dependent singularly perturbed differential-difference convection-diffusion problem.

Kumar [16] proposed the numerical solution using the Shishkin mesh for a class of time-dependent singularly perturbed convection-diffusion problems containing retarded terms. Ranjan and Prasad [17] employed an exponentially fitted three-term finite difference method to approximate the solution of a singularly perturbed differential equation with small shifts. Rao and Chakravarthy [18] presented exponentially fitted finite difference methods for a class of time-dependent singularly perturbed one-dimensional convection diffusion problems with small shifts.

Kiltu et al. [19] presented a method for solving singularly perturbed delay reaction-diffusion equations with layers or oscillatory behavior. Tefera et al. [20] introduced a new fitted operator method for a singularly perturbed parabolic convection-diffusion problem. Vivek and Rao [21] presented one-dimensional singularly perturbed parabolic partial differential difference equations with mixed shifts using fitted operator spline in compression and adaptive spline methods. Chakravarthy et al. [22] presented the numerical solution using spline in tension on a uniform mesh for second-order singularly perturbed delay differential equations. Hassen and Duressa [23] developed a non-classical numerical method to solve a class of singularly perturbed delay parabolic convection-diffusion problems with Dirichlet boundary conditions.

It is widely recognized that classical numerical approaches yield erroneous results when applied to problems with smaller values of the perturbation parameter. Also, the nonlinearity in the equation poses challenges during discretization. Hence effective numerical techniques are needed to solve these equations, whose precision remains constant regardless of the perturbation parameter and also converges consistently. This motivates us to develop parameter-uniformly convergent numerical methods for singularly perturbed nonlinear boundary value problems with negative shift. In the present study we use the methods described in [24] and [25] for solving nonlinear boundary value problems and develop fitted operator finite difference methods for the linearized problem.

The following is the content arrangement in this paper. In Section 1, we provide a description of the problem being investigated and outline the necessary smoothness requirements for functions. Also, the quasilinearization technique is discussed, by which, the nonlinear differential equation is linearized and a sequence of linear equations is obtained and the convergence of the sequence of equations is studied. In Section 2, a modified form of the sequence of linear equations is obtained and some basic properties of the analytical solution are discussed. The numerical schemes for these linear problems are described in Section 3 and also the convergence analysis of the methods is carried out. Section 4 presents the numerical results for the test problems, while Section 5 provides the discussion and conclusions.

1 Statement of the problem

The following nonlinear differential-difference equation with a singular perturbation is considered:

$$\epsilon y'' = f\left(\chi, y(\chi), y'(\chi-\delta)\right), \; 0 \le \chi \le 1 \tag{1}$$

subject to the interval and the boundary conditions

$$y(\chi) = \phi(\chi); \; -\delta \le \chi \le 0, \text{ and } y(1) = \beta, \tag{2}$$

where $0 < \epsilon \ll 1$ is a singular perturbation parameter and δ is the shift parameter such that $0 < \delta < \epsilon$. The function $y(\chi)$ in (1)–(2) is assumed to be continuous across the interval $[0,1]$ and has continuous derivatives over the open interval $(0,1)$.

It is also assumed that $f\left(\chi, y(\chi), y'(\chi-\delta)\right) = \hat{f}(\chi, y, z)$, which is a smooth function that satisfies the following conditions:

- $\frac{\partial \hat{f}}{\partial y} > 0, \quad \frac{\partial \hat{f}}{\partial z} \le 0, \quad \frac{\partial \hat{f}}{\partial y} - \frac{\partial \hat{f}}{\partial z} \ge M > 0$, where M is a constant with a positive value.
- The growth constraint $\hat{f}(\chi, y, z) = O(z^2)$ as $z \to \infty$ for all real y and z and all $\chi \in [0,1]$.

The problem (1)–(2) has a unique solution when $\delta = 0$ and the conditions stated above are satisfied.

1.1 The Quasilinearization Technique

We apply the quasilinearization process [26] described below to develop a numerical scheme for the nonlinear problem (1)–(2):

- An initial guess $y^{(0)}(\chi)$ for the solution of the problem (1) is considered, which also satisfies the interval and boundary conditions

$$y^{(0)}(\chi) = \phi(\chi), \; -\delta \le \chi \le 0, \; y^{(0)}(1) = \beta.$$

- Then the following sequence of boundary value problems is obtained:

$$\epsilon y''^{(k+1)}(\chi) = f\left(\chi, y^{(k)}(\chi), y'^{(k)}(\chi-\delta)\right), \tag{3}$$

  with $y^{(k+1)}(\chi) = \phi(\chi), \; -\delta \le \chi \le 0, \; y^{(k+1)}(1) = \beta$.
- After expanding the right-hand side of equation (3) in terms of Taylor series around $y^{(k)}$, we get

$$\begin{aligned} \epsilon y''^{(k+1)}(\chi) =& f\left(\chi, y^{(k)}(\chi), y'^{(k)}(\chi-\delta)\right) + \left(y'^{(k+1)}(\chi-\delta) - y'^{(k)}(\chi-\delta)\right)\left(\frac{\partial f}{\partial y'}\right)^{(k)} \\ &+ \left(y^{(k+1)}(\chi) - y^{(k)}(\chi)\right)\left(\frac{\partial f}{\partial y}\right)^{(k)} + \dots \end{aligned} \tag{4}$$

- We get the recurrence relation for the singularly perturbed linear differential-difference equations by modifying the terms of (4).

$$\begin{aligned} &\epsilon y''^{(k+1)}(\chi) - \left(\frac{\partial f}{\partial y'}\right)^{(k)} y'^{(k+1)}(\chi-\delta) - \left(\frac{\partial f}{\partial y}\right)^{(k)} y^{(k+1)}(\chi) = f^{(k)}\left(\chi, y^{(k)}(\chi), y'^{(k)}(\chi-\delta)\right) \\ &- \left(\frac{\partial f}{\partial y}\right)^{(k)} y^{(k)}(\chi) - \left(\frac{\partial f}{\partial y'}\right)^{(k)} y'^{(k)}(\chi-\delta), \text{ with } y^{(k+1)}(\chi) = \phi(\chi), \; -\delta \le \chi \le 0, \; y^{(k+1)}(1) = \beta. \end{aligned} \tag{5}$$

- For simplicity, we denote

$$a^{(k)}(\chi) = -\left(\frac{\partial f}{\partial y'}\right)^{(k)}, \quad b^{(k)}(\chi) = -\left(\frac{\partial f}{\partial y}\right)^{(k)}$$

and

$$r^{(k)}(\chi) = f^{(k)}\left(\chi, y^{(k)}(\chi), y'^{(k)}(\chi-\delta)\right) - \left(\frac{\partial f}{\partial y}\right)^{k} y^{(k)}(\chi) - \left(\frac{\partial f}{\partial y'}\right)^{(k)} y'^{(k)}(\chi-\delta)$$

so that (5) can be written as

$$\epsilon y''^{(k+1)}(\chi) + a^{(k)}(\chi) y'^{(k+1)}(\chi-\delta) + b^{(k)}(\chi) y^{(k+1)}(\chi) = r^{(k)}(\chi), \tag{6}$$

with the interval and boundary conditions

$$y^{(k+1)}(\chi) = \phi(\chi), \ -\delta \le \chi \le 0, \quad y^{(k+1)}(1) = \beta. \tag{7}$$

Therefore, for $k = 0, 1, 2, \ldots$, we solve the sequence of linear equations (6) subject to the interval and boundary conditions (7) rather than the original nonlinear problem (1)–(2). The solutions $y^{(k)}(\chi)$ converge to the solution $y(\chi)$ of the original nonlinear differential equation for large values of k. However, numerically, we require that $|y^{(k+1)}(\chi) - y^{(k)}(\chi)| < \mu, \ 0 \le \chi \le 1$, where μ is the imposed tolerance. The iterations can be terminated if the aforementioned requirements are satisfied, resulting in the solution to equations (1)–(2).

The convergence of the sequence of solutions of (6)–(7) is given in the form of the Lemma below:

Lemma 1. If $\langle \psi^{(k)} \rangle$ is the sequence of solutions of (6)–(7), $k = 0, 1, 2, \ldots$, then

$$\max_{0 \le x \le 1} \left| \left(\psi^{(k+1)} - \psi^{(k)}\right)(\chi) \right| \le \kappa_1 \max_{0 \le \chi \le 1} \left(\psi^{(k)} - \psi^{(k-1)}\right)^2, \ \kappa_1 < 1.$$

Proof. Consider the sequence of solutions $\langle \psi^{(k)} \rangle, \ k = 0, 1, 2, \ldots$ obtained by quasilinearization of (1)–(2).

With $\psi^{(0)}(\chi)$ as the initial approximation, equation(3) yields:

$$\epsilon \psi''^{(k)}(\chi) = f^{(k-1)} + \left(\psi^{(k)} - \psi^{(k-1)}\right) f_\psi(\psi^{(k-1)}), \tag{8}$$

where $f^{(k)} = f\left(x, \psi^{(k)}(\chi), \psi'^{(k)}(\chi-\delta)\right)$ and $f_\psi = \frac{\partial f}{\partial \psi}$.

From equation (3) and equation (8), we have

$$\epsilon \left(\psi^{(k+1)} - \psi^{(k)}\right)''(\chi) = f^{(k)} - f^{(k-1)} - \left(\psi^{(k)} - \psi^{(k-1)}\right) f_\psi(\psi^{(k-1)}) + \left(\psi^{(k+1)} - \psi^{(k)}\right) f_\psi(\psi^{(k)}). \tag{9}$$

Equation (9) is a second order dfferential equation in $\left(\psi^{(k+1)} - \psi^{(k)}\right)$. By using Green's function, equation (9) will be equivalent to the following integral form

$$\begin{aligned}\epsilon \left(\psi^{(k+1)} - \psi^{(k)}\right)(\chi) = \int_0^1 \mathcal{G}(\chi, s) \Big[& f^{(k)} - f^{(k-1)} - \left(\psi^{(k)} - \psi^{(k-1)}\right) f_\psi(\psi^{(k-1)}) \\ & + \left(\psi^{(k+1)} - \psi^{(k)}\right) f_\psi(\psi^{(k)}) \Big] ds.\end{aligned} \tag{10}$$

The Green's function $\mathcal{G}(\chi, s)$ is determined by

$$\mathcal{G}(\chi, s) = \begin{cases} (\chi-1)s, \ 0 \le s \le \chi \le 1, \\ \chi(s-1), \ 0 \le \chi \le s \le 1, \end{cases}$$

where $\max\limits_{\chi,s} |\mathcal{G}(\chi, s)| = 1/4$.

Applying the mean value theorem [4], we can conclude that

$$f^{(k)} - f^{(k-1)} = \left(\psi^{(k)} - \psi^{(k-1)}\right) f_\psi(\psi^{(k-1)}) + \frac{\left(\psi^{(k)} - \psi^{(k-1)}\right)^2}{2} f_{\psi\psi}(\theta), \tag{11}$$

where $\psi^{(k-1)} \le \theta \le \psi^{(k)}$.

By replacing the expression (10) with equation (11), we obtain

$$\epsilon\left(\psi^{(k+1)} - \psi^{(k)}\right) = \int_0^1 \mathcal{G}(\chi, s)\Big[\left(\psi^{(k)} - \psi^{(k-1)}\right)^2 f_{\psi\psi}(\theta)/2 + \left(\psi^{(k+1)} - \psi^{(k)}\right) f_\psi(\psi^{(k)})\Big] ds. \tag{12}$$

Let $\max_{\|\psi\| \le 1} f_\psi(\psi) = a_1, \ \max_{\|\psi\| \le 1} f_{\psi\psi}(\psi) = a_2$. By evaluating the largest absolute value over the entire domain of interest for both sides of equation (12), we obtain

$$\begin{aligned} \max_{0 \le \chi \le 1} |\left(\psi^{(k+1)} - \psi^{(k)}\right)(\chi)| \le & \frac{1}{4\epsilon}\int_0^1 \Big[\max_{0 \le \chi \le 1} \frac{\left(\psi^{(k)} - \psi^{(k-1)}\right)^2}{2} \max_{0 \le \chi \le 1} |f_{\psi\psi}(\theta)| \\ & + \max_{0 \le \chi \le 1} |\left(\psi^{(k+1)} - \psi^{(k)}\right)| \max_{0 \le \chi \le 1} |f_\psi(\psi^{(k)})|\Big] ds. \end{aligned}$$

Through the process of simplification, we obtain the following result:

$$\max_{0 \le \chi \le 1} \left|\left(\psi^{(k+1)} - \psi^{(k)}\right)(\chi)\right| \le \kappa_1 \max_{0 \le \chi \le 1} \left(\psi^{(k)} - \psi^{(k-1)}\right)^2,$$

since $\kappa_1 = a_2/(8\epsilon(1 - a_1/4\epsilon)) < 1$.

This demonstrates that the series $\langle \psi^{(k)} \rangle$ of linear equations converges quadratically if $\kappa_1 < 1$, thus satisfying the lemma.

2 Continuous problem

Assuming that the delay parameter δ is smaller than the perturbation parameter ϵ, the expression $y'^{(k+1)}(\chi - \delta)$ can be expanded using Taylor's series and hence the sequence of problems (6)–(7) is transformed into a sequence of boundary value problems:

$$(\epsilon - \delta a^{(k)}(\chi)) y''^{(k+1)}(\chi) + a^{(k)}(\chi) y'^{(k+1)}(\chi) + b^{(k)}(\chi) y^{(k+1)}(\chi) = r^{(k)}(\chi), \tag{13}$$

subject to

$$y^{(k+1)}(0) = \phi(0), \ y^{(k+1)}(1) = \beta. \tag{14}$$

For $\delta = 0$, the solution of the above problem shows a boundary layer on either the left or right side of the interval, depending on the sign of the coefficient $a^{(k)}(\chi)$. Specifically, if $a^{(k)}(\chi) > 0$, the boundary layer is on the left side, and if $a^{(k)}(\chi) < 0$, the boundary layer is on the right side. Here, we consider $\delta \ne 0$ but comparable to ϵ, to study the effects on the boundary layer behavior of the solution to the problem (1). We presuppose that $a^{(k)}(\chi) \ge \mathscr{M} > 0$ and $b^{(k)}(\chi) \le -\theta < 0$ for positive constants $\mathscr{M}, \theta$.

We define the operator

$$\mathscr{L}(\pi(\chi)) \equiv \mu^{(k)}(\chi))\pi''^{(k+1)}(\chi) + a^{(k)}(\chi)\pi'^{(k+1)}(\chi) + b^{(k)}(\chi)\pi^{(k+1)}(\chi) = r^{(k)}(\chi),$$

where $\mu^{(k)}(\chi)) = (\epsilon - \delta a^{(k)}(\chi))$, and hence from the following Lemma, show that $\mathscr{L}(\pi(\chi))$ satisfies the minimum principle:

Lemma 2. If $\pi(s)$ is a smooth function satisfying the conditions $\pi(0) \geq 0$ and $\pi(1) \geq 0$, then $\mathscr{L}\left(\pi^{(k+1)}(s)\right) \geq 0$ for s in the interval $[0,1]$, whenever $\pi^{(k+1)}(s) \leq 0$ for s in the interval $(0,1)$.

Proof. Let us consider $z \in [0,1]$ be such that $\pi^{(k+1)}(z) = \min_{s\in[0,1]} \pi^{(k+1)}(s)$ and suppose that $\pi^{(k+1)}(z) < 0$. It is evident that $z \notin \{0,1\}$. Consequently, $\pi'^{(k+1)}(z) = 0$ and $\pi''^{(k+1)}(z) \geq 0$.

So we have, $\mathscr{L}\pi^{(k+1)}(z) = \mu^{(k)}(z)\pi''^{(k+1)}(z) + a^{(k)}(z)\pi'^{(k+1)}(z) + b^{(k)}(z)\pi^{(k+1)}(z) > 0$. This contradicts our assumption that $\pi^{(k+1)}(z) < 0$.

Thus, it follows that $\pi^{(k+1)}(z) \geq 0$ so $\pi^{(k+1)}(s) \geq 0$ for every s in the interval $[0,1]$.

The following lemma provides the stability estimate for the solution of the continuous problem (13).

Lemma 3. Let $y^{(k+1)}(\chi)$ denote the solution to equations (13) and (14), then $\|y^{(k+1)}\| \leq \theta^{-1}\|r^{(k)}\| + \max(|\phi_0|, |\beta|)$, here $\|.\|$ is $\mathscr{L}_\infty$ norm given by $\|y^{(k+1)}\| = \max_{s\in[0,1]} |y^{(k+1)}(s)|$.

Proof. We define two barrier functions $\psi^{\pm}$ as follows:

$$\psi^{\pm}(\chi) = \Theta^{-1}\|r^{(k)}\| + \max(|\phi_0|, |\beta|) \pm y^{(k+1)}(\chi).$$

Then we have $\psi^{\pm}(0) = \Theta^{-1}\|r^{(k)}\| + \max(|\phi_0|, |\beta|) \pm y^{(k+1)}(0),$ since $y^{(k+1)}(0) = \phi_0 \geq 0$. Furthermore $\psi^{\pm}(1) = \Theta^{-1}\|r^{(k)}\| + \max(|\phi_0|, |\beta|) \pm y^{(k+1)}(1),$ since $y^{(k+1)}(1) = \beta \geq 0$.

Then also

$$\begin{aligned}\mathscr{L}\psi^{\pm}(\chi) =& \mu^{(k)}(\chi)(\psi^{\pm}(\chi))'' + a^{(k)}(\chi)(\psi^{\pm}(\chi))' + b^{(k)}(\chi)\psi^{\pm}(\chi)\\ =& b^{(k)}(\chi)\left[\Theta^{-1}\|r^{(k)}\| + \max(|\phi_0|, |\beta|)\right] \pm \mathscr{L}y^{(k+1)}(\chi)\\ =& b^{(k)}(\chi)\left[\Theta^{-1}\|r^{(k)}\| + \max(|\phi_0|, |\beta|)\right] \pm r^{(k)}(\chi).\end{aligned}$$

Since we have $b^{(k)}(\chi)\Theta^{-1} \leq -1, b^{(k)}(\chi) \leq -\Theta < 0$, using in the above inequality, we will get

$$\mathscr{L}\psi^{\pm}(\chi) \leq \left(-\|r\| \pm r^{(k)}(\chi)\right) + b^{(k)}(\chi)\max(|\phi_0|, |\beta|) \leq 0 \quad \forall\, \chi \in (0,1),$$

where $\|r\| \geq r^{(k)}(\chi)$.

According to the minimal principle [27], it is evident that $\psi^{\pm}(\chi) \geq 0$ for all $\chi \in (0,1)$. Therefore, the desired estimate is satisfied.

Considering $y^{(k+1)}(\chi) = \mathcal{Y}(\chi)$, we show an approximation for the discrete solution, in the following lemma:

Lemma 4. Let $\mathcal{Y}(\chi) = (\mathcal{Y}_0)^{outer} + (\mathcal{Y}_0)^{inner}$ be the zeroth order estimation to the solution of equations (13) and (14). The term $(\mathcal{Y}_0)^{outer}$ represents the zeroth order estimation of the solution in the outer region, while $(\mathcal{Y}_0)^{inner}$ represents the zeroth order estimation in the boundary layer region. Following that, for a given positive integer j,

$$\lim_{h\to 0} \mathcal{Y}(jh) \approx (\mathcal{Y}_0)^{outer}(0) + (\phi(0) - (\mathcal{Y}_0)^{outer}(0))e^{-a^{(k)}(0)j\rho}, \text{ with } \rho = \frac{h}{\mu^{(k)}(0)}.$$

Proof. For $\epsilon = 0$, the reduced problem (13) is as follows:

$$a^{(k)}(\chi)\big((\mathcal{Y}_0)^{outer}(\chi)\big)' + b^{(k)}(\mathcal{Y}_0)^{outer}(\chi) = r^{(k)}(\chi), \quad (\mathcal{Y}_0)^{outer}(1) = \beta$$

and the boundary layer region problem is

$$\begin{aligned}&\big((\mathcal{Y}_0)^{inner}(\hat{\chi})\big)'' + a^{(k)}(0)\left((\mathcal{Y}_0)^{inner}(\hat{\chi})\right)' = 0,\\ &(\mathcal{Y}_0)^{inner}(0) = \phi_0 - (\mathcal{Y}_0)^{outer}(0), (\mathcal{Y}_0)^{inner}(\infty) = 0, \text{ where } \hat{\chi} = \frac{\chi}{\epsilon - \delta\mathscr{M}}.\end{aligned}$$

According to the theory of singular perturbations [28], the zeroth order asymptotic approximations to the problem's solution are

$$\mathcal{Y}(\chi) = (\mathcal{Y}_0)^{outer}(\chi) + \frac{a^{(k)}(0)}{a^{(k)}(\chi)}(\phi_0 - (\mathcal{Y}_0)^{outer}(0))e^{-\int\limits_0^{\chi}\frac{a^{(k)}(\chi)}{\mu^{(k)}(\chi)}d\chi}.$$

Given that the coefficients are locally constant on a small grid,

$$\mathcal{Y}(\chi) = (\mathcal{Y}_0)^{outer}(\chi) + (\phi_0 - (\mathcal{Y}_0)^{outer}(0))e^{-\frac{a^{(k)}(0)}{\mu^{(k)}(0)}\chi}$$

and hence, at the mesh points,

$$\begin{aligned}\mathcal{Y}(\chi_j) &= (\mathcal{Y}_0)^{outer}(\chi_j) + (\phi_0 - (\mathcal{Y}_0)^{outer}(0))e^{-\frac{a^{(k)}(0)}{\mu^{(k)}(0)}\chi_j}, \quad j = 0, 1, \ldots, N,\\ \mathcal{Y}(jh) &= (\mathcal{Y}_0)^{outer}(jh) + (\phi_0 - (\mathcal{Y}_0)^{outer}(0))e^{-\frac{a^{(k)}(0)}{\mu^{(k)}(0)}jh}.\end{aligned}$$

Therefore, $\lim\limits_{h\to 0}\mathcal{Y}(jh) = (\mathcal{Y}_0)^{outer}(0) + (\phi_0 - (\mathcal{Y}_0)^{outer}(0))e^{-a^{(k)}(0)j\rho}$ regarding $j = 0, 1, \ldots, N$, for $\rho = \frac{h}{\mu^{(k)}(0)}$.

3 Numerical schemes

We rewrite Eq.(13) as

$$\mu^{(k)}(\chi)y''^{(k+1)}(\chi) = g(\chi, y^{(k+1)}, y'^{(k+1)}) \tag{15}$$

$$\text{with } \; g(\chi, y^{(k+1)}, y'^{(k+1)}) = r^{(k)}(\chi) - a^{(k)}(\chi)y'^{(k+1)}(\chi) - b^{(k)}(\chi)y^{(k+1)}(\chi). \tag{16}$$

Now, we will partition the interval $[0, 1]$ into N equal segments with a constant mesh length h. This means that the mesh points $\chi_n = nh,\ n = 0, 1, 2, \ldots, N$.

3.1 Exponentially Fitted Higher Order Method-1 (EFHOM-1)

We use the finite difference technique proposed by Chawla [24] for solving the general non-linear boundary value problem, which may be expressed as $u'' = g(x, u, u')$.

$$\begin{aligned}\overline{u}'_s =& \frac{(u_{s+1} - u_{s-1})}{2h},\ \overline{u}'_{s+1} = \frac{(3u_{s+1} - 4u_s - u_{s-1})}{2h},\ \overline{u}'_{s-1} = \frac{(-u_{s+1} + 4u_s - 3u_{s-1})}{2h},\\ \overline{\overline{u}}'_s =& \overline{u}'_s - \frac{h}{20}(\overline{g}_{s+1} - \overline{g}_{s-1}),\ \frac{u_{s+1} - 2u_s + u_{s-1}}{h^2} = \frac{1}{12}\left(\overline{g}_{s+1} + 10\overline{\overline{g}}_s + \overline{g}_{s-1}\right),\end{aligned}$$

where $\overline{\overline{g}}_s = g(x_s, u_s, \overline{\overline{u}}')$ and $\overline{\overline{g}}_{s\pm 1} = g(x_{s\pm 1}, u_{s\pm 1}, \overline{\overline{u}}'_{s\pm 1})$.

Now, we introduce the fitting parameter $\sigma(\rho)$ for the second derivative and apply the preceding approach to equation (15), replacing $a^{(k)}(\chi)$, $b^{(k)}(\chi)$ and $r^{(k)}(\chi)$ for convenience, with $a(\chi)$, $b(\chi)$ and $r(\chi)$ respectively. In this context, we consider the initial approximation $y^{(0)}(\chi)$ as the solution to the reduced problem of (13)–(14). The succeeding approximations $\{y^{(k)}(\chi)\}_{k=0}^{\infty}$ are then determined by (15)–(16). We get the tridiagonal scheme

$$E_n y_{n-1}^{(k+1)} + F_n y_n^{(k+1)} + G_n y_{n+1}^{(k+1)} = H_n, \quad n = 1, 2, \ldots, N-1, \tag{17}$$

here

$$\begin{aligned}
E_n =& \mu_n\sigma_n + \frac{ha_{n+1}}{24} - \frac{5ha_n}{12} + \frac{h^2a_na_{n+1}}{48} + \frac{h^2a_na_{n-1}}{16} - \frac{h^3a_nb_{n-1}}{24} - \frac{ha_{n-1}}{8} - \frac{h^2b_{n-1}}{12},\\
F_n =& -2\mu_n\sigma_n - \frac{ha_{n+1}}{6} - \frac{h^2a_na_{n+1}}{12} - \frac{h^2a_na_{n-1}}{12} + \frac{5h^2b_n}{6} + \frac{ha_{n-1}}{6},\\
G_n =& \mu_n\sigma_n + \frac{ha_{n+1}}{8} + \frac{h^2b_{n+1}}{12} + \frac{5ha_n}{12} + \frac{h^2a_na_{n+1}}{16} + \frac{h^3a_nb_{n+1}}{24} + \frac{h^2a_na_{n-1}}{48} - \frac{ha_{n-1}}{24},\\
H_n =& \frac{h^2}{12}\Big\{(1 - \frac{ha_n}{2})r_{n-1} + 10r_n + (1 + \frac{ha_n}{2})r_{n+1}\Big\}.
\end{aligned}$$

The tri-diagonal scheme mentioned above, which refers to EFHOM-1, together with the boundary conditions (14), is solved using the Thomas Algorithm. Applying Lemma 4, we obtain the fitting parameter: $\sigma(\rho) = \frac{a(0)\rho}{2}\coth\left(\frac{a(0)\rho}{2}\right)$. It represents a constant fitting parameter. Generally, we take the variable fitting parameter into account, i.e,

$$\sigma_n = \frac{a(\chi_n)\rho_n}{2}\coth\left(\frac{a(\chi_n)\rho_n}{2}\right), \quad \text{where} \quad \rho_n = \frac{h}{\mu_n}. \tag{18}$$

3.1.1 Convergence Analysis for EFHOM-1

Multiplying h both sides in the tridiagonal scheme (17) using boundary conditions, the system of equations is represented in matrix form as

$$(\mathbb{D} + \mathbb{J})\mathcal{W} + \mathcal{K} + \mathcal{T}(h) = 0, \tag{19}$$

here

$$\mathbb{D} = \left(\tfrac{\mu_n\sigma_n}{h}, \quad -\tfrac{\mu_n\sigma_n}{h}, \quad \tfrac{\mu_n\sigma_n}{h}\right) = \begin{pmatrix} \frac{-2\mu_1\sigma_1}{h} & \frac{\mu_1\sigma_1}{h} & 0 & \cdots & 0\\ \frac{\mu_2\sigma_2}{h} & \frac{-2\mu_2\sigma_2}{h} & \frac{\mu_2\sigma_2}{h} & \cdots & 0\\ 0 & . & . & \cdots & .\\ . & . & . & \cdots & .\\ 0 & \cdots & 0 & \frac{\mu_{N-1}\sigma_{N-1}}{h} & \frac{-2\mu_{N-1}\sigma_{N-1}}{h} \end{pmatrix},$$

$$\mathbb{J} = \left(\check{p}_m, \quad p_m, \quad \hat{p}_m\right) = \begin{pmatrix} p_1 & \hat{p}_1 & 0 & \cdots & 0\\ \check{p}_2 & p_2 & \hat{p}_2 & \cdots & 0\\ 0 & . & . & \cdots & .\\ . & . & . & \cdots & .\\ 0 & \cdots & 0 & \check{p}_{N-1} & p_{N-1} \end{pmatrix}, \quad \mathcal{T}(h) = O(h^4),$$

$$\check{p}_n = \frac{1}{12}\left[hb_{n-1} + \frac{a_{n+1}}{2} - 5a_n - \frac{h^2a_nb_{n-1}}{2} - \frac{3a_{n-1}}{2} + \frac{ha_n(a_{n+1} + 3a_{n-1})}{4}\right],$$

$$p_n = \left[\frac{5hb_n}{6} - \frac{(a_{n+1} - a_{n-1})}{6} - \frac{ha_n(a_{n+1} + a_{n-1})}{12}\right],$$

$$\hat{p}_n = \frac{1}{12}\left[hb_{n+1} + \frac{3a_{n+1}}{2} + 5a_n + \frac{h^2 a_n b_{n+1}}{2} - \frac{a_{n-1}}{2} + \frac{ha_n(3a_{n+1}+a_{n-1})}{4}\right]$$

and

$$\mathcal{K} = \left[\mathcal{k}_1 + \left(\frac{\mu_1\sigma_1}{h} + r_1\right)\phi_0, \mathcal{k}_2, \mathcal{k}_3, \ldots, \mathcal{k}_{N-2}, \mathcal{k}_{N-1} + \left(\frac{\mu_{N-1}\sigma_{N-1}}{h} + t_{N-1}\right)\beta\right]^T,$$

where

$$\mathcal{K}_n = -\frac{1}{12}\left[h(r_{n+1} + 10r_n + r_{n-1}) + \frac{h^2 a_n(r_{n+1} - r_{n-1})}{2}\right],\ n = 1, 2, \ldots, N-1.$$

Here, $\mathcal{W} = [\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{N-1}]^T$, the truncation errors occured at the mesh points $\mathcal{T}(h) = [\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{N-1}]^T$.

Let $W = [\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_{N-1}]^T \cong \mathcal{Y}$ which at first fulfills the equation

$$(\mathbb{D} + \mathbb{J})\overline{w} + \mathcal{K} = 0. \tag{20}$$

Let $e_n = \overline{w}_n - \mathcal{W}_n$, $\quad n = 1, 2, \ldots, N-1$ denote the discretization error. This will ensure that $\mathcal{E} = [e_1, e_2, \ldots, e_{N-1}]^T = \overline{w} - \mathcal{W}$.

Now subtracting (20) from (19), we get

$$(\mathbb{D} + \mathbb{J})\mathcal{E} = \mathcal{T}(h). \tag{21}$$

Considering $|a(\chi)| \le c_1$, $|b(\chi)| \le c_2$, for positive constants c_1, c_2 and $\mathbb{J}_{i,j}$, the $(i,j)^{th}$ element of the matrix $\mathbb{J}$, we have

$$\begin{aligned}
|\mathbb{J}_{n,n+1}| &= |\hat{p}_n| \le \frac{1}{12}\left(hc_2 + 6c_1 + \frac{h^2 c_1 c_2}{2} + hc_1^2\right),\\
|\mathbb{J}_{n,n-1}| &= |\check{p}_n| \le \frac{1}{12}\left(hc_2 - 6c_1 - \frac{h^2 c_1 c_2}{2} + hc_1^2\right).
\end{aligned}$$

For sufficiently small h, we have

$$\frac{\mu_n\sigma_n}{h} + |\mathbb{J}_{n,n+1}| \le \frac{c_1}{12}\left(\frac{\mu_n\sigma_n}{h} + 1\right) \ne 0,\ n = 1, 2, \ldots, N-2,$$

$$\frac{\mu_n\sigma_n}{h} + |\mathbb{J}_{n,n-1}| \le \frac{c_1}{12}\left(\frac{\mu_n\sigma_n}{h} + 1\right) \ne 0,\ n = 1, 2, \ldots, N-1.$$

Therefore, the matrix is irreducible according to Varga's research [29], ensuring the stability and robustness of the numerical scheme (17).

Define $\mathcal{S}_n$ as the sum of the elements in the n^{th} row of the matrix $(\mathbb{D} + \mathbb{J})$, then we have

$$\begin{aligned}
\mathcal{S}_n &= -\frac{\mu_n\sigma_n}{h} + \frac{5hb_n}{6} - \frac{ha_{n+1}}{24} + \frac{5a_n}{12} + \frac{a_{n-1}}{8} + \frac{hb_{n+1}}{12} - \frac{ha_n a_{n+1}}{48} - \frac{ha_n a_{n-1}}{16} + \frac{h^2 a_n b_{n+1}}{24},\ n = 1,\\
\mathcal{S}_n &= -\frac{\mu_n\sigma_n}{h} + \frac{hb_{n-1}}{12} - \frac{a_{n+1}}{8} - \frac{5a_n}{12} + \frac{a_{n-1}}{24} - \frac{ha_n a_{n+1}}{16} - \frac{ha_n a_{n-1}}{48} - \frac{h^2 a_n b_{n-1}}{24} + \frac{5hb_n}{6},\ n = N-1,\\
\mathcal{S}_n &= \frac{h(b_{n-1} + 10b_n + b_{n+1})}{12} + \frac{h^2 a_n(b_{n+1} - b_{n-1})}{24},\quad \text{for } n = 2, 3, \ldots, N-2.
\end{aligned}$$

Let $c_{1^*} = \min|a(\chi)|$, $c_1^* = \max|a(\chi)|$ and $c_{2^*} = \min|b(\chi)|$, $c_2^* = \max|b(\chi)|$, then $0 < c_{1^*} \le c_1 \le c_1^*$ and $0 < c_{2^*} \le c_2 \le c_2^*$.

It can be readily confirmed that the sum of $\mathbb{D}$ and $\mathbb{J}$, denoted as $(\mathbb{D} + \mathbb{J})$, exhibits monotonic behavior, as stated in references [29] and [30].

From this $(\mathbb{D}+\mathbb{J})^{-1}$ exists because $(\mathbb{D}+\mathbb{J})^{-1} \geq 0$. Based on the error equation (21), we deduce that

$$||\mathcal{E}|| = ||(\mathbb{D}+\mathbb{J})^{-1}|| \cdot ||\mathcal{T}||.$$

When h is small enough, we have

$$\begin{aligned} &\mathcal{S}_n > \frac{h^2}{24} c_1 c_2 \quad \text{for} \quad n = 1, N-1 \quad \text{and} \\ &\mathcal{S}_n > \frac{h^2}{24} c_1 c \quad \text{for} \quad n = 2, 3, \ldots, N-2, \text{ where } c = |b_{n+1} - b_{n-1}|. \end{aligned} \tag{22}$$

Let consider $(\mathbb{D}+\mathbb{J})^{-1}_{(n,i)}$ be the $(n,i)^{th}$ element of $(\mathbb{D}+\mathbb{J})^{-1}$ and we define

$$||(\mathbb{D}+\mathbb{J})^{-1}|| = \max_{1 \leq n \leq N-1} \sum_{i=1}^{N-1} (\mathbb{D}+\mathbb{J})^{-1}_{n,i} \quad \text{and} \quad ||\mathcal{T}(h)|| = \max_{1 \leq n \leq N-1} |\mathcal{T}_n|,$$

since $(\mathbb{D}+\mathbb{J})^{-1}_{(n,i)} \geq 0$ and $\sum_{i=1}^{N-1} (\mathbb{D}+\mathbb{J})^{-1}_{(n,i)} \cdot \mathcal{S}_i = 1$ for $n = 1, 2, \ldots, N-1$.

One has $(\mathbb{D}+\mathbb{J})^{-1}_{(n,1)} \leq \frac{1}{\mathcal{S}_1} \leq \frac{24}{h^2 c_1 c_2^2}$ and $(\mathbb{D}+\mathbb{J})^{-1}_{(n,N-1)} \leq \frac{1}{\mathcal{S}_{N-1}} \leq \frac{24}{h^2 c_1 c_2^2}$.

Additionally $\sum_{i=2}^{N-2} (\mathbb{D}+\mathbb{J})^{-1}_{(n,i)} \leq \frac{1}{\min\limits_{2 \leq i \leq N-2} \mathcal{S}_i} \leq \frac{24}{h^2 c_1 c}$ for $n = 1, 2, \ldots, N-1$.

So, using equations (21) and (22), we get

$$||\mathcal{E}|| = \frac{24}{h^2} \left[\frac{1}{c_1 c_2} + \frac{1}{c c_1} + \frac{1}{c_1 c_2} \right] \times O(h^4) = O(h^2).$$

This demonstrates the convergence of the finite difference scheme (17).

3.2 Exponentially Fitted Higher Order Method-2 (EFHOM-2)

In this section, we consider the finite difference method proposed by Chawla [25] for the general nonlinear boundary value problem of the form $w'' = g(x, w, w')$, as illustrated below:

$$\begin{aligned}
&\overline{w}'_s = \frac{1}{2h}[w_{s+1} - w_{s-1}], \quad \overline{w}'_{s+1} = \frac{1}{2h}[3w_{s+1} - 4w_s + w_{s-1}], \quad \overline{w}'_{s-1} = \frac{1}{2h}[-w_{s+1} + 4w_s - 3w_{s-1}], \\
&\overline{\overline{w}}'_{s+1} = \frac{1}{2h}[w_{s+1} - w_{s-1}] + \frac{h}{3}[2\overline{g}_s + \overline{g}_{s+1}], \quad \overline{\overline{w}}'_{s-1} = \frac{1}{2h}[w_{s+1} - w_{s-1}] - \frac{h}{3}[2\overline{g}_s + \overline{g}_{s-1}], \\
&\overline{w}_{s+\frac{1}{2}} = \frac{1}{32}[15w_{s+1} + 18w_s - w_{s-1}] - \frac{h^2}{64}[3\overline{g}_{s+1} + 4\overline{g}_s - \overline{g}_{s-1}], \\
&\overline{w}_{s-\frac{1}{2}} = \frac{1}{32}[-w_{s+1} + 18w_s + 15w_{s-1}] - \frac{h^2}{64}[-\overline{g}_{s+1} + 4\overline{g}_s + 3\overline{g}_{s-1}], \\
&\overline{\overline{w}}'_{s+\frac{1}{2}} = \frac{1}{4h}[5w_{s+1} - 6w_s + w_{s-1}] - \frac{h}{48}[3\overline{g}_{s+1} + 8\overline{g}_s + \overline{g}_{s-1}], \\
&\overline{\overline{w}}'_{s-\frac{1}{2}} = \frac{1}{4h}[-w_{s+1} + 6w_s - 5w_{s-1}] - \frac{h}{48}[\overline{g}_{s+1} + 8\overline{g}_s + 3\overline{g}_{s-1}], \\
&\hat{w}'_s = \overline{w}'_s + h\left[\frac{1}{78}(\overline{g}_{s+1} - \overline{g}_{s-1}) - \frac{1}{52}(\overline{\overline{g}}_{s+1} - \overline{\overline{g}}_{s-1}) - \frac{2}{13}(\overline{\overline{g}}_{s+\frac{1}{2}} - \overline{\overline{g}}_{s-\frac{1}{2}})\right], \\
&[w_{s-1} - 2w_s + w_{s+1}] = \frac{h^2}{60}\left[26\hat{g}_s + \overline{\overline{g}}_{s+1} + \overline{\overline{g}}_{s-1} + 16(\overline{\overline{g}}_{s+\frac{1}{2}} + \overline{\overline{g}}_{s-\frac{1}{2}})\right],
\end{aligned}$$

where

$$\overline{\overline{g}}_{s\pm1} = g(x_{s\pm1}, w_{s\pm1}, \overline{\overline{w}}'_{s\pm1}), \quad \overline{\overline{g}}_{s\pm\frac{1}{2}} = g(x_{s\pm\frac{1}{2}}, w_{s\pm\frac{1}{2}}, \overline{\overline{w}}'_{s\pm\frac{1}{2}}), \quad \hat{g}_s = g(x_s, w_s, \hat{w}'_s).$$

Now, we introduce the fitting parameter $\sigma(\rho)$ to represent the second derivative and then apply the previously described approach to equation (15). The solution of the simplified problem of (13)–(14) is considered to be the initial approximation $y^{(0)}(\chi)$, and the succeeding approximations $\{y^{(k)}(\chi)\}_{k=0}^{\infty}$ are determined by (15)–(16). We will get the tridiagonal scheme

$$E_n y_{n-1}^{(k+1)} + F_n y_n^{(k+1)} + G_n y_{n+1}^{(k+1)} = H_n, \quad n = 1, 2, \ldots, N-1, \tag{23}$$

where

$$\begin{aligned}
E_n =& \mu_n\sigma_n + \frac{ha_{n+1/2}}{15} - \frac{ha_{n+1}}{120} - \frac{ha_{n-1/2}}{3} - \frac{ha_{n-1}}{120} + \frac{h^2b_{n-1}}{60} + \frac{h^2b_{n-1/2}}{8} - \frac{h^2b_{n+1/2}}{120} - \frac{13ha_n}{60} \\
& - \frac{h^2a_{n-1}^2}{120} - \frac{h^2a_{n+1}^2}{360} - \frac{7h^2a_{n-1}a_n}{720} - \frac{h^2a_{n+1}a_n}{720} + \frac{19h^2a_{n-1/2}a_n}{180} - \frac{h^2a_{n+1/2}a_n}{180} - \frac{h^3b_{n-1}a_n}{360} \\
& - \frac{19h^3b_{n-1/2}a_n}{480} - \frac{h^3b_{n+1/2}a_n}{96} + \frac{h^2a_{n-1}a_{n-1/2}}{40} - \frac{h^2a_{n-1}a_{n+1/2}}{120} - \frac{h^2a_{n+1}a_{n-1/2}}{360} \\
& + \frac{h^2a_{n+1}a_{n+1/2}}{120} + \frac{h^3a_{n-1}a_n^2}{360} + \frac{h^3a_{n+1}a_n^2}{360} + \frac{h^3a_{n-1}^2a_n}{240} - \frac{h^3a_{n+1}^2a_n}{720} - \frac{h^3a_{n-1/2}a_n^2}{180} \\
& - \frac{h^3a_{n+1/2}a_n^2}{180} + \frac{h^3a_{n-1}b_{n-1}}{180} - \frac{3h^3a_{n-1}b_{n-1/2}}{160} + \frac{h^3a_{n-1}b_{n+1/2}}{160} - \frac{h^3a_{n+1}b_{n-1/2}}{480} \\
& + \frac{h^3a_{n+1}b_{n+1/2}}{160} - \frac{h^3a_{n-1/2}b_{n-1}}{60} + \frac{h^3a_{n+1/2}b_{n-1}}{180} + \frac{h^4b_{n-1/2}a_n^2}{480} - \frac{h^4b_{n+1/2}a_n^2}{480} \\
& + \frac{h^4b_{n-1}b_{n-1/2}}{80} - \frac{h^4b_{n-1}b_{n+1/2}}{240} - \frac{h^3a_{n-1}a_{n-1/2}a_n}{160} - \frac{h^3a_{n-1}a_{n+1/2}a_n}{480} + \frac{h^3a_{n+1}a_{n-1/2}a_n}{1440} \\
& + \frac{h^3a_{n+1}a_{n+1/2}a_n}{480} - \frac{h^4a_{n-1}b_{n-1}a_n}{360} + \frac{3h^4a_{n-1}b_{n-1/2}a_n}{640} + \frac{h^4a_{n-1}b_{n+1/2}a_n}{640} + \frac{h^4a_{n+1}b_{n-1/2}a_n}{1920} \\
& + \frac{h^4a_{n+1}b_{n+1/2}a_n}{640} + \frac{h^4a_{n-1/2}b_{n-1}a_n}{240} + \frac{h^4a_{n+1/2}b_{n-1}a_n}{720} - \frac{h^5b_{n-1}b_{n-1/2}a_n}{320} - \frac{h^5b_{n-1}b_{n+1/2}a_n}{960},
\end{aligned}$$

$$\begin{aligned}
F_n =& -2\mu_n\sigma_n + \frac{2ha_{n-1/2}}{5} - \frac{2ha_{n+1/2}}{5} + \frac{13h^2b_n}{30} + \frac{3h^2b_{n-1/2}}{20} + \frac{3h^2b_{n+1/2}}{20} + \frac{h^2a_{n-1}^2}{90} + \frac{h^2a_{n+1}^2}{90} \\
& + \frac{h^2a_{n-1}a_n}{90} + \frac{h^2a_{n+1}a_n}{90} - \frac{h^2a_{n-1/2}a_n}{10} - \frac{h^2a_{n+1/2}a_n}{10} + \frac{h^3a_{n-1}b_n}{90} - \frac{h^3a_{n+1}b_n}{90} - \frac{2h^3a_{n-1/2}b_n}{45} \\
& + \frac{2h^3a_{n+1/2}b_n}{45} - \frac{3h^3b_{n-1/2}a_n}{80} + \frac{3h^3b_{n+1/2}b_n}{80} + \frac{h^4b_{n-1/2}b_n}{60} + \frac{h^4b_{n+1/2}b_n}{60} - \frac{h^2a_{n-1}a_{n-1/2}}{30} \\
& + \frac{h^2a_{n-1}a_{n+1/2}}{90} + \frac{h^2a_{n+1}a_{n-1/2}}{90} - \frac{h^2a_{n+1}a_{n+1/2}}{30} - \frac{h^3a_{n-1}^2a_n}{180} + \frac{h^3a_{n+1}^2a_n}{180} + \frac{h^3a_{n-1}b_{n-1/2}}{40} \\
& - \frac{h^3a_{n-1}b_{n+1/2}}{120} + \frac{h^3a_{n+1}b_{n-1/2}}{120} - \frac{h^3a_{n+1}b_{n+1/2}}{40} - \frac{h^4a_{n-1}a_nb_n}{180} - \frac{h^4a_{n+1}a_nb_n}{180} + \frac{h^4a_{n-1/2}a_nb_n}{90} \\
& + \frac{h^4a_{n+1/2}a_nb_n}{90} - \frac{h^5b_{n-1/2}a_nb_n}{240} + \frac{h^5b_{n+1/2}a_nb_n}{240} + \frac{h^3a_{n-1}a_{n-1/2}a_n}{120} + \frac{h^3a_{n-1}a_{n+1/2}a_n}{360} \\
& - \frac{h^3a_{n+1}a_{n-1/2}a_n}{360} - \frac{h^3a_{n+1}a_{n+1/2}a_n}{120} - \frac{h^4a_{n-1}b_{n-1/2}a_n}{160} - \frac{h^4a_{n-1}b_{n+1/2}a_n}{480} - \frac{h^4a_{n+1}b_{n-1/2}a_n}{480} \\
& - \frac{h^4a_{n+1}b_{n+1/2}a_n}{160},
\end{aligned}$$

$$
\begin{aligned}
G_n =&\mu_n\sigma_n + \frac{ha_{n-1}}{120} + \frac{ha_{n+1}}{120} - \frac{ha_{n-1/2}}{15} + \frac{ha_{n+1/2}}{3} + \frac{h^2b_{n+1}}{60} - \frac{h^2b_{n-1/2}}{120} + \frac{h^2b_{n+1/2}}{8} + \frac{13ha_n}{60} \\
&- \frac{h^2a_{n-1}^2}{360} - \frac{h^2a_{n+1}^2}{120} - \frac{h^2a_{n-1}a_n}{720} - \frac{7h^2a_{n+1}a_n}{720} - \frac{h^2a_{n-1/2}a_n}{180} + \frac{19h^2a_{n+1/2}a_n}{180} + \frac{h^3b_{n+1}a_n}{360} \\
&+ \frac{h^3b_{n-1/2}a_n}{96} + \frac{19h^3b_{n+1/2}a_n}{480} + \frac{h^2a_{n-1}a_{n-1/2}}{120} - \frac{h^2a_{n-1}a_{n+1/2}}{360} - \frac{h^2a_{n+1}a_{n-1/2}}{120} \\
&+ \frac{h^2a_{n+1}a_{n+1/2}}{40} - \frac{h^3a_{n-1}a_n^2}{360} - \frac{h^3a_{n+1}a_n^2}{360} + \frac{h^3a_{n-1}^2a_n}{720} - \frac{h^3a_{n+1}^2a_n}{240} + \frac{h^3a_{n-1/2}a_n^2}{180} \\
&+ \frac{h^3a_{n+1/2}a_n^2}{180} - \frac{h^3a_{n+1}b_{n+1}}{180} - \frac{h^3a_{n-1}b_{n-1/2}}{160} + \frac{h^3a_{n-1}b_{n+1/2}}{480} - \frac{h^3a_{n+1}b_{n-1/2}}{160} \\
&+ \frac{3h^3a_{n+1}b_{n+1/2}}{160} - \frac{h^3a_{n-1/2}b_{n+1}}{180} + \frac{h^3a_{n+1/2}b_{n+1}}{60} - \frac{h^4b_{n-1/2}a_n^2}{480} + \frac{h^4b_{n+1/2}a_n^2}{480} \\
&- \frac{h^4b_{n+1}b_{n-1/2}}{240} + \frac{h^4b_{n+1}b_{n+1/2}}{80} - \frac{h^3a_{n-1}a_{n-1/2}a_n}{480} - \frac{h^3a_{n-1}a_{n+1/2}a_n}{1440} + \frac{h^3a_{n+1}a_{n-1/2}a_n}{480} \\
&+ \frac{h^3a_{n+1}a_{n+1/2}a_n}{160} - \frac{h^4a_{n+1}b_{n+1}a_n}{360} + \frac{h^4a_{n-1}b_{n-1/2}a_n}{640} + \frac{h^4a_{n-1}b_{n+1/2}a_n}{1920} + \frac{h^4a_{n+1}b_{n-1/2}a_n}{640} \\
&+ \frac{3h^4a_{n+1}b_{n+1/2}a_n}{640} + \frac{h^4a_{n-1/2}b_{n+1}a_n}{720} + \frac{h^4a_{n+1/2}b_{n+1}a_n}{240} + \frac{h^5b_{n+1}b_{n-1/2}a_n}{960} + \frac{h^5b_{n+1}b_{n+1/2}a_n}{320},
\end{aligned}
$$

and

$$
\begin{aligned}
H_n =&\frac{13h^2r_n}{30} + \frac{h^2r_{n-1}}{60} + \frac{h^2r_{n+1}}{60} + \frac{4h^2r_{n-1/2}}{15} + \frac{4h^2r_{n+1/2}}{15} + \frac{h^3a_{n-1}r_n}{90} - \frac{h^3a_{n+1}r_n}{90} - \frac{h^3r_{n-1}a_n}{360} \\
&+ \frac{h^3r_{n+1}a_n}{360} - \frac{2h^3a_{n-1/2}r_n}{45} + \frac{2h^3a_{n+1/2}r_n}{45} - \frac{h^3r_{n-1/2}a_n}{15} + \frac{h^3r_{n+1/2}a_n}{15} + \frac{h^4b_{n-1/2}r_n}{60} \\
&+ \frac{h^4b_{n+1/2}r_n}{60} + \frac{h^3a_{n-1}r_{n-1}}{180} - \frac{h^3a_{n+1}r_{n+1}}{180} - \frac{h^3a_{n-1/2}r_{n-1}}{60} - \frac{h^3a_{n-1/2}r_{n+1}}{180} + \frac{h^3a_{n+1/2}r_{n-1}}{180} \\
&+ \frac{h^3a_{n+1/2}r_{n+1}}{60} + \frac{h^4b_{n-1/2}r_{n-1}}{80} - \frac{h^4b_{n-1/2}r_{n+1}}{240} - \frac{h^4b_{n+1/2}r_{n-1}}{240} + \frac{h^4b_{n+1/2}r_{n+1}}{80} - \frac{h^4a_{n-1}a_nr_n}{180} \\
&- \frac{h^4a_{n+1}a_nr_n}{180} + \frac{h^4a_{n-1/2}a_nr_n}{90} + \frac{h^4a_{n+1/2}a_nr_n}{90} - \frac{h^5b_{n-1/2}a_nr_n}{240} + \frac{h^5b_{n+1/2}a_nr_n}{240} - \frac{h^4a_{n-1}r_{n-1}a_n}{360} \\
&- \frac{h^4a_{n+1}r_{n+1}a_n}{360} + \frac{h^4a_{n-1/2}r_{n-1}a_n}{240} + \frac{h^4a_{n-1/2}r_{n+1}a_n}{720} + \frac{h^4a_{n+1/2}r_{n-1}a_n}{720} + \frac{h^4a_{n+1/2}r_{n+1}a_n}{240} \\
&- \frac{h^5b_{n-1/2}r_{n-1}a_n}{320} + \frac{h^5b_{n-1/2}r_{n+1}a_n}{960} - \frac{h^5b_{n+1/2}r_{n-1}a_n}{960} + \frac{h^5b_{n+1/2}r_{n+1}b_n}{320}.
\end{aligned}
$$

The tri-diagonal scheme mentioned above, which refers to EFHOM-2, when combined with the boundary conditions (14), is solved using the Thomas Algorithm. Here, we also employ lemma 4 to determine the fitting parameter, which is equivalent to equation (18).

Note. Similar to Section 3.1.1, the convergence of the finite difference scheme (23) can be demonstrated and proved to be of second order.

4 *Numerical results*

To test the efficiency of EFHOM-1 and EFHOM-2, we have applied them to two nonlinear singularly perturbed differential-difference equations with small negative shift.

Given that the precise solutions for these problems are not known for various values of δ, the maximum absolute errors for the test problems are determined using the double mesh approach as follows:

$$E_N = \max_{0\leq n\leq N} |y_n^N - y_{2n}^{2N}|.$$

The numerical rate of convergence for both examples has been calculated by the formula

$$R_N = \frac{log|E_N/E_{2N}|}{log2}.$$

Example 1. $\epsilon y''(\chi) + 2y'(\chi-\delta) - e^{y(\chi)} = 0$, along with the interval and boundary conditions $y(\chi) = 0,\ -\delta \leq \chi \leq 0,\ y(1) = 0.$

The sequence of linear equations obtained by applying quasilnearization technique is given by the following recurrence relation

$$\epsilon {y''}^{(k+1)}(\chi) + 2{y'}^{(k+1)}(\chi-\delta) - e^{y^{(k)}} y^{(k+1)}(\chi) = e^{y^{(k)}}(1 - y^{(k)}),$$

with the interval and boundary conditions $y^{(k+1)}(\chi) = 0,\ -\delta \leq \chi \leq 0,\ y^{(k+1)}(1) = 0.$

Example 2. $\epsilon y''(\chi) + y(\chi)y'(\chi-\delta) - y(\chi) = 0$, along with the interval and boundary conditions $y(\chi) = 1,\ -\delta \leq \chi \leq 0,\ y(1) = 1.$

The sequence of linear equations obtained by applying quasilnearization technique is given by the following recurrence relation

$$\epsilon {y''}^{(k+1)}(\chi) + y^{(k)}(\chi){y'}^{(k+1)}(\chi-\delta) + (-1 + {y'}^{(k)}(\chi-\delta))y^{(k+1)}(\chi) = y^{(k)}(\chi){y'}^{(k)}(\chi-\delta),$$

with the interval and boundary conditions $y^{(k+1)}(\chi) = 1,\ -\delta \leq \chi \leq 0,\ \ y^{(k+1)}(1) = 1.$

5 *Conclusion*

In this paper, higher order fitted finite difference methods EFHOM-1 and EFHOM-2 are presented for singularly perturbed nonlinear differential-difference equation with small negative shift.

The quasilinearization approach is utilized to transform the nonlinear problem into a series of linear problems. Higher order fitted finite difference methods are applied to this sequence of linear problems. It is theoretically proved that for all values of ϵ, the methods presented are uniformly convergent with order 2.

The maximum absolute errors and the numerical rates of convergence for the solutions of the problems 1 and 2 are calculated by the numerical schemes given in Section 3.1 and Section 3.2 and are tabulated in Tables 1–4. The results in Tables 1 and 2 are compared with [3] and [4] respectively and are found to be in good agreement. The impact of a shift on the problem's solution's layer behavior is shown in Figures 1–4 obtained by the numerical method given in Section 3.1 and Section 3.2. The graphical illustrations described in [4] also reveal similar boundary layer behavior to that depicted in the present paper. The log-log plot of maximum point-wise errors for Examples 1 and 2 using the methods EFHOM-1 and EFHOM-2 are plotted in Figures (5-6) respectively. These graphs illustrate the decay of the maximum absolute errors as the number of mesh points increases, thereby demonstrating the convergence of the proposed methods.

The effectiveness of the proposed methods and the implications of the shifts on the solution's layer behavior are both investigated. The present method offers significant advantages for solving the nonlinear singularly perturbed differential-difference equations based on the comprehensive numerical work on both examples.

Table 1

Maximum absolute errors E_N for Example 1 with $\delta = 0.4\epsilon$

$\epsilon \downarrow N \rightarrow$	100	200	300	400	500
EFHOM-1					
10^{-2}	8.8642E-04	3.9927E-04	2.2482E-04	1.4085E-04	9.4522E-05
10^{-3}	8.9469E-04	4.4893E-04	2.9964E-04	2.2486E-04	1.7994E-04
10^{-4}	8.9469E-04	4.4893E-04	2.9964E-04	2.2486E-04	1.7995E-04
10^{-5}	8.9469E-04	4.4893E-04	2.9964E-04	2.2486E-04	1.7995E-04
10^{-6}	8.9469E-04	4.4893E-04	2.9964E-04	2.2486E-04	1.7995E-04
10^{-7}	8.9469E-04	4.4893E-04	2.9964E-04	2.2486E-04	1.7995E-04
10^{-8}	8.9469E-04	4.4893E-04	2.9964E-04	2.2486E-04	1.7995E-04
10^{-9}	8.9469E-04	4.4893E-04	2.9964E-04	2.2486E-04	1.7995E-04
10^{-10}	8.9469E-04	4.4893E-04	2.9964E-04	2.2486E-04	1.7995E-04
E_N	8.9469E-04	4.4893E-04	2.9964E-04	2.2486E-04	1.7995E-04
EFHOM-2					
10^{-2}	9.5217E-04	4.2995E-04	2.4284E-04	1.5255E-04	1.0260E-04
10^{-3}	9.6088E-04	4.8237E-04	3.2201E-04	2.4167E-04	1.9340E-04
10^{-4}	9.6088E-04	4.8237E-04	3.2201E-04	2.4167E-04	1.9341E-04
10^{-5}	9.6088E-04	4.8237E-04	3.2201E-04	2.4167E-04	1.9341E-04
10^{-6}	9.6088E-04	4.8237E-04	3.2201E-04	2.4167E-04	1.9341E-04
10^{-7}	9.6088E-04	4.8237E-04	3.2201E-04	2.4167E-04	1.9341E-04
10^{-8}	9.6088E-04	4.8237E-04	3.2201E-04	2.4167E-04	1.9341E-04
10^{-9}	9.6088E-04	4.8237E-04	3.2201E-04	2.4167E-04	1.9341E-04
10^{-10}	9.6088E-04	4.8237E-04	3.2201E-04	2.4167E-04	1.9341E-04
E_N	9.6088E-04	4.8237E-04	3.2201E-04	2.4167E-04	1.9341E-04
Results in [3] for $\epsilon \in \{10^{-1}, 10^{-2}, \ldots, 10^{-8}\}$					
$N \rightarrow$	64	128	256	512	1024
E_N	5.655E-02	1.725E-02	5.420E-03	1.645E-03	4.817E-04

Table 2

Maximum absolute errors E_N for Example 2 with $\delta = 0.5\epsilon$

$\epsilon \downarrow N \rightarrow$	100	200	300	400	500
EFHOM-1					
2^{-2}	2.4910E-05	6.2125E-06	2.7599E-06	1.5522E-06	9.9333E-07
2^{-4}	8.2732E-05	2.0085E-05	8.8775E-06	4.9840E-06	3.1869E-06
2^{-6}	4.6993E-04	8.3055E-05	3.5223E-05	1.9452E-05	1.2280E-05
2^{-8}	2.1723E-03	6.0716E-04	2.3510E-04	1.1519E-04	6.5517E-05
2^{-10}	2.4798E-03	1.2370E-03	7.9298E-04	5.4352E-04	3.8222E-04
2^{-12}	2.4799E-03	1.2449E-03	8.3108E-04	6.2371E-04	4.9905E-04
2^{-14}	2.4799E-03	1.2449E-03	8.3108E-04	6.2373E-04	4.9919E-04
2^{-16}	2.4799E-03	1.2449E-03	8.3108E-04	6.2373E-04	4.9919E-04
2^{-18}	2.4799E-03	1.2449E-03	8.3108E-04	6.2373E-04	4.9919E-04
2^{-20}	2.4799E-03	1.2449E-03	8.3108E-04	6.2373E-04	4.9919E-04
E_N	2.4799E-03	1.2449E-03	8.3108E-04	6.2373E-04	4.9919E-04
EFHOM-2					
2^{-2}	3.5710E-05	8.9102E-06	3.9587E-06	2.2265E-06	1.4249E-06
2^{-4}	1.2693E-04	3.1031E-05	1.3734E-05	7.7141E-06	4.9337E-06
2^{-6}	6.7116E-04	1.2730E-04	5.4967E-05	3.0569E-05	1.9363E-05
2^{-8}	2.6279E-03	7.8308E-04	3.2323E-04	1.6559E-04	9.6959E-05
2^{-10}	2.9723E-03	1.4836E-03	9.5342E-04	6.5845E-04	4.6894E-04
2^{-12}	2.9724E-03	1.4931E-03	9.9691E-04	7.4823E-04	5.9872E-04
2^{-14}	2.9724E-03	1.4931E-03	9.9691E-04	7.4826E-04	5.9889E-04
2^{-16}	2.9724E-03	1.4931E-03	9.9691E-04	7.4826E-04	5.9889E-04
2^{-18}	2.9724E-03	1.4931E-03	9.9691E-04	7.4826E-04	5.9889E-04
2^{-20}	2.9724E-03	1.4931E-03	9.9691E-04	7.4826E-04	5.9889E-04
E_N	2.9724E-03	1.4931E-03	9.9691E-04	7.4826E-04	5.9889E-04
Results in [4] for $\epsilon \in \{2^{-1}, 2^{-2}, \ldots, 2^{-32}\}$					
$N \rightarrow$	32	64	128	256	512
E_N	4.84E-02	8.33E-03	1.87E-03	4.45E-03	1.09E-04

Table 3

The numerical rate of convergence R_N for the Example 1 for different values of δ with $\epsilon = 0.01$

$\delta \downarrow N \rightarrow$	100	200	300	400	500
EFHOM-1					
10^{-3}	1.9332	2.1167	1.9830	2.0290	2.0017
10^{-4}	2.0592	2.0810	2.0335	2.0081	2.0124
10^{-5}	2.0703	2.0783	2.0325	2.0025	2.0120
10^{-6}	2.0713	2.0781	2.0324	2.0020	2.0120
10^{-7}	2.0715	2.0781	2.0324	2.0019	2.0120
10^{-8}	2.0715	2.0781	2.0324	2.0019	2.0120
10^{-9}	2.0715	2.0781	2.0324	2.0019	2.0120
10^{-10}	2.0715	2.0781	2.0324	2.0019	2.0120
EFHOM-2					
10^{-3}	1.9237	2.1207	1.9919	2.0301	2.0069
10^{-4}	2.0504	2.0835	2.0321	2.0135	2.0129
10^{-5}	2.0616	2.0808	2.0338	2.0077	2.0125
10^{-6}	2.0627	2.0805	2.0337	2.0072	2.0124
10^{-7}	2.0628	2.0805	2.0337	2.0071	2.0124
10^{-8}	2.0628	2.0805	2.0337	2.0071	2.0124
10^{-9}	2.0628	2.0805	2.0337	2.0071	2.0124
10^{-10}	2.0628	2.0805	2.0337	2.0071	2.0124

Table 4

The numerical rate of convergence R_N for the Example 2 for different values of δ with $\epsilon = 0.01$

$\delta \downarrow N \rightarrow$	100	200	300	400	500
EFHOM-1					
10^{-3}	2.2226	2.0594	2.0199	2.0053	2.0015
10^{-4}	2.1767	2.0464	2.0208	2.0118	2.0075
10^{-5}	2.1729	2.0454	2.0204	2.0115	2.0074
10^{-6}	2.1725	2.0453	2.0203	2.0115	2.0073
10^{-7}	2.1725	2.0453	2.0203	2.0115	2.0073
10^{-8}	2.1725	2.0453	2.0203	2.0115	2.0073
10^{-9}	2.1725	2.0453	2.0203	2.0115	2.0073
10^{-10}	2.1725	2.0453	2.0203	2.0115	2.0073
EFHOM-2					
10^{-3}	2.2026	2.0536	2.0172	2.0037	2.0006
10^{-4}	2.1622	2.0424	2.0190	2.0107	2.0069
10^{-5}	2.1589	2.0414	2.0186	2.0105	2.0067
10^{-6}	2.1585	2.0414	2.0185	2.0105	2.0067
10^{-7}	2.1585	2.0413	2.0185	2.0105	2.0067
10^{-8}	2.1585	2.0413	2.0185	2.0105	2.0067
10^{-9}	2.1585	2.0413	2.0185	2.0105	2.0067
10^{-10}	2.1585	2.0413	2.0185	2.0105	2.0067

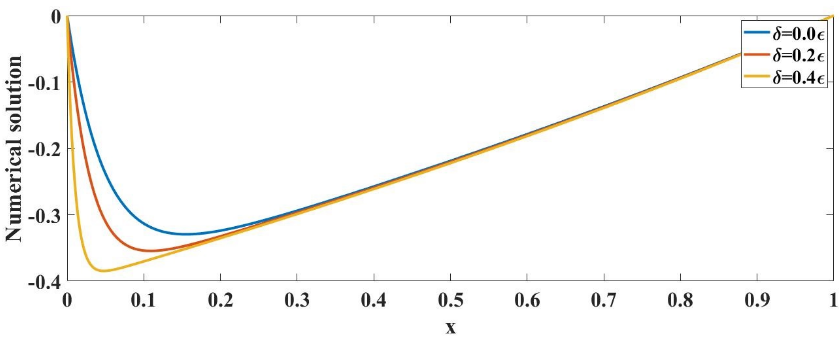

Figure 1. Numerical solution for the given example 1 when $\epsilon = 0.1$ in EFHOM-1 (3.1)

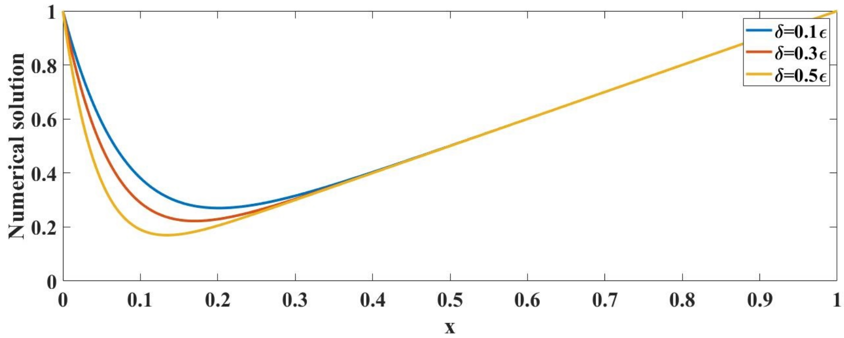

Figure 2. Numerical solution for the given example 2 when $\epsilon = 0.1$ in EFHOM-1 (3.1)

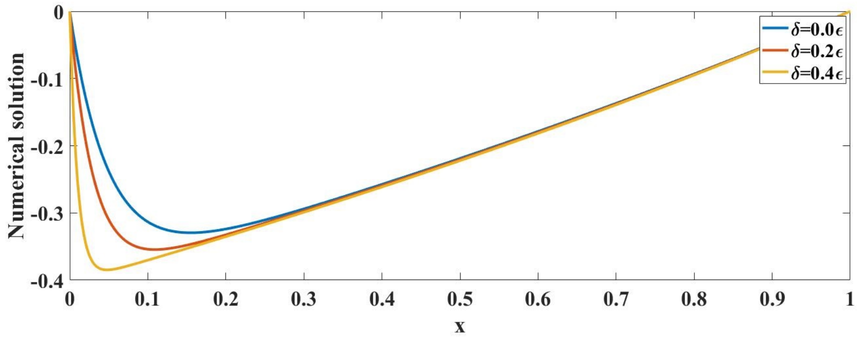

Figure 3. Numerical solution for the given example 1 when $\epsilon = 0.1$ in EFHOM-2 (3.2)

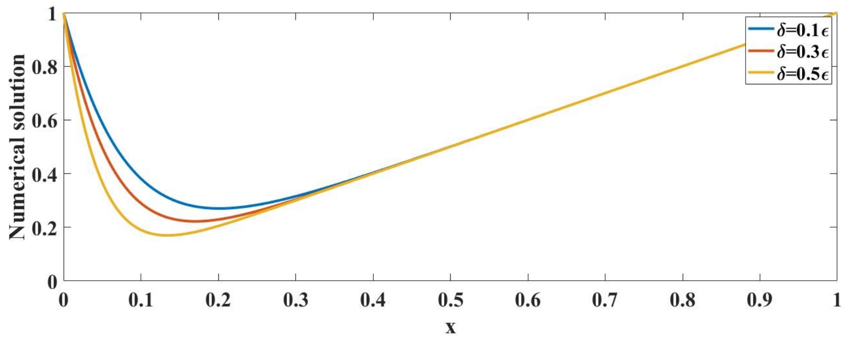

Figure 4. Numerical solution for the given example 2 when $\epsilon = 0.1$ in EFHOM-2 (3.2)

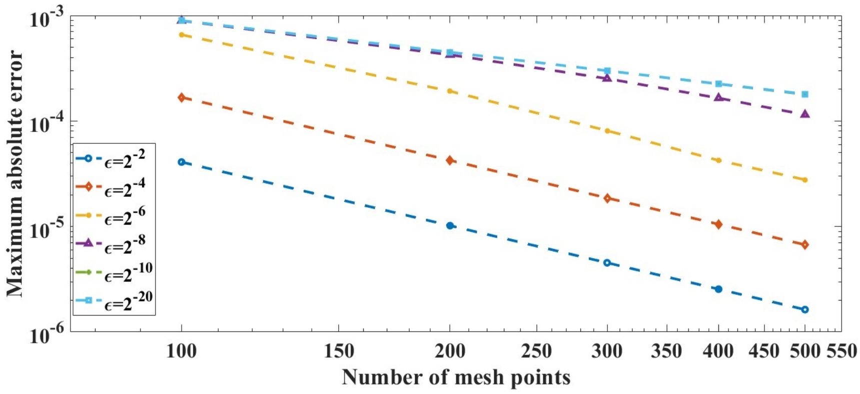

Figure 5. Log-log plot for Example 1 in EFHOM-1 (3.1)

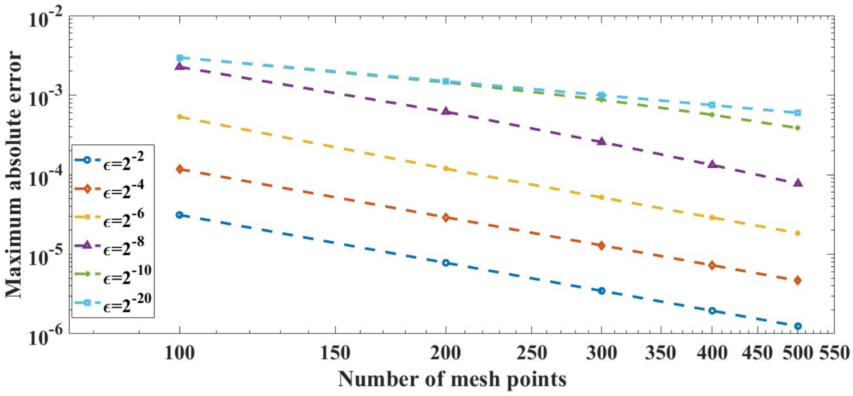

Figure 6. Log-log plot for Example 2 in EFHOM-2 (3.2)

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Authors' contributions

The authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2025M2/208-221 *Research article*

Non-polynomial spline method for singularly perturbed differential difference equations with delay and advance terms

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In this study, we introduced a non-polynomial spline technique to address singularly perturbed differential difference equations involving both delay and advanced parameters. This method exhibits a linear rate of convergence, and we have thoroughly analyzed its convergence properties. To demonstrate the effectiveness of this approach, we provided two numerical examples. We presented the maximum absolute errors in tabular form and displayed the pointwise absolute errors using graphical representations. Additionally, we included tables showing the numerically obtained rate of convergence.

Keywords: singular perturbation problems, numerical methods, delay parameter, advanced parameter, non-polynomial spline.

2020 Mathematics Subject Classification: 65L06, 65L11, 65L20, 65L50.

Introduction

Singularly perturbed problems are a class of differential equations characterized by the presence of a small parameter, which causes the solutions to exhibit rapid changes over small regions of the domain. When these problems incorporate delay (time-lagged terms) and advanced (time-advanced terms) parameters, they form a more complex subclass of differential equations that present unique challenges and opportunities for both theoretical and applied mathematics. The importance of singularly perturbed problems with delay and advanced parameters stems from their ability to model real-world phenomena where current states are influenced by past and future states. They capture dynamics occurring at multiple temporal or spatial scales, providing a more comprehensive understanding of the systems. The inclusion of delay and advanced terms models systems where interactions are not instantaneous but depend on previous or future states, reflecting more realistic scenarios. Also, they allow for the study of stability and transitions in systems. The scope of singularly perturbed problems with delay and advanced parameters is broad and interdisciplinary, encompassing mathematical theory, control theory, biological systems, engineering and so on. The design of robust control algorithms for systems with feedback delays and modeling data flow and congestion where delays are inherent are done in engineering systems. Their applications in biological and medical sciences include epidemiology, neural dynamics etc. These equations play a vital role in economics and finance for market dynamics and supply chains. Climate models and ecological systems can also be studied using these equations. Singularly perturbed problems with delay and advanced parameters are crucial for accurately modeling and analyzing systems where time-dependent interactions play a significant role. Their importance spans various fields, including engineering, biology, economics, and environmental science. The scope of these problems is vast, involving complex mathematical techniques and interdisciplinary applications, making them a significant area of study in both theoretical and applied research. These problems offer a powerful framework for understanding and designing future systems with intricate dynamics

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and time-dependent interactions. Their ability to capture delays, advancements, and rapid transitions makes them invaluable tools for various scientific and technological advancements. Understanding and solving these problems provide deeper insights into the behavior of complex systems, offering robust solutions to real-world challenges.

Numerous researchers have delved into the realm of singularly perturbed small delayed problems of reaction diffusion type. A fourth order exponentially fitted numerical scheme is used in [1] to find the solution of the aforementioned problem. The works in [2] and [3] have also dealt with these problems. Trigonometric cubic B-spline function is utilized in [4] and then the matrix system obtained is solved using Thomas Algorithm. The authors in [5] used non-polynomial spline approach to solve SPDDEs of reaction- diffusion type, whereas a new approach is done in [6] for these type of problems. Furthermore, research has been conducted on convection diffusion type problems as well. Some others concentrated on singularly perturbed large delay differential equations and the systems of these problems. Cubic spline in tension scheme is used in [7] to solve SPDDEs with large delay, whereas the authors in [8] used non-polynomial spline approach to solve them. Convection diffusion type problems under this category is worked in [9] and the researchers in [10] approximated these problems involving integral as end boundary condition by exploring compression technique. SPDDEs of reaction diffusion type with large delay is also addressed by the researchers [11–13]. Non-polynomial spline technique is used [14] to deal with the system of SPDDEs with large delay. The studies in [15] and [16] also concentrated on system of SPDDEs with different techniques. Also, few studies have been done in the area of singularly perturbed delay differential equations involving delay and advanced parameters in the interval $(0, 1)$. They considered the SPDDE with small negative and positive shifts. The authors in [17] used mixed finite difference method via domain decomposition and [18] used numerical integration technique to handle these problems. A meshless approach is explored in [19] to arrive at the numerical approximation of the problem and they have used multiquadric radial basis functions collocation method, which is then coupled with residual subsampling algorithm for support adaptivity. A successive complementary expansion method is used by the researchers in [20]. Many other methods [21–23] have explored SPDDEs with small mixed shifts.

Here, we are considering the problem with large shifts, that is, we consider the following singular perturbation problem involving delay and advanced parameters:

$$\mathcal{H}\kappa \equiv -\xi\kappa''(x) + p(\eta)\kappa'(x) + q(x)\kappa(x) + r(x)\kappa(x-1) + s(x)\kappa(x+1) = t(x), \ x \in \omega = (0, 3), \tag{1}$$

satisfying

$$\begin{aligned} \kappa(x) &= u(x), \quad x \in [-1, 0], \\ \kappa(x) &= v(x), \quad x \in [3, 4], \end{aligned} \tag{2}$$

where $0 < \xi \leq 1$ and u, v are functions defined on $[-1, 0]$ and $[3, 4]$ respectively. It is assumed that $p(x) \geq \bar{p} \geq \bar{\bar{p}} > 0$, $q(x) \geq \bar{q} \geq 0$, $\bar{r} \leq r(x) \leq 0$, $s(x) \geq \bar{s} \geq 0$, $\bar{q} + \bar{r} \geq \bar{\beta} > 0$ and the coefficients are smooth functions on $\bar{\gamma}$.

1 *Analytical Results*

Equations (1) and (2) can be written as:

$$\mathcal{H}\kappa \equiv \mathcal{F}(x),$$

where

$$\mathcal{H}\kappa = \begin{cases} \mathcal{H}_1\kappa(x) = -\xi\kappa''(x) + p(x)\kappa'(x) + q(x)\kappa(x) + s(x)\kappa(x+1), & x \in (0, 1], \\ \mathcal{H}_2\kappa(x) = -\xi\kappa''(x) + p(x)\kappa'(x) + q(x)\kappa(x) + r(x)\kappa(x-1) + s(x)\kappa(x+1), & x \in (1, 2], \\ \mathcal{H}_3\kappa(x) = -\xi\kappa''(x) + p(x)\kappa'(x) + q(x)\kappa(x) + r(x)\kappa(x-1), & x \in (2, 3], \end{cases}$$

$$\mathcal{F}(x) = \begin{cases} t(x) - r(x)\kappa(x-1), & x \in (0,1], \\ t(x), & x \in (1,2], \\ t(x) - s(x)\kappa(x+1), & x \in (2,3], \end{cases}$$

with boundary conditions,

$$\kappa(1^-) = \kappa(1^+),\ \kappa'(1^-) = \kappa'(1^+),\ \kappa(x) = u(x),\ x \in [-1,0],$$
$$\kappa(2^-) = \kappa(2^+),\ \kappa'(2^-) = \kappa'(2^+),\ \kappa(x) = v(x),\ x \in [3,4].$$

Throughout the work, we take $\gamma = (0,3)$, $\bar{\gamma} = [0,3]$, $\gamma_1 = (0,1)$, $\gamma_2 = (1,2)$, $\gamma_3 = (2,3)$, $\gamma^* = \gamma_1 \cup \gamma_2 \cup \gamma_3$ and $\varpi = C^0(\bar{\gamma}) \cap C^1(\gamma) \cap C^2(\gamma^*)$.

Lemma 1. Let $\omega(x)$ be any function in $C^0(\bar{\gamma}) \cap C^1(\gamma) \cap C^2(\gamma^*)$ such that $\omega(0) \geq 0$, $\omega(3) \geq 0$, $\mathcal{H}_1\omega(x) \geq 0$, for all $x \in \gamma_1$, $\mathcal{H}_2\omega(x) \geq 0$, for all $x \in \gamma_2$, $\mathcal{H}_3\omega(x) \geq 0$, for all $x \in \gamma_3$, $[\omega'](1) \leq 0$ and $[\omega'](2) \leq 0$, then $\omega(x) \geq 0$, $\forall x \in \bar{\gamma}$.

Proof. Define the function $a(x)$ as

$$a(x) = \begin{cases} \frac{1+3x}{12}, & x \in [0,1], \\ \frac{1+x}{6}, & x \in [1,2], \\ \frac{4+x}{12}, & x \in [2,3]. \end{cases}$$

It is clear that $a(x)$ is positive for all x in $\bar{\gamma}$. Also note that $\mathcal{H}a(x) > 0$ for all $x \in \gamma^*$; $a(0)$ and $a(3)$ are strictly positive and $[a]'(1)$ and $[a]'(2)$ are less than zero. Consider the set $\left\{\frac{-\omega(x)}{a(x)}\right\}$ and let its maximum be denoted by M in $\bar{\gamma}$. Then $\exists x_0 \in \bar{\gamma}$ satisfying $\omega(x_0) + Ma(x_0) = 0$ and $\omega(x_0) + Ma(x_0) \geq 0$ for all $x \in \bar{\gamma}$. This implies that $(\omega + Ma)(x_0)$ gives minimum value.

If $M < 0$, then $\omega(x) \geq 0$. If $M > 0$, then the function $a(x)$ to be non-negative is not possible. We consider the following cases to prove the same.

Case (a): $x_0 = 0$.

Then,

$$0 < (\omega + Ma)(0) = \omega(0) + Ma(0) = 0,$$

which is a contradiction.

Case (b): $x_0 \in \gamma_1$.

Then,

$$0 < \mathcal{H}_1(\omega + Ma)(x_0) = -\xi(\omega + Ma)''(x_0) + p(x_0)(\omega + Ma)'(x_0) \\ + q(x_0)(\omega + Ma)(x_0) + s(x_0)(\omega + Ma)(x_0 + 1) \leq 0,$$

which is a contradiction.

Case (c): $x_0 = 1$.

Then,

$$0 \leq [(\omega + Ma)'](1) = [\omega'](1) + M[a'](1) < 0,$$

it is a contradiction.

Case (d): $x_0 \in \gamma_2$.

Then,

$$0 < \mathcal{H}_2(\omega + Ma)(x_0) = -\xi(\omega + Ma)''(x_0) + p(x_0)(\omega + Ma)'(x_0) \\ + q(x_0)(\omega + Ma)(x_0) + r(x_0)(\omega + Ma)(x_0 - 1) + s(x)(\omega + Ma)(x_0 + 1) \leq 0,$$

it is again a contradiction.

Case (e): $x_0 = 2$.

Then,

$$0 \le [(\omega + Ma)'](2) = [\omega'](2) + M[a'](2) < 0,$$

which is a contradiction.

Case (f): $x_0 \in \gamma_3$.

Then,

$$0 < \mathcal{H}_3(\omega + Ma)(x_0) = -\xi(\omega + Ma)''(x_0) + p(x_0)(\omega + Ma)'(x_0) \\ + q(x_0)(\omega + Ma)(x_0) + r(x_0)(\omega + Ma)(x_0 - 1) \le 0,$$

which is a contradiction.

Case (g): $x_0 = 3$.

Then,

$$0 \le [(\omega + Ma)](3) = \omega(3) + Ma(3) = 0,$$

which is again a contradiction.

By considering all the above cases, the proof is complete.

Lemma 2. The solution $\kappa(x)$ for the problem (1)–(2) satisfies the bound:

$$\mid \kappa(x) \mid \le \bar{C} \max\{\sup_{r\in\gamma^*} \mid \mathcal{H}\kappa(x) \mid, \mid \kappa(0) \mid, \mid \kappa(3) \mid\}, \quad x \in \bar{\bar{\gamma}}.$$

Proof. Let us define two new functions, denoted by ϑ^+ and ϑ^-. They are defined as:

$$\vartheta^{\pm}(x) = \bar{C}\bar{M}a(x) \pm \kappa(x),\ x \in \bar{\gamma},$$

where $\bar{M}$ denotes the maximum of the set $\{\sup_{r\in\gamma^*} \mid \mathcal{H}\kappa(x) \mid, \mid \kappa(0) \mid, \mid \kappa(3) \mid\}$ and $a(x)$ is the function in the above lemma. By using these newly defined functions within the previously mentioned lemma, we will get the desired result.

Lemma 3. Let $\kappa(x)$ be the solution for the problem (1)–(2). Then

$$\|\kappa^{(k)}(x)\|_{\gamma^*} \le C\xi^{-k},\ \text{for } k = 1, 2, 3, 4.$$

Proof. First, we prove that $\kappa'(x)$ is bounded on ω_1. Consider $\mathcal{H}_1(x)$ and integrate its both sides, then

$$\begin{aligned} -\xi(\kappa'(x) - \kappa'(0)) &= -[p(x)\kappa(x) - p(0)\kappa(0)] + \int_0^x p'(x)\kappa(x)dx - \int_0^x q(x)\kappa(x)dx \\ &\quad - \int_0^x s(x)\kappa(x+1)dx + \int_0^x (t(x) - r(x))u(x-1)dx, \\ \xi\kappa'(0) &= \xi\kappa'(x) - [p(x)\kappa(x) - p(0)\kappa(0)] + \int_0^x p'(x)\kappa(x)dx - \int_0^x q(x)\kappa(x)dx \\ &\quad - \int_0^x s(x)\kappa(x+1)dx + \int_0^x (t(x) - r(x))u(x-1)dx. \end{aligned}$$

Using mean value theorem, we get $\mid \xi\kappa'(x) \mid \le C(\| \kappa(x) \|, \| t(x) \|, \| u \|_{[-1,0]})$, for some x in $(0, \xi)$ and $\mid \kappa'(0) \mid \le C(\| \kappa(x) \| + \| t(x) \| + \| u(x) \|)$. Hence, we have $\mid \kappa'(x) \mid \le C \max\{\| \kappa(x) \|, \| t(x) \|, \| u \|\}$, x in ω_1.

Now consider $\mathcal{H}_2(x)$ and integrate its both sides, then

$$\begin{aligned}
-\xi(\kappa'(x)-\kappa'(0)) &= -[p(x)\kappa(x)-p(0)\kappa(0)]+\int_0^x p'(x)\kappa(x)dx-\int_0^x q(x)\kappa(x)dx\\
&\quad -\int_0^x r(x)\kappa(x-1)dx-\int_0^x s(x)\kappa(x+1)dx+\int_0^x t(x)dx,\\
\xi\kappa'(0) &= \xi\kappa'(x)-[p(x)\kappa(x)-p(0)\kappa(0)]+\int_0^x p'(x)\kappa(x)dx-\int_0^x q(x)\kappa(x)dx\\
&\quad -\int_0^x r(x)\kappa(x-1)dx-\int_0^x s(x)\kappa(x+1)dx+\int_0^x t(x)dx.
\end{aligned}$$

Using mean value theorem, we get $\mid \xi\kappa'(x) \mid \leq C(\| \kappa(x) \|, \| t(x) \|, \| u \|_{[-1,0]})$, for some x in $(0,\xi)$ and $\mid \kappa'(0) \mid \leq C(\| \kappa(x) \| + \| t(x) \| + \| u(x) \|)$. Hence, we have $\mid \kappa'(x) \mid \leq C\max\{\| \kappa(x) \|, \| t(x) \|, \| u \|\}$, x in ω_2.

Now consider $\mathcal{H}_3(x)$ and integrate its both sides, then

$$\begin{aligned}
-\xi(\kappa'(x)-\kappa'(0)) &= -[p(x)\kappa(x)-p(0)\kappa(0)]+\int_0^x p'(x)\kappa(x)dx-\int_0^x q(x)\kappa(x)dx\\
&\quad -\int_0^x r(x)\kappa(x-1)dx+\int_0^x (t(x)-s(x)v(x+1))dx\\
\xi\kappa'(0) &= \xi\kappa'(x)-[p(x)\kappa(x)-p(0)\kappa(0)]+\int_0^x p'(x)\kappa(x)dx-\int_0^x q(x)\kappa(x)dx\\
&\quad -\int_0^x r(x)\kappa(x-1)dx+\int_0^x (t(x)-s(x)v(x+1))dx.
\end{aligned}$$

Using mean value theorem, we get $\mid \xi\kappa'(x) \mid \leq C(\| \kappa(x) \|, \| t(x) \|, \| u \|_{[-1,0]})$, for some x in $(0,\xi)$ and $\mid \kappa'(0) \mid \leq C(\| \kappa(x) \| + \| t(x) \| + \| u(x) \|)$. Hence, we have $\mid \kappa'(x) \mid \leq C\max\{\| \kappa(x) \|, \| t(x) \|, \| u \|\}$, x in ω_3.

Similarly, we can prove for $k=2,3,4$. Hence $\|\kappa^{(k)}(x)\|_{\gamma^*} \leq C\xi^{-k}$, for $k=1,2,3,4$.

2 Non-Polynomial Spline Approach

Divide the interval $[0,3]$ into $3N$ mesh intervals with $x_0=0$, $x_N=1$, $x_{2N}=2$ and $x_{3N}=3$.

The non-polynomial spline [5] for each interval $[x_i,x_{i+1}]$ has the following form:

$$\begin{aligned}
S_{np}(x) = j_i\sin\alpha(x-x_i)+k_i\cos\alpha(x-x_i)+l_i(\exp(\alpha(x-x_i))-\exp(-\alpha(x-x_i)))\\
+m_i(\exp(\alpha(x-x_i))+\exp(-\alpha(x-x_i))),
\end{aligned}$$

where j_i, k_i, l_i, m_i are the unknown quantities and $\alpha \neq 0$ is the parameter which is used to increase the accuracy of the scheme. From (1), we can write as:

$$\xi\kappa''(x) = p(\eta)\kappa'(x)+q(x)\kappa(x)+r(x)\kappa(x-1)+s(x)\kappa(x+1)-t(x).$$

At $x=x_i$, let $\kappa(x_i)=\kappa_i$, $p(x_i)=p_i$, $q(x_i)=q_i$, $s(x_i)=s_i$, $t(x_i)=t_i$. Then equation (3) becomes

$$\xi\kappa_i'' = p_i\kappa_i'+q_i\kappa_i+r_i\kappa(x_i-1)+s_i\kappa(x_i+1)-t_i.$$

Now, we find the unknowns j_i, k_i, l_i, m_i with the help of following conditions:

$$S_{np}(x_i)=\kappa_i,\ S_{np}(x_{i+1})=\kappa_{i+1},\ S''_{np}(x_i)=\mathcal{I}_i,\ S''_{np}(x_{i+1})=\mathcal{I}_{i+1}.$$

Finally, we obtain

$$\begin{array}{ll} j_i = \frac{\kappa_{i+1}\alpha^2 - \mathcal{I}_{i+1} + \cos\beta(\mathcal{I}_i - \alpha^2\kappa_i)}{2\alpha^2\sin\beta}, & k_i = \frac{-\mathcal{I}_i + \alpha^2\kappa_i}{2\alpha^2}, \\ l_i = \frac{2\alpha^2\kappa_{i+1} + 2\mathcal{I}_{i+1} - (e^\beta + e^{-\beta}(\mathcal{I}_i + \alpha^2\kappa_i)}{4\alpha^2(e^\beta - e^{-\beta})}, & m_i = \frac{\mathcal{I}_i + \alpha^2\kappa_i}{4\alpha^2}, \end{array} \tag{3}$$

where β is given by αh. Now, we use the following continuity condition: $S'_{np-1}(x_i) = S'_{np}(x_i)$ to get

$$j_{i-1}\cos\beta + k_{i-1}\sin\beta + l_{i-1}(e^\beta - e^{-\beta}) + m_{i-1}(e^\beta + e^{-\beta}) = j_i + 2l_i.$$

Substituting the values of $j_{i-1}, k_{i-1}, l_{i-1}$ and m_{i-1} using (3) in the above equation, we obtain

$$\begin{aligned} &\left(\frac{-\csc\beta}{2} - \frac{1}{e^\beta - e^{-\beta}}\right)\kappa_{i-1} + \left(\tan\beta + \frac{e^\beta + e^{-\beta}}{e^\beta - e^{-\beta}}\right)\kappa_i \\ &\qquad + \left(\frac{-\csc\beta}{2} - \frac{1}{e^\beta - e^{-\beta}}\right)\kappa_{i+1} = \left(\frac{-h^2\csc\beta}{2\beta^2} - \frac{h^2}{(e^\beta - e^{-\beta})\beta^2}\right)\mathcal{I}_{i-1} \\ &\qquad\qquad + \left(\frac{h^2\tan\beta}{\beta^2} + \frac{h^2(e^\beta + e^{-\beta})}{\beta^2(e^\beta - e^{-\beta})}\right)\mathcal{I}_i + \left(\frac{-h^2\csc\beta}{2\beta^2} - \frac{h^2}{(e^\beta - e^{-\beta})\beta^2}\right)\mathcal{I}_{i+1}. \end{aligned} \tag{4}$$

Equation (4) can be finally written as:

$$\kappa_{i-1} + \theta_1\kappa_i + \kappa_{i+1} = h^2(\theta_2\mathcal{I}_{i-1} + \theta_3\mathcal{I}_i + \theta_2\mathcal{I}_{i+1}), \tag{5}$$

where

$$\begin{aligned} \theta_1 &= \frac{-2e^\beta(\cos\beta + \sin\beta) - 2e^{-\beta}(\sin\beta - \cos\beta)}{e^\beta - e^{-\beta} + 2\sin(\beta)}, \\ \theta_2 &= \frac{e^\beta - e^{-\beta} - 2\sin\beta}{\beta^2(e^\beta - e^{-\beta} + 2\sin\beta)}, \\ \theta_3 &= \frac{-2e^\beta(\cos\beta - \sin\beta) + 2e^{-\beta}(\sin\beta + \cos\beta)}{\beta^2(e^\beta - e^{-\beta} + 2\sin(\beta))}. \end{aligned}$$

Note that β tends to zero as h approaches zero. As a result, θ_1 tends to -2, θ_2 tends to $\frac{1}{6}$, θ_3 tends to $\frac{2}{3}$. Also, we write $\mathcal{I}_{i-1}$, $\mathcal{I}_i$ and $\mathcal{I}_{i+1}$ as follows:

$$\begin{aligned} \mathcal{I}_{i-1} &= \mathcal{I}(x_{i-1}) = \tfrac{1}{\xi}\left(p_{i-1}\kappa'_{i-1} + q_{i-1}\kappa_{i-1} + r_{i-1}\kappa(x_{i-1} - 1) + s_{i-1}\kappa(x_{i-1} + 1) - t_{i-1}\right), \\ \mathcal{I}_i &= \mathcal{I}(x_i) = \tfrac{1}{\xi}\left(p_i\kappa'_i + q_i\kappa_i + r_i\kappa(x_i - 1) + s_i\kappa(x_i + 1) - t_i\right), \\ \mathcal{I}_{i+1} &= \mathcal{I}(x_{i+1}) = \tfrac{1}{\xi}\left(p_{i+1}\kappa'_{i+1} + q_{i+1}\kappa_{i+1} + r_{i+1}\kappa(x_{i+1} - 1) + s_{i+1}\kappa(x_{i+1} + 1) - t_{i+1}\right). \end{aligned} \tag{6}$$

We make use of equations in (6) in equation (5) to obtain:

$$\begin{aligned} \xi\left(\kappa_{i-1} + \theta_1\kappa_i + \kappa_{i+1}\right) = h^2 \big[&\theta_2\left(p_{i-1}\kappa'_{i-1} + q_{i-1}\kappa_{i-1} + r_{i-1}\kappa(x_{i-1} - 1) + s_{i-1}\kappa(x_{i-1} + 1) - t_{i-1}\right) \\ &+\theta_3\left(p_i\kappa'_i + q_i\kappa_i + r_i\kappa(x_i - 1) + s_i\kappa(x_i + 1) - t_i\right) \\ &+\theta_2\left(p_{i+1}\kappa'_{i+1} + q_{i+1}\kappa_{i+1} + r_{i+1}\kappa(x_{i+1} - 1) + s_{i+1}\kappa(x_{i+1} + 1) - t_{i+1}\right)\big]. \end{aligned}$$

We make use of the following Taylor series approximations for κ'_i, κ'_{i-1} and κ'_{i+1}:

$$\kappa'_i \simeq \frac{\kappa_{i+1} - \kappa_{i-1}}{2h}, \quad \kappa'_{i-1} \simeq \frac{-\kappa_{i+1} + 4\kappa_i - 3\kappa_{i-1}}{2h}, \quad \kappa'_{i+1} \simeq \frac{3\kappa_{i+1} - 4\kappa_i + \kappa_{i-1}}{2h}.$$

Finally, we arrive at the following system of equations:

$$\left(\xi + \frac{3h\theta_2 p_{i-1}}{2} - h^2\theta_2 q_{i-1} + \frac{h\theta_3 p_i}{2} - \frac{h\theta_3 p_{i+1}}{2}\right)\kappa_{i-1} + \left(\xi\alpha - 2h\theta_2 p_{i-1} - h^2\theta_3 q_i + 2h\theta_2 p_{i+1}\right)\kappa_i$$
$$+\left(\xi + \frac{h\theta_2 p_{i-1}}{2} - h^2\theta_2 q_{i+1} - \frac{h\theta_3 p_i}{2} - \frac{3h\theta_2 p_{i+1}}{2}\right)\kappa_{i+1} + \left(-h^2\theta_2 r_{i-1}\right)\kappa_{i-1-N} + \left(-h^2\theta_3 r_i\right)\kappa_{i-N}$$
$$+\left(-h^2\theta_2 r_{i+1}\right)\kappa_{i+1-N} + \left(-h^2\theta_2 s_{i-1}\right)\kappa_{i-1+N} + \left(-h^2\theta_3 s_i\right)\kappa_{i+N} + \left(-h^2\theta_2 s_{i+1}\right)\kappa_{i+1+N}$$
$$= -h^2\left(\theta_2 t_{i-1} + \theta_3 t_i + \theta_2 t_{i+1}\right), \ i = 1(1)3N-1.$$

We may assume that the coefficients are locally constant because we are examining the differential equations on suitably small subintervals. By applying the Taylor series expansion for $p(x), q(x)$ about the point $x = 0$ by restricting to their first terms and from the theory of singular perturbation, an approximation for the solution of the homogeneous problem (1) is of the form

$$\kappa(x) = \kappa_0(x) + \frac{p(0)}{p(x)}\left(\beta - \kappa_0(0)\right)e^{-\left(\frac{p(0)}{\xi} - \frac{q(0)}{p(0)}\right)x} + O(\xi), \tag{7}$$

where $\kappa_0(x)$ is the solution of the reduced problem. Taking the limit as $h \to 0$ in (7) and putting $\rho = \frac{h}{\xi}$, we get

$$\begin{aligned}
\lim_{h\to 0}\kappa(ih - h) &= \kappa_0(0) + \left(\beta - \kappa_0(0)\right)e^{-(\delta(i-1)\rho)},\\
\lim_{h\to 0}\kappa(ih) &= \kappa_0(0) + \left(\beta - \kappa_0(0)\right)e^{-(\delta i\rho)},\\
\lim_{h\to 0}\kappa(ih + h) &= \kappa_0(0) + \left(\beta - \kappa_0(0)\right)e^{-(\delta(i+1)\rho)}.
\end{aligned}$$

Substituting these limiting values in the obtained scheme, we get the fitting parameter $\sigma = \frac{\rho p_i}{2}\coth(\frac{\rho\delta}{2})$, where $\rho = \frac{h}{\xi}$ and $\delta = \frac{p(0)^2 - q(0)\xi}{p(0)}$. Incorporating this fitting factor in the above scheme, we obtain:

$$\bar{E}_i\kappa_{i-1} + \bar{F}_i\kappa_i + \bar{G}_i\kappa_{i+1} + \bar{A}_i\kappa_{i-1-N} + \bar{B}_i\kappa_{i-N} + \bar{C}_i\kappa_{i+1-N} + \bar{H}_i\kappa_{i-1+N} + \bar{I}_i\kappa_{i+N} + \bar{J}_i\kappa_{i+1+N} = \bar{D}_i, \tag{8}$$

where

$$\begin{aligned}
\bar{E}_i &= \xi\sigma + \frac{3h\theta_2 p_{i-1}}{2} - h^2\theta_2 q_{i-1} + \frac{h\theta_3 p_i}{2} - \frac{h\theta_2 p_{i+1}}{2},\\
\bar{F}_i &= \xi\theta_1\sigma - 2h\theta_2 p_{i-1} - h^2\theta_3 q_i + 2h\theta_2 p_{i+1},\\
\bar{G}_i &= \xi\sigma + \frac{h\theta_2 p_{i-1}}{2} - h^2\theta_2 q_{i+1} - \frac{h\theta_3 p_i}{2} - \frac{3h\theta_2 p_{i+1}}{2},\\
\bar{D}_i &= -h^2\left(\theta_2 t_{i-1} + \theta_3 t_i + \theta_2 t_{i+1}\right),
\end{aligned}$$

$$\begin{array}{lllllllll}
\bar{A}_i &=& -h^2\theta_2 r_{i-1}, & \bar{B}_i &=& -h^2\theta_3 r_i, & \bar{C}_i &=& -h^2\theta_2 r_{i+1},\\
\bar{H}_i &=& -h^2\theta_2 s_{i-1}, & \bar{I}_i &=& -h^2\theta_3 s_i, & \bar{J}_i &=& -h^2\theta_2 s_{i+1}.
\end{array}$$

3 Convergence Analysis

The scheme in (8) can be written in matrix form as:

$$T^*U = V, \tag{9}$$

where $T^* = (t_{ij})$ is a $(3N - 1 \times 3N - 1)$ matrix.

Truncation error obtained is $W_i(h) = \frac{h^3}{24} X^* + O(h^4)$, where X^* is given by $\left[\frac{p_{i-1}q_i\kappa_i^{(3)}}{3} + \frac{p_{i+1}\kappa_i^{(3)}}{3} - \frac{10p_i\kappa_i^{(3)}}{6} + \frac{hp_ip_{i+1}\kappa_i^{(3)}}{6} + \frac{hp_ip_{i-1}\kappa_i^{(3)}}{6}\right]$. $T^*U = V$ can also be written as:

$$T^*\overline{U} - W(h) = V, \tag{10}$$

in error form, where $\overline{U} = \begin{pmatrix}\overline{u}_1 & \overline{u}_2 & . & . & . & \overline{u}_{3N-1}\end{pmatrix}^t$ and $W(h) = \begin{pmatrix}w_1 & w_2 & . & . & . & w_{3N-1}\end{pmatrix}^t$ denote the exact solution and truncation error respectively. Let

$$\overline{E}_r = \overline{U} - U = \begin{pmatrix}\bar{e}_1 & \bar{e}_2 & . & . & . & \bar{e}_{3N-1}\end{pmatrix}^t$$

denote the error obtained by numerical approximation. Equations (9) and (10) give

$$T^*\overline{E}_r = W(h). \tag{11}$$

Let $\bar{S}_i$ denote the $i-$th row sum of the matrix T^*. Then

$$\begin{aligned}
\bar{S}_1 &= \bar{F}_i + \bar{G}_i + \bar{H}_i + \bar{I}_i + \bar{J}_i \\
&= -\xi\sigma - \frac{h}{2}\left(3\theta_2 p_{i-1} + \theta_3 p_i - \theta_2 p_{i+1}\right) - h^2\left(\theta_3 s_i + \theta_2 s_{i+1}\right), \\
\bar{S}_i &= \bar{E}_i + \bar{F}_i + \bar{G}_i + \bar{H}_i + \bar{I}_i + \bar{J}_i \\
&= -h^2\left(\theta_2 s_{i-1} + \theta_3 s_i + \theta_2 s_{i+1}\right), \ i = 2(1)N-1, \\
\bar{S}_N &= \bar{E}_i + \bar{F}_i + \bar{G}_i + \bar{C}_i + \bar{H}_i + \bar{I}_i + \bar{J}_i \\
&= -h^2\left(\theta_2 r_{i+1} + \theta_2 s_{i-1} + \theta_3 s_i + \theta_2 s_{i+1}\right), \\
\bar{S}_{N+1} &= \bar{E}_i + \bar{F}_i + \bar{G}_i + \bar{B}_i + \bar{C}_i + \bar{H}_i + \bar{I}_i + \bar{J}_i \\
&= -h^2\left(\theta_3 r_i + \theta_2 r_{i+1} + \theta_2 s_{i-1} + \theta_3 s_i + \theta_2 s_{i+1}\right), \\
\bar{S}_i &= \bar{E}_i + \bar{F}_i + \bar{G}_i + \bar{A}_i + \bar{B}_i + \bar{C}_i + \bar{H}_i + \bar{I}_i + \bar{J}_i \\
&= -h^2\left(\theta_2 r_{i-1} + \theta_3 r_i + \theta_2 r_{i+1} + \theta_2 s_{i-1} + \theta_3 s_i + \theta_2 s_{i+1}\right), \\
\bar{S}_{2N-1} &= \bar{E}_i + \bar{F}_i + \bar{G}_i + \bar{A}_i + \bar{B}_i + \bar{C}_i + \bar{H}_i + \bar{I}_i \\
&= -h^2\left(\theta_2 r_{i-1} + \theta_3 r_i + \theta_2 r_{i+1} + \theta_2 s_{i-1} + \theta_3 s_i\right), \\
\bar{S}_{2N} &= \bar{E}_i + \bar{F}_i + \bar{G}_i + \bar{A}_i + \bar{B}_i + \bar{C}_i + \bar{H}_i \\
&= -h^2\left(\theta_2 r_{i-1} + \theta_3 r_i + \theta_2 r_{i+1} + \theta_2 s_{i-1}\right), \\
\bar{S}_i &= \bar{E}_i + \bar{F}_i + \bar{G}_i + \bar{A}_i + \bar{B}_i + \bar{C}_i \\
&= -h^2\left(\theta_2 r_{i-1} + \theta_3 r_i + \theta_2 r_{i+1}\right), \ i = 2N+1(1)3N-1.
\end{aligned}$$

As we make the calculations progressively more detailed (by letting the mesh size approach zero), a key property of the matrix T^* emerges. It becomes both monotone and irreducible. These properties together guarantee the existence of the inverse of the matrix, with its elements greater than zero. Then

$$\overline{E}_r \ = \ T^{*-1}W(h), \tag{12}$$

which implies

$$\|\overline{E}_r\| \ \leq \ \|T^{*-1}\|\|W(h)\|. \tag{13}$$

If (j,i)-th element of the inverse matrix of T^* is denoted by $\bar{t}_{j,i}$, then, for $j = 1(1)3N-1$,

$$\sum_{i=1}^{3N-1} \bar{t}_{j,i}\bar{S}_i = 1.$$

So,

$$\sum_{i=1}^{3N-1} \bar{t}_{j,i} \leq \frac{1}{\min_{1\leq i\leq 3N-1} \bar{S}_i} \leq \frac{1}{h^2 \mid \bar{B}_i \mid}. \tag{14}$$

From equations (11), (12), (13) and (14), we have

$$\bar{e}_i = \sum_{j=1}^{3N-1} \bar{t}_{j,i} w_i(h), \quad i = 1(1)3N-1,$$

which implies

$$\begin{aligned} \bar{e}_i &\leq \left(\sum_{j=1}^{3N-1} \bar{v}_{j,i}\right) \max_{1\leq i\leq 3N-1} \mid w_i(h) \mid \\ &\leq \frac{1}{h^2 \mid \bar{B}_{i^*} \mid} \times \frac{h^3 X^*}{2} = O(h), \end{aligned}$$

where X^* is a constant which is independent on h. So $\|\overline{E}_r\| = O(h)$. Hence, our technique is of linear rate of convergence.

4 Numerical Experiments

The schemes (8) give a system of $(3N-1)$ equations in $(3N+1)$ variables, which can be reduced to the system of $(3N-1)$ equations in $(3N-1)$ variables as follows.

$$\begin{aligned} \bar{F}_i\kappa_i + \bar{G}_i\kappa_{i+1} + \bar{H}_i\kappa_{i-1+N} + \bar{I}_i\kappa_{i+N} + \bar{J}_i\kappa_{i+1+N} = \bar{D}_i - \bar{A}_i u_{i-1-N} - \bar{B}_i u_{i-N} \\ - \bar{C}_i u_{i+1-N} - \bar{E}_i u_{i-1}, \text{ for } i = 1, \end{aligned}$$

$$\begin{aligned} \bar{E}_i\kappa_{i-1} + \bar{F}_i\kappa_i + \bar{G}_i\kappa_{i+1} + \bar{H}_i\kappa_{i-1+N} + \bar{I}_i\kappa_{i+N} + \bar{J}_i\kappa_{i+1+N} = \bar{D}_i - \bar{A}_i u_{i-1-N} - \bar{B}_i u_{i-N} \\ - \bar{C}_i u_{i+1-N}, \text{ for } 2 \leq i \leq N-1, \end{aligned}$$

$$\bar{E}_i\kappa_{i-1} + \bar{F}_i\kappa_i + \bar{G}_i\kappa_{i+1} + \bar{C}_i\kappa_1 + \bar{H}_i\kappa_{2N-1} + \bar{I}_i\kappa_{2N} + \bar{J}_i\kappa_{2N+1} = \bar{D}_i - \bar{A}_i u_{-1} - \bar{B}_i u_0, \text{ for } i = N,$$

$$\bar{E}_i\kappa_{i-1} + \bar{F}_i\kappa_i + \bar{G}_i\kappa_{i+1} + \bar{B}_i\kappa_1 + \bar{C}_i\kappa_2 + \bar{H}_i\kappa_{2N} + \bar{I}_i\kappa_{2N+1} + \bar{J}_i\kappa_{2N+2} = \bar{D}_i - \bar{A}_i u_0, \text{ for } i = N+1,$$

$$\begin{aligned} \bar{E}_i\kappa_{i-1} + \bar{F}_i\kappa_i + \bar{G}_i\kappa_{i+1} + \bar{A}_i\kappa_{i-1-N} + \bar{B}_i\kappa_{i-N} + \bar{C}_i\kappa_{i+1-N} \\ + \bar{H}_i\kappa_{i-1+N} + \bar{I}_i\kappa_{i+N} + \bar{J}_i\kappa_{i+1+N} = \bar{D}_i, \text{ for } N+2 \leq i \leq 2N-2, \end{aligned}$$

$$\begin{aligned} \bar{E}_i\kappa_{i-1} + \bar{F}_i\kappa_i + \bar{G}_i\kappa_{i+1} + \bar{A}_i\kappa_{i-1-N} + \bar{B}_i\kappa_{i-N} + \bar{C}_i\kappa_{i+1-N} \\ + \bar{H}_i\kappa_{i-1+N} + \bar{I}_i\kappa_{i+N} = \bar{D}_i - \bar{J}_i v_{3N}, \text{ for } i = 2N-1, \end{aligned}$$

$$\begin{aligned} \bar{E}_i\kappa_{i-1} + \bar{F}_i\kappa_i + \bar{G}_i\kappa_{i+1} + \bar{A}_i\kappa_{i-1-N} + \bar{B}_i\kappa_{i-N} + \bar{C}_i\kappa_{i+1-N} \\ + \bar{H}_i\kappa_{i-1+N} = \bar{D}_i - \bar{I}_i v_{3N} - \bar{J}_i v_{3N+1}, \text{ for } i = 2N, \end{aligned}$$

$$\bar{E}_i\kappa_{i-1} + \bar{F}_i\kappa_i + \bar{G}_i\kappa_{i+1} + \bar{A}_i\kappa_{i-1-N} + \bar{B}_i\kappa_{i-N} + \bar{C}_i\kappa_{i+1-N} = \bar{D}_i - \bar{H}_i v_{i-1+N} - \bar{I}_i v_{i+N} - \bar{J}_i v_{i+1+N}, \text{ for } i = 2N+1 \le i \le 3N-1.$$

We make use of the function $v(x)$ in the interval $[3,4]$. This is then solved using Gauss elimination method with the help of MATLAB R2022a mathematical software.

We tested our proposed method through two numerical experiments and confirmed it works well. The solution is presented in tables for various levels of perturbation values by changing the level of mesh size in the simulations. A technique called the double-mesh principle is used on these examples to find the maximum absolute errors.

$$E^N = \max_i \mid \kappa_i^N - \kappa_{2i}^{2N} \mid .$$

Rate of Convergence

We employ the double mesh principle to determine the convergence rate, denoted by $\bar{\rho}$ and defined as:

$$\bar{\rho} = \frac{\log(E_h/E_{h/2})}{\log 2}.$$

Example 1. To demonstrate the capabilities of this method, we analyze the following problem:

$$-\xi\kappa''(x) + 5\kappa'(x) + 2\kappa(x) - \kappa(x-1) + \kappa(x+1) = 1, \ x \in \gamma = (0,3),$$

satisfying

$$\begin{aligned} \kappa(x) &= 1, \ x \in [-1,0], \\ \kappa(x) &= 2, \ x \in [3,4]. \end{aligned}$$

This section delves deeper into the results for the first example. Table 1 shows how the solution's accuracy (maximum absolute error) changes as a key parameter (the perturbation parameter denoted by ξ) is varied across different scales (from 10^{-3} to 10^{-10}). The table also considered solutions obtained using different mesh sizes (N values ranging from 16 to 512). For reference, Figure 1(a) visualizes the solution when the perturbation parameter is set to 10^{-4} (one specific data point from Table 1) and Figure 2(a) depicts the point-wise absolute error of this example for varying N values. Going a step further, Table 2 calculates the rate of convergence for this example. By analyzing both Table 2 and the theoretical convergence analysis, we can see that the proposed method converges at a linear rate. In simpler terms, the error reduces proportionally as the mesh size is refined.

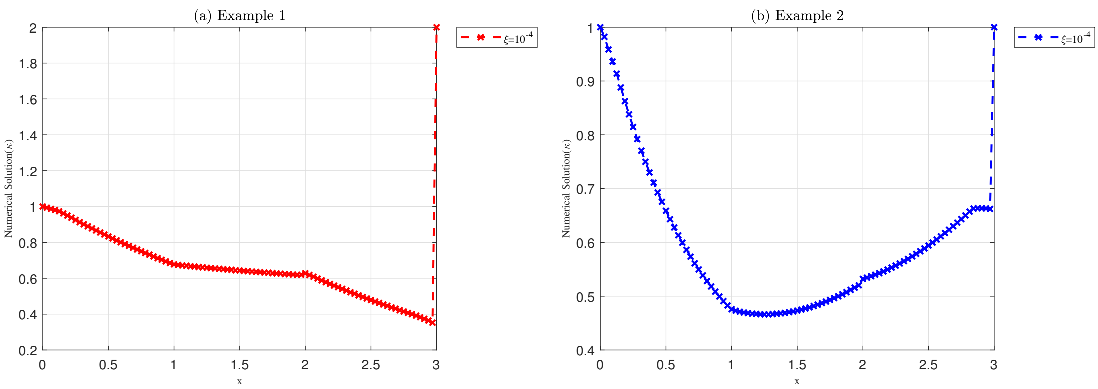

Figure 1. The numerical solution of Example 1 and Example 2 for $\xi = 10^{-4}$

Table 1

The maximum absolute error of Example 1 for different values of ξ

ξ	N					
	16	32	64	128	256	512
10^{-3}	2.1481e-02	1.1610e-02	6.0264e-03	3.0690e-03	1.5485e-03	7.7920e-04
10^{-4}	2.1481e-02	1.1610e-02	6.0264e-03	3.0690e-03	1.5485e-03	7.7775e-04
10^{-5}	2.1481e-02	1.1610e-02	6.0264e-03	3.0690e-03	1.5485e-03	7.7775e-04
10^{-6}	2.1481e-02	1.1610e-02	6.0264e-03	3.0690e-03	1.5485e-03	7.7775e-04
10^{-7}	2.1481e-02	1.1610e-02	6.0264e-03	3.0690e-03	1.5485e-03	7.7775e-04
10^{-8}	2.1481e-02	1.1610e-02	6.0264e-03	3.0690e-03	1.5485e-03	7.7775e-04
10^{-9}	2.1481e-02	1.1610e-02	6.0264e-03	3.0690e-03	1.5485e-03	7.7775e-04
10^{-10}	2.1481e-02	1.1610e-02	6.0264e-03	3.0690e-03	1.5485e-03	7.7775e-04
E^N	2.1481e-02	1.1610e-02	6.0264e-03	3.0690e-03	1.5485e-03	7.7775e-04

Table 2

Rate of convergence of Example 1 for $\xi = 10^{-4}$

	h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
	$1/16$	$1/32$	2.1481e-02	$1/64$	1.1610e-02	0.8877
	$1/32$	$1/64$	1.1610e-02	$1/128$	6.0264e-03	0.9460
	$1/64$	$1/128$	6.0264e-03	$1/256$	3.0690e-03	0.9735

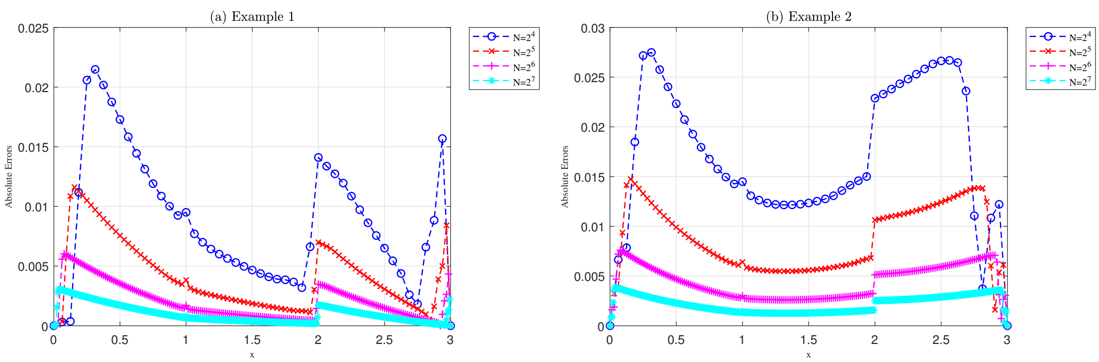

Figure 2. The point-wise absolute errors of Example 1 and Example 2 for different values of N

Example 2. To demonstrate the capabilities of this method, we analyze the following problem:

$$-\xi\kappa''(x) + (x+5)\kappa'(x) + 2\kappa(x) - \kappa(x-1) + x\kappa(x+1) = x, x \in \gamma = (0,3),$$

satisfying

$$\begin{aligned} \kappa(x) &= 1, \ x \in [-1,0], \\ \kappa(x) &= 1, \ x \in [3,4]. \end{aligned}$$

This section delves deeper into the results for the first example. Table 3 shows how the solution's accuracy (maximum absolute error) changes as a key parameter (the perturbation parameter denoted by ξ) is varied across different scales (from 10^{-3} to 10^{-10}). The table also considered solutions obtained using different mesh sizes (N values ranging from 16 to 512). For reference, Figure 1(b) visualizes the solution when the perturbation parameter is set to 10^{-4} (one specific data point from Table 3) and Figure 2(b) depicts the point-wise absolute error of this example for varying N values. Going a step further, Table 4 calculates the rate of convergence for this example. By analyzing both Table 4 and the theoretical convergence analysis, we can see that the proposed method converges at a linear rate. In simpler terms, the error reduces proportionally as the mesh size is refined.

Table 3

The maximum absolute error of Example 2 for different values of ξ

ξ	N					
	16	32	64	128	256	512
10^{-3}	2.7489e-02	1.4740e-02	7.6052e-03	3.8568e-03	1.9394e-03	2.1622e-03
10^{-4}	2.7489e-02	1.4740e-02	7.6052e-03	3.8568e-03	1.9411e-03	9.7360e-03
10^{-5}	2.7489e-02	1.4740e-02	7.6052e-03	3.8568e-03	1.9411e-03	9.7360e-03
10^{-6}	2.7489e-02	1.4740e-02	7.6052e-03	3.8568e-03	1.9411e-03	9.7360e-03
10^{-7}	2.7489e-02	1.4740e-02	7.6052e-03	3.8568e-03	1.9411e-03	9.7360e-03
10^{-8}	2.7489e-02	1.4740e-02	7.6052e-03	3.8568e-03	1.9411e-03	9.7360e-03
10^{-9}	2.7489e-02	1.4740e-02	7.6052e-03	3.8568e-03	1.9411e-03	9.7360e-03
10^{-10}	2.7489e-02	1.4740e-02	7.6052e-03	3.8568e-03	1.9411e-03	9.7360e-03
E^N	2.7489e-02	1.4740e-02	7.6052e-03	3.8568e-03	1.9411e-03	9.7360e-03

Table 4

Rate of convergence of Example 2 for $\xi = 10^{-4}$

	h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
	1/16	1/32	2.7489e-02	1/64	1.4740e-02	0.8992
	1/32	1/64	1.4740e-02	1/128	7.6052e-03	0.9546
	1/64	1/128	7.6052e-03	1/256	3.8568e-03	0.9796

Conclusion

This study tackles a specific class of differential equations: singularly perturbed differential difference equations with mixed shifts. These equations involve delays (where the solution depends on a past state) and advances (where it depends on a future state). The authors propose a solution method using non-polynomial splines. This approach is unique because it incorporates a fitting factor, that can be adjusted within different intervals of the problem. To demonstrate the effectiveness of this method, two examples are presented. For each example, corresponding figures (Figure 1 and Figure 2) visualize the numerical solution alongside the point-wise absolute errors. Additionally, tables (Table 1 and Table 3) summarize the maximum absolute errors for each example. These results all point towards convergence of the numerical solution to the true solution. In other words, as the mesh size in the calculations is reduced, the error gets smaller. Interestingly, both theoretical analysis and the numerical results suggest a convergence rate of one (see Tables 2 and 4), which is a desirable property in numerical methods.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2025M2/222-240 *Research article*

Variational method of numerical solution of the inverse problem of gas lift oil production process

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This paper proposes a constructive method for numerically solving direct and inverse problems arising in the gas-lift oil production process, which is described by a hyperbolic system of differential equations. To solve the direct problem, a second-order difference scheme is used, which ensures stability and accuracy of calculations in the space-time domain. The inverse problem is formulated as an optimal control problem, where the minimization of the objective functional is carried out using the gradient method. The calculation of the gradient of the objective function is based on the constructed adjoint problem using the Lagrange identity and the duality principle, which guarantees the mathematical rigor of the approach. Numerical experiments confirmed the efficiency of the proposed method for solving the inverse problem and optimizing the input parameters of the gas lift process. The adjoint problem contains valuable information about the solution of the direct problem, so the gradients of the functional are equal to the solution of the adjoint problem and its first derivative with respect to time at $t = 0$. Numerical calculations show that the values of the minimized functional decrease monotonically and remain bounded below. This means that the used iterative method converges. Additional conditions set at $T = 0$ for the direct problem are used to formulate the condition of the adjoint problem. The developed algorithm contributes to the development of the numerical implementation of the adjoint optimization method of the inverse problem for a hyperbolic equation. The problem of the type under study is of great practical importance and can be used to calculate the intensity of the gas lift process of oil production.

Keywords: gas lift process of oil production, hyperbolic equation, conjugate equation, inverse problem, optimal control, gradient method, numerical methods.

2020 Mathematics Subject Classification: 49N45.

Introduction

The gas lift process is an oil production technology in which gas is injected into the annular space of a well to reduce the density of the gas-liquid mixture (GLM), facilitating its rise to the surface.

The model proposed in [1–3] describes the dynamics of gas and gas-liquid mixtures in the designated regions using a system of hyperbolic partial differential equations. These equations take into account key parameters such as pressure P and gas volume flow rate Q, which significantly affect the transport of the mixture.

The method of lines allows to reduce a hyperbolic system of partial differential equations to a system of ordinary differential equations. Subsequently, the optimal control problem is formulated, the goal of which is to increase the volume of oil production with minimal gas costs. The control parameters in this case are the pressure or volume of the injected gas.

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In [4], a mathematical method for optimizing oil recovery from gas-lift wells is presented. The main objective is to determine the economically feasible level of production and reduce operating costs by minimizing gas consumption. Based on the collected well performance data using the PIPESIM application, performance curves were modeled. A multi-criteria model was then developed for optimization.

In the study [5], a gas lift distribution method using genetic algorithms was studied to enhance oil recovery.

Modern methods for solving inverse problems in mathematical physics often involve formulating them as optimal control problems. In particular, such problems are formulated to control the behavior of objects described by partial differential equations. The goal is to transfer the system from one state to another by influencing its parameters. Such problems were first considered by J.L. Lions [6, 7].

The right side of the equation and the boundary conditions can act as control. The methods of adjoint equations are described in detail in the studies of V.I. Agoshkov [8].

Boundary control is a separate class of problems in which control is implemented through boundary conditions. In the works of V.A. Il'in and E.I. Moiseev [9], boundary control problems for string vibration equations were investigated, where control functions were constructed to transfer the system from the initial state to the final state in a certain time.

G.I. Marchuk [10] gave a definition of adjoint operators and equations, and described their application in mathematical modeling. Studies of linear operators in Hilbert spaces are widely covered in various monographs.

The method of fictitious regions for optimization problems was proposed in [11]. It allows constructing difference schemes in an extended region, which leads to the convergence of the solution to the desired one. Minimization of the Lagrange functional was carried out using the conjugate gradient method.

In [12], the Cauchy problem for the Helmholtz equation was investigated. The main attention was paid to reducing the problem to a boundary inverse problem. The proposed method was based on optimization using the Landweber and Nesterov methods.

In the article by A.V. Arguchintsev and V.P. Poplevko [13], the problem of optimal control of a system of semilinear hyperbolic equations with boundary conditions specified through ordinary differential equations with delay was considered.

The work [14] considers the development of methods for solving optimal control problems in the class of smooth control actions, taking into account the constraints characteristic of inverse problems of mathematical physics.

In the work of N.M. Temirbekov [15], the Bublov-Galerkin projection method with bases in the form of Legendre wavelets was used to solve the Fredholm integral equation of the first kind. In the Galerkin method, Legendre wavelets are used as basis functions and a system of linear algebraic equations is obtained to determine the expansion coefficients. This system of linear algebraic equations is solved by the conjugate gradient method.

Inverse problems of various types occur in many areas, including everyday practice. The papers [16, 17] are devoted to the study of the application of numerical methods to solving problems related to acoustic equations, with a special emphasis on problems that have significant practical significance, for example, in the field of medical imaging and theoretical acoustics.

In the study [18], an optimization method was proposed for solving the Cauchy problem, which is based on the minimization of a functional that includes both the problem data and the regularization term. The use of numerical schemes within this method allowed stabilizing the solution process and minimizing the influence of errors. This method is one of the first numerical methods for solving boundary inverse problems and demonstrates significant advantages in the context of ill-posed problems.

Paper [19] presents the construction of an algorithm and a numerical solution of the inverse problem for the acoustics equation using the gradient method. The authors investigate the correctness of the problem statement, propose a numerical solution method, and analyze the convergence and accuracy of the solutions obtained. The main focus is on the application of gradient optimization methods to recover unknown parameters in an acoustic model. The results show the effectiveness of the proposed algorithm, as well as the possibility of its use in practical problems of acoustic analysis.

The paper [20] considers numerical modeling for improved prediction of the transport of pollutants in the atmosphere of industrial regions. The authors are developing a mathematical model and a numerical algorithm that allows taking into account the complex dynamic processes of diffusion and convection of pollutants in the air.

The purpose of this article is to develop a finite-difference method for solving direct and inverse initial-boundary value problems for a hyperbolic equation with discontinuous and rapidly changing coefficients. Given additional conditions on the solution and its first derivative with respect to time at $T = 0$, it is necessary to determine the initial conditions on the solution and its first derivative at $t = 0$.

For this purpose, the inverse problem is considered as a variational one and minimization of the functional by the gradient method leads to a conjugate retrospective problem. An algorithm for the numerical implementation of the gradient method is developed. Numerous calculations of the problem are given. The results show the convergence of the iterative process. The values of the functional decrease monotonically, and the initial conditions of the direct problem are restored in accordance with the values of the additional conditions, which corresponds to the physical formulation of the problem.

1 Statement of the direct and inverse problem

The mathematical model of the operation of a gas lift well is described by the following system of linear differential equations [1–3]:

$$\frac{\partial P}{\partial t} = -\frac{c^2}{\overline{F}} \cdot \frac{\partial Q}{\partial x}, \tag{1}$$

$$\frac{\partial Q}{\partial t} = -\overline{F} \cdot \frac{\partial P}{\partial x} - 2a \cdot Q, \quad t \geq 0, \ x \in (0, 2l), \tag{2}$$

where t is time, x is a coordinate along the well depth, P is pressure, Q is volumetric gas flow rate, $\overline{F}$ is a cross-sectional area of the well, c is speed of sound in liquid, l is well depth, a is a coefficient depending on input parameters.

Parameters c, $\overline{F}$, and a may have different values in different parts of the well.

The model parameters depend on the well section:

$$c = \begin{cases} c_1, & x \in (0, l), \\ c_2, & x \in (l, 2l), \end{cases} \qquad \overline{F} = \begin{cases} \overline{F}_1, & x \in (0, l), \\ \overline{F}_2, & x \in (l, 2l), \end{cases} \qquad a = \begin{cases} a_1, & x \in (0, l), \\ a_2, & x \in (l, 2l). \end{cases}$$

Initial conditions:

$$P(0, x) = P^0(x), \quad Q(0, x) = Q^0(x), \tag{3}$$

where $P^0(x)$ and $Q^0(x)$ are the initial distribution of pressure and volumetric flow rate of gas.

Boundary conditions:

The following conditions are set at the boundaries of the domain:

1. At the wellhead ($x = 0$):

$$P(t,0)=P_0(t), \quad Q(t,0)=Q_0(t). \tag{4}$$

2. At the domain boundary ($x=l$):

$$P(t,l+0)=P(t,l-0)+P_{res}(t), \quad Q(t,l+0)=Q(t,l-0)+Q_{res}(t), \tag{5}$$

where $P_{res}(t)$ and $Q_{res}(t)$ are the pressure and flow rate of gas from the formation.

3. At the well outlet ($x=2l$):

$$P(t,2l)=P_{out}(t), \quad Q(t,2l)=Q_{out}(t), \tag{6}$$

where Q_{res} is volumetric flow rate from the reservoir, P_{res} is reservoir pressure, $P^0(x)$ is initial gas pressure, $Q^0(x)$ is initial gas volumetric flow rate, $P_{out}(t)$ is outlet pressure, $Q_{out}(t)$ is volumetric flow rate at the outlet.

To transform the system of first-order hyperbolic equations (1), (2) into one second-order equation:

1. Calculate the derivative with respect to x from (1):

$$\frac{\partial^2 P}{\partial t \partial x}=\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial t}\right)=\frac{\partial}{\partial x}\left(-\frac{c^2}{\overline{\overline{F}}}\cdot\frac{\partial Q}{\partial x}\right)=-\frac{c^2}{\overline{\overline{F}}}\cdot\frac{\partial^2 Q}{\partial x^2}. \tag{7}$$

2. We calculate the derivative with respect to t from equation (2):

$$\frac{\partial^2 Q}{\partial t^2}=-\overline{F}\cdot\frac{\partial^2 P}{\partial t\partial x}-2a\cdot\frac{\partial Q}{\partial t}. \tag{8}$$

3. Substitute (7) into (8):

$$\frac{\partial^2 Q}{\partial t^2}+2a\frac{\partial Q}{\partial t}=c^2\frac{\partial^2 Q}{\partial x^2}, \quad x\in(0,2l), \quad t\in(0,T). \tag{9}$$

Initial and boundary conditions (3)–(6) remain unchanged

$$Q(0,x)=Q^0(x), \quad \frac{\partial Q}{\partial t}(0,x)=G^0(x), \quad x\in[0,2l], \tag{10}$$

$$Q(t,0){=}Q_0\,(t), \quad Q(t,2l){=}Q_{out}\,(t), \quad t{\in}\,[0,T], \tag{11}$$

$$Q(t,l+0)=Q(t,l-0)+Q_{res}(t), \quad t\in[0,T], \tag{12}$$

where $G^0(x)$ is the initial velocity of gas displacement.

$G^0(x)$ is expressed as follows:

$$G^0(x)=-\overline{F}\cdot\frac{\partial P^0(x)}{\partial x}-2aQ^0(x).$$

The direct problem is to find the function $Q(t,x)$ based on the given functions: $Q^0\,(x)$, $G^0\,(x)$, $Q_0\,(t)$, $Q_{res}\,(t)$, $Q_{out}\,(t)$.

The inverse problem is formulated as finding the initial velocity of gas displacement $G^0(x)$ based on the known parameters of the direct problem (9)–(12) and the additional condition:

$$Q(T,x)=Q^{(1)}(x), \quad \frac{\partial Q}{\partial t}(T,x)=Q^{(2)}(x), \quad x\in[0,2l]. \tag{13}$$

Solving the inverse problem involves using the method conjugate equations and minimization of the objective functional, which allows us to restore $G^0(x)$, guaranteeing the fulfillment of all conditions of the problem.

Now we will reduce the equation (9) to an invariant form to get rid of the first-order derivative.

We make the following notation $Q(t,x) = V(t,x)\cdot e^{\alpha t}$ and substitute into (9) and initial boundary conditions (10)–(12)

$$\frac{\partial^2 V}{\partial t^2} + (2\alpha + 2a)\cdot\frac{\partial V}{\partial t} + \left(\alpha^2 + 2a\cdot\alpha\right)V = c^2\frac{\partial^2 V}{\partial x^2}.$$

We have the initial $H^0(x) = G^0(x) + \alpha\cdot Q^0(x)$, boundary $V_0(t) = Q_0(t)\cdot e^{-\alpha t}$, $V_{out}(t) = Q_{out}(t)\cdot e^{-\alpha t}$, $V_{res}(t) = Q_{res}(t)\cdot e^{-\alpha t}$ and additional conditions $V(T,x) = V^{(1)}(x)$.

We equate the coefficient $\alpha = -a$.

Then problem (9)–(12), (13) is transformed into the following problem

$$\frac{\partial^2 V}{\partial t^2} - a^2 V = c^2\frac{\partial^2 V}{\partial x^2}, \quad x\in(0,2l), \quad t\in(0,T), \tag{14}$$

with the initial conditions

$$V(0,x) = Q^0(x), \quad \frac{\partial V}{\partial t}(0,x) = H^0(x), \quad x\in[0,2l] \tag{15}$$

and the boundary conditions respectively

$$V(t,0)=V_0(t), \quad V(t,2l)=V_{out}(t), \quad t\in[0,T], \tag{16}$$

$$V(t,l+0) = V(t,l-0) + V_{res}(t), \quad t\in[0,T], \tag{17}$$

where $H^0(x) = G^0(x) + a\cdot Q^0(x)$, $V_0(t) = Q_0(t)\cdot e^{at}$, $V_{out}(t) = Q_{out}(t)\cdot e^{at}$, $V_{res}(t) = Q_{res}(t)\cdot e^{at}$.

In this direct problem, we need to find $V(t,x)$ given the functions $H^0(x)$, $Q^0(x)$, $V_0(t)$, $V_{res}(t)$, $V_{out}(t)$.

We have additional conditions

$$V(T,x) = V^{(1)}(x) \quad \text{at} \quad t=T, \ x\in[0,2l], \tag{18}$$

$$\frac{\partial V}{\partial t}(T,x) = V^{(2)}(x) \quad \text{at} \quad t=T, \ x\in[0,2l]. \tag{19}$$

In the inverse problem, it is necessary to find $H^0(x)$ from the retrospective problem (14), (16), (17) with the additional condition (18).

2 Statement of the variational problem

The inverse problem of mathematical physics is often reduced to the optimal control problem, which allows it to be solved using variational methods. In this case, it is required to find the initial values $Q^0(x)$ and $H^0(x)$ belonging to the space $W_2^2(0,2l)$, such that the solutions of problem (14)–(17), (18), (19) are as close as possible to the given value $V^{(1)}(x)$ at $t=T$.

To evaluate how closely the solution $V(T,x)$ matches the desired value $V^{(1)}(x)$, we define the objective functional:

$$J\left(H^0,Q^0\right) = \int\limits_0^{2l}\left[V\left(T,x;Q^0(x)\right) - V^{(1)}(x)\right]^2 dx + \int\limits_0^{2l}\left[V_t\left(T,x;H^0(x)\right) - V^{(2)}(x)\right]^2 dx \to \min, \tag{20}$$

Here the notations $V\left(T,x;Q^0\left(x\right)\right)$ and $V_t\left(T,x;H^0\left(x\right)\right)$ means the dependence of the solution $V\left(T,x\right)$ to problem (14)–(17), (18) on the initial conditions.

The goal is to find functions $Q^0(x)$ and $H^0\left(x\right)$ that minimize the objective functional $J\left(Q^0,H^0\right)$:

$$J\left(Q^0,H^0\right)\rightarrow \min,\quad Q^0(x)\in G_{ad}^{(1)},\,H^0(x)\in G_{ad}^{(2)},$$

where $G_{ad}^{(1)}\subset W_2^1(0,2l), G_{ad}^{(2)}\subset W_2^2(0,2l)$ is the set of admissible values.

We minimize the functional using the gradient method [21].

To minimize the objective functional $J(Q^0,H^0)$, the iterative gradient method is used. The value of the function $Q^0(x)$, $H^0(x)$ is updated at each iteration using the following formula:

$$Q_{n+1}^0=Q_n^0-\alpha\cdot J'\left(Q_n^0\right),\quad H_{n+1}^0=H_n^0-\alpha\cdot J'\left(H_n^0\right),\quad n=0,1,2,\dots\ ,$$

where $Q_n^0(x)$, $H_n^0(x)$ are approximations at the n-th iteration; α is a relaxation parameter. The relaxation parameter determines different gradient methods, and its choice is important.

$J'(Q_n^0(x))$ is a gradient of the functional $J(Q^0)$ with respect to $Q^0(x)$, $J'(H_n^0(x))$ is a gradient of the functional $J(H^0)$ with respect to $H^0(x)$.

Let us define the first variation of the entire functional (20)

$$\delta J\left(Q^0,H^0\right)=J\left(Q^0+\delta Q^0,\,H^0+\delta H^0\right)-J\left(Q^0,H^0\right)=$$

$$=\int_0^{2l}\left[V\left(T,x;Q^0\left(x\right)+\delta Q^0\right)-V^{(1)}\left(x\right)\right]^2dx+\int_0^{2l}\left[V_t\left(T,x;H^0\left(x\right)+\delta H\right)-V^{(2)}\left(x\right)\right]^2dx-$$

$$-\int_0^{2l}\left[V\left(T,x;Q^0\right)-V^{(1)}\left(x\right)\right]^2dx-\int_0^{2l}\left[V_t\left(T,x;H^0\left(x\right)\right)-V^{(2)}\left(x\right)\right]^2dx.$$

Considering that

$$V\left(T,x;Q^0\left(x\right)+\delta Q^0\right)=V\left(T,x;Q^0\left(x\right)\right)+\delta V\left(T,x;\delta Q^0\left(x\right)\right),$$

$$V_t\left(T,x;H^0\left(x\right)+\delta H^0\right)=V_t\left(T,x;H^0\left(x\right)\right)+\delta V_t\left(T,x;\delta H^0\left(x\right)\right).$$

We have

$$\begin{gathered}\delta J\left(Q^0,H^0\right)\approx\int_0^{2l}\delta V\left(T,x;\delta Q^0\left(x\right)\right)\cdot 2\cdot\left[V\left(T,x;Q^0\left(x\right)\right)-V^{(1)}\left(x\right)\right]dx+\\ +\int_0^{2l}\delta V_t\left(T,x;\delta H^0\left(x\right)\right)\cdot 2\cdot\left[V_t\left(T,x;H^0\left(x\right)\right)-V^{(2)}\left(x\right)\right]dx.\end{gathered}\qquad(21)$$

Due to the smallness of the terms containing $\delta V^2\left(T,x;\delta Q^0\left(x\right)\right)$, $\left[\dfrac{\partial\delta V}{\partial t}\left(T,x;\delta H^0\left(x\right)\right)\right]^2$ they can be neglected.

On the other hand, in accordance with the definition of the Frechet derivative, the equality is satisfied:

$$\delta J\left(Q^0\right) =< J'Q^0, \delta Q^0 >, \quad \delta J\left(H^0\right) =< J'H^0, \delta H^0 > . \tag{22}$$

Let us introduce the following notations:

$$\widetilde{V}\left(t,x\right){=}V\left(t,x;H^0+\delta H^0\right), \qquad V\left(t,x\right)=V\left(t,x;H^0\right), \qquad \delta V\left(t,x\right)=\widetilde{V}\left(t,x\right)-V\left(t,x\right).$$

Let us consider the perturbed problem corresponding to problem (14)–(17):

$$\frac{\partial^2\widetilde{V}}{\partial t^2}-a^2\widetilde{V}=c^2\frac{\partial^2\widetilde{V}}{\partial x^2}, \qquad x\in(0,2l)\,, \quad t\in(0,T), \tag{23}$$

with initial conditions:

$$\widetilde{V}\left(0,x\right)=Q^0\left(x\right)+\delta Q^0\left(x\right), \quad \frac{\partial\widetilde{V}}{\partial t}(0,x)=H^0\left(x\right)+\delta H^0, \quad x\in[0,2l]\,, \tag{24}$$

boundary conditions:

$$\widetilde{V}\left(t,0\right){=}V_0\left(t\right), \quad \widetilde{V}\left(t,2l\right){=}V_{out}\left(t\right), \quad t{\in}\left[0,T\right], \tag{25}$$

and the consistency condition:

$$\widetilde{V}\left(t,l+0\right)=\widetilde{V}\left(t,l-0\right)+V_{res}\left(t\right), \quad t\in[0,T]\,. \tag{26}$$

Let us subtract problem (23)–(26) from problem (14)–(17) to obtain the equation for $\delta V\left(T,x;\delta H^0\right)$:

$$\frac{\partial^2\delta V}{\partial t^2}-a^2\delta V=c^2\frac{\partial^2\delta V}{\partial x^2}, \quad x\in(0,2l)\,, \quad t\in(0,T), \tag{27}$$

$$\delta V\left(0,x\right)=\delta Q^0\left(x\right), \quad \frac{\partial\delta V}{\partial t}(0,x)=\delta H^0\left(x\right), \quad x\in[0,2l]\,, \tag{28}$$

$$\delta V\left(t,0\right)=0, \;\; \delta V\left(t,2l\right)=0, \quad t\in[0,T]\,, \tag{29}$$

$$\delta V\left(t,l-0\right)=\delta V\left(t,l+0\right), \quad t\in[0,T]\,. \tag{30}$$

Let us consider an expression that is identically equal to zero obtained from (27) by multiplying by the still unknown function $V^*\left(t,x\right)$ and integrating over t and over x.

$$(A\delta V,V^*)=\int\limits_0^T\int\limits_0^{2l}\left[\frac{\partial^2\delta V}{\partial t^2}-a^2\delta V-c^2\frac{\partial^2\delta V}{\partial x^2}\right]\cdot V^*dxdt\equiv 0,$$

where $AV=\dfrac{\partial^2 V}{\partial t^2}-a^2V-c^2\dfrac{\partial^2 V}{\partial x^2}$.

Let's perform integration by parts of this expression:

$$\int\limits_0^T\int\limits_0^{2l}\left[\frac{\partial^2\delta V}{\partial t^2}-a^2\delta V-c^2\frac{\partial^2\delta V}{\partial x^2}\right]\cdot V^*dxdt=$$

$$=\int_0^{2l}\left[\frac{\partial\delta V}{\partial t}\cdot V^*\Big|_0^T-\int_0^T\frac{\partial\delta V}{\partial t}\cdot\frac{\partial V^*}{\partial t}dt\right]dx-a^2\int_0^T\int_0^{2l}\delta V\cdot V^*dxdt-$$

$$-c^2\int_0^T\left[\frac{\partial\delta V}{\partial x}V^*\Big|_0^{2l}-\int_0^{2l}\frac{\partial\delta V}{\partial x}\cdot\frac{\partial V^*}{\partial x}dx\right]dt.$$

We again perform integration by parts for the original expression:

$$\left(A\delta V,V^*\right)=\int_0^T\int_0^{2l}\left[\frac{\partial^2V^*}{\partial t^2}-a^2V^*-c^2\frac{\partial^2V^*}{\partial x^2}\right]\delta V dxdt+\int_0^{2l}\left[\frac{\partial\delta V}{\partial t}(T,x)\,V^*(T,x)-\right.$$

$$\left.-\frac{\partial\delta V}{\partial t}(0,x)\,V^*(0,x)-\delta V(T,x)\,\frac{\partial V^*}{\partial t}(T,x)+\delta V(0,x)\,\frac{\partial V^*}{\partial t}(0,x)\right]dx-$$

$$-c^2\int_0^T\left[-\frac{\partial\delta V}{\partial x}(t,0)\,V^*(t,0)+\frac{\partial\delta V}{\partial x}(t,2l)\,V^*(t,2l)+\right.$$

$$\left.+\delta V(t,0)\,\frac{\partial V^*}{\partial x}(t,0)-\delta V(t,2l)\,\frac{\partial V^*}{\partial x}(t,2l)\right]dt.$$

Taking into account the boundary conditions (28)–(30), we write the expression as follows:

$$\left(\delta V,A^*V^*\right)=\int_0^T\int_0^{2l}\left[\frac{\partial^2V^*}{\partial t^2}-a^2V^*-c^2\frac{\partial^2V^*}{\partial x^2}\right]\delta V dxdt+\int_0^{2l}\left[\frac{\partial\delta V}{\partial t}(T,x)\,V^*(T,x)-\right.$$

$$\left.-\frac{\partial\delta V}{\partial t}(0,x)\,V^*(0,x)-\delta V(T,x)\,\frac{\partial V^*}{\partial t}(T,x)+\delta V(0,x)\,\frac{\partial V^*}{\partial t}(0,x)\right]dx-$$

$$-c^2\int_0^T\left[-\frac{\partial\delta V}{\partial x}(t,0)\,V^*(t,0)+\frac{\partial\delta V}{\partial x}(t,2l)\,V^*(t,2l)\right]dt. \tag{31}$$

To satisfy the Lagrange identity, the requirements that all non-integral terms be equal to zero, as well as the conditions for the variations of the functional (21) and the Frechet derivatives (22), lead to the following conjugate problem.

$$A^*V^*=\frac{\partial^2V^*}{\partial t^2}-a^2V^*-c^2\frac{\partial^2V^*}{\partial x^2}=0, \tag{32}$$

$$V^*(T,x)=2\left[V\left(T,x;Q^0\right)-V^{(1)}(x)\right],\quad \frac{\partial V^*}{\partial t}(T,x)=2\left[V_t\left(T,x;H^0\right)-V^{(2)}(x)\right], \tag{33}$$

$$V^*(t,0)=0,\quad V^*(t,2l)=0. \tag{34}$$

These arguments on the expression (31) lead to the following lemma.

Lemma. Let $Q^0,\ Q^0+\delta Q^0\in G_{ad}^{(1)},\quad H^0,\ H^0+\delta H^0\in G_{ad}^{(2)}$ be given elements.

If $V\left(t,x;Q^{0}\left(x\right)\right)$ is a solution to problem (14)–(17), and $V^{*}\left(t,x;Q^{0}\right)$ is a solution to adjoint problem (32)–(34), then the following identity holds

$$\begin{gathered}\int\limits_{0}^{2l}\delta\frac{\partial\delta V}{\partial t}\left(T,x\right)V^{*}\left(T,x\right)dx-\int\limits_{0}^{2l}\frac{\partial\delta V}{\partial t}\left(0,x\right)V^{*}\left(0,x\right)dx=\\=\int\limits_{0}^{2l}\delta V\left(T,x\right)\frac{\partial V^{*}}{\partial t}\left(T,x\right)dx-\int\limits_{0}^{2l}\delta V\left(0,x\right)\frac{\partial V^{*}}{\partial t}\left(0,x\right)dx.\end{gathered}\tag{35}$$

From condition (35), taking into account boundary conditions (28), (33) and the definition of the Frechet derivative (22), we obtain the form

$$J'\left(Q^{0}\right)=-V^{*}\left(0,x\right),\quad J'\left(H^{0}\right)=-\frac{\partial V^{*}}{\partial t}\left(0,x\right).\tag{36}$$

Adjoint problem (32)–(34), the formulas for the gradients of the functional (36) follow from the Lagrange principle:

$$\left(A\delta V,V^{*}\right)=\left(\delta V,A^{*}V^{*}\right)$$

and is called the principle of duality.

3 Algorithm for solving a variational problem

1. Set the initial approximation $Q_0^0(x)$, $H_0^0(x)$.
2. Assume that Q_n^0 and H_n^0 are already known, then solve direct problem (14)–(17).
3. We calculate the value of the functional

$$J\left(Q_n^0,H_n^0\right)=\int\limits_{0}^{2l}\left[V\left(T,x;Q_n^0\right)-V^{(1)}\left(x\right)\right]^2dx+\int\limits_{0}^{2l}\left[\frac{\partial V}{\partial t}\left(T,x;H^0\left(x\right)\right)-V^{(2)}\left(x\right)\right]^2dx.$$

4. If the current value of the functional $J\left(Q_n^0,H_n^0\right)$ is not small enough, then we solve adjoint problem (32)–(34).
5. Calculate the gradient of functional (36).
6. We calculate the next approximation

$$Q_{n+1}^0=Q_n^0-\alpha\cdot J'\left(Q_n^0\right),\quad H_{n+1}^0=H_n^0-\alpha\cdot J'\left(H_n^0\right).$$

7. We return to step 2 using the updated Q_{n+1}^0, H_{n+1}^0.

4 Numerical solution of the direct problem

4.1 Scheme for solving the direct problem

We approximate the problem (14)–(17) using a uniform grid. Let N_t be the number of grid nodes in time on the interval $[0,\,T]$, and N_x be the number of grid nodes in space on the interval $[0,\,2l]$.

Let us construct in the domain $\Omega=((0,2l)\times(0,T))$ a regular grid ω_h with steps $\tau=T/N_t$, $h=2l/N_x$, where N_t, N_x are positive integers.

In the grid $\overline{\omega}_{h\tau}=\left\{x=ih,\quad t=k\tau,\quad i=\overline{0,\quad N_x},\quad k=\overline{0,\quad N_t}\right\}$, we formulate the difference problem. Accordingly, we write the approximation of problem (14)–(17) as follows: we replace the derivatives included in equation (14) by the formulas

$$\frac{\partial^2 V}{\partial t^2} \sim y_{\bar{t}t}^k = \frac{y_i^{k+1} - 2y_i^k + y_i^{k-1}}{\tau^2},$$

$$\frac{\partial^2 V}{\partial x^2} \sim \Lambda y^k = y_{\bar{x}x}^k = \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{h^2}.$$

Let's consider a family of schemes with weights

$$y_{\bar{t}t}^k = c^2 \Lambda \left(\sigma y^{n+1} + (1-2\sigma)\, y^n + \sigma y^{n-1}\right) + a^2 \cdot y_i^n, \tag{37}$$

$$y_0 = y_0\,(t)\,, y_{Nx} = y_{out}\,(t)\,, \quad t \in [0, T]\,,$$
$$y^0\,(x) = Q^0\,(x)\,, \quad y_t\,(0,\,x) = \tilde{H}^0\,(x)\,.$$

The boundary conditions and the first initial condition on the grid $\omega_{h\tau}$ are satisfied exactly. We choose $\tilde{H}^0\,(x)$ so that the approximation error $\tilde{H}^0\,(x) - \dfrac{\partial V\,(0,\,x)}{\partial t}$ is $O\left(\tau^2\right)$.

Thus, the difference problem is posed. To determine y_i^{n+1}, we obtain the boundary value problem

$$c^2 \sigma \cdot \gamma^2 \cdot \left(y_{i+1}^{n+1} + y_{i-1}^{n+1}\right) - \left(1 + 2c^2 \sigma \gamma^2\right) y_i^{n+1} = -F_i, \quad 0 < i < N_x, \tag{38}$$

$$y_0 = V_0(t^{t+1}), \quad y_{Nx} = V_{out}(t^{t+1}), \quad \gamma = \frac{\tau}{h}, \tag{39}$$

$$F_i = \left(2y_i^n - y_i^{n-1}\right) + a^2 \tau^2 y_i^n + c^2 \tau^2\,(1-2\sigma) \cdot y_{\bar{x}x}^n + c^2 \tau^2 \sigma \cdot \gamma^2 y_{\bar{x}x}^{n-1},$$

$$F_i = \left(2 + a^2 \tau^2\right) y_i^n - y_i^{n-1} + c^2 \tau^2\,(1-2\sigma) \cdot y_{\bar{x}x}^n + c^2 \tau^2 y_{\bar{x}x}^{n-1}. \tag{40}$$

Problem (38)–(40) is solved by the sweep method. The sweep is stable for $\sigma > 0$.

The approximation error for scheme (38) will be $\psi = O\left(\tau^2 + h^2\right)$, provided that the second initial $V_t\,(0, x) = \tilde{H}^0\,(x)$ is approximated by the second order. If we put

$$\tilde{H}_0\,(x) = H_0\,(x) + 0{,}5\tau \cdot \left(a^2 Q_0\,(x) + c^2\,(Q_0\,(x))''\right),$$

then $\tilde{H}_0\,(x) - V_t\,(x) = O\left(\tau^2\right)$.

Next

$$y^0 = Q^0\,(x_i)\,, \quad y_i^1 = y_i^0 + h \cdot \tilde{H}_0\,(x_i)\,, \quad i = 1,\ ...,\ N_x.$$

4.2 Scheme of the solution of the adjoint problem.

Analogously to the direct one, we approximate the problem (34)–(36) using a uniform grid. Accordingly, we write the approximation of the problem (34)–(36) as follows:

$$y_{\bar{t}t}^* = c^2 \Lambda \left(\sigma y^{*n+1} + (1-2\sigma)\, y^{*n} + \sigma y^{*n-1}\right) + a^2 \cdot y_i^{*n}, \tag{41}$$

$$y_i^{*N_t} = 0, \tag{42}$$

$$y_0^{*n} = 0, y_{N_x}^{*n} = 0, \quad n = N_t - 1, \quad N_t - 2, ..., 1, \tag{43}$$

where $V(x) = 2\left(V\left(T, x; H^0\right) - V^{(1)}(x)\right)$.

To determine y_i^{*n-1} from (41), we obtain a three-layer difference problem

$$c^2\sigma\Lambda y^{*n-1} - y^{*n-1}/\tau^2 = \frac{-\,ay_i^{*n+1} - 2y_i^n}{\tau^2} - c^2\Lambda\left(\sigma y^{*n+1} + (1-2\sigma)\,y^{*n}\right) + a^2\cdot y_i^{*n}. \tag{44}$$

Multiplying (44) by τ^2 and expanding the difference operator Λ, we have

$$c^2\sigma\cdot\gamma^2\cdot\left(y_{i+1}^{*n-1} + y_{i-1}^{*n-1}\right) - \left(1+2c^2\sigma\gamma^2\right)y_i^{*n-1} = -F_i, \quad 0 < i < N_x, \tag{45}$$

where $F_i = \left(2y_i^n - y_i^{n-1}\right) + a^2\tau^2 y_i^{*n} + c^2\tau^2\sigma y^{*n+1} + c^2\tau^2\,(1-2\sigma)\,y^{*n}$.

Problem (45) with conditions (42), (43) is also solved by the sweep method [22]. From (45) it is clear that the sweep is stable for $\sigma > 0$.

5 *Numerical solution of the problem*

5.1 *Computational experiment*

The following initial data were set for the computational experiment: $Q_0 = 0.21$ (m^3/s) is volumetric flow rate of the injected gas, $\bar{P}_0 = 1.0355$ is initial pressure distribution, $\overline{P}_{pl} = 0.2195$ is formation pressure, $Q_{pl} = 0.001$ (m^3/s) is volumetric flow rate from the formation, $\overline{P}_{out} = 1$ is outlet pressure, $Q_{out} = 0.2$ (m^3/s) is volumetric flow rate at the outlet, $\bar{l} = 1$ is well depth, $\lambda_1 = 0.01$ is hydraulic resistance in the ring, $\lambda_2 = 0.23$ is hydraulic resistance in the well, $d_1 \approx 0.1353$ (m) is effective diameter of the well annular space, $d_2 = 0.073$ (m) is diameter of the inner well, $\rho_1 = 0.75$ (kg/m^3) is gas density, $\rho_2 = 700$ (kg/m^3) is oil density, $g = 9.8$ (m/s^2) is acceleration due to gravity, $c_1 = 331$ (m/s) is C-speed of sound in the annular space, $c_2 = 850$ (m/s) is speed of sound in the well, $r_1 = \dfrac{d_1}{2}$ (m) is radii of the annular space, $r_2 = \dfrac{d_2}{2}$ (m) is radii of the inner well, $F_1 = \pi r_1^2$ (m^2) is cross-sectional area of the annular space of the well, $F_2 = \pi r_2^2$ (m^2) is cross-sectional area of the inner well, $w_1 = \dfrac{Q_0}{F_1\rho_1}$ (m/s) is averaged over the cross-section velocity of the mixture in the annulus, $w_2 = \dfrac{Q_0}{F_2\rho_2}$ (m/s) is averaged over the cross-section velocity of the mixture in the well; coefficients in the ring and the inner well: $a_1 = \dfrac{g}{2w_1} + \dfrac{\lambda_1 w_1}{4d_1}$, $a_2 = \dfrac{g}{2w_2} + \dfrac{\lambda_2 w_2}{4d_2}$.

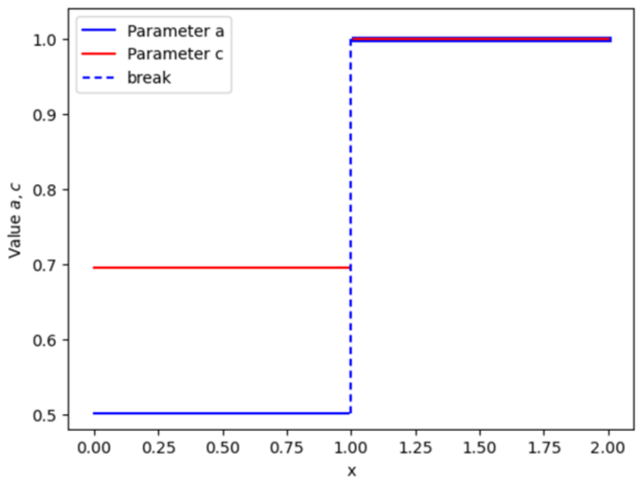

Figure 1. Graphs of functions a and c

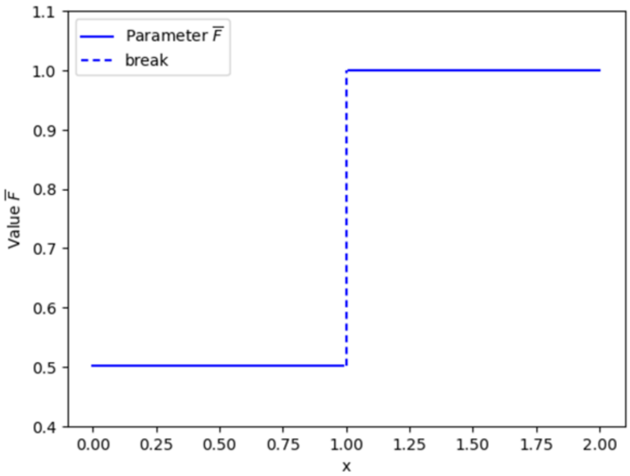

Figure 2. Graph of functions $\overline{F}$

The graphs of the functions a, c and $\overline{F}$ are shown in Figures 1 and 2. They show the change in these parameters along the wellbore depth and reflect the physical conditions in the annular space and in the wellbore.

We computed the test problem using the following parameters:
$l = 1$, $T = 0.001$, $N_x = 21$, $N_t = 21$, grid steps $h = \frac{2l}{N_x}$ and $\tau = \frac{T}{N_t}$, gradient descent step $\alpha = 0.09$.

The initial conditions $Q^0(x)$ and $P^0(x)$ for the direct problem were specified as linear functions:

$$Q^0(x) = Q_0 + \frac{Q_{out} - Q_0}{2l} \cdot x,$$

$$P^0(x) = P_0 + \frac{P_{out} - P_0}{2l} \cdot x.$$

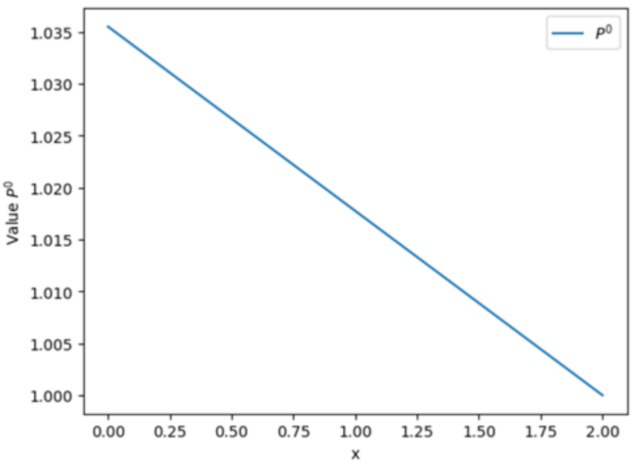

Figure 3. Graph of functions $P^0\,(x)$

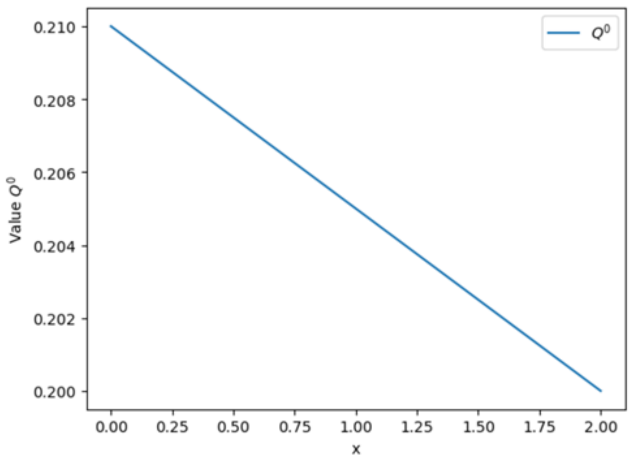

Figure 4. Graph of functions $Q^0\,(x)$

Figures 3 and 4 show the graphs of the function $P^0\,(x)$, $Q^0\,(x)$.

The additional condition for the inverse problem was set as follows:

$$V^{(1)}(T,x) = \left(-x^2 + q\cdot x + r\right)\cdot e^{aT},$$

$$V^{(2)}(T,x) = a\cdot\left(-x^2 + q\cdot x + r\right)\cdot e^{aT},$$

where $q = \dfrac{Q_{out} - Q_0 + 4l^2}{2l}$, $r = Q_0$.

For the numerical solution of the direct and adjoint problem in the difference schemes (37), (41), the weight coefficient σ is chosen equal to 1. The program is implemented in Python 3.13.2. The library for working with multidimensional arrays numpy and matplotlib.pyplot were used to output graphs.

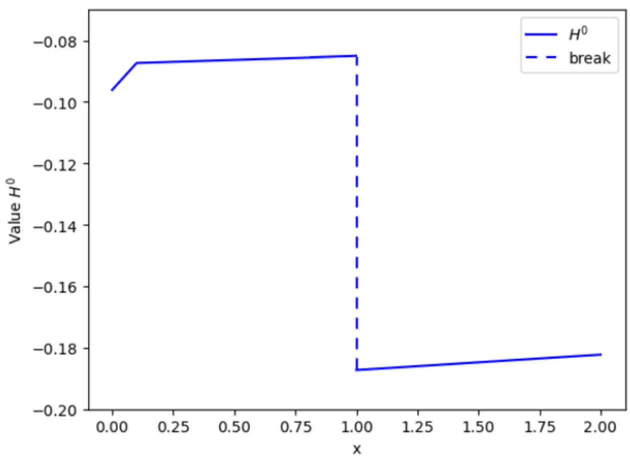

Figure 5. Initial approximation $H_0\,(x)$

The initial approximation of the sought function H_n^0 is given in the form of piecewise constant functions as shown in Figure 5. The graph shows a stepwise distribution associated with a sharp change in the functions a in the middle of the region. This is due to the fact that the values of the physical parameters at depth l change when moving from the annular space of the outer well to the production well.

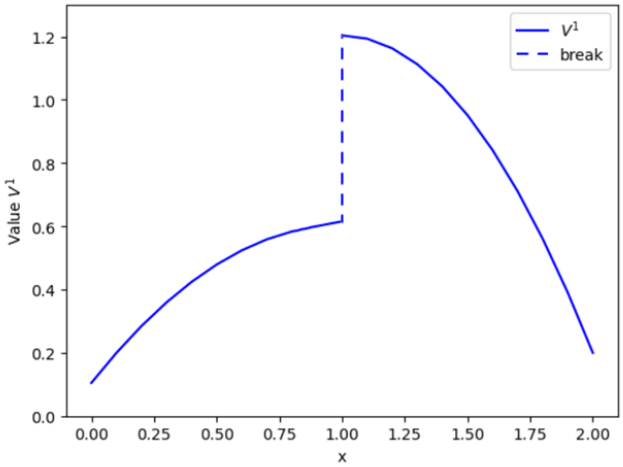

Figure 6. Graph of functions $V^{(1)}(x)$

Figure 6 shows a graph of the change in the values of $V^{(1)}(x)$.

In the iterative process, the value of the functional J decreases monotonically and reaches the value $\|J_n - J_{n-1}\| \leq \varepsilon$, $\varepsilon = 1 \times 10^{-7}$ at $n = 66$ iterations. The graph of the decrease in the value of the functional is shown in Figure 7.

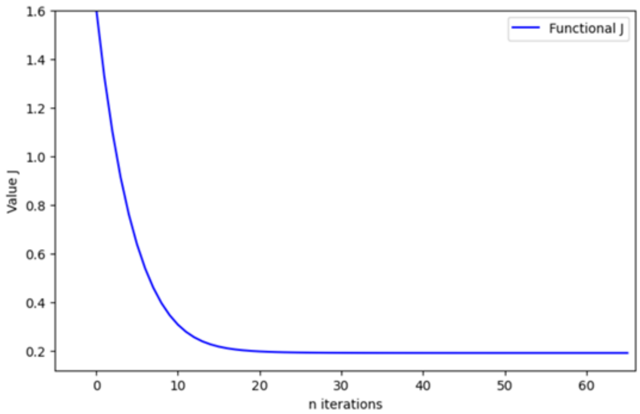

Figure 7. Graph of decreasing functional J

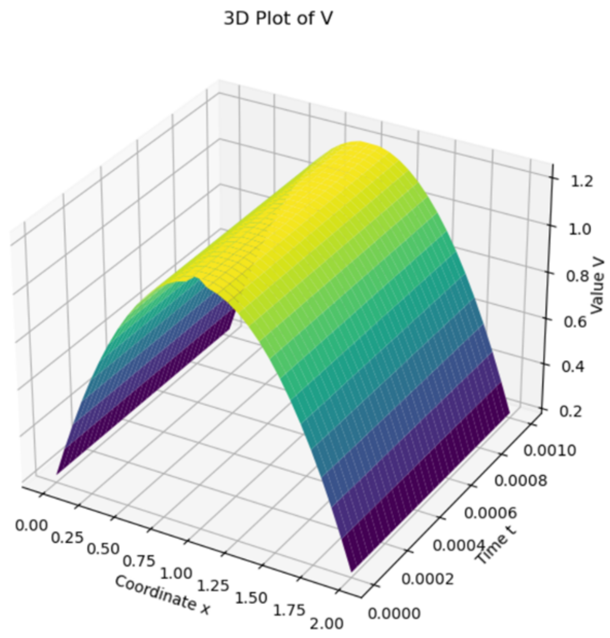

Figure 8. Graph of the solution of the direct problem of functions $V\,(t, x)$

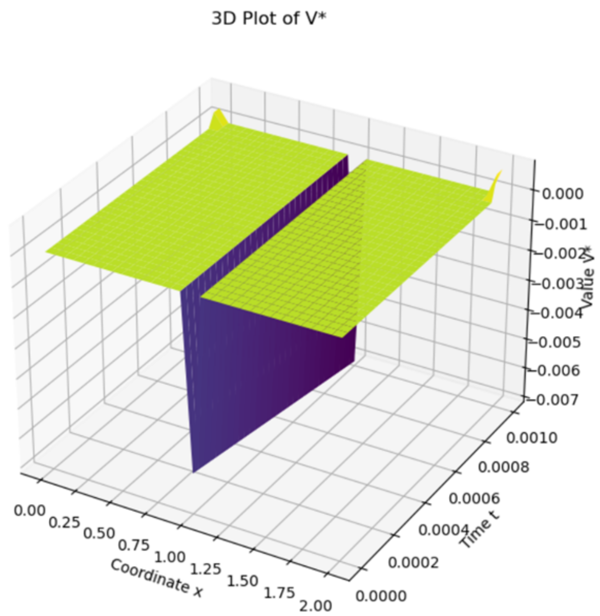

Figure 9. Graph of the solution of the conjugate problem of functions $V^*\,(t, x)$

Figures 8 and 9 show three-dimensional graphs of the functions V and V^*.

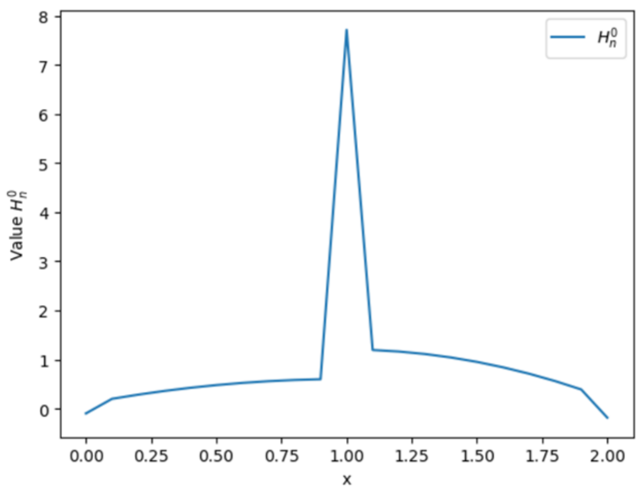

Figure 10. H_n^0 graphs

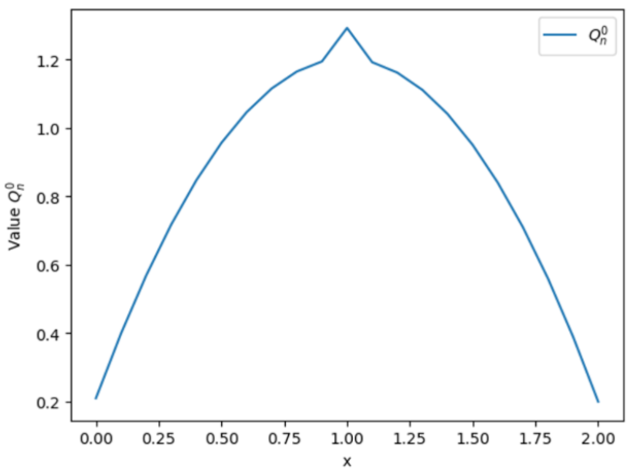

Figure 11. Q_n^0 graphs

Figures 10 and 11 show the graphs of the desired functions H_n^0 and Q_n^0 for 66 iterations.

It is known that the adjoint problem carries valuable information about the solution of the direct problem. This property is confirmed by numerical calculations, since the gradients of the functional for determining the initial conditions of the direct problem at each iteration were chosen as the solution of the adjoint problem at $t = 0$, i.e.,

$$J'\left(H_n^0\right) = V^*\left(0, x\right), J'\left(Q_n^0\right) = \frac{\partial V^*}{\partial t}\left(0, x\right).$$

The numerical calculations performed confirm the effectiveness of the proposed algorithm for modeling the gas lift process of oil production.

Conclusion

In this paper, we present a numerical method for solving direct and inverse problems associated with the gas-lift oil production process using the adjoint equation method. The mathematical model of the process is represented by a hyperbolic equation. The inverse problem is formulated as a problem of restoring the initial condition based on additional information about the solution at $t = T$. To solve it, an optimal control method is used, including minimization of the objective functional using the gradient method, where the gradient of the functional is determined through the solution of the adjoint retrospective problem.

The numerical experiment results demonstrate the effectiveness of the proposed method for inverse problems in the gas-lift process. The plotted graphs demonstrate that with the correct choice of parameters, such as the gradient descent step, it is possible to achieve high accuracy of the solution in a relatively small number of iterations.

The proposed method has practical value, as it enables optimization of gas-lift well operating parameters, reducing gas injection costs while maximizing oil recovery. In the future, the method can be expanded to analyze the influence of various boundary conditions and nonlinear effects, and also adapted to multidimensional models of the gas-lift process, which will expand its scope of application in the oil industry.

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Author Contributions

N.M. Temirbekov collected and analyzed data, and led manuscript preparation. A.K. Turarov assisted in data collection and analysis. N.M. Temirbekov served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2025M2/241-251 *Research article*

Hyper-Generalized Weakly Symmetric Para-Sasakian Manifolds and Their Geometric Properties

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This paper examines para-Sasakian manifolds that satisfy a hyper-generalized weakly symmetric curvature condition. The conditions under which such a manifold with a hyper-generalized weakly symmetric curvature condition satisfies the η-Einstein manifold are established. Furthermore, the geometric behavior of a hyper-generalized weakly symmetric para-Sasakian manifold admitting quarter-symmetric metric connection is analyzed.

Keywords: hyper-generalized, para-Sasakian manifold, Quarter-symmetric metric connection.

2020 Mathematics Subject Classification: 53C05, 53C25.

Introduction

The study of symmetric and generalized symmetric manifolds play an important role in differential geometry due to their rich structures and diverse applications in both mathematics and theoretical physics. Among the various extensions of, symmetric spaces, the concept of weakly symmetric manifolds, which was first introduced by Tamassy and Binh [1], has been a focal point. This concept has been studied by many geometers (for details, please see [2–9]).

A non-flat, n-dimensional Riemannian manifold (M^n, g) $n > 2$ is called a generalized weakly symmetric manifold [10] if its curvature tensor R of type $(0,4)$ is non zero and satisfies the condition:

$$\begin{aligned}(\nabla_U R)(V,X,Y,Z) &= A(U)R(V,X,Y,Z) + B(V)R(U,X,Y,Z) + B(X)R(V,U,Y,Z)\\ &+ D(Y)R(V,X,U,Z) + D(Z)R(V,X,Y,U) + \alpha(U)G(V,X,Y,Z)\\ &+ \beta(V)G(U,X,Y,Z) + \beta(X)G(V,U,Y,Z) + \gamma(Y)G(V,X,U,Z)\\ &+ \gamma(Z)G(V,X,Y,U),\end{aligned}$$

where $G(X,\ Y)Z = g(Y,\ Z)X - g(X,\ Z)Y$, and $A,\ B,\ D,\ \alpha,\ \beta,\ \gamma$ are non zero 1-forms defined as follows: $A(X) = g(X,\theta_1)$, $B(X) = g(X,\phi_1)$, $D(X) = g(X,\pi_1)$, $\alpha(X) = g(X,\theta_2)$, $\beta(X) = g(X,\phi_2)$ and $\gamma(X) = g(X,\pi_2)$, in which $\theta_1,\ \phi_1,\ \pi_1,\ \theta_2,\ \phi_2$ and π_2 are associated vector fields of $A,\ B,\ D,\ \alpha,\ \beta$ and γ respectively.

Keeping in tune with Shaikh and Patra [11], we shall call a Riemannian manifold of dimension n, hyper-generalized weakly symmetric if it admits the equation:

$$\begin{aligned}(\nabla_U R)(V,X,Y,Z) &= A(U)R(V,X,Y,Z) + B(V)R(U,X,Y,Z) + B(X)R(V,U,Y,Z)\\ &+ D(Y)R(V,X,U,Z) + D(Z)R(V,X,Y,U)\\ &+ \alpha(U)(g\wedge S)(V,X,Y,Z) + \beta(V)(g\wedge S)(U,X,Y,Z)\\ &+ \beta(X)(g\wedge S)(V,U,Y,Z) + \gamma(Y)(g\wedge S)(V,X,U,Z)\\ &+ \gamma(Z)(g\wedge S)(V,X,Y,U),\end{aligned}$$

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where

$$(g \wedge S)(V,X,Y,Z) = g(V,Z)S(X,Y) + g(X,Y)S(V,Z) - g(V,Y)S(X,Z) \\ - g(X,Z)S(V,Y), \tag{1}$$

and A, B, D, α, β, γ are non zero 1-forms defined as follows: $A(X) = g(X,\theta_1)$, $B(X) = g(X,\phi_1)$, $D(X) = g(X,\pi_1)$, $\alpha(X) = g(X,\theta_2)$, $\beta(X) = g(X,\phi_2)$ and $\gamma(X) = g(X,\pi_2)$.

The $(H(GWS))_n$ manifolds exhibit various properties, aligning with different types of symmetric spaces.

In recent years, Blaga et al. [12] studied hyper generalized pseudo Q-symmetric semi-Riemannian manifold. And Bakshi et al. [13] investigated the existence of hyper-generalized weakly symmetric Lorentzian para-Sasakian manifold.

The concept of an almost para-contact structure was introduced by Sato [14] as an analogue to the almost contact structure, which has been widely studied in differential geometry. While an almost contact manifold is always of odd dimension, an almost para-contact manifold can exist in both odd and even dimensions, making it more versatile structure in geometric analysis.

Kaneyuki and Williams [15] further developed this idea by investigating the almost para-contact structure on pseudo-Riemannian manifolds. Recently, there has been a growing interest in almost para-contact geometry, in particular, para-Sasakian geometry, due to its connections with the theory of para-Kähler manifolds. The study of almost para-contact and para-Sasakian structures is also growing traction because of their applications in pseudo-Riemannian geometry and mathematical physics. Almost para-contact structures facilitate the exploration of new geometric invariants and curvature conditions that difer significantly from their contact counter parts. In recent year, Bulut and İnselöz [16] studied para-Sasaki-like Riemannian manifolds with generalized symmetric metric connection. Extending this, we study para-Sasakian manifold whose curvature tensor satisfies the hyper-generalized weakly symmetric condition.

Golab [17] extended the concept of semi-symmetric connection by introducing the notion of a quarter-symmetric connection within differentiable manifolds equipped with an affine connection. This idea was subsequently explored further by researchers such as Mondal and De [18], Rastogi [19, 20], Yano and Imai [21], among others.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M,g) is defined as a quarter-symmetric connection [17], if its torsion tensor T of the connection $\tilde{\nabla}$, given by

$$T(U,V) = \tilde{\nabla}_U V - \tilde{\nabla}_V U - [U,V],$$

satisfies the condition

$$T(U,V) = \eta(V)\phi U - \eta(U)\phi V,$$

where η is a 1-form and ϕ is a (1,1)-tensor field.

In special case, where $\phi U = U$, the quarter-symmetric connection reduces to the semi-symmetric connection [22, 23], thus showing that the concept of a quarter-symmetric connection generalizes the idea of the semi-symmetric connection.

When a quarter-symmetric connection $\tilde{\nabla}$ satisfies

$$(\tilde{\nabla}_U g)(V,W) = 0,$$

it is known as a quarter-symmetric metric connection. Otherwise it is referred to as a quarter-symmetric non metric connection.

The structure of this paper is as follows: in Section 2, we define para-Sasakian manifold and present some known results of para-Sasakian manifold. Then in the next section, we discuss hyper-generalized para-Sasakian manifold and derive some relations of the 1-forms. In the next two sections, we discuss the conditions under which a hyper-generalized weakly symmetric para-Sasakian manifold admitting Codazzi type of Ricci tensor and recurrent tensor becomes an η-Einstein manifold.

1 Para-Sasakian Manifold

Consider M as an n-dimensional almost para-contact metric manifold admitting an almost para-contact metric structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric. Then [24]

$$\phi^2 U = U - \eta(U)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi U) = 0,$$

$$g(\phi U, \phi V) = -g(U, V) + \eta(U)\eta(V), \quad g(\phi U, V) = -g(U, \phi V), \quad g(U, \xi) = \eta(U),$$

for all vector fields U, V on TM.

An almost para-contact manifold is called a para-Sasakian manifold if

$$(\nabla_U \phi)V = -g(U, V)\xi + \eta(V)U,$$

$$d\eta = 0 \quad \text{and} \quad \nabla_U \xi = -\phi U.$$

In a para-Sasakian manifold equipped with the structure (ϕ, ξ, η, g), the following relations also hold [24]

$$(\nabla_U \eta)V = g(U, \phi V),$$

$$\eta(R(U, V)W) = g(U, W)\eta(V) - g(V, W)\eta(U),$$

$$R(\xi, U)V = -g(U, V)\xi + \eta(V)U,$$

$$R(U, V)\xi = \eta(U)V - \eta(V)U,$$

$$S(U, \xi) = -(n-1)\eta(U),$$

$$Q\xi = -(n-1)\xi,$$

for all $U,\ V \in TM$, where S is the Ricci tensor, R is a Riemannian curvature tensor.

Let us consider a quarter-symmetric metric connection $\tilde{\nabla}$ on a para-Sasakian manifold [25] given by

$$\tilde{\nabla}_U V = \nabla_U V + \eta(V)\phi U - g(\phi U, V)\xi. \tag{2}$$

The curvature tensor $\tilde{R}$ associated with the quarter-symmetric connection relates to the curvature tensor R of the Levi-Civita connection by [26]:

$$\begin{aligned}\tilde{R}(U, V)W = R(U, V)W &+ 3g(\phi U, W)\phi V - 3g(\phi V, W)\phi U + [\eta(U)V - \eta(V)U]\eta(W)\\ &- [g(V, W)\eta(U) - g(U, W)\eta(V)]\xi,\end{aligned} \tag{3}$$

which yields

$$\tilde{S}(V, W) = S(V, W) + 2g(V, W) - (n+1)\eta(V)\eta(W) - 3tr\phi g(\phi V, W),$$

where $\tilde{S}$ and S are Ricci tensor of $\tilde{\nabla}$ and ∇, respectively.

Definition 1. A para-Sasakian manifold M is said to be an η-Einstein manifold, if its Ricci tensor S is of the form

$$S(X,\ Y) = \alpha g(X,\ Y) + \beta\eta(X)\eta(Y)$$

for any vector fields X and Y, where α and β are constants. If $\beta = 0$, then the manifold M^{2n+1} is an Einstein manifold.

2 Hyper-Generalized Weakly Symmetric Para-Sasakian Manifolds

A para-Sasakian manifold is considered to be hyper-generalized weakly symmetric if it satisfies the following curvature condition [11]

$$
\begin{aligned}
(\nabla_U R)(V,X,Y,W) = {} & A(U)R(V,X,Y,W) + B(V)R(U,X,Y,W) + B(X)R(V,U,Y,W) \\
& + D(Y)R(V,X,U,W) + D(W)R(V,X,Y,U) \\
& + \alpha(U)(g\wedge S)(V,X,Y,W) + \beta(V)(g\wedge S)(U,X,Y,W) \\
& + \beta(X)(g\wedge S)(V,U,Y,W) + \gamma(Y)(g\wedge S)(V,X,U,W) \\
& + \gamma(W)(g\wedge S)(V,X,Y,U),
\end{aligned}
\tag{4}
$$

where $g \wedge S$ is defined in (1).

By applying (1) and then contracting it with U and V in (4), we obtain

$$
\begin{aligned}
(\nabla_U S)(V,W) = {} & A(U)S(V,W) + B(V)S(U,W) + D(W)S(U,V) + B(R(U,V)W) \\
& + D(R(U,W)V) + \alpha(U)[(n-2)S(V,W) + rg(V,W)] \\
& + \beta(V)[(n-2)S(U,W) + rg(U,W)] + \gamma(W)[(n-2)S(U,V) + rg(U,V)] \\
& + \tilde{\beta}(U)g(V,W) + \beta(U)S(V,W) - \beta(V)S(U,W) - \tilde{\beta}(V)g(U,W) \\
& + \tilde{\gamma}(U)g(V,W) + \gamma(U)S(V,W) - \tilde{\gamma}(W)g(V,U) - \gamma(W)S(V,U).
\end{aligned}
\tag{5}
$$

By further contracting the above equation, we get

$$
\begin{aligned}
dr(U) = {} & rA(U) + 2\tilde{B}(U) + 2\tilde{D}(U) + 2(n-1)r\alpha(U) + 2(n-2)[\tilde{\beta(U)} + \tilde{\gamma}(U)] \\
& + 2r[\beta(U) + \gamma(U)],
\end{aligned}
\tag{6}
$$

where $\tilde{B}(U) = S(U,\ \phi_1)$, $\tilde{D}(U) = S(U,\pi_1)$, $\tilde{\beta}(U) = S(U,\ \phi_2)$ and $\tilde{\gamma}(U) = S(U,\ \pi_2)$. Assuming that the scalar curvature of a hyper-generalized weakly symmetric para-Sasakian manifold is a non-zero constant (to avoid flat manifold and to preserve the para-Sasakian Structure), equation (6) reduces to

$$
\begin{aligned}
& r[A(U) + 2(n-1)\alpha(U) + 2\beta(U) + 2\gamma(U)] \\
& = -2\tilde{B}(U) - 2\tilde{D}(U) - 2(n-2)[\tilde{\beta(U)} + \tilde{\gamma}(U)].
\end{aligned}
\tag{7}
$$

This leads to the following result:

Theorem 1. Let M be a hyper-generalized weakly symmetric para-Sasakian manifold with non-zero constant scalar curvature. The 1-forms are then related by the equation (7).

Putting $V = \xi$ in (5), we get

$$
\begin{aligned}
(\nabla_U S)(\xi,W) = {} & -(n-1)A(U)\eta(W) + B(\xi)S(U,W) - (n-1)D(W)\eta(U) + B(\xi)g(U,W) \\
& - \eta(W)B(U) + D(W)\eta(U) - \eta(W)D(U) - (n-1)(n-2)\alpha(U)\eta(W) \\
& + r\eta(W)\alpha(U) + (n-2)\beta(\xi)S(U,W) + r\beta(\xi)g(U,W) - \tilde{\beta}(\xi)g(U,W) \\
& + r\gamma(W)\eta(U) + \tilde{\beta}(U)\eta(W) - (n-1)\beta(U)\eta(W) - \beta(\xi)S(U,W) + \tilde{\gamma}(U)\eta(W) \\
& - (n-1)\gamma(U)\eta(W) - \tilde{\gamma}(W)\eta(U) + (n-1)\gamma(W)\eta(U) \\
& - (n-1)(n-2)\gamma(W)\eta(U).
\end{aligned}
\tag{8}
$$

Setting $W=\xi$ in (5), we have

$$\begin{aligned}(\nabla_U S)(V,\xi) &= -(n-1)A(U)\eta(V) - (n-1)B(V)\eta(U) + D(\xi)S(U,V) + B(V)\eta(U)\\ &- B(U)\eta(V) + D(\xi)g(U,V) - D(U)\eta(V) - (n-1)(n-2)\alpha(U)\eta(V)\\ &+ r\alpha(U)\eta(V) - (n-1)(n-2)\beta(V)\eta(U) + r\beta(V)\eta(U) + r\gamma(\xi)g(U,V)\\ &+ (n-2)\gamma(\xi)S(U,V) + \tilde{\beta}(U)\eta(V) - (n-1)\beta(U)\eta(V) - \tilde{\beta}(V)\eta(U)\\ &+ (n-1)\beta(V)\eta(U) + \tilde{\gamma}(U)\eta(V) - (n-1)\gamma(U)\eta(V) - \tilde{\gamma}(\xi)\eta(V)\\ &- \gamma(\xi)S(U,V).\end{aligned} \tag{9}$$

Replacing $V=W$ in the above equation, we obtain

$$\begin{aligned}(\nabla_U S)(W,\xi) &= -(n-1)A(U)\eta(W) - (n-1)B(W)\eta(U) + D(\xi)S(U,W) + B(W)\eta(U)\\ &- B(U)\eta(W) + D(\xi)g(U,W) - D(U)\eta(W) - (n-1)(n-2)\alpha(U)\eta(W)\\ &+ r\alpha(U)\eta(W) - (n-1)(n-2)\beta(W)\eta(U) + r\beta(W)\eta(U) + r\gamma(\xi)g(U,W)\\ &+ (n-2)\gamma(\xi)S(U,W) + \tilde{\beta}(U)\eta(W) - (n-1)\beta(U)\eta(W) - \tilde{\beta}(W)\eta(U)\\ &+ (n-1)\beta(W)\eta(U) + \tilde{\gamma}(U)\eta(W) - (n-1)\gamma(U)\eta(W) - \tilde{\gamma}(\xi)\eta(W)\\ &- \gamma(\xi)S(U,W).\end{aligned} \tag{10}$$

Comparing (8) and (10), we have

$$\begin{aligned}&[B(\xi) + (n-3)\beta(\xi) - D(\xi) - (n-1)\gamma(\xi)]S(U,W)\\ &= [(n-2)D(Z) + (n-2)^2\gamma(W) - r\gamma(W) - (n-2)B(W) + (n-1)(n-3)\beta(W) + r\beta(W)\\ &- \tilde{\beta}(W)]\eta(U) + [\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]g(U,W).\end{aligned} \tag{11}$$

Putting $U=\xi$ in (11), we get

$$\begin{aligned}&(n-2)D(W) + (n-2)^2\gamma(W) - r\gamma(W) - (n-2)B(W) + (n-1)(n-3)\beta(W)\\ &+ r\beta(W) - \tilde{\beta}(W) = [(n-1)B(\xi) + (n-1)(n-3)\beta(\xi) - (n-1)D(\xi)\\ &- (n-1)^2\gamma(\xi)\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]\eta(W).\end{aligned} \tag{12}$$

Substituting (12) in (11), we deduce

$$\begin{aligned}&[B(\xi) + (n-3)\beta(\xi) - D(\xi) - (n-1)\gamma(\xi)]S(U,W)\\ &= [(n-1)B(\xi) + (n-1)(n-3)\beta(\xi) - (n-1)D(\xi)\\ &- (n-1)^2\gamma(\xi)\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]\eta(U)\eta(W)\\ &+ [\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]g(U,W).\end{aligned}$$

Thus, we can state the following theorem:

Theorem 2. A hyper-generalized para-Sasakian manifold is an η-Einstein manifold if

$$B(\xi) + (n-3)\beta(\xi) \neq D(\xi) + (n-1)\gamma(\xi).$$

We can also observe that,

$$(\nabla_X S)(Y,\xi) = -(n-1)g(X,\phi Y) + S(Y,\phi X). \tag{13}$$

From (9) and (13), we derive

$$\begin{aligned}
&-(n-1)g(U,\phi V)+S(V,\phi U)\\
&=-(n-1)A(U)\eta(V)-(n-1)B(V)\eta(U)+D(\xi)S(U,V)+B(V)\eta(U)\\
&-B(U)\eta(V)+D(\xi)g(U,V)-D(U)\eta(V)-(n-1)(n-2)\alpha(U)\eta(V)\\
&+r\alpha(U)\eta(V)-(n-1)(n-2)\beta(V)\eta(U)+r\beta(V)\eta(U)+r\gamma(\xi)g(U,V)\\
&+(n-2)\gamma(\xi)S(U,V)+\tilde{\beta}(U)\eta(V)-(n-1)\beta(U)\eta(V)-\tilde{\beta}(V)\eta(U)\\
&+(n-1)\beta(V)\eta(U)+\tilde{\gamma}(U)\eta(V)-(n-1)\gamma(U)\eta(V)-\tilde{\gamma}(\xi)\eta(V)-\gamma(\xi)S(U,V).
\end{aligned} \tag{14}$$

Setting $U=V=\xi$ in (14) yields

$$\begin{aligned}
&(n-1)[A(\xi)+B(\xi)+D(\xi)+(n-2)\alpha(\xi)+(n-2)\beta(\xi)+(n-2)\gamma(\xi)]\\
&-r[\alpha(\xi)+\beta(\xi)+\gamma(\xi)]=0.
\end{aligned} \tag{15}$$

The above equation expresses a relationship between these functions along ξ (reeb vectors), indicating how the curvature properties interact under hyper-generalized weak symmetric. Hence, the following theorem can be stated:

Theorem 3. In para-Sasakian manifold with hyper-generalized weakly symmetric curvature condition, the relations (15) holds.

3 Codazzi Type Of Ricci Tensor

A Ricci tensor is of Codazzi type when its covariant derivative is symmetric with respect to its indices, i.e, for any vector fields U, V, W it satisfies

$$(\nabla_U S)(V,W)=(\nabla_W S)(V,U), \tag{16}$$

where ∇ is the Levi-Civita connection of the manifold's metric.

In view of (5) and (16), we have

$$\begin{aligned}
0=&[A(U)-D(U)+(n-2)\alpha(U)-(n-4)\gamma(U)+\beta(U)]S(V,W)\\
&+[D(W)+(n-4)\gamma(W)-A(W)-(n-2)\alpha(W)-\beta(W)]S(V,U)\\
&+[r\alpha(U)+\tilde{\beta}(U)+2\tilde{\gamma}(U)-r\gamma(U)]g(V,W)\\
&+[r\gamma(W)-\tilde{\gamma}(W)-r\alpha(W)-\tilde{\beta}(W)-\tilde{\gamma}(W)]g(V,U)\\
&+B(R(U,V)W)+2D(R(U,W)V)-B(R(W,V)U).
\end{aligned} \tag{17}$$

Now by substituting $W=\xi$ in (17), we obtain

$$\begin{aligned}
0=&-(n-1)[A(U)-D(U)+(n-2)\alpha(U)-(n-4)\gamma(U)+\beta(U)]\eta(V)\\
&+[D(\xi)+(n-4)\gamma(\xi)-A(\xi)-(n-2)\alpha(\xi)-\beta(\xi)]S(V,U)\\
&+[r\alpha(U)+\tilde{\beta}(U)+2\tilde{\gamma}(U)-r\gamma(U)]\eta(V)\\
&+[r\gamma(\xi)-\tilde{\gamma}(\xi)-r\alpha(\xi)-\tilde{\beta}(\xi)-\tilde{\gamma}(\xi)]g(V,U)\\
&-B(V)\eta(U)+B(U)\eta(V)-2D(U)\eta(V)+2D(\xi)g(U,V)\\
&+B(\xi)g(V,U)-B(V)\eta(U).
\end{aligned} \tag{18}$$

Putting $U=\xi$ in (18), we get

$$B(V)=B(\xi)\eta(V). \tag{19}$$

Replacing $V=\xi$ in (18), we deduce

$$\begin{aligned}
&(n-1)[A(U)-D(U)+(n-2)\alpha(U)-(n-4)\gamma(U)+\beta(U)]\\
&-[r\alpha(U)+\tilde{\beta}(U)+2\tilde{\gamma}(U)-r\gamma(U)]-B(U)+2D(U)\\
&=\{-(n-1)[D(\xi)+(n-4)\gamma(\xi)-A(\xi)-(n-2)\alpha(\xi)-\beta(\xi)]\\
&+[r\gamma(\xi)-2\tilde{\gamma}(\xi)-r\alpha(\xi)-\tilde{\beta}(\xi)-B(\xi)+D(\xi)]\}\eta(U).
\end{aligned} \tag{20}$$

Using (19) and (20) in (18), we obtain

$$\begin{aligned}
&[D(\xi)+(n-4)\gamma(\xi)-A(\xi)-(n-2)\alpha(\xi)-\beta(\xi)]S(U,V)\\
&=\{-(n-1)[D(\xi)+(n-4)\gamma(\xi)-A(\xi)-(n-2)\alpha(\xi)-\beta(\xi)]\\
&+[r\gamma(\xi)-2\tilde{\gamma}(\xi)-r\alpha(\xi)-\tilde{\beta}(\xi)-B(\xi)+D(\xi)]\}\eta(U)\eta(V)\\
&-[r\gamma(\xi)-2\tilde{\gamma}(\xi)-r\alpha(\xi)-\tilde{\beta}(\xi)-B(\xi)+D(\xi)]g(U,V).
\end{aligned}$$

From this, we can present the following theorem:

Theorem 4. A hyper-generalized weakly symmetric para-Sasakian manifold is η-Einstein if it admits Codazzi type of Ricci tensor, provided that

$$D(\xi)+(n-4)\gamma(\xi)=A(\xi)+(n-2)\alpha(\xi)+\beta(\xi).$$

4 *Recurrent Ricci Tensor*

If the hyper-generalized weakly symmetric para-Sasakian manifold has a recurrent Ricci tensor then

$$(\nabla_U S)(V,W)=\lambda(U)S(V,W). \tag{21}$$

In view of (5) and (21), we have

$$\begin{aligned}
\lambda(U)S(V,W)&=A(U)S(V,W)+B(V)S(U,W)+D(W)S(U,V)+B(R(U,V)W)\\
&+D(R(U,W)V)+\alpha(U)[(n-2)S(V,W)+rg(V,W)]\\
&+\beta(V)[(n-2)S(U,W)+rg(U,W)]+\gamma(W)[(n-2)S(U,V)+rg(U,V)]\\
&+\tilde{\beta}(U)g(V,W)+\beta(U)S(V,W)-\beta(V)S(U,W)-\tilde{\beta}(V)g(U,W)\\
&+\tilde{\gamma}(U)g(V,W)+\gamma(U)S(V,W)-\tilde{\gamma}(W)g(V,U)-\gamma(W)S(V,U).
\end{aligned} \tag{22}$$

Placing $W=\xi$ in (22), we obtain

$$\begin{aligned}
-(n-1)\lambda(U)\eta(V)&=-(n-1)A(U)\eta(V)-(n-1)B(V)\eta(U)+D(\xi)S(U,V)\\
&+B(V)\eta(U)-B(U)\eta(V)+D(\xi)g(U,V)-D(U)\eta(V)\\
&-(n-1)(n-2)\alpha(U)\eta(V)+r\alpha(U)\eta(V)+\tilde{\beta}(U)\eta(V)\\
&+r\beta(V)\eta(U)+(n-2)\gamma(\xi)S(U,V)+r\gamma(\xi)g(U,V)\\
&-(n-1)\beta(U)\eta(V)+(n-1)\beta(V)\eta(U)-\tilde{\beta}(V)\eta(U)\\
&+\tilde{\gamma}(U)\eta(V)-(n-1)\gamma(U)\eta(V)-\tilde{\gamma}(\xi)g(V,U)-\gamma(\xi)S(V,U)\\
&-(n-1)(n-2)\beta(V)\eta(U).
\end{aligned} \tag{23}$$

Putting $U=\xi$ in (23), we have

$$\begin{aligned}
&-(n-2)B(V)-(n-1)(n-3)\beta(V)+r\beta(V)-\tilde{\beta}(V)\\
&=[-(n-1)\lambda(\xi)+(n-1)A(\xi)+(n-1)D(\xi)+B(\xi)+(n-1)(n-2)\alpha(\xi)-r\alpha(\xi)\\
&-(n-1)(n-2)\gamma(\xi)-r\gamma(\xi)-\tilde{\beta}(\xi)+(n-1)\beta(\xi)]\eta(V).
\end{aligned} \tag{24}$$

Considering $V = \xi$ in (23), we get

$$\begin{aligned}&(n-1)A(U) - (n-1)\lambda(U) + B(U) + D(U) + (n-1)(n-2)\alpha(U)\\&- r\alpha(U) + (n-1)\beta(U) - \tilde{\beta}(U) - \tilde{\gamma}(U) + (n-1)\gamma(U)\\&= [-(n-2)\beta(\xi) - (n-2)D(\xi) - (n-1)(n-2)\beta(\xi) + r\beta(\xi)\\&- (n-1)^2\gamma(\xi) + r\gamma(\xi) - \tilde{\beta}(\xi) - \tilde{\gamma}(\xi)]\eta(U).\end{aligned} \tag{25}$$

Now, setting $U = V = \xi$ in (23), we deduce

$$\begin{aligned}&-(n-1)[A(\xi) + B(\xi) + D(\xi) + (n-2)\alpha(\xi) + (n-2)\beta(\xi) + (n-2)\gamma(\xi) - \lambda(\xi)]\\&+ r[\alpha(\xi) + \beta(\xi) + \gamma(\xi)] = 0.\end{aligned} \tag{26}$$

Putting the value of (24), (25), (26) in (23), we obtain

$$\begin{aligned}&[D(\xi) + (n-3)\gamma(\xi)]S(U,V)\\&= [B(\xi) + (n-3)D(\xi) - (n-1)(n-3)\gamma(\xi) + (n-1)\beta(\xi) + r\gamma(\xi)]\eta(U)\eta(V)\\&- [D(\xi) + (r-1)\gamma(\xi)]g(U,V).\end{aligned}$$

Theorem 5. In a hyper-generalized weakly symmetric para-Sasakian manifold, if the Ricci tensor is recurrent then the manifold becomes an η-Einstein, provided that

$$D(\xi) \neq (n-3)\gamma(\xi).$$

5 *Hyper-generalized Weakly Symmetric Para-Sasakian Manifold Admitting a Quarter-Symmetric Metric Connection*

For a $[(H(GWS))_n, \tilde{\nabla}]$ [11], we have

$$\begin{aligned}(\nabla_U \tilde{R})(V,X,Y,W) &= A(U)\tilde{R}(V,X,Y,W) + B(V)\tilde{R}(U,X,Y,W) + B(X)R(V,U,Y,W)\\&+ D(Y)\tilde{R}(V,X,U,W) + D(W)\tilde{R}(V,X,Y,U)\\&+ \alpha(U)(g \wedge \tilde{S})(V,X,Y,W) + \beta(V)(g \wedge \tilde{S})(U,X,Y,W)\\&+ \beta(X)(g \wedge \tilde{S})(V,U,Y,W) + \gamma(Y)(g \wedge \tilde{S})(V,X,U,W)\\&+ \gamma(W)(g \wedge \tilde{S})(V,X,Y,U),\end{aligned} \tag{27}$$

for all $X, Y, U, V, W \in TM$. Making use of (2), we can find

$$\begin{aligned}(\tilde{\nabla}_U \tilde{R})(V,X,Y,W) &= (\nabla_U \tilde{R})(V,X,Y,W) + \eta(\tilde{R}(V,X)Y)g(\phi U,W) - \eta(V)\tilde{R}(\phi U,X,Y,W)\\&- \eta(X)\tilde{R}(V,\phi U,Y,W) - \eta(Y)\tilde{R}(V,X,\phi U,W) - \eta(Y)\tilde{R}(V,X,Y,\phi U)\\&+ g(\phi U,V)\tilde{R}(\xi,X,Y,W) + g(\phi U,X)\tilde{R}(V,\xi,Y,W)\\&+ g(\phi U,Y)\tilde{R}(V,X,\xi,W) + g(\phi U,W)\tilde{R}(V,X,Y,\xi).\end{aligned}$$

Now, using (3) in the foregoing equation, we have

$$\begin{aligned}
(\tilde{\nabla}_U \tilde{R})(V,X,Y,W) &= (\nabla_U R)(V,X,Y,W) + 3(\nabla_U g)(\phi V,Y)g(\phi X,W) \\
&+ 3g(\phi V,Y)(\nabla_U g)(\phi X,W) - 3(\nabla_U g)(\phi X,Y)g(\phi V,W) \\
&- 3g(\phi X,Y)\nabla_U g)(\phi V,W) + (\nabla_U \eta)(V)g(X,W)\eta(Y) \\
&+ \eta(V)g(X,W)(\nabla_U \eta)(Y) - (\nabla_U \eta)(X)g(V,W)\eta(Y) \\
&- \eta(X)g(V,W)(\nabla_U \eta)(W) - g(X,Y)(\nabla_U \eta)(V)\eta(W) \\
&- g(X,Y)\eta(V)(\nabla_U \eta)(W) + g(V,Y)(\nabla_U \eta)(X)\eta(W) \\
&+ g(V,Y)\eta(X)(\nabla_U \eta)(W) - [\eta(R(V,X,Y)) - \eta(V)g(X,Y) \\
&+ \eta(X)g(V,Y)]g(\phi U,W) - \eta(V)[R(\phi U,X,Y,W) \\
&+ 3g(\phi U,\phi Y)g(\phi X,W) - 3g(\phi V,Y)g(\phi V,\phi W) - \eta(X)g(\phi U,W)\eta(Y) \\
&- \eta(X)g(\phi U,Y)\eta(W)] - \eta(X)[R(V,\phi U,Y,W) \\
&+ 3g(\phi V,Y)g(\phi U,\phi W) - 3g(\phi U,\phi Y)g(\phi V,W) + \eta(V)g(\phi U,W)\eta(Y) \\
&- \eta(V)g(\phi U,Y)\eta(W)] - \eta(Y)[R(V,X,\phi U,W) + 3g(\phi V,\phi U)g(\phi X,W) \\
&- 3g(\phi X,\phi U)g(\phi V,W) - \eta(V)g(X,\phi U)\eta(W) - \eta(X)g(V,\phi U)\eta(W)] \\
&- \eta(W)[R(V,X,\phi U) + 3g(\phi V,Y)g(\phi X,\phi U) \\
&- 3g(\phi X,Y)g(\phi V,\phi U) - \eta(V)g(X,\phi U)\eta(Y) - \eta(X)g(V,\phi U)\eta(Y)] \\
&+ g(\phi U,V)[R(\xi,X,Y,W) + g(X,W)\eta(Y) - g(X,Y)\eta(W)] \\
&+ g(\phi U,X)[R(V,\xi,Y,W) + g(X,W)\eta(V) - g(V,W)\eta(X)] \\
&+ g(\phi U,Y)[R(V,X,\xi,W) + g(X,W)\eta(V) - g(V,W)\eta(Y)] \\
&+ g(\phi U,W)[R(V,X,Y,\xi) - g(X,Y)\eta(V) + g(V,Y)\eta(X)].
\end{aligned} \tag{28}$$

A hyper-generalized weakly symmetric para-Sasakian manifold with quarter-symmetric metric connection $\tilde{\nabla}$ simplifies to a hyper-generalized weakly symmetric para-Sasakian manifold with a Riemannian metric connection ∇, provided the following condition is satisfied:

$$(\tilde{\nabla}_U \tilde{R})(V,X,Y,W) = (\nabla_U \tilde{R})(V,X,Y,W). \tag{29}$$

Using (3), (27) and (28) in (29), it yields

$$A(\xi) + B(\xi) + D(\xi) = 0,$$

for $U = X = Y = \xi$. From the above, we can state the following:

Theorem 6. A $[(H(GWS))_n, \tilde{\nabla}]$ is a $[(H(GWS))_n, \nabla]$, if the following relations hold

$$A(\xi) + B(\xi) + D(\xi) = 0.$$

Conclusion

In this paper, we have explored the geometric properties of hyper-generalized weakly symmetric (H(GWS)) para-Sasakian manifolds. We have shown that, under certain conditions, (H(GWS)) para-Sasakian manifolds can reduce to simpler forms, such as η-Einstein manifolds, when specific relations between their 1-forms are satisfied. The study also highlights the role of quarter-symmetric metric connections, showing how these connections influence the manifold's curvature and overall geometric behavior. This work not only advances the understanding of para-Sasakian geometry but also provides a framework for exploring further extensions and applications of weakly symmetric manifolds in differential geometry and theoretical physics.

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Author Contributions

Bidyabati Thangjam: Conceptualization, Methodology, Formal Analysis, Writing-Original draft, Writing-Review and Editing, Validation.

M.S. Devi: Methodology, Review, Validation.

Conflict of Interest

The authors declare no conflict of interest.

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Analyzing Restrained Pitchfork Domination Across Path-Related Graph Structures

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Let $G = (V, E)$ be a finite, simple, and undirected graph without an isolated vertex. A dominating subset $D \subseteq V(G)$ is a restrained pitchfork dominating set if $1 \leq |N(u) \cap V - D| \leq 2$ for every $u \in D$ and every vertex not in D is adjacent to at least one vertex in the same set. The cardinality of a minimum restrained pitchfork dominating set is the restrained pitchfork domination number $\gamma_{rpf}(G)$. In the course of this investigation, we undertake an examination of the restrained pitchfork domination number within various path-related graphs. This analysis encompasses a range of graph structures, including the coconut tree, double star, banana tree, binomial tree, thorn path, thorn graph, and the square of the path denoted as P_n.

Keywords: Domination, restrained domination, pitchfork domination, restrained pitchfork domination, path graph.

2020 Mathematics Subject Classification: 05C69, 05C38.

Introduction

Graph theory provides a fundamental framework for understanding and analyzing various systems and networks, ranging from social networks to biological pathways to communication networks. A graph $G = (V, E)$ comprises a set V of vertices (or nodes) and a set E of edges (or connections) that link pairs of vertices. The order of a graph, denoted as n, represents the number of vertices in the graph, while the size, denoted as m, indicates the number of edges. For basic and detailed concepts, we refer [1, 2].

Dominating sets play a crucial role in graph theory, offering insights into the structure and connectivity of graphs. A dominating set $D \subseteq V(G)$ within a graph G ensures that every vertex not in D is adjacent to at least one vertex in D. A dominating set D is considered minimal if no proper subset of D retains the dominating property. The cardinality of the smallest dominating set in a graph G is known as the domination number $\gamma(G)$, representing a fundamental parameter of the graph's structure.

In certain contexts, such as when studying path-related graphs or tree structures [3, 4], additional constraints on dominating sets may be considered. A dominating subset D is deemed restrained if each vertex outside of D is adjacent to at least one vertex within D. Furthermore, a specialized form of dominating set, known as a pitchfork dominating set, imposes stricter conditions: each vertex within the dominating set must dominate at least one vertex and at most two vertices outside of the set [5, 6].

The concept of restrained domination has garnered attention [7,8], particularly in the study of path-related graphs, as explored by Vaidya [9, 10]. Additionally, research on restrained domination in tree structures has been well-documented. These investigations highlight the significance of understanding and characterizing various types of dominating sets in different graph structures, shedding light on their properties and implications in diverse applications.

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1 Restrained pitchfork domination in paths

Definition 1. Let $G = (V, E)$ be a finite, simple, undirected graph without isolated vertices. A dominating subset D of $V(G)$ is a restrained pitchfork dominating set if $1 \leq |N(u) \cap (V - D)| \leq 2$ for every $u \in D$, and every vertex in $V - D$ is adjacent to at least one vertex in $V - D$. D is minimal if it has no proper restrained pitchfork dominating subset. The restrained pitchfork domination number is denoted by $\gamma_{rpf}(G)$, which is the minimum cardinality of a minimal restrained pitchfork dominating set.

Observation 1. Let G be a graph with restrained pitchfork domination number $\gamma_{rpf}(G)$ and a restrained pitchfork dominating set D. Then:

(i) $\gamma_{rpf} \geq 1$.

(ii) The degree of each vertex is greater than or equal to 1 for every $u \in D$.

(iii) Each restrained pitchfork dominating set has a vertex of degree one that belongs to it.

Definition 2. [11] (see Fig. 1) For any positive integers n and m greater than 2, the coconut tree graph $CT(m, n)$ is constructed by appending n additional pendant edges at the final vertex of the path P_m.

Definition 3. [3] The double star graph $ST(m, n)$ is formed by connecting the centres of two stars, $ST(m)$ and $ST(n)$, thereby creating an edge between them.

Definition 4. [3] (see Fig. 2) A banana tree, denoted as $B(m, n)$, is obtained by linking one leaf from each of m copies of an n-star network to a new single root vertex, represented by v.

Definition 5. [3](see Fig. 3) The binomial tree B_n of order zero consists of a single node R if $n = 0$. For $n > 0$, B_n includes the root R and n subtrees $B_0, B_1, \ldots, B_{n-1}$.

Definition 6. [3](see Fig. 4) A thorn path $P_{n,p,k}$ is created by adding p neighbors to each non-terminal vertex of the path P_n, and k neighbors to each terminal vertex.

Definition 7. [3](see Fig. 5) A thorn rod $P_{n,m}$ consists of terminal vertices of degree m at both ends and a linear network with n vertices in between.

Definition 8. [3](see Fig. 6) The square of a graph G, denoted as G^2, shares the same vertex set as G and includes an edge between any two vertices u and v if the distance between them in G is less than 3.

2 Main Results

Theorem 1. Let $CT(m, n)$ be a coconut tree where $m \geqslant 2$, $n \geqslant 3$ then

$$\gamma_{rpf} CT(m,n) = \begin{cases} m + [\frac{n}{3}] & \text{for} \quad n \equiv 0 \pmod 3, \\ m + \lfloor \frac{n-2}{3} \rfloor + 1 & \text{for} \quad n \equiv 1 \pmod 3, \\ m + \lfloor \frac{n-3}{3} \rfloor + 3 & \text{for} \quad n \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $CT(m, n)$ be a coconut tree with the dominating set D. As it is a restrained dominating set all the pendent vertices are in D. Since it has m pendent edges that are adjacent to the n^{th} vertex of p_n. It has m pendent vertices and all of m are in D. Now we consider only P_n. There are three cases in D. Hence D is of any one of the forms. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ be the set of vertices. Hence

$$D = \begin{cases} v_{3i+1} & i = 0, 1, \ldots, \frac{n}{3} - 1 \quad \text{for} \quad n \equiv 0 \pmod 3, \\ v_{3i+1} & i = \left\{0, 1, \ldots, \lceil \frac{n}{3} \rceil - 2\right\} \cup \{v_{n-2}\} \quad \text{for} \quad n \equiv 1 \pmod 3, \\ v_{3i+1} & i = \left\{0, 1, \ldots, \lceil \frac{n}{3} \rceil - 3\right\} \cup \{v_{n-2}, v_{n-3}, v_{n-6}\} \quad \text{for} \quad n \equiv 2 \pmod 3. \end{cases}$$

Case (i): If $n \equiv 0 \pmod 3$.

Let us divide the vertex sets into $\frac{n}{3}$ subsets. It contains $\frac{n}{3}$ subsets, each containing 3 elements. From that, we take the first element. Hence, we get $m + [\frac{n}{3}]$.

Case (ii): If $n \equiv 1 \pmod 3$.

Let us divide the vertex sets into $\frac{n}{3}$ subsets. It contains $\lfloor \frac{n-2}{3} \rfloor$ subsets, each contains 3 elements. From that, we take the first vertex. Since v_n cannot be in D, from the remaining subsets, we take v_{n-2}. Hence, we get $m + \lfloor \frac{n-2}{3} \rfloor + 1$.

Case (iii): If $n \equiv 2 \pmod 3$.

Let us divide the vertex sets into $\frac{n}{3}$ subsets. It contains $\lfloor \frac{n-3}{3} \rfloor$ subsets, each contains 3 elements. From that, we take the first vertex. Since v_n cannot be in D, from the remaining subsets we take v_{n-2}, v_{n-3}, v_{n-6}. Hence, we get $m + \lfloor \frac{n-3}{3} \rfloor + 3$.

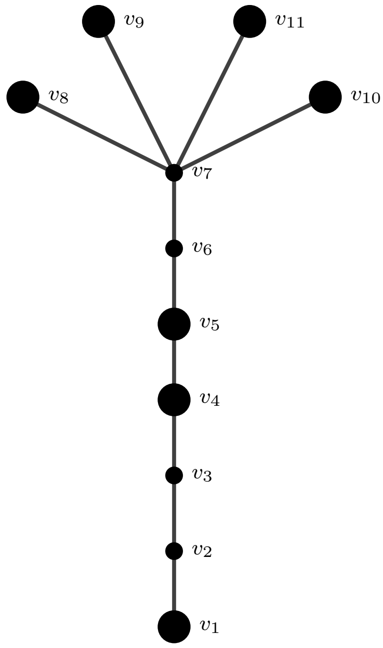

Figure 1. $\gamma_{rpf} CT(7, 4)$

Theorem 2. Let $ST(m, n)$ be a double star graph with $m, n \geqslant 1$, then $\gamma_{rpf} ST(m, n) = m + n$.

Proof. Since this graph contains $m + n$ pendent vertices, hence the result.

Theorem 3. Let $B(m, n)$ be a banana graph, then $\gamma_{rpf} B(m, n) = m(n-2) + 1$ if and only if $m = 2$.

Proof. Let the banana tree comprise star graphs and its root vertex v_0. Let the vertex set of each (m) star graph be $\{v_1, v_2, \ldots, v_n\}$. Suppose that one of the vertices, say v_3, has degree n, and all other vertices have degree one. Among them, one pendent vertex, say (v_1), is adjacent to the root vertex v_0 whose degree is two. Hence each star graph has $(n-2)$ pendent vertices and these belong to D. Moreover, v_0 is in D. Hence $m(n-2) + 1$.

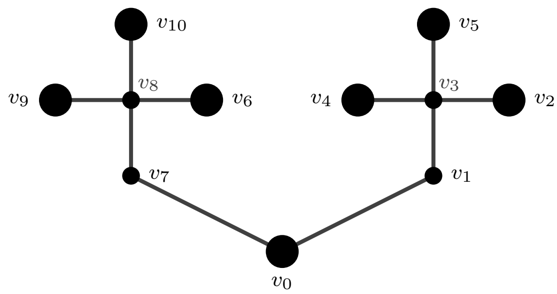

Figure 2. $\gamma_{rpf}B(2,5)$

Theorem 4. Let B_n be a binomial tree with $n \geqslant 2$, then $\gamma_{rpf}Bn = 2(n-1)$.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices in B_n. Since B_n can be formed from two copies of B_{n-1}, each with $(n-1)$ children at the root, all of which have degree one. Obviously, these two $(n-1)$ vertices belong to D. Hence, $2(n-1)$.

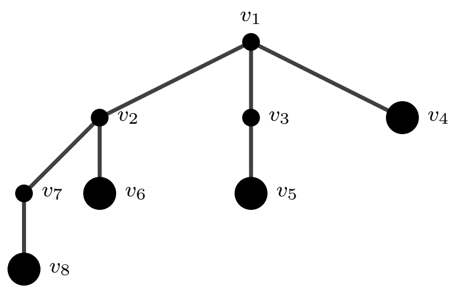

Figure 3. $\gamma_{rpf}B_3$

Theorem 5. Let $P_{n,p,k}$ be a thorn path graph then

$$\gamma_{rpf}(P_{n,p,k}) = \begin{cases} 2k & \text{for} \quad n = 2, \\ 2k + (n-2)p & \text{for} \quad n \geqslant 3. \end{cases}$$

Proof. Case (i): If $n = 2$.

Let $P_{n,p.k}$ be a thorn path graph. Now we are adding k vertices to the terminal vertices. Hence, we get $2k$ pendent vertices which are all in D.

Case (ii): If $n \geqslant 3$.

In this case, there are $(n-2)$ non terminal vertices and 2 terminal vertices. Then each of $(n-2)$ non terminal vertices has p pendent vertices, and each of two terminal vertices is attached to k pendent vertices. Thus it contains $2k + (n-2)p$ vertices in D.

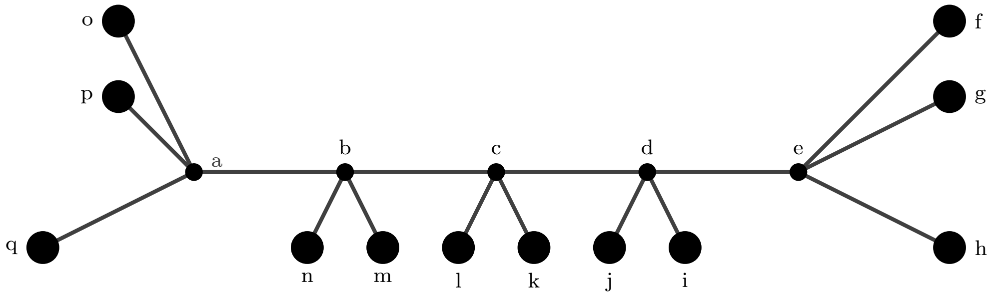

Figure 4. $\gamma_{rpf}P_{5,3,2}$

Theorem 6. Let $(P_{n,m})$ be Thorn rod graph then

$$\gamma_{rpf}(P_{n,m}) = \begin{cases} 2m & \text{for} \quad n = 2, \\ 2m + \lfloor \frac{n-1}{3} \rfloor + 1 & \text{for} \quad n \equiv 0 \pmod 3, \\ 2m + \lfloor \frac{n-5}{3} \rfloor + 3 & \text{for} \quad n \equiv 1 \pmod 3, \\ 2m + \lfloor \frac{n}{3} \rfloor & \text{for} \quad n \equiv 2 \pmod 3. \end{cases}$$

Proof. *Case (i):* If $n = 2$.

It is obvious, that all the pendent vertices belong to D.

Case (ii): If $n \equiv 0 \pmod 3$.

Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices in P_n. Moreover, v_1 and v_n are adjacent to m pendent vertices. Then it has $2m$ pendent vertices. Hence, v_1 and v_n are not in D. Therefore v_{n-2} is in D. Thus, D is in any one of the form

$$D = \begin{cases} v_{3i+3} & i = \left\{0, 1, \ldots, \frac{n}{3} - 1\right\} \cup \{v_{n-2}\} \quad \text{for} \quad n \equiv 0 \pmod 3, \\ v_{3i+3} & i = \left\{0, 1, \ldots, \lfloor \frac{n-5}{3} \rfloor - 1\right\} \cup \{v_{n-2}, v_{n-3}, v_{n-6}\} \quad \text{for} \quad n \equiv 1 \pmod 3, \\ v_{3i+3} & i = \left\{0, 1, \ldots, \lfloor \frac{n}{3} \rfloor\right\} \quad \text{for} \quad n \equiv 2 \pmod 3. \end{cases}$$

Let us divide the vertex set into $\frac{n}{3}$ subsets. Since v_{n-2} is in D, from $\lfloor \frac{n-1}{3} \rfloor$ subsets we take the last vertex. Hence D becomes $2m + \lfloor \frac{n-1}{3} \rfloor + 1$.

Case (iii): If $n \equiv 1 \pmod 3$.

Let us divide the vertex set into $\frac{n}{3}$ subsets. Since v_{n-2} is always included in D, we can consider only $\lfloor \frac{n-5}{3} \rfloor$ subsets. From that set, we take one (the last) vertex. Still, it does not satisfy our condition, so we also take v_{n-3}, v_{n-6}. Then we get $D = 2m + \lfloor \frac{n-5}{3} \rfloor + 3$.

Case (iv): If $n \equiv 2 \pmod 3$.

Let us divide $P(V_n)$ into $\frac{n}{3}$ subsets. It has $\lfloor \frac{n}{3} \rfloor$ subsets. From each subset, we take one (the last) vertex. Hence, D is $2m + \lfloor \frac{n}{3} \rfloor$.

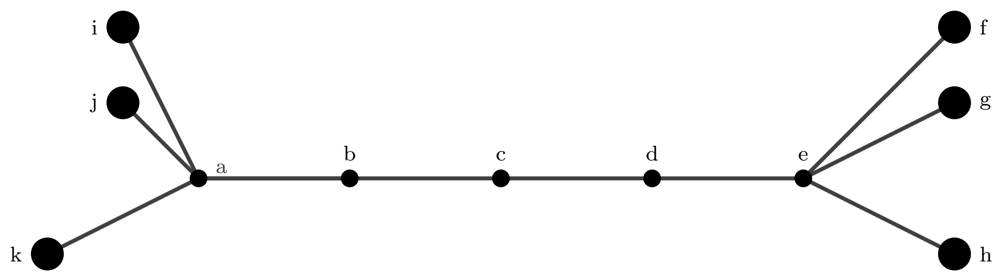

Figure 5. $\gamma_{rpf} P_{5,3}$

Theorem 7. Let the square of the path graph be P_n^2 with $n \geqslant 3$, then

$$\gamma_{rpf}(P_n^2) = \begin{cases} 1 & \text{for} \quad n = 3, \\ 2 & \text{for} \quad n = 4, 5, 6, \\ 3 & \text{for} \quad n = 7, \\ 4 & \text{for} \quad n = 8, \\ 4[\frac{n}{8}] & \text{for} \quad n \equiv 0, 1 \pmod 8, \\ 4\lfloor \frac{n}{8} \rfloor & \text{for} \quad n \equiv 2 \pmod 8, \\ 4\lfloor \frac{n}{8} \rfloor + 1 & \text{for} \quad n \equiv 1, 3, 4, 5 \pmod 8, \\ 4\lfloor \frac{n}{8} \rfloor + 2 & \text{for} \quad n \equiv 1, 6 \pmod 8, \\ 4\lfloor \frac{n}{8} \rfloor + 3 & \text{for} \quad n \equiv 1, 7 \pmod 8. \end{cases}$$

Proof. Case (i): If $n = 3$. It is obvious.

Case (ii): If $n = 4, 5, 6$. Then v_1 and v_n are in the dominating set.

Case (iii): If $n = 7$. Here v_1, v_{n-1}, and v_n are vertices, which satisfy our conditions and hence, they belong to D.

Case (iv): If $n = 7$. Here v_1, v_2, v_{n-1}, and v_n are vertices, which satisfy our conditions and hence, they belong to D.

Case (v): If $n \equiv 1 \pmod 8$.

Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices. Let us divide the vertex set into $\frac{n}{8}$ subsets. Here $\lfloor \frac{n}{8} \rfloor$ subsets contain 4 dominating vertices and the remaining subsets may be P_1, P_2, P_3, P_4, P_5, P_6 or P_7. Now we consider the following cases:

Case (a): If $n \equiv 1 \pmod 8$ and the remaining subset is either P_1 or P_2, then each $\lfloor \frac{n}{8} \rfloor$ has 4 vertices. Therefore D is of the form $v_{8i+1}, v_{8i+5}, v_{8i+6}, v_{8i+7}$, $i = 0, 1, \ldots, \lfloor \frac{n-1}{8} \rfloor - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor$.

Case (b): If $n \equiv 1 \pmod 8$ and the remaining subset is P_3, then each $\lfloor \frac{n}{8} \rfloor$ has 4 vertices. Therefore D is of the form $v_{8i+1}, v_{8i+5}, v_{8i+6}, v_{8i+7} \cup \{v_n\}$, $i = 0, 1, \ldots, \lfloor \frac{n-1}{8} \rfloor - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 1$.

Case (c): If $n \equiv 1 \pmod 8$ and the remaining subset is either P_4 or P_5, then each $\lfloor \frac{n}{8} \rfloor$ has 4 vertices. Therefore D is of the form $v_{8i+1}, v_{8i+5}, v_{8i+6}, v_{8i+7} \cup \{v_n, v_{n-3}\}$, $i = 0, 1, \ldots, \lfloor \frac{n-1}{8} \rfloor - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 2$.

Case (d): If $n \equiv 1 \pmod 8$ and the remaining subset is either P_6 or P_7, then each $\lfloor \frac{n}{8} \rfloor$ has 4 vertices. Therefore D is of the form $v_{8i+1}, v_{8i+5}, v_{8i+6}, v_{8i+7} \cup \{v_n, v_{n-1}, v_{n-5}\}$, $i = 0, 1, \ldots, \lfloor \frac{n-1}{8} \rfloor - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 3$.

Case (vi): If $n \equiv 0 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets, each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+2}, v_{8i+7}, v_{8i+8}$, $i = 0, 1, \ldots, \frac{n}{8} - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor$.

Case (vii): If $n \equiv 2 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets, each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8}$, $i = 0, 1, \ldots, \frac{n-2}{8} - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor$.

Case (viii): If $n \equiv 3 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8} \cup \{v_n\}$, $i = 0, 1, \ldots, \frac{n-3}{8} - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 1$.

Case (ix): If $n \equiv 4 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8} \cup \{v_n\}$, $i = 0, 1, \ldots, \frac{n-4}{8} - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 1$.

Case (ix): If $n \equiv 5 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ and each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8} \cup \{v_n\}$, $i = 0, 1, \ldots, \frac{n-5}{8} - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 1$.

Case (x): If $n \equiv 6 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets each containing 8 vertices. From each subset, we take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8} \cup \{v_n, v_{n-1}\}$, $i = 0, 1, \ldots, \frac{n-6}{8} - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 2$.

Case (xi): If $n \equiv 7 \pmod 8$, let us divide $V(P_n)$ into $\frac{n}{8}$ subsets each containing 8 vertices. From each subset, take 4 vertices of the form $v_{8i+1}, v_{8i+6}, v_{8i+7}, v_{8i+8} \cup \{v_n, v_{n-1}, v_{n-5}\}$, $i = 0, 1, \ldots, \frac{n-6}{8} - 1$. Hence, D is $4\lfloor \frac{n}{8} \rfloor + 3$.

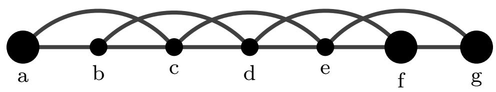

Figure 6. $\gamma_{rpf} {P_7}^2$

Conclusion

In this study, we examined the concept of restrained pitchfork domination across various path-related graph structures, establishing key results for their domination numbers. Through rigorous mathematical analysis, we derived explicit formulations for the restrained pitchfork domination number in structures such as the coconut tree, double star, banana tree, binomial tree, thorn path, thorn rod, and the square of a path. The results obtained contribute to the broader understanding of domination in graph theory, particularly in specialized graph classes.

The findings presented in this paper not only provide theoretical insights but also hold potential for applications in network optimization, communication systems, and combinatorial optimization problems where controlled domination constraints are relevant. The scientific novelty of this work lies in the extension of existing domination parameters by incorporating restrained pitchfork constraints, thereby refining structural characterizations of these graphs.

Future research in this area can explore variations of restrained pitchfork domination in more complex graph families, including weighted graphs and directed graphs. Additionally, investigating algorithmic approaches to efficiently compute restrained pitchfork domination numbers in large-scale graphs remains an open direction for further study.

Author Contributions

P. Vijayalakshmi collected and analyzed data, and led manuscript preparation. K. Karuppasamy served as the principal investigator of the research and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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On the properties for families of function classes over harmonic intervals and their embedding relation with Besov spaces

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The article is dedicated to the issues of studying the approximation of functions by trigonometric polynomials with a spectrum from special sets. In this paper, these special sets are harmonic intervals. To study the approximation of functions over harmonic intervals, families of function classes have been created, designed as a subsidiary tool. These families of function classes are characterized through the best approximations of functions by trigonometric polynomials over such sets and are used in the research. For these families of function classes, their properties and the connection with classical Besov spaces are shown. The results of the study are presented in the form of theorems and lemmas. In carrying out the research presented in the article, the main apparatus for proving theorems are the fundamentals of approximation theory, the method of real interpolation of spaces, and the fundamentals of the theory of embedding classes of functions and functional spaces. The article is destined for mathematicians and can be used by researchers and specialists whose interests lie in the indicated areas of mathematics.

Keywords: harmonic interval, spectrum, best approximation of a function by trigonometric polynomials with a spectrum from harmonic intervals, family of classes of functions, Besov spaces, embedding theorems.

2020 Mathematics Subject Classification: 42A10, 46B70.

Introduction

In approximation theory, the spectrum of approximating functions plays an important role. This spectrum can be selected from sets of different shapes, types, contours, structures, locations, etc. The spectrum can contain a wide variety of sets [1, 2].

A person's perception of information occurs through his sense organs, readings from various devices of measuring and observation, etc. And they all have a limited finite, not infinite, range. Based on this, when modeling diverse, multifaceted practical and applied problems [3–5], it is necessary to find a solution on some finite set that reflects this finite range. Special sets called harmonic intervals I_k^N [6, 7], where N defines exactly such a finite range of perception, to some extent help to conduct research and solve such problems.

The approximative properties of a function are usually characterized by the magnitude of the best approximation or the speed of approximation by a linear method. Often, during the course of research, there is a need to create auxiliary elements and tools. One of them is the families of classes of functions $\{B_{p,q,N}^N\}_N$, the definition of which is expressed in terms of the best approximations of functions by trigonometric polynomials with spectrum from sets I_k^N. These families of function classes are used, for example, in the theorems on the boundedness of the partial sum operator of the Fourier series, etc.

In carrying out the proofs of the statements described in the article, the fundamentals of approximation theory [8–10], the method of real interpolation of spaces [11–13] and the fundamentals of embedding theory [14] for classes of functions and function spaces are used.

In the article the properties of families of function classes $\left\{B_{p,q,N}^r\right\}_N$ and the relationship of these families of function classes to the classical Besov spaces [15, 16] are studied.

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1 Definitions and auxiliary results

We construct families of classes of functions related to the best approximations of functions by trigonometric polynomials with a spectrum from harmonic intervals.

Definition 1. [17] Let $1 \leq p, q \leq \infty$, $r > 0$, $N \in \mathrm{N}$, $f \in L_p\left[0; 2\pi\right)$. The family of function classes is denoted by $\left\{B^r_{p,q,N}\right\}_N$ and is defined in terms of the best approximations over harmonic intervals using the relation

$$B^r_{p,q,N} = \left\{f : \|f\|_{B^r_{p,q,N}} < \infty\right\},$$

where

$$\|f\|_{B^r_{p,q,N}} = \left(\sum_{k=1}^{N} k^{rq-1} \left(E^N_{k-1}\left(f\right)_p\right)^q\right)^{\frac{1}{q}}.$$

Definition 2. [17] Let A^N and B^N be two classes of functions depending on a parameter N. The class of functions A^N is embedded in the class of functions B^N and denote it by

$$A^N \hookrightarrow B^N$$

if the following conditions are met:

1) $A^N \subset B^N$;

2) there is a positive parameter C such that the inequality

$$\|f\|_{B^N} \leq C\,\|f\|_{A^N}$$

is correct for any $f \in A^N$, and the parameter C does not depend on f and N.

Definition 3. [17] Let two families of function classes $\left\{A^N\right\}_N$ and $\left\{B^N\right\}_N$ be given, where $N \in \mathrm{N}$ and $\left\{A^N\right\}_N \bigcap \left\{B^N\right\}_N = \varnothing$. The classes of functions $\left\{A^N\right\}_N$ and $\left\{B^N\right\}_N$ is said to be equivalent

$$\|f\|_{A^N} \sim \|f\|_{B^N}$$

if there are positive parameters C_1, C_2 such that for any $f \in A^N$ the relation

$$C_1\,\|f\|_{B^N} \leq \|f\|_{A^N} \leq C_2\,\|f\|_{B^N}$$

is correct. Moreover, the parameters C_1, C_2 do not depend on f and N. In this case, it is assumed that the families of function classes $\left\{A^N\right\}_N$ and $\left\{B^N\right\}_N$ coincide

$$\left\{A^N\right\}_N = \left\{B^N\right\}_N.$$

Definition 4. [11] Let (A_0, A_1) be an interpolation pair. For any t such that $0 < t < \infty$, the functional is defined by the equality

$$K\left(t, a; \bar{A}\right) = \inf_{a=a_0+a_1}\left(\|a_0\|_{A_0} + t\,\|a_1\|_{A_1}\right),$$

where $a = A_0 + A_1$, and $\bar{A}$=(A_0, A_1) is a compatible pair of spaces. This functional is called Petre's K-functional or simply Petre's functional.

Definition 5. [11] Let (A_0, A_1) be an interpolation pair, $0 < \theta < 1$. For 1$\leq q < \infty$, we have

$$\bar{A}_{\theta,q} = \left(A_0, A_1\right)_{\theta,q} =$$

$$= \left\{a :\, a \in A_0 + A_1, \quad \|a\|_{\bar{A}_{\theta,q}} = \left(\int_0^{\infty} \left[t^{-\theta} K\left(t, a; \bar{A}\right)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty\right\},$$

and for $q = \infty$ the equality

$$\bar{A}_{\theta,\infty} = (A_0, A_1)_{\theta,\infty} = \{a : a \in A_0 + A_1,\ \|a\|_{\bar{A}_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta} K\left(t, a; \bar{A}\right) < \infty\}$$

holds. The space $\bar{A}_{\theta,q}$ is defined by the following relation

$$\bar{A}_{\theta,q} = \{a : \|a\|_{\bar{A}_{\theta,q}} < \infty\}.$$

Theorem 1. [7] Let $m \in \mathrm{N}$, $1 \le p, q, q_0, q_1 \le \infty$, $0 < \theta < 1$, $r > 0$, $r_0 > r_1 > 0$. Then there is the embedding of the form

$$B^r_{p,q,2^m} \hookrightarrow \left(B^{r_0}_{p,q_0,2^m}; B^{r_1}_{p,q_1,2^m}\right)_{\theta,q},$$

where $r = (1-\theta)\, r_0 + \theta r_1$.

Theorem 2. [7] Let $N \in \mathrm{N}$, $1 \le p, q, q_1 \le \infty$, $r > 0$, then the following embedding

$$B^r_{p,q,N} \hookrightarrow B^r_{p,q_1,N}$$

is performed for $q < q_1$.

Theorem 3. [7] Let $f \in B^r_{p,q,2^m}$, $m \in \mathrm{N}$, $\sum\limits_{v\in \mathrm{Z}} a_v e^{ivx}$ be the trigonometric Fourier series of the function *f*. Then for $1 \le p, q \le \infty$, $r > 0$ the following relation

$$\|f\|_{B^r_{p,q,2^m}} \sim \left(\sum_{k=1}^{m} 2^{rqk} \left(\delta_k\left(f\right)_p\right)^q\right)^{\frac{1}{q}}$$

is correct, where

$$\delta_k\left(f\right)_p = \left\|\sum_{s\in \mathrm{Z}} \sum_{\tau=2^{k-1}}^{2^k-1} a_{\tau+s\cdot 2^m} \cdot e^{i(\tau+s\cdot 2^m)x}\right\|_p.$$

Lemma 1. [6] Let $B = [-k, k]$ be the segment in Z; $k, d, h \in \mathrm{N}$, $k < h$, $\left\{I^{h,d}_B\right\}_{d=0}^{\infty}$ be the sequence of harmonic segments in Z, converging to the harmonic interval I^h_B, where $I^h_B = \bigcup\limits_{v=-\infty}^{\infty} [B + vh]$. If $f \in L_p\,[0;\ 2\pi)$, $1 \le p \le \infty$, $\sum\limits_{v\in \mathrm{Z}} a_v e^{ivx}$ is the Fourier series of the function *f*, then the sequence of partial sums of the Fourier series of the function f over harmonic segments

$$S^{h,d}_B\left(f\right) = \sum_{v\in I^{h,d}_B} a_v e^{ivx}$$

converges in the space $L_p\,[0;\ 2\pi)$ as $d \to \infty$ to the function

$$S^h_B\left(f\right) = \frac{1}{h}\sum_{r=0}^{h-1} f\left(x + \frac{2\pi r}{h}\right) D_B\left(\frac{2\pi r}{h}\right), \tag{1}$$

where $D_B\left(x\right) = \sum\limits_{m\in B} e^{imx}$ is the Dirichlet kernel corresponding to the segment B from Z, and its Fourier series is the function $\sum\limits_{v\in I^h_B} a_v e^{ivx}$.

Theorem 4. [7] Let $m, d, N \in \mathrm{N}$, $f \in L_p[0;2\pi)$, $1 < p < \infty$, $S_m^N(f)$ and $E_m^N(f)_p$ be the partial sum of the Fourier series and the best approximation of the function f over the harmonic interval I_m^N, respectively, then the following relation is correct

$$E_m^N(f)_p \sim \left\| f - S_m^N(f) \right\|_p .$$

Theorem 5. [7] Let $f \in B^r_{p,q,2^m}, m \in \mathrm{N}$, then for $1 \le p, q \le \infty$, $r > 0$, we have

$$\|f\|_{B^r_{p,q,2^m}} \sim \left(\sum_{k=1}^{m} 2^{rqk} \left(E^{2^m}_{2^k-1}(f)_p \right)^q \right)^{\frac{1}{q}} .$$

2 Properties for families of function classes $\left\{ B^r_{p,q,N} \right\}_N$

Embedding Theorem 6 is the inverse of Embedding Theorem 1 for the same parameter values.

Theorem 6. Let $m \in \mathrm{N}$, $1 \le p, q, q_0, q_1 \le \infty$, $0 < \theta < 1$, $r > 0$, $r_0 > r_1 > 0$, then the embedding of the following type

$$\left(B^{r_0}_{p,q_0,2^m}; B^{r_1}_{p,q_1,2^m} \right)_{\theta,q} \hookrightarrow B^r_{p,q,2^m}$$

is satisfied, where $r = (1-\theta)\, r_0 + \theta\, r_1$.

Proof. From Theorem 2 for families of function classes $\left\{ B^r_{p,q,N} \right\}_N$, it follows that

$$B^r_{p,q,N} \hookrightarrow B^r_{p,\infty,N}.$$

Then, to prove the statement of the theorem, it is sufficient to prove the following embedding

$$\left(B^{r_0}_{p,\infty,2^m}; B^{r_1}_{p,\infty,2^m} \right)_{\theta,q} \hookrightarrow B^r_{p,q,2^m}.$$

Let $f = f_0 + f_1$ be an arbitrary representation of a function f, where $f_0 \in B^{r_0}_{p,\infty,2^m}$, $f_1 \in B^{r_1}_{p,\infty,2^m}$. Using Theorem 3 and Definition 1, we estimate the following expression

$$2^{r_0 k} \delta_k(f)_p \le 2^{r_0 k} \delta_k(f_0)_p + 2^{(r_0 - r_1)k} 2^{r_1 k} \delta_k(f_1)_p \le$$

$$\le \|f_0\|_{B^{r_0}_{p,\infty,2^m}} + 2^{(r_0-r_1)k} \|f_1\|_{B^{r_1}_{p,\infty,2^m}} .$$

Given the arbitrariness of the representation for the function f and using Definition 4, we receive the ratio

$$2^{r_0 k} \delta_k(f)_p \le \inf_{f = f_0 + f_1} \left(\|f_0\|_{B^{r_0}_{p,\infty,2^m}} + 2^{(r_0-r_1)k} \|f_1\|_{B^{r_1}_{p,\infty,2^m}} \right) =$$

$$= K\left(2^{(r_0-r_1)k}, f; B^{r_0}_{p,\infty,2^m}, B^{r_1}_{p,\infty,2^m} \right),$$

where K is Petre's functional.

Therefore, we obtain the following inequality

$$\|f\|_{B^r_{p,q,2^m}} \le \left(\sum_{k=1}^{m} 2^{rqk} \left(\delta_k(f)_p \right)^q \right)^{\frac{1}{q}} \le$$

$$\le \left(\sum_{k=1}^{m} 2^{(r-r_0)qk} \left\{ K\left(2^{(r_0-r_1)k}, f; B^{r_0}_{p,\infty,2^m}, B^{r_1}_{p,\infty,2^m} \right) \right\}^q \right)^{\frac{1}{q}} . \qquad (2)$$

Taking into account that $r - r_0 = -\theta\,(r_0 - r_1)$, considering the partition of the interval $(0;\infty)$ into half-intervals $\left[2^{k(r_0-r_1)}; 2^{(k+1)(r_0-r_1)}\right)$, $k \in \mathrm{Z}$, and using Definition 5, we transform inequality (2) in the following way

$$\|f\|_{B^r_{p,q,2^m}} \le \left(\sum_{k\in \mathrm{Z}} 2^{-\theta(r_0-r_1)qk}\left\{K\left(2^{(r_0-r_1)k},\ f;\ B^{r_0}_{p,\infty,2^m},\ B^{r_1}_{p,\infty,2^m}\right)\right\}^q\right)^{\frac{1}{q}} \le$$

$$\le \left(\sum_{k\in Z}\left\{2^{-\theta(r_0-r_1)k} K\left(2^{(r_0-r_1)k}, f;\ B^{r_0}_{p,\infty,2^m},\ B^{r_1}_{p,\infty,2^m}\right)\right\}^q \times\right.$$

$$\left.\times 2^{-(r_0-r_1)k}\left(2^{(r_0-r_1)(k+1)} - 2^{(r_0-r_1)k}\right)\right)^{\frac{1}{q}} \le$$

$$\le \left(\int_0^\infty \left\{t^{-\theta} K\left(t, f;\ B^{r_0}_{p,\infty,2^m},\ B^{r_1}_{p,\infty,2^m}\right)\right\}^q \frac{dt}{t}\right)^{\frac{1}{q}} = \|f\|_{\left(B^{r_0}_{p,\infty,2^m}; B^{r_1}_{p,\infty,2^m}\right)_{\theta,q}}.$$

As a result, we have

$$\|f\|_{B^r_{p,\infty,2^m}} \le C\,\|f\|_{\left(B^{r_0}_{p,\infty,2^m}; B^{r_1}_{p,\infty,2^m}\right)_{\theta,q}}.$$

This inequality determines the required statement of the theorem. The theorem is proved.

The following corollary follows directly from Theorems 1, 6 and Definitions 2, 3.

Corollary 1. Let $m \in \mathrm{N}$, $1 \le p, q, q_0, q_1 \le \infty$, $0 < \theta < 1$, $r > 0$, $r_0 > r_1 > 0$, then the equality

$$\left(B^{r_0}_{p,q_0,2^m}; B^{r_1}_{p,q_1,2^m}\right)_{\theta,q} = B^r_{p,q,2^m}$$

is correct, where $r = (1-\theta)\,r_0 + \theta\,r_1$.

3 Relationship for the families of function classes with Besov spaces

The following theorems show the connection between families of function classes $\left\{B^r_{p,q,N}\right\}_N$ and the classical Besov spaces.

Theorem 7. [17] If $N \in \mathrm{N}$, $1 \le p, q, \le \infty$, $r > 0$, then the following relation is fulfilled

$$\bigcap_{N=1}^{\infty} B^r_{p,q,N} = B^r_{p,q}.$$

Theorem 8. Let $1 \le p \le q \le \infty$, $1 \le r \le \infty$, $m \in \mathrm{N}$, $\alpha > 0$, $\beta > 0$, $\alpha - \beta = \frac{1}{p} - \frac{1}{q}$, then for any value of m the following embedding

$$B^\alpha_{p,r} \hookrightarrow B^\beta_{q,r,2^m}$$

holds, that is, the inequality

$$\|f\|_{B^\beta_{q,r,2^m}} \le C\,\|f\|_{B^\alpha_{p,r}}$$

is satisfied, where C is a constant that does not depend on f and m.

Proof. First, we show that the following inequality

$$\|f\|_{B^\beta_{q,\infty,2^m}} \le \|f\|_{B^\beta_{q,\infty}} \tag{3}$$

holds. Indeed, the above inequality is correct, since it follows from the relation

$$\|f\|_{B^{\beta}_{q,\infty,2^m}} = \max_{1\le k\le m} 2^{\beta k}\, E^{2^m}_{2^{k-1}}(f)_q \le \sup_{k\ge 1} 2^{\beta k}\, E_{2^k}(f)_q = \|f\|_{B^{\beta}_{q,\infty}}\,.$$

According to the Besov embedding theorem [16], for $1 \le p \le q \le \infty$, $\alpha - \beta = \frac{1}{p} - \frac{1}{q}$, we have such inequality

$$\|f\|_{B^{\beta}_{q,\infty}} \le C\,\|f\|_{B^{\alpha}_{p,\infty}}\,, \tag{4}$$

where C is a constant that does not depend on f and m. Based on (3) and (4), we get an inequality of the form

$$\|f\|_{B^{\beta_i}_{q,\infty,2^m}} \le C_i\,\|f\|_{B^{\alpha_i}_{p,\infty}}\,, \tag{5}$$

where C_i are constants that do not depend on f and m, $i = 0,\ 1$.

With the parameters α, β, p, q specified in the conditions of the theorem, we define the operator I as follows

$$If = f.$$

Let us take pairs (α_0, α_1) and (β_0, β_1), where the parameters α_0, β_0, α_1, β_1 satisfy the conditions

$$\alpha_0 < \alpha < \alpha_1,\ \beta_0 < \beta < \beta_1,\ \alpha_0 - \beta_0 = \alpha_1 - \beta_1 = \frac{1}{p} - \frac{1}{q}.$$

Taking into account inequality (5), we have

$$I : B^{\alpha_0}_{p,\infty} \to B^{\beta_0}_{q,\infty,2^m}$$

with the norm C_0, and

$$I : B^{\alpha_1}_{p,\infty} \to B^{\beta_1}_{q,\infty,2^m}$$

with the norm C_1. Then, by the interpolation theorem [11], we obtain

$$I : \left(B^{\alpha_0}_{p,\infty}; B^{\alpha_1}_{p,\infty}\right)_{\theta,r} \to \left(B^{\beta_0}_{q,\infty,2^m}; B^{\beta_1}_{q,\infty,2^m}\right)_{\theta,r}. \tag{6}$$

According to Petre's theorem [11], the equality

$$\left(B^{\alpha_0}_{p,\infty}; B^{\alpha_1}_{p,\infty}\right)_{\theta,r} = B^{\alpha_\theta}_{p,r} \tag{7}$$

is valid, where

$$\alpha_\theta = (1-\theta)\,\alpha_0 + \theta\alpha_1,$$

$0 < \theta < 1$. Using a similar equality for families of classes of functions $\left\{B^{r}_{p,q,N}\right\}_N$, we have the following relation

$$\left(B^{\beta_0}_{q,\infty,2^m}; B^{\beta_1}_{q,\infty,2^m}\right)_{\theta,r} = B^{\beta_\theta}_{q,r,2^m}, \tag{8}$$

where

$$\beta_\theta = (1-\theta)\,\beta_0 + \theta\beta_1,$$

$0 < \theta < 1$. Since

$$\alpha_\theta - \beta_\theta = (1-\theta)\,(\alpha_0 - \beta_0) + \theta\,(\alpha_1 - \beta_1) = \frac{1}{p} - \frac{1}{q},$$

then there exists $\theta \in (0;1)$ such that

$$\beta_\theta = \beta \ \ \Rightarrow\ \ \alpha_\theta = \alpha. \tag{9}$$

As a result, taking into account (7)–(9), relation (6) determines that

$$I : B^{\alpha}_{p,r} \to B^{\beta}_{q,r,2^m},$$

and besides

$$\|f\|_{B^{\beta}_{q,r,2^m}} \le C\,\|f\|_{B^{\alpha}_{p,r}}\,.$$

This inequality proves the statement of the theorem. The theorem is proved.

In particular, the following lemma is correct for the conditions of Theorem 6.

Lemma 2. Let $1 \le p \le q \le \infty$, $1 \le r \le \infty$; $m, n \in \mathrm{N}$, $\alpha > 0$, $\beta > 0$, $\alpha - \beta = \frac{1}{p} - \frac{1}{q}$. For functions f that are summable to the p-th power, the inequality

$$\left(\sum_{k=1}^{m} 2^{\beta\, rk} \left(\left\| f - \frac{1}{2^{m+1}} \sum_{n=0}^{2^{m+1}-1} f\left(x + \frac{\pi\, n}{2^m}\right) D_{2^k-1}\left(\frac{\pi\, n}{2^m}\right)\right\|_q\right)^r\right)^{\frac{1}{r}} \le C\,\|f\|_{B^{\alpha}_{p,r}}$$

holds, where $D_{2^k-1}\left(\frac{\pi\, n}{2^m}\right)$ is the Dirichlet kernel.

Proof. By successively applying (1), Theorems 4 and 5, we obtain the relation

$$\left(\sum_{k=1}^{m} 2^{\beta rk} \left(\left\| f - \frac{1}{2^{m+1}} \sum_{n=0}^{2^{m+1}-1} f\left(x + \frac{\pi n}{2^m}\right) D_{2^k-1}\left(\frac{\pi n}{2^m}\right)\right\|_q\right)^r\right)^{\frac{1}{r}} =$$

$$= \left(\sum_{k=1}^{m} 2^{\beta\, rk} \left(\left\| f - S^{2^m}_{2^k-1}(f)\right\|_q\right)^r\right)^{\frac{1}{r}} \sim$$

$$\sim \left(\sum_{k=1}^{m} 2^{\beta\, rk} \left(E^{2^m}_{2^k-1}(f)_q\right)^r\right)^{\frac{1}{r}} \sim \|f\|_{B^{\beta}_{q,r,2^m}}.$$

From this relation, taking into account the ratio

$$\|f\|_{B^{\beta}_{q,r,2^m}} \le C\,\|f\|_{B^{\alpha}_{p,r}}\,,$$

the required inequality follows. The lemma is proved.

Conclusion

In the article the properties of families of function classes $\left\{B^{r}_{p,q,N}\right\}_N$ are presented. Embedding theorems for the specified families of function classes and the lemma on estimating the norm of the Besov space are proved. The embedding theorem showing the relationship of these families of function classes to the classical Besov spaces is proved by the method of real interpolation. The results presented in the article will be used in future directions of research on the approximation of functions over harmonic intervals.

Author Contributions

L.A. Serikova conducted the research presented in the article under the scientific supervision of G.A. Yessenbayeva. All authors contributed equally to this work and participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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