ISSN 2518-7929 (Print)
ISSN 2663-5011 (Online)

BULLETIN
OF THE KARAGANDA UNIVERSITY

MATHEMATICS
SERIES

No. 1(117)/2025

ISSN 2518-7929 (Print)
ISSN 2663–5011(Online)
Индексі 74618
Индекс 74618

ҚАРАҒАНДЫ УНИВЕРСИТЕТІНІҢ **ХАБАРШЫСЫ**

ВЕСТНИК КАРАГАНДИНСКОГО УНИВЕРСИТЕТА

BULLETIN OF THE KARAGANDA UNIVERSITY

МАТЕМАТИКА сериясы

Серия МАТЕМАТИКА

MATHEMATICS Series

No. 1(117)/2025

1996 жылдан бастап шығады
Издается с 1996 года
Founded in 1996

Жылына 4 рет шығады
Выходит 4 раза в год
Published 4 times a year

Қарағанды / Караганда / Karaganda
2025

Postal address: 28, University Str., Karaganda, 100024, Kazakhstan.
E-mail: vestnikku@gmail.com. Web-site: mathematics-vestnik.ksu.kz

Executive Editor
PhD **G.B. Sarzhanova**

Proofreader
I.N. Murtazina

Computer layout
K.A. Foroastyanova

Bulletin of the Karaganda University. Mathematics Series.
ISSN 2518-7929 (Print). ISSN 2663–5011 (Online).

Proprietary: NLC «Karagandy University of the name of academician E.A. Buketov».

Registered by the Ministry of Culture and Information of the Republic of Kazakhstan.
Rediscount certificate No. KZ83VPY00089812 dated 29.03.2024.

Signed in print 30.03.2025. Format 60×84 1/8. Photocopier paper. Volume 26,25 p.sh. Circulation 200 copies.
Price upon request. Order № 6.

Printed in the Publishing house of NLC «Karagandy University of the name of academician E.A. Buketov».
28, University Str., Karaganda, 100024, Kazakhstan. Tel.: (7212) 35-63-16. E-mail: izd_karu@buketov.edu.kz

CONTENT

MATHEMATICS

MATHEMATICS

https://doi.org/10.31489/2025M1/4-11 *Research article*

On Graded J_{gr}-Prime Submodules

M. Alnimer, K. Al-Zoubi*, M. Al-Dolat

Jordan University of Science and Technology, Irbid, Jordan
(E-mail: mfalnimer21@sci.just.edu.jo, kfzoubi@just.edu.jo, mmaldolat@just.edu.jo)

In this paper, several results concerning graded $\mathfrak{J}_{gr}$-prime submodules over a commutative graded ring were obtained. For example, we give characterization of graded $\mathfrak{J}_{gr}$-prime submodules and results related to residual of graded $\mathfrak{J}_{gr}$-prime submodules. Also, the relations between graded $\mathfrak{J}_{gr}$-prime submodules and graded prime submodules of $\mathfrak{D}$ were studied. In addition, we present the necessary and sufficient condition for graded submodules to be graded $\mathfrak{J}_{gr}$-prime submodules.

Keywords: graded $\mathfrak{J}_{gr}$-prime submodule, graded prime submodule, graded submodule.

2020 Mathematics Subject Classification: 13A02, 16W50.

Introduction

The study of graded rings and modules has attracted the attentions of many researchers for a long time due to their important applications in many fields in such as geometry and physics. For example, graded Lie algebra plays a significant role in differential geometry such as Frolicher-Nijenhuis as well as Nijenhuis-Richardson bracket [1]. In addition, they solve many physical problems related to supermanifolds, supersymmetries and quantizations of systems with symmetry [2, 3].

In recent years, graded prime submodules have attracted the attention of many mathematicians, for example [4–8]. In addition, many other generalizations of graded prime have been investigated. For example, in [9], the authors introduce the concept of graded weakly prime submodules of graded modules as a generalization of graded prime submodule. In [10] Al-Zoubi and Alghueiri mentioned the concept of graded $\mathfrak{J}_{gr}$-prime submodules. Here, we discuss the concept of graded $\mathfrak{J}_{gr}$-prime submodule and we study several results concerning it. For example, we characterize graded $\mathfrak{J}_{gr}$-prime submodules. Also, the relations between graded $\mathfrak{J}_{gr}$-prime submodules and graded prime submodules were studied. In addition, the necessary and sufficient condition for graded submodules to be graded $\mathfrak{J}_{gr}$-prime submodules were investigated.

*Corresponding author. *E-mail: kfzoubi@just.edu.jo*
Received: 26 May 2024; *Accepted:* 21 November 2024.

1 Preliminaries

Throughout this article, we assume that $\mathfrak{A}$ is a commutative $\mathfrak{G}$-graded ring with identity and $\mathfrak{D}$ is a unitary graded $\mathcal{A}$-module. A left $\mathfrak{A}$-module $\mathfrak{D}$ is called a graded $\mathfrak{A}$-module if there exists a family of additive subgroups $\{\mathfrak{D}_\alpha\}_{\alpha\in\mathfrak{G}}$ of $\mathfrak{D}$ such that $\mathfrak{D} = \bigoplus_{\alpha\in\mathfrak{G}} \mathfrak{D}_\alpha$ and $\mathfrak{A}_\alpha\ \mathfrak{D}_\beta \subseteq \mathfrak{D}_{\alpha\beta}$ for all $\alpha, \beta \in \mathfrak{G}$. Also if an element of $\mathfrak{D}$ belongs to $\cup_{\alpha\in\mathfrak{G}}\ \mathfrak{D}_\alpha = h(\mathfrak{D})$, then it is called a homogeneous. Let $\mathfrak{A} = \bigoplus_{\alpha\in\mathfrak{G}}$ $\mathfrak{A}_\alpha$ be a $\mathfrak{G}$-graded ring. A submodule $\mathcal{V}$ of $\mathfrak{D}$ is said to be a graded submodule of $\mathfrak{D}$ if $\mathcal{V} = \bigoplus_{\alpha\in\mathfrak{G}}$ $(\mathcal{V} \cap \mathfrak{D}_\alpha) := \bigoplus_{\alpha\in\mathfrak{G}} \mathcal{V}_\alpha$. In this case, $\mathcal{V}_\alpha$ is called the α-component of $\mathcal{V}$ [11, 12]. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring and $\mathfrak{D}$ — a graded $\mathfrak{A}$-module. A graded submodule $\mathcal{V}$ of $\mathfrak{D}$ is said to be a graded maximal (briefly, Gr-maximal) submodule if $\mathcal{V} \neq \mathfrak{D}$ and if there is a graded submodule L of $\mathfrak{D}$ such that $\mathcal{V} \subseteq L \subseteq \mathfrak{D}$, then $\mathcal{V} = L$ or $L = \mathfrak{D}$ [13]. The graded Jacobson radical of a graded module $\mathfrak{D}$, denoted by $\mathfrak{J}_{gr}(\mathfrak{D})$, is defined to be the intersection of all Gr-maximal submodules of $\mathfrak{D}$, if $\mathfrak{D}$ has no Gr-maximal submodule then we shall take, by definition, $\mathfrak{J}_{gr}(\mathfrak{D}) = \mathfrak{D}$ [12]. A proper graded submodule $\mathcal{V}$ of $\mathfrak{D}$ is called a graded prime submodule if whenever $rm \in \mathcal{V}$ where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, then $r \in (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$ or $m \in \mathcal{V}$ [6]. A proper graded submodule $\mathcal{V}$ of $\mathfrak{D}$ is called a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ if whenever $r_g \in h(\mathfrak{A})$ and $m_\lambda \in h(\mathfrak{D})$ with $r_g\ m_\lambda \in \mathcal{V}$, then either $m_\lambda \in \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$ or $r_g \in (\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, where $\mathfrak{J}_{gr}(\mathfrak{D})$ is the graded Jacobson radical of $\mathfrak{D}$ [10].

2 Results

Theorem 1. If $\mathcal{V}$ is a graded prime submodule of $\mathfrak{D}$, then $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Let $rm \in \mathcal{V}$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, since $\mathcal{V}$ is a graded prime submodule of $\mathfrak{D}$, then $r \in (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$ or $m \in \mathcal{V}$. If $r \in (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$, then $rM \subseteq \mathcal{V}$, but $\mathcal{V} \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, thus $rM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, it follows that $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. If $m \in \mathcal{V}$, since $\mathcal{V} \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, then $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Hence $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

In the following example, it is shown that the converse of Theorem 1 is not necessarily true.

Example 1. Let $\mathfrak{G} = \mathbb{Z}_2$, $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \mathbb{Z}$, $\mathfrak{A}_1 = \{0\}$. Let $\mathfrak{D} = \mathbb{Z}_{12}$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \mathbb{Z}_{12}$ and $\mathfrak{D}_1 = \{\overline{0}\}$. Now, consider $\mathcal{V} = \{\overline{0}, \overline{4}, \overline{8}\} = \langle\overline{4}\rangle$ be a graded submodule of $\mathbb{Z}_{12}$. Then $\mathcal{V}$ is not graded prime submodule of $\mathfrak{D}$, since there exist $2 \in h(\mathfrak{A})$ and $\overline{2} \in h(\mathfrak{D})$ such that $2 \cdot \overline{2} = \overline{4} \in \mathcal{V}$, but $\overline{2} \notin \mathcal{V}$ and $2 \notin (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D}) = 4\mathbb{Z}$. However, an easy computation shows that $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Example 2. Let $\mathfrak{G} = \mathbb{Z}_2$, $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \mathbb{Z}$, $\mathfrak{A}_1 = \{0\}$, and $\mathfrak{D} = \mathbb{Z} \times \mathbb{Z}$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \mathbb{Z} \times \mathbb{Z}$, $\mathfrak{D}_1 = \{(0, 0)\}$. The graded submodule $\mathcal{V} = 2\mathbb{Z} \times \langle 0\rangle$ is not graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$. Since $(6, 0) = 2(3, 0) \in \mathcal{V}$, but $(3, 0) \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathcal{V} + \{(0, 0)\} = \mathcal{V}$ and $2 \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D}) = (2\mathbb{Z} \times \langle 0\rangle :_{\mathfrak{A}} \mathbb{Z} \times \mathbb{Z}) = \langle 0\rangle$, hence $\mathcal{V} = 2\mathbb{Z} \times \langle 0\rangle$ is not graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Remark 1. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring and $\mathfrak{D}$ a graded $\mathfrak{A}$-module.

1) If $\mathfrak{J}_{gr}(\mathfrak{D}) = 0$, then every graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ is a graded prime submodule of $\mathfrak{D}$.

2) If $\mathfrak{J}_{gr}(\mathfrak{D})$ is contained in every graded submodule of $\mathfrak{D}$, then every graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ is a graded prime submodule of $\mathfrak{D}$.

A graded $\mathfrak{A}$-module $\mathfrak{D}$ is called a Gr-torsion free if whenever $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$ with $rm = 0$, then either $r = 0$ or $m = 0$ [5].

The following theorem characterizes graded $\mathfrak{J}_{gr}$-prime submodules.

Theorem 2. Let $\mathcal{V}$ be a proper graded submodule of $\mathfrak{D}$ and $P = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Then the following statements are equivalent:

1) $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule.

2) For every graded submodule $\mathcal{K}$ of $\mathfrak{D}$ and for every graded ideal $\mathcal{U}$ of $\mathfrak{A}$ such that $\mathcal{U}\mathcal{K} \subseteq \mathcal{V}$ implies that either $\mathcal{K} \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ or $\mathcal{U} \subseteq P = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$.

3) $\mathfrak{D}/(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}))$ is a Gr-torsion free $\mathfrak{A}/P$-module.

4) The graded submodule $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle) = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, for each $r \in h(\mathfrak{A}) - P$.

5) The graded ideal $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle) = P$, for each $x \in h(\mathfrak{D}) - (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}))$.

Proof. (1)⇒(2) Let $\mathcal{K}$ be a graded submodule of $\mathfrak{D}$ and $\mathcal{U}$ be a graded ideal of $\mathfrak{A}$ such that $\mathcal{U}\mathcal{K} \subseteq \mathcal{V}$. Suppose $\mathcal{K} \nsubseteq \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$, then there exists $k \in \mathcal{K}\cap h(\mathfrak{D}) - (\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}))$. Let $i \in \mathcal{U}\cap h(\mathfrak{A})$. Since $k \in \mathcal{K}$, then $ik \in \mathcal{U}\mathcal{K} \subseteq \mathcal{V}$, so $ik \in \mathcal{V}$. But $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule, then either $i \in (\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ or $k \in \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$. But $k \notin \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$, thus $i \in (\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Hence $\mathcal{U} \subseteq (\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = P$.

(2)⇒(3) Assume that $(r+P)(m+\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})) = \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$ and $r+P \neq P$, where $r+P \in h(\mathfrak{A}/P)$ and $m+\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) \in h(\mathfrak{D}/(\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})))$. Then $rm+\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) = \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$, thus $rm \in \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$ it follows that $\langle r\rangle\langle m\rangle \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, by hypothesis, we get either $\langle m\rangle \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) + \mathfrak{J}_{gr}(\mathfrak{D})$ or $\langle r\rangle \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. That is either $\langle m\rangle \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ or $\langle r\rangle \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. If $\langle r\rangle \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = P$, then $r \in P$, thus $r + P = P$ as a contradiction. So we have $\langle m\rangle \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ implies $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Hence $m + \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Therefore, $\mathfrak{D}/(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}))$ is a Gr-torsion free $\mathfrak{A}/P$-module.

(3)⇒(4) Let $r \in h(\mathfrak{A})-P$ and let $m \in (\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle)\cap h(\mathfrak{D})$. Then $\langle r\rangle m \subseteq \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$ it follows that $rm \in \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$. Thus $(r+P)(m+\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})) = \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$, since $r \notin P$ and $\mathfrak{D}/(\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}))$ is a Gr-torsion free $\mathfrak{A}/P$-module we get $m+\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) = \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$, thus $m \in \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$. Hence $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ for each $r \in h(\mathfrak{A}) - P$. Now, let $m \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) \cap h(\mathfrak{D})$ and $r \in h(\mathfrak{A}) - P$, then $rm \in \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$ it follows that $\langle r\rangle m \subseteq \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$, thus $m \in (\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle)$. Hence $\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) \subseteq (\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle)$ for each $r \in h(\mathfrak{A}) - P$.

(4)⇒(5) Let $x \in h(\mathfrak{D}) - (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}))$. Let $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle) \cap h(\mathfrak{A})$. Suppose the contrary, $r \notin P$. Since $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle) \cap h(\mathfrak{A})$, then $r\langle x\rangle \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ it follows that $rx \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, thus $\langle r\rangle x \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. That is $x \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle)$ but by hypothesis $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{D}} \langle r\rangle) = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, for each $r \in h(\mathfrak{A}) - P$, so we get $x \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ a contradiction. Hence, $r \in P$. Therefore, $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle) \subseteq P$. Now, let $r \in P \cap h(\mathfrak{A}) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) \cap h(\mathfrak{A})$. Then $rM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. In particular, $rx \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, thus $r\langle x\rangle \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ implies $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle)$. Hence $P \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle)$. Therefore $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x\rangle) = P$.

(5)⇒(1) Let $rm \in \mathcal{V}$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$. Suppose $m \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, we need to prove that $r \in P$. Since $rm \in \mathcal{V} \subseteq \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$, then $r\langle m\rangle \subseteq \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$ it follows that $r \in (\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle m\rangle)$, apply hypothesis, we have $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle m\rangle) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = P$, hence $r \in P$. Therefore, $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Theorem 3. If $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, then $(\mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ is a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathfrak{A}$.

Proof. We show that P is a graded prime ideal of $\mathfrak{A}$, where $P = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Let $ab \in P$, where $a, b \in h(\mathfrak{A})$. Suppose $a \notin P$, then there exists $x \in h(\mathfrak{D})$ such that $ax \notin \mathcal{V}+\mathfrak{J}_{gr}(\mathfrak{D})$. Since $ab \in P$, then $abM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. In particular, $b(ax) \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Thus $b(ax) + \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, it follows that $(b + P)(ax + \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Since $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule, by Theorem 2, we get $\mathfrak{D}/(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}))$ is a Gr-torsion free $\mathfrak{A}/P$-module. But $ax \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, then $b + P = P$, so we have $b \in P$. Therefore, $P = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ is a graded prime ideal of $\mathfrak{A}$, then P is a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathfrak{A}$, by Theorem 1.

A graded ring $\mathfrak{A}$ is called a graded integral domain if whenever $ab = 0$, where a, $b \in h(\mathfrak{A})$, then either $a = 0$ or $b = 0$ [10].

In the following example, it is shown that the converse of Theorem 3 is not necessarily true.

Example 3. Let $\mathfrak{G} = \mathbb{Z}_2$, $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \mathbb{Z}$, $\mathfrak{A}_1 = \{0\}$, and $\mathfrak{D} = \mathbb{Z} \times \mathbb{Z}$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \mathbb{Z} \times \mathbb{Z}$, $\mathfrak{D}_1 = \{(0,0)\}$. The graded submodule $\mathcal{V} = 2\mathbb{Z} \times \langle 0 \rangle$ is not graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, by Example 2. However, $P = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (2\mathbb{Z} \times \langle 0 \rangle :_{\mathfrak{A}} \mathbb{Z} \times \mathbb{Z}) = \langle 0 \rangle$ is a graded prime ideal of $\mathbb{Z}$. Since if $ab \in P = \langle 0 \rangle$, where $a, b \in h(\mathbb{Z})$, then $ab = 0$ implies either $a = 0$ or $b = 0$ as $\mathbb{Z}$ is a graded integral domain. Thus $a \in P$ or $b \in P$, by Theorem 1, we have P is a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathbb{Z}$.

The following example shows that the residual of graded $\mathfrak{J}_{gr}$-prime submodule is not necessarily a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathfrak{A}$.

Example 4. Let $\mathfrak{G} = \mathbb{Z}_2$, $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \mathbb{Z}$ and $\mathfrak{A}_1 = \{0\}$. Let $\mathfrak{D} = \mathbb{Z}_{12}$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \mathbb{Z}_{12}$ and $\mathfrak{D}_1 = \{\overline{0}\}$. Consider $\mathcal{V} = \{\overline{0}, \overline{4}, \overline{8}\} = \langle \overline{4} \rangle$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathbb{Z}$-module $\mathbb{Z}_{12}$, but $(\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12})$ is not graded $\mathfrak{J}_{gr}$-prime ideal of $\mathbb{Z}$, since there exists $2 \in h(\mathbb{Z})$ such that $2 \cdot 2 = 4 \in (\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12})$, but $2 \notin (\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12}) + \mathfrak{J}_{gr}(\mathbb{Z}) = (\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12}) + \{0\} = (\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12}) = 4\mathbb{Z}$ and $2 \notin ((\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12}) + \mathfrak{J}_{gr}(\mathbb{Z}) :_{\mathbb{Z}} \mathbb{Z}) = ((\mathcal{V} :_{\mathbb{Z}} \mathbb{Z}_{12}) :_{\mathbb{Z}} \mathbb{Z})$.

Theorem 4. If $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ with $\mathfrak{J}_{gr}(\mathfrak{D}) \subseteq \mathcal{V}$, then $(\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$ is a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathfrak{A}$.

Proof. Since $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, then $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ is a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathfrak{A}$ by Theorem 3. But $\mathfrak{J}_{gr}(\mathfrak{D}) \subseteq \mathcal{V}$, thus $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$. Therefore, $(\mathcal{V} :_{\mathfrak{A}} \mathfrak{D})$ is a graded $\mathfrak{J}_{gr}$-prime ideal of $\mathfrak{A}$.

Theorem 5. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ — a graded $\mathfrak{A}$-module and $\mathcal{V}$ — a proper graded submodule of $\mathfrak{D}$. Then the following statements are equivalent:

1) $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

2) $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle c \rangle)$ for each $c \in h(\mathfrak{D}) - (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}))$.

3) $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$ for each graded submodule $\mathcal{K}$ of $\mathfrak{D}$ such that $\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) \subsetneq \mathcal{K}$.

Proof. (1)⇒(2) By Theorem 2.

(2) ⇒(3) Let $\mathcal{K}$ be a graded submodule of $\mathfrak{D}$ such that $\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) \subsetneq \mathcal{K}$. It is clear that $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$ since if $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) \cap h(\mathfrak{A})$, then $rM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, but $\mathcal{K} \subseteq \mathfrak{D}$ implies $rK \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, thus $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$, hence $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$. Now, let $s \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K}) \cap h(\mathfrak{A})$, then $sK \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, but $\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) \subsetneq \mathcal{K}$ so there exists $x \in \mathcal{K} \cap h(\mathfrak{D})$ and $x \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. In particular $sx \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, it follows that $s\langle x \rangle \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ implies $s \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x \rangle)$ but by hypothesis we have $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \langle x \rangle) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, so $s \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, hence $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K}) \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Therefore, $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$ for each $\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) \subsetneq \mathcal{K}$.

(3)⇒(1) Let $rm \in \mathcal{V}$ and $m \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$. Take $\mathcal{K} = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) + \langle m \rangle$, where $\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) \subsetneq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) + \langle m \rangle$ (since $m \in \mathcal{K} - (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}))$), it follows that $rK = r(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) + r\langle m \rangle \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) + \mathcal{V} = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, so $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K})$. But by hypothesis, we have $(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathcal{K}) = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, thus $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Therefore, $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

A proper graded submodule $\mathcal{V}$ is called a graded small (Gr-small) of $\mathfrak{D}$ if $\mathfrak{D} = \mathcal{V} + L$ for some graded submodule L of $\mathfrak{D}$ implies that $L = \mathfrak{D}$. A graded $\mathfrak{A}$-module $\mathfrak{D}$ is said to be a graded hollow (Gr-hollow) module if every proper graded submodule $\mathcal{V}$ of $\mathfrak{D}$ is a Gr-small [13].

Theorem 6. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ a Gr-hollow $\mathfrak{A}$-module and $\mathfrak{J}_{gr}(\mathfrak{D})$ a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, then every proper graded submodule of $\mathfrak{D}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Let $\mathcal{V}$ be a proper graded submodule of $\mathfrak{D}$ and let $rm \in \mathcal{V}$ where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$. Since $\mathfrak{D}$ is a Gr-hollow then $\mathcal{V}$ is a Gr-small, so $rm \in \mathcal{V} \subseteq \sum\{A : A \text{ is a } Gr\text{-small}\} = \mathfrak{J}_{gr}(\mathfrak{D})$ by [14; Theorem 2.10]. But $\mathfrak{J}_{gr}(\mathfrak{D})$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$. Thus either $m \in \mathfrak{J}_{gr}(\mathfrak{D}) + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathfrak{J}_{gr}(\mathfrak{D}) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ or $rM \subseteq \mathfrak{J}_{gr}(\mathfrak{D}) + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathfrak{J}_{gr}(\mathfrak{D}) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. So either $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ or $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Therefore, $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

A nonempty subset $S \subseteq h(\mathfrak{A})$ of a $\mathfrak{G}$-graded ring $\mathfrak{A}$ is called multiplicatively closed subset (briefly, $m.c.s.$) of $\mathfrak{A}$ if $0 \notin S$, $1 \in S$ and $x \cdot y \in S$ for all $x, y \in S$. Let $S \subseteq h(\mathfrak{A})$ be a multiplicatively closed subset of $\mathfrak{A}$ and $\mathcal{V}$ be a graded submodule of $\mathfrak{D}$ then $\mathcal{V}(S) = \{x \in \mathfrak{D} : \text{there exists } t \in S \text{ such that } tx \in \mathcal{V}\}$ be a graded submodule of $\mathfrak{D}$ is said to be the component of $\mathcal{V}$ determined by S, or simply the S-component of $\mathcal{V}$. We conclude from definition $\mathcal{V} \subseteq \mathcal{V}(S)$.

Lemma 1. Let P be a proper graded ideal of $\mathfrak{A}$. Then P is a graded prime ideal of a graded ring $\mathfrak{A}$ if and only if $h(\mathfrak{A}) - P$ is a $m.c.s.$ of $\mathfrak{A}$.

Proof. Let P is a proper graded submodule of $\mathfrak{D}$, then $0 \in P$, $1 \notin P$ (if $1 \in P$, then $P = \mathfrak{D}$, thus P is not proper a contradiction) and since P is a graded prime ideal of $\mathfrak{A}$, we have $0 \notin h(\mathfrak{A}) - P$, $1 \in h(\mathfrak{A}) - P$ and $ab \in h(\mathfrak{A}) - P$ for each $a, b \in h(\mathfrak{A}) - P$. Therefore, $h(\mathfrak{A}) - P$ is a $m.c.s.$ of $\mathfrak{A}$. Conversely, suppose the contrary, P is not graded prime ideal of $\mathfrak{A}$, then there exist $x, y \in h(\mathfrak{A}) - P$ with $xy \in P$. Since $h(\mathfrak{A}) - P$ is $m.c.s.$ of $\mathfrak{A}$, then $xy \in h(\mathfrak{A}) - P$ which is a contradiction.

Theorem 7. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ — a graded $\mathfrak{A}$-module and $\mathcal{V}$ — a graded submodule of $\mathfrak{D}$. Then $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ if and only if the graded ideal $P = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ is a graded prime of $\mathfrak{A}$ and $\mathcal{V}(S) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ for each $S \subseteq h(\mathfrak{A})$ a $m.c.s.$ of $\mathfrak{A}$ such that $S \cap P = \phi$.

Proof. Let $\mathcal{V}$ be a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ and $S \subseteq h(\mathfrak{A})$ be $m.c.s.$ of $\mathfrak{A}$ with $S \cap P = \phi$. Let $ab \in P$, where $a, b \in h(\mathfrak{A})$. Suppose $a \notin P$, then there exists $x \in h(\mathfrak{D})$ such that $ax \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Since $ab \in P$, then $abM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. In particular, $b(ax) \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, thus $b(ax) + \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, it follows that $(b+P)(ax + \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) = \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Since $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule, by Theorem 2, we get $\mathfrak{D}/(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}))$ is a Gr-torsion free $\mathfrak{A}/P$-module. But $ax \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, then $b + P = P$, so we have $b \in P$. Therefore, $P = (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ is a graded prime ideal of $\mathfrak{A}$. Now, let $a \in \mathcal{V}(S) \cap h(\mathfrak{D})$, then there exists $s \in S$ such that $sa \in \mathcal{V}$. Since $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, $S \cap (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D}) = \phi$ and $s \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, we have $a \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Hence $\mathcal{V}(S) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Conversely, suppose not, let $rm \in \mathcal{V}$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, but $r \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ and $m \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Assume that P is a graded prime ideal of $\mathfrak{A}$, by Lemma 1, we have $h(\mathfrak{A}) - P$ is a $m.c.s.$ of $\mathfrak{A}$. Since $(h(\mathfrak{A}) - P) \cap P = \phi$, by assumption we have $\mathcal{V}(h(\mathfrak{A}) - P) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. But $m \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ and $\mathcal{V}(h(\mathfrak{A}) - P) \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ it follows that $m \notin \mathcal{V}(h(\mathfrak{A}) - P)$. This yields that for each $s \in h(\mathfrak{A}) - P$ we have $sm \notin \mathcal{V}$, but $r \in h(\mathfrak{A}) - P$, then $rm \notin \mathcal{V}$ which is a contradiction. Therefore, $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

The following example shows that the intersection of two graded $\mathfrak{J}_{gr}$-prime submodules needs, not to be a graded $\mathfrak{J}_{gr}$-prime submodule.

Example 5. Let $\mathfrak{G} = \mathbb{Z}_2$ and $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \mathbb{Z}$ and $\mathfrak{A}_1 = \{0\}$. Let $\mathfrak{D} = \mathbb{Z}_6$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \mathbb{Z}_6$ and $\mathfrak{D}_1 = \{\overline{0}\}$. Consider $\mathcal{V} = \langle\overline{2}\rangle = \{\overline{0}, \overline{2}, \overline{4}\}$ and $L = \langle\overline{3}\rangle = \{\overline{0}, \overline{3}\}$ are graded submodules of $\mathbb{Z}_6$. Then $\mathcal{V} \cap L = \langle\overline{0}\rangle$ is not a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, since there exist $3 \in h(\mathbb{Z})$ and $\overline{2} \in h(\mathbb{Z}_6)$ such that $3 \cdot \overline{2} = \overline{0} \in \mathcal{V} \cap L$, but $\overline{2} \notin (\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathbb{Z}_6) = \langle\overline{0}\rangle + \langle\overline{0}\rangle = \langle\overline{0}\rangle$ and $3 \notin ((\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathbb{Z}_6) :_{\mathbb{Z}} \mathbb{Z}_6) = 6\mathbb{Z}$. However, an easy computation and using the definition of graded $\mathfrak{J}_{gr}$-prime submodule to show that $\mathcal{V}$ and L are graded $\mathfrak{J}_{gr}$-prime submodules of $\mathfrak{D}$.

The next theorem shows that the intersection of two graded $\mathfrak{J}_{gr}$-prime submodules is a graded $\mathfrak{J}_{gr}$-prime submodule under conditions.

Theorem 8. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ a graded $\mathfrak{A}$-module and $\mathcal{V}$, L be two graded submodules of $\mathfrak{D}$ such that $\mathcal{V} \subseteq \mathfrak{J}_{gr}(\mathfrak{D})$ or $L \subseteq \mathfrak{J}_{gr}(\mathfrak{D})$. If $\mathcal{V}$ and L are graded $\mathfrak{J}_{gr}$-prime submodules of $\mathfrak{D}$, then $\mathcal{V} \cap L$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Assume that $\mathcal{V}$ and L are graded $\mathfrak{J}_{gr}$-prime submodules of $\mathfrak{D}$. Let $rm \in \mathcal{V} \cap L$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, then $rm \in \mathcal{V}$ and $rm \in L$. If $\mathcal{V} \subseteq \mathfrak{J}_{gr}(\mathfrak{D})$, since $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, then either $rM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = \mathfrak{J}_{gr}(\mathfrak{D}) \subseteq (\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathfrak{D})$ or $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = = \mathfrak{J}_{gr}(\mathfrak{D}) \subseteq (\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathfrak{D})$. Thus either $r \in ((\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ or $m \in (\mathcal{V} \cap L) + \mathfrak{J}_{gr}(\mathfrak{D})$. Hence $\mathcal{V} \cap L$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$. Similarly, If $L \subseteq \mathfrak{J}_{gr}(\mathfrak{D})$, we get $\mathcal{V} \cap L$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Theorem 9. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ and $\mathfrak{D}'$ be two graded $\mathfrak{A}$-modules and $\mathcal{V}$, $\mathcal{V}'$ be two proper graded submodules of $\mathfrak{D}$, $\mathfrak{D}'$,respectively. If $\mathcal{V} \times \mathcal{V}'$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D} \times \mathfrak{D}'$, then $\mathcal{V}$ and $\mathcal{V}'$ are graded $\mathfrak{J}_{gr}$-prime submodules of $\mathfrak{D}$ and $\mathfrak{D}'$, respectively.

Proof. To prove $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, let $rm \in \mathcal{V}$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, then $r(m,0) \in \mathcal{V} \times \mathcal{V}'$ as $r(m,0) = (rm,0) \in \mathcal{V} \times \mathcal{V}'$. Since $\mathcal{V} \times \mathcal{V}'$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D} \times \mathfrak{D}'$, so either $r \in ((\mathcal{V} \times \mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D} \times \mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D} \times \mathfrak{D}')$ or $(m,0) \in (\mathcal{V} \times \mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D} \times \mathfrak{D}')$. If $r \in ((\mathcal{V} \times \mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D} \times \mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D} \times \mathfrak{D}')$, then $r(\mathfrak{D} \times \mathfrak{D}') \subseteq (\mathcal{V} \times \mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D} \times \mathfrak{D}') = (\mathcal{V} \times \mathcal{V}') + + (\mathfrak{J}_{gr}(\mathfrak{D}) \times \mathfrak{J}_{gr}(\mathfrak{D}'))$, it follows that $(rM \times rM') \subseteq (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) \times (\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}'))$, so $rM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ and $rM' \subseteq \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$. This implies that $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ and $r \in (\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D}')$. If $(m,0) \in (\mathcal{V} \times \mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D} \times \mathfrak{D}')$, then $(m,0) \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) \times (\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$. Thus $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$ and $0 \in \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$. Hence $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$. In a similar manner, we can prove that $\mathcal{V}'$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}'$.

Theorem 10. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ and $\mathfrak{D}'$ be two graded $\mathfrak{A}$-modules and $f : \mathfrak{D} \longrightarrow \mathfrak{D}'$ be a graded epimorphism. If $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ containing $kerf$, then $f(\mathcal{V})$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}'$.

Proof. Since $\mathcal{V}$ is a proper graded submodule of $\mathfrak{D}$, by [15; Lemma 4.8], we have $f(\mathcal{V})$ is a proper graded submodule of $\mathfrak{D}'$. Let $rm' \in f(\mathcal{V})$, where $r \in h(\mathfrak{A})$ and $m' \in h(\mathfrak{D}')$, since f is onto and $m' \in h(\mathfrak{D}')$, then there exists $m \in h(\mathfrak{D})$ such that $f(m) = m'$. Thus $rm' = rf(m) = f(rm) \in f(\mathcal{V})$, so there exists $n \in \mathcal{V} \cap h(\mathfrak{D})$ such that $f(rm) = f(n)$, thus $f(rm-n) = 0$, it follows that $rm-n \in kerf \subseteq \mathcal{V}$ so $rm + \mathcal{V} = n + \mathcal{V} = \mathcal{V}$. That is $rm \in \mathcal{V}$, but $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$, then either $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ or $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathcal{V})$. If $r \in (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, then $rM \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, thus $f(rM) \subseteq f(\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})) = f(\mathcal{V}) + f(\mathfrak{J}_{gr}(\mathfrak{D}))$, implies that $rf(\mathfrak{D}) = rM' \subseteq f(\mathcal{V}) + f(\mathfrak{J}_{gr}(\mathfrak{D}))$, by [14; Theorem 2.12], we get $f(\mathfrak{J}_{gr}(\mathfrak{D})) \subseteq \mathfrak{J}_{gr}(\mathfrak{D}')$. So $rM' \subseteq f(\mathcal{V}) + \mathfrak{J}_{gr}(\mathfrak{D}')$, then $r \in (f(\mathcal{V}) + \mathfrak{J}_{gr}(\mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D}')$. If $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$, then $f(m) \in f(\mathcal{V}) + f(\mathfrak{J}_{gr}(\mathfrak{D}))$, but $f(m) = m'$, by [14; Theorem 2.12], we have $m' \in f(\mathcal{V}) + f(\mathfrak{J}_{gr}(\mathfrak{D})) \subseteq f(\mathcal{V}) + \mathfrak{J}_{gr}(\mathfrak{D}')$. Hence $f(\mathcal{V})$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}'$.

Theorem 11. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ and $\mathfrak{D}'$ be a graded $\mathfrak{A}$-modules. Let $f : \mathfrak{D} \longrightarrow \mathfrak{D}'$ be a graded epimorphism with $kerf$ is a Gr-small submodule of $\mathfrak{D}$. If $\mathcal{V}'$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}'$, then $f^{-1}(\mathcal{V}')$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Since $\mathcal{V}'$ is a proper graded submodule of $\mathfrak{D}'$, by [15; Lemma 5.2], we have $f^{-1}(\mathcal{V}')$ is a proper graded submodule of $\mathfrak{D}$. Let $rm \in f^{-1}(\mathcal{V}')$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, then $f(rm) \in \mathcal{V}'$, thus $rf(m) \in \mathcal{V}'$ since $\mathcal{V}'$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}'$, then either $r \in (\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D}')$ or $f(m) \in \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$. If $r \in (\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}') :_{\mathfrak{A}} \mathfrak{D}')$, then $rM' \subseteq \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$ since f is a graded epimorphism, then f is onto, so $\mathfrak{D}' = f(\mathfrak{D})$ implies that $rf(\mathfrak{D}) \subseteq \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$ then $f(rM) \subseteq \mathcal{V}' + + \mathfrak{J}_{gr}(\mathfrak{D}')$, it follows that $rM \subseteq f^{-1}(\mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')) = f^{-1}(\mathcal{V}') + f^{-1}(\mathfrak{J}_{gr}(\mathfrak{D}'))$, since f is a graded epimorphism and $kerf$ is a Gr-small of $\mathfrak{D}$ [14; Theorem 2.12], we get $f(\mathfrak{J}_{gr}(\mathfrak{D})) = \mathfrak{J}_{gr}(\mathfrak{D}')$. Thus $\mathfrak{J}_{gr}(\mathfrak{D}) = f^{-1}(\mathfrak{J}_{gr}(\mathfrak{D}'))$, so $rM \subseteq f^{-1}(\mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D})$, it follows that $r \in (f^{-1}(\mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. If

$f(m) \in \mathcal{V}' + \mathfrak{J}_{gr}(\mathfrak{D}')$, then $m \in f^{-1}(\mathcal{V}') + f^{-1}(\mathfrak{J}_{gr}(\mathfrak{D}')) = f^{-1}(\mathcal{V}') + \mathfrak{J}_{gr}(\mathfrak{D})$. Hence $f^{-1}(\mathcal{V}')$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Corollary 1. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ a graded $\mathfrak{A}$-module and $\mathcal{V}$, $\mathcal{K}$ proper graded submodules of $\mathfrak{D}$ such that $\mathcal{K} \subseteq \mathcal{V}$ and $ker f$ is Gr-small of $\mathfrak{D}$. If $\mathcal{V}/\mathcal{K}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}/\mathcal{K}$, then $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Define $f : \mathfrak{D} \longrightarrow \mathfrak{D}/\mathcal{K}$ by $f(x) = x + \mathcal{K}$. Then f is a graded epimorphism, so by Theorem 11, we get $f^{-1}(\mathcal{V}/\mathcal{K}) = \mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Recall that a proper graded submodule $\mathcal{V}$ of $\mathfrak{D}$ is called a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$, if $\mathcal{V} \cap \mathcal{U}\mathfrak{D} = \mathcal{U}\mathcal{V} + (\mathfrak{J}_{gr}(\mathfrak{D}) \cap \mathcal{V} \cap \mathcal{U}\mathfrak{D})$ for each proper graded ideal $\mathcal{U}$ of $\mathfrak{A}$, see [14; Definition 2.19].

The following example shows that a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$ not necessarily a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Example 6. Let $\mathfrak{A} = \mathbb{Z}$ be a $\mathfrak{G}$-graded ring with $\mathfrak{A}_0 = \{0\}$ and $\mathfrak{A}_1 = \mathbb{Z}$, where $\mathfrak{G} = \mathbb{Z}_2$. Let $\mathfrak{D} = \mathbb{Z}_6$ be a graded $\mathfrak{A}$-module with $\mathfrak{D}_0 = \{\overline{0}\}$ and $\mathfrak{D}_1 = \mathbb{Z}_6$. $\mathcal{V} = \{\overline{0}\}$ is a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$. However $\mathcal{V}$ is not graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ since there exist $3 \in h(\mathfrak{A})$ and $\overline{2} \in h(\mathfrak{D})$ such that $3 \cdot \overline{2} = \overline{0} \in \mathcal{V}$ but $3 \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$ and $\overline{2} \notin \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) = \{\overline{0}\}$.

The next theorem shows that a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$ is a graded $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$ with under some conditions.

Theorem 12. Let $\mathfrak{A}$ be a $\mathfrak{G}$-graded ring, $\mathfrak{D}$ a Gr-torsion free $\mathfrak{A}$-module and $\mathcal{V}$ a proper graded submodule of $\mathfrak{D}$ with $\mathfrak{J}_{gr}(\mathfrak{D}) = \{0\}$. If $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$, then $\mathcal{V}$ is a $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$.

Proof. Let $rm \in \mathcal{V}$, where $r \in h(\mathfrak{A})$ and $m \in h(\mathfrak{D})$, assume that $r \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$. Thus $rm \in \mathcal{V} \cap \langle r \rangle \mathfrak{D} = \langle r \rangle \mathcal{V} + (\mathfrak{J}_{gr}(\mathfrak{D}) \cap \mathcal{V} \cap \langle r \rangle \mathfrak{D})$ as $\mathcal{V}$ is a graded $\mathfrak{J}_{gr}$-pure submodule of $\mathfrak{D}$. But $\mathfrak{J}_{gr}(\mathfrak{D}) = \{0\}$. Thus $rm \in \langle r \rangle \mathcal{V}$, it follows that there exists $n \in \mathcal{V} \cap h(\mathfrak{D})$ and $r' \in h(\mathfrak{A})$ such that $rm = rr'n$. Thus $rm - rr'n = 0$ implies $r(m - r'n) = 0$. Since $\mathfrak{D}$ is a Gr-torsion free and $r \notin (\mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D}) :_{\mathfrak{A}} \mathfrak{D})$, then $m = r'n \in \mathcal{V} \subseteq \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Hence $m \in \mathcal{V} + \mathfrak{J}_{gr}(\mathfrak{D})$. Therefore, $\mathcal{V}$ is a $\mathfrak{J}_{gr}$-prime submodule of $\mathfrak{D}$

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Kolář, I., Slovák, J., & Michor, P.W. (2013). *Natural Operations in Differential Geometry.* Springer Science and Business Media. https://doi.org/10.1007/978-3-662-02950-3

2 Deligne, P. (1999). *Quantum Fields and Strings: A course for Mathematicians.* AMS IAS.

3 Rogers, A. (2007). *Supermanifolds: Theory and Applications.* World Sci. Publ.

4 Atani, S.E., & Farzalipour, F. (2007). On graded secondary modules. *Turk. J. Math., 4*(31), 371–378.

5 Atani, S.E. (2006). On graded prime submodules. *Chiang Mai J. Sci., 1*(33), 3–7.

6 Al-Zoubi, K., & Qarqaz, F. (2018). An Intersection condition for graded prime submodules in Gr-multiplication modules. *Math. Reports, 20*(70), 329–336.

7 Al-Zoubi, K. (2015). Some properties of graded 2-prime submodules. *Asian-Eur. J. Math., 8*(2), 1550016-1–1550016-5. https://doi.org/10.1142/S1793557115500163

8 Oral, K.H., Tekir, U., & Agargun, A.G. (2011). On Graded prime and primary submodules. *Turk. J. Math., 35*(2), 159-167. https://doi.org/10.3906/mat-0904-11

9 Atani, S.E. (2006). On graded weakly prime submodules. *International Mathematical Forum, 1*(2), 61–66.

10 Al-Zoubi, K., & Alghueiri, S. (2011). On graded J_gr-2-absorbing and graded weakly J_{gr}-2-absorbing submodules of graded modules over graded commutative rings. *Int. J. Math. Comput. Sci., 16*(4), 1169–1178.

11 Nastasescu, C., & Van Oystaeyen, F. (1982). *Graded and filtered rings and modules.* Lecture notes in mathematics 758, Berlin-New York: Springer-Verlag.

12 Nastasescu, C., & Van Oystaeyen, F. (2004). *Methods of Graded Rings.* LNM 1836. Berlin-Heidelberg: Springer-Verlag.

13 Al-Zoubi, K., & Al-Qderat, A. (2017). Some properties of graded comultiplication modules. *Open Math., 15*, 187–192. https://doi.org/10.1515/math-2017-0016

14 Al-Zoubi, K., & Alghueiri, S. (2021). On graded J_{gr}-semiprime submodules. *Ital. J. Pure Appl. Math., 46*, 361–369.

15 Atani, S.E., & Saraei, F.E.K. (2010). Graded modules which satisfy the Gr-radical formula. *Thai J. Math., 1*(8), 161–170.

*Author Information**

Malak Alnimer — Master's Degree, Lecturer, Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O.Box 3030, Irbid 22110, Jordan; e-mail: *mfalnimer21@sci.just.edu.jo*; https://orcid.org/0009-0005-5600-8647

Khaldoun Al-Zoubi (*corresponding author*) — Doctor of mathematical sciences, Professor, Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O.Box 3030, Irbid 22110, Jordan; e-mail: *kfzoubi@just.edu.jo*; https://orcid.org/0000-0001-6082-4480

Mohammed Al-Dolat — Doctor of mathematical sciences, Professor, Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O.Box 3030, Irbid 22110, Jordan; e-mail: *mmaldolat@just.edu.jo*; https://orcid.org/0000-0003-2738-2072

*The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 12–23

https://doi.org/10.31489/2025M1/12-23 *Research article*

On the behavior of solutions of a doubly nonlinear degenerate parabolic system with nonlinear sources and absorptions with variable densities

M.M. Aripov[1], O.X. Atabaev[2,*], A.M. Al-Marashi[3]

[1] *National University of Uzbekistan, Tashkent, Uzbekistan;*
[2] *Andijan State Technical Institute, Andijan, Uzbekistan;*
[3] *King Abdulaziz University, Jeddah, Saudi Arabia*
(E-mail: aripovmirsaid@gmail.com, odiljonatabaev@gmail.com, aalmarashi@kau.edu.sa)

In this paper, the problem of a doubly nonlinear degenerate parabolic system with nonlinear sources and absorption terms not located in a homogeneous medium was considered. It obeys zero Dirichlet boundary conditions in a smooth bounded domain. The comparison principle and self-similar approach was used to study the problem. In this paper, the nonlinear splitting method was used to prove the existence of global and blow-up in finite time solutions. It is shown that the role of the nonlinear source and nonlinear absorption is important for the existence and non-existence of the solution. The results contribute to a broader understanding of nonlinear parabolic systems.

Keywords: doubly nonlinear degenerate parabolic system, nonlinear source, nonlinear absorption, global existence, blow-up in finite time, comparison principle, variable density.

2020 Mathematics Subject Classification: 35A01, 35B44, 35K57, 35K65.

Introduction

In this paper, we consider the following doubly nonlinear degenerate parabolic system with both nonlinear sources and absorptions with variable densities:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \nabla\left(|x|^{n_1} u^{m_1-1}\left|\nabla u^{k_1}\right|^{p_1-2}\nabla u\right) + v^{q_1} - \alpha_1 u^{r_1}, \quad x\in\Omega, t>0,\\
&\frac{\partial v}{\partial t} = \nabla\left(|x|^{n_2} v^{m_2-1}\left|\nabla v^{k_2}\right|^{p_2-2}\nabla v\right) + u^{q_2} - \alpha_2 v^{r_2}, \quad x\in\Omega, t>0,\\
&u(x,t) = v(x,t) = 0, \quad x\in\partial\Omega, t>0,\\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x\in\Omega,
\end{aligned} \tag{1}$$

where $p_i \geq 2$, $k_i, m_i \geq 1$, $n_i, q_i, r_i, \alpha_i \geq 0$ $(i=\overline{1,2})$ and Ω is a bounded domain of R^N, $N\geq 1$ with a smooth boundary $\partial\Omega$. The initial data $u_0(x), v_0(x) \in C^{2+\nu}\left(\overline{\Omega}\right)$, with $0<\nu<1$, $u_0(x), v_0(x)\geq 0$ and $u_0(x), v_0(x) \not\equiv 0$.

In recent years, the problem of reaction-diffusion processes with nonlinear interactions has attracted considerable attention because it arises in such fields as biology, chemistry and physics. Population dynamics, chemical reactions, heat transfer and other phenomena can be predicted if the conditions for the existence of global and blow-up in finite time solutions are known. For example, in a biological context, the presence of a variable density term means changing population density or resources that are not distributed uniformly. For more detailed information on physical models describing with the above and similar equations, we refer to literature [1–7] and references therein.

*Corresponding author. *E-mail: odiljonatabaev@gmail.com*
Received: 5 September 2024; *Accepted:* 3 December 2024.

However, there are few studies on doubly nonlinear degenerate parabolic systems including both reaction and absorption terms in an inhomogeneous medium.

Doubly nonlinear degenerate parabolic equations and systems have been studied by many scientists (see [5–12] and references therein). Especially, when $n_i = 0$, $p_i = 2$ and $m_i = k_i = 1$ $(i = \overline{1,2})$ the system (1) reduces to following semilinear form:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \Delta u + v^{q_1} - \alpha_1 u^{r_1}, \quad x \in \Omega, t > 0, \\
&\frac{\partial v}{\partial t} = \Delta v + u^{q_2} - \alpha_2 v^{r_2}, \quad x \in \Omega, t > 0, \\
&u\,(x,t) = v\,(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\
&u\,(x,0) = u_0\,(x)\,, \quad v\,(x,0) = v_0\,(x)\,, \quad x \in \Omega.
\end{aligned} \tag{2}$$

In particular, Bedjaoui and Souplet [3] used the comparison principle to show the global solvability of problem (2). Authors of [4–8, 13–15] analyzed the blow-up properties of solutions to system (2) without absorption terms, when $\alpha_1 = \alpha_2 = 0$. The critical Fujita exponent for solutions with blow-up was found by the authors of [13–15]. Aripov and Bobokandov [8] obtained estimates of solutions and fronts (free boundaries) for equations with a single absorption term in an inhomogeneous medium.

Recently, many authors have addressed the problems with variable densities [8, 11, 12, 15–21]. Zhou et al. [9, 10] determined the global existence and blow-up of solutions to a degenerate singular parabolic system. The authors obtained the blow-up set and uniform blow-up conditions using the comparison principle and asymptotic analysis methods. Kong et al. [5] established uniform blow-up profiles for the weakly absorbed case of a semilinear parabolic system.

Anh et al. [20] and Niu et al. [21] investigated the long-time behavior of solutions to the following degenerate parabolic equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \nabla \left(\sigma(x) \left|\nabla u\right|^{p-2} \nabla u \right) + g - f(x,u), \quad x \in \Omega \times R^+, \\
&u\,(x,t) = 0, \quad x \in \partial\Omega \times R^+, \\
&u\,(x,0) = u_0\,(x)\,, \quad x \in \Omega.
\end{aligned} \tag{3}$$

The existence of a global attractor in L^q was shown using some estimates of the solution.

Other related works includes [22–29], where the authors studied doubly degenerate parabolic equations with nonlinear sources and absorption terms when $k_i = m_i$ $(i = \overline{1,2})$

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \nabla \left(\left|\nabla u^{m_1}\right|^{p_1-2} \nabla u^{m_1} \right) + v^{q_1} - \alpha_1 u^{r_1}, \quad x \in \Omega, t > 0, \\
&\frac{\partial v}{\partial t} = \nabla \left(\left|\nabla v^{m_2}\right|^{p_2-2} \nabla v^{m_2} \right) + u^{q_2} - \alpha_2 v^{r_2}, \quad x \in \Omega, t > 0, \\
&u\,(x,t) = v\,(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\
&u\,(x,0) = u_0\,(x)\,, \quad v\,(x,0) = v_0\,(x)\,, \quad x \in \Omega.
\end{aligned}$$

This is also known as the p-Laplacian system when $k_i = m_i = 1$ $(i = \overline{1,2})$. Xiulan Wu [30] provided criteria for the global existence or the finite-time blow-up of solutions to (3).

This study extends the results to the more general case of a doubly nonlinear degenerate parabolic system including variable density terms. This provides a more precise understanding of the behavior of the physical phenomena described by the system (3).

The rest of the paper is organized as follows. Section 1 gives preliminary notations and main results. Section 2 is devoted to the existence of global and finite-time exploding solutions. Finally, conclusions and observations are discussed.

1 Preliminaries and Main results

Degenerate equations may not have classical solutions, so we define weak upper and weak lower solutions. In this paper we denote $Q_T = \Omega \times (0, T)$.

Definition 1. We call a non-negative function $(u, v) \in \left[C^{2,1}(Q_T) \cap C(Q_T)\right]^2$ a weak upper solution (a weak lower solution) of problem (1) in Q_T if the following fulfills:

$$\begin{aligned}
&\frac{\partial u}{\partial t} \geq (\leq) \nabla \left(|x|^{n_1} u^{m_1 - 1} \left| \nabla u^{k_1} \right|^{p_1 - 2} \nabla u \right) + v^{q_1} - \alpha_1 u^{r_1}, \quad x \in \Omega, t > 0,\\
&\frac{\partial v}{\partial t} \geq (\leq) \nabla \left(|x|^{n_2} v^{m_2 - 1} \left| \nabla v^{k_2} \right|^{p_2 - 2} \nabla v \right) + u^{q_2} - \alpha_2 v^{r_2}, \quad x \in \Omega, t > 0,\\
&u(x,t) \geq v(x,t) \geq 0, \quad x \in \partial\Omega, t > 0,\\
&u(x,0) \geq u_0(x), \quad v(x,0) \geq v_0(x), \quad x \in \Omega.
\end{aligned}$$

We also say that (u, v) is a weak solution of problem (1) in Q_T if (u, v) is both weak upper and weak lower solution of (1) in Q_T. Moreover, (u, v) is a global solution of problem (1) if it is a solution of (1) in Q_T for any $T > 0$, and any solution (u, v) blows up in the sense of the L^∞ norm if $T < \infty$:

$$\lim_{t \to T} \left(\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty \right) = \infty.$$

In order to state our results, we introduce some useful symbols. Let $\varphi(x)$ and $\psi(x)$ satisfy the following elliptic problem respectively:

$$-\nabla \left(|x|^{n_1} \varphi^{m_1 - 1} \left| \nabla \varphi^{k_1} \right|^{p_1 - 2} \nabla \varphi \right) = 1, \quad x \in \Omega, \quad \varphi(x) = 1, \quad x \in \partial\Omega, \tag{4}$$

$$-\nabla \left(|x|^{n_2} \psi^{m_1 - 1} \left| \nabla \psi^{k_1} \right|^{p_1 - 2} \nabla \psi \right) = 1, \quad x \in \Omega, \quad \psi(x) = 1, \quad x \in \partial\Omega. \tag{5}$$

It is known [19] that (4) and (5) have unique solutions with the following properties:

$$M_1 = \max_{x \in \overline{\Omega}} \varphi(x) < \infty, \quad M_2 = \max_{x \in \overline{\Omega}} \psi(x) < \infty,$$

$$\varphi(x), \psi(x) > 1 \text{ in } \Omega, \quad \nabla\varphi\nu < 0, \ \nabla\psi\nu < 0 \text{ on } \partial\Omega.$$

To simplify notation, we also let $\mu_1 = \max\{m_1 + k_1(p_1 - 2), r_1\}$, $\mu_2 = \max\{m_2 + k_2(p_2 - 2), r_2\}$.

Theorem 1. Let $n_1 n_2 < p_1 p_2$. If $q_1 q_2 < \mu_1 \mu_2$, then all nonnegative solutions of problem (1) are global.

Theorem 2. Let $n_1 n_2 < p_1 p_2$ and $q_1 q_2 = \mu_1 \mu_2$, then:

1. if $r_1 > m_1 + k_1(p_1 - 2)$, $r_2 > m_2 + k_2(p_2 - 2)$, which is $q_1 q_2 = r_1 r_2$ and if α_1, α_2 are sufficiently large, then nonnegative solutions of problem (1) blows up in finite time, and exists globally for small initial values;
2. if $r_1 < m_1 + k_1(p_1 - 2)$, $r_2 < m_2 + k_2(p_2 - 2)$, thus $q_1 q_2 = (m_1 + k_1(p_1 - 2))(m_2 + k_2(p_2 - 2))$, then all nonnegative solutions of problem (1) are global for small initial values;
3. suppose $r_1 < m_1 + k_1(p_1 - 2)$, $r_2 > m_2 + k_2(p_2 - 2)$, thus $q_1 q_2 = (m_1 + k_1(p_1 - 2)) r_2$, then there is a non-negative blow-up in finite time solution of problem (1) for large initial data;
4. suppose $r_1 > m_1 + k_1(p_1 - 2)$, $r_2 < m_2 + k_2(p_2 - 2)$, thus $q_1 q_2 = r_1(m_2 + k_2(p_2 - 2))$, then there is a non-negative blow-up in finite time solution of problem (1) for large initial data.

Theorem 3. Suppose $n_1 n_2 < p_1 p_2$. If $q_1 q_2 > \mu_1 \mu_2$, then nonnegative solutions of problem (1) blows up in finite time for sufficiently large initial data and exists globally for small initial values.

2 Global existence and Blow-up

Here we give the proof of global existence and blow-up solutions using the comparison principle. Self-similar approach and nonlinear splitting methods are used to construct comparable solutions. We start with Theorem 1.

Proof of Theorem 1. We divide the proof of Theorem 1 into 4 cases:

Case 1: When $\mu_1 = m_1 + k_1(p_2 - 2)$, $\mu_2 = m_2 + k_2(p_2 - 2)$, thus $q_1 q_2 < (m_1 + k_1(p_2 - 2)) \times$ $\times (m_2 + k_2(p_2 - 2))$ and $n_1 n_2 < p_1 p_2$. We have $u \le w$ and $v \le z$, where (w, z) satisfies

$$\begin{aligned}
&\frac{\partial w}{\partial t} = \nabla\left(|x|^{n_1} w^{m_1-1}\left|\nabla w^{k_1}\right|^{p_1-2}\nabla w\right) + z^{q_1}, \quad x \in \Omega, t > 0,\\
&\frac{\partial z}{\partial t} = \nabla\left(|x|^{n_2} z^{m_2-1}\left|\nabla z^{k_2}\right|^{p_2-2}\nabla z\right) + w^{q_2}, \quad x \in \Omega, t > 0,\\
&w(x,t) = z(x,t) = 0, \quad x \in \partial\Omega, t > 0,\\
&w(x,0) = w_0(x), \quad z(x,0) = z_0(x), \quad x \in \Omega
\end{aligned}$$

by comparison principle and from [3, 20], it follows that (w, z) is global and so it is (u, v).

Case 2: When $\mu_1 = r_1$, $\mu_2 = r_2$, thus $q_1 q_2 < r_1 r_2$ and $n_1 n_2 < p_1 p_2$. Let $(\overline{u}, \overline{v}) = (A_1, A_2)$, where $A_1 \ge \max_{x\in\overline{\Omega}} u_0(x)$, $A_2 \ge \max_{x\in\overline{\Omega}} v_0(x)$ and A_1, A_2 will be determined later. After some calculations, we have

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial t} - \nabla\left(|x|^{n_1} \overline{u}^{m_1-1}\left|\nabla \overline{u}^{k_1}\right|^{p_1-2}\nabla \overline{u}\right) - \overline{v}^{q_1} + \alpha_1 \overline{u}^{r_1} = \alpha_1 A_1^{r_1} - A_2^{q_1},\\
&\frac{\partial \overline{v}}{\partial t} - \nabla\left(|x|^{n_2} \overline{v}^{m_2-1}\left|\nabla \overline{v}^{k_2}\right|^{p_2-2}\nabla \overline{v}\right) - \overline{u}^{q_2} + \alpha_2 \overline{v}^{r_2} = \alpha_2 A_2^{r_2} - A_1^{q_2}.
\end{aligned}$$

$(\overline{u}, \overline{v}) = (A_1, A_2)$ is a time-independent upper solution of problem (1) if

$$\alpha_1 A_1^{r_1} \ge A_2^{q_1} \quad \textit{and} \quad \alpha_2 A_2^{r_2} \ge A_1^{q_2},$$

i.e.

$$A_2^{\frac{q_1}{r_1}} \alpha_1^{-\frac{1}{r_1}} \le A_1 \le A_2^{\frac{r_2}{q_2}} \alpha_2^{\frac{1}{q_2}}. \tag{6}$$

Since $q_1 q_2 < r_1 r_2$, then there exist A_1, A_2 satisfying (6).

Case 3: When $\mu_1 = r_1$, $\mu_2 = m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$, we have $q_1 q_2 < r_1(m_2 + k_2(p_2 - 2)))$. Let $(\overline{u}, \overline{v}) = (A_1, A_2\psi(x))$, where $\psi(x)$ from (5) and we can choose $A_1 \ge \max_{x\in\overline{\Omega}} u_0(x)$ and $A_2 \ge \max_{x\in\overline{\Omega}} v_0(x)$ satisfying

$$A_1^{\overline{\frac{q_2}{m_2+k_2(p_2-2)}}} \le A_2 \le \frac{1}{M_2}\left(\alpha_1 A_1^{r_1}\right)^{\frac{1}{q_1}}.$$

After direct computation, we have

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial t} - \nabla\left(|x|^{n_1} \overline{u}^{m_1-1}\left|\nabla \overline{u}^{k_1}\right|^{p_1-2}\nabla \overline{u}\right) - \overline{v}^{q_1} + \alpha_1 \overline{u}^{r_1} \ge 0,\\
&\frac{\partial \overline{v}}{\partial t} - \nabla\left(|x|^{n_2} \overline{v}^{m_2-1}\left|\nabla \overline{v}^{k_2}\right|^{p_2-2}\nabla \overline{v}\right) - \overline{u}^{q_2} + \alpha_2 \overline{v}^{r_2} \ge 0.
\end{aligned} \tag{7}$$

So, $(\overline{u}, \overline{v}) = (A, B\psi(x))$ is a upper solution for system (1)

Case 4: If $\mu_1 = m_1 + k_1(p_1 - 2)$, $\mu_2 = r_2$, we have $q_1 q_2 < r_2(m_1 + k_1(p_1 - 2))$ and $n_1 n_2 < p_1 p_2$, we let $(\overline{u}, \overline{v}) = (A_1(\varphi(x) + 1), A_2)$, where $\varphi(x)$ from (4) and choose such $A_1 \geq \max_{x\in\overline{\Omega}} u_0(x)$ and $A_2 \geq \max_{x\in\overline{\Omega}} v_0(x)$ satisfying

$$A_2^{\frac{q_1}{m_1+k_1(p_1-2)}} \leq A_1 \leq \frac{1}{M_1} \left(\alpha_2 A_2^{r_2}\right)^{\frac{1}{q_2}}.$$

Then system of inequalities (7) is fulfilled.

Proof of Theorem 1 is completed.

Proof of Theorem 2. We consider 4 cases to prove Theorem 2:

Case 1: When $r_1 > m_1 + k_1(p_1 - 2)$, $r_2 > m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$, we know $q_1 q_2 = r_1 r_2$. We can choose A_1 and A_2 sufficiently large, fulfilling $A_1 \geq \max_{x\in\overline{\Omega}} u_0(x)$, $A_2 \geq \max_{x\in\overline{\Omega}} v_0(x)$ and

$$\alpha_1^{-\frac{1}{r_1}} A_2^{\frac{q_1}{r_1}} \leq A_1 \leq \alpha_2^{\frac{1}{q_2}} A_2^{\frac{r_2}{q_2}}.$$

It is clear, that $(\overline{u}, \overline{v}) = (A_1, A_2)$ is a weak upper solution of problem (1). It is easy to check, by comparison principle that the solution $(\overline{u}, \overline{v}) = (A_1, A_2)$ of problem (1) is global.

Now we need to prove our blow-up conclusion. Assume that Ω contains the origin. Denote

$$\underline{u}(x,t) = (T-t)^{-\gamma_1} U(\xi_1), \quad \underline{v}(x,t) = (T-t)^{-\gamma_2} V(\xi_2),$$

where $U(\xi_1) = \left(A^{\frac{p_1-n_1}{p_1-1}} - \xi_1^{\frac{p_1-n_1}{p_1-1}}\right)^{\frac{p_1-1}{m_1-1+k_1(p_1-2)}}$, $V(\xi_2) = \left(A^{\frac{p_2-n_2}{p_2-1}} - \xi_2^{\frac{p_2-n_2}{p_2-1}}\right)^{\frac{p_2-1}{m_2-1+k_2(p_2-2)}}$, $\xi_1 = \frac{|x|}{(T-t)^{\beta_1}}$, $\xi_2 = \frac{|x|}{(T-t)^{\beta_2}}$, $\beta_1 = \frac{1-\gamma_1(m_1+k_1(p_1-2)-1)}{p_1} > 0$, $\beta_2 = \frac{1-\gamma_2(m_2+k_2(p_2-2)-1)}{p_2} > 0$ and $\gamma_1, \gamma_2, A, T > 0$ are to be determined later. Note that $B_{AT^\beta}(0)$ contains the support of $\underline{u}(x,t)$ and $\underline{v}(x,t)$, where $\beta = \max \beta_1, \beta_2$ if $T > 1$; $\beta = \min \beta_1, \beta_2$ if $T \leq 1$, which is included in Ω if T is sufficiently small.

After a direct computation, we obtain

$$\frac{\partial \underline{u}}{\partial t} = (T-t)^{-\gamma_1 - 1} \left(\gamma_1 U(\xi_1) + \beta_1 \xi_1 \frac{dU}{d\xi_1}\right)$$

$$\nabla \left(|x|^{n_1} \underline{u}^{m_1-1} \left|\nabla \underline{u}^{k_1}\right|^{p-2} \nabla \underline{u}\right) = (T-t)^{-\gamma_1(m_1+k_1(p_1-2))-\beta_1 p_1}$$
$$\xi_1^{n_1} \left(k_1^{p_1-2} N b_1^{p_1-1} U(\xi_1) + k_1^{p_1-2} b_1^{p_1-1} \xi_1 \frac{dU}{d\xi_1}\right),$$

$$\frac{\partial \underline{v}}{\partial t} = (T-t)^{-\gamma_2 - 1} \left(\gamma_2 V(\xi_2) + \beta_2 \xi_2 \frac{dV}{d\xi_2}\right)$$

$$\nabla \left(|x|^{n_2} \underline{v}^{m_2-1} \left|\nabla \underline{v}^{k_2}\right|^{p-2} \nabla \underline{v}\right) = (T-t)^{-\gamma_2(m_2+k_2(p_2-2))-\beta_2 p_2}$$
$$\xi_2^{n_2} \left(k_2^{p_2-2} N b_2^{p_2-1} V(\xi_2) + k_2^{p_2-2} b_2^{p_2-1} \xi_2 \frac{dV}{d\xi_2}\right),$$

where $b_1 = \frac{p_1}{m_1+k_1(p_1-2)-1}$ and $b_2 = \frac{p_2}{m_2+k_2(p_2-2)-1}$.

We need to find suitable parameters such that

$$(T-t)^{-\gamma_1-1}\left(\gamma_1 U(\xi_1) + \beta_1 \xi_1 \frac{dU}{d\xi_1} - k_1^{p_1-2} N b_1^{p_1-1} \xi_1^{n_1} U(\xi_1) - k_1^{p_1-2} b_1^{p_1-1} \xi_1^{n_1+1} \frac{dU}{d\xi_1}\right) + \\ + (T-t)^{-r_1\gamma_1} U^{r_1}(\xi_1) \leq (T-t)^{-q_1\gamma_2} V^{q_1}(\xi_2) \tag{8}$$

$$(T-t)^{-\gamma_2-1}\left(\gamma_2 V(\xi_2)+\beta_2\xi_2\frac{dV}{d\xi_2}-k_2^{p_2-2}Nb_2^{p_2-1}\xi_2^{n_2}V(\xi_2)-k_2^{p_2-2}b_2^{p_2-1}\xi_2^{n_2+1}\frac{dV}{d\xi_2}\right)+ \\ +(T-t)^{-r_2\gamma_2}V^{r_2}(\xi_2)\le (T-t)^{-q_2\gamma_1}V^{q_2}(\xi_1). \tag{9}$$

Note that U, V are continuous for C^2 except for $\xi_1 = A$, $\xi_2 = A$ where U', V' has a positive jump. Therefore, to obtain a lower solution of (1), we will prove (8) and (9) pointwise for $\xi_1 > 0$, with $\xi_1 \neq A$. It is easy to see that

$$\frac{dU}{d\xi_1}=-\frac{p_1}{m_1+k_1(p_1-2)-1}\xi_1^{\frac{p_1+n_1}{p_1-1}-1}\left(A^{\frac{p_1+n_1}{p_1-1}}-\xi_1^{\frac{p_1+n_1}{p_1-1}}\right)^{\frac{p_1-1}{m_1+k_1(p_1-2)-1}-1},$$

$$\frac{dV}{d\xi_2}=-\frac{p_2}{m_2+k_2(p_2-2)-1}\xi_2^{\frac{p_2+n_2}{p_2-1}-1}\left(A^{\frac{p_2+n_2}{p_2-1}}-\xi_2^{\frac{p_2+n_2}{p_2-1}}\right)^{\frac{p_2-1}{m_2+k_2(p_2-2)-1}-1}$$

and (8) is trivial for $\xi_1 \geq A$. A simple computation shows that (8) is satisfied. We distinguish two steps for $0 < \xi_1 < \theta_1 A$ and $\theta_1 A < \xi_1 < A$, where

$$\theta_1=\left(\frac{\gamma_1+k_1^{p_1-2}Nb_1^{p_1-1}}{\gamma_1+k_1^{p_1-2}Nb_1^{p_1-1}+\beta_1 b_1+k_1^{p_1-2}b_1^{p_1}}\right)^{\frac{p_1-1}{p_1+n_1}}<1.$$

Step 1. For $\theta_1 A < \xi_1 < A$, we have

$$\gamma_1 U(\xi_1)+\beta\xi_1\frac{dU}{d\xi_1}-k_1^{p_1-2}Nb_1^{p_1-1}\xi_1^{n_1}U(\xi_1)-k_1^{p_1-2}b_1^{p_1-1}\xi_1^{n_1+1}\frac{dU}{d\xi_1}$$

$$=\left(A^{\frac{p_1+n_1}{p_1-1}}-\xi_1^{\frac{p_1+n_1}{p_1-1}}\right)^{\frac{p_1-1}{m_1+k_1(p_1-2)-1}-1}$$

$$\left(\left(\gamma_1+k_1^{p_1-2}Nb_1^{p_1-1}\right)A^{\frac{p_1+n_1}{p_1-1}}-\left(\gamma_1+k^{p_1-2}Nb_1^{p_1-1}+\beta_1 b_1+k_1^{p_1-2}b_1^{p_1}\right)\xi_1^{\frac{p_1+n_1}{p_1-1}}\right)\le$$

$$\le\left(A^{\frac{p_1+n_1}{p_1-1}}-\xi_1^{\frac{p_1+n_1}{p_1-1}}\right)^{\frac{p_1-1}{m_1+k_1(p_1-2)-1}-1}$$

$$\left(\left(\gamma_1+k_1^{p_1-2}Nb_1^{p_1-1}\right)A^{\frac{p_1+n_1}{p_1-1}}-\left(\gamma_1+k_1^{p_1-2}Nb_1^{p_1-1}+\beta_1 b_1+k_1^{p_1-2}b_1^{p_1}\right)(\theta_1 A)^{\frac{p_1+n_1}{p_1-1}}\right)$$

$$\le-\beta_1 b_1(\theta_1 A)^{\frac{p_1+n_1}{p_1-1}}\left(A^{\frac{p_1+n_1}{p_1-1}}-\xi_1^{\frac{p_1+n_1}{p_1-1}}\right)_+^{\frac{p_1-1}{m_1+k_1(p_1-2)-1}-1}.$$

Step 2. For $0 < \xi_1 \leq \theta_1 A$, the inequality

$$U(\xi_1)\ge\left(1-\theta_1^{\frac{p_1+n_1}{p_1-1}}\right)^{\frac{p_1-1}{m_1+k_1(p_1-2)-1}}A^{\frac{p_1}{m_1+k_1(p_1-2)-1}}>0,$$

$$V(\xi_2)\ge\left(1-\theta_1^{\frac{p_2+n_2}{p_2-1}}\right)^{\frac{p_2-1}{m_2+k_2(p_2-2)-1}}A^{\frac{p_2}{m_2+k_2(p_2-2)-1}}>0$$

holds. It follows from $\gamma_2 q_1 > \gamma_1 + 1$ that

$$(T-t)^{-\gamma_1-1}\left(\gamma_1 U(\xi_1)+\beta_1\xi_1\frac{dU}{d\xi_1}-k^{p_1-2}Nb_1^{p_1-1}\xi_1^{n_1}U(\xi_1)-k_1^{p_1-2}b_1^{p_1-1}\xi_1^{n_1+1}\frac{dU}{d\xi_1}\right)$$

$$\le\frac{p_1-1}{p_1}(T-t)^{-q_1\gamma_2}U^{q_1}(\xi_1)$$

if T is sufficiently small. If

$$\alpha_1 (T-t)^{-r_1\gamma_1} U^{r_1}(\xi_1) \le \frac{p_1-1}{p_1} (T-t)^{-q_1\gamma_2} V^{\gamma_2}(\xi_2), \tag{10}$$

then (8) holds.
Similarly, if

$$\alpha_2 (T-t)^{-r_2\gamma_2} V^{r_2}(\xi_2) \le \frac{p_2-1}{p_2} (T-t)^{-q_2\gamma_1} U^{q_2}(\xi_1), \tag{11}$$

$\gamma_1 q_2 > \gamma_2 + 1$ and T is sufficiently small, then (9) holds.
Next, we choose suitable γ_1 γ_2 to satisfy (10) and (11). It is easy to see that there is $\gamma_1 > \frac{q_1+1}{q_1q_2-1}$ and $\gamma_2 > \frac{q_2+1}{q_1q_2-1}$, fulfilling the inequalities

$$\gamma_2 q_1 > \gamma_1 + 1, \quad \gamma_1 q_2 > \gamma_2 + 1. \tag{12}$$

If $q_1q_2 = r_1r_2$, then we choose some large γ_1 and γ_2 satisfying (12), and $q_1 = \frac{\gamma_1}{\gamma_2} r_1$, hence $q_2 = \frac{\gamma_2}{\gamma_1} r_2$. Consequently, by (10) and (11), for sufficiently small α_1 and α_2 the following hold

$$\alpha_1 U^{r_1} \le \frac{p_1-1}{p_1} V^{q_1}, \quad \alpha_2 V^{r_2} \quad \le \frac{p_2-1}{p_2} U^{q_2}. \tag{13}$$

Hence, $(\underline{u}, \underline{v})$ is a blow-up lower solution of problem (1) with sufficiently large initial data $(\underline{u}_0, \underline{v}_0)$.

Case 2: When $r_1 < m_1 + k_1(p_1-2)$, $r_2 < m_2 + k_2(p_2-2)$ and $n_1n_2 < p_1p_2$, we have $q_1q_2 = (m_1 + k_1(p_1-2))(m_2 + k_2(p_2-2))$. We choose A_1 and A_2 satisfying

$$(A_2 \psi(x))^{\frac{q_1}{m_1+k_1(p_1-2)}} \le A_1 \le \left(A_2 \varphi^{-q_2}(x)\right)^{\frac{m_2+k_2(p_2-2)}{q_2}}.$$

The solution is global for small initial data, because $(\overline{u}, \overline{v}) = (A_1\varphi(x), A_2\psi(x))$, where $\varphi(x)$, $\psi(x)$ satisfy (4), (5) respectively, is a global upper solution of problem (1).

Cases 3 and 4: Cases 3 and 4 are proved similarly to Case 2.

Proof of Theorem 3. We consider two main cases: large initial values and small initial data for theorem. First, we consider the case of large initial values and prove that the solution blows up in finite time. Then, we consider the case of small initial data and show that the solutions exist globally. For each, we break into four subcases based on different conditions of μ_1 and μ_2.

Case 1: 1. When $\mu_1 = m_1 + k_1(p_1-2)$, $\mu_2 = m_2 + k_2(p_2-2)$, that is $q_1q_2 > (m_1 + k_1(p_1-2))(m_2 + k_2(p_2-2))$ and $n_1n_2 < p_1p_2$. Let $(\overline{u}, \overline{v}) = (A_1\varphi(x), A_2\psi(x))$, where $\varphi(x)$, $\psi(x)$ satisfy (4), (5) respectively. Choosing then

$$A_1 = \frac{1}{2}\left((A_2M_2)^{\frac{q_1}{m_1+k_1(p_1-2)}} + A_2^{\frac{m_2+k_2(p_2-1)}{q_2}} M_1^{-1} \right),$$

$$A_2 = \left(M_1^{m_1+k_1(p_1-2)} M_2^{q_1}\right)^{-\frac{q_2}{q_1q_2-(m_1+k_1(p_1-2))(m_2+k_2(p_2-1))}}.$$

Therefore, $(\overline{u}, \overline{v})$ is a global upper solution for problem (1) if $A_1 \ge \max_{x\in\overline{\Omega}} u_0(0)$ and $A_2 \ge \max_{x\in\overline{\Omega}} v_0(0)$ for small initial values.

2. When $\mu_1 = r_1$, $\mu_2 = r_2$ and $n_1 n_2 < p_1 p_2$, that is $q_1 q_2 > r_1 r_2$ choosing
$$A_1 = \frac{1}{2}\left(\alpha_1^{-\frac{1}{r_1}} A_2^{\frac{q_1}{r_1}} + \alpha_2^{\frac{1}{q_2}} A_2^{\frac{r_2}{q_2}}\right) \quad \text{and} \quad A_2 = \left(\alpha_1^{q_2}\alpha_2^{r_1}\right)^{\frac{1}{q_1 q_2 - r_1 r_2}},$$
then $(\overline{u}, \overline{v}) = (A_1, A_2)$ is a global upper solution for problem (1), if $A_1 \geq \max_{x\in\overline{\Omega}} u_0(0)$ and $A_2 \geq \max_{x\in\overline{\Omega}} v_0(0)$ for small initial values.
3. When $\mu_1 = r_1$, $\mu_2 = m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$, that is $q_1 q_2 > r_1(m_2 + k_2(p_2 - 2))$. Let $(\overline{u}, \overline{v}) = (A_1, A_2\psi(x))$, where $\psi(x)$ satisfies (5). Then we choose
$$A_1 = \left(\alpha_1 M_2^{q_1}\right)^{-\frac{m_2 + k_2(p_2-2)}{q_1 q_2 - r_1(m_2 + k_2(p_2-1))}}, \quad A_2 = \frac{1}{2}\left(A_1^{\frac{q_2}{m_2 + k_2(p_2-2)}} + \left(\alpha_1 A_1^{r_1}\right)^{\frac{1}{q_1}} \frac{1}{M_2}\right).$$
Therefore, $(\overline{u}, \overline{v})$ is a global upper solution for problem (1), if $A_1 \geq \max_{x\in\overline{\Omega}} u_0(0)$ and $A_2 \geq \max_{x\in\overline{\Omega}} v_0(0)$.
4. When $\mu_1 = m_1 + k_1(p_1 - 2)$, $\mu_2 = r_2$ and $n_1 n_2 < p_1 p_2$, that is $q_1 q_2 > r_2(m_1 + k_1(p_1 - 2))$. Let $(\overline{u}, \overline{v}) = (A_1\varphi(x), A_2)$, where $\varphi(x)$ satisfies (4). We choose
$$A_1 = \frac{1}{2}\left(A_2^{\frac{q_1}{m_1 + k_1(p_1-2)}} + \left(\alpha_2 A_2^{r_2}\right)^{\frac{1}{q_2}} \frac{1}{M_1}\right), \quad A_2 = \left(\alpha_2 M_1^{q_2}\right)^{-\frac{m_1 + k_1(p_1-2)}{q_1 q_2 - r_2(m_1 + k_1(p_1-2))}}.$$
Therefore, $(\overline{u}, \overline{v})$ is a global upper solution for problem (1), if $A_1 \geq \max_{x\in\overline{\Omega}} u_0(0)$ and $A_2 \geq \max_{x\in\overline{\Omega}} v_0(0)$.

Case 2: Next, consider large initial values. We construct a blow-up lower solution and use the comparison principle. Let $w(x) > 0$ be a continuous function and $w(x)|_{\partial\Omega} = 0$. We assume, that $0 \in \Omega$ and $w(0) > 0$.

1. Let $r_1 \geq m_1 + k_1(p_1 - 2)$ and $r_2 \geq m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$, hence $q_1 q_1 > r_1 r_2$. The proof is similar to that in [1]. However, we give more details. In order to fulfill (13), we require
$$\begin{aligned} &\gamma_2 q_1 > \gamma_1 + 1, \quad \gamma_2 q_1 > \gamma_1 r_1, \\ &\gamma_1 q_2 > \gamma_2 + 1 > \gamma_2 r_2. \end{aligned} \tag{14}$$
We set $\lambda = \gamma_1/\gamma_2$, then by (14),
$$\frac{r_2}{q_2} < \lambda < \frac{q_1}{r_1}, \quad r_2 - 1 < \frac{1}{\gamma_2} < \min\{q_1 - \lambda, \lambda q_2 - 1\}.$$
If $\lambda \leq \frac{q_1+1}{q_2+1}$, then $\min\{q_1 - \lambda, \lambda q_2 - 1\} = \lambda q_2 - 1$. We assume, that
$$\frac{r_2}{q_2} < \frac{q_1 + 1}{q_2 + 1}. \tag{15}$$
Since $q_1 q_2 > r_1 r_2$, (15) holds or
$$\frac{r_1}{q_1} < \frac{q_2 + 1}{q_1 + 1}. \tag{16}$$
If (16) holds, we just exchange the roles of functions u and v in problem (1). Therefore, we need to guarantee, that (15) holds.

To fulfill (14), we have to find a suitable λ from
$$\frac{r_2}{q_2} < \lambda < \min\left\{\frac{q_1 + 1}{q_2 + 1}, \frac{q_1}{r_1}\right\}.$$

It is possible since $\frac{r_2}{q_2} < \frac{q_1}{r_1}$ and $\alpha_2 > 0$, such that

$$0 < r_2 - 1 < \frac{1}{\gamma_2} < \lambda q_2 - 1.$$

Thus, (13) holds. Therefore, $(\underline{u}, \underline{v})$ is a lower solution of (1) for $r_1 \geq m_1 + k_1(p_1 - 2)$ and $r_2 > m_2 + k_2(p_2 - 2)$.

2. If $r_1 < m_1 + k_1(p_1 - 2)$ and $r_2 \geq m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$, hence $q_1 q_2 > r_2 (m_1 + k_1(p_1 - 2))$. Any solution of (1) is an upper solution to the following homogeneous Dirichlet problem

$$\begin{aligned} &\frac{\partial u}{\partial t} \geq \nabla \left(|x|^{n_1} u^{m_1 - 1} \left| \nabla u^{k_1} \right|^{p_1 - 2} \nabla u \right) + v^{q_1} - \alpha_1 u^{r_1} - \alpha_1, \quad x \in \Omega, t > 0, \\ &\frac{\partial v}{\partial t} \geq \nabla \left(|x|^{n_2} v^{m_2 - 1} \left| \nabla v^{k_2} \right|^{p_2 - 2} \nabla v \right) + u^{q_2} - \alpha_2 v^{r_2}, \quad x \in \Omega, t > 0. \end{aligned} \tag{17}$$

Following $q_1 q_2 > r_2 (m_1 + k_1(p_1 - 2))$, similarly to the above proof, one can see that $(\underline{u}, \underline{v})$ is still a lower solution of (17) for appropriate u_0 and v_0. It means that (u, v) blows up.

The subcases (3) $r_1 \geq m_1 + k_1(p_1 - 2)$, $r_2 < m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$ and (4) $r_1 < m_1 + k_1(p_1 - 2)$, $r_2 < m_2 + k_2(p_2 - 2)$ and $n_1 n_2 < p_1 p_2$ can be treated in a similar way.

Proof of Theorem 3 is completed.

Conclusion

This paper considers the problem of a doubly nonlinear degenerate parabolic system with nonlinear source and absorption terms with variable density. We extend existing results on the global existence and blow-up of solutions to the case with variable density in the diffusion term. Since the global existence and blow-up property of solutions allow us to predict or control the future of processes, the obtained results are significant and contribute to a concise understanding of doubly nonlinear degenerate parabolic problems. Future directions of research include investigating the problem in wider settings and studying other qualitative properties of the solutions.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Samarskii, A.A., Galaktionov, V.A., Kurdyumov, S.P., & d Mikhailov, A.P. (1995). *Blow-Up in Quasilinear Parabolic Equations.* De Gruyter. https://doi.org/10.1515/9783110889864

2 Aripov, M., & Sadullaeva, S. (2020). *Computer simulation of nonlinear diffusion processes.* Tashkent, National University Press.

3 Bedjaoui, N., & Souplet, P. (2002). Critical blowup exponents for a system of reaction-diffusion equations with absorption. *Zeitschrift für angewandte Mathematik und Physik ZAMP, 53*, 197–210. https://doi.org/10.1007/S00033-002-8152-9

4 Chaouai, Z., & El Hachimi, A. (2020). Qualitative properties of weak solutions for p-laplacian equations with nonlocal source and gradient absorption. *Bulletin of the Korean Mathematical Society, 4*(57), 1003–1031. https://doi.org/10.4134/BKMS.B190720

5 Liu, B., & Xia, H. (2022). Blow-up solutions in a Cauchy problem of parabolic equations with spatial coefficients. *Bull. Iran. Math. Soc., 48*, 1051–1074. https://doi.org/10.1007/s41980-021-00562-2

6 Kong, L., Wang, X., & Zhao, X. (2012). Asymptotic analysis to a parabolic system with weighted localized sources and inner absorptions. *Arch. Math., 99*, 375–386. https://doi.org/10.1007/s00013-012-0433-8

7 Xu, X., & Cheng, T. (2020). Extinction and Decay Estimates of Solutions for a Non-Newton Polytropic Filtration System. *Bull. Malays. Math. Sci., 43*, 2399–2415. https://doi.org/10.1007/s40840-019-00814-9

8 Aripov, M., & Bobokandov, M. (2023). The Cauchy Problem for a Doubly Nonlinear Parabolic Equation with Variable Density and Nonlinear Time-Dependent Absorption. *J. Math. Sci., 277*, 355–365. https://doi.org/10.1007/s10958-023-06840-0

9 Zhou, J. (2014). Global existence and blowup for a degenerate and singular parabolic system with nonlocal source and absorptions. *Z. Angew. Math. Phys., 65*, 449–469. https://doi.org/10.1007/s00033-013-0342-0

10 Dong, Z., & Zhou, J. (2017). Blow-up of solutions to a parabolic system with nonlocal source. *Applicable Analysis, 95*(5), 825–841. https://doi.org/10.1080/00036811.2017.1292351

11 Liu, D., & Liu, Ch. (2022). On the global existence and extinction behavior for a polytropic filtration equation with variable coefficients. *Electronic Research Archive 30*(2), 425–439. https://doi.org/10.3934/era.2022022

12 Xue, Y.Z. (2023). Lower bounds of blow up time for a class of slow reaction diffusion equations with inner absorption terms. *Wuhan University Journal of Natural Sciences,* 28(5), 373–378. https://doi.org/10.1051/wujns/2023285373

13 Sun, X., Liu, B., & Li, F. (2022). Fujita exponents for a space-time weighted parabolic problem in bounded domains. *Nonlinear Analysis: Real World Applications, 62*. https://doi.org/10.1016/j.nonrwa.2021.103358

14 Li, L.-L., Sun, H.-R., & Zhang, Q.-G. (2017). Existence and nonexistence of global solutions for a semilinear reaction-diffusion system. *Journal of Mathematical Analysis and Applications, 1*(445), 97–124. https://doi.org/10.1016/j.jmaa.2016.07.067

15 Liu, B., & Lin, H. (2019). A Cauchy problem of spatial-weighted reaction-diffusion equations, *Applied Mathematics Letters, 92*, 128–133. https://doi.org/10.1016/j.aml.2019.01.021

16 Jiang, W., & Xie, J. (2022). Extinction of solutions to a diffusion problem with nonlinear sources and variable density. *Results in Applied Mathematics, 13*, 1–8. https://doi.org/10.1016/j.rinam.2022.100242

17 Liu, B., Li, F., & Zhao, Z. (2019). Non-simultaneous blow-up profile and boundary layer estimate in nonlinear parabolic problems. *Applicable Analysis, 2*(100), 417–427. https://doi.org/10.1080/00036811.2019.1608959

18 Li, H., Han, Y., & Gao, W. (2015). Critical extinction exponents for polytropic filtration equations with nonlocal source and absorption. *Acta Mathematica Scientia, 2*(35), 366–374. https://doi.org/10.1016/S0252-9602(15)60008-X

19 Cung, T.A., & Le, T.T. (2013). Long-time behavior for semilinear degenerate parabolic equations on RN. *Communications of the Korean Mathematical Society, 4*(28), 751–766. https://doi.org/10.4134/CKMS.2013.28.4.751

20 Anh, C.T., & Ke, T.D. (2010). On quasilinear parabolic equations involving weighted p-Laplacian operators. *Nonlinear Differ. Equ. Appl., 17*, 195–212. https://doi.org/10.1007/s00030-009-0048-3

21 Niu, W., Meng, Q., & Chai, X. (2020). Asymptotic behavior for nonlinear degenerate parabolic equations with irregular data. *Applicable Analysis, 16*(100), 3391–3405. https://doi.org/10.1080/00036811.2020.1721470

22 Wang, J. (2015). Critical Exponents in a Doubly Degenerate Nonlinear Parabolic System with Inner Absorptions. *Bull. Malays. Math. Sci. Soc., 38*, 415–435. https://doi.org/10.1007/s40840-014-0028-6a

23 Wang, Y., & Yin, J. (2013). Critical extinction exponents for a polytropic filtration equation with absorption and source. *Mathematical Methods in the Applied Sciences, 36*(12), 1591–1597. https://doi.org/10.1002/mma.2708

24 Sheng-Chen, F. (2009). Critical exponent for a system of slow diffusion equations with both reaction and absorption terms. *Taiwanese J. Math., 1*(13), 307–316. https://doi.org/10.11650/twjm/1500405285

25 Aripov, M.M., Rakhmonov, Z.R., & Alimov, A.A. (2024). On the behaviors of solutions of a nonlinear diffusion system with a source and nonlinear boundary conditions. *Bulletin of the Karaganda University. Mathematics Series,* 1(113), 28–45. https://doi.org/10.31489/2024m1/28-45

26 Youpeng, C. (2007). Global blow-up for a heat system with localized sources and absorptions. *Applied Mathematics–A Journal of Chinese Universities, 22*, 213–225. http://doi.org/10.1007/s11766-007-0210-9

27 Lu, H., Wu, J., & Liu, W. (2022). Analysis of Solutions to a Parabolic System with Absorption. *Symmetry, 14*(6), 1274. https://doi.org/10.3390/sym14061274

28 Aripov, M., & Atabaev, O. (2024). Qualitative behavior of solutions of doubly nonlinear parabolic problem with nonlinear source and absorption terms. *Bull. Inst. Math., 1*(7), 21–32.

29 Xiang, Z., Chen, Q., & Mu, C. (2007). Blow-up rate estimates for a system of reaction-diffusion equations with absorption. *J. Korean Math. Soc., 4*(44), 779–786. http://doi.org/10.4134/JKMS.2007.44.4.779

30 Wu, X.L., & Gao, W. (2014). Global existence and blow-up solution for doubly degenerate parabolic system with nonlocal sources and inner absorptions. *Mathematical Methods in the Applied Sciences, 37*. http://doi.org/10.1002/mma.2813

*Author Information**

Mirsaid Mirsiddikovich Aripov — Doctor of physical and mathematical sciences, Professor, Professor of Applied Mathematics and Computer Analysis Department, National University of Uzbekistan, 4 University street, Tashkent, 100174, Uzbekistan; e-mail: *mirsaidaripov@gmail.com*; https://orcid.org/0000-0001-5207-8852

Odiljon Xusniddin o'g'li Atabaev (*corresponding author*) — Lecturer at Information Technologies Department, Andijan State Technical Institute, 56 Boburshokh street, Andijan, 170119, Uzbekistan; e-mail: odiljonatabaev@gmail.com; https://orcid.org/0000-0002-5989-2141

*The author's name is presented in the order: First, Middle and Last Names.

Abdullah Meadh M. AL-Marashi — Professor, Department of Statistics, Faculty of Sciences, King Abdulaziz University, Jeddah, Saudi Arabia; e-mail: *aalmarashi@kau.edu.sa*; https://orcid.org/0000-0003-3201-8961

https://doi.org/10.31489/2025M1/24-33 *Research article*

On the stability of the third order partial differential equation with time delay

A. Ashyralyev[1,2,3], S. Ibrahim[4,*], E. Hincal[4]

[1] *Bahcesehir University, Istanbul, Turkey;*
[2] *Peoples' Friendship University of Russia (RUDN University), Moscow, Russian Federation;*
[3] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[4] *Near East University, Nicosia, TRNC, Mersin 10, Turkey*
(E-mail: allaberen.ashyralyev@eng.bau.edu.tr, ibrahim.suleiman@neu.edu.tr, evren.hincal@neu.edu.tr)

In this paper, the initial value problem for a third-order partial differential equation with time delay within a Hilbert space was analyzed. We establish a key theorem regarding the stability of this problem. Additionally, we demonstrate how this stability theorem can be applied to the third-order partial differential equation with time delay.

Keywords: stability, third order partial differential equations, time delay.

2020 Mathematics Subject Classification: 35G10, 35L90, 58D25.

Introduction

In physics, various problems give rise to third order partial differential equations (PDEs). In various branches of engineering and science, such as applied mathematics, these problems have become a key research area. Within the last 10 decades, interest towards nonlocal and local boundary value problems (BVPs) for PDEs with space and time variables have increased significantly. Nonlocal and local BVPs for third order PDEs have been investigated widely in a lot literature (for instance, see [1–3]).

One of the most frequently occurring phenomena in various engineering applications is time delay (TD). A typical instance with regards to control theory can be seen in sampled-data control process.

Applications and theory of nonlinear and linear third-order differential and difference equations comprising a delay term were investigated widely (for instance, see [4–11], and the included references).

Lastly, applications and theory of PDEs of the same order having delay operator term with respect to the other operator term were studied for parabolic differential equations with delay term (for example, see [12–18], and the included references).

However, the stability theory of third-order PDEs having a delay term is not well developed. In this paper, our aim is to study the initial value problem (IVP) for the third order PDE having TD

$$\begin{cases} \frac{d^3y(s)}{ds^3} + B\frac{dy(s)}{ds} = cBy(s-z) + h(s), \ 0 < s < \infty, \\ \\ y(s) = k(s), \ -z \leq s \leq 0 \end{cases} \tag{1}$$

in G, a Hilbert space, having self-adjoint positive definite operator (SAPDO) $B,\ B \geq \lambda I$, where $\lambda > 0$. Here $k(s)$ defined on $[-z, 0]$ is the given abstract continuous function (ACF) with values in $D(B),\ h(s)$ defined on $(0, \infty)$ is the given ACF having values in G, and $c \in R^1$.

The structure of the paper is as follows. In Section 1, we establish the main theorem on the stability of problem (1). Section 2 presents theorems on stability estimates for the solutions of three problems involving third-order PDEs. Finally, Section 3 provides the conclusion.

*Corresponding author. *E-mail: ibrahim.suleiman@neu.edu.tr*

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP05131225).

Received: 17 June 2024; *Accepted:* 26 November 2024.

1 Main theorem on Stability

If conditions i, ii, iii below are met, then a function $y(s)$ is considered a solution to problem (1):

i. $y(s)$ is twice continuously differentiable over the interval $[0,\infty)$, with the derivative at $s=0$ taken as the unilateral derivative.

ii. The derivative $\frac{dy(s)}{ds}$ lies in $D(B)$ for every $s\in[0,\infty)$, and the function $B\frac{dy(s)}{ds}$ is continuous throughout the interval $[0,\infty)$.

iii. $y(s)$ satisfies the primary equation and the initial conditions described in (1).

Throughout this paper, let $\{E(s), s\geq 0\}$ be an operator function, where $E(s)=\cos(sB^{\frac{1}{2}})$, and is defined by the formula

$$E(s)=\frac{e^{isB^{\frac{1}{2}}}+e^{-isB^{\frac{1}{2}}}}{2}. \tag{2}$$

From the operator function $T(s)=B^{-\frac{1}{2}}\sin(sB^{\frac{1}{2}})$, where $T(s)=\int_0^s E(p)\,dp$, it follows that

$$T(s)=B^{-\frac{1}{2}}\frac{e^{isB^{\frac{1}{2}}}-e^{-isB^{\frac{1}{2}}}}{2i}. \tag{3}$$

We refer to [19] for the theory of cosine operator functions. We now present an important lemma below.

Lemma 1.1. The estimates that follows holds for $s\geq 0$:

$$\left\|\exp\left\{\pm isB^{\frac{1}{2}}\right\}\right\|_{G\to G}\leq 1,\quad ||E(s)||_{G\to G}\leq 1,\quad \left\|B^{\frac{1}{2}}T(s)\right\|_{G\to G}\leq 1. \tag{4}$$

The proof of the lemma above depends on the spectral representation of unit SAPDO B.

Moreover, for all $\frac{dx(s)}{ds}\in D(B)$ we can write

$$\frac{d^3x(s)}{ds^3}+B\frac{dx(s)}{ds}=\left(\frac{d^2}{ds^2}+B\right)\frac{d}{ds}x(s).$$

Therefore, problem (1) be rewritten as the equivalent IVP

$$\begin{cases} \dfrac{dy(s)}{ds}=x(s),\\ \dfrac{d^2x(s)}{ds^2}+Bx(s)=cBy(s-z)+h(s),\ 0<s<\infty,\\ y(s)=k(s), -z\leq s\leq 0\end{cases}$$

for the system of linear differential equations. Integrating these equations, we can write

$$\begin{cases} y(s)=y(0)+\int_0^s x(r)dr,\\ x(s)=E\left(s\right)x\left(0\right)+T\left(s\right)x'\left(0\right)+\int_0^s T\left(s-r\right)\left[cBk(r-z)+h(r)\right]dr\end{cases}$$

for all $s\in[0,z]$ and

$$\begin{cases} y(s)=y(mz)+\int_{mz}^s x(r)dr,\\ x(s)=E\left(s-mz\right)x\left(mz\right)+T\left(s-mz\right)x'\left(mz\right)+\int_{mz}^s T\left(s-r\right)\left[cBy(r-z)+h(r)\right]dr\end{cases}$$

for all $s\in\left[mz,(m+1)\,z\right]$, $m=1,2,\ldots$

Applying (2) and (3), we can write

$$\int_0^s T(r)drx = -B^{-1}\left(E\left(s\right)-I\right)x.$$

From that and equation $\frac{dy(s)}{ds}=x(s)$ it follows $x\left(mz\right)=y'\left(mz\right), x'\left(mz\right)=y''\left(mz\right)$ and

$$y(s)=\begin{cases} y\left(0\right)+T\left(s\right)y'\left(0\right)-B^{-1}\left(E\left(s\right)-I\right)y''\left(0\right)+ \\ +\int_0^s B^{-1}\left(I-E\left(s-r\right)\right)\left[cBk(r-z)+h(r)\right]dr,\ s\in\left[0,z\right], \\ y(mz)+T\left(s-mz\right)y'\left(mz\right)-B^{-1}\left(E\left(s-mz\right)-I\right)y''\left(mz\right)+ \\ +\int_{mz}^s B^{-1}\left(I-E\left(s-r\right)\right)\left[cBy(r-z)+h(r)\right]dr,\ s\in\left[mz,\ \left(m+1\right)z\right],\ m=1,\ldots \end{cases} \tag{5}$$

The main theorem is formulated below.

Theorem 1. Assume that $k(s)$ be a twice continuously differentiable function and $k^0(s)\in D(B^{(3)/2})$, $k^1(s)\in D(B^{(2)/2})$, $k^2(s)\in D(B^{(1)/2})$. Then the following estimates hold for the solution of problem (1):

$$\max_{0\le s\le z}\left\|B^{\frac12}\frac{d^2y(s)}{ds^2}\right\|_G,\ \max_{0\le s\le z}\left\|B\frac{dy(s)}{ds}\right\|_G,\frac12\ \max_{0\le s\le z}\left\|B^{\frac32}y(s)\right\|_G \tag{6}$$

$$\le\left(2+|c|\,z\right)a_0+\int_0^z\left\|B^{\frac12}h(r)\right\|_G dr,$$

$$a_0=\max\left\{\max_{-z\le s\le 0}\left\|B^{\frac12}\frac{d^2k(s)}{ds^2}\right\|_G,\ \max_{-z\le s\le 0}\left\|B\frac{dk(s)}{ds}\right\|_G,\ \max_{-z\le s\le 0}\left\|B^{\frac32}k(s)\right\|_G\right\},$$

$$\max_{mz\le s\le(m+1)z}\left\|B^{\frac12}\frac{d^2y(s)}{ds^2}\right\|_G,\ \max_{mz\le s\le(m+1)z}\left\|B\frac{dy(s)}{ds}\right\|_G,\frac12\ \max_{mz\le s\le(m+1)z}\left\|B^{\frac32}y(s)\right\|_G \tag{7}$$

$$\le\left(2+|c|\,z\right)a_m+\sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac12}h(r)\right\|_G dr,$$

$$a_m=\max\left\{\max_{(m-1)z\le s\le mz}\left\|B^{\frac12}\frac{d^2y(s)}{ds^2}\right\|_G,\ \max_{(m-1)z\le s\le mz}\left\|B\frac{dy(s)}{ds}\right\|_G,\ \max_{(m-1)z\le s\le mz}\left\|B^{\frac32}y(s)\right\|_G\right\},$$

$$m=1,2,\ldots$$

Proof. Let $s\in[0,z]$. Then, applying (5), we get

$$y(s)=k\left(0\right)+T\left(s\right)k'\left(0\right)-B^{-1}\left(E\left(s\right)-I\right)k''\left(0\right)$$

$$+\int_0^s B^{-1}\left(I-E\left(s-r\right)\right)\left[cBk(r-z)+h(r)\right]dr,$$

$$B\frac{dy(s)}{ds}=E\left(s\right)Bk'\left(0\right)+T\left(s\right)Bk''\left(0\right)$$

$$+\int_0^s BT\left(s-r\right)\left[cBk(r-z)+h(r)\right]dr,$$

$$B^{\frac12}\frac{d^2y(s)}{ds^2}=-B^{\frac12}T\left(s\right)Bk'\left(0\right)+E\left(s\right)B^{\frac12}k''\left(0\right)$$

$$+\int_0^s B^{\frac{1}{2}} E\left(s-r\right)\left[cBk(r-z)+h(r)\right]dr.$$

Using these formulas, estimates (4) and the triangle inequality, we get

$$\left\|B^{\frac{3}{2}}y(s)\right\|_G \le \left\|B^{\frac{3}{2}}k(0)\right\|_G + \left\|Bk'\left(0\right)\right\|_G + 2\left\|B^{\frac{1}{2}}k''\left(0\right)\right\|_G$$

$$+2\left|c\right| z \max_{-z\le s\le 0}\left\|B^{\frac{3}{2}}k(s)\right\|_G + 2\int_0^s \left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\le 2\left(2+\left|c\right| z\right)a_0 + 2\int_0^z \left\|B^{\frac{1}{2}}h(r)\right\|_G dr,$$

$$\left\|B\frac{dy(s)}{ds}\right\|_G \le \left\|Bk'\left(0\right)\right\|_G + \left\|B^{\frac{1}{2}}k''\left(0\right)\right\|_G$$

$$+\left|c\right| z \max_{-z\le s\le 0}\left\|B^{\frac{3}{2}}k(s)\right\|_G + \int_0^s \left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\le \left(2+\left|c\right| z\right)a_0 + \int_0^z \left\|B^{\frac{1}{2}}h(r)\right\|_G dr,$$

$$\left\|B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2}\right\|_G \le \left\|B^{\frac{1}{2}}k''\left(0\right)\right\|_G$$

$$+\left|c\right| z \max_{-z\le s\le 0}\left\|B^{\frac{3}{2}}k(s)\right\|_G + \int_0^s \left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\le \left(2+\left|c\right| z\right)a_0 + \int_0^z \left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

for $s\in[0,z]$. From that estimate (6) follows. Let $s\in[mz,(m+1)\,z],\ m=1,2,\dots$ Then, applying (5), we get

$$y(s) = y(mz) + T\left(s-mz\right)y'\left(mz\right) - B^{-1}\left(D\left(s-mz\right)-I\right)y''\left(mz\right)$$

$$+\int_{mz}^s B^{-1}\left(I-D\left(s-r\right)\right)\left[cBy(r-z)+h(r)\right]dr,$$

$$B\frac{dy(s)}{ds} = D\left(s-mz\right)By'\left(mz\right) + T\left(s-mz\right)By''\left(mz\right)$$

$$+\int_{mz}^s BT\left(s-r\right)\left[cBy(r-z)+h(r)\right]dr,$$

$$B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2} = -B^{\frac{1}{2}}T\left(s-mz\right)By'\left(mz\right) + D\left(s-mz\right)B^{\frac{1}{2}}y''\left(mz\right)$$

$$+\int_{mz}^s B^{\frac{1}{2}}D\left(s-r\right)\left[cBy(r-z)+h(r)\right]dr.$$

Using these formulas, estimates (4) and the triangle inequality, we get

$$\left\|B^{\frac{3}{2}}y(s)\right\|_G \le \left\|B^{\frac{3}{2}}y(mz)\right\|_G + \left\|By'\left(mz\right)\right\|_G + 2\left\|B^{\frac{1}{2}}y''\left(mz\right)\right\|_G$$

$$+2\left|c\right| z \max_{(m-1)z\le s\le mz}\left\|B^{\frac{3}{2}}y(s)\right\|_G + 2\sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\leq 2\left(2+|c|\,z\right)a_m+2\sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\left\|B\frac{dy(s)}{ds}\right\|_G\leq \left\|By'\left(mz\right)\right\|_G+\left\|B^{\frac{1}{2}}y''\left(mz\right)\right\|_G$$

$$+|c|\,z\max_{(m-1)z\leq s\leq mz}\left\|B^{\frac{3}{2}}y(s)\right\|_G+\sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\leq \left(2+|c|\,z\right)a_m+\sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\left\|B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2}\right\|_G\leq \left\|By'\left(mz\right)\right\|_G+\left\|B^{\frac{1}{2}}y''\left(mz\right)\right\|_G$$

$$+|c|\,z\max_{(m-1)z\leq s\leq mz}\left\|B^{\frac{3}{2}}y(s)\right\|_G+\sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\leq \left(2+|c|\,z\right)a_m+\sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

for $s\in[mz,(m+1)\,z],\ m=1,2,\ldots$ Estimate (7) follows from it. Theorem 1 is proved.

According to Theorem 1, the following stability estimate holds for the solution of problem (1):

$$\max_{0\leq s\leq (m+1)z}\left\|B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2}\right\|_G,\ \max_{0\leq s\leq (m+1)z}\left\|B\frac{dy(s)}{ds}\right\|_G,\frac{1}{2}\max_{0\leq s\leq (m+1)z}\left\|B^{\frac{3}{2}}y(s)\right\|_G$$

$$\leq \left(2+|c|\,z\right)^m a_0+\sum_{j=1}^{m}\left(2+|c|\,z\right)^{m-j}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr.$$

.

2 Applications

The applications of Theorem 1 are considered in this section.

First, the initial nonlocal BVP for the third order PDE with TD

$$\begin{cases}\frac{\partial^3 x(s,u)}{\partial s^3}-\left(b(u)x_{su}(s,u)\right)_u+\rho x_s(s,u)=c\left(-\left(b(u)x_u(s-z,u)\right)_u+\rho x(s-z,u)\right)+h(s,u),\\ 0<s<\infty,\ 0<u<1,\\ x(s,u)=k(s,u),\ -z\leq s\leq 0,\ 0\leq u\leq 1,\\ x(s,0)=x(s,1),\ x_u(s,0)=x_u(s,1),\ 0\leq s<\infty\end{cases}\tag{8}$$

is considered. Problem (8) has a unique solution $x(s,u)$, under compatibility conditions, for the smooth functions $b(u)\geq b>0,\ u\in(0,1),\ \rho>0,\ b(1)=b(0),\ k(s,u)\ -z\leq s\leq 0,\ 0\leq u\leq 1,\ h(s,u),$ $0<s<\infty$, $0<u<1$, and $c\in R^1$. This allows us to reduce the BVP (8) to the IVP (1) in a Hilbert space $G=L_2[0,1]$ with a SAPDO B^u defined by the formula:

$$B^u x(u)=-(b(u)x_u)_u+\rho x$$

with domain

$$D(B^u) = \{x(u) : x(u),\ x_u(u),\ (b(u)x_u)_u \in L_2[0,1],\ x(1) = x(0),\ x_u(1) = x_u(0)\}\,.$$

By utilizing the symmetry property of the spatial operator B^u, domain of which is $D(B^u) \subset W_2^2[0,1]$, and incorporating the estimates from Theorem 1, the following theorem concerning the stability of problem (8) is obtained.

Theorem 2. The solutions to problem (8) satisfy the stability estimates that follow:

$$\max_{0\le s\le mz}\left\|\frac{d^2y(s,\cdot)}{ds^2}\right\|_{W_2^1[0,1]},\ \max_{0\le s\le mz}\left\|\frac{dy(s,\cdot)}{ds}\right\|_{W_2^2[0,1]},\ \frac{1}{2}\ \max_{0\le s\le mz}\|y(s,\cdot)\|_{W_2^3[0,1]}$$

$$\le M_1\left[(2+|c|z)^m\,a_0+\sum_{j=1}^{m}(2+|c|z)^{m-j}\int\limits_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r,\cdot)\right\|_{L_2[0,1]}dr\right],$$

$$a_0 = \max\left\{\max_{-z\le s\le 0}\left\|\frac{d^2k(s,\cdot)}{ds^2}\right\|_{W_2^1[0,1]},\ \max_{-z\le s\le 0}\left\|\frac{dk(s,\cdot)}{ds}\right\|_{W_2^2[0,1]},\ \max_{-z\le s\le 0}\|k(s,\cdot)\|_{W_2^3[0,1]}\right\},$$

where M_1 does not depend on $k(s,u)$ and $h(s,u)$.

In this context, $W_2^1[0,1]$, $W_2^2[0,1]$ and $W_2^3[0,1]$ are Sobolev spaces consisting of all square integrable functions $\phi(u)$ defined on the interval $[0,1]$, endowed with the following norm:

$$\|\phi\|_{W_2^\zeta[0,1]} = \left(\int\limits_0^1\sum_{j=0}^{\zeta}\left(\phi^{(j)}(u)\right)^2du\right)^{\frac{1}{2}},\quad \zeta = 1,2,3.$$

Next, let Ω represent the unit open cube in the n-dimensional Euclidean space R^n, where $u = (u_1,\ldots,u_n)$ and $0 < u_\zeta < 1$ for $\zeta = 1,\ldots,n$. The boundary of this domain is denoted by P, and we define $\overline{\Omega} = \Omega\cup P$. Within the domain $[0,\infty)\times\Omega$, we consider the initial BVP for a third-order multi-dimensional PDE with a TD, subject to Dirichlet boundary conditions.

$$\begin{cases}\frac{\partial^3x(s,u)}{\partial s^3}-\sum\limits_{t=1}^{n}\left(b_t(u)x_{su_t}(s,u)\right)_{u_t} = -c\sum\limits_{t=1}^{n}\left(b_t(u)x_{u_t}(s-z,u)\right)_{u_t},\\ 0<s<\infty,\ u\in\Omega,\\ x(s,u) = k(s,u),\ -z\le s\le 0,\ u\in\overline{\Omega},\\ x(s,u) = 0,\ u\in P,\ 0\le s<\infty\end{cases}\tag{9}$$

is considered. Here $b_t(u)\ge b>0$, $(u\in\Omega)$, $k(s,u)$, $-z\le s\le 0$, $u\in\overline{\Omega}$, $h(s,u)$, $0<s<\infty$, $u\in\Omega$ are given smooth functions, and $c\in R^1$.

We consider the Hilbert space $L_2(\overline{\Omega})$ of all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|\ h\ \|_{L_2(\overline{\Omega})} = \left(\int\limits_{u\in\overline{\Omega}}|h(u)|^2du_1\cdots du_n\right)^{\frac{1}{2}}.$$

Problem (9) has a unique solution $x(s,u)$, under compatibility conditions, for the smooth functions $b(u)\ge b>0$, $u\in\Omega$, $\rho>0$, $k(s,u)$, $-z\le s\le 0$, $u\in\overline{\Omega}$, $h(s,u)$, $0<s<\infty$, $u\in\Omega$, and $c\in R^1$. With this problem (9) can be reduced to the IVP (1) in the Hilbert space $G = L_2(\overline{\Omega})$ with a SAPDO B^u defined by the formula

$$B^u x(u) = -\sum_{t=1}^{n}(b_t(u)x_{u_t})_{u_t} \tag{10}$$

with domain

$$D(B^u) = \{x(u) : x(u), x_{u_t}(u), (b_t(u)x_{u_t})_{u_t} \in L_2(\Omega), \quad 1 \le t \le n, \quad x(u) = 0, u \in P\}.$$

As a result, we can establish the following theorem concerning the stability of problem (9).

Theorem 3. The following stability estimates are derived for the solutions of problem (9):

$$\max_{0\le s\le mz}\left\|\frac{d^2y(s,\cdot)}{ds^2}\right\|_{W_2^1(\overline{\Omega})}, \max_{0\le s\le mz}\left\|\frac{dy(s,\cdot)}{ds}\right\|_{W_2^2(\overline{\Omega})}, \frac{1}{2}\max_{0\le s\le mz}\|y(s,\cdot)\|_{W_2^3(\overline{\Omega})}$$

$$\le M_2\left[(2+|c|\,z)^n b_0 + \sum_{j=1}^{m}(2+|c|\,z)^{m-j}\int_{(j-1)\omega}^{jz}\left\|B^{\frac{1}{2}}h(s,\cdot)\right\|_{L_2(\overline{\Omega})}ds\right],$$

$$b_0 = \max\left\{\max_{-z\le s\le 0}\left\|\frac{d^2k(s,\cdot)}{ds^2}\right\|_{W_2^1(\overline{\Omega})}, \max_{-z\le s\le 0}\left\|\frac{dk(s,\cdot)}{ds}\right\|_{W_2^2(\overline{\Omega})}, \max_{-z\le s\le 0}\|k(s,\cdot)\|_{W_2^3(\overline{\Omega})}\right\},$$

where M_2 does not depend on $k(s,u)$ and $h(s,u)$. Here, $W_2^1(\overline{\Omega}), W_2^2(\overline{\Omega})$ and $W_2^3(\overline{\Omega})$ are Sobolev spaces of all square integrable functions $\phi(u)$ defined on $\overline{\Omega}$, equipped with the norm

$$\|\phi\|_{W_2^\zeta(\overline{\Omega})} = \left(\int\cdots\int_{u\in\overline{\Omega}}\sum_{j=0}^{\zeta}\sum_{t=1}^{n}\left(\phi_{\underbrace{u_t\cdots u_t}_{j\text{ times}}}(u)\right)^2 du_1\cdots du_n\right)^{\frac{1}{2}}.$$

The proof of Theorem 3 is based on Theorem 1 and the symmetry property of the operator B^u defined by formula (10) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 4. For the solution of the elliptic differential problem [10]:

$$\begin{cases} B^u x(u) = \mu(u), u \in \Omega, \\ x(u) = 0, u \in P, \end{cases}$$

the following coercivity inequality holds:

$$\sum_{t=1}^{m}\|x_{u_t u_t}\|_{L_2(\overline{\Omega})} \le M_3\|\mu\|_{L_2(\overline{\Omega})}.$$

Here, M_3 does not depend on $\mu(u)$.

Third, in $[0,\infty)\times\Omega$, the BVP for the multi-dimensional Schrödinger equation with TD and Neumann boundary condition is considered:

$$\begin{cases} \frac{\partial^3 x(s,u)}{\partial s^3} - \sum\limits_{t=1}^{m}(b_t(u)x_{su_t}(s,u))_{u_t} + \rho x_s(s,u) = c\left(-\sum\limits_{t=1}^{m}(b_t(u)x_{u_t}(s-z,u))_{u_t} + \rho x(s-z,u)\right), \\ 0 < s < \infty, u \in \Omega, \\ x(s,u) = k(s,u), -z \le s \le 0, u \in \overline{\Omega}, \\ \frac{\partial x(s,u)}{\partial \vec{m}} = 0, u \in P, 0 \le s < \infty. \end{cases} \tag{11}$$

Here, $\vec{m}$ is the normal vector to P, $b_t(u) \geq b > 0, (u \in \Omega)$, $k(s,u)$, $-z \leq s \leq 0, 0 \leq u \leq 1$ and $h(s,u), 0 < s < \infty, 0 < u < 1$ are given smooth functions, and $c \in R^1$.

Problem (11) has a unique solution $x(s,u)$, under compatibility conditions, for the smooth functions $\varphi(u)$ and $b_t(u)$. This enables us to simplify problem (11) into the IVP in the Hilbert space $G = L_2(\overline{\Omega})$ with a SAPDO B^u, defined by the following expression:

$$B^u x(u) = -\sum_{t=1}^{m} (b_t(u) x_{u_t})_{u_t} + \rho x$$

having domain:

$$D(B^u) = \left\{ x(u) : x(u), x_{u_t}(u), (b_t(u) x_{u_t})_{u_t} \in L_2(\overline{\Omega}),\ 1 \leq t \leq m,\ \frac{\partial x(u)}{\partial \vec{m}} = 0,\ u \in P \right\}.$$

Therefore, estimates of Theorem 1 with $G = L_2(\overline{\Omega})$ allow us to state the following theorem on stability of problem (11).

Theorem 5. The following stability estimates hold for the solutions of problem (11):

$$\max_{0 \leq s \leq mz} \left\| \frac{d^2 y(s,\cdot)}{ds^2} \right\|_{W_2^1(\overline{\Omega})}, \max_{0 \leq s \leq mz} \left\| \frac{dy(s,\cdot)}{ds} \right\|_{W_2^2(\overline{\Omega})}, \frac{1}{2} \max_{0 \leq s \leq mz} \|y(s,\cdot)\|_{W_2^3(\overline{\Omega})}$$

$$\leq M_4 \left[(2 + |c|\, z)^m b_0 + \sum_{j=1}^{m} (2 + |c|\, z)^{m-j} \int_{(j-1)z}^{jz} \left\| B^{\frac{1}{2}} h(r,\cdot) \right\|_{L_2(\overline{\Omega})} dr \right],$$

$$b_0 = \max \left\{ \max_{-z \leq s \leq 0} \left\| \frac{d^2 k(s,\cdot)}{ds^2} \right\|_{W_2^1(\overline{\Omega})}, \max_{-z \leq s \leq 0} \left\| \frac{dk(s,\cdot)}{ds} \right\|_{W_2^2(\overline{\Omega})}, \max_{-z \leq s \leq 0} \|k(s,\cdot)\|_{W_2^3(\overline{\Omega})} \right\},$$

where M_4 does not depend on $\varphi(u)$.

The proof of Theorem 5 is based on the stability estimates from Theorem 1, where $G = L_2(\overline{\Omega})$, as well as the symmetry property of the operator B^u defined in formula (11) together with the next theorem regarding the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 6. For the solution of the elliptic differential problem [20],

$$\begin{cases} B^u x(u) = \mu(u), & u \in \Omega, \\ \frac{\partial x(u)}{\partial \vec{m}} = 0, & u \in P, \end{cases}$$

the coercivity inequality that follows holds:

$$\sum_{t=1}^{m} \|x_{u_t u_t}\|_{L_2(\overline{\Omega})} \leq M_5 \|\mu\|_{L_2(\overline{\Omega})}.$$

Here, M_5 is independent of $\mu(u)$.

3 Conclusion

In this paper, we examine the IVP for a third-order PDE with TD in a Hilbert space. We establish a key theorem concerning the stability of this problem and demonstrate its applications. Additionally, some of the results discussed here, albeit without proofs, were previously published in [21].

Using this method, we can investigate the IVP for the nonlinear third order PDE with TD

$$\begin{cases} \frac{d^3y(s)}{ds^3} + B\frac{dy(s)}{ds} = h(s, y(s-z)), \ 0 < s < \infty, \\ y(s) = k(s), \ -z \le s \le 0 \end{cases}$$

in G, a Hilbert space, having SAPDO B, $B \geq \lambda I$, where $\lambda > 0$. Here $k(s)$ defined on $[-z, 0]$ is the given ACF with values in $D(B)$.

Acknowledgments

We would like to thank the following institutions for their support. The publication has been prepared with the support of the "RUDN University Program 5-100". This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP05131225).

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Amirov, S., & Kozhanov, A.I. (2013). Mixed boundary value problem for a class of strongly nonlinear Sobolev-type equations of higher order. *Doklady Mathematics, 88*(5), 446–448. https://doi.org/10.1134/S1064562413040236

2 Ashyralyev, A., & Belakroum, K. (2022). A stable difference scheme for a third-order partial differential equation. *Journal of Mathematical Sciences, 260*(4), 399–417. https://doi.org/10.1007/s10958-022-05702-5

3 Niu, J., & Li, P. (2014). Numerical algorithm for the third-order partial differential equation with three-point boundary value problem. *Abstract and Applied Analysis, 2014*(40), 1–7. https://doi.org/10.1155/2014/630671

4 Afuwape, A.U., & Omeike, M.O. (2010). Stability and boundedness of solutions of a kind of third-order delay differential equations. *Journal of Computational and Applied Mathematics, 29*(3), 329–342. https://doi.org/10.1590/S1807-03022010000300001

5 Al Themairi, A., Qaraad, B., Bazighifan, O., & Nonlaopon, K. (2022). Third-order neutral differential equations with damping and distributed delay: new asymptotic properties of solutions. *Symmetry, 14*(10), 1–13. https://doi.org/10.3390/sym14102192

6 Ashyralyev, A., & Ibrahim, S. (2023). A numerical algorithm for the third-order delay partial differential equation with involution and Neumann boundary condition. *AIP Conference Proceedings, 2781*(1), 1–6. https://doi.org/10.1063/5.0145923

7 Baculíková, B., Džurina, J., & Rogovchenko, Y.V. (2012). Oscillation of third-order trinomial delay differential equations. *Applied Mathematics and Computation, 218*(13), 7023–7033. https://doi.org/10.1016/j.amc.2011.12.049

8 Cahlon, B., & Schmidt, D. (2006). Stability criteria for certain third-order delay differential equations. *Journal of Computational and Applied Mathematics, 188*(2), 319–335. https://doi.org/10.1016/j.cam.2005.04.034

9 Grace, S.R. (2015). Oscillation criteria for a third-order nonlinear delay differential equations with time delay. *Opuscula Mathematica, 35*(4), 485–497. http://dx.doi.org/10.7494/OpMath.2015.35.4.485

10 Pikina, P.A. (2017). Predictive time optimal algorithm for a third-order dynamical system with delay. *Journal of Physics: Conference Series, 891*(1), 9–11. https://dx.doi.org/10.1088/1742-6596/891/1/012278

11 Xiang, H. (2016). Oscillation of the third-order nonlinear neutral differential equations with distributed time delay. *Italian Journal of Pure and Applied Mathematics, 36*(2016), 769–782.

12 Agirseven, D. (2012). Approximate solutions of delay parabolic equations with the Dirichlet condition. *Abstract and Applied Analysis, 2012*, 682752. https://doi.org/10.1155/2012/ 682752

13 Ardito, A., & Ricciardi, P. (1980). Existence and regularity for linear delay partial differential equations. *Nonlinear Analysis, 4*(2), 411–414. https://doi.org/10.1016/0362-546X(80)90064-4

14 Ashyralyev, A., & Agirseven, D. (2013). On convergence of difference schemes for delay parabolic equations. *Computers and Mathematics with Applications, 66*(7), 1232–1244. https://doi.org/10.1016/j.camwa.2013.07.018

15 Ashyralyev, A., & Agirseven, D. (2014). Well-posedness of delay parabolic difference equations. *Advances in Difference Equations, 2014*(18), 1–20. https://doi.org/10.1186/1687-1847-2014-18

16 Ashyralyev, A., & Sobolevskii, P.E. (2004). *New difference schemes for partial differential equations. Operator theory advances and applications.* Basel, Boston, Berlin: Birkhäuser Verlag Press.

17 Blasio, G.D. (2003). Delay differential equations with unbounded operators acting on delay terms. *Nonlinear Analysis, 52*(1), 1–18. https://doi.org/10.1016/S0362-546X(01)00868-9

18 Sinestrari, E. (1984). On a class of retarded partial differential equations. *Mathematische Zeitschrift, 186*(1), 223–246. https://doi.org/10.1007/BF01161806

19 Fattorini, H.O. (1985). Second-order linear differential equations in Banach spaces. Amsterdam: Elsevier Science Publishers B.V.

20 Sobolevskii, P.E., & Ashyralyev, A. (1975). *Difference methods for the approximate solution of differential equations.* Voronezh, Russia: Voronezh State University Press.

21 Ashyralyev, A., Hincal, E., & Ibrahim, S. (2018). Stability of the third-order partial differential equations with time delay. *AIP Conference Proceedings, 1997*(1), 020086. https://doi.org/10.1063/1.5049080

Author Information[*]

Allaberen Ashyralyev — Doctor of mathematical sciences, Professor, Lecturer, Department of Mathematics, Bahcesehir University, Istanbul, 34353, Turkey; Department of Mathematics, Peoples' Friendship University of Russia (RUDN University), Moscow, 117198, Russian Federation; Institute of Mathematics and Mathematical Modeling, Almaty, 050010, Kazakhstan; e-mail: *allaberen.ashyralyev@eng.bau.edu.tr*; https://orcid.org/0000-0002-4153-6624

Suleiman Ibrahim (*corresponding author*) — Doctor of mathematical sciences, Assistant Professor, Lecturer, Department of Mathematics, Faculty of Arts and Sciences, Near East University, Nicosia, TRNC, Mersin 10, Turkey; e-mail: *ibrahim.suleiman@neu.edu.tr*; https://orcid.org/0009-0009-8913-7217

Evren Hincal — Doctor of mathematical sciences, Professor, Head of the Department of Mathematics, Faculty of Arts and Sciences, Near East University, Nicosia, TRNC, Mersin 10, Turkey; e-mail: *evren.hincal@neu.edu.tr*; https://orcid.org/0000-0001-6175-1455

[*]The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2025M1/34-45 Research article

Numerical-analytical method for solving initial-boundary value problem for loaded parabolic equation

A.T. Assanova[1,2], Zh.M. Kadirbayeva[1,2,*], S.K. Kuanysh[3]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Kazakh National Women's Teacher Training University, Almaty, Kazakhstan;*
[3] *Al-Farabi Kazakh National University Almaty, Kazakhstan*
(E-mail: assanova@math.kz, zhkadirbayeva@gmail.com, saule270898@gmail.com)

An initial-boundary value problem for a loaded parabolic equation in a rectangular domain was considered. By discretization with respect to a spatial variable, the problem under study is reduced to the initial problem for a system of loaded ordinary differential equations. Based on the previously obtained results of Dzhumabaev and Assanova, an estimate for the solution of the original initial-boundary value problem for a loaded parabolic equation was established. An auxiliary initial problem for a system of loaded ordinary differential equations is solved by the Dzhumabaev parameterization method. Conditions of the unique solvability of the considering problem are obtained and algorithms for finding a solution are constructed. The results are illustrated with a numerical example.

Keywords: loaded parabolic equations, initial-boundary value problem, solvability conditions, parameterization method, polygonal method, numerical solution.

2020 Mathematics Subject Classification: 35K20, 34K10, 35R10, 65M22.

Introduction

The parabolic partial differential equations play a very important role in many branches of science and engineering. Applied problems involving boundary value problems for parabolic equations include: thermal analysis in engineering, groundwater flow, climate modeling, biological processes, chemical reactions, environmental engineering, material science and financial engineering [1–5].

Loaded parabolic equations are a type of parabolic partial differential equations that include additional terms or conditions representing external influences or interactions, which can vary over time and space. These "loads" can be functions or integrals that are added to the standard parabolic equation. Loaded parabolic equations often arise in various physical and engineering applications where external sources, sinks, or other dynamic interactions need to be modeled [6]. These equations are used to model more complex systems where simple diffusion or heat conduction is modified by additional processes such as external forces, reaction terms, or other dynamic effects. For information on various boundary value problems for loaded parabolic differential equations, refer to works [7–11].

A boundary value problem for a linear parabolic equation without loading was considered in works [12, 13]. Using the polygonal method, estimates of the solution and their derivatives were obtained in terms of the equation coefficients and boundary conditions [12]. Coefficient estimates of solutions and the first derivative with respect to x of a linear boundary value problem for a parabolic equation with one spatial variable were obtained [13].

*Corresponding author. *E-mail: zhkadirbayeva@gmail.com*

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19675193).

Received: 25 July 2024; *Accepted:* 12 November 2024.

The numerical research of boundary value problems for a parabolic equations with and without loading are of great interest due to their broad range of applications. Several methods for solving these problems have been developed [14–18].

This work is aimed at development of methods for solving the initial-boundary value problem for parabolic equation, proposed in [12, 13] to the initial-boundary value problem for a loaded parabolic equation of the following form

$$\frac{\partial u}{\partial t} = p(t,x)\frac{\partial^2 u}{\partial x^2} + q(t,x)u(t,x) + \sum_{j=1}^{m+1} k_j(t,x)u(\xi_j,x) + f(t,x), \quad (t,x) \in \Omega = (0,T) \times (0,\omega), \tag{1}$$

$$u(0,x) = \varphi(x), \qquad x \in [0,\omega], \tag{2}$$

$$u(t,0) = \psi_0(t), \qquad u(t,\omega) = \psi_1(t), \qquad t \in [0,T], \tag{3}$$

where functions $p(t,x) > 0$, $q(t,x) \le 0$, $m \in \mathbb{N}$, $k_j(t,x)$, $j = \overline{1,m+1}$, $f(t,x)$ are continuous with respect to t and Holder continuous with respect to x; functions $\varphi(x)$, $\psi_0(t)$, $\psi_1(t)$ are sufficiently smooth and $\psi_0(0) = \varphi(0)$, $\psi_1(0) = \varphi(\omega)$ matching conditions are performed.

A solution to problem (1)–(3) is a function $u(t,x)$, continuous on $\bar{\Omega} = [0,T] \times [0,\omega]$ that has continuous first-order partial derivatives with respect to t and continuous second-order partial derivatives with respect to x. It satisfies the loaded differential equation (1) and boundary conditions (2), (3).

By discretizing with respect to the spatial variable x, problem (1)–(3) transforms to a problem for systems of loaded ordinary differential equations. An auxiliary problem for a system of loaded ordinary differential equations will be investigated. Based on the properties of solutions to the auxiliary problem, estimates for the solution of the original initial-boundary value problem for the loaded parabolic equation will be established. In this case, the approach proposed in works [12, 13] will be used. A numerical method for solving the initial problem for systems of loaded ordinary differential equations is also proposed. A numerical implementation of the algorithm for the initial-boundary value problem for a loaded parabolic equation is presented. The error between the exact solution of the problem under consideration and its numerical discrete solution has been established.

1 *Problem formulation*

We take $\forall\tau$ and produce a discretization by variable x: $x_i = i\tau$, $i = \overline{0,N}$, $N\tau = \omega$.

We present the following notations: $u_i(t) = u(t,i\tau)$, $p_i(t) = p(t,i\tau)$, $q_i(t) = q(t,i\tau)$, $k_j^i(t) = k_j(t,i\tau)$, $j = \overline{1,m}$, $f_i(t) = f(t,i\tau)$, $\varphi_i = \varphi(i\tau)$, $i = \overline{0,N}$.

Using these notations, the problem (1)–(3) is transformed into the following problem

$$\frac{du_i}{dt} = p_i(t)\frac{u_{i+1} - 2u_i + u_{i-1}}{\tau^2} + q_i(t)u_i + \sum_{j=1}^{m+1} k_j^i(t)u_i(\xi_j) + f_i(t), \quad i = \overline{1,N-1}, \tag{4}$$

$$u_i(0) = \varphi_i, \qquad i = \overline{1,N-1}, \tag{5}$$

$$u_0(0) = \varphi_0, \qquad u_N(0) = \varphi_N, \qquad u_0(t) = \psi_0(t), \qquad u_N(t) = \psi_1(t), \qquad t \in [0,T]. \tag{6}$$

Here from relation (6) it is clear that functions $u_0(t)$ and $u_N(t)$ are known.

Due to the linearity of the system for $\forall\tau > 0$ there is a unique solution to problem (4)–(6) defined on $[0,T]$: $\{u_1(t), u_2(t), \ldots, u_{N-1}(t)\}$. Relating the functions u_{i+1}, u_{i-1} to the right side of each i-th equation of system (4), we apply the estimate from work [19]

$$\|u_i\| = \max_{t\in[0,T]}\{|u_i(t)|\} \le \max\left\{|\varphi_i|, \frac{1}{2}\left\|\frac{u_{i-1}(t)}{1+\frac{|q_i(t)|\tau^2}{2p_i(t)}}\right\| + \frac{1}{2}\left\|\frac{u_{i+1}(t)}{1+\frac{|q_i(t)|\tau^2}{2p_i(t)}}\right\|\right.$$
$$\left. + \frac{1}{2}\left\|\frac{f_i(t)}{p_i(t)\left[1+\frac{|q_i(t)|\tau^2}{2p_i(t)}\right]}\right\|\tau^2 + \frac{1}{2}\left\|\frac{\sum\limits_{j=1}^{m+1} k_j^i(t)u_i(\xi_j)}{p_i(t)\left[1+\frac{|q_i(t)|\tau^2}{2p_i(t)}\right]}\right\|\tau^2\right\}$$
$$\le \max\left\{|\varphi_i|, \frac{1}{2}\|u_{i-1}\| + \frac{1}{2}\|u_{i+1}\| + \frac{1}{2}\max_{t\in[0,T]}\left\|\frac{f_i(t)}{p_i(t)}\right\|\tau^2 + \frac{1}{2}\max_{t\in[0,T]}\left\|\frac{\sum\limits_{j=1}^{m+1} k_j^i(t)u_i(\xi_j)}{p_i(t)}\right\|\tau^2\right\}.$$

From here it is easy to obtain the following inequality

$$\eta_i \le \frac{1}{2}\eta_{i-1} + \frac{1}{2}\eta_{i+1} + \frac{1}{2}\max_{t\in[0,T]}\left\|\frac{f_i(t)}{p_i(t)}\right\|\tau^2 + \frac{1}{2}\max_{t\in[0,T]}\sum_{j=1}^{m+1}\left\|\frac{k_j^i(t)}{p_i(t)}\right\|\cdot\|u_i\|\,\tau^2, \quad i=\overline{1,N-1}, \tag{7}$$

where $\eta_i = \max\{\hat{\varphi}, \|u_i\|\}$, $\hat{\varphi} = \max\limits_{i=\overline{1,N}}\{|\varphi_i|\}$.

Similarly to [12, 13], using the sweep down and up from (7), we get

$$\eta_i \le \frac{N-i}{N}\eta_0 + \frac{i}{N}\eta_N + \frac{N-i}{N}\sum_{l=1}^{i}\max_{t\in[0,T]}\left\|l\frac{f_l(t)}{p_l(t)}\right\|\tau^2 + \frac{i}{N}\sum_{l=i+1}^{N-1}\max_{t\in[0,T]}\left\|(N-l)\frac{f_l(t)}{p_l(t)}\right\|\tau^2$$
$$+ \frac{N-i}{N}\sum_{l=1}^{i}\sum_{j=1}^{m+1}\max_{t\in[0,T]}\left\|l\frac{k_j^l(t)}{p_l(t)}\right\|\tau^2\cdot\eta_i + \frac{i}{N}\sum_{l=i+1}^{N-1}\sum_{j=1}^{m+1}\max_{t\in[0,T]}\left\|(N-l)\frac{k_j^l(t)}{p_l(t)}\right\|\tau^2\cdot\eta_i.$$

Considering that $\|u_i\| \le \eta_i$, we set

$$\|u_i\| \le \frac{N-i}{N}\max\{\hat{\varphi}, \|\psi_0\|\} + \frac{i}{N}\max\{\hat{\varphi}, \|\psi_1\|\} + \frac{N-i}{N}\sum_{l=1}^{i}\max_{t\in[0,T]}\left\|l\frac{f_l(t)}{p_l(t)}\right\|\tau^2$$
$$+ \frac{i}{N}\sum_{l=i+1}^{N-1}\max_{t\in[0,T]}\left\|(N-l)\frac{f_l(t)}{p_l(t)}\right\|\tau^2 + \Theta_i\|u_i\|, \tag{8}$$

where $\Theta_i = \frac{N-i}{N}\sum\limits_{l=1}^{i}\sum\limits_{j=1}^{m+1}\max\limits_{t\in[0,T]}\left\|l\frac{k_j^l(t)}{p_l(t)}\right\|\tau^2 + \frac{i}{N}\sum\limits_{l=i+1}^{N-1}\sum\limits_{j=1}^{m+1}\max\limits_{t\in[0,T]}\left\|(N-l)\frac{k_j^l(t)}{p_l(t)}\right\|\tau^2 < 1$.

Then

$$\|u_i\| \le \frac{1}{1-\Theta_i}\left\{\frac{N-i}{N}\max\{\varphi, \|\psi_0\|\} + \frac{i}{N}\max\{\varphi, \|\psi_1\|\}\right.$$
$$\left. + \frac{N-i}{N}\sum_{l=1}^{i}\max_{t\in[0,T]}\left\|l\frac{f_l(t)}{p_l(t)}\right\|\tau^2 + \frac{i}{N}\sum_{l=i+1}^{N-1}\max_{t\in[0,T]}\left\|(N-l)\frac{f_l(t)}{p_l(t)}\right\|\tau^2\right\}.$$

From inequality (8), reasoning in the same way as in [13] and [19], we obtain the validity of the following statement:

Theorem 1. Let

a) the assumptions with respect to the data of problem (1)–(3) be fulfilled;

b) the inequality $\Theta(x) = \frac{\omega-x}{\omega}\int\limits_0^x z\sum\limits_{j=1}^{m+1}\max\limits_{t\in[0,T]}\left\|\frac{k_j(t,z)}{p(t,z)}\right\|dz + \frac{x}{\omega}\int\limits_x^\omega(\omega-z)\sum\limits_{j=1}^{m+1}\max\limits_{t\in[0,T]}\left\|\frac{k_j(t,z)}{p(t,z)}\right\|dz < 1$ be valid for all $x\in[0,\omega]$.

Then the problem (1)–(3) has a unique classical solution $u^*(t,x)$, and for it the following inequality holds:

$$\max_{t\in[0,T]}|u^*(t,x)| \le \frac{\omega-x}{\omega[1-\Theta(x)]}\max\Big\{\max_{x\in[0,\omega]}|\varphi(x)|, \max_{t\in[0,T]}|\psi_0(t)|\Big\}$$
$$+\frac{x}{\omega[1-\Theta(x)]}\max\Big\{\max_{x\in[0,\omega]}|\varphi(x)|, \max_{t\in[0,T]}|\psi_1(t)|\Big\}$$
$$+\frac{\omega-x}{\omega[1-\Theta(x)]}\int\limits_0^x z\max_{t\in[0,T]}\left\|\frac{f(t,z)}{p(t,z)}\right\|dz + \frac{x}{\omega[1-\Theta(x)]}\int\limits_x^\omega(\omega-z)\max_{t\in[0,T]}\left\|\frac{f(t,z)}{p(t,z)}\right\|dz.$$

The proof of Theorem 1, with minor modifications, follows the same principles as the proof of Theorem in [13].

Thus, we have established an estimate for the solution of the original initial-boundary value problem of the loaded parabolic equation (1)–(3).

Substituting $u_0(t)$ and $u_N(t)$ into the system of loaded equations (4), the discretized problem (4)–(6) can be written in the following matrix-vector form

$$\frac{d\tilde u}{dt} = \mathcal{A}(t)\tilde u + \sum_{j=1}^{m+1}\mathcal{K}_j(t)\tilde u(\xi_j) + \mathcal{F}(t), \quad \tilde u\in\mathbb{R}^{N-1}, \quad t\in[0,T), \tag{9}$$

$$\tilde u(0) = \Phi, \qquad \Phi\in\mathbb{R}^{N-1}, \tag{10}$$

where $\tilde u(t) = \big(u_1(t), u_2(t), \dots, u_{N-1}(t)\big)$ is unknown function, the $(N-1)\times(N-1)$ matrices $\mathcal{A}(t)$, $\mathcal{K}_j(t)$, $j=\overline{1,m+1}$, and $(N-1)$ vector-function $\mathcal{F}(t)$ are continuous on $[0,T]$, $0=\xi_0<\xi_1<\ldots<\xi_m<\xi_{m+1}=T$. Here

$$\mathcal{A}(t) = \begin{bmatrix} -\frac{2p_1(t)}{\tau^2}+q_1(t) & \frac{p_1(t)}{\tau^2} & 0 & \dots & 0 \\ \frac{p_2(t)}{\tau^2} & -\frac{2p_2(t)}{\tau^2}+q_2(t) & \frac{p_2(t)}{\tau^2} & \dots & 0 \\ 0 & \frac{p_3(t)}{\tau^2} & -\frac{2p_3(t)}{\tau^2}+q_3(t) & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & -\frac{2p_{N-1}(t)}{\tau^2}+q_{N-1}(t) \end{bmatrix},$$

$$\mathcal{K}_j(t) = \begin{bmatrix} k_j^1(t) & 0 & 0 & \dots & 0 \\ 0 & k_j^2(t) & 0 & \dots & 0 \\ 0 & 0 & k_j^3(t) & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & k_j^{N-1}(t) \end{bmatrix}, \quad j=\overline{1,m+1},$$

$$\mathcal{F}(t) = \begin{bmatrix} \frac{p_1(t)}{\tau^2}\psi_0(t)+f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ \frac{p_{N-1}(t)}{\tau^2}\psi_1(t)+f_{N-1}(t) \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_{N-1} \end{bmatrix}.$$

A solution to problem (9), (10) is a vector function $\tilde{u}(t)$, which is continuous on $[0,T]$ and continuously differentiable on $(0,T)$. This function satisfies the loaded differential equation (9) and the condition (10).

2 *Solving problem* (9), (10) *by using the parameterization method*

We will use the approach proposed in [20–24] to solve the initial value problem for loaded differential equations (9), (10). This approach relies on the algorithms of the Dzhumabaev parametrization method [19,25] and numerical methods. The implementation and efficiency of this method for finding analytical and numerical solutions to boundary value problems for various differential equations are shown in [26–32].

The interval $[0,T]$ is partitioned into subintervals by loading points: $[0,T) = \bigcup\limits_{s=1}^{m+1}[\xi_{s-1},\xi_s)$.

Define the space $C([0,T],\xi_s,\mathbb{R}^{(N-1)(m+1)})$ consisting of system functions $\tilde{u}[t] = (\tilde{u}_1(t),\tilde{u}_2(t),\ldots,\tilde{u}_{m+1}(t))$, where each $\tilde{u}_s : [\xi_{s-1},\xi_s) \to \mathbb{R}^{N-1}$ is continuous on $[\xi_{s-1},\xi_s)$ and has finite left-sided limits $\lim\limits_{t\to\xi_s-0}\tilde{u}_s(t)$ for all $s=\overline{1,m+1}$. The norm on this space is defined as $||\tilde{u}[\cdot]||_2 = \max\limits_{s=\overline{1,m+1}}\sup\limits_{t\in[\xi_{s-1},\xi_s)}||\tilde{u}_s(t)||$.

The restriction of the function $\tilde{u}(t)$ to the interval $[\xi_{s-1},\xi_s)$ is denoted by $\tilde{u}_s(t)$, meaning $\tilde{u}_s(t) = \tilde{u}(t)$ for $t\in[\xi_{s-1},\xi_s)$, $s=\overline{1,m+1}$. We introduce additional parameters $\mu_s = \tilde{u}_{s+1}(\xi_s)$, $s=\overline{1,m}$, $\mu_{m+1} = \tilde{u}(\xi_{m+1})$. By making the substitution $\tilde{u}_1(t) = v_1(t)+\Phi$ on $[\xi_0,\xi_1)$ and $\tilde{u}_s(t) = v_s(t)+\mu_{s-1}$ on each interval $[\xi_{s-1},\xi_s)$, $s=\overline{2,m+1}$, we obtain multi-point initial value problem with parameters

$$\frac{dv_1}{dt} = \mathcal{A}(t)(v_1+\Phi) + \sum_{j=1}^{m+1}\mathcal{K}_j(t)\mu_j + \mathcal{F}(t), \quad t\in[\xi_0,\xi_1), \tag{11}$$

$$\frac{dv_s}{dt} = \mathcal{A}(t)(v_s+\mu_{s-1}) + \sum_{j=1}^{m+1}\mathcal{K}_j(t)\mu_j + \mathcal{F}(t), \quad t\in[\xi_{s-1},\xi_s), \quad s=\overline{2,m+1}, \tag{12}$$

$$v_s(\xi_{s-1}) = 0, \qquad s=\overline{1,m+1}, \tag{13}$$

$$\Phi + \lim_{t\to\xi_1-0} v_1(t) = \mu_1, \tag{14}$$

$$\mu_{s-1} + \lim_{t\to\xi_s-0} v_s(t) = \mu_s, \qquad s=\overline{2,m+1}. \tag{15}$$

A pair $(v^*[t],\mu^*)$, where the elements are $v^*[t] = \big(v_1^*(t),v_2^*(t),\ldots,v_{m+1}^*(t)\big)\in C([0,T],\xi_s,R^{(N-1)(m+1)})$, $\mu^* = (\mu_1^*,\mu_2^*,\ldots,\mu_{m+1}^*)\in R^{(N-1)(m+1)}$, is said to be a solution to problem (11)–(15) if the functions $v_s^*(t)$, $s=\overline{1,m+1}$, are continuously differentiable on $[\xi_{s-1},\xi_s)$ and satisfy (11), (12) and additional conditions (14), (15) with $\mu_j=\mu_j^*$, $j=\overline{1,m+1}$, and initial conditions (13).

Problems (9), (10) and (11)–(15) are equivalent. If the $\tilde{u}^*(t)$ is a solution of problem (9), (10), then the pair $(v^*[t],\mu^*)$, where $v^*[t] = \Big(\tilde{u}^*(t)-\Phi,\tilde{u}^*(t)-\tilde{u}^*(\xi_1),\ldots,\tilde{u}^*(t)-\tilde{u}^*(\xi_m)\Big)$, and $\mu^* = \Big(\tilde{u}^*(\xi_1),\tilde{u}^*(\xi_2),\ldots,\tilde{u}^*(\xi_m),\tilde{u}^*(\xi_{m+1})\Big)$, is a solution to problem (11)–(15). Conversely, if the pair $(\widetilde{v}[t],\widetilde{\mu})$ with elements $\widetilde{v}[t] = \big(\widetilde{v}_1(t),\widetilde{v}_2(t),\ldots,\widetilde{v}_{m+1}(t)\big)\in C([0,T],\xi_s,R^{(N-1)(m+1)})$, $\widetilde{\mu} = (\widetilde{\mu}_1,\widetilde{\mu}_2,\ldots,\widetilde{\mu}_{m+1})\in R^{(N-1)(m+1)}$ is a solution to problem (11)–(15), then the function $\widetilde{\tilde{u}}(t)$ defined by the equalities $\widetilde{\tilde{u}}(t) = \widetilde{v}_1(t)+\Phi$, $t\in[\xi_0,\xi_1)$, $\widetilde{\tilde{u}}(t) = \widetilde{v}_s(t)+\widetilde{\mu}_{s-1}$, $t\in[\xi_{s-1},\xi_s)$, $s=\overline{2,m+1}$, and $\widetilde{\tilde{u}}(T) = \widetilde{\mu}_{m+1}$, will be the solution of the original problem (9), (10).

By employing the fundamental matrix $\mathcal{X}_s(t)$ of differential equation $\frac{d\tilde{u}}{dt} = \mathcal{A}(t)\tilde{u}$ on $[\xi_{s-1}, \xi_s]$, $s = \overline{1, m+1}$, we transform the solution of an initial value problem for a differential equations with parameters (11)–(13) into an equivalent system of integral equations:

$$\begin{gathered} v_1(t) = \mathcal{X}_1(t) \int\limits_{\xi_0}^{t} \mathcal{X}_1^{-1}(\eta)\mathcal{A}(\eta)d\eta \cdot \Phi + \mathcal{X}_1(t) \int\limits_{\xi_0}^{t} \mathcal{X}_1^{-1}(\eta) \sum_{j=1}^{m+1} \mathcal{K}_j(\eta)d\eta\mu_j \\ + \mathcal{X}_1(t) \int\limits_{\xi_0}^{t} \mathcal{X}_1^{-1}(\eta)\mathcal{F}(\eta)d\eta, \quad t \in [\xi_0, \xi_1), \end{gathered} \tag{16}$$

$$\begin{gathered} v_s(t) = \mathcal{X}_s(t) \int\limits_{\xi_{s-1}}^{t} \mathcal{X}_s^{-1}(\eta)\mathcal{A}(\eta)d\eta \cdot \mu_{s-1} + \mathcal{X}_s(t) \int\limits_{\xi_{s-1}}^{t} \mathcal{X}_s^{-1}(\eta) \sum_{j=1}^{m+1} \mathcal{K}_j(\eta)d\eta\mu_j \\ + \mathcal{X}_s(t) \int\limits_{\xi_{s-1}}^{t} \mathcal{X}_s^{-1}(\eta)\mathcal{F}(\eta)d\eta, \quad t \in [\xi_{s-1}, \xi_s), \quad s = \overline{2, m+1}. \end{gathered} \tag{17}$$

By substituting the respective expressions from (16), (17) into the conditions (14) and (15), we get a system of linear algebraic equations with respect to the parameters μ_s, $s = \overline{1, m+1}$:

$$\begin{gathered} \mathcal{X}_1(\xi_1) \int\limits_{\xi_0}^{\xi_1} \mathcal{X}_1^{-1}(\eta) \sum_{j=1}^{m+1} \mathcal{K}_j(\eta)d\eta\mu_j - \mu_1 \\ = -\Phi - \mathcal{X}_1(\xi_1) \int\limits_{\xi_0}^{\xi_1} \mathcal{X}_1^{-1}(\eta)\mathcal{A}(\eta)d\eta \cdot \Phi - \mathcal{X}_1(\xi_1) \int\limits_{\xi_0}^{\xi_1} \mathcal{X}_1^{-1}(\eta)\mathcal{F}(\eta)d\eta, \end{gathered} \tag{18}$$

$$\begin{gathered} \mu_{s-1} + \mathcal{X}_s(\xi_s) \int\limits_{\xi_{s-1}}^{\xi_s} \mathcal{X}_s^{-1}(\eta)\mathcal{A}(\eta)d\eta \cdot \mu_{s-1} - \mu_s + \mathcal{X}_s(\xi_s) \int\limits_{\xi_{s-1}}^{\xi_s} \mathcal{X}_s^{-1}(\eta) \sum_{j=1}^{m+1} \mathcal{K}_j(\eta)d\eta\mu_j \\ = -\mathcal{X}_s(\xi_s) \int\limits_{\xi_{s-1}}^{\xi_s} \mathcal{X}_s^{-1}(\eta)\mathcal{F}(\eta)d\eta, \quad t \in [\xi_{s-1}, \xi_s), \quad s = \overline{2, m+1}. \end{gathered} \tag{19}$$

Unknown parameters μ_s, $s = \overline{1, m+1}$ can be found using the system (18), (19). Using $O \in \mathbb{R}^{N-1,N-1}$ zero matrix, $I \in \mathbb{R}^{N-1,N-1}$ identity matrix and

$$y_s(B, t) = \mathcal{X}_s(t) \int\limits_{\xi_{s-1}}^{t} \mathcal{X}_s^{-1}(\eta)B(\eta)d\eta, \quad s = \overline{1, m}$$

notations, we write system (18), (19) in the following form

$$Q_*(\xi)\mu = F_*(\xi), \quad \mu \in \mathbb{R}^{(N-1)(m+1)}, \tag{20}$$

where

$$Q_*(\xi) = \begin{bmatrix} y_1(\mathcal{K}_1, \xi_1) - I & y_1(\mathcal{K}_2, \xi_1) & y_1(\mathcal{K}_3, \xi_1) & \dots & y_1(\mathcal{K}_{m+1}, \xi_1) \\ I + y_2(\mathcal{A}, \xi_2) + y_2(\mathcal{K}_1, \xi_2) & y_2(\mathcal{K}_2, \xi_2) - I & y_2(\mathcal{K}_3, \xi_2) & \dots & y_2(\mathcal{K}_{m+1}, \xi_2) \\ y_3(\mathcal{K}_1, \xi_3) & I + y_3(\mathcal{A}, \xi_3) + y_3(\mathcal{K}_2, \xi_3) & y_3(\mathcal{K}_3, \xi_3) - I & \dots & y_3(\mathcal{K}_{m+1}, \xi_3) \\ \dots & \dots & \dots & \ddots & \dots \\ y_{m+1}(\mathcal{K}_1, \xi_{m+1}) & y_{m+1}(\mathcal{K}_2, \xi_{m+1}) & y_{m+1}(\mathcal{K}_3, \xi_{m+1}) & \dots & y_{m+1}(\mathcal{K}_{m+1}, \xi_{m+1}) - I \end{bmatrix},$$

$$F_*(\xi) = \Big(-\Phi - y_1(\mathcal{A},\xi_1)\Phi - y_1(\mathcal{F},\xi_1), -y_2(\mathcal{F},\xi_2), \ldots, -y_m(\mathcal{F},\xi_m), -y_{m+1}(\mathcal{F},\xi_{m+1})\Big)'.$$

It can be readily shown that solving the boundary value problem (9), (10) is equivalent to solving the system (20).

Theorem 2. Let the matrix $Q_*(\xi) : \mathbb{R}^{(N-1)(m+1)} \to \mathbb{R}^{(N-1)(m+1)}$ be invertible. Then, for any $\mathcal{F}(t)$ and $\Phi \in \mathbb{R}^{(N-1)}$, the problem (9), (10) has a unique solution $\tilde{u}^\star(t)$ and this solution satisfies the estimate

$$\|\tilde{u}^\star\|_1 \le M \max(\|\Phi\|, \|\mathcal{F}\|_1),$$

$$M = e^{\alpha\bar{\xi}}\bar{\xi}\Big\{\alpha \max\Big(1, \gamma(\xi)\Big[1 + e^{\alpha\bar{\xi}}\bar{\xi}\alpha + e^{\alpha\bar{\xi}}\bar{\xi}\Big]\Big) + \Big(\sum_{j=1}^{m+1}\beta_j\gamma(\xi) + 1\Big)\Big[1 + e^{\alpha\bar{\xi}}\bar{\xi}\alpha + e^{\alpha\bar{\xi}}\bar{\xi}\Big]\Big\}$$
$$+ \gamma(\xi)\Big[1 + e^{\alpha\bar{\xi}}\bar{\xi}\alpha + e^{\alpha\bar{\xi}}\bar{\xi}\Big],$$

where $\gamma(\xi) = \|[Q_*(\xi)]^{-1}\|$, $\alpha = \max\limits_{t\in[0,T]}\|\mathcal{A}(t)\|$, $\beta_j = \max\limits_{t\in[0,T]}\|\mathcal{K}_j(t)\|$, $j = \overline{1,m+1}$, $\bar{\xi} = \max\limits_{s=\overline{1,m+1}}(\xi_s - \xi_{s-1})$, $\|\tilde{u}^\star\|_1 = \max\limits_{t\in[0,T]}\|\tilde{u}^\star(t)\|$.

The proof of Theorem 2, with minor modifications, follows the same principles as the proof of Theorem 1.1. in [32].

3 *Algorithm for numerical solving of problem* (9), (10) *and* (1)–(3)

The proposed numerical algorithm is based on the construction and solving of system (20). The coefficients and the right-hand side of this system (20) are found as solutions to Cauchy problems.

Algorithm for numerical solving of problem (9), (10):

1. Assume we have a partition: $0 = \xi_0 < \xi_1 < ... < \xi_m < \xi_{m+1} = T$. Divide every interval $[\xi_{s-1}, \xi_s]$, $s = \overline{1,m+1}$, with step $h_s = (\xi_s - \xi_{s-1})/l$, $l \in \mathbb{N}$, $s = \overline{1,m+1}$.

2. To determine the values of matrix $Q_*(\xi)$ and the vector $F_*(\xi)$ in system (20) we compute the values $y_s(\mathcal{A}, h_s)$, $y_s(\mathcal{K}_j, h_s)$, $j = \overline{1,m+1}$, $y_s(\mathcal{F}, h_s)$, $s = \overline{1,m+1}$, using the Runge Kutta RK4 Method with step size h_s in each subinterval.

3. Solve the system of linear algebraic equations

$$Q_*^{\tilde{h}}(\xi)\mu = F_*^{\tilde{h}}(\xi), \quad \mu \in R^{(N-1)(m+1)}, \tag{21}$$

here $\tilde{u}^{\tilde{h}_r}(\xi_s) = \mu_s^{\tilde{h}}$, $s = \overline{1,m+1}$.

4. To define the values of approximate solution at the remaining points, we solve the Cauchy problems

$$\frac{d\tilde{u}}{dt} = \mathcal{A}(t)\tilde{u} + \sum_{j=1}^{m+1}\mathcal{K}_j(t)\mu_j^{\tilde{h}} + \mathcal{F}(t), \quad \tilde{u}(0) = \Phi, \quad t \in [\xi_0, \xi_1], \tag{22}$$

$$\frac{d\tilde{u}}{dt} = \mathcal{A}(t)\tilde{u} + \sum_{j=1}^{m+1}\mathcal{K}_j(t)\mu_j^{\tilde{h}} + \mathcal{F}(t), \quad \tilde{u}(\xi_{s-1}) = \mu_{s-1}^{\tilde{h}}, \quad t \in [\xi_{s-1}, \xi_s), \quad s = \overline{2,m+1}. \tag{23}$$

Solving Cauchy problems (22), (23) also using the Runge Kutta RK4 Method, we obtain a numerical solution to linear initial-boundary value problem for loaded differential equations (9), (10).

If $\tilde{u}^\star(\xi_j^{\tilde{h}}) = \big(u_1^\star(\xi_j^{\tilde{h}}), u_2^\star(\xi_j^{\tilde{h}}), \ldots, u_{N-1}^\star(\xi_j^{\tilde{h}})\big)'$, $j = \overline{0,(m+1)l}$ is a numerical solution to linear initial value problem for loaded differential equations (9), (10), then $u^\star(\xi_j^{\tilde{h}}, s\tau) = u_s^\star(\xi_j^{\tilde{h}})$, $s = \overline{1,N-1}$ will be a numerical solution to linear initial value problem for loaded differential equation of parabolic type (1)–(3).

4 Example

To provide a clear overview of our investigation, we selected a specific test problem. Thus, the numerical method discussed in earlier sections were applied to the following initial-boundary value problem for a loaded parabolic equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= xt\frac{\partial^2 u}{\partial x^2} - (x+2t)u(t,x)\\ &+ xtu(0.02,x) + 2x^2tu(0.04,x) + (x+t)u(0.06,x) + 4xtu(0.08,x) + 5xt^2u(0.1,x)\\ &+ (4x^3+8tx^2-8tx)cos25\pi t - 100\pi x^2 sin25\pi t + \Big(2+\frac{7}{2}x - x^2\Big)t^2 + \frac{97}{50}x - \frac{3}{25}x^2\\ &+ \Big(8x^4 - \frac{404}{25}x^3 + \frac{31}{25}x^2 + \frac{27}{50}x - \frac{3}{50}\Big)t + 1, \quad (t,x)\in(0,0.1)\times(0,0.5),\end{aligned} \tag{24}$$

$$u(0,x) = 4x^2, \qquad x\in[0,0.5], \tag{25}$$

$$u(t,0) = t, \qquad u(t,\omega) = 2t + cos25\pi t, \qquad t\in[0,0.1]. \tag{26}$$

The analytical solution of the given problem (24)–(26) is $\hat{u}(t,x) = 2xt + 4x^2cos25\pi t + t$.

We take $\tau = 0.1$ and produce a discretization by x: $x_i = i\tau$, $i=\overline{0,5}$. Using this, the problem (24)–(26) is replaced by the following boundary value problem for loaded differential equation

$$\frac{d\tilde{u}}{dt} = \mathcal{A}(t)\tilde{u} + \sum_{j=1}^{5}\mathcal{K}_j(t)\tilde{u}(\xi_j) + \mathcal{F}(t), \quad t\in(0,0.1), \tag{27}$$

$$\tilde{u}(0) = \Phi, \quad \tilde{u}\in\mathbb{R}^4. \tag{28}$$

Here $\xi_1 = 0.02$, $\xi_2 = 0.04$, $\xi_3 = 0.06$, $\xi_4 = 0.08$, $\xi_5 = 0.1$,

$$\tilde{u} = \begin{pmatrix} u_1(t)\\ u_2(t)\\ u_3(t)\\ u_4(t)\end{pmatrix}, \mathcal{A}(t) = \begin{pmatrix} -22t-0.1 & 10t & 0 & 0\\ 20t & -42t-0.2 & 20t & 0\\ 0 & 30t & -62t-0.3 & 30t\\ 0 & 0 & 40t & -82t-0.4\end{pmatrix}, \Phi = \begin{pmatrix}0.04\\0.16\\0.36\\0.64\end{pmatrix},$$

$$\mathcal{K}_j(t) = \begin{pmatrix} j\cdot 0.1^j\cdot t & 0 & 0 & 0\\ 0 & j\cdot 0.2^j\cdot t & 0 & 0\\ 0 & 0 & j\cdot 0.3^j\cdot t & 0\\ 0 & 0 & 0 & j\cdot 0.4^j\cdot t\end{pmatrix}, \ j=\overline{1,2},$$

$$\mathcal{K}_3(t) = \begin{pmatrix} 0.1+t & 0 & 0 & 0\\ 0 & 0.2+t & 0 & 0\\ 0 & 0 & 0.3+t & 0\\ 0 & 0 & 0 & 0.4+t\end{pmatrix},$$

$$\mathcal{K}_i(t) = \begin{pmatrix} i\cdot 0.1\cdot t^{i-3} & 0 & 0 & 0\\ 0 & i\cdot 0.2\cdot t^{i-3} & 0 & 0\\ 0 & 0 & i\cdot 0.3\cdot t^{i-3} & 0\\ 0 & 0 & 0 & i\cdot 0.4\cdot t^{i-3}\end{pmatrix}, \ i=\overline{4,5},$$

$$\mathcal{F}(t) = \begin{pmatrix} \Big(\frac{1}{250}-\frac{18t}{25}\Big)cos25\pi t - \pi sin25\pi t + \frac{617}{50}t^2 - \frac{28t}{3125} + \frac{1491}{1250}\\ \Big(\frac{4}{125}-\frac{32t}{25}\Big)cos25\pi t - 4\pi sin25\pi t + \frac{133}{50}t^2 - \frac{59t}{3125} + \frac{1729}{1250}\\ \Big(\frac{27}{250}-\frac{42t}{25}\Big)cos25\pi t - 9\pi sin25\pi t + \frac{74}{25}t^2 - \frac{987t}{6250} + \frac{982}{625}\\ \Big(\frac{32}{125}+\frac{952t}{25}\Big)cos25\pi t - 16\pi sin25\pi t + \frac{2081}{25}t^2 - \frac{2969t}{6250} + \frac{1098}{625}\end{pmatrix}.$$

To solve linear initial value problem for loaded differential equations (27), (28) we will use the algorithm of Dzhumabaev parameterization method. According to the scheme of this method, the interval $[0,0.1)$ is partitioned into subintervals by loading points: $[0,0.1) = \bigcup_{s=1}^{5}[\xi_{s-1},\xi_s)$. Using the above proposed algorithm, we compose system (21) and by solving this system (21) we find the numerical values of the unknown parameters μ:

$$\mu_1 = \begin{pmatrix} 0.02399999 \\ 0.02799997 \\ 0.03199992 \\ 0.03599988 \end{pmatrix}, \ \mu_2 = \begin{pmatrix} 0.00799998 \\ -0.10400007 \\ -0.29600015 \\ -0.56800025 \end{pmatrix}, \ \mu_3 = \begin{pmatrix} 0.07199999 \\ 0.08399997 \\ 0.09599992 \\ 0.10799986 \end{pmatrix},$$

$$\mu_4 = \begin{pmatrix} 0.136 \\ 0.272 \\ 0.488 \\ 0.78400001 \end{pmatrix}, \ \mu_5 = \begin{pmatrix} 0.11999999 \\ 0.13999996 \\ 0.15999993 \\ 0.17999994 \end{pmatrix}.$$

We find the values of approximate solution at the remaining points of the subintervals of problem (27), (28) by solving Cauchy problems (22), (23). Then $u^\star(\widetilde{\xi_j^h}, s\tau) = u_s^\star(\widetilde{\xi_j^h})$, $s = \overline{1,4}$, $j = \overline{0,50}$, $\tau = 0.1$, will be a numerical solution to linear boundary value problem for loaded differential equation of parabolic type (24)–(26).

The graph of the exact solution $\hat{u}(t,x)$ and the found numerical solutions $u^\star(t,x)$ of the boundary value problem for a parabolic equation (24)–(26) are shown in Figure.

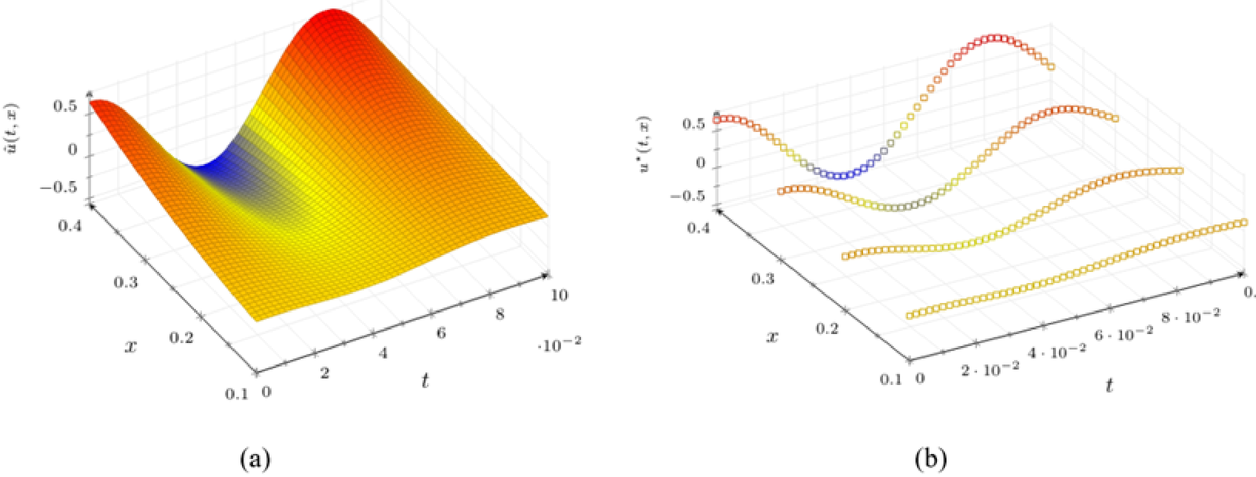

Figure. Exact (a) and numerical (b) solutions for example

For the difference of the corresponding values of the exact $\hat{u}(t,x)$ and constructed solutions $u^\star(t,x)$ of the boundary value problem for a parabolic equation (24)–(26) the following estimate is true:

$$\max_{j=\overline{0,50},i=\overline{1,4}} \|\hat{u}(\widetilde{\xi_j^h}, x_i) - u^\star(\widetilde{\xi_j^h}, x_i)\| < 0.0000003.$$

Conclusion

The initial-boundary value problem for a loaded parabolic equation in a rectangular domain is investigated. Using discretization with respect to the variable x, the problem under consideration is reduced to the initial problem for a system of loaded ordinary differential equations. Using the results of works [12, 13], an estimate for the solution of the initial-boundary value problem for a loaded parabolic equation is established. The parameterization method is used to solve the initial problem for a system of loaded ordinary differential equations. Algorithms for finding a solution to the problem under study are constructed and their convergence is shown. Conditions for the unique solvability of the initial problem for a system of loaded ordinary differential equations are established. Further, the proposed approach will be applied to solving boundary value problems for a loaded parabolic equation.

Acknowledgments

The authors thank the referees for his/her careful reading of the manuscript and useful suggestions.

Funding information

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19675193).

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

This work does not have any conflicts of interest.

References

1 Nakhushev, A.M. (1995). *Uravneniia matematicheskoi biologii [Equations of mathematical biology]*. Nalchik: Vysch.sh. [in Russian].

2 Friedman, A. (2008). *Partial Differential Equations of Parabolic Type.* Dover Publications.

3 Tian, Y., & Weng, P. (2009). Spreading speed and wavefronts for parabolic functional differential equations with spatio-temporal delays *Nonl. Anal.: Theory, Meth. Appl, 71*, 3374–3388.

4 Knabner, P., & Angermann, L. (2021). *Numerical Methods for Elliptic and Parabolic Partial Differential Equations.* Springer Cham.

5 Tikhonov, A.N. (2013). *Equations of Mathematical Physics.* Courier Corporation.

6 Nakhushev, A.M. (2012). *Nagruzhennye uravneniia i ikh primenenie [Loaded equations and their applications]*. Moscow: Nauka [in Russian].

7 Dzhenaliev, M.T., & Ramazanov, M.I. (2010). *Nagruzhennye uravneniia kak vozmushcheniia differentsialnykh uravnenii [Loaded equations as perturbations of differential equations]*. Almaty: Fylym [in Russian].

8 Jenaliyev, M.T., & Kassymbekova, A.S. (2018). On the boundary value problem for the loaded parabolic equations with irregular coefficients. *Bulletin of the Karaganda University. Mathematics Series,* 1(89), 20–27. https://doi.org/10.31489/2018m1/20-27

9 Jaruszewska-Walczak, D. (2012). Monotone iterative methods for infinite systems of parabolic functional differential equations. *Nonl. Anal.: Theory, Meth. Appl., 75*(10), 4051–4061. https://doi.org/10.1016/j.na.2012.02.021

10 Attaev, A.Kh., Ramazanov, M.I., & Omarov, M.T. (2022). On the correctness of boundary value problems for the two-dimensional loaded parabolic equation. *Bulletin of the Karaganda university. Mathematics Series,* 4(108), 34–41. https://doi.org/10.31489/2022m/34-41

11 Akhmanova, D.M., Shamatayeva, N.K., & Kasymova, L.Zh. (2020). On boundary value problems for essentially loaded parabolic equations in bounded domains. *Bulletin of the Karaganda University. Mathematics Series,* 2(98), 6–14. https://doi.org/10.31489/2020m2/6-14

12 Dzhumabaev, D.S. (1983). Obosnovanie metoda lomanykh dlia odnoi kraevoi zadachi lineinogo parabolicheskogo uravneniia [Substantiation of the poliline method for a single boundary value problem of a linear parabolic equation]. *Izvestiia AN KazSSR. Seriia fiziko-matematicheskaia — News of the Academy of Sciences of the Kazakh SSR. Physics and Mathematics Series, 1*, 8–11 [in Russian].

13 Assanova, A.T., & Dzhumabaev, D.S. (2000). Ob otsenkakh reshenii i ikh proizvodnykh kraevoi zadachi dlia parabolicheskogo uravneniia [On estimates of solutions and their derivatives of the boundary value problem for the parabolic equation]. *Izvestiia MON RK, NAN RK. Seriia fiziko-matematicheskaia — News of the Ministry of Education and Science of the Republic of Kazakhstan, National Academy of Sciences of the Republic of Kazakhstan. Physics and Mathematics Series, 5*, 3–8 [in Russian].

14 Ashyralyev, A., Ashyralyyev, C., & Ahmed, A.M.S. (2023). Numerical solution of the boundary value problems for the parabolic equation with involution. *Bulletin of the Karaganda University. Mathematics Series,* 1(109), 48–57. https://doi.org/10.31489/2023m1/48-57

15 Assanova, A.T., Bakirova, E.A., & Kadirbayeva, Zh.M. (2019). Numerically approximate method for solving of a control problem for integro-differential equations of parabolic type. *News of the NAS RK. Phys.-Math. Series, 6*(328), 14–24. https://doi.org/10.32014/2019.2518-1726.69

16 Dehghan, M. (2003). Numerical solution of a parabolic equation with non-local boundary specifications. *Applied Mathematics and Computation, 145*(1), 185–194. https://doi.org/10.1016/S0096-3003(02)00479-4

17 Ignjatovic, M. (2015). On solving parabolic equation with homogeneous boundary and integral initial conditions. *Matematicki Vesnik, 4*, 258–268.

18 El-Gamel, M., El-Baghdady, G.I., & El-Hady, M.A. (2022). Highly Efficient Method for Solving Parabolic PDE with Nonlocal Boundary Conditions. *Applied Mathematics, 13*, 101–119. https://doi.org/10.4236/am.2022.132009

19 Dzhumabaev, D.S. (1982). Ob odnom metode issledovaniia obyknovennykh differentsialnykh uravnenii [About one method of investigating ordinary differential equations]. *Izvestiia AN KazSSR. Seriia fiziko-matematicheskaia — News of the Academy of Sciences of the Kazakh SSR. Physics and Mathematics Series, 3*, 1–4 [in Russian].

20 Dzhumabaev, D.S. (2016). On one approach to solve the linear boundary value problems for Fredholm integrodifferential equations. *J. Comput. Appl. Math, 294*, 342–357. http://dx.doi.org/10.1016/j.cam.2015.08.023

21 Assanova, A.T., Imanchiyev, A.E., & Kadirbayeva, Zh.M. (2020). A nonlocal problem for loaded partial differential equations of fourth order. *Bulletin of the Karaganda University. Mathematics Series,* 1(97), 6–16. https://doi.org/10.31489/2020m1/6-16

22 Bakirova, E.A., Kadirbayeva, Zh.M., & Tleulesova, A.B. (2017). On one algorithm for finding a solution to a two-point boundary value problem for loaded differential equations with impulse effect. *Bulletin of the Karaganda University. Mathematics Series,* 3(87), 43–50. https://doi.org/10.31489/2017m3/43-50

23 Kadirbayeva, Zh.M., & Dzhumabaev, A.D. (2020). Numerical implementation of solving a control problem for loaded differential equations with multi-point condition. *Bulletin of the Karaganda University. Mathematics Series,* 4(100), 81–91. https://doi.org/10.31489/2020m4/81-91

24 Uteshova, R., Kadirbayeva, Z., Marat, G., & Minglibayeva, B. (2022). A novel numerical im-

plementation for solving problem for loaded depcag. *International Journal of Mathematics and Physics, 13*(2), 50–57. https://doi.org/10.26577/ijmph.2022.v13.i2.07

25 Dzhumabayev, D. (1989). Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. *U.S.S.R. Comp. Math. Math. Phys., 29*(1), 34–36.

26 Assanova, A.T., Iskakova, N.B., & Orumbayeva, N.T. (2018). Well-posedness of a periodic boundary value problem for the system of hyperbolic equations with delayed argument. *Bulletin of the Karaganda University. Mathematics Series,* 1(89), 8–14. https://doi.org/10.31489/2018m1/8-14

27 Assanova, A.T., Imanchiyev, A.E., & Kadirbayeva, Zh.M. (2019) Solvability of nonlocal problems for systems of Sobolev-type differential equations with a multipoint condition. *Russian Mathematics, 63*(12), 1–12. https://doi.org/10.3103/S1066369X19120016

28 Assanova, A.T., Kadirbayeva, Zh.M., Medetbekova, R.A., & Mynbayeva, S.T. (2024). Problem for differential-algebraic equations with a significant loads. *Bulletin of the Karaganda University. Mathematics Series,* 1(113), 46–59. https://doi.org/10.31489/2024m1/46-59

29 Bakirova, E.A., Assanova, A.T., & Kadirbayeva, Zh.M. (2021). A problem with parameter for the integro-differential equations. *Mathematical Modelling and Analysis, 26*(1), 34–54. https://doi.org/10.3846/mma.2021.11977

30 Abdimanapova, P.B., & Temesheva, S.M. (2023). Well-posedness criteria for one family of boundary value problems. *Bulletin of the Karaganda University. Mathematics Series,* 4(112), 5–20. https://doi.org/10.31489/2023m4/5-20

31 Bakirova, E.A., Iskakova, N.B., & Kadirbayeva, Z.M. (2023). Numerical implementation for solving the boundary value problem for impulsive integro-differential equations with parameter *KazNU Bulletin. Mathematics, Mechanics, Computer Science Series, 119*(3), 19–29. https://doi.org/10.26577/JMMCS2023v119i3a2

32 Assanova, A.T., Imanchiev A.E., & Kadirbayeva, Zh.M. (2018). Numerical solution of systems of loaded ordinary differential equations with multipoint conditions. *Computational mathematics and mathematical physics, 58*(4), 508–516. https://doi.org/10.1134/S096554251804005X

*Author Information**

Anar Turmaganbetkyzy Assanova — Doctor of physical and mathematical sciences, Professor, Head of the Department of Differential Equations and Dynamical Systems, Institute of Mathematics and Mathematical Modeling, 28 Shevshenko street, Almaty, 050010; Professor of Department of Mathematics, Kazakh National Women's Teacher Training University, 99 Ayteke bi street, Almaty, 050026, Kazakhstan; e-mail: *assanova@math.kz*; https://orcid.org/0000-0001-8697-8920

Zhazira Muratbekovna Kadirbayeva (corresponding author) — Candidate of physical and mathematical sciences, Associate Professor, Chief Researcher of the Department of Differential Equations and Dynamical Systems, Institute of Mathematics and Mathematical Modeling, 28 Shevshenko street, 050010; Associate Professor of Department of Mathematics, Kazakh National Women's Teacher Training University, 99 Ayteke bi street, Almaty, 050026, Kazakhstan; e-mail: *zhkadirbayeva@gmail.com*; https://orcid.org/0000-0001-8861-4100

Saule Kaldarbekovna Kuanysh — Doctoral student, Teacher at the Department of Mathematics, Al-Farabi Kazakh National University, 71/23 Al-Farabi Avenue, Almaty, 050040, Kazakhstan; e-mail: *saule270898@gmail.com*; https://orcid.org/0009-0006-0975-1662

*The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 46–52

https://doi.org/10.31489/2025M1/46-52 *Research article*

On the solvability of the Goursat problem for one class of loaded second-order hyperbolic equations

A.Kh. Attaev*

Institute of Applied Mathematics and Automation of Kabardin-Balkar Scientific Center of RAS, Nalchik, Russia (E-mail: attaev.anatoly@yandex.ru)

In this paper the solvability of the Goursat problems for two locally loaded second-order hyperbolic equations with a wave operator in the main part was explored. The loaded terms for both equations have the same trace, namely a part of one of the characteristics for the given hyperbolic operator, but the trace-forming maps are different. Moreover, in the first case, any point of the domain under consideration and the corresponding point of the load trace always lie on a straight line, which is a characteristic. In the second case, this does not work. It turns out that in the first case the Goursat problem is Voltairean, and in the second case it is Fredholmian.

Keywords: loaded equation, Goursat problem, characteristics of a hyperbolic equation, Volterra equation of the second kind, Fredholm equation of the second kind with a spectral parameter, eigenvalue, eigenfunction.

2020 Mathematics Subject Classification: 35L10.

Introduction

Mathematical modeling of many very important processes in various fields of natural science leads to boundary value problems for loaded partial differential equations. The main questions arising in the theory of boundary value problems for ordinary partial differential equations remain the same for boundary value problems for loaded partial differential equations. However, loads in equations makes their own adjustments to the formulation of the study and the correctness of a particular boundary value problem. For the first time, mentioning of a load eliminating the inequality of characteristics in the second Darboux problem for one second order degenerate hyperbolic equation belongs to A.M. Nakhushev in [1; 87]. Fundamental results on the effects of load, including spectral ones, for significantly loaded parabolic equations of arbitrary order were obtained by M.T. Dzhenaliev and M.I. Ramazanov. The main results in this direction are presented in their joint monograph [2]. Examples of the application of loaded differential equations, the main results of research in this field, as well as a numerous references on this subject are given in monographs [1–3].

We note works [4–9] devoted to the Goursat problem for loaded hyperbolic equations. The works [10–20] are devoted to boundary value problems for equations of the mixed type, boundary value problems with periodic boundary conditions, initial value problems and boundary control problems for loaded partial differential equations with both characteristic and non-characteristic load.

In this paper, research objects are two loaded second-order hyperbolic equations with a wave operator in the main part

$$Lu = \lambda u\left(\frac{\alpha(x+ky)+l}{2}, \frac{\alpha(x+ky)-l}{2}\right), \tag{1}$$

where

$$Lu = u_{xx} - u_{yy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u, \quad k = 1, -1, \quad \lambda = const.$$

It should be noted that the traces of the equations for $y = 0$ coincide.

*Corresponding author. *E-mail: attaev.anatoly@yandex.ru*

Received: 2 April 2024; *Accepted:* 4 November 2024.

1 Main part

Assume $\Omega = \{(x,y) : 0 < x - y < l, 0 < x + y < l\}$.

Goursat problem. In the domain Ω, find a solution to the equation (1) by class $C(\bar{\Omega}) \cap C^2(\Omega)$, satisfying the boundary conditions

$$u\left(\frac{x}{2}, -\frac{x}{2}\right) = \varphi(x), \quad 0 \le x \le l, \tag{2}$$

$$u\left(\frac{x}{2}, \frac{x}{2}\right) = \psi(x), \quad 0 \le x \le l, \tag{3}$$

where $\bar{\Omega}$ is the boundary for Ω.

It is assumed that

$$\alpha(x) \le x,\ \alpha(0) = 0,\ \alpha(l) = l,\ 0 \le x \le l, \tag{4}$$

$$a, b \in C^1(\Omega),\ \ c \in C(\bar{\Omega}),\ \ \varphi, \psi, \alpha \in C(\bar{J}) \cap C^2(J), \tag{5}$$

where $\bar{J}$ is the boundary for the interval $J = (0,l)$.

Goursat problem (2), (3) for the equation (1) is studied separately for $k = 1$ and $k = -1$.

2 Case $k = 1$

Theorem 1. Goursat problem (2), (3) for equation (1) at $k = 1$ is solvable in a unique way.

First, we present the proof of the theorem for $a \equiv b \equiv c \equiv 0$, $\alpha(x+y) = x+y$, that is, for the model equation

$$u_{xx} - u_{yy} = \lambda u\left(\frac{x+y+l}{2}, \frac{x+y-l}{2}\right) \tag{6}$$

with the characteristic variables $\xi = x - y$, $\eta = x + y$, equation (6) and boundary conditions (2), (3) take the form

$$v_{\xi\eta} = \frac{\lambda}{4} v(l,\eta), \tag{7}$$

$$v(\xi, 0) = \varphi(\xi),\ \ 0 \le \xi \le l, \tag{8}$$

$$v(0, \eta) = \psi(\eta),\ \ 0 \le \eta \le l, \tag{9}$$

where $v(\xi,\eta) = u\left(\frac{\eta+\xi}{2}, \frac{\eta-\xi}{2}\right)$.

The domain Ω goes into the rectangular domain $\Omega_0 = \{(\xi,\eta) : 0 < \xi < l, 0 < \eta < l\}$.

Suppose there is a solution to Goursat problem (8), (9) for equation (7), then it is easy to see that $v(\xi,\eta)$ is a solution to the following loaded integral equation

$$v(\xi,\eta) - \frac{\lambda}{4}\xi \int\limits_0^\eta v(l,t)dt = \varphi(\xi) + \psi(\eta) - \varphi(0). \tag{10}$$

Assuming $\xi = l$ by (10) we get

$$\int\limits_0^\eta v(l,t)dt = \int\limits_0^\eta e^{\frac{\lambda}{4}(\eta - t)}[\varphi(l) - \varphi(0) + \psi(t)]dt.$$

Substituting the resulting value for the integral into (10) and returning to the original variables, we have

$$u(x,y)=\varphi(x-y)+\psi(x+y)-\varphi(0)+\frac{\lambda}{4}(x-y)\int\limits_0^{x+y} e^{\frac{\lambda}{4}(x+y-t)}[\varphi(l)-\varphi(0)+\psi(t)]dt.$$

Let's go back to equation (1) with $k=1$. In the characteristic variables $\xi=x-y$, $\eta=x+y$ equation (1) is written as follows

$$L_1 v \equiv \frac{\lambda}{4}v(l,\alpha(\eta)), \tag{11}$$

where

$$L_1 v\equiv v_{\xi\eta}+A(\xi,\eta)v_\xi+B(\xi,\eta)v_\eta+C(\xi,\eta)v, \quad a+b=4A,\ a-b=4B,\ c=4C.$$

Condition (5) guarantees the Riemann function $R(\xi_0,\eta_0;\xi,\eta)$ for equation (11). Assume that the right side of the equation is known. Applying the well-known formula [21] for solving the Goursat problem, to find $v(\xi,\eta)$ we obtain the following integral equation

$$v(\xi,\eta)=R(\xi,0;\xi,\eta)\varphi(\xi)+R(0,\eta;\xi,\eta)\psi(\eta)-R(0,0;\xi,\eta)\varphi(0)+$$
$$+\int\limits_0^{\xi}\left[B(t,0)R(t,0;\xi,\eta)-\frac{\partial}{\partial t}R(t,0;\xi,\eta)\right]\varphi(t)dt+$$
$$+\int\limits_0^{\eta}\left[A(0,\tau)R(0,\tau;\xi,\eta)-\frac{\partial}{\partial \tau}R(0,\tau;\xi,\eta)\right]\psi(\tau)d\tau+$$
$$+\frac{\lambda}{4}\int\limits_0^{\xi}dt\int\limits_0^{\eta}R(t,\tau;\xi,\eta)v(l,\alpha(\tau))d\tau. \tag{12}$$

Changing the order of integration in the double integral and passing to the limit at $\xi\to l$ in (12), to find $v(l,t)$, we obtain the following integral equation

$$v(l,\eta)=\frac{\lambda}{4}\int\limits_0^{\eta}K(\eta,\tau)v(l,\alpha(\tau))d\tau=F(\eta), \tag{13}$$

$$K(\eta,\tau)=\int\limits_0^{l}R(t,\tau;l,\eta)dt,$$

$$F(\eta)=\int\limits_0^{l}\left[B(t,0)R(t,0;l,\eta)-\frac{\partial}{\partial \tau}R(t,0;l,\eta)\right]\varphi(t)dt+$$
$$+\int\limits_0^{l}\left[A(0,\tau)R(t,\tau;l,\eta)-\frac{\partial}{\partial \tau}R(0,\tau;l,\eta)\right]\psi(\tau)d\tau+$$
$$+R(l,0;l,\eta)\varphi(l)+R(0,\eta;l,\eta)\psi(\eta)-R(0,0;l,\eta)\varphi(0).$$

Substituting $\alpha(\eta)$ into (13) instead of η, we obtain

$$v(l,\alpha(\eta)) - \frac{\lambda}{4}\int\limits_0^{\alpha(\eta)} K(\alpha(\eta),\tau)v(l,\alpha(\tau))d\tau = F(\alpha(\eta)).$$

Condition (4) guarantees the existence of a unique solution $v(l,\alpha(\eta)) \in C(\bar{J})$. After finding $v(l,\alpha(\eta))$, a unique solution to Goursat problem (2), (3) for equation (1) at $k=1$ in the domain Ω due to (12) is given by the formula

$$u(x,y) = u_0(x,y) + \frac{\lambda}{4}\int\limits_0^{\xi}\int\limits_0^{\eta} R(t,\tau;\xi,\eta)v(l,\alpha(\tau))d\tau dt,$$

where $u_0(x,y)$ is a solution to the same problem for equation (1) at $\lambda = 0$.

3 The case with $k=-1$

Theorem 2. Assume $|\lambda| < \frac{4}{Ml}$, where $M = \max_{\bar{\Omega}_0}\left|\int\limits_0^{\alpha(s)} R(t,\tau;l,\alpha(s))d\tau\right|$, $R(\xi_0,\eta_0;\xi,\eta)$ is the Riemann operator for L_1, then Goursat problem (2), (3) for equation (1) at $k=-1$ is solvable in a unique way.

Proof of Theorem 2. In the characteristic variables $\xi = x-y$, $\eta = x+y$ equation (1) for $k=-1$ takes the form

$$L_1 v \equiv \frac{\lambda}{4} v(l,\alpha(\xi)).$$

Applying the same reasoning as for $k=1$, to find $v(l,\alpha(\xi))$, we obtain the following Fredholm integral equation with the spectral parameter

$$v(l,\alpha(\xi)) - \lambda\int\limits_0^{l} K(\alpha(\xi),t)v(l,\alpha(t))dt = F(\alpha(\xi)), \tag{14}$$

where

$$K(\alpha(\xi),t) = \frac{1}{4}\int\limits_0^{\alpha(\xi)} R(t,\tau;l,\alpha(\xi))d\tau.$$

By virtue of continuity $K(\alpha(\xi),t)$ and $F(\alpha(\xi))$ respectively in $\bar{\Omega}_0$ and $\bar{J}$, in $\bar{\Omega}_0$ for $|K(\alpha(\xi),t)|$, we get some maximum value M, $|F(\alpha(\xi))|$ has some maximum value M_1.

Under these conditions, solution (14) can be obtained, for example, by the method of successive substitutions in the form of an absolutely and uniformly convergent series.

Let us now consider the case when $a \equiv b \equiv c \equiv 0$, $\alpha(x-y) = x-y$, that is, the Goursat problem (2), (3) for the equation

$$u_{xx} - u_{yy} = \lambda u\left(\frac{x-y+l}{2}, \frac{x-y-l}{2}\right). \tag{15}$$

Theorem 3. Goursat problem (2), (3) for equation (15) is solvable in a unique way for all $\lambda \neq \frac{8}{l^2}$.

In the case when $\lambda = \frac{8}{l^2}$:

1) The homogeneous problem corresponding to problem (2), (3) for equation (13) has an infinite number of solutions

$$u(x,y) = C(x+y)(x-y)^2,$$

where C is the arbitrary constant.

2) An inhomogeneous problem is solvable if and only if $\int\limits_0^l [\varphi(l)-\varphi(0)+\psi(t)]dt = 0$. If this condition is satisfied, it also has infinitely many solutions.

Proof of Theorem 3. With the characteristic variables $\xi = x-y$, $\eta = x+y$ equation (15) takes the form

$$v_{\xi\eta} = \frac{\lambda}{4} v(l,\xi). \tag{16}$$

Integrating (16) over ξ ranging from 0 to ξ, and then over η ranging from 0 to η, we verify that $v(\xi,\eta)$ is the solution to the following loaded integral equation

$$v(\xi,\eta) - \frac{\lambda}{4}\eta \int\limits_0^\xi v(l,t)dt = \varphi(\xi) + \psi(\eta) - \varphi(0).$$

Replacing ξ by l and η by ξ in the last relation, we obtain

$$v(l,\xi) - \frac{\lambda}{4}\xi \int\limits_0^l v(l,t)dt = \varphi(l) + \psi(\xi) - \varphi(0). \tag{17}$$

Equation (17) is the simplest integral equation with a spectral parameter and a degenerate kernel.

Since the degenerate kernel consists of one term, the corresponding system becomes a single equation

$$q_1 = \frac{\lambda}{4}\left(\int\limits_0^l t dt\right) q_1 + \int\limits_0^l \left[\varphi(l) + \psi(t) - \varphi(0)\right] dt,$$

where $q_1 = \int\limits_0^l v(l,t)dt$, then the result of the Theorem 3 follows directly.

Conclusion

If we consider the Goursat problem

$$u\left(\frac{x}{2},\frac{x}{2}\right) = \varphi(x),\ u\left(\frac{l+x}{2},\frac{l-x}{2}\right) = \psi(x),\ 0 \le x \le l \tag{18}$$

for equation (15), then it is obvious that equation (17) takes the form

$$v(l,\xi) - \frac{\lambda}{4}\int\limits_0^l (\xi - l)v(l,t)dt = \varphi(l) + \psi(\xi) - \varphi(0),$$

whence it follows that it is uniquely solvable for all $\lambda \neq -\frac{l^2}{8}$. We find that for $\lambda = \frac{l^2}{8}$ Goursat problem (2), (3) for equation (15) is not correct, and Goursat problem (18) for of the same equation is correct, but for $\lambda = -\frac{l^2}{8}$, vice versa.

As a consequence, equation (15) can be called an example of an equation for which the effect of inequality of characteristics as carriers of Goursat data takes place.

Conflict of Interest

The author declares no conflict of interest.

References

1 Kal'menov, T.Sh., & Soldatov, A.P. (Eds.) (2024). *Izbrannye glavy teorii kraevykh zadach so smeshcheniem, teorii nagruzhennykh uravnenii i drobnogo ischisleniia [Selected chapters on the theory of boundary value problems with displacement, the theory of loaded equations and fractional calculus].* Moscow: RAS [in Russian].

2 Jenaliyev, M.T., & Ramazanov, M.I. (2010). *Nagruzhennye uravneniia kak vozmushcheniia differentsialnykh uravnenii: monografiia [Loaded equations as perturbations of differential equations: monograph.* Almaty: Gylym [in Russian].

3 Nakhushev, A.M. (2012). *Nagruzhennye uravneniia i ikh primenenie: monografiia [Loaded equations and their applications: monograph.* Moscow: Nauka [in Russian].

4 Kaziev, V.M. (1981). Goursat problem for one loaded integro-differential equation. *Differ. equations, 17*(2), 313–319. Retriened from http://mi.mathnet.ru/de4195

5 Ogorodnikov, E.N. (2003). Nekotorye kharakteristicheskie zadachi dlia sistem nagruzhennykh differentsialnykh uravnenii i ikh sviaz s nelokalnymi kraevymi zadachami [Some characteristic problems for loaded systems of differential equations and their relationship with non-local boundary value problems]. *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Seriia «Fiziko-matematicheskie nauki» — Bulletin of Samara State Technical University. Series "Physical and Mathematical Sciences", 19*, 22–28 [in Russian]. https://doi.org/10.14498/vsgtu134

6 Attaev, A.Kh. (2008). Zadacha Gursa dlia lokalno-nagruzhennogo uravneniia so stepennym parabolicheskim vyrozhdeniem [Goursat problem for a locally loaded equation with power parabolic degeneration]. *Doklady Adygskoi (Cherkesskoi) Mezhdunarodnoi Akademii Nauk — Reports of the Adyghe (Circassian) International Academy of Sciences, 10*(2), 14–16 [in Russian].

7 Attaev, A.Kh. (2016). Zadacha Gursa dlia nagruzhennogo vyrozhdaiushchegosia giperbolicheskogo uravneniia vtorogo poriadka s operatorom Gellerstedta v glavnoi chasti [Goursat problem for a loaded degenerate second-order hyperbolic equation with the Gellerstedt operator in the principal part]. *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Seriia "Fiziko-matematicheskie nauki" – Bulletin of Samara State Technical University. Series "Physical and Mathematical Sciences", 20*(1), 7–21 [in Russian]. https://doi.org/10.14498/vsgtu1452

8 Attaev, A.Kh. (2013). Zadacha s dannymi na parallelnykh kharakteristikakh dlia nagruzhennogo volnovogo uravneniia [Problem with data on parallel characteristics for loaded wave equation]. *Doklady Adygskoi (Cherkesskoi) Mezhdunarodnoi Akademii Nauk — Reports of the Adyghe (Circassian) International Academy of Sciences, 15*(2), 25–27 [in Russian].

9 Attaev, A.Kh. (2014). Zadacha Gursa dlia nagruzhennogo giperbolicheskogo uravneniia [Goursat problem for a loaded hyperbolic equation]. *Doklady Adygskoi (Cherkesskoi) Mezhdunarodnoi Akademii Nauk — Reports of the Adyghe (Circassian) International Academy of Sciences, 16*(3), 9–12 [in Russian].

10 Islomov, B.I., & Xolbekov, J.A. (2021). Ob odnoi nelokalnoi kraevoi zadache dlia nagruzhennogo parabolo-giperbolicheskogo uravneniia s tremia liniiami izmeneniia tipa [On a nonlocal boundary-value problem for a loaded parabolic-hyperbolic equation with three lines of degeneracy]. *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Seriia «Fiziko-matematicheskie nauki» — Bulletin of Samara State Technical University. Series "Physical and Mathematical Sciences", 25*(3), 407–422 [in Russian]. https://doi.org/10.14498/vsgtu1822

11 Islomov, B.I., & Abdullaev, O.Kh. (2020). Zadachi tipa Gellerstedta dlia nagruzhennogo uravneniia parabolo-giperbolicheskogo tipa s operatorami Kaputo i Erdeili–Kobera drobnogo poriadka [Gellerstedt type problem for the loaded parabolic-hyperbolic type equation with Caputo and Erdelyi-Kober operators of fractional order]. *Izvestiia vysshikh uchebnykh zavedenii. Matematika – News of higher educational institutions. Mathematics, 10*, 33–46 [in Russian]. https://doi.org/10.26907/0021-3446-2020-10-33-46

12 Islomov, B.I., & Juraev, F.M. (2022). Local boundary value problems for a loaded equation of parabolic-hyperbolic type degenerating inside the domain. *Ufa Math. J., 14*(1), 37–51. https://doi.org/10.13108/2022-14-1-37

13 Sabitov, K.B. (2015). Initial-boundary problem for parabolic-hyperbolic equation with loaded summands. *Russian Math. (Iz. VUZ), 59*(6), 23–33. https://doi.org/10.3103/S1066369X15060055

14 Islomov, B., & Baltaeva, U.I. (2015). Boundary value problems for a third-order loaded parabolic-hyperbolic equation with variable coefficients. *Electron. J. Diff. Equ., 2015*(221), 1–10.

15 Mamchuev, M.O. (2016). Solutions of the main boundary value problems for a loaded second-order parabolic equation with constant coefficients. *Diff Equat, 52*, 789–797. https://doi.org/10.1134/S0012266116060094

16 Dzhamalov, S.Z., & Ashurov, R.R. (2018). On a nonlocal boundary-value problem for second kind second-order mixed type loaded equation in a rectangle. *Uzbek Math. J.*, (3), 63–72. https://doi.org/10.29229/uzmj.2018-3-6

17 Dzhenaliev, M.T. (2001). Loaded equations with periodic boundary conditions. *Differ. Equ., 37*(1), 51–57. https://doi.org/10.1023/A:1019268231282

18 Pskhu, A.V., Ramazanov, M.I., & Kosmakova, M.T. (2023). Boundary value problem for a loaded fractional diffusion equation. *Turkish Journal of Mathematics, 47*(5), Article 21. https://doi.org/10.55730/1300-0098.3450

19 Attaev, A.Kh. (2017). The Cauchy problem for the Mc Kendrick-von Foerster loaded equation. *International Journal of Pure and Applied Mathematics, 113*(4), 569–574.

20 Attaev, A.Kh. (2020). Boundary Control Problem for a Loaded String Vibration Equation. *Diff. Equat., 56*, 635–640. https://doi.org/10.1134/S0012266120050080

21 Sabitov, K.B. (2021). *Uravneniia matematicheskoi fiziki [Equations of Mathematical Physics].* Moscow: Fizmatlit, 352 [in Russian].

*Author Information**

Anatoliy Khuseevich Attaev — PhD (Phys. & Math.), Associate Professor, Head of Department of Equations of Mixed Type, Institute of Applied Mathematics and Automation KBSC RAS, 89A Shortanova St., Nalchik, 360000, Russia; e-mail:, *attaev.anatoly@yandex.ru*; https://orcid.org/0000-0001-5864-6283

*The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 53–62

https://doi.org/10.31489/2025M1/53-62 *Research article*

Gagliardo–Nirenberg type inequalities for smoothness spaces related to Morrey spaces over n-dimensional torus

Sh.A. Balgimbayeva*, A.K. Janabilova

Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
(E-mail: sholpan.balgyn@gmail.com, zh.aizere2001@gmail.com)

In the paper, the Gagliardo–Nirenberg type inequalities for smoothness spaces $B^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ of Nikol'skii–Besov type and spaces $F^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ of Lizorkin–Triebel type both related to Morrey spaces over n-dimensional torus for some range of the parameters s, p, q, τ were proved. These spaces are natural analogues of the spaces $B^{s\tau}_{pq}(\mathbb{R}^n)$ and $F^{s\tau}_{pq}(\mathbb{R}^n)$ in the case of multidimensional torus $\mathbb{T}^n$. The main results of the article are two theorems, each of which proves the Gagliardo–Nirenberg type inequality for the Lizorkin–Triebel type spaces or the Nikol'skii–Besov type spaces respectively.

Keywords: Nikol'skii–Besov/Lizorkin–Triebel smoothness spaces related to Morrey space, multidimensional torus, Gagliardo–Nirenberg type inequalities.

2020 Mathematics Subject Classification: 42A35, 41A63, 41A55.

Introduction

Multiplicative and additive inequalities for (partial) derivatives of functions play crucial role in different areas of Analysis and Applied Mathematics, in particular, in Analysis of Partial Differential Equations.

Multiplicative and additive inequalities for derivatives of functions in single variable (on an axis, a semi-axis, a segment, or a unit circle) with exact constants are an extensive section of modern function theory, originated from the classical works of J. Hadamard and A.N. Kolmogorov. The development of this field can be traced in surveys [1, 2].

In the case of functions in several variables, E. Gagliardo and L. Nirenberg proved important inequality, nowadays known as the Gagliardo–Nirenberg inequality (see [3; ch. III, sect. 15]):

Proposition 1. Let function u belong to $L_q(\mathbb{R}^n)$ and such that all its (distributional) derivatives of order $l(\in \mathbb{N})$ belong to $L_r(\mathbb{R}^n)$, with $1 \le q, r \le \infty$. Then for $0 \le j < l$, the following inequality

$$\sum_{|\alpha|=j} \|\partial^\alpha f \,|\, L_p(\mathbb{R}^n)\| \le C \|f \,|\, L_q(\mathbb{R}^n)\|^{1-\theta} \Big(\sum_{|\alpha|=l} \|\partial^\alpha f \,|\, L_r(\mathbb{R}^n)\| \Big)^{\theta} \tag{1}$$

holds, where $\frac{1}{p} = \frac{j}{n} + (1-\theta)\frac{1}{q} + \theta\big(\frac{1}{r} - \frac{l}{n}\big)$ for all θ in the interval $[\frac{j}{l}, 1]$ (the positive constant C depending only on n, l, j, q, r, θ), with the following exceptional cases:

1. If $j = 0$, $rl < n$, $q = \infty$, then we make the additional assumption that either u tends to zero at infinity or $u \in L_{q^*}(\mathbb{R}^n)$ for some finite $q^* > 0$.

2. If $1 < r < \infty$ and $l - j - \frac{n}{r}$ is a nonnegative integer then inequality (1) holds only for θ satisfying $\frac{j}{l} \le \theta < 1$.

*Corresponding author. *E-mail: sholpan.balgyn@gmail.com*

This research was supported by Ministry of Science and Higher Education, Republic of Kazakhstan, under grant AP14869246.

Received: 30 August 2024; *Accepted:* 31 October 2024.

The multiplicative inequality (1) is equivalent to the corresponding additive inequality with an arbitrary parameter $\varepsilon > 0$:

$$\sum_{|\alpha|=j} \|\partial^\alpha f \mid L_p(\mathbb{R}^n)\| \le C\Big(\varepsilon^{-\frac{1}{1-\theta}} \|f \mid L_q(\mathbb{R}^n)\| + \varepsilon^{\frac{1}{\theta}} \sum_{|\alpha|=l} \|\partial^\alpha f \mid L_r(\mathbb{R}^n)\|\Big), \quad \forall \varepsilon > 0.$$

Note that under some particular assumptions, the inequality (1) and its additive analogue for some special cases of mixed L_p-, L_q- and L_r-norms were established by V.P. Il'in, L. Nirenberg and others; further, M. Troisi, V.A. Solonnikov and others obtained analogues of inequality (1) for the anisotropic case of specifying differential properties of functions in L_r (see details and general results in [3; ch. III, sect. 15]).

The classical Gagliardo–Nirenberg inequalities and their generalizations mentioned above are a very useful tool in connection with partial differential equations (see, for example, the monograph [3]). For this reason, there is also some interest in their analogues in various non-classical situations.

In 2001 H. Brezis and P. Mironescu [4] proved the following Gagliardo–Nirenberg type inequalities for the (isotropic) Lizorkin–Triebel spaces.

Proposition 2. (i) Let a tempered distribution f belongs to both Lizorkin–Triebel spaces $F^{s_0}_{p_0q_0}(\mathbb{R}^n)$ and $F^{s_1}_{p_1q_1}(\mathbb{R}^n)$, with $0 < p_0,\, p_1 < \infty$, $0 < q_0$, $q_1 \le \infty$, $-\infty < s_0 < s_1 < \infty$. Then for any $\theta : 0 < \theta < 1$ and $q : 0 < q \le \infty$, the following inequality

$$\|f \mid F^s_{pq}(\mathbb{R}^n)\| \le C\|f \mid F^{s_0}_{p_0q_0}(\mathbb{R}^n)\|^{1-\theta}\|f \mid F^{s_1}_{p_1q_1}(\mathbb{R}^n)\|^{\theta} \tag{2}$$

holds, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\quad s = (1-\theta)s_0 + \theta s_1$ (the positive constant C depending only on n, s_0, s_1, p_0, p_1, q_0, q_1, q, θ).

(ii) Let a tempered distribution f belongs to both the Lizorkin–Triebel spaces $F^{s_0}_{p_0q_0}(\mathbb{R}^n)$ and $F^{s_1}_{\infty\infty}(\mathbb{R}^n)$, with $0 < p_0 < \infty$, $0 < q_0 \le \infty, -\infty < s_0 \ne s_1 < \infty$. Then for any $\theta : 0 < \theta < 1$ and $q : 0 < q \le \infty$, the following inequality

$$\|f \mid F^s_{pq}(\mathbb{R}^n)\| \le C\|f \mid F^{s_0}_{p_0q_0}(\mathbb{R}^n)\|^{1-\theta}\|f \mid F^{s_1}_{\infty\infty}(\mathbb{R}^n)\|^{\theta} \tag{3}$$

holds, where $\frac{1}{p} = \frac{1-\theta}{p_0}$, $s = (1-\theta)s_0 + \theta s_1$ (the positive constant C depending only on $n, s_0, s_1, p_0, q_0, q, \theta$).

The analogues of the inequalities (2) and (3) for the (isotropic) Besov–Nikol'skii spaces are as follows.

Proposition 3. Let a tempered distribution f belong to both Nikol'skii–Besov spaces $B^{s_0}_{p_0q_0}(\mathbb{R}^n)$ and $B^{s_1}_{p_1q_1}(\mathbb{R}^n)$, with $0 < p_0$, $p_1 < \infty$, $0 < q_0$, $q_1 \le \infty$, $-\infty < s_0 < s_1 < \infty$. Then for any $\theta : 0 < \theta < 1$, the following inequality

$$\|f \mid B^s_{pq}(\mathbb{R}^n)\| \le C\|f \mid B^{s_0}_{p_0q_0}(\mathbb{R}^n)\|^{1-\theta}\|f \mid B^{s_1}_{p_1q_1}(\mathbb{R}^n)\|^{\theta} \tag{4}$$

holds, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $s = (1-\theta)s_0 + \theta s_1$ (the positive constant C depending only on $n, s_0, s_1, p_0, p_1, q_0, q_1, \theta$).

The inequality (4) is a classical result of J. Peetre, proved in middle of 1960s.

Note that for the Nikol'skii–Besov spaces the Gagliardo–Nirenberg type inequality (4) is established for full natural range of parameters $n, s_0, s_1, p_0, p_1,\ q_0, q_1, \theta$, in contrast to that inequality for the Lizorkin–Triebel spaces: here, there is a gap for the case where $0 < p_0 < \infty$, $\quad 0 < q_0 \le \infty$, $p_1 = \infty$, $\quad 0 < q_1 < \infty$.

Moreover, as can be seen from the inequalities (2) and (3), the result (Gagliardo–Nirenberg type inequality for the Lizorkin–Triebel spaces) is completely independent of the values of the "microscopic" parameters q, q_0, q_1. Unlike Lizorkin–Triebel type spaces, in the inequality (4) the parameter q is strictly connected with q_0 and q_1 like the other parameters.

The gap mentioned above was fulfilled by W. Sickel [5]:

Proposition 4. Let a tempered distribution f belongs to $F^{s_0}_{p_0q_0}(\mathbb{R}^n)\cap F^{s_1}_{\infty q_1}(\mathbb{R}^n)$, with $0<p_0<\infty$, $0<q_0\le\infty$, $0<q_1<\infty$, $-\infty<s_0\ne s_1<\infty$. Then for any $\theta:0<\theta<1$ and $q:0<q\le\infty$, the following inequality

$$\|f\mid F^{s}_{pq}(\mathbb{R}^n)\|\le C\|f\mid F^{s_0}_{p_0q_0}(\mathbb{R}^n)\|^{1-\theta}\|f\mid F^{s_1}_{\infty\infty}(\mathbb{R}^n)\|^{\theta} \tag{5}$$

holds, where $\frac{1}{p}=\frac{1-\theta}{p_0}$, $s=(1-\theta)s_0+\theta s_1$ (the positive constant C depending only on $n,s_0,s_1,p_0,q_0,q,\theta$).

In fact, W. Sickel established the Gagliardo–Nirenberg type inequalities for two scales (of Nikol'skii–Besov) $B^{s\tau}_{pq}(\mathbb{R}^n)$ and (Lizorkin–Triebel) $F^{s\tau}_{pq}(\mathbb{R}^n)$ (with additional real parameter τ) of smoothness spaces related to Morrey spaces over whole Euclidean space $\mathbb{R}^n$ in full range of parameters involved. The inequalities from [5] contain the inequalities (2)–(5) as special cases because $B^{s0}_{pq}(\mathbb{R}^n)\equiv B^{s}_{pq}(\mathbb{R}^n)$ and $F^{s0}_{pq}(\mathbb{R}^n)\equiv B^{s}_{pq}(\mathbb{R}^n)$ in sense of equivalent (quasi)norms.

Goal of the paper is to prove the Gagliardo–Nirenberg type inequalities for the spaces $B^{s\tau}_{pq}(\mathbb{T}^n)$ and $F^{s\tau}_{pq}(\mathbb{T}^n)$, which are natural analogues of the spaces $B^{s\tau}_{pq}(\mathbb{R}^n)$ and $F^{s\tau}_{pq}(\mathbb{R}^n)$ in the case of multidimensional torus $\mathbb{T}^n$.

The rest of the paper is organized as follows. In Section 2 we introduce some notation, define the spaces of distributions $B^{s\tau}_{pq}(\mathbb{R}^n)$, $F^{s\tau}_{pq}(\mathbb{R}^n)$, $B^{s\tau}_{pq}(\mathbb{T}^n)$ and $F^{s\tau}_{pq}(\mathbb{T}^n)$ and formulate main results of the paper (Theorems 1 and 2). Section 3 contains the proof of crucial Lemma. Finally, in Section 4, we give proofs of Theorems 1 and 2.

1 *The Gagliardo–Nirenberg type inequalities for the smoothness spaces related to Morrey spaces*

First we introduce some notation and give definitions of (the two scales of) spaces of distributions under consideration.

Let $n\in\mathbb{N}$, $n\ge 2$, $\mathrm{z}_n=\{1,\dots,n\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $x=(x_1,\dots,x_n), y=(y_1,\dots,y_n)\in\mathbb{R}^n$, we put $xy=x_1y_1+\ldots+x_ny_n$, $|x|=|x_1|+\ldots+|x_n|$, $|x|_\infty=\max(|x_\nu|:\nu\in \mathrm{z}_n)$; $x\le y$ $(x<y)$ $\Leftrightarrow x_\nu\le y_\nu$ $(x_\nu<y_\nu)$ for all $\nu\in \mathrm{z}_n$.

Let $\mathcal{S}:=\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'=\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of test functions and tempered distributions respectively; $\widehat{f}$ is Fourier transform for $f\in\mathcal{S}'(\mathbb{R}^n)$; in particular, for $\varphi\in\mathcal{S}$,

$$\widehat{\varphi}(\xi)=\int_{\mathbb{R}^n}\varphi(x)e^{-2\pi i\,\xi x}dx.$$

For $0<p\le\infty$ and a measurable set $G\subset\mathbb{R}^n$, as usual, let $L_p(G)$ be the space of functions $f:G\to\mathbb{C}$ integrable in sense of Lebesgue to the power p (essentially bounded if $p=\infty$) over G, endowed with standard (quasi)norm (norm if $p\ge 1$)

$$\|\,f\,|\,L_p(G)\,\|=\Big(\int_G|\,f(x)\,|^p dx\Big)^{\frac{1}{p}}\ (p<\infty),$$

$$\|\,f\,|\,L_\infty(G)\,\|=\text{ess sup}(|\,f(x)\,|:x\in G).$$

For $0<q\le\infty$ let $\ell_q:=\ell_q(\mathbb{N}_0)$ be the space of (complex–valued) sequences $(c_j)=(c_j:j\in\mathbb{N}_0)$ with finite standard (quasi)norm (norm if $q\ge 1$) $\|(c_j)\,|\,\ell_q\|$.

Further, let $\ell_q(L_p(G))$ ($L_p(G;\ell_q)$ respectively) be the space of function sequences $(g_j(x))=(g_j(x):k\in\mathbb{N}_0)$ $(x\in G)$ with finite (quasi)norm (norm if $p,q\ge 1$)

$$\|\,(g_j(x))\,|\,\ell_q(L_p(G))\,\|=\|\,(\,\|\,g_j\,|\,L_p(G)\|)\,|\,\ell_q\,\|,$$

$$(\|\,(g_j(x))\,|\,L_p(G;\ell_q)\,\|=\|\,\|\,(g_j(\cdot))\,|\,\ell_q\,\|\,|\,L_p(G)\|$$

respectively).

We choose a test function $\eta_0 \in \mathcal{S}$ such that

$$0 \le \widehat{\eta}_0(\xi) \le 1, \ \ \xi \in \mathbb{R}^n; \ \ \widehat{\eta}_0(\xi) = 1 \ \text{if} \ |\xi|_\infty \le 1; \ \operatorname{supp} \widehat{\eta}_0 = \{\xi \in \mathbb{R}^n \,|\, |\xi|_\infty \le 2\}.$$

Put $\widehat{\eta}(\xi) = \widehat{\eta}_0(2^{-1}\xi) - \widehat{\eta}_0(\xi)$, $\widehat{\eta}_j(\xi) := \widehat{\eta}_j(\xi) = \widehat{\eta}(2^{1-j}\xi)$, $j \in \mathbb{N}$. Then

$$\sum_{j=0}^{\infty} \widehat{\eta}_j(\xi) \equiv 1, \ \ \xi \in \mathbb{R}^n,$$

i.e. $\{\widehat{\eta}_j(\xi) \,|\, j \in \mathbb{N}_0\}$ is the resolution of unity over $\mathbb{R}^n$. It is clear that

$$\eta(x) = 2^n \eta_0(2x) - \eta_0(x), \ \eta_j(x) := 2^{(j-1)n} \eta(2^{j-1}x), \ \ j \in \mathbb{N}.$$

We define the operator Δ_j^η on $\mathcal{S}'$ as follows: for $f \in \mathcal{S}'$ put

$$\Delta_j^\eta(f, x) = f * \eta_j(x) = \langle f, \eta_j(x - \cdot)\rangle.$$

Let $\mathcal{Q}$ be the set of all dyadic cubes in $\mathbb{R}^n$ of the form

$$Q = Q_{j\xi} = \{\, x \in \mathbb{R}^n \, : \, 2^j x - \xi \in [0,1)^n \,\} \ (j \in \mathbb{Z}, \xi \in \mathbb{Z}^n).$$

Denote by $j(Q)(= j)$ and $|Q|(= 2^{-jm})$ "level" and the volume of the cube $Q = Q_{j\xi}$ respectively.

Now we recall important definition of the Lizorkin–Triebel space $F^s_{\infty\, q}(\mathbb{R}^n)$ $(0 < q < \infty)$, invented by M. Frazier and B. Jawerth [6].

Definition 1. Let $s \in \mathbb{R}$, $0 < q < \infty$. The space $F^s_{\infty\, q} := F^s_{\infty\, q}(\mathbb{R}^n)$ consists of all distributions $f \in \mathcal{S}'$ for which (quasi)norm

$$\| \, f \,|\, F^s_{\infty\, q} \,\| \, = \Big(\sup_{Q \in \mathcal{Q}: j(Q) \ge 0} \frac{1}{|Q|} \int_Q \sum_{j=j(Q)}^{\infty} |2^{sj} \Delta_j^\eta(f, x)|^q dx \Big)^{1/q}$$

is finite.

Further, denote by $\widetilde{\mathcal{S}}' := \mathcal{S}'(\mathbb{T}^n)$ the space of all distributions $f \in \mathcal{S}'$, 1–periodic in each variable (i.e. such that $\langle f, \varphi(\cdot + \xi)\rangle = \langle f, \varphi\rangle$ for all $\varphi \in \mathcal{S}$ and any $\xi \in \mathbb{Z}^n$), and $\tilde{\mathcal{S}} := \mathcal{S}(\mathbb{T}^n)$ the space of all infinitely differentiable functions over $\mathbb{T}^n$ endowed with the topology of uniform convergence of all partial derivatives over $\mathbb{T}^n$. Then $\mathcal{S}'(\mathbb{T}^n)$ is identified naturally with the space topologically dual to $\mathcal{S}(\mathbb{T}^n)$. It is known that $f \in \tilde{\mathcal{S}}'$ if and only if $\operatorname{supp} \widehat{f} \subset \mathbb{Z}^n$, i.e. $\widehat{f} = 0$ on open set $\mathbb{R}^n \backslash \mathbb{Z}^n$. Here $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ is n-dimensional torus.

Let $g : \mathbb{R}^n \to \mathbb{C}$ be an arbitrary function, then its periodization $\widetilde{g} : \mathbb{T}^n \to \mathbb{C}$ is defined as (at least formal) sum of series $\sum\limits_{\xi \in \mathbb{Z}^n} g(x + \xi)$.

By the Poisson summation formula it is easy to verify that for $\varphi \in \mathcal{S}$, $\widetilde{\varphi} \in \tilde{\mathcal{S}}$, and, moreover, $\widetilde{\varphi}(x) = \sum_{\xi \in \mathbb{Z}^n} \widehat{\varphi}(\xi) e^{2\pi i \xi x}$.

Now we define operators $\widetilde{\Delta}_j^\eta$ on $\widetilde{\mathcal{S}}'$ $(j \in \mathbb{N}_0)$ as follows: for $f \in \tilde{\mathcal{S}}'$, put

$$\widetilde{\Delta}_j^\eta(f, x) = f * \widetilde{\eta}_j(x) = \langle f, \widetilde{\eta}_j(x - \cdot)\rangle = \sum_{\xi \in \mathbb{Z}^n} \widehat{\eta}_j(\xi) \widehat{f}(\xi) e^{2\pi i \, \xi x}.$$

Let $(\mathbf{0} = (0, \ldots, 0), \mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n)$

$$\widetilde{\mathcal{Q}} = \{Q \in \mathcal{Q} \,|\, Q \subset Q_0 := [0,1)^n\} = \{Q_{j\xi} \,|\, j \in \mathbb{N}_0, \, \xi \in \mathbb{Z}^n : \mathbf{0} \le \xi < 2^j \mathbf{1}\}.$$

In analogy with 1, we give

Definition 2. Let $s \in \mathbb{R}$, $0 < q < \infty$. The Lizorkin–Triebel space $\widetilde{F}^s_{\infty\, q} := F^s_{\infty\, q}(\mathbb{T}^n)$ consists of all distributions $f \in \widetilde{\mathcal{S}}'$, for which (quasi)norm

$$\| f \,|\, \widetilde{F}^s_{\infty\, q} \| = \Big(\sup_{Q \in \widetilde{\mathcal{Q}}} \frac{1}{|Q|} \int_Q \sum_{j=j(Q)}^{\infty} |2^{sj} \widetilde{\Delta}^{\eta}_j(f,x)|^q dx \Big)^{1/q}$$

is finite.

Now we recall definitions of two scales (of Nikol'skii–Besov type) $B^{s\,\tau}_{p\,q}(\mathbb{R}^n)$ and (Lizorkin–Triebel type) $F^{s\,\tau}_{p\,q}(\mathbb{R}^n)$ of (inhomogeneous) smoothness spaces related to Morrey spaces and their periodic analogues $B^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ and $F^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ (below $t_+ := \max\{0, t\}$ if $t \in \mathbb{R}$).

Definition 3. Let $s, \tau \in \mathbb{R}$, $0 < p, q \le \infty$. Then

I. the Nikol'skii–Besov type space $B^{s\,\tau}_{p\,q} := B^{s\,\tau}_{p\,q}(\mathbb{R}^n)$ consists of all distributions $f \in \mathcal{S}'$, for which (quasi)norm

$$\| f \,|\, B^{s\,\tau}_{p\,q} \| = \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^{\tau}} \| (2^{sj} \Delta^{\eta}_j(f,x)(j+1-j(Q))^0_+) \,|\, \ell_q(L_p(Q)) \|$$

is finite;

II. the Lizorkin–Triebel type space $F^{s\,\tau}_{p\,q} := F^{s\,\tau}_{p\,q}(\mathbb{R}^n)$ $(p < \infty)$ consists of all distributions $f \in \mathcal{S}'$, for which (quasi)norm

$$\| f \,|\, F^{s\,\tau}_{p\,q} \| = \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^{\tau}} \| (2^{sj} \Delta^{\eta}_j(f,x)(j+1-j(Q))^0_+) \,|\, L_p(Q; \ell_q) \|$$

is finite.

Remark 1. Inhomogeneous spaces $B^{s\,\tau}_{p\,q}$ and $F^{s\,\tau}_{p\,q}$ were introduced in [7] and have been studied thoroughly (see, in particular, [5, 7–10]). We also noted that (local) Morrey spaces and Nikol'skii–Besov–Morrey and Lizorkin–Triebel–Morrey spaces have been attracting a lot of attention, see, for instance, [5, 7–14].

Definition 4. Let $s, \tau \in \mathbb{R}$, $0 < p, q \le \infty$. Then

I. the Nikol'skii–Besov type space $\widetilde{B}^{s\,\tau}_{p\,q} := B^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ consists of all distributions $f \in \tilde{\mathcal{S}}'$, for which (quasi)norm

$$\| f \,|\, B^{s\,\tau}_{p\,q}(\mathbb{T}^n) \| = \sup_{Q \in \widetilde{\mathcal{Q}}} \frac{1}{|Q|^{\tau}} \| (2^{sj} \widetilde{\Delta}^{\eta}_j(f,x)(j+1-j(Q))^0_+) \,|\, \ell_q(L_p(Q)) \|$$

is finite;

II. the Lizorkin–Triebel type space $\widetilde{F}^{s\,\tau}_{p\,q} := F^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ $(p < \infty)$ consists of all distributions $f \in \tilde{\mathcal{S}}'$, for which (quasi)norm

$$\| f \,|\, F^{s\,\tau}_{p\,q}(\mathbb{T}^n) \| = \sup_{Q \in \widetilde{\mathcal{Q}}} \frac{1}{|Q|^{\tau}} \| (2^{sj} \widetilde{\Delta}^{\eta}_j(f,x)(j+1-j(Q))^0_+) \,|\, L_p(Q; \ell_q) \|$$

is finite.

Remark 2. Obviously, the spaces $\widetilde{B}^{s\,0}_{p\,q}$ and $\widetilde{F}^{s\,0}_{p\,q}$ coincide with the isotropic periodic Nikol'skii–Besov spaces $\widetilde{B}^s_{p\,q}$ and Lizorkin–Triebel spaces $\widetilde{F}^s_{p\,q}$ respectively. Furthermore, it is not hard to see that for any $\tau \le 0$, we have coincidence $\widetilde{B}^{s\,\tau}_{p\,q} = \widetilde{B}^s_{p\,q}$ and $F^{s\,\tau}_{p\,q} = \widetilde{F}^s_{p\,q}$ in sense of equivalent (quasi)norms, in contrast to the spaces $B^{s\,\tau}_{p\,q}$ and $F^{s\,\tau}_{p\,q}$: as known, $B^{s\,\tau}_{p\,q} = \{0\}$ and $F^{s\,\tau}_{p\,q} = \{0\}$ when $\tau < 0$ (see [7]).

We noted that periodic Morrey spaces and Nikol'skii–Besov–Morrey and Lizorkin–Triebel–Morrey spaces have been attracting increasing attention as well, see, for instance, [15–18].

First, we consider the Gagliardo–Nirenberg type inequalities for the Lizorkin–Triebel type spaces $F^{s\,\tau}_{p\,q}(\mathbb{T}^n)$.

Theorem 1. Let $0 < q_0, q_1 \le \infty,\ -\infty < s_0 < s_1 < \infty,\ \tau_0, \tau_1 \ge 0$.

(i) Let $0 < p_0, p_1 < \infty$. Then for any $0 < \theta < 1$ and $0 < q \le \infty$, there exists constant $C > 0$ such that the inequality

$$\|f \mid F^{s\,\tau}_{p\,q}(\mathbb{T}^n)\| \le C\|f \mid F^{s_0\,\tau_0}_{p_0\,q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid F^{s_1\,\tau_1}_{p_1\,q_1}(\mathbb{T}^n)\|^{\theta}$$

is satisfied for all $f \in \mathcal{S}'(\mathbb{T}^n)$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\tau = (1-\theta)\tau_0 + \theta\tau_1$, $s = (1-\theta)s_0 + \theta s_1$.

(ii) Let $0 < p_0 < \infty$. Then for any $0 < \theta < 1$ and $0 < q \le \infty$, there exists constant $C > 0$ such that the inequality

$$\|f \mid F^{s\tau}_{pq}(\mathbb{T}^n)\| \le C\|f \mid F^{s_0\tau_0}_{p_0q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid B^{s_1\tau_1}_{\infty\infty}(\mathbb{T}^n)\|^{\theta}$$

holds for all $f \in \mathcal{S}'(\mathbb{T}^n)$, where $\frac{1}{p} = \frac{1-\theta}{p_0}$, $\tau = (1-\theta)\tau_0 + \theta\tau_1$, $s = (1-\theta)s_0 + \theta s_1$.

Remark 3. The proof given below is due to H. Brezis and P. Mironescu [4] for $\tau = 0$ and W. Sickel [5] for $\tau > 0$ in the non-periodic case of $\mathbb{R}^n$.

The Gagliardo–Nirenberg type inequalities for the Nikol'skii–Besov type spaces $B^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ are as follows.

Theorem 2. Let $0 < p_0, p_1 \le \infty$, $0 < q_0, q_1 \le \infty$, $-\infty < s_0 < s_1 < \infty$, $\tau_0,\ \tau_1 \ge 0$. Then for any $0 < \theta < 1$, there exists constant $C > 0$ such that the inequality

$$\|f \mid B^{s\,\tau}_{p\,q}(\mathbb{T}^n)\| \le C\|f \mid B^{s_0\,\tau_0}_{p_0\,q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid B^{s_1\,\tau_1}_{p_1\,q_1}(\mathbb{T}^n)\|^{\theta}$$

is valid for all $f \in \mathcal{S}'(\mathbb{T}^n)$ where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\tau = (1-\theta)\tau_0 + \theta\tau_1$, $s = (1-\theta)s_0 + \theta s_1$.

Remark 4. If we replace $\mathbb{T}^n$ by $\mathbb{R}^n$ in Theorems 1 and 2, we obtain an exact formulation of above–mentioned W. Sickel's results for $B^{s\,\tau}_{p\,q}(\mathbb{R}^n)$ and $F^{s\,\tau}_{p\,q}(\mathbb{R}^n)$.

2 Crucial Lemma

Key ingredient in what follows is the following inequality of F. Oru (see Lemma 3.7 in [4]).

Lemma 1. Let $0 < \theta < 1$, $-\infty < s_0, s_1 < \infty$, $s = (1-\theta)s_0 + \theta s_1$, $0 < q \le \infty$. Then there exists $C = C(s_0, s_1, \theta, q) > 0$ such that for any sequence $(a_j)_j$ of complex numbers the inequality

$$\|(2^{sj}a_j)_j \mid \ell_q\| \le C\|(2^{s_0 j}a_j)_j \mid \ell_\infty\|^{1-\theta}\|(2^{s_1 j}a_j)_j \mid \ell_\infty\|^{\theta} \tag{6}$$

holds true.

For completeness, we present the proof of Lemma 1 from [4].

Proof. Let $C_1 = \sup 2^{s_1 j} \mid a_j \mid$, $C_2 = \sup 2^{s_2 j} \mid a_j \mid$, so that $C_1 \le C_2$. We will assume that $C_1 > 0$, otherwise there is nothing to prove. Since $s_1 < s_2$, there exists some $j_0 > 0$ such that

$$\min\left\{\frac{C_1}{2^{s_1 j}}, \frac{C_2}{2^{s_2 j}}\right\} = \begin{cases} \frac{C_1}{2^{s_1 j}}, & j \le j_0, \\ \frac{C_2}{2^{s_2 j}}, & j > j_0. \end{cases}$$

Since $\frac{C_1}{2^{s_1 j}} \le \frac{C_2}{2^{s_2 j}}$ and $\frac{C_2}{2^{s_2(j_0+1)}} \le \frac{C_1}{2^{s_1(j_0+1)}}$, we find

$$C_2 \sim C_1 2^{(s_2-s_1)j_0}.$$

Therefore,

$$\|(2^{s_1 j}a_j) \mid \ell_\infty\|^{\theta}\|(2^{s_2 j}a_j) \mid \ell_\infty\|^{1-\theta} \sim C_1 2^{(s_2-s_1)j_0(1-\theta)}. \tag{7}$$

On the other hand, we have $a_j \le \min\left\{\frac{C_1}{2^{s_1 j}}, \frac{C_2}{2^{s_2 j}}\right\}$, so that

$$a_j \le \frac{C_1}{2^{s_1 j}} \ \text{ for } 0 \le j \le j_0, \quad a_j \le \frac{C_2}{2^{s_2 j}} \text{ for } j > j_0.$$

It follows that

$$\|(2^s_j a_j) \,|\, \ell_q\| \le \left(\sum_{j \le j_0} C_1^q 2^{(s-s_1)jq} + \sum_{j > j_0} C_2^q 2^{(s-s_2)jq} \right)^{1/q} \le$$

$$\le C \left(\sum_{j \le j_0} C_1^q 2^{(s-s_1)jq} + \sum_{j > j_0} C_1^q 2^{-\theta(s_2-s_1)jq+(s_2-s_1)j_0 q} \right)^{1/q}.$$

Therefore,

$$\|(2^s_j a_j) \,|\, \ell_q\| \le C C_1 2^{(s_2-s_1)j_0(1-\theta)}.$$

Finally, we find that the inequality

$$\|(2^{sj} a_j) \,|\, \ell_q\| \le C C_1 2^{(s_2-s_1)j_0(1-\theta)}. \tag{8}$$

Now (6) follows from (7) and (8). Thus, Lemma 1 is completely proved.

3 Proofs of Theorems 1 and 2

Proof of Theorem 1.

Proof. As mentioned above, the line of argument follows [4]. First, we prove `(i)`. Successively applying Lemma 1 with $a_j = |\widetilde{\Delta}^\eta_j(f,x)|$, Holder's integral inequality (with exponents $P_0 = \frac{p_0}{(1-\theta)p}$ and $P_1 = \frac{p_1}{\theta p}$) and Jensen's inequalities ($\|\cdot \,|\, \ell_{q_0}\| \ge \|\cdot \,|\, \ell_\infty\|$ and $\|\cdot \,|\, \ell_{q_1}\| \ge \|\cdot \,|\, \ell_\infty\|$), we find

$$\|f \mid F^{s\tau}_{pq}(\mathbb{T}^n)\| \le c\|f \mid F^{s_0\tau_0}_{p_0\infty}(\mathbb{T}^n)\|^{1-\theta}\|f \mid F^{s_1\tau_1}_{p_1\infty}(\mathbb{T}^n)\|^\theta \le$$

$$\le C\|f \mid F^{s_0\tau_0}_{p_0 q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid F^{s_1\tau_1}_{p_1 q_1}(\mathbb{T}^n)\|^\theta,$$

thus part `(i)` is established.

Now we turn to proof of part `(ii)`. It follows from the condition that $p_1 = \infty$, $p_0 < p < \infty$. Therefore, successively applying Lemma 1 with $a_j = |\widetilde{\Delta}^\eta_j(f,x)|$, the inequality $\| g \,|\, L_p(Q)\| \le (\| g \,|\, L_{p_0}(Q)\|)^{1-\theta}(\| g \,|\, L_\infty(Q)\|)^\theta$ and further arguing as in case `(i)`, we obtain

$$\|f \mid F^{s\tau}_{pq}(\mathbb{T}^n)\| \le c\|f \mid F^{s_0\tau_0}_{p_0\infty}(\mathbb{T}^n)\|^{1-\theta}\|f \mid F^{s_1\tau_1}_{\infty\infty}(\mathbb{T}^n)\|^\theta \le$$

$$\le C\|f \mid F^{s_0\tau_0}_{p_0 q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid F^{s_1\tau_1}_{\infty\infty}(\mathbb{T}^n)\|^\theta \equiv$$

$$\equiv C\|f \mid F^{s_0\tau_0}_{p_0 q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid B^{s_1\tau_1}_{\infty\infty}(\mathbb{T}^n)\|^\theta,$$

Thus, part `(ii)` is also obtained.

Proof of Theorem 2.

Proof. Here, successively applying the Holder inequality for integrals (with exponents $P_0 = \frac{p_0}{(1-\theta)p}$ and $P_1 = \frac{p_1}{\theta p}$), the Holder inequality for series $\left(\text{with exponents } Q_0 = \frac{q_0}{(1-\theta)q} \text{ and } Q_1 = \frac{q_1}{\theta q}\right)$ and using elementary properties of suprema, we obtain

$$\|f \mid B^{s\tau}_{pq}(\mathbb{T}^n)\| \equiv \sup_{Q \in \widetilde{Q}} \frac{1}{|Q|^\tau} \left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{jsp} |\widetilde{\Delta}^\eta_j(f,x)|^p dx \right]^{q/p} \right\}^{1/q} \le$$

$$\leq \sup_{Q\in\widetilde{Q}} \frac{1}{|Q|^{\tau}}\left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{js_0p_0}|\widetilde{\Delta}_j^{\eta}(f,x)|^{p_0}dx\right]^{(q(1-\theta))/p_0} \left[\int_Q 2^{js_1p_1}|\widetilde{\Delta}_j^{\eta}(f,x)|^{p_1}dx\right]^{q\theta/p_1}\right\}^{1/q} \leq$$

$$\leq \sup_{Q\in\widetilde{Q}} \frac{1}{|Q|^{\tau}}\left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{js_0p_0}|\widetilde{\Delta}_j^{\eta}(f,x)|^{p_0}dx\right]^{q_0/p_0}\right\}^{(1-\theta)/q_0} \times$$

$$\times \sup_{Q\in\widetilde{Q}} \frac{1}{|Q|^{\tau_1\theta}}\left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{js_1p_1}|\widetilde{\Delta}_j^{\eta}(f,x)|^{p_1}dx\right]^{q_1/p_1}\right\}^{\theta/q_1} \leq$$

$$\leq \sup_{Q\in\widetilde{Q}} \frac{1}{|Q|^{\tau_0(1-\theta)}}\left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{js_0p_0}|\widetilde{\Delta}_j^{\eta}(f,x)|^{p_0}dx\right]^{q_0/p_0}\right\}^{(1-\theta)/q_0} \times$$

$$\times \sup_{Q\in\widetilde{Q}} \frac{1}{|Q|^{\tau_1\theta}}\left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{js_1p_1}|\widetilde{\Delta}_j^{\eta}(f,x)|^{p_1}dx\right]^{q_1/p_1}\right\}^{\theta/q_1} =$$

$$= \|f \mid B^{s_0,\tau_0}_{p_0,q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid B^{s_1,\tau_1}_{p_1,q_1}(\mathbb{T}^n)\|^{\theta}.$$

Thus, Theorem 2 is completely proved as well.

Acknowledgments

The authors are grateful to the reviewer for his attentive attitude and comments, thanks to which the article has changed for the better.

Author Contributions

All authors contributed equally to this work. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Arestov, V.V., & Akopyan, R.R. (2020). Zadacha Stechkina o nailuchshem priblizhenii neogranichennogo operatora ogranichennymi i rodstvennye zadachi [Stechkin's problem on the best approximation of unbounded operators by bounded ones and related problems.] *Trudy Instituta Matematiki i Mekhaniki UrO RAN, 26*(4), 7–31 [in Russian]. https://doi.org/10.21538/0134-4889-2020-26-4-7-31

2 Arestov, V.V. (2023). Approximation of differentiation operators by bounded linear operators in Lebesgue spaces on the axis and related problems in the spaces of (p, q)-multipliers and their predual spaces. *Ural Mathematical Journal, 9*(2), 4–27. https://doi.org/10.15826/umj.2023.2.001

3 Besov, O.V., Ilyin, V.P., & Nikol'skii, S.M. (1996). *Integralnye predstavleniia funktsii i teoremy vlozheniia [Integral representations of functions and embedding theorems]. 2nd ed., revised and enlarged.* Moscow: Nauka [in Russian].

4 Brezis, H., & Mironescu, P. (2001). Gagliardo–Nirenberg, composition and products in fractional Sobolev spaces. *Journal of Evolution Equations, 1*, 387–404. https://doi.org/10.1007/PL00001378

5 Sickel, W. (2013). Smoothness spaces related to Morrey spaces — a survey. II. *Eurasian Mathematical Journal, 4*(1), 82–124.

6 Frazier, M., & Jawerth, B. (1990). A discrete transform and decompositions of distribution spaces. *Journal of Functional Analysis, 93*, 34–171. https://doi.org/10.1016/0022-1236(90)90137-A

7 Yuan, W., Sickel, W., & Yang, D. (2010). *Morrey and Campanato meet Besov, Lizorkin and Triebel: Lecture Notes in Math.* (Vol. 2005). Berlin: Springer. https://doi.org/10.1007/978-3-642-14606-0

8 Sickel, W. (2012). Smoothness spaces related to Morrey spaces — a survey. I. *Eurasian Mathematical Journal, 3*, 104–143.

9 Yuan, W., Haroske, D.D., Moura, S.D., Skrzypczak, L., & Yang, D. (2015). Limiting embeddings in smoothness Morrey spaces, continuity envelopes and applications. *Journal of approximation theory, 192*, 306–335. https://doi.org/10.1016/j.jat.2014.12.006

10 Haroske, D.D., & Triebel, H. (2023). Morrey smoothness spaces: A new approach. *Science China Mathematics, 66*(6), 1301–1358. https://doi.org/10.1007/s11425-021-1960-0

11 Haroske, D.D., & Moura, S.D. (2016). Some specific unboundedness property in smoothness Morrey spaces. The non-existence of growth envelopes in the subcritical case. *Acta Mathematica Sinica, English Series, 32*, 137–152. https://doi.org/10.1007/s10114-016-5104-4

12 Haroske, D.D., & Skrzypczak, L. (2020). Entropy numbers of compact embeddings of smoothness Morrey spaces on bounded domains. *Journal of approximation theory, 256*, 105424. https://doi.org/10.1016/j.jat.2020.105424

13 Haroske, D.D., Moura, S.D., & Skrzypczak, L. (2020). Some Embeddings of Morrey Spaces with Critical Smoothness. *Journal of Fourier Analysis and Applications, 26*, Article number 50. https://doi.org/10.1007/s00041-020-09758-2

14 Senouci, M.A. (2021). Boundedness of the generalized Riemann–Liouville operator in local Morrey type spaces. *Eurasian Mathematical Journal, 14*(4), 63–68. https://doi.org/10.32523/2077-9879-2023-14-4-63-68

15 Baituyakova, Z., & Sickel, W. (2017). Strong summability of Fourier series and generalized Morrey spaces. *Analysis Mathematica, 43*(3), 371–414. https://doi.org/10.1007/s10476-017-0401-4

16 Burenkov, V.I., & Joseph, D.J. (2024). Inequalities for trigonometric polynomials in periodic Morrey spaces. *Eurasian Mathematical Journal, 15*(2), 92–100. https://doi.org/10.32523/2077-9879-2024-15-2-92-100

17 Sautbekova, M., & Sickel, W. (2014). Strong summability of Fourier series and Morrey spaces. *Analysis Mathematica, 40*(1), 31–62. https://doi.org/10.1007/s10476-014-0104-z

18 Balgimbayeva, Sh., & Bazarkhanov, D. (2024). Optimal cubature formulas for Morrey type function classes on multidimensional torus. *Eurasian Mathematical Journal, 15*(3), 25–37. https://doi.org/10.32523/2077-9879-2024-15-3-25-37

*Author Information**

Sholpan Albanovna Balgimbayeva (*corresponding author*) — Candidate of physical and mathematical sciences, Associate Professor, Leading Researcher, Department of Function Theory and Functional Analysis, Institute of Mathematics and Mathematical Modeling, 28 Shevchenko str., Almaty, 050010, Kazakhstan; e-mail: *sholpan.balgyn@gmail.com*; https://orcid.org/0000-0001-5770-0161

Aizere Kendebaykyzy Janabilova — Junior Researcher, Department of Function Theory and Functional Analysis, Institute of Mathematics and Mathematical Modeling, 28 Shevchenko str., Almaty, 050010, Kazakhstan; e-mail: *zh.aizere2001@gmail.com*

*The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 63–70

https://doi.org/10.31489/2025M1/63-70 *Research article*

Solitary Wave Solutions of the coupled Kawahara Equation

K. Bharatha[1,*], R. Rangarajan[2], C.J. Neethu[1]

[1] *St. Philomena's College, Mysore, India;*
[2] *University of Mysore, Mysore, India*
(E-mail: bharatha.k12@gmail.com, rajra63@gmail.com, neethucjohnson02@gmail.com)

The field of nonlinear differential equations have made significant contribution in understanding nonlinear dynamics and its complex phenomenon. One such evolution equation is Kawahara equation, which has gained its importance in plasma physics and allied fields. Many researchers are interested to work on their soliton, multi-solitons solutions and to study other properties such as stability, integrability, conservation laws and so on. The aim of the paper is to study the Coupled Kawahara equation and to deduce its soliton solutions. The coupled equation is treated with the ansatz method and the tanh method to compute soliton solutions. The novelty of this work is to demonstrate the fact, that the derived system efficiently gives two governing equations admitting solitary wave solutions. Further, in the coupled equation, one equation has the nonlinear term vv_x addition to the Kawahara equation, while the other is the modified Kawahara equation. Scope for future works is also highlighted.

Keywords: Evolution Equation, Bounded solutions, the Ansatz method, the Tanh method.

2020 Mathematics Subject Classification: 35L55, 35Q51.

Introduction

The study of nonlinear dynamics has significantly advanced our understanding of various physical phenomena through nonlinear partial differential equations. One prominent area within this field is solitary wave theory. The concept of solitary waves was first observed empirically by John Scott Russell in 1844 [1]. Later, in 1965, Korteweg and de Vries formulated the mathematical representation of these waves, now known as the KdV equation. This third-order nonlinear differential equation, involving spatial derivatives, has found extensive applications in areas such as shallow water wave theory, ocean engineering, optics, and related disciplines [2–5].

The Kawahara equation is a significant evolution equation used to model various physical phenomena, including plasma dynamics and gravity waves on viscous liquid surfaces. It also describes magneto-acoustic wave behavior in plasma and the dynamics of long water waves beneath ice-covered surfaces [3–9]. Essentially, the Kawahara equation extends the KdV equation by incorporating a fifth-order term. However, unlike the KdV equation, it is not integrable, as it does not appear in Hietarinta's classification of integrable systems [10].

In [8], the governing model for waves in dispersive media was introduced. In [11], travelling wave solutions for the Kawahara equation and its modified form were derived. A comparison of two numerical approaches for solving the Kawahara equation was presented in [12]. Solitary wave solutions for the modified Kawahara equation were explored in [13], while the soliton solution for the generalized Kawahara equation was provided in [14]. Additionally, solitary wave solutions for the Hirota-Satsuma coupled KdV equation have been studied in [15, 16].

There are many methods to solve the nonlinear evolution equations namely, the Adomian decomposition method, the Homotopy perturbation method, the Hirota's Bilinear method, the Bilinear neural network method, Lie symmetry analysis, the tanh method and so on [17–19].

*Corresponding author. *E-mail: bharatha.k12@gmail.com*
Received: 13 November 2023; *Accepted:* 10 December 2024.

In our work, first we obtain the coupled Kawahara equation by transforming it into a function of complex variable. Further, the solitons of the coupled equation are computed using the ansatz method and the tanh method. Computed solutions are simulated using Maple. Further, analysis of solutions is carried out, which conveys that the transformed coupled equation resembles Kawahara type and the modified Kawahara equation.

1 Soliton Solutions

The Kawahara equation is the extension of KdV equation with higher order dispersion term p_{xxxxx} which reads as [4, 8, 14] :

$$p_t + 6pp_x + p_{xxx} - p_{xxxxx} = 0\,;\;\; x, t > 0 \in \mathbb{R}. \tag{1}$$

By considering,

$$p(x,t) = u(x,t) + i\; v(x,t),$$

in equation (1) results in the coupled Kawahara equation:

$$u_t + 6uu_x - 6vv_x + u_{xxx} - u_{xxxxx} = 0, \tag{2}$$

$$v_t + 6uv_x + 6vu_x + v_{xxx} - v_{xxxxx} = 0. \tag{3}$$

If $v = 0$, the above system of equations (2) and (3) will reduce to the well known Kawahara equation.

To study this coupled system and its soliton solution, we use the ansatz method and the tanh method in the following section.

1.1 The ansatz method

As noted earlier, $v = 0$ results in the Kawahara equation and its soliton solution is of the form sech^4 we begin with the ansatz,

$$u(x,t) = A\ \mathrm{sech}^M k(x - ct), \tag{4}$$

$$v(x,t) = B\ \mathrm{sech}^N k(x - ct), \tag{5}$$

where $M, N \in \mathbb{N}$; $k, c, A > 0$ and $B > 0$ are scalars from the field $\mathbb{R}$.

The restriction of A and B to be positive is to retain the coupled system.

By balancing the higher nonlinear term and higher linear term, we obtain $M = 4$ and $N = 2$. Substituting this M and N in equation (4) and (5), we get

$$\begin{aligned} u(x,t) &= A\ \mathrm{sech}^4 k(x - ct), \\ v(x,t) &= B\ \mathrm{sech}^2 k(x - ct). \end{aligned} \tag{6}$$

Now, substituting the above expressions u, v of (6) into the equation (2), we obtain,

$$\begin{aligned} &[Ac - 6A^2 + 14Ak^2 + 376Ak^4 + 3B^2] \\ &+ [6A^2 - 30Ak^2 - 120Ak^4]\ \tanh^2 k(x - ct) \\ &+ [6A^2 - 1680Ak^4]\ \tanh^2 k(x - ct)\ \mathrm{sech}^2 k(x - ct) = 0. \end{aligned}$$

As the set $\{1, \tanh^2 k(x - ct), \tanh^2 k(x - ct)\ \mathrm{sech}^2 k(x - ct)\}$ is linearly independent, it leads to,

$$\begin{aligned} Ac - 6A^2 + 14Ak^2 + 376Ak^4 + 3B^2 &= 0, \\ 6A^2 - 30Ak^2 - 120Ak^4 &= 0, \\ 6A^2 - 1680Ak^4 &= 0. \end{aligned}$$

Solving the above system gives the values of A and k as

$$A = 280k^4,\ k = \pm\frac{1}{2\sqrt{13}},\ \text{and}\ c = \frac{-1014B^2}{35} + \frac{36}{169},\ \text{with}\ c < \frac{36}{169}.$$

Therefore, solutions are

$$\begin{aligned} u_1(x,t) &= 280k^4\text{sech}^4\left\{\pm\frac{1}{2\sqrt{13}}\left(x - \Big(\frac{-1014B^2}{35} + \frac{36}{169}\Big)t\right)\right\} \text{ and} \\ v_1(x,t) &= B\ \text{sech}^2\left\{\pm\frac{1}{2\sqrt{13}}\left(x - \Big(\frac{-1014B^2}{35} + \frac{36}{169}\Big)t\right)\right\}. \end{aligned} \tag{7}$$

Now, using u and v of (6) in equation (3), we observe that

$$\begin{aligned} &[Bc - 18AB + 8Bk^2 + 136Bk^4] \\ &+ [18AB - 12Bk^2 - 120Bk^4]\ \tanh^2 k(x-ct) \\ &+ [18AB - 360Bk^4]\tanh^2 k(x-ct)\ \text{sech}^2 k(x-ct)) = 0. \end{aligned}$$

This in turn implies the system of equations

$$\begin{aligned} Bc - 18AB + 8Bk^2 + 136Bk^4 &= 0, \\ 18AB - 12Bk^2 - 120Bk^4 &= 0, \\ 18AB - 360Bk^4 &= 0. \end{aligned}$$

Solving the above system, we obtain

$$A = 20k^4,\ B = B,\ c = \frac{4}{25} \text{ and } k = \pm\frac{1}{2\sqrt{5}}.$$

Therefore, the corresponding solutions for above values are given by

$$\begin{aligned} u_2(x,t) &= 20k^4\text{sech}^4\left(\pm\frac{1}{2\sqrt{5}}\left(x - \frac{4}{25}t\right)\right) \text{ and} \\ v_2(x,t) &= B\ \text{sech}^2\left(\pm\frac{1}{2\sqrt{5}}\left(x - \frac{4}{25}t\right)\right). \end{aligned} \tag{8}$$

1.2 The tanh method

In this subsection, we replicate the soliton solutions that are obtained using the ansatz method using the tanh method. For more details refer [4, 20–23].

By introducing a new variable $z = x - ct$ in (2) and (3), we obtain

$$-cU^{(1)} + 6UU^{(1)} - 6VV^{(1)} + U^{(3)} - U^{(5)} = 0, \tag{9}$$

$$-cV^{(1)} + 6(UV)^{(1)} + V^{(3)} - V^{(5)} = 0. \tag{10}$$

The above equations (9) and (10) can be integrated once to get,

$$-cU + 3U^2 - 3V^2 + U^{(2)} - U^{(4)} = 0, \tag{11}$$

$$-cV + 3UV + V^{(2)} - V^{(4)} = 0. \tag{12}$$

Let $U(Y) = \sum_{j=0}^{M} a_j Y^j$ and $V(Y) = \sum_{j=0}^{N} b_j Y^j$, where $U^{(n)} = \dfrac{d^n U}{dZ^n}$, $Y = \tanh Z$, $M, N \in \mathbb{N}$, a_j and b_j are real coefficients need to be determined.

By balancing the power of highest order of the derivative and nonlinear terms: $U^{(4)}$ and U^2 of (11), we obtain $M = 4$.

Analogously, balancing powers for the equation (12), we obtain $N = 2$. So, we have

$$U(Y) = a_0 + a_1 Y + a_2 Y^2 + a_3 Y^3 + a_4 Y^4, \tag{13}$$

$$V(Y) = b_0 + b_1 Y + b_2 Y^2. \tag{14}$$

Using equation (13) in (9), we obtain the following system of equations:

$$\begin{aligned}
-ca_0 + 3a_0^2 - 3b_0^2 + 2a_2k^2 + 16a_4k^4 &= 0,\\
-ca_1 + 6a_0a_1 + 6a_3k^2 - 2a_1k^2 - 16a_1k^4 + 120a_4k^4 &= 0,\\
-ca_2 + 6a_0a_2 + 3a_1^2 + 12a_4k^2 - 8a_2k^2 - 136a_2k^4 + 480a_4k^4 &= 0,\\
-ca_3 + 6a_0a_3 + 6a_1a_2 + 2a_1k^2 - 18a_3k^2 - 576a_3k^4 + 40a_1k^4 &= 0,\\
-ca_4 + 6a_0a_4 + 6a_1a_3 + 3a_2^2 + 6a_2k^2 - 32a_4k^2 - 1696a_4k^4 + 240a_2k^4 &= 0,\\
6a_1a_4 + 6a_2a_3 + 12a_3k^4 - 24a_1k^4 + 816a_3k^4 &= 0,\\
6a_2a_4 + 3a_3^2 + 20a_4k^2 - 120a_2k^4 + 2080a_4k^4 &= 0,\\
6a_3a_4 - 360a_3k^4 &= 0,\\
3a_4^2 - 840a_4k^4 &= 0.
\end{aligned}$$

Solving the above system of equations, we obtain

$$\begin{aligned}
&c = -\frac{1}{39}\left[\frac{495040k^8 + 31360k^6 + 280k^4 + 117b_0^2 - 117a_0^2}{a_0}\right],\\
&a_1 = 0, a_2 = -\frac{1120}{3}k^4 - \frac{140}{39}k^2, a_3 = 0, a_4 = 280k^4,\\
&b_1 = 0, a_0 \neq 0, b_2, \text{ and } b_0 \text{ are arbitrary constants.}
\end{aligned}$$

By fixing $a_0 = a_4$, $a_2 = -2a_4$ and $b_2 = -b_0$, results in $k = \pm\dfrac{1}{2\sqrt{13}}$ and $c = \dfrac{36}{169} - \dfrac{1014}{35}b_0^2$, which agrees with the ansatz method.

$$u_1(x,t) = 280k^4 \text{sech}^4\left\{\pm\frac{1}{2\sqrt{13}}\left(x - \left(\frac{-1014b_0^2}{35} + \frac{36}{169}\right)t\right)\right\} \text{ and}$$

$$v_1(x,t) = b_0 \text{ sech}^2\left\{\pm\frac{1}{2\sqrt{13}}\left(x - \left(\frac{-1014b_0^2}{35} + \frac{36}{169}\right)t\right)\right\}.$$

Now, using (14) in (12), we obtain system of equations:

$$\begin{aligned}
-cb_0 + 6a_0b_0 + 2b_2k^2 + 16b_2k^4 &= 0,\\
-cb_1 + 6a_0b_1 + 6a_1b_0 - 2b_1k^2 - 16b_1k^4 &= 0,\\
-cb_2 + 6a_0b_2 + 6a_1b_1 + 6a_2b_0 - 8b_2k^2 - 136b_2k^4 &= 0,\\
6(a_1b_2 + a_2b_1 + b_0a_3) + 2b_1k^2 + 40b_1k^4 &= 0,\\
6(a_4b_0 + a_2b_2 + a_3b_1 + b_2k^2) + 240b_2k^4 &= 0,\\
6(a_3b_2 + a_4b_1) - 24b_1k^4 &= 0,\\
6a_4b_2 - 120b_2k^4 &= 0.
\end{aligned}$$

Solving the above system of equations, we obtain

$$a_0 = \frac{8}{3}k^4 + \frac{1}{3}k^2 + \frac{1}{6}c, a_1 = 0, a_2 = -(20k^2+1)k^2,$$
$$a_3 = 0,\ a_4 = 20k^4,\ b_1 = 0,$$
$$b_0,\ b_2,\ \text{and } c \text{ are arbitrary constants.}$$

By fixing $a_0 = a_4$ and $a_2 = -2a_4$, will result in $k = \pm\frac{1}{2\sqrt{5}}$ and $c = \frac{4}{25}$. Hence,

$$u_2(x,t) = 20k^4\text{sech}^4\left(\pm\frac{1}{2\sqrt{5}}\left(x - \frac{4}{25}t\right)\right) \text{ and}$$
$$v_2(x,t) = b_0\ \text{sech}^2\left(\pm\frac{1}{2\sqrt{5}}\left(x - \frac{4}{25}t\right)\right).$$

2 Plots of the Solutions

In this subsection, we simulate the solutions of the coupled equation using maple package [24].

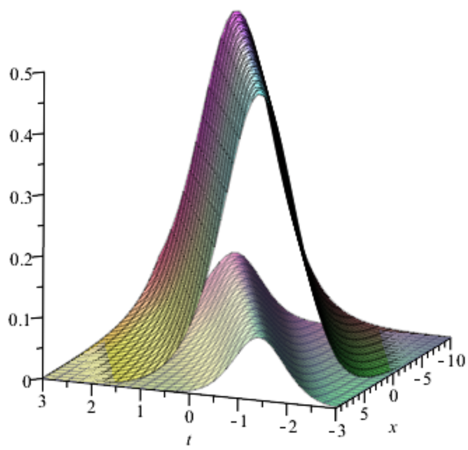

Figure 1. Plots of (7) with $b = \frac{1}{2}$

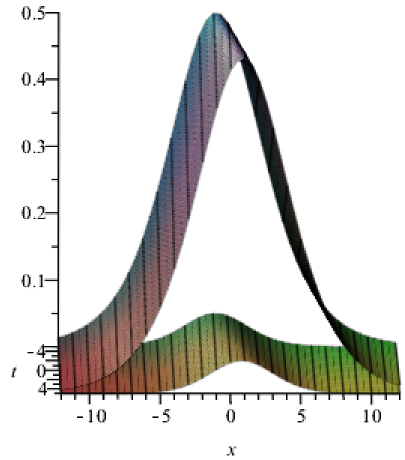

Figure 2. Plots of (8) with $b = \frac{1}{2}$

3 Analysis of Solutions

We observe that, the solutions $u(x,t)$ and $v(x,t)$ are related by $u = \frac{A}{B^2}v^2$, then equation (3) will reduce to $v_t + 18\frac{A}{B^2}v^2v_x + v_{xxx} - v_{xxxxx} = 0$.

So, the coupled equation (2) and (3) will be of the form,

$$u_t + 6uu_x - 6vv_x + u_{xxx} - u_{xxxxx} = 0, \tag{15}$$

$$v_t + 18\frac{A}{B^2}v^2v_x + v_{xxx} - v_{xxxxx} = 0. \tag{16}$$

Equation (15) is the Kawahara type equation with the additional term vv_x to the Kawahara equation and equation (16) is the modified Kawahara equation. Further, one can observe that the solutions simulated in Figure 1 and Figure 2, for a particular choice $b = \frac{1}{2}$. The solution given in equation (7) indicated by Figure 1 has a slightly high amplitude compared to the solution given by equation (8), which is depicted in Figure 2.

4 Discussion

In conclusion, transforming the Kawahara equation to a coupled system results in two different governing equations in which one is Kawahara type equation and the other is the modified Kawahara equation. As a scope for further work, one can compute the other solutions such as periodic solution, shock solution and singular solution to the discussed system.

Author Contributions

R. Rangarajan suggested the equation and validated the computed solutions. K. Bharatha contributed to the application of the tanh method and established the conditions for obtaining the coupled system. C.J. Neethu developed the ansatz method and performed the simulation of the solutions. All authors participated in revising the manuscript and approved its final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Russell, J.S. (1845). Report on Waves. *Report of the 14th meeting of the British Association for the advancement of Science.* York, London, 311–391.

2 Ablowitz, M.J., & Clarkson, P.A. (1991). *Solitons, nonlinear evolution equations and inverse scattering.* Cambridge University Press, Cambridge.

3 Drazin, P.G., & Johnson, R.S. (1992). *Solitons: an Introduction.* Cambridge University Press.

4 Wazwaz, A.M. (2009). *Partial Differential Equations and Solitary Waves Theory.* Springer.

5 Hosseini, K., Mirzazadeh, M., Salahshour, S., Baleanu, D., & Zafar, A. (2022). Specific wave structure of fifth-order nonlinear water wave equation. *Journal of Ocean Engineering and Science, 7*, 462–466. https://doi.org/10.1016/j.joes.2021.09.019

6 Hounkonnou, M.N., & Mahaman, M.K. (2008). Symmetry, integrability and solutions of the Kawahara equation. *SUT Journal of Mathematics, 44*(1), 39–53. https://doi.org/10.55937/sut/1217621941

7 Kalliadasis, S., Ruyer-Quil, C., Scheid, B., & Velarde, M.G. (2012). *Falling Liquid Films.* Springer-Verlag.

8 Kawahara, T. (1972). Oscillatory Solitary Waves in Dispersive Media. *Journal of the Physical Society of Japan, 33*(1), 260–264. https://doi.org/10.1143/JPSJ.33.260

9 Polyanin, A.D., & Zaitsev, V.F. (2004). *Handbook of nonlinear partial differential equations*, CRC Press.

10 Hietarinta, J. (1987). A search for bilinear equations passing Hirota's three soliton condition I. KdV type bilinear equations. *Journal of Mathematical Physics, 28*(8), 1732–1742. https://doi.org/10.1063/1.527815

11 Sirendaoreji (2004). New Exact travelling wave solutions for the Kawahara equation and the modified Kawahara equations. *Chaos Solitons Fractals, 19*(1), 147–150. https://doi.org/10.1016/S0960-0779(03)00102-4

12 Kaya, D., & Al-Khaled, K. (2007). A numerical Comparison of a Kawahara equation. *Physics Letters A, 363*(5-6), 433–439. https://doi.org/10.1016/j.physleta.2006.11.055

13 Wazwaz, A.M. (2007). New solitary wave solutions to the modified Kawahara equation. *Physics Letters A, 360*, 588–592. https://doi.org/10.1016/j.physleta.2006.08.068

14 Biswas, A. (2009). Solitary wave solution for the generalized Kawahara equation. *Applied Mathematics Letters, 22*(2), 208–210. https://doi.org/10.1016/j.aml.2008.03.011

15 Hirota, R., & Satsuma, J. (1981). Soliton solutions of a coupled Kotreweg-de Vries Equation. *Physics Letters A, 85*(8-9), 407–408. https://doi.org/10.1016/0375-9601(81)90423-0

16 Bharatha, K., & Rangarajan, R. (2023). A New Perturbation method to Solve Hirota-Satsuma Coupled KdV Equation. *AIP Conference Proceedings, 2875*(1). https://doi.org/10.1063/5.0154093

17 Rangarajan, R., & Bharatha, K. (2024). A KdV-Type equation: Lax Pair and travelling wave solution. *The Mathematics Student, 93*(1-2), 03–13.

18 Bharatha, K., & Rangarajan, R. (2023). Multi-Soliton Solutions to the Generalized Boussinesq Equation of Tenth Order. *Computational methods for differential equations, 11*(4), 727–737. https://doi.org/10.22034/cmde.2023.55261.2297

19 Tuan, N.M., Koonprasert, S., Sirisubtawee, S., & Meesad, P. (2024). The bilinear neural network method for solving Benney-Luke equation. *Partial Differential Equations in Applied Mathematics, 10*. https://doi.org/10.1016/j.padiff.2024.100682

20 Malfliet, W., & Hereman, W. (1996). The tanh method: I, Exact solutions of nonlinear evolution and wave equations. *Physica Scripta, 54*(6), 563–568.

21 Malfliet, W. (2004). The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations. *Journal of Computational and Applied Mathematics, 164-165*, 529–541. https://doi.org/10.1016/S0377-0427(03)00645-9

22 Wazwaz, A.M. (2004). The tanh method for travelling wave solutions of nonlinear equations. *Applied Mathematics and Computation, 154*(3), 713–723. https://doi.org/10.1016/S0096-3003(03)00745-8

23 Wazwaz, A.M. (2007). The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations. *Applied Mathematics and Computation, 184*(2), 1002–1014. https://doi.org/10.1016/j.amc.2006.07.002

24 Shingareva, I., & Celaya, C.L. (2011). *Solving Nonlinear Partial Differential Equations with Maple and Mathematica*. Springer.

Author Information

Kalegowda Bharatha (*corresponding author*) — Doctor of Philosophy in Mathematics, Assistant Professor, Department of Mathematics, St. Philomena's College, Mysore, 570015, India; e-mail: *bharatha.k12@gmail.com*

Raghavachar Rangarajan — Doctor of Philosophy in Mathematics, Senior Professor, Department of Studies in Mathematics, University of Mysore, 570006, India; e-mail: *rajra63@gmail.com*; https://orcid.org/0000-0001-6251-5518

Chakkallakal Johnson Neethu — Post Graduate in Mathematics, Assistant Professor, Department of Mathematics, St. Philomena's College, Mysore, 570015, India; e-mail: *neethucjohnson02@gmail.com*

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 71–80

https://doi.org/10.31489/2025M1/71-80 *Research article*

Solution of the model problem of heat conduction with Bessel operator

N.K. Gulmanov*, S.S. Kopbalina, A.O. Tanin

Karaganda Buketov University, Karaganda, Kazakhstan
(E-mail: gulmanov.nurtay@gmail.com, kopbalina@mail.ru, tan.alibek@mail.ru)

In this work, a model boundary value problem for a parabolic equation with a Bessel operator was investigated. The solution to the problem under consideration is sought as a sum of thermal potentials: the double-layer and volume potentials, which reduces the problem to a Volterra integral equation of the second kind. The questions of existence and uniqueness of the obtained integral equation were investigated. The existence condition for the solution to the given problem was found. It is shown that if this condition is fulfilled, the problem has a single solution. The problem considered in this paper is called a model problem because the region in which the solution of the problem is sought is cylindrical and its results will be used in solving boundary value problems for the parabolic equation in noncylindrical regions having different order of degeneracy of the solution region to a point at the initial moment of time.

Keywords: heat equation, boundary value problem, Bessel operator, cylindrical domain, double layer thermal potential, thermal volume potential, Volterra integral equation, Laplace transform, homogeneous and inhomogeneous integral equation, resolvent.

2020 Mathematics Subject Classification: 35K05, 45D99.

Introduction

In modern conditions, the rapid advancement of contact technology and increasing electrical device speeds make precise temperature field measurement in contact systems particularly important. In addition, it is important to study the dynamics of temperature field changes in time. When studying temperature processes in high-current contacts, it is necessary to take into account changes in the dimensions of the contact area, which occur both under the influence of electrodynamic forces and due to melting of the contact material at high temperatures.

During the electrode opening process, the temperature at the contact surface reaches the melting point, resulting in the formation of a liquid metal bridge between the electrodes. As further opening occurs, the bridge separates, causing material transfer from one electrode to the other. This process, known as bridge erosion, can significantly affect the performance of the contact system.

A distinctive characteristic of such problems, from a mathematical perspective, is the presence of a movable boundary in the solution domain, along with the fact that, at the initial moment, the contacts are closed, causing the solution domain to degenerate into a point. The solution of such thermal problems requires the application of generalized thermal potentials and the subsequent transformation of the initial boundary value problem to Volterra-type integral equations. In some cases, for example, when the order of degeneration of the region to a point is high enough, the integral equations will be singular, namely, the classical method of successive approximations is not applicable to them [1–14].

Earlier we considered boundary value problems for parabolic equations with Bessel operator in the domain $Q = \{(r,t)|\, 0 < r < t^{\omega},\ t > 0\}$ at $\omega > \frac{1}{2}$. The problem considered in this paper is called a

*Corresponding author. *E-mail: gulmanov.nurtay@gmail.com*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488729, 2024–2026).

Received: 31 August 2024; *Accepted:* 10 December 2024.

model problem because the domain in which the solution of the problem is sought is cylindrical and its solution will be used in solving the problem in the case when the boundary of the domain will change according to the law $x = t^{\omega},\ 0 < \omega < \frac{1}{2}$.

1 Problem statement

In the region $Q = \{(r,t)|\ 0 < r < 1,\ 0 < t < T\}$, the following boundary value problem is considered:

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{1-2\beta}{r} \cdot \frac{\partial u}{\partial r} + a^2 \cdot \frac{\partial^2 u}{\partial r^2} + f(r,t), \tag{1}$$

$$u(r,t)|_{r=0} = 0,\ \ t > 0, \tag{2}$$

$$u(r,t)|_{r=1} = 0,\ \ t > 0, \tag{3}$$

$$u(r,t)|_{t=0} = 0, \tag{4}$$

where $0 < \beta < 1,\ f(r,t)$ is a given function.

2 Fundamental solution for equation (1)

In the domain $Q^{\infty} = \{(r,t)|\ r > 0,\ t > 0\}$ consider the boundary value problem for the homogeneous equation

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{1-2\beta}{r} \cdot \frac{\partial u}{\partial r} + a^2 \cdot \frac{\partial^2 u}{\partial r^2}, \tag{5}$$

corresponding to the inhomogeneous equation (1) of the basic boundary value problem, at boundary conditions

$$u(r,t)|_{r=0} = 0,\ \ t > 0, \tag{6}$$

$$u(r,t)|_{r=\infty} = 0,\ \ t > 0, \tag{7}$$

and the initial condition

$$u(r,t)|_{t=0} = \frac{\delta(r-\xi)}{r^{1-2\beta}}, \tag{8}$$

where $\delta(z)$ is the Dirac delta function, $\xi > 0$. Applying to the problem (5)–(8) the Laplace transform on the variable t, we obtain the boundary value problem for the ordinary differential equation

$$\frac{\partial^2 \hat{u}}{\partial r^2} + \frac{1-2\beta}{r} \cdot \frac{\partial \hat{u}}{\partial r} - \frac{p}{a^2} \cdot \hat{u} = -\frac{\delta(r-\xi)}{a^2 r^{1-2\beta}} \tag{9}$$

with boundary conditions

$$\hat{u}(r,p)|_{r=0} = 0, \tag{10}$$

$$\hat{u}(r,p)|_{r=\infty} = 0. \tag{11}$$

This boundary value problem (9)–(11) has a single solution $\hat{u}(r,p) = \hat{G}(r,p,\xi)$, where

$$\hat{G}(r,p,\xi) = \begin{cases} \frac{r^{\beta}\cdot\xi^{\beta}}{a^2} \cdot K_{\beta}\left(\frac{\xi\sqrt{p}}{a}\right) \cdot I_{\beta}\left(\frac{r\sqrt{p}}{a}\right),\ 0 < r < \xi, \\ \frac{\xi^{\beta}\cdot r^{\beta}}{a^2} \cdot I_{\beta}\left(\frac{\xi\sqrt{p}}{a}\right) \cdot K_{\beta}\left(\frac{r\sqrt{p}}{a}\right),\ \xi < r < \infty, \end{cases}$$

where $I_{\beta}(z),\ K_{\beta}(z)$ are cylindrical functions of imaginary argument of order β (Infeld and McDonald functions). The function $\hat{G}(r,p,\xi)$ belongs to the class of Laplace transform images. Performing its inversion [15; 350], we obtain

$$G(r,\xi,t) = \frac{1}{2a^2} \cdot \frac{r^{\beta} \cdot \xi^{1-\beta}}{t} \cdot \exp\left[-\frac{r^2+\xi^2}{4a^2 t}\right] \cdot I_{\beta}\left(\frac{r\xi}{2a^2 t}\right).$$

Let us replace the variable t in the function $G(r,\xi,t)$ by $(t-\tau)$, then

$$G(r,\xi,t-\tau)=\frac{1}{2a^2}\cdot\frac{r^{\beta}\cdot\xi^{1-\beta}}{t-\tau}\cdot\exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right]\cdot I_{\beta}\left(\frac{r\xi}{2a^2(t-\tau)}\right),$$

which has the following properties:

$$\lim_{r\to 0}G(r,\xi,t-\tau)\ =0,\quad \tau<t,\quad \xi>0,$$

$$\lim_{r\to \infty}G(r,\xi,t-\tau)\ =0,\quad \tau<t,\quad \xi>0,$$

$$\lim_{\tau\to t}G(r,\xi,t-\tau)\ =0,\quad r\neq\xi,$$

$$\lim_{\tau\to t}\int_0^{\infty}G(r,\xi,t-\tau)\cdot r^{1-2\beta}dr\ =1.$$

This function will be used to construct the thermal potential of the double layer in the domain $Q=\{(r,t)|\ 0<r<\infty,\ 0<t<T\}$:

$$W(r,t)=2a^2\int_0^t\left.\frac{\partial G(r,\xi,t-\tau)}{\partial\xi}\right|_{\xi=1}\cdot g(\tau)d\tau,$$

and thermal volume potential in the region $Q=\{(r,t)|\ 0<r<\infty,\ 0<t<T\}$:

$$F(r,t)=\int_0^t d\tau\int_0^1 f(\xi,\tau)\cdot G(r,\xi,t-\tau)\cdot\xi^{1-2\beta}d\xi.$$

Remark 1. The density $f(r,t)$ is defined and continuous in the domain $\{(r,t)|\ 0<r\leq 1,\ 0<t<T\}$, and inside the domain there is an estimate:

$$|f(r,t)|\leq M\cdot r^{\gamma},\ \ M=\text{const},\ \ \gamma>-2+\beta. \tag{12}$$

The following properties are valid for the function $F(r,t)$.

1. The function $F(r,t)$ is defined and continuous in the domain $Q=\{(r,t)|\ 0<r<\infty,\ 0<t<T\}$ and for any values $t>0$ the equality is true

$$\lim_{r\to 0}F(r,t)\ =0.$$

2. Everywhere in the region $Q=\{(r,t)|\ 0<r<\infty,\ 0<t<T\}$ there exists and is continuous the derivative $\frac{\partial F}{\partial r}$, which is defined as follows:

$$\frac{\partial F}{\partial r}=\int_0^t d\tau\int_0^1 f(\xi,\tau)\cdot\frac{\partial G(r,\xi,t-\tau)}{\partial r}\cdot\xi^{1-2\beta}d\xi.$$

3. In the domain $Q=\{(r,t)|\ 0<r<\infty,\ 0<t<T\}$ there exists and is continuous the derivative $\frac{\partial F}{\partial t}$, which is defined by the equality

$$\frac{\partial F}{\partial t}=\int_0^t d\tau\int_0^1 f(\xi,\tau)\cdot\frac{\partial G(r,\xi,t-\tau)}{\partial t}\cdot\xi^{1-2\beta}d\xi+f(r,t).$$

3 Reduction of the boundary value problem (1)–(4) to the Volterra integral equation

As we found out, the fundamental solution for equation (1) is the function

$$G(r,\xi,t-\tau)=\frac{1}{2a^2}\cdot\frac{r^{\beta}\cdot\xi^{1-\beta}}{t-\tau}\cdot\exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right]\cdot I_{\beta}\left(\frac{r\xi}{2a^2(t-\tau)}\right),$$

where ξ is a parameter, $0<\beta<1$, $I_{\beta}(z)$ is a modified Bessel function of order β. The solution of problem (1)–(4) is found as a sum of the thermal double-layer potential and the volume potential:

$$u(r,t)=\int_0^t \left.\frac{\partial G(r,\xi,t-\tau)}{\partial\xi}\right|_{\xi=1}\mu(\tau)d\tau+\int_0^t \left.\frac{\partial G(r,\xi,t-\tau)}{\partial\xi}\right|_{\xi=0}\nu(\tau)d\tau+F(r,t),$$

where

$$F(r,t)=\int_0^t d\tau\int_0^1 f(\xi,\tau)\cdot G(r,\xi,t-\tau)\cdot\xi^{1-2\beta}d\xi,$$

and the densities $\mu(t)$ and $\nu(t)$ are to be defined. Using the fact that

$$\left.\frac{\partial G(r,\xi,t-\tau)}{\partial\xi}\right|_{\xi=0}=\frac{1}{(2a^2)^{\beta+1}}\cdot\frac{r^{2\beta}}{2^{\beta}(t-\tau)^{\beta+1}}\cdot\frac{1}{\beta\Gamma(\beta)}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right],$$

and

$$\left.\frac{\partial G(r,\xi,t-\tau)}{\partial\xi}\right|_{\xi=1}=\frac{r^{\beta}(r-1)}{4a^4(t-\tau)^2}\cdot\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\cdot\exp\left[-\frac{r}{2a^2(t-\tau)}\right]I_{\beta}\left(\frac{r}{2a^2(t-\tau)}\right)+$$
$$+\frac{r^{\beta+1}}{4a^4(t-\tau)^2}\cdot\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r}{2a^2(t-\tau)}\right]I_{\beta-1,\beta}\left(\frac{r}{2a^2(t-\tau)}\right)+$$
$$+\frac{r^{\beta}(1-2\beta)}{2a^2(t-\tau)}\cdot\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\cdot\exp\left[-\frac{r}{2a^2(t-\tau)}\right]\cdot I_{\beta}\left(\frac{r}{2a^2(t-\tau)}\right),$$

where the designation

$$I_{\beta-1,\beta}(z)=I_{\beta-1}(z)-I_{\beta}(z),$$

we obtain the integral representation of the solution of the equation:

$$\begin{aligned}u(r,t)=\int_0^t\Bigg\{&\frac{r^{\beta}(r-1)}{4a^4t(t-\tau)^2}\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r}{2a^2(t-\tau)}\right]I_{\beta}\left(\frac{r}{2a^2(t-\tau)}\right)+\\&+\frac{r^{\beta+1}}{4a^4(t-\tau)^2}\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r}{2a^2(t-\tau)}\right]I_{\beta-1,\beta}\left(\frac{r}{2a^2(t-\tau)}\right)+\\&+\frac{r^{\beta}(1-2\beta)}{2a^2(t-\tau)}\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r}{2a^2(t-\tau)}\right]\cdot I_{\beta}\left(\frac{r}{2a^2(t-\tau)}\right)\Bigg\}\mu(\tau)d\tau+\\&+\int_0^t\frac{1}{(2a^2)^{\beta+1}}\cdot\frac{r^{2\beta}}{2^{\beta}(t-\tau)^{\beta+1}}\cdot\frac{1}{\beta\Gamma(\beta)}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\cdot\nu(\tau)d\tau+F(r,t),\end{aligned}\tag{13}$$

where

$$\mu(t)\in L_{\infty}(0,\infty).\tag{14}$$

Using the boundary condition (2) for (13), we determine that the density

$$\nu(t) = 0.$$

Then

$$u(r,t) = \int_0^t \left\{ \frac{r^\beta (r-1)}{4a^4 (t-\tau)^2} \exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r}{2a^2(t-\tau)}\right] I_\beta \left(\frac{r}{2a^2(t-\tau)}\right) + \right.$$
$$+ \frac{r^{\beta+1}}{4a^4(t-\tau)^2} \exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r}{2a^2(t-\tau)}\right] I_{\beta-1,\beta} \left(\frac{r}{2a^2(t-\tau)}\right) + \tag{15}$$
$$\left. + \frac{r^\beta(1-2\beta)}{2a^2(t-\tau)} \exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r}{2a^2(t-\tau)}\right] \cdot I_\beta \left(\frac{r}{2a^2(t-\tau)}\right) \right\} \mu(\tau) d\tau +$$
$$+ F(r,t).$$

Using the boundary condition (3), we obtain the integral equation with respect to the unknown density $\mu(t)$:

$$\mu(t) - \int_0^t \sum_{i=1}^{2} N_i(t,\tau) \cdot \mu(\tau) d\tau = f(t), \tag{16}$$

where

$$N_1(t,\tau) = \frac{1-2\beta}{t-\tau} \exp\left[-\frac{1}{2a^2(t-\tau)}\right] \cdot I_\beta \left(\frac{1}{2a^2(t-\tau)}\right),$$

$$N_2(t,\tau) = \frac{1}{2a^2(t-\tau)^2} \exp\left[-\frac{1}{2a^2(t-\tau)}\right] I_{\beta-1,\beta} \left(\frac{1}{2a^2(t-\tau)}\right),$$

$$f(t) = F(t,t).$$

The solution of this integral equation, if it exists in the class of functions (14), is singular and can be found by the method of successive approximations, since the estimates [16] are valid:

$$0 < e^{-z} \cdot I_\beta(z) < \frac{C_1}{\sqrt{z}}, \ 0 < e^{-z} \cdot I_{\beta-1,\beta}(z) < \frac{C_1}{\sqrt{z^3}}, \ C_1, \ C_2 = \text{const}.$$

To clarify the question of existence of a solution to equation (16), we use the method of integral Laplace transform.

4 Solution of the integral equation (16)

Let us apply the Laplace transform to both parts of the integral equation (16):

$$\widehat{\mu}(p) \left\{ 1 - \left[\widehat{N_1}(p) + \widehat{N_2}(p) \right] \right\} = \hat{f}(p), \ \text{Re} p > 0,$$

$$\widehat{\mu}(p) = \frac{\hat{f}(p)}{1 - \left[\widehat{N_1}(p) + \widehat{N_2}(p) \right]}. \tag{17}$$

In order to find the image of the function $\widehat{N_1}(p) + \widehat{N_2}(p)$ we will use:
1) formula (29.169) [15; 350];

2) the property: let $f(t) \risingdotseq \hat{f}(p)$, then $\frac{1}{t}f(t) \risingdotseq \int_p^\infty \hat{f}(p)dp$ [17; 506]. Then we have:

$$\widehat{N_1}(p) = 2(1-2\beta)K_\beta\left(\frac{\sqrt{p}}{a}\right)I_\beta\left(\frac{\sqrt{p}}{a}\right), \ \mathrm{Re}p > 0,$$

$$\widehat{N_2}(p) = \frac{1}{a^2}\int_p^\infty \left[K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right)I_{\beta-1}\left(\frac{\sqrt{p}}{a}\right) - K_\beta\left(\frac{\sqrt{p}}{a}\right)I_\beta\left(\frac{\sqrt{p}}{a}\right)\right]dp =$$

$$= 1 - 2\frac{\sqrt{p}}{a}I_\beta\left(\frac{\sqrt{p}}{a}\right)K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right), \ \mathrm{Re}p > 0.$$

Let us show that the homogeneous integral equation

$$\mu(t) - \int_0^t \sum_{i=1}^2 N_i(t,\tau)\cdot\mu(\tau)d\tau = 0 \tag{18}$$

has only zero solution in the class of functions $\mu(t) \in L_\infty(0,\infty)$. For this purpose, let us find the roots of the equation

$$1 - \left[\widehat{N_1}(p) + \widehat{N_2}(p)\right] = 0$$

or

$$2I_\beta\left(\frac{\sqrt{p}}{a}\right)\cdot\left\{\frac{\sqrt{p}}{a}K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right) - (1-2\beta)K_\beta\left(\frac{\sqrt{p}}{a}\right)\right\} = 0. \tag{19}$$

Let $I_\beta\left(\frac{\sqrt{p}}{a}\right) = 0$ in equality (19). According to the definition of Bessel function of imaginary argument $I_\beta\left(\frac{\sqrt{p}}{a}\right) = e^{-\frac{\pi}{2}\beta i}J_\beta\left(\frac{i\sqrt{p}}{a}\right)$, where $J_\beta\left(\frac{i\sqrt{p}}{a}\right)$ is a cylindrical Bessel function of the first kind. The function $J_\beta\left(\frac{i\sqrt{p}}{a}\right)$ has infinitely many valid roots for any valid β; if $\beta > -1$, all its roots are valid and equal to $i\frac{\sqrt{p_k}}{a} = \alpha_k, \ p_k = -a^2\alpha_k^2, \ \alpha_k \in \mathbb{R}, \ k \in \mathbb{Z}\setminus\{0\}$ [18], which contradicts the $\mathrm{Re}p > 0$ condition.

It is clear that the second multiplier at $\frac{1}{2} < \beta < 1$

$$\frac{\sqrt{p}}{a}K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right) - (1-2\beta)K_\beta\left(\frac{\sqrt{p}}{a}\right) \neq 0,$$

and at $0 < \beta < \frac{1}{2}$ it has a single root $p = p_0 > 0$. It follows that in this case the solution of the homogeneous equation (18) is the function $\mu_0(t) = C\cdot e^{p_0t}$, which does not belong to the class (14). Thus, it is shown that the homogeneous integral equation (18) has only zero solution.

It follows from equality (17) at $0 < \beta < \frac{1}{2}$ that if the function $\hat{f}(p)$ goes to zero at the point p_0, then the expression

$$\frac{\hat{f}(p)}{1 - \left[\widehat{N_1}(p) + \widehat{N_2}(p)\right]}$$

has no poles and in this case equation (16) will have a single solution in the class of functions (14). Thus, for solvability of equation (16) at $0 < \beta < \frac{1}{2}$ it is necessary and sufficient to fulfill the condition

$$\int_0^\infty e^{-p_0t}f(t)dt = 0.$$

If $\frac{1}{2} < \beta < 1$, then equation (16) is unconditionally solvable.

Let this condition be satisfied. Let us find the solution of the inhomogeneous integral equation. For this purpose, let us represent (17) in the following form:

$$\widehat{\mu}(p) = \hat{f}(p) + \hat{R}(p) \cdot \hat{f}(p),$$

where

$$\hat{R}(p) = \frac{\widehat{N_1}(p) + \widehat{N_2}(p)}{1 - \left[\widehat{N_1}(p) + \widehat{N_2}(p)\right]} = \frac{1 - 2I_\beta\left(\frac{\sqrt{p}}{a}\right)\left[\frac{\sqrt{p}}{a}K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right) - (1-2\beta)K_\beta\left(\frac{\sqrt{p}}{a}\right)\right]}{2I_\beta\left(\frac{\sqrt{p}}{a}\right)\left[\frac{\sqrt{p}}{a}K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right) - (1-2\beta)K_\beta\left(\frac{\sqrt{p}}{a}\right)\right]}.$$

Let us use the properties of [15; 191]:

1. If $\varphi(t) \risingdotseq \widehat{\varphi}(p)$, then

$$\varphi(\alpha t) \risingdotseq \frac{1}{\alpha}\widehat{\varphi}(\frac{p}{\alpha}), \ \alpha > 0. \tag{20}$$

2. If $\widehat{\varphi}(p) \risingdotseq \varphi(t)$, then

$$\widehat{\varphi}(\sqrt{p}) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{t^{\frac{3}{2}}}\int_0^\infty \tau \cdot e^{-\frac{\tau^2}{4t}}\varphi(\tau)d\tau. \tag{21}$$

For convenience we introduce the notation $\frac{\sqrt{p}}{a} = z$ and find the original expression

$$\widehat{R^*}(z) = \frac{1 - 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right]}{2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right]}.$$

According to [17; 519]:

$$\widehat{R^*}(z) = \frac{A(z)}{B(z)} \risingdotseq \sum_{-\infty}^{+\infty}\frac{A(z_k)}{B'(z_k)}e^{-z_k y},$$

where z_k are zeros of the function

$$B(z) = 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right].$$

1) Let $y_\beta(z) = zK_{\beta-1}(z) - (1-2\beta)K_\beta(z) = 0$. This equation, as noted earlier, has one root z_0 at $0 < \beta < \frac{1}{2}$.

2) Let $I_\beta(z) = e^{-\frac{\pi}{2}\beta i}J_\beta(iz) = 0$. Therefore, $iz_k = \alpha_k$ or $z_k = -i\alpha_k$, where $\alpha_k \in \mathbb{R}$.

Then

$$\widehat{R^*}(z) = \frac{A(z)}{B(z)} \risingdotseq \sum_{-\infty}^{+\infty}\frac{A(z_k)}{B'(z_k)}e^{-z_k y} = \sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{A(z_k)}{B'(z_k)}e^{-z_k y} + \frac{A(z_0)}{B'(z_0)}e^{-z_0 y} = R^*_-(y),$$

where

$$B(z) = 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right],$$

$$B'(z) = 2I_{\beta-1}(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right] + 2(1-2\beta)I_\beta(z)K_{\beta-1}(z) + \\ + \left(\frac{4\beta(1-2\beta)}{z} - 2z\right)I_\beta(z)K_\beta(z).$$

Thus, we obtained that at $0 < \beta < \frac{1}{2}$:

$$R^*(y) = \sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{e^{-z_k y}}{2I_{\beta-1}(z_k)\left[z_kK_{\beta-1}(z_k) - (1-2\beta)K_\beta(z_k)\right]} + \frac{e^{-z_0 y}}{2I_\beta(z_0)K_{\beta-1}(z_0)\left[1 - \frac{1}{1-2\beta}z_0^2\right]}. \tag{22}$$

Let us introduce the following notations:

$$A_{\beta,k} = \frac{1}{2I_{\beta-1}(z_k)\left[z_k K_{\beta-1}(z_k) - (1-2\beta)K_\beta(z_k)\right]}, \quad A_{\beta,0} = \frac{1}{2I_\beta(z_0)K_{\beta-1}(z_0)\left[1 - \frac{1}{1-2\beta}z_0^2\right]}.$$

From equality (22) and properties of (20) and (21) we have:

$$\hat{R}\left(\frac{\sqrt{p}}{a}\right) \risingdotseq R(t) = \frac{a^2}{2\sqrt{\pi}t^{\frac{3}{2}}} \cdot \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \cdot \int_0^\infty \tau e^{-\frac{\tau^2}{4t} - ia^2\alpha_k\tau} d\tau + \frac{a^2}{2\sqrt{\pi}t^{\frac{3}{2}}} \cdot A_{\beta,0} \cdot \int_0^\infty \tau e^{-\frac{\tau^2}{4t} - z_0 a^2\tau} d\tau,$$

where α_k are zeros of the function $J_\beta(z)$. For the resolvent $R(t)$ the following estimation is valid

$$R(t) \le \frac{a^2\pi}{4\sqrt{t}}.$$

Remark 2. At $0 < \beta < \frac{1}{2}$ it follows from the equality (17) that for solvability of the integral equation (15) it is necessary and sufficient to fulfill the condition

$$\int_0^\infty e^{-p_0 t} f(t)dt = 0, \tag{23}$$

where $f(t) = \lim_{r\to t} F(r,t)$.

Theorem 1. For any function $f(t) \in C(0,T)$, equation (16) has a single solution if $\frac{1}{2} < \beta < 1$. When $0 < \beta < \frac{1}{2}$, it is necessary and sufficient for the solvability of the integral equation (16) that condition (23) is satisfied. In this case, for any function $f(t) \in C(0,T)$, the integral equation (16) has a single solution.

Remark 3. If at $0 < \beta < \frac{1}{2}$ condition (23) is not satisfied, then equation (16) has no solutions in the chosen class of functions. However, this result does not contradict the well-known fact that the Volterra equation always has a single solution. Equation (16) belongs to the class of Volterra-type equations of the second kind and, therefore, in case the condition (23) is not satisfied, it will also be solvable, but in a wider space of functions with exponential growth.

5 Solution of the boundary value problem (1)–(4)

Theorem 2. For any function $f(r,t)$ from the class (12), the boundary value problem (1)–(4):

1) for $\frac{1}{2} < \beta < 1$ it has a single solution $u(r,t) \in C(0,T)$;

2) when $0 < \beta < \frac{1}{2}$, it is necessary and sufficient to fulfill condition (23) for the existence of a solution. If this condition is satisfied, the problem has a single solution in the class of functions $u(r,t) \in C(0,T)$.

Conclusion

In this work we study a model boundary value problem for a parabolic equation with a Bessel operator. The existence condition for the solution to this problem at $0 < \beta < \frac{1}{2}$ is found. It is shown that if this condition is fulfilled, the problem has a single solution. If $\frac{1}{2} < \beta < 1$, the problem is unconditionally solvable. The results of this work will be used in solving boundary value problems for parabolic equations in non-cylindrical regions having different order of degeneration of the solution region to a point at the initial moment of time.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Kavokin, A.A., Kulakhmetova, A.T., & Shpadi, Y.R. (2018). Application of thermal potentials to the solution of the problem of heat conduction in a region degenerates at the initial moment. *Filomat, 32*(3), 825–836. https://doi.org/10.2298/FIL1803825K

2 Amangaliyeva, M.M., Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity. *Differential Equations, 47*(2), 231–243. https://doi.org/10.1134/S0012266111020091

3 Jenaliyev, M.T, Amangaliyeva, M.M, Kosmakova, M.T, & Ramazanov, M.I. (2015). On a Volterra equation of the second kind with "incompressible" kernel. *Advances in Difference Equations, 71*, 1–14.

4 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). On one homogeneous problem for the heat equation in an infinite angular domain. *Sib. Math. Jour., 56*(6), 982–995. https://doi.org/10.1134/S0037446615060038

5 Jenaliyev, M.T., & Ramazanov, M.I. (2018). On a homogeneous parabolic problem in an infinite corner domain. *Filomat, 32*(3), 965–974. https://doi.org/10.2298/FIL1803965J

6 Ramazanov, M.I., & Gulmanov, N.K. (2021). O singuliarnom integralnom uravnenii Volterra kraevoi zadachi teploprovodnosti v vyrozhdaiushcheisia oblasti [On the singular Volterra integral equation of the boundary value problem for heat conduction in a degenerating domain]. *Vestnik Udmurtskogo universiteta. Matematika. Mekhanika. Kompiuternye nauki — The Bulletin of Udmurt University. Mathematics. Mechanics. Computer Science, 31*(2), 241–252 [in Russian]. https://doi.org/10.35634/vm210206

7 Ramazanov, M.I., Jenaliyev, M.T., & Gulmanov, N.K. (2022). Solution of the boundary value problem of heat conduction in a cone. *Opuscula Mathematica, 42*(1), 75–91. https://doi.org/10.7494/OpMath.2022.42.1.75

8 Ramazanov, M.I., & Gulmanov, N.K. (2021). Solution of a two-dimensional boundary value problem of heat conduction in a degenerating domain. *Journal of Mathematics, Mechanics and Computer Science, 111*(3), 65–78. https://doi.org/10.26577/JMMCS.2021.v111.i3.06

9 Pskhu, A.V., Ramazanov, M.I., Gulmanov, N.K., & Iskakov, S.A. (2022). Boundary value problem for fractional diffusion equation in a curvilinear angle domain. *Bulletin of the Karaganda university. Mathematics Series,* 1(105), 83–95. https://doi.org/10.31489/2022M1/83-95

10 Ramazanov, M.I., Kosmakova, M.T., & Tuleutaeva, Z.M. (2021). On the Solvability of the Dirichlet Problem for the Heat Equation in a Degenerating Domain. *Lobachevskii Journal of Mathematicsthis, 42*, 3715–3725. https://doi.org/10.1134/S1995080222030179

11 Jenaliyev, M.T., Assetov, A.A., & Yergaliyev, M.G. (2021). On the Solvability of the Burgers Equation with Dynamic Boundary Conditions in a Degenerating Domain. *Lobachevskii Journal of Mathematics, 42*, 3661–3674. https://doi.org/10.1134/S199508022203012X

12 Yuldashev, T.K., & Rakhmonov, F.D. (2021). On a Benney-Luke Type Differential Equation with Nonlinear Boundary Value Conditions. *Lobachevskii Journal of Mathematicsthis, 42*, 3761–3772. https://doi.org/10.1134/S1995080222030210

13 Zarifzoda, S.K., Yuldashev, T.K., & Djumakhon, I. (2021). Volterra-Type Integro-Differential Equations with Two-Point Singular Differential Operator. *Lobachevskii Journal of Mathematics, 42*, 3784–3792. https://doi.org/10.1134/S1995080222030234

14 Yumagulov, M.G., Ibragimova, L.S., & Belova, A.S. (2021). First Approximation Formulas in the Problem of Perturbation of Definite and Indefinite Multipliers of Linear Hamiltonian Systems. *Lobachevskii Journal of Mathematics, 42*, 3773–3783. https://doi.org/10.1134/S1995080222030222

15 Ditkin, V.A., & Prudnikov, A.P. (1965). *Spravochnik po operatsionnomu ischisleniiu [Handbook of Operational Calculus]*. Moscow: High School [in Russian].

16 Gradshteyn, I.S., & Ryzhik, I.M. (2014). *Table of Integrals, Series, and Products.* Academic Press.

17 Lavrent'ev, M.A., & Shabat, B.V. (1973). *Metody teorii funktsii kompleksnogo peremennogo [Methods of the theory of functions of a complex variable]*. Moscow: Nauka [in Russian].

18 Watson, G.N. (1949). *Teoriia besselevykh funktsii [Theory of Bessel functions]*. Moscow: IL [in Russian].

Author Information

Nurtay Kudaibergenovich Gulmanov (*corresponding author*) — PhD, Senior Lecturer of the Department of Mathematical Analysis and Differential Equations, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *gulmanov.nurtay@gmail.com*; https://orcid.org/0000-0002-4159-1551

Saltanat Serikovna Kopbalina —PhD student, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *kopbalina@mail.ru*; https://orcid.org/0009-0003-5468-5973

Alibek Orlanovich Tanin — PhD, Associative Professor of the Department of Mathematical Analysis and Differential Equations, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *tan.alibek@mail.ru*; https://orcid.org/0000-0002-5687-3711

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 81–91

https://doi.org/10.31489/2025M1/81-91 *Research article*

Solution of nonlocal boundary value problems for the heat equation with discontinuous coefficients, in the case of two discontinuity points

U.K. Koilyshov[1,2,*], M.A. Sadybekov[1,2], K.A. Beisenbayeva[3]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Al-Farabi Kazakh National University, Almaty, Kazakhstan;*
[3] *Academy of Logistics and Transport, Almaty, Kazakhstan*
(E-mail: koylyshov@math.kz, sadybekov@math.kz, beisenbaeva@mail.ru)

In this paper, the solution of the initial-boundary value problem for the heat equation with a discontinuous coefficient under periodic or antiperiodic boundary conditions in the case of two discontinuity points is substantiated using the method of separation of variables. Using the replacement, the problem under consideration is reduced to a self-adjoint problem. By means of the Fourier method, this problem is reduced to the corresponding spectral problem. Then, the eigenvalues and eigenfunctions of this self-adjoint spectral problem are found. In conclusion, the main theorem on the existence and uniqueness of the classical solution to the problem under consideration is proved. The peculiarity of the problem under consideration is the non-local boundary conditions and the presence of two discontinuity points, which have not been considered before. The authors were able to find eigenvalues explicitly and construct eigenfunctions. This technique is also applicable in the case of more than two discontinuity points. The solution obtained in explicit form can be further used for numerical calculations.

Keywords: Heat equation with discontinuous coefficients, spectral problem, non-self-adjoint problem, Riesz basis, classical solution, Fourier method.

2020 Mathematics Subject Classification: 35K10, 35K20, 35P10.

Introduction. Problem statement and research metods

We consider an initial boundary value problem for the heat equation with a discontinuity constant coefficient

$$\frac{\partial u_j}{\partial t} = k_j^2 \frac{\partial^2 u_j}{\partial x^2} \tag{1}$$

in the domain $\Omega = \cup\Omega_j$, $\Omega_j = \{(x,t) : l_{j-1} < x < l_j,\ 0 < t < T\}$ $(j = 1, 2, 3)$, with the initial condition

$$u(x, 0) = \varphi(x), \quad l_0 \leq x \leq l_3, \tag{2}$$

boundary conditions of the form

$$\begin{cases} u_1(l_0, t) + e^{i\pi\theta} u_3(l_3, t) = 0, \\ k_1 \dfrac{\partial u_1(l_0, t)}{\partial x} + e^{i\pi\theta} k_3 \dfrac{\partial u_3(l_3, t)}{\partial x} = 0, \end{cases} \quad 0 \leq t \leq T \tag{3}$$

and with conjugation conditions

$$u_j(l_j - 0, t) = u_{j+1}(l_j + 0, t), \quad 0 \leq t \leq T, \quad j = 1, 2, \tag{4}$$

*Corresponding author. *E-mail: koylyshov@math.kz*

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19679487).

Received: 12 July 2024; *Accepted:* 26 November 2024.

$$k_j\frac{\partial u_1(l_j-0,t)}{\partial x}=k_{j+1}\frac{\partial u_{j+1}(l_j+0,t)}{\partial x},\quad 0\le t\le T,\quad j=1,2,\tag{5}$$

where $l_0<l_1<l_2<l_3$, the coefficients $k_j>0,\ \theta=1,2$.

Parabolic equations with discontinuous coefficients have been studied quite well [1–3]. In these works, the correctness of various initial-boundary value problems for a parabolic equation with discontinuous coefficients was proved using the Green function method and method of thermal potentials. In the absence of a discontinuity, the spectral theory of these problems has been constructed almost completely [4–6]. In [7], some properties of the eigenfunctions of the Sturm-Liouville operator with discontinuous coefficients were studied. In the case of a discontinuous coefficient, the spectral theory of such problems is considered in [8–12].

Works devoted to solving problems of multilayer diffusion should be especially noted. Mathematical models of diffusion in layered materials arise in many industrial, ecological, biological, medical applications and the theory of thermal conductivity of composite materials. Diffusion in several layers is used in a wide range of heat and mass transfer areas [13–21].

Let W be the linear variety of functions from the class $u(x,t)\in C(\overline{\Omega})\cap C^{2,1}(\overline{\Omega_1})\cap C^{2,1}(\overline{\Omega_2})\cap C^{2,1}(\overline{\Omega_3})$ which satisfy all conditions (2)–(4). A function $u(x,t)$ from the class $u(x,t)\in W$ will be called a classical solution to problem (1)–(5) if: 1) it is continuous in the domain $\overline{\Omega}$; 2) it has continuous first-order derivatives with respect to t and continuous second-order derivatives with respect to x in the domain; 3) it satisfies equation (1) and all conditions (2)–(5) in the usual, continuous sense.

First let's consider the case $\theta=1$. After the next replacement $u_j(x,t)=v_j(y,t)$, where

$$y=\begin{cases}\dfrac{x-l_0}{k_1}, & l_0<x<l_1,\\ \dfrac{x-l_1}{k_2}, & l_1<x<l_2,\\ \dfrac{x-l_2}{k_3}, & l_2<x<l_3,\end{cases}\tag{6}$$

problem (1)–(5) take the following form:

$$\frac{\partial v_j}{\partial t}=\frac{\partial^2 v_j}{\partial y^2}\tag{7}$$

in the domain $D_j=\{(y,t):0<y<h_j,0<t<T\}$ $(j=1,2,3)$,

$$v_j(y,0)=\psi_j(y),\quad 0\le y\le h_j,\tag{8}$$

$$\begin{cases}v_1(0,t)-v_3(h_3,t)=0,\\ \dfrac{\partial v_1(0,t)}{\partial y}-\dfrac{\partial v_3(h_3,t)}{\partial y}=0,\end{cases}\quad 0\le t\le T,\tag{9}$$

$$v_1(h_1,t)=v_2(0,t),\quad v_2(h_2,t)=v_3(0,t),\quad 0\le t\le T,\tag{10}$$

$$\frac{\partial v_1(h_1,t)}{\partial y}=\frac{\partial v_2(0,t)}{\partial y},\quad \frac{\partial v_2(h_2,t)}{\partial y}=\frac{\partial v_3(0,t)}{\partial y},\quad 0\le t\le T,\tag{11}$$

where

$$h_j=\frac{l_j-l_{j-1}}{k_j},\quad \psi_j(y)=\varphi_j(k_jy+l_{j-1}),\quad j=1,2,3.\tag{12}$$

To solve problem (7)–(11), we apply the Fourier method: $v_j(y,t)=Y_j(y)\cdot T(t)\neq 0$.

Substituting $v_j(y,t)=Y_j(y)\cdot T(t)$ into equation (7) and conditions (8)–(11), and separating the variables, we obtain the following spectral problem

$$LY(y) = \begin{cases} -Y''(x), & 0 < y < h_1 \\ -Y''(x), & 0 < y < h_2 \\ -Y''(x), & 0 < y < h_3 \end{cases} = \lambda Y(y), \tag{13}$$

$$\begin{cases} Y_1(0) - Y_3(h_3) = 0, \\ Y_1'(0) - Y_3'(h_3) = 0, \end{cases} \tag{14}$$

$$Y_1(h_1) = Y_2(0), \quad Y_2(h_2) = Y_3(0), \quad Y_1'(h_1) = Y_2'(0), \quad Y_2'(h_2) = Y_3'(0). \tag{15}$$

The function $T(t)$ is a solution to the equation

$$T'(t) + \lambda T(t) = 0.$$

The following holds:

Lemma 1. Spectral problem (13)–(15) is self-adjoint.

The proof is carried out by direct calculation.

Now we will find the eigenvalues and construct the eigenfunctions of spectral problem (13)–(15). The general solution to equation (13) has the form:

$$\begin{cases} Y_1(y) = c_1 cos\sqrt{\lambda}y + c_2 sin\sqrt{\lambda}y, \quad 0 < y < h_1, \\ Y_2(y) = c_3 cos\sqrt{\lambda}y + c_4 sin\sqrt{\lambda}y, \quad 0 < y < h_2, \\ Y_3(y) = c_5 cos\sqrt{\lambda}(h_3 - y) + c_6 sin\sqrt{\lambda}(h_3 - y), \quad 0 < y < h_3, \end{cases} \tag{16}$$

where c_j are arbitrary constants ($j = 1, 2, 3, 4, 5, 6$).

Substituting general solution (16) into boundary conditions (14) and conjugation conditions (15) we obtain the following system

$$\begin{cases} c_1 = c_5, \\ c_2 = -c_6, \\ c_1 \cos\sqrt{\lambda}h_1 + c_2 \sin\sqrt{\lambda}h_1 = c_3, \\ -c_1 \sin\sqrt{\lambda}h_1 + c_2 \cos\sqrt{\lambda}h_1 = c_4, \\ c_3 \cos\sqrt{\lambda}h_2 + c_4 \sin\sqrt{\lambda}h_2 = c_5 \cos\sqrt{\lambda}h_3 + c_6 \sin\sqrt{\lambda}h_3, \\ -c_3 \sin\sqrt{\lambda}h_2 + c_4 \cos\sqrt{\lambda}h_2 = c_5 \sin\sqrt{\lambda}h_3 - c_6 \cos\sqrt{\lambda}h_3. \end{cases}$$

The characteristic determinant of the system has the form:

$$\Delta(\lambda) = 2 - 2\cos(s_3\sqrt{\lambda}) = 0,$$

where $s_3 = \sum\limits_{j=1}^{3} h_j = \sum\limits_{j=1}^{3} \dfrac{l_j - l_{j-1}}{k_j}$. From the last equation we find the eigenvalues of problem (13)–(15):

$$\lambda_n = \left(\frac{2\pi n}{s_3}\right)^2, \quad n = 0, 1, 2, ... \tag{17}$$

Since these eigenvalues are twofold, the following eigenfunctions correspond to them:

$$Y_n(y) = C \begin{cases} cos\left(\dfrac{2\pi n}{s_3}y\right), & 0 < y < h_1, \\ cos\left(\dfrac{2\pi n}{s_3}(h_2 + h_3 - y)\right), & 0 < y < h_2, \\ cos\left(\dfrac{2\pi n}{s_3}(h_3 - y)\right), & 0 < y < h_3, \end{cases} \tag{18}$$

$$\widetilde{Y}_n(y) = C\begin{cases} sin\left(\dfrac{2\pi n}{s_3}y\right), & 0<y<h_1,\\ sin\left(\dfrac{2\pi n}{s_3}(y-h_2-h_3)\right), & 0<y<h_2,\\ -sin\left(\dfrac{2\pi n}{s_3}(h_3-y)\right), & 0<x<h_3. \end{cases} \tag{19}$$

Lemma 2. The system of eigenfunctions (18)-(19) forms an orthonormal basis.

The proof follows from the general theory of self-adjoint problems. From the normalization condition it is not difficult to find $C=\sqrt{\dfrac{2}{s_3}}$.

From Lemma 2 it follows that the solution to problem (7)–(11) can be written in the following form:

$$v_j(y,t) = \sum_{n=0}^{\infty}\left(\varphi_n Y_n(y) + \widetilde{\varphi}_n\widetilde{Y}_n(y)\right)e^{-\lambda_n t},$$

where

$$\varphi_n = \int_0^{h_1}\psi_1(\eta)Y_n(\eta)d\eta + \int_0^{h_2}\psi_2(\eta)Y_n(\eta)d\eta + \int_0^{h_3}\psi_3(\eta)Y_n(\eta)d\eta, \tag{20}$$

$$\widetilde{\varphi}_n = \int_0^{h_1}\psi_1(\eta)\widetilde{Y}_n(\eta)d\eta + \int_0^{h_2}\psi_2(\eta)\widetilde{Y}_n(\eta)d\eta + \int_0^{h_3}\psi_3(\eta)\widetilde{Y}_n(\eta)d\eta. \tag{21}$$

Let us transform formula (20). In each integral we make the following replacements, respectively: $\eta=\dfrac{\xi-l_{j-1}}{k_j}$, $d\eta=\dfrac{d\xi}{k_j}$, $(j=1,2,3)$. Taking into account formula (12), we obtain

$$\varphi_n = \frac{1}{k_1}\int_{l_0}^{l_1}\varphi_1(\xi)Y_n(\frac{\xi-l_0}{k_1})d\xi + \frac{1}{k_2}\int_{l_1}^{l_2}\varphi_2(\xi)Y_n(\frac{\xi-l_1}{k_2})d\xi + \frac{1}{k_3}\int_{l_2}^{l_3}\varphi_3(\xi)Y_n(\frac{\xi-l_2}{k_3})d\xi. \tag{22}$$

Similarly, transforming formula (21), we have

$$\widetilde{\varphi}_n = \frac{1}{k_1}\int_{l_0}^{l_1}\varphi_1(\xi)\widetilde{Y}_n(\frac{\xi-l_0}{k_1})d\xi + \frac{1}{k_2}\int_{l_1}^{l_2}\varphi_2(\xi)\widetilde{Y}_n(\frac{\xi-l_1}{k_2})d\xi + \frac{1}{k_3}\int_{l_2}^{l_3}\varphi_3(\xi)\widetilde{Y}_n(\frac{\xi-l_2}{k_3})d\xi. \tag{23}$$

If we move to the initial variable using formula (6), then formulas (18)–(19) take the form:

$$Y_n(y) = \sqrt{\frac{2}{s_3}}\begin{cases} Y_n(\dfrac{x-l_0}{k_1}), & l_0<x<l_1,\\ Y_n(\dfrac{x-l_1}{k_2}), & l_1<x<l_2,\\ Y_n(\dfrac{x-l_2}{k_3}), & l_2<x<l_3, \end{cases} = \sqrt{\frac{2}{s_3}}\begin{cases} cos\left(\dfrac{2\pi n}{s_3}\left(\dfrac{x-l_0}{k_1}\right)\right), & l_0<x<l_1,\\ cos\left(\dfrac{2\pi n}{s_3}\left(\dfrac{l_2-x}{k_2}+\dfrac{l_3-l_2}{k_3}\right)\right), & l_1<x<l_2,\\ cos\left(\dfrac{2\pi n}{s_3}\left(\dfrac{l_3-x}{k_3}\right)\right), & l_2<x<l_3, \end{cases}$$

$$\widetilde{Y}_n(y) = \sqrt{\frac{2}{s_3}}\begin{cases} \widetilde{Y}_n(\dfrac{x-l_0}{k_1}), & l_0<x<l_1,\\ \widetilde{Y}_n(\dfrac{x-l_1}{k_2}), & l_1<x<l_2,\\ \widetilde{Y}_n(\dfrac{x-l_2}{k_3}), & l_2<x<l_3, \end{cases} = \sqrt{\frac{2}{s_3}}\begin{cases} sin\left(\dfrac{2\pi n}{s_3}\left(\dfrac{x-l_0}{k_1}\right)\right), & l_0<x<l_1,\\ sin\left(\dfrac{2\pi n}{s_3}\left(\dfrac{x-l_2}{k_2}+\dfrac{l_2-l_3}{k_3}\right)\right), & l_1<x<l_2,\\ sin\left(\dfrac{2\pi n}{s_3}\left(\dfrac{x-l_3}{k_3}\right)\right), & l_2<x<l_3. \end{cases}$$

We redesignate the last formulas as follows: $Y_n(y) = X_n(x), \widetilde{Y}_n(y) = \widetilde{X}_n(x)$. Then

$$X_n(x) = \sqrt{\frac{2}{s_3}} \begin{cases} cos\left(\frac{2\pi n}{s_3}\left(\frac{x-l_0}{k_1}\right)\right), & l_0 < x < l_1, \\ cos\left(\frac{2\pi n}{s_3}\left(\frac{l_2-x}{k_2} + \frac{l_3-l_2}{k_3}\right)\right), & l_1 < x < l_2, \\ cos\left(\frac{2\pi n}{s_3}\left(\frac{l_3-x}{k_3}\right)\right), & l_2 < x < l_3, \end{cases}$$

$$\widetilde{X}_n(x) = \sqrt{\frac{2}{s_3}} \begin{cases} sin\left(\frac{2\pi n}{s_3}\left(\frac{x-l_0}{k_1}\right)\right), & l_0 < x < l_1, \\ sin\left(\frac{2\pi n}{s_3}\left(\frac{x-l_2}{k_2} + \frac{l_2-l_3}{k_3}\right)\right), & l_1 < x < l_2, \\ sin\left(\frac{2\pi n}{s_3}\left(\frac{x-l_3}{k_3}\right)\right), & l_2 < x < l_3. \end{cases}$$

Since the system of eigenfunctions $\{Y_n(y), \widetilde{Y}_n(y)\}$ forms a basis, the functions $\{X_n(x), \widetilde{X}_n(x)\}$ also form a basis. Formulas (22)-(23) have the form:

$$\varphi_n = \frac{1}{k_1}\int_{l_0}^{l_1} \varphi_1(\xi)X_n(\xi)d\xi + \frac{1}{k_2}\int_{l_1}^{l_2} \varphi_2(\xi)X_n(\xi)d\xi + \frac{1}{k_3}\int_{l_2}^{l_3} \varphi_3(\xi)X_n(\xi)d\xi, \tag{24}$$

$$\widetilde{\varphi}_n = \frac{1}{k_1}\int_{l_0}^{l_1} \varphi_1(\xi)\widetilde{X}_n(\xi)d\xi + \frac{1}{k_2}\int_{l_1}^{l_2} \varphi_2(\xi)\widetilde{X}_n(\xi)d\xi + \frac{1}{k_3}\int_{l_2}^{l_3} \varphi_3(\xi)\widetilde{X}_n(\xi)d\xi. \tag{25}$$

Now let's prove the main theorem.

Theorem. Let $\varphi(x)$ be a continuously differentiable function satisfying the conditions $\varphi(l_0) = \varphi(l_3)$, $k_1\varphi'(l_0) = k_3\varphi'(l_3)$, $\varphi(l_j - 0) = \varphi(l_j + 0)$, $k_j\varphi'(l_j - 0) = k_{j+1}\varphi'(l_j + 0)$ $(j = 1, 2)$.

Then the function

$$u(x,t) = \sum_{n=0}^{\infty} \left(\varphi_n X_n(x) + \widetilde{\varphi}_n \widetilde{X}_n(x)\right) e^{-\lambda_n t}, \tag{26}$$

where the coefficients are determined by formulas (24)-(25), is the only classical solution of (1)–(5).

Proof. First we prove the existence of solution (26). Since $\left\{X_n(x), \widetilde{X}_n(x)\right\}$ the eigenfunctions and λ_n eigenvalues of problem (13)–(15), then it is easy to verify that the function $u(x,t)$ determined by formula (26) satisfies the equation, initial condition, boundary conditions and pairing conditions of problem (1)–(5). Series (26) is the sum of functions

$$u_n(x,t) = \left(\varphi_n X_n(x) + \widetilde{\varphi}_n \widetilde{X}_n(x)\right) e^{-\lambda_n t}. \tag{27}$$

Let us show that when $t \geq \varepsilon > 0$ (ε is any positive number) the series $\sum_{n=0}^{\infty} u_n(x,t)$, $\sum_{n=0}^{\infty} \frac{\partial u_n}{\partial t}$, $\sum_{n=0}^{\infty} \frac{\partial^2 u_n}{\partial x^2}$ converges uniformly. Obviously, $|\varphi| \leq M_1$ then from formula (27) it follows that $\left\{|\varphi_n|, |\widetilde{\varphi}_n|\right\} \leq M_2$. Then from equality (27) and from the following equalities

$$\frac{\partial u_n}{\partial t} = \left(-\lambda_n X_n(x)\varphi_n - \lambda_n \widetilde{X}_n(x)\widetilde{\varphi}_n\right) e^{-\lambda_n t}, \quad \frac{\partial^2 u_n}{\partial x^2} = \frac{\lambda_n}{k_j^2}\left(-X_n(x)\varphi_n - \widetilde{X}_n(x)\widetilde{\varphi}_n\right) e^{-\lambda_n t},$$

we get

$$|u_n(x,t)| \le M_3 e^{-\lambda_n \varepsilon}, \quad \left\{ \left| \frac{\partial u_n}{\partial t} \right|, \left| \frac{\partial^2 u_n}{\partial x^2} \right| \right\} \le M_4 \lambda_n e^{-\lambda_n \varepsilon},$$

where constants M_3, M_4 positive and does not depend on n. Taking into account formula (17), we have

$$\left\{ \sum_{n=1}^{\infty} \left| u_n(x,t) \right|, \sum_{n=1}^{\infty} \left| \frac{\partial u_n}{\partial t} \right|, \sum_{n=1}^{\infty} \left| \frac{\partial^2 u_n}{\partial x^2} \right| \right\} \le \sum_{n=1}^{\infty} M n^2 e^{-\left(\frac{2\pi n}{s_3}\right)^2 \varepsilon},$$

where constant $M > 0$, and does not depend on n. Since the series $\sum\limits_{n=1}^{\infty} M n^2 e^{-\left(\frac{2\pi n}{s_3}\right)^2 \varepsilon}$ an absolutely convergent series, hence, according to Weierstrass's test, the series $\left\{ \sum\limits_{n=0}^{\infty} \left| u_n(x,t) \right|, \sum\limits_{n=0}^{\infty} \left| \frac{\partial u_n}{\partial t} \right|, \sum\limits_{n=0}^{\infty} \left| \frac{\partial^2 u_n}{\partial x^2} \right| \right\}$ converge uniformly for $t \ge \varepsilon$ and are continuous for $t \ge \varepsilon$ the functions $u(x,t)$, $\frac{\partial u(x,t)}{\partial t}$, $\frac{\partial^2 u(x,t)}{\partial x^2}$.

Now we need to prove that series (26) converges uniformly everywhere in $\overline{\Omega}$. Note that the n-term of the series (26) is dominated by the sum $|\varphi_n| + |\widetilde{\varphi}_n|$. Integrating by parts the integral in formula (24), we obtain

$$|\varphi_n| \le \frac{C_1 s_3}{2\pi} \cdot \frac{|\alpha_n|}{n}, \quad |\widetilde{\varphi}_n| \le \frac{C_1 s_3}{2\pi} \cdot \frac{|\widetilde{\alpha}_n|}{n}, \quad C_1 = \max\left(\sqrt{k_1}, \sqrt{k_2}, \sqrt{k_3}\right),$$

where $\alpha_n = \frac{1}{\sqrt{k_1}} \int\limits_{l_0}^{l_3} \varphi'(\xi) X_n(\xi) d\xi$, $\widetilde{\alpha}_n = \frac{1}{\sqrt{k_2}} \int\limits_{l_0}^{l_3} \varphi'(\xi) \widetilde{X}_n(\xi) d\xi$ are Fourier coefficients of functions $\varphi'(x)$ on a segment $[l_0, l_3]$. Taking into account the inequality $ab \le \frac{1}{2}(a^2 + b^2)$, we have

$$|\varphi_n| + |\widetilde{\varphi}_n| \le \frac{C_1 s_3}{4\pi} \cdot \left(\alpha_n^2 + \widetilde{\alpha}_n^2 + \frac{2}{n^2} \right).$$

Using the Bessel inequality

$$\sum_{n=0}^{\infty} \left(\alpha_n^2 + \widetilde{\alpha}_n^2 \right) \le \|\varphi'\|^2$$

and the well-known equality $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we get $\sum\limits_{n=0}^{\infty} \left(|\varphi_n| + |\widetilde{\varphi}_n| \right) \le C$.

Thus, the majorizing series is absolutely convergent, this means series (26) converges uniformly in $\overline{\Omega}$ and defines a continuous function $u(x,t)$ in $\overline{\Omega}$. Thus, we proved the existence of a solution. Now let's prove uniqueness. Let's assume there are two solutions $\widetilde{v}(x,t)$, $\widehat{v}(x,t)$. Then for the function $v(x,t) = \widetilde{v}(x,t) - \widehat{v}(x,t)$, we have the following *problem* C:

$$\frac{\partial v}{\partial t} = k_j^2 \frac{\partial^2 v}{\partial x^2},$$

$$v(x,0) = 0, \quad l_0 \le x \le l_3,$$

$$\begin{cases} v(l_0,t) - v(l_3,t) = 0, \\ k_1 \frac{\partial v(l_0,t)}{\partial x} - k_3 \frac{\partial v(l_3,t)}{\partial x} = 0, \end{cases} \qquad 0 \le t \le T,$$

$$\begin{cases} v(l_j-0,t)=v(l_j+0,t), \\ k_j\dfrac{\partial v(l_j-0,t)}{\partial x}=k_{j+1}\dfrac{\partial v(l_j+0,t)}{\partial x}, \end{cases} \qquad j=1,2.$$

The solution to this *problem* C can be represented in the form of an expansion in terms of the basis $\{X_n(x),\widetilde{X}_n(x)\}$ and it has the form:

$$v(x,t)=\sum_{n=0}^{\infty}\big(A_n(t)X_n(x)+\widetilde{A}_n(t)\widetilde{X}_n(x)\big). \tag{28}$$

The coefficients $A_n(t)$ and $\widetilde{A}_n(t)$ are determined by the formulas

$$A_n(t)=\frac{1}{k_1}\int_{l_0}^{l_1}v(\xi,t)X_n(\xi)d\xi+\frac{1}{k_2}\int_{l_1}^{l_2}v(\xi,t)X_n(\xi)d\xi+\frac{1}{k_3}\int_{l_2}^{l_3}v(\xi,t)X_n(\xi)d\xi, \tag{29}$$

$$\widetilde{A}_n=\frac{1}{k_1}\int_{l_0}^{l_1}v(\xi,t)\widetilde{X}_n(\xi)d\xi+\frac{1}{k_2}\int_{l_1}^{l_2}v(\xi,t)\widetilde{X}_n(\xi)d\xi+\frac{1}{k_3}\int_{l_2}^{l_3}v(\xi,t)\widetilde{X}_n(\xi)d\xi. \tag{30}$$

First, we transform formula (29). Differentiating with respect to the variable t, we obtain

$$A_n'(t)=\frac{1}{k_1}\int_{l_0}^{l_1}\frac{\partial v(\xi,t)}{\partial t}X_n(\xi)d\xi+\frac{1}{k_2}\int_{l_1}^{l_2}\frac{\partial v(\xi,t)}{\partial t}X_n(\xi)d\xi+\frac{1}{k_3}\int_{l_2}^{l_3}\frac{\partial v(\xi,t)}{\partial t}X_n(\xi)d\xi=$$

$$=k_1\int_{l_0}^{l_1}\frac{\partial^2 v(\xi,t)}{\partial\xi^2}\cos\left(\frac{2\pi n}{s_3}\left(\frac{\xi-l_0}{k_1}\right)\right)d\xi+k_2\int_{l_1}^{l_2}\frac{\partial^2 v(\xi,t)}{\partial\xi^2}\cos\left(\frac{2\pi n}{s_3}\left(\frac{l_2-\xi}{k_2}+\frac{l_3-l_2}{k_3}\right)\right)\xi+$$

$$+k_3\int_{l_2}^{l_3}\frac{\partial^2 v(\xi,t)}{\partial\xi^2}\cos\left(\frac{2\pi n}{s_3}\left(\frac{l_3-\xi}{k_3}\right)\right)d\xi.$$

Integrating by parts twice and using the boundary conditions and conjugation conditions, we have

$$A_n'(t)=-\left(\frac{2\pi n}{s_3}\right)^2\frac{1}{k_1}\int_{l_0}^{l_1}v(x,t)\cos\left(\frac{2\pi n}{s_3}\left(\frac{x-l_0}{k_1}\right)\right)dx-$$

$$-\left(\frac{2\pi n}{s_3}\right)^2\frac{1}{k_2}\int_{l_1}^{l_2}v(x,t)\cos\left(\frac{2\pi n}{s_3}\left(\frac{l_2-x}{k_2}+\frac{l_3-l_2}{k_3}\right)\right)dx-$$

$$-\left(\frac{2\pi n}{s_3}\right)^2\frac{1}{k_3}\int_{l_2}^{l_3}v(x,t)\cos\left(\frac{2\pi n}{s_3}\left(\frac{l_3-x}{k_3}\right)\right)dx=$$

$$=-\lambda_n\int_{l_0}^{l_3}v(x,t)X_n(x)dx=-\lambda_nA_n(t).$$

Therefore $A_n(t) = c_n e^{-\lambda_n t}$. Transforming in a similar way, we obtain for the coefficient $\widetilde{A}_n(t)$.

$$\widetilde{A}_n'(t) = -\lambda_n \widetilde{A}_n(t) \quad \Rightarrow \quad \widetilde{A}_n(t) = \widetilde{c}_n e^{-\lambda_n t}.$$

Substituting the found $A_n(t)$ and $\widetilde{A}_n(t)$ into formula (29)-(30), we obtain

$$\frac{1}{k_1}\int\limits_{l_0}^{l_1} v(\xi,t)X_n(\xi)d\xi + \frac{1}{k_2}\int\limits_{l_1}^{l_2} v(\xi,t)X_n(\xi)d\xi + \frac{1}{k_3}\int\limits_{l_2}^{l_3} v(\xi,t)X_n(\xi)d\xi = c_n e^{-\lambda_n t}, \tag{31}$$

$$\frac{1}{k_1}\int\limits_{l_0}^{l_1} v(\xi,t)\widetilde{X}_n(\xi)d\xi + \frac{1}{k_2}\int\limits_{l_1}^{l_2} v(\xi,t)\widetilde{X}_n(\xi)d\xi + \frac{1}{k_3}\int\limits_{l_2}^{l_3} v(\xi,t)\widetilde{X}_n(\xi)d\xi = \widetilde{c}_n e^{-\lambda_n t}. \tag{32}$$

Passing to the limit $t \to 0$ in equality (31)-(32) what is possible due to continuity $v(x,t)$ in $\overline{\Omega}$, we have

$$0 = A_n(0) = c_n, \quad 0 = \widetilde{A}_n(0) = \widetilde{c}_n,$$

therefore $c_n = 0$, $\widetilde{c}_n = 0$.

Then from formula (28), we obtain $v(x,t) = 0$, it follows from this that $\widetilde{v}(x,t) = \widehat{v}(x,t)$. The theorem is proved.

Now consider the case $\theta = 2$.

Then, after applying the method of separation of variables, we obtain the following spectral problem

$$X_j''(x) + \frac{\lambda}{k_j^2}X_j(x) = 0, \quad l_{j-1} < x < l_j, \quad j = 1,2,3, \tag{33}$$

$$\begin{cases} X_1(l_0) + X_3(l_3) = 0, \\ k_1 X_1'(l_0) + k_3 X_3'(l_3) = 0, \end{cases} \tag{34}$$

$$X_j(l_j - 0) = X_{j+1}(l_j + 0), \quad k_j X_j'(l_j - 0) = k_{j+1}X_{j+1}'(l_j + 0), \quad j = 1,2. \tag{35}$$

The eigenvalues of problem (33)–(35) have the form: $\lambda_n = \left(\frac{(2n+1)\pi}{s_3}\right)^2$, $n = 0,1,2,...$

The following eigenfunctions correspond to these eigenvalues.

$$X_n(x) = \sqrt{\frac{2}{s_3}}\begin{cases} \cos\left(\frac{(2n+1)\pi}{s_3}\left(\frac{x-l_0}{k_1}\right)\right), & l_0 < x < l_1, \\ -\cos\left(\frac{(2n+1)\pi}{s_3}\left(\frac{l_2-x}{k_2} + \frac{l_3-l_2}{k_3}\right)\right), & l_1 < x < l_2, \\ -\cos\left(\frac{(2n+1)\pi}{s_3}\left(\frac{l_3-x}{k_3}\right)\right), & l_2 < x < l_3, \end{cases}$$

$$\widetilde{X}_n(x) = \sqrt{\frac{2}{s_3}}\begin{cases} \sin\left(\frac{(2n+1)\pi}{s_3}\left(\frac{x-l_0}{k_1}\right)\right), & l_0 < x < l_1, \\ \sin\left(\frac{(2n+1)\pi}{s_3}\left(\frac{l_2-x}{k_2} + \frac{l_3-l_2}{k_3}\right)\right), & l_1 < x < l_2, \\ \sin\left(\frac{(2n+1)\pi}{s_3}\left(\frac{l_3-x}{k_3}\right)\right), & l_2 < x < l_3. \end{cases}$$

All other calculations, including the proof of the theorem, are carried out in a similar way.

Conclusion

The method proposed in this article can be used in the case of n break points, where $n \geq 3$, and for the more general case of the conjugation condition (in this work, the ideal contact condition is considered). The solution to the problem is found in explicit form, which allows it to be used for numerical solution.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

Acknowledgments

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19679487).

References

1 Samarskiy, A.A. (1958). Uravneniia parabolicheskogo tipa s razryvnymi koeffitsientami [Parabolic equations with discontinuous coefficients]. *Doklady AN SSSR — Reports of the USSR Academy of Sciences 2*(121), 225–228 [in Russian].

2 Kim, Ye.I., & Baymukhanov, B.B. (1961). O raspredelenii temperatury v kusochno-odnorodnoi polu-beskonechnoi plastinke [On the temperature distribution in a piecewise homogeneous semi-infinite plate]. *Doklady AN SSSR — Reports of the USSR Academy of Sciences, 2*(140), 333–336 [in Russian].

3 Kamynin, L.I. (1962). O metode potentsialov dlia parabolicheskogo uravneniia s razryvnymi koeffitsientami [On the potential method for a parabolic equation with discontinuous coefficients]. *Doklady AN SSSR — Reports of the USSR Academy of Sciences, 6*(145), 1213–1216 [in Russian].

4 Ionkin, N.I., & Moiseyev, Ye.I. (1979). O zadache dlia uravneniia teploprovodnosti s dvutochechnymi kraevymi usloviiami [On the problem for the heat equation with two-point boundary values conditions]. *Differentsialnye uravneniia — Differential equations, 7*(15), 1284–1295 [in Russian].

5 Orazov, I., & Sadybekov, M.A. (2012). Ob odnoi nelokalnoi zadache opredeleniia temperatury i plotnosti istochnikov tepla [About one nonlocal problem of determining the temperature and density of heat sources]. *Izvestiia vuzov. Matematika — News of universities. Mathematics, 2*, 70–75 [in Russian].

6 Sadybekov, M.A. (2017). Initial-Boundary Value Problem for a Heat Equation with not Strongly Regular Boundary Conditions. *Functional Analysis in Interdisciplinary Applications — Springer Proceedings in Mathematics Statistics, 216*, 330–348. https://doi.org/10.1007/978-3-319-67053-9_32

7 Il'in, V.A. (1977). O skhodimosti razlozhenii po sobstvennym funktsiiam v tochkakh razryva koeffitsientov differentsialnogo operatora [On the convergence of expansions in eigenfunctions at discontinuity points of the coefficients of a differential operator]. *Matematicheskie zametki — Math Notes, 5*(22), 679–698 [in Russian]. https://doi.org/10.1007/BF01098352

8 Sadybekov, M.A., & Koilyshov, U.K. (2022). Two-phase tasks thermal conductivity with boundary conditions of the Sturm type. *Sixth International Conference on Analysis and Applied Mathematics. Abstract book of the conference ICAAM*, Antalya, Turkey.

9 Sadybekov, M.A., & Koilyshov, U.K. (2021). Nachalno-kraevye zadachi dlia uravneniia teploprovodnosti s kusochno-postoiannym koeffitsientom [Initial boundary value problems for the heat equation with piecewise constant coefficient]. *Izvestiia Mezhdunarodnogo kazakhsko-turetskogo Universiteta im. Kh.A.Yasavi. Seriia matematika, fizika, informatika — News of the Akhmet Yassawi Kazakh-turkish international universitaty. Mathematics, physics, computer series, 16*(1), 7–16 [in Russian].

10 Koilyshov, U.K., & Sadybekov, M.A. (2023). Solution of initial-boundary value problems for the heat equation with discontinuous coefficients arising in financial mathematics. *International Mathematical Conference "Functional Analysis in Interdisciplinary Applications".* Antalya, Turkiye.

11 Koilyshov, U.K., & Sadybekov, M.A. (2022). Solution of boundary value problems for the heat equation with a piecewise constant coefficient using the Fourier method. *Kazakh Mathematical Journal, 2*(22), 16–27. https://doi.org/10.70474/drms4076

12 Koilyshov, U.K., & Sadybekov, M.A. (2024). Two-phase heat conduction problems with Sturm-type boundary conditions. *Boundary Value Problems, 149.* https://doi.org/10.1186/s13661-024-01964-x

13 Carr, E.J., & March, N.G. (2018). Semi-analitical solution of multilayer diffusion problems with time-varying boundary conditions and general inter face conditions. *Applied Mathematics and Computation, 333*, 286–303. https://doi.org/10.1016/j.amc.2018.03.095

14 Rodrigo, M.R., & Worthy, A.L. (2016). Solution of multilayer diffusion problems via the Laplace transform. *Journal of Mathematical Analysis and Applications, 444*, 475–502. https://doi.org/10.1016/j.jmaa.2016.06.042

15 Carr, E.J., & Turner, I.W. (2016). A semi-analytical solution for multilayer diffusion in a composite medium consisting of a large number of layers. *Applied Mathematical Modelling, 15-16*(40), 7034–7050. https://doi.org/10.1016/j.apm.2016.02.041

16 Carr, E.J., Turner, I.W., & Perre, P. (2017). Macroscale modelling of multilayer diffusion: Using volume averaging to correct the boundary conditions. *Applied Mathematical Modelling, 47*, 600–618. https://doi.org/10.1016/j.apm.2017.03.044

17 Carr, E.J., Perre, P., & Turner, I.W. (2016). The extended distributed microstructure model for gradient-driven transport: a two-scale model for bypassing effective parameters. *J. Comput. Phys., 327*, 810–829. https://doi.org/10.1016/j.jcp.2016.10.004

18 Mantzavinos, D., Papadomanolaki, M.G., Saridakis,Y.G., & Sifalakis, A.G. (2016). Fokas transform method for a brain tumor invasion model with heterogeneous diffusion in 1 + 1 dimensions. *Applied Numerical Mathematics, 104*, 47–61. https://doi.org/10.1016/j.apnum.2014.09.006

19 Sheils, N.E., & Deconinck, B. (2016). Initial-to-Interface Maps for the Heat Equation on Composite Domains. *Studies in Applied Mathematics, 1*(137), 140–154. https://doi.org/10.1111/sapm.12138

20 Asvestas, M., Sifalakis, A.G., Papadopoulou, E.P., & Saridakis, Y.G. (2014). Fokas method for a multi-domain linear reaction-di usion equation with discontinuous di usivity. *Publishing Journal of Physics: Conference Series, 490.* https://doi.org/10.1088/1742-6596/490/1/012143

21 Itkin, A., Lipton, A., & Muravey, D. (2022). *Multilayer heat equations: Application to finance, 1*(1), 99–135. https://doi.org/10.3934/fmf.2021004

*Author Information**

Umbetkul Kurmankulovich Koilyshov (*corresponding author*) — Candidate of physical and mathematical sciences, Associate professor, Institute of Mathematics and Mathematical Modeling, 28 Shevshenko Avenue, Almaty, 050010, Kazakhstan; e-mail: *koylyshov@math.kz*; https://orcid.org/0000-0003-1752-7848

Makhmud Abdysametovich Sadybekov — Doctor of physical and mathematical sciences, professor, Institute of Mathematics and Mathematical Modeling, 28 Shevshenko Avenue, Almaty, 050010, Kazakhstan; e-mail: *sadybekov@math.kz*; https://orcid.org/0000-0001-8450-8191

Kulnyar Adambaevna Beisenbayeva — Candidate of physical and mathematical sciences, Associate professor, Academy of Logistics and Transport, 97 Shevshenko Avenue, Almaty, 050010, Kazakhstan; e-mail: *beisenbaeva@mail.ru*; https://orcid.org/0009-0004-3496-5289

*The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 92–103

https://doi.org/10.31489/2025M1/92-103 *Research article*

BVP for the heat equation with a fractional integro-differentiation operator

M.T. Kosmakova[1,2], N.T. Orumbayeva[1,2,*], D.M. Akhmanova[2], L.Zh. Kasymova[3]

[1] *Institute of Applied Mathematics, Karaganda, Kazakhstan;*
[2] *Karaganda Buketov University, Karaganda, Kazakhstan;*
[3] *Abylkas Saginov Karaganda Technical University, Karaganda, Kazakhstan*
(E-mail: svetlanamir578@gmail.com, orumbayevan@mail.ru, danna.67@mail.ru, l.kasymova2017@mail.ru)

A boundary value problem for a loaded heat conduction equation is considered, when the loaded term has the form of a fractional Riemann-Liouville derivative with respect to a spatial variable, and the loading point moves with a variable velocity. The problem is reduced to a Volterra integral equation of the second kind, the kernel of which contains a special function, namely, a Wright-type function. The kernel of the resulting integral equation is estimated, and it is shown, under certain restrictions on the line along which the load moves, that the kernel of the equation has a weak singularity, which is the basis for the assertion that the loaded term in the equation of the problem is a weak perturbation of its differential part. The study is based on the asymptotic behavior of the Wright function at infinity and at zero.

Keywords: loaded heat equation, fractional derivative, Volterra integral equation, Wright function.

2020 Mathematics Subject Classification: 45D05, 35K20.

Introduction

The heat conduction equation plays a key role in modeling thermal processes in various physical systems. In the classical formulation, it describes the temperature distribution in a medium subject to heat transfer. However, to more accurately account for complex physical effects, such as anomalous diffusion or material memory, generalized models are introduced that include additional terms, for example, containing a fractional integro-differentiation operator.

Fractional derivatives, unlike integer derivatives, make it possible to take into account memory effects and nonlocality of processes. Their application in heat conduction modeling has been actively developing in recent decades. The works [1, 2] consider the fundamentals of the theory of fractional calculus and its applications in mathematical physics. The application of fractional derivatives in heat conduction equations was investigated in [3], where it was shown that such models describe anomalous diffusion processes well. Fractional derivatives can also take into account spatial correlations and coordinate nonlocality in systems where the influence on the state at a given point in space depends not only on neighboring points, but also on more distant ones [4].

Boundary value problems for heat equations with fractional derivatives represent a separate area of research. They require the development of new approaches, since the presence of a fractional term leads to a complication of the mathematical structure of the problem. In [5], the spectral properties of operators with fractional derivatives are analyzed, and in [6, 7] boundary conditions for fractional models are studied.

Problems with loaded terms involving fractional derivatives are of particular interest. These problems arise in the context of modeling processes with heat sources or sinks that depend on time or spatial

[*]Corresponding author. *E-mail: orumbayevan@mail.ru*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488740, Grant No. AP23488729, 2024–2026.)

Received: 10 September 2024; *Accepted:* 13 December 2024.

coordinates. A loaded differential equation is an equation with a loaded term, which can contain differential or integrodifferential operators. This loaded term can be expressed as a function containing both the variables themselves and their derivatives.

Loaded equations allow you to model more complex physical or mathematical systems that cannot always be described by simple equations. For example, in problems of mathematical physics or control theory, loaded differential equations can be used to take into account the influence of external factors or additional conditions on the dynamics of the system. It is obvious that the presence of a loaded term gives rise to new, still unexplored problems in the theory of boundary value problems, therefore there is a need to develop new methods for solving the evolving theory of loaded differential equations [8].

Loaded differential equations can be considered as weak or strong perturbations of differential equations. In some cases, boundary value problems remain correct in natural classes of functions, where the loaded term is interpreted as a weak perturbation [9]. If the uniqueness of the solution to the boundary value problem is violated, then the load can be considered as a strong perturbation [10]. It turns out that the nature of the load (weak or strong perturbation) depends both on the order of the derivatives included in the loaded (perturbed) part of the operator, and on the manifold on which the trace of the desired function is specified.

The study of boundary value problems with loaded terms, presented in the form of integrals or fractional derivatives, can lead to different results depending on the specifics of the equation and the conditions of the problem. There may also be difficulties associated with the analysis and evaluation of integral operators in the resulting integral equations, since their kernels contain special functions. In [11, 12], the intervals for changing the order of the fractional derivative, that is contained in the loaded term, are determined, for which the theorems of existence and uniqueness of solutions to boundary value problems and arising integral equations are valid. We also note that the boundary value problems of heat conduction and the Volterra integral equations arising in their study with singularities in the kernel, similar to the singularities in this paper, were considered in [13, 14].

Also, integral equations with singularities in the kernel arise when studying boundary value problems in non-cylindrical domains that degenerate into a point at the initial moment of time [15–20].

Fractional derivatives in equations add new aspects and difficulties in the study of boundary value problems, since they take into account not only the previous state of the system, but also its history. The fractional order differentiation operation is a combination of differentiation and integration operations. Recently, work has appeared on the study of inverse boundary value problems with a load of fractional order. In [21], the inverse problem with a nonlinear gluing condition for a loaded equation of parabolic-hyperbolic type is studied for solvability. The problem is reduced to the study of the nonlinear Fredholm integral equation of the second kind. In [22], as an application of the analyticity of the solution, the uniqueness of an inverse problem in determining the fractional orders in the multi-term time-fractional diffusion equations from one interior point observation is established.

This paper examines a boundary value problem (BVP) defined in the open right upper quadrant. The problem is transformed into an integral equation, which, in certain instances, takes the form of a pseudo-Volterra type. The solvability of this equation is influenced by the order of differentiation in the loaded term and the behavior of the load line near the origin. In Section 1, we introduce some necessary definitions and mathematical preliminaries of fractional calculus, special functions and boundary value problems which will be needed in the forthcoming Sections. The problem statement for a heat equation with a loaded term as the Riemann-Liouville fractional derivative in the right upper quadrant (x, t) is given in Section 2. The initial conditions are homogeneous. Process of reducing a boundary value problem to an integral equation is the content of Section 3. In Section 4, we estimate the integral equation's kernel and establish conditions under which it has a weak singularity. Estimating the integral equation's kernel is based on the asymptotic behavior of the Wright function at infinity and at zero. This implies the solvability conditions for the BVP which are provided in Section 5.

In Section 5 the main results is formulated.

1 Preliminaries

Definition 1. [23] Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville integral of the order β is defined as follows

$$_rD_{a,t}^{-\beta}f(t) = \frac{1}{\Gamma(\beta)}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-\beta}}d\tau, \quad \beta, a \in R, \quad \beta > 0. \tag{1}$$

Definition 2. Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville derivative of the order β is defined as follows

$$_rD_{a,t}^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}}d\tau, \quad \beta, a \in R, \quad n-1 < \beta < n. \tag{2}$$

From formula (2) it follows that

$$_rD_{a,t}^{0}f(t) = f(t), \quad _rD_{a,t}^{n}f(t) = f^{(n)}(t), \quad n \in N.$$

Taking into account formula (1), formula (2) can be rewritten as

$$_rD_{a,t}^{\beta}f(t) = \frac{d^n}{dt^n}{}_rD_{a,t}^{\beta-n}f(t), \quad \beta, a \in R, \quad n-1 < \beta < n.$$

Information about the Mittag-Leffler function and the Wright function is taken from [24, 25].

Definition 3. The entire function of the form

$$E_{\lambda,\mu}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\lambda n + \mu)}, \quad \lambda > 0, \quad \mu \in C \tag{3}$$

is called the Mittag-Leffler function.

Definition 4. The entire function of the form

$$\phi(\lambda, \mu; z) = \sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in C \tag{4}$$

is called the Wright function.

Definition 5. A Wright-type function is a function $e_{\alpha,\beta}^{\mu,\delta}(z)$ defined by the contour integral and the Mittag-Leffler function (3)

$$e_{\alpha,\beta}^{\mu,\delta}(z) = \frac{1}{2\pi i}\int_{\gamma(r,\omega\pi)} e^t t^{-\delta} E_{\alpha,\mu}\left(zt^{\beta}\right)dt,$$

where $\gamma(r, \omega\pi)$ is the Hankel contour, the value of ω is chosen such that

$$1 - \omega\beta > \frac{\alpha}{2}, \quad \frac{1}{2} < \omega \le 1. \tag{5}$$

Inequalities (5) are always satisfied when

$$0 < \alpha < 2, \quad 0 < \alpha + \beta < 2, \quad \beta < 1, \quad \delta + \beta > 0.$$

For $\alpha > \beta$, $\alpha > 0$, for any $z \in \mathbb{C}$ the Wright-type function can be represented as a series

$$e_{\alpha,\beta}^{\mu,\delta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \mu)\Gamma(\delta - \beta n)}, \quad \mu \in \mathbb{C}, \quad \delta \in \mathbb{C}.$$

When $\alpha = \mu = 1$ it coincides with the Wright function:

$$e_{1,\beta}^{1,\delta}(z) = \phi(-\beta, \delta, z). \tag{6}$$

For a Wright-type function, the following autotransformation formula is valid:

$$e_{\alpha,\beta}^{\mu-\alpha,\delta+\beta}(z) = z e_{\alpha,\beta}^{\mu,\delta}(z) + \frac{1}{\Gamma(\mu-\alpha)\Gamma(\delta+\beta)}. \tag{7}$$

If $\pi \geq |\arg z| > \pi(\alpha+\beta)/2 + \varepsilon,\ \varepsilon > 0,\ k = 0, 1, 2, \ldots$, then the following limit relations are valid for large absolute values of z:

$$\begin{aligned} &\lim_{|z|\to\infty} e_{\alpha,\beta}^{\mu,\delta}(z) = 0, \\ &\lim_{|z|\to\infty} z e_{\alpha,\beta}^{\mu,\delta}(z) = -\frac{1}{\Gamma(\mu-\alpha)\Gamma(\delta+\beta)}. \end{aligned} \tag{8}$$

Let $c \in \mathbb{C}$. If $\mu > 0$, then

$$D_{0x}^{\nu} x^{\mu-1} e_{\alpha,\beta}^{\mu,\delta}\left(c x^{\alpha}\right) = x^{\mu-\nu-1} e_{\alpha,\beta}^{\mu-\nu,\delta}\left(c x^{\alpha}\right). \tag{9}$$

When $\mu = 0$, the following formula is valid

$$D_{0x}^{\nu} \frac{1}{x} e_{\alpha,\beta}^{0,\delta}\left(c x^{\alpha}\right) = x^{-\nu-1} e_{\alpha,\beta}^{-\nu,\delta}\left(c x^{\alpha}\right) - \frac{x^{-\nu-1}}{\Gamma(-\nu)\Gamma(\delta)}.$$

When $\nu = n \in \mathbb{N}$, formula (9) is valid for all $\mu \in \mathbb{R}$

$$\frac{d^n}{dx^n} x^{\mu-1} e_{\alpha,\beta}^{\mu,\delta}\left(c x^{\alpha}\right) = x^{\mu-n-1} e_{\alpha,\beta}^{\mu-n,\delta}\left(c x^{\alpha}\right).$$

The following equalities hold

$$\int_0^{\infty} \frac{1}{t} e_{\alpha,\beta}^{0,\delta}(-\lambda t) dt = -\frac{\alpha}{\Gamma(\delta)'}, \quad \int_0^{\infty} \frac{1}{t} e_{\alpha,\beta}^{\mu,0}(-\lambda t) dt = \frac{\beta}{\Gamma(\mu)}.$$

Also the formulas for differentiating a Wright type function are valid

$$\frac{d}{dz} e_{\alpha,\beta}^{\mu,\delta}(z) = \frac{1}{\alpha z}\left[e_{\alpha,\beta}^{\mu-1,\delta}(z) + (1-\mu) e_{\alpha,\beta}^{\mu,\delta}(z)\right].$$

It's known [26; 57] that in the domain $Q = \{(x,t) \mid x > 0, \quad t > 0\}$ the solution to the boundary value problem of heat conduction

$$u_t = a^2 u_{xx} + F\left(x, t\right),$$

$$u\left|_{t=0}\right. = f(x), \quad u\left|_{x=0}\right. = g(x)$$

is described by the formula

$$u\left(x, t\right) = \int_0^{\infty} G\left(x, \xi, t\right)\, f(\xi)\, d\xi + \int_0^{t} H\left(x, t-\tau\right)\, g(\tau)\, d\tau +$$

$$+\int_0^t \int_0^\infty G(x,\xi,t-\tau)\, F(\xi,\tau)\, d\xi d\tau, \tag{10}$$

where

$$G(x,\,\xi,\,t) = \frac{1}{2\sqrt{\pi\, a\, t}} \left\{ \exp\left(-\frac{(x-\xi)^2}{4\,a\,t}\right) - \exp\left(-\frac{(x+\xi)^2}{4\,a\,t}\right) \right\},$$

$$H(x,\,t) = \frac{1}{2\sqrt{\pi\, a}\, t^{3/2}} \exp\left(-\frac{x^2}{4\,a\,t}\right).$$

The Green function $G(x,\xi,t)$ satisfies the relation

$$\int_0^\infty G(x,\xi,t)\, d\xi = \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right),$$

where $\operatorname{erf}(z)$ is the error integral.

2 The problem's statement

In the domain $Q = \{(x,t) : x > 0, t > 0\}$, we consider a BVP

$$u_t = u_{xx} - \lambda \left\{ {}_rD^\beta_{0,x} u(x,t) \right\} \big|_{x=\gamma(t)} + f(x,t), \tag{11}$$

$$u(x,0) = 0, \quad u(0,t) = 0, \tag{12}$$

where λ is a complex parameter, ${}_rD^\beta_{0,\,t}\, u(x,\,t)$ is the Riemann-Liouville derivative (2) of an order β, $1 < \beta < 2$, $\gamma(t)$ is a continuous increasing function, $\gamma(0) = 0$.

The problem is studied in the class of continuous functions.

For the right side of the equation, we require the following conditions to be satisfied:

$$f(x,t) \in L_\infty(A) \cap C(B), \tag{13}$$

where $A = \{(x,t) \,|\, x > 0,\ t \in [0,T]\}$, $B = \{(x,t) \,|\, x > 0,\ t \geq 0\}$, $T = const > 0$,

$$f_1(x,t) = \int_0^t \int_0^\infty G(x,\xi,t-\tau)\, f(\xi,\tau)\, d\xi d\tau \in L_1\ \ (x > 0). \tag{14}$$

Let us introduce the notation

$$D^\nu_{at} g(t) = \frac{1}{\Gamma(-\nu)} \int_a^t \frac{g(\xi)\, d\xi}{(t-\xi)^{\nu+1}},\ \ \nu < 0.$$

When $\nu = 0$ $D^0_{at} g(t) = g(t)$, then

$$D^\nu_{at} g(t) = \frac{d^n}{dt^n} D^{\nu-n}_{at} g(t), \quad n-1 < 0 \leq n, \quad n \in N.$$

We consider the fractional derivative in the Riemann-Liouville sense with respect to the spatial variable. If $a = 0$, $n = 2$, $\nu = \beta \Rightarrow$

$${}_rD^\beta_{0x} u(x,t) = \frac{d^2}{dx^2} D^{\beta-2}_{0x} u(x,t) \tag{15}$$

or

$${}_rD^\beta_{0x} u(x,t) = \frac{d^2}{dx^2} \left(\frac{1}{\Gamma(2-\beta)} \int_0^x \frac{u(x,\xi)\, d\xi}{(x-\xi)^{\beta-1}} \right). \tag{16}$$

The derivative in the loaded term of equation (11) is determined by the formula (16).

3 Reducing the BVP to an integral equation

According to the formula (10) a solution to BVP (11)-(12) can be represented as

$$u(x,t) = -\lambda \int_0^t \int_0^\infty G(x,\xi,t-\tau)\,\mu(\tau)\,d\xi d\tau + f_1(x,t), \tag{17}$$

where

$$\mu(t) = \left\{ {}_rD^{\beta}_{0,x} u(x,t) \right\} \big|_{x=\gamma(t)}, \tag{18}$$

$$f_1(x,t) = \int_0^t \int_0^{+\infty} G(x,\xi,t-\tau)\, f(\xi,\tau)\,d\xi d\tau. \tag{19}$$

In [9] it was proved formulas

$$e^{-\xi^2} = \sqrt{\pi}\phi\left(-\frac{1}{2},\frac{1}{2},-2\xi\right), \tag{20}$$

where $\phi\,(\phi(\lambda,\mu;z))$ is the Wright function (4),

$$\operatorname{erf}(z) = 2\int_0^z \phi\left(-\frac{1}{2},\frac{1}{2},-2\xi\right) d\xi = 1-\phi\left(-\frac{1}{2},1,-2z\right). \tag{21}$$

Then, taking into account formulas (20) and (21) representation (17) can be rewritten as:

$$u(x,t) = -\lambda \int_0^t K\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu(\tau)\,d\tau + f_1(x,t), \tag{22}$$

where

$$K\left(\frac{x}{2\sqrt{t-\tau}}\right) = 1-\phi\left(-\frac{1}{2},1,-\frac{x}{\sqrt{t-\tau}}\right) \tag{23}$$

and $\mu(t)$ and $f_1(t)$ are defined by formulas (18) and (19) respectively.

To (22) we apply the fractional integro-differentiation operator by formula (15). Taking into account formulas (23), (6), (7), and (9), we obtain, when $1<\beta<2$:

$${}_rD^{\beta}_{0x}\left(\mathrm{K}\left(\frac{x}{2\sqrt{t-\tau}}\right)\right) = x^{-\beta}\left(\frac{1}{\Gamma(1-\beta)} - e^{1-\beta,1}_{1,\frac{1}{2}}\left(-\frac{x}{\sqrt{t-\tau}}\right)\right) = \frac{x^{1-\beta}}{\sqrt{t-\tau}} e^{2-\beta,\frac{1}{2}}_{1,\frac{1}{2}}\left(-\frac{x}{\sqrt{t-\tau}}\right).$$

Indeed, according to the law of composition, we have

$$D^{\beta}_{0x}(f(x)) = D^1 D^{\beta-1}_{0x}(f(x)) \Rightarrow$$

$${}_rD^{\beta-1}_{0x}\left(K\left(\frac{x}{2\sqrt{t-\tau}}\right)\right) = {}_rD^{\beta-1}_{0x}\left(1-\Phi\left(-\frac{1}{2},1;-\frac{x}{\sqrt{t-\tau}}\right)\right) =$$

$$= {}_rD^{\beta-1}_{0x}\left(1-e^{1,1}_{1,\frac{1}{2}}\left(-\frac{x}{\sqrt{t-\tau}}\right)\right) = x^{1-\beta}\left(\frac{1}{\Gamma(2-\beta)} - e^{2-\beta,1}_{1,\frac{1}{2}}\left(-\frac{x}{\sqrt{t-\tau}}\right)\right).$$

Then, taking into account the autotransformation formula (7), we get

$$D^1\left(\frac{x^{1-\beta}}{\Gamma(2-\beta)} - x^{1-\beta} e^{2-\beta,1}_{1,\frac{1}{2}}\left(-\frac{x}{\sqrt{t-\tau}}\right)\right) =$$

$$= \frac{1-\beta}{\Gamma(2-\beta)}x^{-\beta} - x^{-\beta} e^{1-\beta,1}_{1,\frac{1}{2}}\left(-\frac{x}{\sqrt{t-\tau}}\right) = \frac{x^{1-\beta}}{\sqrt{t-\tau}} e^{2-\beta,\frac{1}{2}}_{1,\frac{1}{2}}\left(-\frac{x}{\sqrt{t-\tau}}\right).$$

Thus, BVP (11)-(12) is reduced to a Volterra integral equation of the second kind

$$\mu(t) + \lambda \int_0^t K_\beta(t,\tau)\mu(\tau)d\tau = f_2(t), \tag{24}$$

with a kernel

$$K_\beta(t,\tau) = \frac{(\gamma(t))^{1-\beta}}{\sqrt{t-\tau}} e^{2-\beta,\frac{1}{2}}_{1,\frac{1}{2}}\left(-\frac{\gamma(t)}{\sqrt{t-\tau}}\right), \quad 1<\beta<2, \tag{25}$$

and with the right part

$$f_2\left(t\right) = \left\{ {}_r D^\beta_{0,x} f_1\left(x,t\right)\right\} |_{x=\gamma(t)} \,. \tag{26}$$

4 Research of the integral equation

Since in the given problem (11)-(12) the line, along which the load is moving, has the form $x = \gamma\left(t\right)$, and $\gamma\left(t\right)$ increases and $\gamma\left(0\right)=0$, then there are different cases of behavior for $\frac{x}{\sqrt{t}}\left|_{x=\gamma(t)}\right.$, when $t \to 0$.

Let $0 < x = \gamma(t) \sim t^\omega$ when $t \to 0,\ \omega > 0$.

Let's introduce a change of variable τ:

$$z = \frac{\gamma(t)}{\sqrt{t-\tau}} \Rightarrow \sqrt{t-\tau} = \frac{\gamma(t)}{z}.$$

Then

$$K_\beta(t,z) = (\gamma(t))^{-\beta} z e^{2-\beta,\frac{1}{2}}_{1,\frac{1}{2}}(-z).$$

Let's consider the following cases:

$$a)\ 0<\omega<\frac{1}{2} \quad \Rightarrow \quad |z| \to +\infty, \quad \text{when} \quad t\to 0.$$

Taking into account the limiting ratio (8), we get

$$\lim_{|z|\to+\infty} z e^{\mu,\nu}_{\alpha,\beta}(-z) = -\frac{1}{\Gamma(\mu-\alpha)\cdot\Gamma(\delta+\beta)} \Rightarrow$$

$$\lim_{t\to 0} K_\beta(t,\tau) = \lim_{t\to 0}(\gamma(t))^{-\beta} z e^{2-\beta,\frac{1}{2}}_{1,\frac{1}{2}}(-z) = \lim_{t\to 0} t^{-\beta\omega}\frac{1}{\Gamma(1-\beta)} = +\infty$$

$$b)\ \omega>\frac{1}{2} \quad \Rightarrow \quad |z|\to 0, \quad \text{when} \quad t\to 0.$$

Taking into account the limiting ratio (8), we get

$$\lim_{t\to 0} e^{2-\beta,\frac{1}{2}}_{1,\frac{1}{2}}\left(-\frac{t^\omega}{\sqrt{t-\tau}}\right) = \frac{1}{\Gamma(2-\beta)\sqrt{\pi}} \Rightarrow K_\beta(t,\tau) \sim \frac{t^{\omega(1-\beta)}}{\sqrt{t-\tau}}, \quad \text{when} \quad t\to 0.$$

The kernel (25) of the integral equation (24) has singularities at $t=0$ and $t=\tau$.

Let us define the conditions under which the integral operator of the equation is compressible in the class of continuous functions. Consider the integral

$$\int_0^t K_\beta(t,\tau)d\tau = t^{\omega(1-\beta)}\sqrt{t} = t^{\omega(1-\beta)+\frac{1}{2}} \underset{t\to 0}{\to} 0$$

if $\omega(1-\beta)+\frac{1}{2}>0 \Rightarrow \omega < \frac{1}{2(\beta-1)}$.

$$c)\ \omega = \frac{1}{2} \Rightarrow K_\beta(t,\tau) \sim \frac{t^{\frac{1-\beta}{2}}}{\sqrt{t-\tau}} e^{2-\beta \frac{1}{2}}_{1,\frac{1}{2}} \left(-\sqrt{\frac{t}{t-\tau}}\right), \quad \text{when} \quad t \to 0.$$

Since $e^{2-\beta \frac{1}{2}}_{1,\frac{1}{2}} \left(-\sqrt{\frac{t}{t-\tau}}\right) \underset{t\to 0}{\to}$ const, then

$$\int_0^t K_\beta(t,\tau) d\tau \sim t^{\frac{1}{2}-\frac{\beta}{2}} \sqrt{t} = t^{1-\frac{\beta}{2}} \underset{t\to 0+}{\to} 0,$$

as $1 < \beta < 2$.

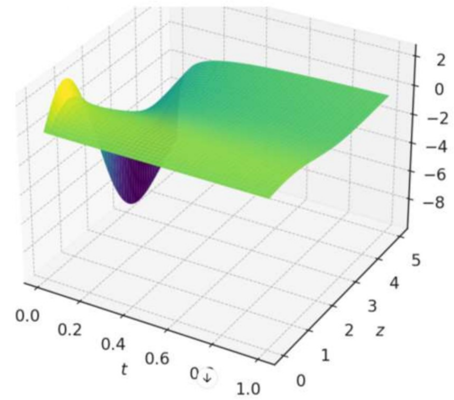

Figure 1. Graph of the kernel

Figure 1 presents the graph of the kernel which shows stability at small time values.

5 *The main results*

So, the following theorem has been proven.

Theorem 1. Integral equation (24) with kernel (25) for $1 < \beta < 2$ and with $\gamma(t) \sim t^\omega$ in the neighborhood of $t = 0$ is $\gamma(t) \sim t^\omega$, $\omega > 0$, $\gamma(0) = 0$ uniquely solvable in the class of continuous functions for any continuous right-hand side $f_2(t)$, if $\omega < \dfrac{1}{2(\beta-1)}$ and $\omega = \dfrac{1}{2}$.

This result coincided with the result obtained in [12].

Let us introduce a class of functions

$$\mathfrak{U} = \Big\{ u \,\big|\, (x\sqrt{t})^{-1} u \in L_\infty(A) \cap C(B); \qquad u_t - u_{xx} \in L_\infty(A) \cap C(B);$$

$$\left\{ {}_r D^\beta_{0,x} u(x,t) \right\} \big|_{x=\gamma(t)} \in C([0;T]), \quad T = const > 0, \quad 1 < \beta < 2 \Big\}, \tag{27}$$

where $A = \{(x,t) \,|\, x > 0,\ t \in [0,T]\}$, $B = \{(x,t) \,|\, x > 0,\ t \geq 0\}$, $T = const > 0$.

Since the solution of the integral equation (24) $\mu(t)$ is a continuous and bounded function under the conditions of Theorem (1), it can be shown that for the solution of problem (11)-(12), which has the form (22), where $f\,(x,t)$ belongs to the class (13), the following estimate is valid

$$|u\,(x,t)\,| \leq C(\lambda)\,x\,\sqrt{t},$$

where $C(\lambda) = C_1|\lambda| + C_2$.

Also it can be shown that function (18) satisfies BVP (11)-(12) and belongs to the class (27).

The following main result follows from Theorem 1:

Theorem 2. Let the function $f\,(x,t)$ satisfy conditions (13) and (14), the function $\mu(t) \in C([0;T])$ be a solution of integral equation (24) with the right-hand side $f_2(t) \in C([0;T])$ defined by formulas (19) and (26). Then BVP (11)-(12) with the load motion law $\gamma(t) \sim t^\omega$ (in the neighborhood of the point $t = 0$) has a unique solution (22) in the class (27), if $\omega < \dfrac{1}{2(\beta-1)}$ and $\omega = \dfrac{1}{2}$.

Conclusion

Under the conditions of the theorem, the kernel (25) of the integral equation (24) has a weak singularity. Therefore, the method of successive approximations can be applied to find a unique solution of the equation (24). Then the corresponding boundary value problems are correct in natural classes of functions, i.e. the loaded term of the posed boundary value problem is a weak perturbation of the differential equation.

Since the problem statement contains a fractional derivative, then the obtained results can be applied in several domains such that:

Thermal processes: the study is particularly relevant to heat conduction problems where the material exhibits memory effects or non-locality. For instance: heat diffusion in heterogeneous materials with varying thermal properties, processes involving spatially moving heat sources or sinks.

Anomalous Diffusion: The fractional derivative approach effectively models systems exhibiting anomalous diffusion, as encountered in porous media, biological tissues with complex transport phenomena.

Engineering Systems: in mechanical and civil engineering, materials with hereditary properties, such as viscoelastic materials, benefit from this approach.

Mathematical Physics: the results are applicable in studying boundary value problems in non-cylindrical domains and domains with degeneracies, enhancing the analysis of complex geometries.

Now we will give a comparison with related studies, incorporating the comparative analysis. References [9, 11] provide foundational insights into the behavior of fractional derivatives in heat equations. Our study extends this by analyzing the effect of weak perturbations caused by the load term. In contrast to [12], which focuses on specific fixed domains, our results address moving load scenarios, offering broader applicability. Prior work, such as [13, 14], emphasizes integral equations with singularities. Our approach diverges by providing a detailed kernel analysis under varying load motion laws, as expressed through. Studies like [21] examine inverse problems for fractional equations but do not address weak perturbations in moving loads. Our results bridge this gap, contributing to a more comprehensive framework.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488740, Grant No. AP23488729, 2024–2026.)

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Podlubny, I. (1999). *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications.* San Diego-Boston-New York-London-Tokyo-Toronto: Academic Press.

2 Mainardi, F. (2012). Fractional calculus: Some basic problems in continuum and statistical mechanics. New York: Springer-Verlag. https://doi.org/10.48550/arXiv.1201.0863

3 Metzler, R., & Klafter, J. (2000). The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Reports, 1*(339), 1–77. https://doi.org/10.1016/S0370-1573(00)00070-3

4 Magin, R. (2004). Fractional Calculus in Bioengineering. *Critical Reviews in Biomedical Engineering, 1*(32), 1–104. https://doi.org/10.1615/critrevbiomedeng.v32.i1.10

5 Kukushkin, M.V. (2018). Spectral properties of fractional differentiation operators. *Electronic Journal of Differential Equations, 2018*(29), 1–24. https://doi.org/10.48550/arXiv.1710.02662

6 Atanackovic, T.M., Pilipovic, S., & Zorica, D. (2011). Distributed-order fractional wave equation on a finite domain. Stress relaxation in a rod. *International Journal of Engineering Science, 2*(49), 175–190. https://doi.org/10.1016/j.ijengsci.2010.11.004

7 Podlubny, I. (2002). Geometric and physical interpretation of fractional integration and fractional differentiation. *Fract. Calculus Appl. Anal., 4*(5), 367–386. https://doi.org/10.48550/arXiv.math/0110241

8 Parasidis, I.N. (2019). Extension method for a class of loaded differential equations with nonlocal integral boundary conditions. *Bulletin of the Karaganda University. Mathematics Series,* 4(96), 58–68. https://doi.org/10.31489/2019M4/58-68

9 Pskhu, A.V., Kosmakova, M.T., Akhmanova, D.M., Kassymova, L.Zh., & Assetov, A.A. (2022). Boundary value problem for the heat equation with a load as the Riemann-Liouville fractional derivative. *Bulletin of the Karaganda University. Mathematics Series,* 1(105), 74–82. https://doi.org/10.31489/2022M1/74-82

10 Ramazanov, M.I., Kosmakova, M.T., Romanovsky, V.G., & Tuleutaeva, Zh.M. (2018). Boundary value problems for essentially-loaded parabolic equation. *Bulletin of the Karaganda University. Mathematics Series,* 4(92), 79–86. https://doi.org/10.31489/2018M4/79-86

11 Kosmakova, M.T., Iskakov, S.A., & Kasymova, L.Zh. (2021). To solving the fractionally loaded heat equation. *Bulletin of the Karaganda University. Mathematics Series,* 1(101), 65–77. https://doi.org/10.31489/2021M1/65-77

12 Kosmakova, M.T., Ramazanov, M.I., & Kasymova, L.Zh. (2021). To Solving the Heat Equation with Fractional Load. *Lobachevskii Journal of Mathematics, 12*(42), 2854–2866. https://doi.org/10.1134/S1995080221120210

13 Kosmakova, M.T. (2016). On an integral equation of the Dirichlet problem for the heat equation in the degenerating domain. *Bulletin of the Karaganda University. Mathematics Series,* 1(81), 62–67. https://mathematics-vestnik.ksu.kz/index.php/mathematics-vestnik/article/view/79/74

14 Kosmakova, M.T., Akhmanova, D.M., Iskakov, S.A., Tuleutaeva, Zh.M., & Kasymova, L.Zh. (2019). Solving one pseudo-Volterra integral equation. *Bulletin of the Karaganda University. Mathematics Series,* 1(93), 72–77. https://doi.org/10.31489/2019m1/72-77

15 Jenaliyev, M.T., Ramazanov, M.I., Kosmakova, M.T., & Tuleutaeva, Zh.M. (2020). On the solution to a two-dimensional heat conduction problem in a degenerate domain. *Eurasian Mathematical Journal, 11*(3), 89–94. https://doi.org/10.32523/2077-9879-2020-11-3-89-94

16 Kosmakova, M.T., Orumbayeva, N.T., Medeubaev, N.K., & Tuleutaeva, Zh.M. (2018). Problems of Heat Conduction with Different Boundary Conditions in Noncylindrical Domains. *AIP Conference Proceedings, 1*(1997). https://doi.org/10.1063/1.5049065

17 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). Uniqueness and non-uniqueness of solutions of the boundary value problems of the heat equation. *AIP Conference Proceedings, 1*(1676). https://doi.org/10.1063/1.4930454

18 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2014). On the spectrum of Volterra integral equation with the incompressible kernel. *AIP Conference Proceedings, 1*(1611), 127–132. https://doi.org/10.1063/1.4893816

19 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2017). On the Solvability of Nonhomogeneous Boundary Value Problem for the Burgers Equation in the Angular Domain and Related Integral Equations. *Springer Proceedings in Mathematics and Statistics, 216,* 123–141. https://doi.org/10.1007/978-3-319-67053-9_12

20 Ramazanov, M.I., Kosmakova, M.T., & Tuleutaeva, Zh.M. (2021). On the Solvability of the Dirichlet Problem for the Heat Equation in a Degenerating Domain. *Lobachevskii Journal of Mathematics, 42*(15), 3715–3725. https://doi.org/10.1134/S1995080222030179

21 Abdullaev, O.Kh., & Yuldashev, T.K. (2023). Inverse Problems for the Loaded Parabolic-Hyperbolic Equation Involves Riemann-Liouville Operator. *Lobachevskii Journal of Mathematics, 3*(44), 1080–1090. https://doi.org/10.1134/S1995080223030034

22 Kubica, A., Ryszewska, K., & Yamamoto, M. (2020). Initial boundary value problems for time-fractional diffusion equations. *Time-Fractional Differential Equations,* 73–108. https://doi.org/10.1007/978-981-15-9066-5_4

23 Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional Integrals and Derivatives. Theory and Application.* Gordon and Breach: New York.

24 Stanković, B. (1970). On the function of E.M. Wright. *Publ. de l'Institut Mathématique, Beograd, Nouvelle Sér. 10*, 113–124.

25 Gorenflo, R., Luchko Y., & Mainardi, F. (1999). Analytical properties and applications of the Wright function. *Fractional Calculus and Applied Analysis, 2*(4), 383–414. https://doi.org/10.48550/arXiv.math-ph/0701069

26 Polyanin, A.D. (2002). *Handbook of Linear Partial Differential Equations for Engineers and Scientists.* Chapman and Hall/CRC: New York-London.

*Author Information**

Minzilya Timerbaevna Kosmakova — PhD, Associate professor, Professor, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *svetlanamir578@gmail.com*; https://orcid.org/0000-0003-4070-0215

Nurgul Tumarbekovna Orumbayeva (*corresponding author*) — Candidate of physical and mathematical sciences, Associate professor, Professor, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *orumbayevan@mail.ru*; https://orcid.org/0000-0003-1714-6850

Danna Maratovna Akhmanova — Candidate of physical and mathematical sciences, Associate professor, Professor, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *danna.67@mail.ru*; https://orcid.org/0000-0003-1040-2495

Laila Zhumazhanovna Kasymova — PhD, Senior Lecturer, Abylkas Saginov Karaganda Technical University, 56 Ave. Nursultan Nazarbayev, Karaganda, 100027, Kazakhstan; e-mail: *l.kasymova2017@mail.ru*; https://orcid.org/0000-0002-4696-867X

*The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 104–117

https://doi.org/10.31489/2025M1/104-117 *Research article*

Almost quasi-Urbanik structures and theories

B.Sh. Kulpeshov[1,2,*], S.V. Sudoplatov[3]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Kazakh British Technical University, Almaty, Kazakhstan;*
[3] *Sobolev Institute of Mathematics, Novosibirsk, Russia*
(E-mail: b.kulpeshov@kbtu.kz, sudoplat@math.nsc.ru)

The notions of almost quasi-Urbanik structures and theories, and studied possibilities for the degrees of quasi-Urbanikness, both for existential and universal cases were introduced. Links of these characteristics and their possible values are described. These values for structures of unary predicates, equivalence relations, linearly ordered, preordered and spherically ordered structures and theories as well as for strongly minimal ones, and for some natural operations including disjoint unions and compositions of structures and theories were studied. A series of examples illustrates possibilities of these characteristics.

Keywords: almost quasi-Urbanik structure, almost quasi-Urbanik theory, degree of quasi-Urbanikness.

2020 Mathematics Subject Classification: 03C64, 03C07.

Introduction

The property of quasi-Urbanikness allows to clarify and describe structural properties in various classes of structures and theories, including strongly minimal ones [1, 2]. These properties can be classified using natural semantic and syntactic characteristics. A series of results on these characteristics are obtained in general [3], for abelian groups [4], for variations of rigidity in general [5] and for ordered structures [6], etc.

In the present paper we continue to study related characteristics introducing the notions of almost quasi-Urbanik structures and theories, and their existential and universal degrees. Possibilities of these degrees are described both in general and for a series of natural structures and theories including structures and theories of unary predicates, equivalence relations, ordered structures and theories, strongly minimal structures and theories, disjoint unions and compositions of structures and theories. We illustrate possibilities of degrees by a series of examples.

The paper is organized as follows. The notions of almost quasi-Urbanik structures and theories, degrees and their spectra are described in general, for unary predicates, and equivalence relations are described in Section 1. In Section 2, degrees of quasi-Urbanikness are described for ordered theories including spherically ordered and some preordered ones. Degrees of quasi-Urbanikness and links for dimensions are studied in Section 3. In Sections 4 and 5, we describe possibilities of degrees of quasi-Urbanikness for disjoin unions and E-definable compositions, respectively. In Section 6, we discuss some general operators transforming a given structure into quasi-Urbanik one.

[*]Corresponding author. *E-mail: b.kulpeshov@kbtu.kz*

This research has been funded by Science Committee of Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. BR20281002), and in the framework of the State Contract of the Sobolev Institute of Mathematics, Project No. FWNF-2022-0012.

Received: 6 August 2024; *Accepted:* 21 November 2024.

1 Almost quasi-Urbanik structures, their theories and degrees

Let L be a countable first-order language. Throughout we consider L-structures and their complete elementary theories; and we use standard model-theoretic notions and notations [7–10].

Following [1], a theory T is called *strongly minimal* if for any formula $\varphi(x,\bar{a})$ of language obtained by adding parameters of $\overline{a}$ (in a model $\mathcal{M} \models T$) to the language of T, either $\varphi(x,\bar{a})$, or $\neg\varphi(x,\bar{a})$ has finitely many solutions.

Following [11], for $n \in \omega \setminus \{0\}$ and a set A, an element b is called *n-algebraic* over A if $a \in \mathrm{acl}(A)$ and it is witnessed by a formula $\varphi(x,\overline{a})$, for $\overline{a} \in A$, with at most n solutions. The set of all n-algebraic elements over A is denoted by $\mathrm{acl}_n(A)$. If $A = \mathrm{acl}_n(A)$, then A is called *n-algebraically* closed. A type p is *n-algebraic* if it is realized by at most n tuples only, i.e., $\deg(p) \leq n$. The complete n-algebraic types $p(x) \in S(A)$ are exactly ones of the form $\mathrm{tp}(a/A)$, where a is n-algebraic over A, i.e., with $\deg(a/A) \leq n$. Here $\deg(a/A) = k \leq n$ defines the *n-degree* $\deg_n(a/A)$ of $\mathrm{tp}(a/A)$ and of a over A. If $\mathrm{acl}(A) = \mathrm{acl}_n(A)$ then minimal such n is called the *degree of algebraization* over the set A and it is denoted by $\deg_{\mathrm{acl}}(A)$. If that n does not exist, then we put $\deg_{\mathrm{acl}}(A) = \infty$. The supremum of values $\deg_{\mathrm{acl}}(A)$ with respect to all sets A of given theory T is denoted by $\deg_{\mathrm{acl}}(T)$ and called the *degree of algebraization* of the theory T.

Following [2], theories T with $\deg_{\mathrm{acl}}(T) = 1$, i.e., with defined $\mathrm{cl}_1(A)$ for any set A of T, are called *quasi-Urbanik*, and the models $\mathcal{M}$ of T are *quasi-Urbanik*, too.

Remark 1. Notice that if a structure $\mathcal{M}$ is quasi-Urbanik it does not guarantee that its theory $T = \mathrm{Th}(\mathcal{M})$ is quasi-Urbanik, too. Indeed, let $\mathcal{M}$ be a strongly minimal structure consisting of infinitely many two-element equivalence classes $E(a)$. Marking one element a in each E-class by a constant c_a, we obtain a syntactically rigid structure $\mathcal{M}'$, with definable $b \in E(a) \setminus \{a\}$ by formulae $E(x, c_a) \wedge \neg x \approx c_a$. At the same time $\mathcal{M}'$ has an elementary extension $\mathcal{N}$ with some unmarked E-classes. These E-classes fail the quasi-Urbanikness of T.

Definition 1. A theory T is called *almost quasi-Urbanik*, if some expansion of T by finitely many constants is quasi-Urbanik, and the models $\mathcal{M}$ of T are *almost quasi-Urbanik*, too. If a finite set A of constants produces a quasi-Urbanik expansion T_A of T then we say that A *witnesses* that T is almost quasi-Urbanik.

The least cardinality of the witnessing set A is called the *quasi-Urbanik $\exists$-degree* of T and it is denoted by $\deg^{\exists}_{\mathrm{qU}}(T)$. If these finite sets A do not exist, we put $\deg^{\exists}_{\mathrm{qU}}(T) = \infty$. The minimal cardinality $n \in \omega$ such that each set A of cardinality n produces the quasi-Urbanik theory T_A is called the *quasi-Urbanik $\forall$-degree* of T and it is denoted by $\deg^{\forall}_{\mathrm{qU}}(T)$. If such n does not exist, then we put $\deg^{\forall}_{\mathrm{qU}}(T) = \infty$. Similarly it is transformed to the models $\mathcal{M}$ of T with quasi-Urbanik $\exists$-degrees $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M})$ and $\forall$-degrees $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M})$.

Clearly, for any theory T, $\deg^{\exists}_{\mathrm{qU}}(T) = 0$ iff $\deg^{\forall}_{\mathrm{qU}}(T) = 0$, and iff T is quasi-Urbanik. Thus, by the definition, any quasi-Urbanik theory is almost quasi-Urbanik.

Besides, for any theory T,

$$\deg^{\exists}_{\mathrm{qU}}(T) \leq \deg^{\forall}_{\mathrm{qU}}(T) \qquad (1)$$

implying that if T is not almost quasi-Urbanik then $\deg^{\exists}_{\mathrm{qU}}(T) = \deg^{\forall}_{\mathrm{qU}}(T) = \infty$, and vice versa.

The following example shows that the difference in the inequality (1) can be arbitrary:

Example 1. Let $\mathcal{M}$ be a structure of an equivalence relation E, $T = \mathrm{Th}(\mathcal{M})$. Clearly, T is quasi-Urbanik iff $\mathcal{M}$ has 0, 1 or infinitely many one-element E-classes, 0 or infinitely many two-element E-classes, and does not have finite E-classes with at least three elements, producing $\deg^{\exists}_{\mathrm{qU}}(T) = \deg^{\forall}_{\mathrm{qU}}(T) = 0$. In particular, $\mathcal{M}$ with zero, one or infinitely many one-element E-classes, zero or infinitely many two-element E-classes, and without n-element E-classes, for $n \geq 3$, is quasi-Urbanik.

A finite value $\deg^{\exists}_{\mathrm{qU}}(T)$ means that we can collect a finite set A containing $m-1$ elements in singletons $E(a)$, if there are $m \in \omega \setminus \{0,1\}$ these singletons, a finite set B containing m elements in pairwise distinct two-element E-classes $E(b)$, if there are $m \in \omega \setminus \{0\}$ these E-classes, and a finite set C containing $n-1$ elements in each E-class $E(a)$ of finite cardinality $n \geq 3$, obtaining $\deg^{\exists}_{\rm qU}(T) = |A| + |B| + |C|$. Here $\deg^{\exists}_{\rm qU}(T) = |C|$ if there are 0, 1 or infinitely many one-element E-classes, and there are 0 or infinitely many two-element E-classes.

At the same time, if $\mathcal{M}$ is not quasi-Urbanik, then $\deg^{\forall}_{\rm qU}(T)$ is finite iff $\mathcal{M}$ is finite. In such a case if $\mathcal{M}$ contains k singletons $E(a)$ and m two-element E-classes, then $\deg^{\forall}_{\rm qU}(T) = k-1$, if $\mathcal{M}$ consists of E-singletons, and $\deg^{\forall}_{\rm qU}(T) = k+2m-1$, if $\mathcal{M}$ consists of one-element and two-element E-classes, and if $\mathcal{M}$ contains n-element E-classes, for $n \geq 3$, then $\deg^{\forall}_{\rm qU}(T) = |M|-1$.

In view of Example 1 we have the following theorem describing possibilities of quasi-Urbanik degrees:

Theorem 1. For any $\mu, \nu \in (\omega\setminus\{0\}) \cup \{\infty\}$ with $\mu \leq \nu$ there is a theory $T_{\mu,\nu}$ such that $\deg^{\exists}_{\rm qU}(T_{\mu,\nu}) = \mu$ and $\deg^{\forall}_{\rm qU}(T_{\mu,\nu}) = \nu$.

For a theory T we denote by $\deg_{2,\rm qU}(T)$ the pair $\left(\deg^{\exists}_{\rm qU}(T), \deg^{\forall}_{\rm qU}(T)\right)$ of quasi-Urbanik degrees for T.

In view of the inequality (1) and Theorem 1 the set

$$\mathrm{DEG}_{2,\rm qU} = \{(0,0)\} \cup \{(\mu,\nu) \in ((\omega \setminus \{0\}) \cup \{\infty\})^2 \mid \mu \leq \nu\} \tag{2}$$

collects the spectrum of all possibilities for $\deg_{2,\rm qU}(T)$.

For a family $\mathcal{T}$ of theories we denote by $\mathrm{DEG}_{2,\rm qU}(\mathcal{T})$ the restriction of $\mathrm{DEG}_{2,\rm qU}$ to the family of theories in $\mathcal{T}$:

$$\mathrm{DEG}_{2,\rm qU}(\mathcal{T}) = \{\deg_{2,\rm qU}(T) \mid T \in \mathcal{T}\}.$$

The operator $\mathrm{DEG}_{2,\rm qU}(\cdot)\colon \mathcal{T} \mapsto \mathrm{DEG}_{2,\rm qU}(\mathcal{T})$ is monotone: indeed, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ then we have $\mathrm{DEG}_{2,\rm qU}(\mathcal{T}_1) \subseteq \mathrm{DEG}_{2,\rm qU}(\mathcal{T}_2)$. Hence, if $\mathrm{DEG}_{2,\rm qU}(\mathcal{T}_1) = \mathrm{DEG}_{2,\rm qU}$ and $\mathrm{DEG}_{2,\rm qU}(\mathcal{T}_1) = \mathrm{DEG}_{2,\rm qU}$ then

$$\mathrm{DEG}_{2,\rm qU}(\mathcal{T}_2) = \mathrm{DEG}_{2,\rm qU}.$$

A natural *question* arises on a description of spectra $\mathrm{DEG}_{2,\rm qU}(\mathcal{T})$ for various families of theories. Below we will give partial answers to this question.

Similarly to theories, for any structure $\mathcal{M}$, $\deg^{\exists}_{\rm qU}(\mathcal{M}) = 0$ iff $\deg^{\forall}_{\rm qU}(\mathcal{M}) = 0$, and iff $\mathcal{M}$ is quasi-Urbanik. Thus, by the definition any quasi-Urbanik structure is almost quasi-Urbanik.

Besides, for any structure $\mathcal{M}$,

$$\deg^{\exists}_{\rm qU}(\mathcal{M}) \leq \deg^{\forall}_{\rm qU}(\mathcal{M}) \tag{3}$$

implying that if $\mathcal{M}$ is not almost quasi-Urbanik then $\deg^{\exists}_{\rm qU}(\mathcal{M}) = \deg^{\forall}_{\rm qU}(\mathcal{M}) = \infty$. Example 1 also illustrates that the difference in the inequality (3) can be arbitrary.

For any model $\mathcal{M}$ of a theory T we have:

$$\deg^{\exists}_{\rm qU}(\mathcal{M}) \leq \deg^{\exists}_{\rm qU}(T) \tag{4}$$

and

$$\deg^{\forall}_{\rm qU}(\mathcal{M}) \leq \deg^{\forall}_{\rm qU}(T). \tag{5}$$

Indeed, if a set A of constants produces quasi-Urbanik theory T_A then its model $\mathcal{M}_A$ is quasi-Urbanik, too. At the same time, as the following example shows, the inequalities (4) and (5) can be strict.

Example 2. Let $\mathcal{M}$ be a strongly minimal structure of an equivalence relation E consisting of infinitely many n-element E-classes such that there is an E-class $E(a)$ elements of which are not marked by constants and all elements in $M \setminus E(a)$ are marked by constants. We have $\deg^{\exists}_{\rm qU}(\mathcal{M}) = n-1$, witnessed by the set $E(a) \setminus \{a\}$, whereas $\deg^{\exists}_{\rm qU}({\rm Th}(\mathcal{M})) = \infty$ since new E-classes in strict elementary extensions of $\mathcal{M}$ fail the quasi-Urbanikness.

If $\mathcal{M}_0$ is an elementary substructure of $\mathcal{M}$ which does not contain $E(a)$ then $\mathcal{M}_0$ is quasi-Urbanik, with $\deg^{\forall}_{\rm qU}(\mathcal{M}_0) = \deg^{\exists}_{\rm qU}(\mathcal{M}_0) = 0$, whereas for $T = {\rm Th}(\mathcal{M}) = {\rm Th}(\mathcal{M}_0)$, $\deg^{\forall}_{\rm qU}(T) = \deg^{\exists}_{\rm qU}(T) = \infty$.

Example 2 illustrates that there are (almost) quasi-Urbanik structures theories of which are not almost quasi-Urbanik.

The list of inequalities (4) and (5) is extended by the following:

$$\deg^{\exists}_{\rm qU}(\mathcal{M}) \leq \deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M}). \tag{6}$$

Indeed, the inequality (6) holds for any structure $\mathcal{M}$ since ${\rm dcl}(A) = M$ implies that ${\rm dcl}(B) = M = {\rm acl}(B)$ for any $B \supseteq M$, i.e. $\mathcal{M}_A$ is quasi-Urbanik, with $\deg^{\exists}_{\rm qU}(\mathcal{M}) \leq |A| = \deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$, where A witnesses the value $\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$.

The inequality (6) can be arbitrarily strict since any unar $\mathcal{M}$ with a successor function $s(x)$ is quasi-Urbanik, with $\deg^{\exists}_{\rm qU}(\mathcal{M})$, whereas that unar can have arbitrarily many connected components. The finite number of connected components equals $\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$, and if there are infinitely many connected components, then $\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M}) = \infty$.

Similarly to the inequality (6) we have the inequality:

$$\deg^{\forall}_{\rm qU}(\mathcal{M}) \leq \deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M}).$$

The following assertion shows that if a theory T is almost quasi-Urbanik, with finite degrees, then the equalities in (4) and (5) hold:

Proposition 1. If a theory T is almost quasi-Urbanik, then each model $\mathcal{M}$ of T is almost quasi-Urbanik, too, with $\deg^{\exists}_{\rm qU}(\mathcal{M}) = \deg^{\exists}_{\rm qU}(T)$. Moreover, if $\deg^{\forall}_{\rm qU}(T)$ is finite, then we have $\deg^{\forall}_{\rm qU}(\mathcal{M})$ is finite, too, with $\deg^{\forall}_{\rm qU}(\mathcal{M}) = \deg^{\forall}_{\rm qU}(T)$.

Proof. Let $\deg^{\exists}_{\rm qU}(T) = n \in \omega$. Then T admits an expansion T_A by a set A of constants, with minimal cardinality n, such that T_A is quasi-Urbanik: $\deg^{\exists}_{\rm qU}(T_A) = 0$. Since a model $\mathcal{M}$ of T is expansible till a model $\mathcal{M}_A$ of T_A and the property $\deg^{\exists}_{\rm qU}(T) = n$ is expressed syntactically containing a description that $(n-1)$-element sets do not produce the quasi-Urbanikness, it is satisfies in $\mathcal{M}$ implying $\deg^{\exists}_{\rm qU}(\mathcal{M}) = n$. Similar arguments witness that if $\deg^{\forall}_{\rm qU}(T) = n \in \omega$, then $\deg^{\forall}_{\rm qU}(\mathcal{M}) = \deg^{\forall}_{\rm qU}(T)$ for any $\mathcal{M} \models T$. □

Let Σ_1 be a signature of both unary predicate symbols and constant symbols.

The following theorem describes the behavior of almost quasi-Urbanikness of theories in the signature Σ_1.

Theorem 2. Let T be a theory of a signature Σ_1, $\mathcal{M} \models T$. Then the following conditions hold:

1) T is quasi-Urbanik iff each algebraic 1-type over $\emptyset$ has a unique realization;

2) T is almost quasi-Urbanik iff T has finitely many algebraic 1-types $p_1, \ldots, p_n$ over $\emptyset$ with at least two realizations; here $\deg^{\exists}_{\rm qU}(T) = \sum\limits_{i=1}^{n}(|p_i(\mathcal{M})| - 1)$;

3) $\deg^{\forall}_{\rm qU}(T) > 0$ is finite iff $\mathcal{M}$ is finite and has an algebraic 1-type $p \in S(\emptyset)$ with at least two realizations; here $\deg^{\forall}_{\rm qU}(T) = |M| - 1$.

Proof. Without loss of generality we assume that constant symbols are replaced by unary predicates with unique solutions. In view of the signature Σ_1 there are no links between elements and algebraic sets are defined by Boolean combinations of given unary predicates such that these Boolean combinations have finitely many solutions. Thus, the quasi-Urbanikness means that these complete Boolean combinations defining $\mathrm{acl}(\emptyset)$ defines singletons producing Item 1.

The almost quasi-Urbanikness of T means that algebraic 1-types p become definable after fixing all their realizations except one for each type p. It confirms Item 2.

If $\deg^{\forall}_{\rm qU}(T) \in \omega \setminus \{0\}$, then T is almost quasi-Urbanik and is not quasi-Urbanik. Using 1) and 2) we find an algebraic 1-type $p \in S(\emptyset)$ with at least two realizations. Now $\mathcal{M}$ is finite and the $(|M|-1)$-element subsets of M confirm the value $\deg^{\exists}_{\rm qU}(T)$ since the smaller quantity can not cover universally $|p(\mathcal{M})|-1$ realizations of p. □

In view of the equality (2) and Theorem 2, we have the following:

Corollary 1. Let $\mathcal{T}$ be the family of theories in signatures of the form Σ_1. Then

$$\mathrm{DEG}_{2,\mathrm{qU}}(\mathcal{T}) = \mathrm{DEG}_{2,\mathrm{qU}}.$$

Clearly, any theory of a finite structure is almost quasi-Urbanik. At the same time, as the following example shows, there are almost quasi-Urbanik theories of infinite structures which are not quasi-Urbanik.

Example 3. Let $\mathcal{M}$ be a countable structure of an equivalence relation E with one two-element E-class $E_0 = \{a, b\}$ and two infinite E-classes E_a and E_b such that $\mathcal{M}$ is supplied by a binary relation $R = \{(a, a') \mid a' \in E_a\} \cup \{(b, b') \mid b' \in E_b\}$. For $T = \mathrm{Th}(\mathcal{M})$, we have $\deg^{\forall}_{\rm qU}(T) = 1$ since $\mathrm{dcl}(\emptyset) = \emptyset$, $E_0 = \mathrm{acl}(\emptyset)$, and $M = \mathrm{dcl}(\{d\})$ for any element $d \in M$, producing $\mathrm{dcl}(A) = \mathrm{acl}(A)$ for any nonempty $A \subseteq M$. $\deg^{\exists}_{\rm qU}(T) = 1$, too. Thus, $\deg_{2,\mathrm{qU}}(T) = (1, 1)$. The theory T is ω-categorical and ω-stable with Morley rank 1 and Morley degree 2: $\mathrm{MR}(T) = 1$, $\deg(T) = 2$.

Below we will show that natural values $\deg^{\forall}_{\rm qU}(T) \geq 1$ can not be realized in the class of strongly minimal theories, i.e. Morley characteristics in Example 3 are minimally possible.

2 Spectra of almost quasi-Urbanikness for ordered structures and their theories

Example 4. Let $\mathcal{M}$ be a structure of an equivalence relation E expanded by a linear order on the quotient $\mathcal{M}/E$, i.e., $\mathcal{M}$ is a preordered set by a preorder $\leq$ such that maximal antichains form the equivalence relation E such that elements in distinct E-classes are $\leq$-comparable. Besides, $E = \leq \cap \geq$.

Clearly, $T = \mathrm{Th}(\mathcal{M})$ is quasi-Urbanik iff E has either one-element or infinite E-classes. Moreover, any linearly ordered structure $\mathcal{M}$ is quasi-Urbanik, i.e. $\deg_{2,\mathrm{qU}}(T) = (0, 0)$.

Since elements of each E-class $E(a)$ are connected by automorphisms over sets of elements in other E-classes, the possibilities of values $\deg^{\exists}_{\rm qU}(\mathcal{M})$ and $\deg^{\forall}_{\rm qU}(\mathcal{M})$ repeat ones in Example 1. Here, if $\mathcal{M}$ has finitely many one-element E-classes then all these E-classes are contained in $\mathrm{dcl}(\emptyset)$.

In particular, $\mathrm{Th}(\mathcal{M})$ is almost quasi-Urbanik with $\deg^{\exists}_{\rm qU}(\mathrm{Th}(\mathcal{M})) > 0$ iff $\mathcal{M}$ has a finite E-class with at least two elements and there are finitely many these E-classes, and $\deg^{\exists}_{\rm qU}(\mathrm{Th}(\mathcal{M})) \in \omega \setminus \{0\}$ iff $\mathcal{M}$ is quasi-Urbanik, or $\mathcal{M}$ is finite with some E-class containing at least two elements.

In view of Example 4 we have the following modification of Theorem 1:

Theorem 3. Let $\mathcal{T}_{\rm po}$ be the family of theories of preordered structures. Then

$$\mathrm{DEG}_{2,\mathrm{qU}}(\mathcal{T}_{\rm po}) = \mathrm{DEG}_{2,\mathrm{qU}}.$$

Definition 2. [12,13]. The following generalization of linear and circular orders produces an *n-ball*, or *n-spherical*, or *n-circular* order relation, for $n \geq 2$, which is described by an n-ary relation K_n satisfying the following conditions:

(nso1) for any even permutation σ on $\{1, 2, \ldots, n\}$,

$$\forall x_1, \ldots, x_n \left(K_n(x_1, x_2, \ldots, x_n) \to K_n\left(x_{\sigma(1)}, x_{\sigma(2)} \ldots, x_{\sigma(n)}\right)\right);$$

(nso2) $$\forall x_1, \ldots, x_n \Bigg((K_n(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \wedge$$

$$\wedge K_n(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)) \leftrightarrow \bigvee_{1 \le k < l \le n} x_k \approx x_l \Bigg)$$

for any $1 \le i < j \le n$;

(nso3) $$\forall x_1, \ldots, x_n \Bigg(K_n(x_1, \ldots, x_n) \to$$

$$\to \forall t \left(\bigvee_{i=1}^{n} K_n(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) \right) \Bigg);$$

(nso4) $$\forall x_1, \ldots, x_n (K_n(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \vee$$

$$\vee K_n(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)),\ 1 \le i < j \le n.$$

The axioms above produce linear orders K_2 and circular orders K_3.

Structures $\mathcal{M} = \langle M, K_n \rangle$ with n-spherical orders K_n are called *n-spherically ordered sets*, or *n-spherical orders*, too. If a structure $\mathcal{M}$ contains a n-spherical order, then $\mathcal{M}$ is called a *n-spherically ordered structure*, or simply *spherically ordered structure* if n is known.

An n-spherically ordered set $\langle A, K_n \rangle$, where $n \ge 2$, is called *dense* if it contains at least two elements and for each $(a_1, a_2, a_3, \ldots, a_n) \in K_n$ with $a_1 \ne a_2$ there is $b \in A \setminus \{a_1, a_2, \ldots, a_n\}$ such that

$$\models K_n(a_1, b, a_3, \ldots, a_n) \wedge K_n(b, a_2, a_3, \ldots, a_n).$$

Following [14], n-spherical orders K_n on infinite sets M witness the strict order property producing unstable structures $\langle M, K_n \rangle$, since fixing $n-2$ distinct coordinates $a_1, \ldots, a_{n-2}$ in the relation K_n, we obtain a linear order on $M \setminus \{a_1, \ldots, a_{n-2}\}$.

As any linearly ordered structure is quasi-Urbanik, we obtain the following:

Theorem 4. Any n-spherically ordered structure $\mathcal{M}$ has an almost quasi-Urbanik theory T with $\deg^{\forall}_{\mathrm{qU}}(T) \le n-2$.

Remark 2. The inequality in Theorem 4 can be strict and can produce the equality. Indeed, if $\mathcal{M}$ is a dense spherical order, then $\mathcal{M}$ is quasi-Urbanik, with $\deg^{\forall}_{\mathrm{qU}}(\mathrm{Th}(\mathcal{M})) = 0$, since $\mathrm{dcl}(A) = \mathrm{acl}(A) = A$ since the theory $\mathrm{Th}(\mathcal{M})$ has quantifier elimination [13] without possibilities to define new algebraic elements, outside A.

At the same time if $\mathcal{M}'$ is an expansion of $\mathcal{M}$ by a unary predicate P_m containing $m > 1$ elements, then we have to fix $n-2$ arbitrary elements in M producing a quasi-Urbanik expansion that implies $\deg^{\forall}_{\mathrm{qU}}(\mathrm{Th}(\mathcal{M}')) = n-2$. Here the value $\deg^{\exists}_{\mathrm{qU}}(\mathrm{Th}(\mathcal{M}'))$ can vary depending on m: it is equal $m-1$ for $m-1 < n-2$, and equals $n-2$ if $m-1 \ge n-2$.

In view of the equality (2), Theorem 4 and Remark 2, we have:

Theorem 5. Let $\mathcal{T}_{\mathrm{so}}$ be the family of theories of spherically ordered structures. Then

$$\mathrm{DEG}_{2,\mathrm{qU}}(\mathcal{T}_{\mathrm{so}}) = \{(0,0)\} \cup \{(m,n) \mid m,n \in \omega \setminus \{0\}, m \le n\} =$$

$$= \mathrm{DEG}_{2,\mathrm{qU}} \setminus \{(\mu, \infty) \mid \mu \in (\omega \setminus \{0\}) \cup \{\infty\}\}.$$

The following examples illustrate possibilities for $\deg_{2,\mathrm{qU}}(T) = (m, n)$ for extensions of spherically ordered structures by two new elements.

Example 5. Let $\mathcal{M}_{1,2}$ be a countable structure of an equivalence relation E with one two-element E-class $E_0 = \{a_1, a_2\}$ and one infinite E-class E_1 such that E_1 is supplied by a binary relation $<$ of dense linear order without endpoints, and $\mathcal{M}$ is supplied by a ternary relation R_3 consisting of triples (a_1, b_1, b_2) and $(a_2.b_2.b_1)$ for any $b_1 < b_2$. For the theory $T_{1,2} = \mathrm{Th}(\mathcal{M}_{1,2})$, we have $\deg^{\exists}_{\mathrm{qU}}(T_{1,2}) = 1$ and $\deg^{\forall}_{\mathrm{qU}}(T_{1,2}) = 2$ since $\mathrm{dcl}(\emptyset) = \emptyset$, $E_0 = \mathrm{acl}(\emptyset)$, $E_0 = \mathrm{dcl}(\{a_i\}) = \mathrm{dcl}(\{a_i, b_1\}) = \mathrm{dcl}(\{b_1, b_2\})$, $i = 1, 2$, $b_1, b_2 \in E_1$, $b_1 \neq b_2$, producing $\mathrm{dcl}(A) = \mathrm{acl}(A)$ for any $A \subseteq M_{1,2}$ with $|A| \geq 2$. Thus $\deg_{2,\mathrm{qU}}(T_{1,2}) = (1, 2)$.

Now we modify the theory $T_{1,2}$ replacing in $\mathcal{M}_{1,2}$ the linear order $<$ by a dense n-spherical order K_n [13], $n \geq 3$, reduced to the strict one K^*_n, i.e. the reduction of the spherical order to the set of tuples with pairwise distinct coordinates. The relation K_n divides the set of n-tuples with pairwise distinct coordinates in E_1 into two parts such that the complement $\overline{K_n}$ of K_n in E^n_1 equals the set of odd permutations of tuples in K^*_n. Instead of R_3 we consider the $(n+1)$-ary relation R_{n+1} collecting tuples $(a_1, \bar{b})$, $\bar{b} \in K^*_n$, and tuples $(a_2, \bar{b})$, $\bar{b} \in \overline{K_n}$. For the obtained structure $\mathcal{M}_{1,n}$ and its theory $T_{1,n}$, we have $\deg^{\exists}_{\mathrm{qU}}(T_{1,n}) = 1$ and $\deg^{\forall}_{\mathrm{qU}}(T_{1,n}) = n$ since $\mathrm{dcl}(\emptyset) = \emptyset$, $E_0 = \mathrm{acl}(\emptyset)$, $E_0 = \mathrm{dcl}(\{a_i\}) = \mathrm{dcl}(\bar{b})$, $i = 1, 2$, $\bar{b} \in E_1$ with pairwise distinct coordinates, $l(\bar{b}) = n$, producing $\mathrm{dcl}(A) = \mathrm{acl}(A)$ for any $A \subseteq M_{1,n}$ with $|A| \geq n$. Thus, $\deg_{2,\mathrm{qU}}(T_{1,n}) = (1, n)$.

3 Spectra of almost quasi-Urbanikness and relative dimensions of algebraic closures for strongly minimal theories

Theorem 6. For any strongly minimal theory T either $\deg^{\forall}_{\mathrm{qU}}(T) = 0$ or $\deg^{\forall}_{\mathrm{qU}}(T) = \infty$.

Proof. Let $\deg^{\forall}_{\mathrm{qU}}(T) = m > 0$, $m \in \omega$. Taking a big saturated model $\mathcal{M}$, we find an algebraic set $A \subset M$ with $|A| = n > 1$ such that A is definable by a complete formula $\varphi(x, \bar{a})$ such that for any m-tuple $\bar{b} \subset M$, A is divided into singletons by formulae $\psi_i(x, \bar{b})$, $i = 1, \ldots, n$. Since the tuples $\bar{b}$ are arbitrary, we can fix $m - 1$ coordinates and take one mobile coordinate, say m-th one, realizing the unique non-algebraic type $p(y)$ over A. Moreover, as the model $\mathcal{M}$ is big enough, elements of A are connected by $\bar{a}$-automorphisms and realizations of $p(y)$ are connected by A-automorphisms. Since A is finite and its elements are connected by $\bar{a}$-automorphisms we can connect elements of A by a fixed formula ψ_i with infinitely many realizations of $p(y)$ such that ψ_i-images with respect to these realizations are unique. As T is strongly minimal, $\mathcal{M}$ can not be divided into two infinite definable parts. Thus, ψ_i-preimages of elements of A should be intersected contradicting the uniqueness of ψ_i-images. □

In view of the equality (2), Theorem 6 and strongly minimal realizations of quasi-Urbanik degrees in Example 1, we obtain the following:

Corollary 2. Let $\mathcal{T}_{\mathrm{sm}}$ be the family of strongly minimal theories. Then

$$\mathrm{DEG}_{2,\mathrm{qU}}(\mathcal{T}_{\mathrm{sm}}) = \{(0,0)\} \cup \{(\mu, \infty) \mid \mu \in (\omega \setminus \{0\}) \cup \{\infty\}\}.$$

Example 6. Let $\mathcal{M} = \langle M, s^1 \rangle$ be a structure, where $s(x)$ is the successor function, and $Th(\mathcal{M})$ has the following axioms:

$$A_1 := \forall z \exists!\, t\; s(z) = t,$$

$$A_2 := \exists x_1 \exists x_2 [x_1 \neq x_2 \wedge \forall y_1\; s(y_1) \neq x_1 \wedge \forall y_2\; s(y_2) \neq x_2 \wedge \forall t (t \neq x_1 \wedge t \neq x_2 \to \exists z\; s(z) = t)],$$

$$A_3 := \forall x_1 \forall x_2 \forall t_1 \forall t_2 [s(x_1) = t_1 \wedge s(x_2) = t_2 \to (x_1 \neq x_2 \leftrightarrow t_1 \neq t_2)].$$

Thus, M consists of two disjoint copies of $\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. It can be established that $\mathrm{Th}(\mathcal{M})$ is a strongly minimal theory. Further, we have: $\mathrm{dcl}(\emptyset) = \emptyset$, $\mathrm{acl}(\emptyset) = M$, and $\mathrm{dcl}(A) = \mathrm{acl}(A)$ for any non-empty $A \subseteq M$. Thus, $\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)$ is infinite, $Th(\mathcal{M})$ is almost quasi-Urbanik and non-quasi-Urbanik.

We say that a set A is *definably independent* if $a \notin \mathrm{dcl}(A \setminus \{a\})$ for any $a \in A$. Denote by $\dim_{dcl}(A)$ the cardinality of maximal definably independent subset of A.

Observe that in Example 6 $\dim_{dcl}(\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)) = 1$.

Consider for every $m \geq 2$ the following sentence:

$$B_m := \exists x_1 \dots \exists x_m [\wedge_{1 \leq i < j \leq m} x_i \neq x_j \wedge \wedge_{i=1}^{m} \forall y\, s(y) \neq x_i \wedge \forall t (\wedge_{i=1}^{m} t \neq x_i \to \exists z\, s(z) = t)].$$

If we consider the structure $\mathcal{M} = \langle M, s^1 \rangle$ with axioms A_1, B_m and A_3, then we have $\dim_{dcl}(\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)) = m - 1$.

Consider for every $m \geq 2$ the following sentence:

$$C_m := \exists x_1 \dots \exists x_m [\wedge_{1 \leq i < j \leq m} x_i \neq x_j \wedge \wedge_{i=1}^{m} \forall y\, s(y) \neq x_i].$$

If we consider the structure $\mathcal{M} = \langle M, s^1 \rangle$ with axioms A_1, A_3 and $\{C_m \mid m \geq 1\}$, then we lose the strong minimality, and $Th(\mathcal{M})$ is an ω-stable quasi-Urbanik theory of Morley rank 2 with $\mathrm{acl}(\emptyset) = \emptyset$.

Thus, we have the following proposition:

Proposition 2. For every natural $m \geq 1$ there exists an almost quasi-Urbanik strongly minimal theory such that $\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)$ is infinite and $\dim_{dcl}(\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)) = m$.

The following example shows that if T is not almost quasi-Urbanik, then $\dim_{dcl}(\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)) = m$ can be infinite.

Example 7. Let $\mathcal{M} = \langle M, E^2 \rangle$ be a strongly minimal structure, where E is an equivalence relation partitioning M into infinitely many n-element E-classes for some $n \geq 3$. Let $\mathcal{M}'$ be an expansion of $\mathcal{M}$ by marking exactly one element from each E-class by a constant. Then we have that both $\mathrm{acl}_{\mathcal{M}'}(\emptyset) \setminus \mathrm{dcl}_{\mathcal{M}'}(\emptyset)$ and $\dim_{dcl}(\mathrm{acl}_{\mathcal{M}'}(\emptyset) \setminus \mathrm{dcl}_{\mathcal{M}'}(\emptyset))$ are infinite, but $\mathrm{Th}(\mathcal{M}')$ is not almost quasi-Urbanik, cf. Example 1. At the same time, following Example 5, the theory $\mathrm{Th}(\mathcal{M}')$ admits a cyclification producing a quasi-Urbanik expansion.

The following example produces a similar effect, as in Example 7, in the class of simple unstable theories.

Example 8. Consider a predicate language L consisting of two unary predicate symbols P and Q, two binary predicate symbols E and R, expanded by countably many constant symbols c_n, $n \in \omega$. We construct a countable structure $\mathcal{M}$ with $M = P \dot\cup Q$, where P and Q are countable, such that E is an equivalence relation dividing P on three-element E-classes E_n with $c_n \in E_n$, $n \in \omega$, and having one-element E-classes on Q. Now we interpret R as a random symmetric binary relation connecting each element of Q with one element in each $E(c_n) \setminus \{c_n\}$, $n \in \omega$.

We have $\mathrm{dcl}_{\mathcal{M}}(\emptyset) = \{c_n \mid n \in \omega\}$, $\mathrm{acl}_{\mathcal{M}}(\emptyset) = \bigcup\limits_{n \in \omega} E(c_n)$, with infinite $\mathrm{acl}_{\mathcal{M}}(\emptyset) \setminus \mathrm{dcl}_{\mathcal{M}}(\emptyset)$. Thus the structure $\mathcal{M}$ is not quasi-Urbanik. At the same time $\mathcal{M}$ is almost quasi-Urbanik, since for any element $a \in Q$ the expansion $\mathcal{M}'$ of $\mathcal{M}$ by the constant c_a for this element allows to define all elements of P: $\mathrm{dcl}_{\mathcal{M}'}(\emptyset) = \mathrm{acl}_{\mathcal{M}'}(\emptyset) = P \cup \{c_a\}$. Moreover, for any $A \subseteq M$, $\mathrm{dcl}_{\mathcal{M}'}(A) = \mathrm{acl}_{\mathcal{M}'}(A) = \mathrm{dcl}_{\mathcal{M}'}(\emptyset) \cup (Q \cap A)$.

Finally we observe that the theory $\mathrm{Th}(\mathcal{M})$ is not almost quasi-Urbanik, since $\mathcal{M}$ has elementary extensions with three-element E-classes which are not marked by constants, and finitely many new constants can not reduce algebraic closures to definable ones for these E-classes.

4 Degrees of quasi-Urbanikness for disjoint unions of structures and their theories

In this section we describe possibilities for degrees of quasi-Urbanikness for disjoint unions of structures and their theories. This description correlates with similar description for degrees of rigidity [5].

Definition 3. [15] The *disjoint union* $\bigsqcup\limits_{n\in\omega}\mathcal{M}_n$ of pairwise disjoint structures $\mathcal{M}_n$ for pairwise disjoint predicate languages Σ_n, $n\in\omega$, is the structure of language $\bigcup\limits_{n\in\omega}\Sigma_n\cup\{P_n^{(1)}\mid n\in\omega\}$ with the universe $\bigsqcup\limits_{n\in\omega}M_n$, $P_n=M_n$, and interpretations of predicate symbols in Σ_n coinciding with their interpretations in $\mathcal{M}_n$, $n\in\omega$. The *disjoint union of theories* T_n for pairwise disjoint languages Σ_n accordingly, $n\in\omega$, is the theory

$$\bigsqcup_{n\in\omega}T_n\rightleftharpoons\mathrm{Th}\left(\bigsqcup_{n\in\omega}\mathcal{M}_n\right),$$

where $\mathcal{M}_n\models T_n$, $n\in\omega$.

Clearly, the theory $\bigsqcup\limits_{n\in\omega}T_n$ does not depend on choice of models $\mathcal{M}_n\models T_n$. Besides, the notion of disjoint union admits reductions to finitely many structures and theories, obtaining the structures $\mathcal{M}_1\sqcup\ldots\sqcup\mathcal{M}_n$ and their theories $T_1\sqcup\ldots\sqcup T_n$.

Theorem 7. For any disjoint predicate structures $\mathcal{M}_1$ and $\mathcal{M}_2$ the following conditions hold:

1. $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_1\sqcup\mathcal{M}_2)=\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_1)+\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_2)$, in particular, $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_1\sqcup\mathcal{M}_2)$ is finite iff $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_1)$ and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_2)$ are finite.

2. $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1\sqcup\mathcal{M}_2)=0$ iff $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1)=0$ and $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_2)=0$, i.e. $\mathcal{M}_1\sqcup\mathcal{M}_2$ is quasi-Urbanik iff $\mathcal{M}_1$ and $\mathcal{M}_2$ are quasi-Urbanik.

3. If $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1\sqcup\mathcal{M}_2)>0$ then it is finite iff $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1)>0$ is finite and $\mathcal{M}_2$ is finite, or $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_2)>0$ is finite and $\mathcal{M}_1$ is finite. Here,

$$\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1\sqcup\mathcal{M}_2)=\max\{|M_1|+\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_2),|M_2|+\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1)\}.$$

Proof word by word repeats the proof of Theorem 2 in [5] replacing degrees of rigidity by degrees of quasi-Urbanikness. □

Theorem 7 immediately implies the following corollaries.

Corollary 3. For any disjoint predicate structures $\mathcal{M}_1$ and $\mathcal{M}_2$ and their theories T_1 and T_2, respectively, the following conditions hold:

1. $\deg^{\exists}_{\mathrm{qU}}(T_1\sqcup T_2)=\deg^{\exists}_{\mathrm{qU}}(T_1)+\deg^{\exists}_{\mathrm{qU}}(T_2)$, in particular, $\deg^{\exists}_{\mathrm{qU}}(T_1\sqcup T_2)$ is finite iff $\deg^{\exists}_{\mathrm{qU}}(T_1)$ and $\deg^{\exists}_{\mathrm{qU}}(T_2)$ are finite.

2. $\deg^{\forall}_{\mathrm{qU}}(T_1\sqcup T_2)=0$ iff $\deg^{\forall}_{\mathrm{qU}}(T_1)=0$ and $\deg^{\forall}_{\mathrm{qU}}(T_2)=0$, i.e. $T_1\sqcup T_2$ is quasi-Urbanik iff T_1 and T_2 are quasi-Urbanik.

3. If $\deg^{\forall}_{\mathrm{qU}}(T_1\sqcup T_2)>0$, then it is finite iff $\deg^{\forall}_{\mathrm{qU}}(T_1)>0$ is finite and $\mathcal{M}_2$ is finite, or $\deg^{\forall}_{\mathrm{qU}}(T_2)>0$ is finite and $\mathcal{M}_1$ is finite. Here,

$$\deg^{\forall}_{\mathrm{qU}}(T_1\sqcup T_2)=\max\{|M_1|+\deg^{\forall}_{\mathrm{qU}}(T_2),|M_2|+\deg^{\forall}_{\mathrm{qU}}(T_1)\}.$$

Corollary 4. Let $\mathcal{T}$ be the family of all theories of form $T_1\sqcup T_2$. Then $\mathrm{DEG}_{2,\mathrm{qU}}(\mathcal{T})=\mathrm{DEG}_{2,\mathrm{qU}}$.

5 *Degrees of quasi-Urbanikness for compositions of structures and their theories*

Recall the notions of composition for structures and theories.

Definition 4. [16] Let $\mathcal{M}$ and $\mathcal{N}$ be structures of relational languages $\Sigma_{\mathcal{M}}$ and $\Sigma_{\mathcal{N}}$ respectively. We define the *composition* $\mathcal{M}[\mathcal{N}]$ of $\mathcal{M}$ and $\mathcal{N}$ satisfying the following conditions:

1) $\Sigma_{\mathcal{M}[\mathcal{N}]}=\Sigma_{\mathcal{M}}\cup\Sigma_{\mathcal{N}}$;

2) $M[N]=M\times N$, where $M[N]$, M, N are universes of $\mathcal{M}[\mathcal{N}]$, $\mathcal{M}$, and $\mathcal{N}$ respectively;

3) if $R\in\Sigma_{\mathcal{M}}\setminus\Sigma_{\mathcal{N}}$, $\mu(R)=n$, then $((a_1,b_1),\ldots,(a_n,b_n))\in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1,\ldots,a_n)\in R_{\mathcal{M}}$;

4) if $R \in \Sigma_{\mathcal{N}} \setminus \Sigma_{\mathcal{M}}$, $\mu(R) = n$, then $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $a_1 = \ldots = a_n$ and $(b_1, \ldots, b_n) \in R_{\mathcal{N}}$;

5) if $R \in \Sigma_{\mathcal{M}} \cap \Sigma_{\mathcal{N}}$, $\mu(R) = n$, then $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1, \ldots, a_n) \in R_{\mathcal{M}}$, or $a_1 = \ldots = a_n$ and $(b_1, \ldots, b_n) \in R_{\mathcal{N}}$.

The theory $T = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is called the *composition* $T_1[T_2]$ of the theories $T_1 = \mathrm{Th}(\mathcal{M})$ and $T_2 = \mathrm{Th}(\mathcal{N})$.

By the definition, the composition $\mathcal{M}[\mathcal{N}]$ is obtained replacing each element of $\mathcal{M}$ by a copy of $\mathcal{N}$.

Definition 5. [16]. The composition $\mathcal{M}[\mathcal{N}]$ is called *E-definable* if $\mathcal{M}[\mathcal{N}]$ has an $\emptyset$-definable equivalence relation E E-classes of which are universes of the copies of $\mathcal{N}$ forming $\mathcal{M}[\mathcal{N}]$.

Proposition 3. Let $\mathcal{M}[\mathcal{N}]$ be an E-definable composition consisting of copies $\mathcal{N}_i$, $i \in I$, of the structure $\mathcal{N}$, $A \subseteq M[N]$. Then:

1) $\mathrm{acl}_{\mathcal{M}[\mathcal{N}]}(A) = \bigcup\limits_i \mathrm{acl}_{\mathcal{N}_i}(A \cap N_i) \cup \bigcup\limits_{N_j/E \in \mathrm{acl}_{\mathcal{M}[\mathcal{N}]/E}(A/E)} \mathrm{acl}_{\mathcal{N}_j}(\emptyset)$;

2) $\mathrm{dcl}_{\mathcal{M}[\mathcal{N}]}(A) = \bigcup\limits_i \mathrm{dcl}_{\mathcal{N}_i}(A \cap N_i) \cup \bigcup\limits_{N_j/E \in \mathrm{dcl}_{\mathcal{M}[\mathcal{N}]/E}(A/E)} \mathrm{dcl}_{\mathcal{N}_j}(\emptyset)$.

Proof. 1. By the definition of E-definable composition, formulae define both E-classes by means of the language for $\mathcal{M}$ and subsets of E-classes by means of the language for $\mathcal{N}$. Therefore formulae in the language for $\mathcal{M}[\mathcal{N}]$ define both algebraic sets of E-classes in the quotient $\mathcal{M}[\mathcal{N}]/E$ and algebraic sets inside copies $\mathcal{N}_i$ of $\mathcal{N}$. Thus $\mathrm{acl}_{\mathcal{M}[\mathcal{N}]}(A)$ is composed by algebraic sets inside copies $\mathcal{N}_i$ containing elements of A and defined by restrictions $A \cap N_i$, and by algebraic sets of E-classes with respect to the quotient A/E in $\mathcal{M}[\mathcal{N}]/E$. In the latter case defining finitely many E-classes containing copies $\mathcal{N}_j$, we collect $\mathrm{acl}_{\mathcal{N}_j}(\emptyset)$, obtaining the required equality.

2. We repeat the arguments above replacing algebraic closures by definable ones. □

The following theorem describes possibilities of $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}])$ with respect to characteristics of given predicate structures $\mathcal{M}$ and $\mathcal{N}$.

Theorem 8. For any E-definable composition $\mathcal{M}[\mathcal{N}]$ the following conditions hold:

1) if $\mathcal{N}$ is finite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = 0$, then $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \deg^{\exists}_{\mathrm{qU}}(\mathcal{M})$;

2) if $\mathcal{N}$ is finite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) > 0$, then $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) \cdot |M|$ for finite $\mathcal{M}$ and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \infty$ for infinite $\mathcal{M}$; these equalities stay valid for infinite $\mathcal{N}$ with positive natural $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N})$;

3) if $\mathcal{N}$ is infinite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = 0$, then $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = 0$;

4) if $\mathcal{N}$ is infinite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = \infty$, then $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \infty$.

Proof. 1. Let $\mathcal{N}$ be finite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = 0$. It implies that for any set $A \subseteq N$, its algebraic closure equals definable one. Now taking elements in each copy of $\mathcal{N}$ laying in $\mathcal{M}[\mathcal{N}]$ such that these copies correspond to elements in $\mathcal{M}$ witnessing $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M})$, we obtain algebraic closures composed by copies of $\mathcal{N}$ correspondent to elements of algebraic closures in $\mathcal{M}$ and reduced to definable ones. Thus, using algebraic sets in algebraic closures described in Proposition 3, we obtain $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \deg^{\exists}_{\mathrm{qU}}(\mathcal{M})$.

2. Let $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = n \in \omega \setminus \{0\}$. Since copies of $\mathcal{N}$ in $\mathcal{M}[\mathcal{N}]$ become quasi-Urbanik marking independently appropriate n elements, we have to mark these elements by constants all together obtaining $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = n \cdot |M|$. If $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = \infty$, then each copy of $\mathcal{N}$ in $\mathcal{M}[\mathcal{N}]$ can not become quasi-Urbanik after marking by finitely many constants implying $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \infty$.

3. Let $\mathcal{N}$ be infinite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = 0$. Then algebraic closures are definable inside any copy of $\mathcal{N}$ in $\mathcal{M}[\mathcal{N}]$ and, as $\mathcal{N}$ be infinite, this property is preserved for links between distinct copies of $\mathcal{N}$ in $\mathcal{M}[\mathcal{N}]$ in view of Proposition 3. Thus $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = 0$.

4. We repeat arguments for Item 2 with infinite $\mathcal{N}$: if $\deg^{\exists}_{\rm qU}(\mathcal{N}) = \infty$, then no finite set can transform an algebraic closure into a definable one implying $\deg^{\exists}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \infty$. □

Remark 3. It is shown in [16] that E-definable compositions $\mathcal{M}[\mathcal{N}]$ uniquely define theories ${\rm Th}(\mathcal{M}[\mathcal{N}])$ by theories ${\rm Th}(\mathcal{M})$ and ${\rm Th}(\mathcal{N})$ and types of elements in copies of $\mathcal{N}$ are defined by types in these copies and types for connections between these copies.

In view of Theorem 8 and Remark 3 we have the following:

Corollary 5. For any E-definable composition $\mathcal{M}[\mathcal{N}]$ and the theories T_1 of $\mathcal{M}$, T_2 of $\mathcal{N}$, and $T_1[T_2]$ of $\mathcal{M}[\mathcal{N}]$ the following conditions hold:

1) if $\mathcal{N}$ is finite and $\deg^{\exists}_{\rm qU}(T) = 0$, then $\deg^{\exists}_{\rm qU}(T_1[T_2]) = \deg^{\exists}_{\rm qU}(T_1)$;

2) if $\mathcal{N}$ is finite and $\deg^{\exists}_{\rm qU}(T_2) > 0$, then $\deg^{\exists}_{\rm qU}(T_1[T_2]) = \deg^{\exists}_{\rm qU}(T_2)\cdot|M|$ for finite $\mathcal{M}$ and $\deg^{\exists}_{\rm qU}(T_1[T_2]) = \infty$ for infinite $\mathcal{M}$; these equalities stay valid for infinite $\mathcal{N}$ with positive natural $\deg^{\exists}_{\rm qU}(T_2)$;

3) if $\mathcal{N}$ is infinite and $\deg^{\exists}_{\rm qU}(T_2) = 0$, then $\deg^{\exists}_{\rm qU}(T_1[T_2]) = 0$;

4) if $\mathcal{N}$ is infinite and $\deg^{\exists}_{\rm qU}(T_2) = \infty$, then $\deg^{\exists}_{\rm qU}(T_1[T_2]) = \infty$.

The following theorem describes possibilities of $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}])$ with respect to characteristics of given predicate structures $\mathcal{M}$ and $\mathcal{N}$.

Theorem 9. For any E-definable composition $\mathcal{M}[\mathcal{N}]$ the following conditions hold:

1) if $\mathcal{N}$ is finite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) = 0$, then $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{M})\cdot|N|$;

2) if $\mathcal{N}$ is finite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) > 0$, then $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{N}) + (|M|-1)|N|$ for finite $\mathcal{M}$ and $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \infty$ for infinite $\mathcal{M}$;

3) if $\mathcal{N}$ is infinite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) = 0$, then $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = 0$;

4) if $\mathcal{N}$ is infinite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) > 0$, then $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{N})$ for $|M| = 1$ and $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \infty$ for $|M| > 1$.

Proof. 1. Let $\mathcal{N}$ be finite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) = 0$. Then the value $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}])$ is reduced to the value $\deg^{\forall}_{\rm qU}(\mathcal{M})$, where each element in a set witnessing this value is replaced by a copy of $\mathcal{N}$. Since all elements of these copies are involved to witness $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}])$, we obtain $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{M})\cdot|N|$.

2. Let $\mathcal{N}$ be finite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) > 0$. In such a case each copy of $\mathcal{N}$ $\mathcal{M}[\mathcal{N}]$ should contain copies of sets witnessing $\deg^{\forall}_{\rm qU}(\mathcal{N})$. Moreover, since elements of one copy can not reduce algebraic closures to definable ones in other copies, the set witnessing the value $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}])$ has to contain all elements in all copies of $\mathcal{N}$ besides one. Thus, $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{N}) + (|M|-1)|N|$ for finite $\mathcal{M}$ and $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \infty$ for infinite $\mathcal{M}$.

3. If $\mathcal{N}$ is infinite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) = 0$, then neither links between elements of $\mathcal{M}$ nor links between elements of $\mathcal{N}$ can give algebraic sets which are not reduced to definable ones. In view of Proposition 3, we obtain $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = 0$.

4. Let $\mathcal{N}$ be infinite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) > 0$. If $|M| = 1$, then $\mathcal{M}[\mathcal{N}]$ is reduced to $\mathcal{N}$ implying $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{N})$. Otherwise algebraic closures in $\mathcal{M}[\mathcal{N}]$ are reduced to algebraic closures inside copies $\mathcal{N}_i$ of $\mathcal{N}$ and finite possibility of $\deg^{\forall}_{\rm qU}(\mathcal{N}_i)$ is witnessed by arbitrary subsets in other copies of $\mathcal{N}$ which are infinite. Then in any case, finite or infinite $\deg^{\forall}_{\rm qU}(\mathcal{N}_i)$, we obtain $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \infty$. □

In view of Theorem 8 and Remark 3 we have the following:

Corollary 6. For any E-definable composition $\mathcal{M}[\mathcal{N}]$ and the theories T_1 of $\mathcal{M}$, T_2 of $\mathcal{N}$, and $T_1[T_2]$ of $\mathcal{M}[\mathcal{N}]$, the following conditions hold:

1) if $\mathcal{N}$ is finite and $\deg^{\forall}_{\mathrm{qU}}(T_2) = 0$, then $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = \deg^{\forall}_{\mathrm{qU}}(T_1) \cdot |N|$;

2) if $\mathcal{N}$ is finite and $\deg^{\forall}_{\mathrm{qU}}(T_2) > 0$, then $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = \deg^{\forall}_{\mathrm{qU}}(T_2) + (|M| - 1)|N|$ for finite $\mathcal{M}$ and $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = \infty$ for infinite $\mathcal{M}$;

3) if $\mathcal{N}$ is infinite and $\deg^{\forall}_{\mathrm{qU}}(T_2) = 0$, then $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = 0$;

4) if $\mathcal{N}$ is infinite and $\deg^{\forall}_{\mathrm{qU}}(T_2) > 0$, then $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = \deg^{\forall}_{\mathrm{qU}}(T_2)$ for $|M| = 1$ and $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = \infty$ for $|M| > 1$.

6 *Quasi-Urbanikization*

Definition 6. An expansion $\mathcal{M}'$ of a structure $\mathcal{M}$ is called a *quasi-Urbanikization* if $\mathcal{M}'$ is quasi-Urbanik. If $T' = \mathrm{Th}(\mathcal{M}')$ is quasi-Urbanik for a quasi-Urbanikization $\mathcal{M}'$ of $\mathcal{M}$, then T' is a *quasi-Urbanikization* of the theory $\mathrm{Th}(\mathcal{M})$.

Remark 4. Let $\mathcal{M}'$ be a *namization*, or a *constantization* of a structure $\mathcal{M}$, i.e. naming each element of $\mathcal{M}$ by constants. Clearly, $\mathcal{M}'$ is a quasi-Urbanikization of $\mathcal{M}$ whereas this property does not guarantee it for the theory $\mathrm{Th}(\mathcal{M}')$, as illustrated in Example 2.

Here, if $\mathcal{M}$ is finite, then any its namization $\mathcal{M}'$ produces a quasi-Urbanikization $\mathrm{Th}(\mathcal{M}')$ of the theory $\mathrm{Th}(\mathcal{M})$.

Remark 5. Let $\mathcal{M}$ be an infinite structure of an equivalence relation E each E-class of which contains n elements. We expand $\mathcal{M}$ by a unary predicate R, choosing unique element in each E-class, and by unary function f forming a cycle on each E-class $E(a)$ and including all elements of $E(a)$. Thus we obtain a quasi-Urbanikization both for $\mathcal{M}$ and for the theory $T = \mathrm{Th}(\mathcal{M})$. The operator producing that quasi-Urbanikization is called the *R-cyclification* of the structure $\mathcal{M}$ and its theory T.

It is essential here that $\mathcal{M}$ is infinite since the considered cyclification preserves $\mathrm{acl}(\emptyset)$ which is not equal to $\mathrm{dcl}(\emptyset) = \emptyset$ for $|M| \in \omega \setminus \{0, 1\}$.

More generally, we can define cyclifications for algebraic $\overline{a}$-complete formulae $\varphi(x, \overline{a})$, introducing $(l(\overline{a})+2)$-ary predicates $R'(x, y, \overline{z})$ such that $R'(x, y, \overline{a})$ defines a cycle on $\varphi(\mathcal{M}, \overline{a})$ of length $|\varphi(\mathcal{M}, \overline{a})|$, as the R-cyclification for E-classes above.

These possibilities of quasi-Urbanikization can be considered as variations of almost quasi-Urbanikness.

In view of Remarks 4 and 5, we have the following:

Proposition 4. Any structure $\mathcal{M}$ admits a quasi-Urbanikization, i.e. $\mathcal{M}$ has a quasi-Urbanik expansion $\mathcal{M}'$.

A natural *question* arises on the possibility of quasi-Urbanikization of an arbitrary theory T.

Conclusion

We introduced the notions of almost quasi-Urbanik structures and theories, and studied possibilities for the degrees of quasi-Urbanikness, both for existential and universal cases. Links of these characteristics and their possible values are described. We studied these values for linearly ordered, preordered and spherically ordered structures and theories as well as for strongly minimal ones, and for some natural operations including disjoint unions and compositions of structures and theories. A series of examples illustrates possibilities of these characteristics. It would be interesting to continue this research, describing possible values of degrees for natural classes of structures and their theories.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Baldwin, J.T., & Lachlan, A.H. (1971). On strongly minimal sets. *Journal of Symbolic Logic, 36*(1), 79–96. https://doi.org/10.2307/2271517

2 Zil'ber, B.I. (1999). Hereditarily transitive groups and quasi-Urbanik structures. *American Mathematical Society Translations: Series 2*(195), 165–186. https://doi.org/10.1090/trans2/195

3 Sudoplatov, S.V. (2021). Formulas and properties, their links and characteristics. *Mathematics, 9*(12), 1391. https://doi.org/10.3390/math9121391

4 Pavlyuk, In.I., & Sudoplatov, S.V. (2021). Formulas and properties for families of theories of Abelian groups. *Bulletin of Irkutsk State University. Series Mathematics, 36*, 95–109. https://doi.org/10.26516/1997-7670.2021.36.95

5 Sudoplatov, S.V. (2024). Variations of rigidity. *Bulletin of Irkutsk State University. Series Mathematics, 47*, 119–136. https://doi.org/10.26516/1997-7670.2024.47.119

6 Kulpeshov, B.Sh., & Sudoplatov, S.V. (2024). Variations of rigidity for ordered theories. *Bulletin of Irkutsk State University. Series Mathematics, 48*, 129–144. https://doi.org/10.26516/1997-7670.2024.48.129

7 Ershov, Yu.L., & Palyutin, E.A. (2011). *Matematicheskaia logika [Mathematical logic].* Fizmatlit [in Russian].

8 Shelah, S. (1990). Classification theory and the number of non-isomorphic models. *The Journal of Symbolic Logic, 58*(3), 1071–1074. https://doi.org/10.2307/2275113

9 Hodges, W. (1993). *Model theory.* Cambridge University Press. http://dx.doi.org/10.2307/2275861

10 Tent, K., & Ziegler, M. (2012). *A Course in Model Theory.* Cambridge University Press. https://doi.org/10.1017/CBO9781139015417

11 Sudoplatov, S.V. (2023). *Algebraic closures and their variations.* arXiv:2307.12536 [math.LO], 16. https://doi.org/10.48550/arXiv.2307.12536

12 Sudoplatov, S.V. (2022). Arities and aritizabilities of first-order theories. *Siberian Electronic Mathematical Reports, 19*(2), 889–901. https://doi.org/10.20944/preprints202112.0392.v1

13 Kulpeshov, B.Sh., & Sudoplatov, S.V. (2023). Spherical orders, properties and countable spectra of their theories. *Siberian Electronic Mathematical Reports, 20*(2), 588–599. https://doi.org/10.48550/arXiv.2208.05097

14 Sudoplatov, S.V. (2023). Minimality conditions, topologies and ranks for spherically ordered theories. *Siberian Electronic Mathematical Reports, 20*(2), 600–615.

15 Woodrow, R.E. (1976). *Theories with a finite number of countable models and a small language.* PhD Thesis. Simon Fraser University.

16 Emel'yanov, D.Yu., Kulpeshov, B.Sh., & Sudoplatov, S.V. (2020). Algebras of binary formulas for compositions of theories. *Algebra and Logic, 59*(4), 295–312. https://doi.org/10.1007/s10469-020-09602-y

Author Information[*]

Beibut Shaiykovich Kulpeshov (*corresponding author*) — Doctor of physical and mathematical sciences, Corresponding Member of National Academy of Sciences of the Republic of Kazakhstan, Professor, Principal Researcher of Institute of Mathematics and Mathematical Modeling, Professor of Kazakh-British Technical University, 28 Shevchenko street, Almaty, 050010, Kazakhstan; e-mail: *b.kulpeshov@kbtu.kz*; https://orcid.org/0000-0002-4242-0463

Sergey Vladimirovich Sudoplatov — Doctor of physical and mathematical sciences, Professor, Principal Researcher of Sobolev Institute of Mathematics, 4 Academician Koptyug Avenue, Novosibirsk, 630090, Russia; e-mail: *sudoplat@math.nsc.ru*; https://orcid.org/0000-0002-3268-9389

[*]The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 118–132

https://doi.org/10.31489/2025M1/118-132 *Research article*

On a boundary value problem for a parabolic-hyperbolic equation of the fourth order

M. Mamajonov

Kokand State Pedagogical Institute, Kokand, Uzbekistan
(E-mail: mirzamamajonov@gmail.com)

In this paper a boundary value problem for a fourth-order equation of parabolic-hyperbolic type within a pentagonal domain was investigated. In the equation under consideration, one characteristic aligns with the Ox axis while the other aligns with the Oy axis. Initially, the problem was examined within the lower triangle of the specified domain. Utilizing a differential equation solution construction method, a solution to the formulated problem was derived. Subsequently, within the rectangles of the domain, employing the continuation method, two relationships between the solution's traces were established. Moreover, from the parabolic segment of the domain, two additional relationships between unknown traces will be derived. Solving this system of four equations enables determination of these traces, thereby resolving the problem.

Keywords: characteristic of the equation, differential and integral equations, method of constructing a solution, boundary value problem, equations of parabolic-hyperbolic type, the line of type changing.

2020 Mathematics Subject Classification: 35C15, 35G15.

Introduction

Intensive research into equations of mixed elliptic-parabolic and parabolic-hyperbolic types is motivated by the fact that, on one hand, these new types of mixed equations have been little studied theoretically, and on the other hand, they are widely used in important issues of mechanics, physics, and technology.

The necessity of considering conjugation problems arises when a parabolic equation is defined in one part of the domain and a hyperbolic equation in another, as emphasized by I.M. Gelfand in 1959 [1]. He provides an example concerning the movement of gas in a channel surrounded by a porous medium: within the channel, the gas movement is described by the wave equation, while outside it, it is governed by the diffusion equation.

One of the earliest works dedicated to the study of boundary value problems for parabolic-hyperbolic equations was conducted by G.M. Struchina [2]. Subsequently, Y.S. Uflyand [3] further explored the problem of electrical oscillation propagation in composite lines, where losses are neglected in the semi-infinite section of the line, treating the remainder of the line as a cable without leakage, through the solution of boundary value problems for parabolic-hyperbolic equations.

Since the 1970s, research on boundary value problems for equations of third, fourth, and higher orders of the parabolic-hyperbolic type has seen intensive development. These boundary value problems were initially explored by T.D. Dzhuraev and his students [4, 5].

Subsequently, research on boundary value problems for third and fourth-order equations [6] and those of higher orders in the parabolic-hyperbolic type has significantly broadened. Currently, it is expanding into areas concerning the complexity of equations, the breadth of their application, and the generalization of problems related to these equations. The investigation has extended to numerous

*Corresponding author. *E-mail: mirzamamajonov@gmail.com*
Received: 7 June 2024; *Accepted:* 21 November 2024.

boundary value problems for such equations across various domains, involving two or three lines of change in type [7–9]. In the works [10, 11], some boundary value problems for fourth-order equations of parabolic-hyperbolic type, similar to equations of type (1) (see below) in a pentagonal region with two lines of type change, were studied. In the works [12, 13], considered boundary value problems for a mixed parabolic-hyperbolic equation with known and unknown dividing lines, as well as nonlocal boundary value problems and problems with a free boundary for parabolic, hyperbolic and mixed parabolic-hyperbolic equations. Work [14] is devoted to the study of boundary value problems and their spectral properties for equations of mixed parabolic-hyperbolic and mixed-composite types. In [15], boundary value problems for linear loaded differential and third-order integro-differential equations of mixed type were posed and studied. Then the study began of a number of different boundary value problems for mixed parabolic-hyperbolic equations of the third and fourth orders in various domains with two and three lines of change of type see, for example, [16–28].

1 Formulation of the problem

In the work [29], an equation of the form

$$\left(a_1\frac{\partial}{\partial x}+b_1\frac{\partial}{\partial y}\right)\left(a_2\frac{\partial}{\partial x}+b_2\frac{\partial}{\partial y}\right)(Lu)=0 \tag{1}$$

in the pentagonal area G, indicated below, where $a_1, b_1, a_2, b_2 \in R$, and $a_i^2+b_i^2 \neq 0$ $(i = 1, 2)$. Depending on the value of the coefficients a_1, b_1, a_2, b_2, a number of boundary value problems are posed for equation (1). In this work, boundary value problems are posed for 21 cases separately. In this present work, we consider the case of 3° $(a_1 = 1,\ b_1 = 0,\ a_2 = 0,\ b_2 = 1)$. Then equation (1) has the form

$$\frac{\partial^2}{\partial x\partial y}(Lu)=0, \tag{2}$$

where $G = G_1 \cup G_2 \cup G_3 \cup G_4 \cup J_1 \cup J_2 \cup J_3$; G_1, G_3, G_4 are rectangles with vertices at points $A\,(0,0)$, $A_0\,(0,1)$, $B_0\,(1,1)$, $B\,(1,0)$; A, A_0, $D_0\,(-1,1)$, $D\,(-1,0)$; B, B_0, $C_0\,(2,1)$, $C\,(2,0)$ respectively, G_2 is a triangle with vertices at points C, D, $E\,(1/2\ ,-3/2\ )$; J_1, J_2, J_3 are open segments with vertices at points C, D; A, A_0; B, B_0 respectively, which are lines of change like equation (2); $u = u\,(x,y)$ is an unknown function,

$$Lu=\begin{cases} u_{1xx}-u_{1y},\ (x,y)\in G_1,\\ u_{ixx}-u_{iyy},\ (x,y)\in G_i\ \ (i=2,3,4)\,. \end{cases}$$

For equation (2), the following problem is formulated:

Problem M. It is required to find a function $u\,(x,y)$, satisfying the following conditions:

1) it is continuous in $\overline{G}$ and in the domain $G\backslash J_1\backslash J_2\backslash J_3$ has continuous derivatives involved in equation (2), and u_x, u_y, u_{xx}, u_{xy} and u_{yy} are continuous up to part of the boundary of the domain G, indicated in the boundary conditions;

2) it satisfies the equation (2) in the domain $G\backslash J_1\backslash J_2\backslash J_3$;

3) it satisfies the following boundary conditions:

$$u(2,\ y)=\varphi_1\,(y)\,,\quad 0\le y\le 1, \tag{3}$$

$$u(-1,\ y)=\varphi_2\,(y)\,,\ \ 0\le y\le 1, \tag{4}$$

$$u_x\,(-1,\ y)=\varphi_4\,(y)\,,\ \ 0\le y\le 1, \tag{5}$$

$$u|_{CE}=\psi_1\,(x)\,,\ \ 1/2\ \le x\le 2, \tag{6}$$

$$u|_{DP} = \psi_2(x), \quad -1 \le x \le -1/2, \tag{7}$$

$$u|_{QE} = \psi_3(x), \quad 0 \le x \le 1/2, \tag{8}$$

$$\left.\frac{\partial u}{\partial n}\right|_{DE} = \psi_4(x), \quad -1 \le x \le 1/2, \tag{9}$$

$$\left.\frac{\partial^2 u}{\partial n^2}\right|_{DE} = \psi_5(x), \quad -1 \le x \le 1/2, \tag{10}$$

$$\left.\frac{\partial u}{\partial n}\right|_{CE} = \psi_6(x), \quad 1/2 \le x \le 2; \tag{11}$$

and

4) the following continuous gluing conditions:

$$u(x, +0) = u(x, -0) = T(x), \quad -1 \le x \le 2, \tag{12}$$

$$u_y(x, +0) = u_y(x, -0) = N(x), \quad -1 \le x \le 2, \tag{13}$$

$$u_{yy}(x, +0) = u_{yy}(x, -0) = M(x), \quad x \in (-1, 0) \cup (0, 1) \cup (1, 2), \tag{14}$$

$$u(+0, y) = u(-0, y) = \tau_4(y), \quad 0 \le y \le 1, \tag{15}$$

$$u_x(+0, y) = u_x(-0, y) = \nu_4(y), \quad 0 \le y \le 1, \tag{16}$$

$$u_{xx}(+0, y) = u_{xx}(-0, y) = \mu_4(y), \quad 0 < y < 1, \tag{17}$$

$$u(1+0, y) = u(1-0, y) = \tau_5(y), \quad 0 \le y \le 1, \tag{18}$$

$$u_x(1+0, y) = u_x(1-0, y) = \nu_5(y), \quad 0 \le y \le 1, \tag{19}$$

$$u_{xx}(1+0, y) = u_{xx}(1-0, y) = \mu_5(y), \quad 0 < y < 1. \tag{20}$$

Here φ_1, φ_2, φ_4 and $\psi_j\left(j = \overline{1,6}\right)$ are given sufficiently smooth functions, n is the internal normal to the line $x - y = 2$ (CE) or $x + y = -1$ (DE), and $P(-1/2, -1/2)$, $Q(0, -1)$. Besides,

$$T(x) = \begin{cases} \tau_2(x), & -1 \le x \le 0, \\ \tau_1(x), & 0 \le x \le 1, \\ \tau_3(x), & 1 \le x \le 2, \end{cases} \quad N(x) = \begin{cases} \nu_2(x), & -1 \le x \le 0, \\ \nu_1(x), & 0 \le x \le 1, \\ \nu_3(x), & 1 \le x \le 2, \end{cases} \quad M(x) = \begin{cases} \mu_2(x), & -1 < x < 0, \\ \mu_1(x), & 0 < x < 1, \\ \mu_3(x), & 1 < x < 2, \end{cases}$$

where τ_i, ν_i, μ_i $\left(i = \overline{1,5}\right)$ are temporarily unknown sufficiently smooth functions.

2 The solution of the problem

The following theorem holds:

Theorem 1. Let $\varphi_1 \in C^4[0, 1]$, $\varphi_2 \in C^4[0, 1]$, $\varphi_4 \in C^3[0, 1]$, $\psi_1 \in C^4[1/2, 2]$, $\psi_2 \in C^4[-1, -1/2]$, $\psi_3 \in C^4[0, 1/2]$, $\psi_4 \in C^3[-1, 1/2]$, $\psi_5 \in C^2[-1, 1/2]$, $\psi_6 \in C^3[1/2, 2]$, and the matching conditions $\varphi_1(0) = \psi_1(2)$, $\varphi_2(0) = \psi_2(-1)$, $\tau_1(0) = \tau_2(0) = \tau_4(0)$, $\tau_1(1) = \tau_3(1) = \tau_5(0)$, $\nu_1(0) = \nu_2(0) = \tau_4'(0)$, $\nu_1(1) = \nu_3(1) = \tau_5'(0)$, $\tau_1'(0) = \nu_4(0)$, $\tau_1'(1) = \nu_5(0)$ are satisfied, then the problem M has a unique solution.

Proof. We shall prove the theorem by the method of constructing a solution. To do this, we rewrite equation (2) as

$$u_{1xx} - u_{1y} = \omega_{11}(x) + \omega_{12}(y), \quad (x, y) \in G_1, \tag{21}$$

$$u_{ixx} - u_{iyy} = \omega_{i1}(x) + \omega_{i2}(y), \quad (x, y) \in G_i \ \ (i = 2, 3, 4), \tag{22}$$

where the notation $u(x, y) = u_i(x, y), \ (x, y) \in G_i \ \left(i = \overline{1, 4}\right)$ is introduced, and $\omega_{i1}(x)$, $\omega_{i2}(y)$ $(i = \overline{1,4})$ are unknown sufficiently smooth functions.

First, consider problem M in the domain G_2. The solution to equation (22) $(i = 2)$, satisfying conditions (12), (13), is represented in the form

$$u_2(x, y) = \frac{1}{2}\left[T(x+y) + T(x-y)\right] + \frac{1}{2}\int\limits_{x-y}^{x+y} N(t)\,dt - \frac{1}{2}\int\limits_0^y d\eta \int\limits_{x-y+\eta}^{x+y-\eta} \omega_{21}(\xi)\,d\xi - \int\limits_0^y (y-\eta)\,\omega_{22}(\eta)\,d\eta. \tag{23}$$

Substituting (23) into conditions (9) and (10) after some calculations, we obtain the following system of equations

$$\omega_{21}(x) + \omega_{22}(-1-x) = \sqrt{2}\psi_4'(x), \ -1 \le x \le 1/2,$$

$$\omega_{21}(x) - \omega_{22}(-1-x) = 2\psi_5(x) - 2T''(-1) - 2N'(-1) - 2\omega_{21}(-1), \ -1 \le x \le 1/2.$$

From this system after some transformations, we find

$$\omega_{21}(x) = \psi_5(x) + \frac{\sqrt{2}}{2}\psi_4'(x) - T''(-1) - N'(-1) - \omega_{21}(-1), \ -1 \le x \le 1/2, \tag{24}$$

$$\omega_{22}(y) = -\psi_5(-1-y) + \frac{\sqrt{2}}{2}\psi_4'(-1-y) + T''(-1) + N'(-1) + \omega_{21}(-1), \ -3/2 \le y \le 0. \tag{25}$$

Adding (24) and (25), we find

$$\begin{gathered}\omega_{21}(x) + \omega_{22}(y) = \left[\psi_5(x) - \psi_5(-1-y)\right] + \tfrac{\sqrt{2}}{2}\left[\psi_4'(x) + \psi_4'(-1-y)\right], \\ -1 \le x \le 1/2, \ -3/2 \le y \le 0.\end{gathered} \tag{26}$$

Now substituting (23) into condition (11), we have

$$\omega_{21}(x) + \omega_{22}(x-2) = -\sqrt{2}\psi_6'(x), \ 1/2 \le x \le 2.$$

Setting in (25) $y = x - 2$ and substituting the obtained equality into the last equality, we find

$$\omega_{21}(x) = -\sqrt{2}\psi_6'(x) + \psi_5(1-x) - \frac{\sqrt{2}}{2}\psi_4'(1-x) - T''(-1) - N'(-1) - \omega_{21}(-1), \ 1/2 \le x \le 2. \tag{27}$$

Hence, adding (25) and (27), we get

$$\begin{gathered}\omega_{21}(x) + \omega_{22}(y) = -\sqrt{2}\psi'_6(x) + \left[\psi_5(1-x) - \psi_5(-1-y)\right] \\ -\tfrac{\sqrt{2}}{2}\left[\psi_4'(1-x) - \psi_4'(-1-y)\right], \quad 1/2 \le x \le 2, \ -3/2 \le y \le 0.\end{gathered} \tag{28}$$

From (26) and (28), it follows that $\psi_4'(1/2) = -\psi'_6(1/2)$.

Thus, we have found the function $\omega_{21}(x) + \omega_{22}(y)$ for $-1 \le x \le 2$, $-3/2 \le y \le 0$ completely. It is determined by formulas (26), (28).

Now, substituting (23) into the condition (6), we arrive at the relation

$$T'(x) + N(x) = \alpha_1(x), \ -1 \le x \le 2, \tag{29}$$

where

$$\alpha_1(x) = \psi_1'\left(\frac{x+2}{2}\right) + \int_0^{\frac{x-2}{2}} \omega_{21}(x-\eta)\, d\eta + \int_0^{\frac{x-2}{2}} \omega_{22}(\eta)\, d\eta.$$

And substituting (23) into condition (7), we get

$$\tau_2'(x) - \nu_2(x) = \delta_1(x), \ -1 \le x \le 0, \tag{30}$$

where

$$\delta_1(x) = \psi_2'\left(\frac{x-1}{2}\right) - \int_0^{-\frac{x+1}{2}} \omega_{21}(x+\eta)\, d\eta - \int_0^{-\frac{x+1}{2}} \omega_{22}(\eta)\, d\eta.$$

Next, substituting (23) into condition (8), we have

$$\tau_3'(x) - \nu_3(x) = \delta_2(x), \ 1 \le x \le 2, \tag{31}$$

where

$$\delta_2(x) = \psi_3'\left(\frac{x-1}{2}\right) - \int_0^{-\frac{x+1}{2}} \omega_{21}(x+\eta)\, d\eta - \int_0^{-\frac{x+1}{2}} \omega_{22}(\eta)\, d\eta.$$

a) For $0 \le x \le 1$ the relation (29) has the form

$$\tau_1'(x) + \nu_1(x) = \alpha_1(x), \ 0 \le x \le 1; \tag{32}$$

b) for $-1 \le x \le 0$,

$$\tau_2'(x) + \nu_2(x) = \alpha_1(x), \ -1 \le x \le 0; \tag{33}$$

c) and when $1 \le x \le 2$,

$$\tau_3'(x) + \nu_3(x) = \alpha_1(x), \ 1 \le x \le 2. \tag{34}$$

Solving the system {(30), (33)}, we find

$$\tau_2'(x) = \frac{1}{2}\left[\alpha_1(x) + \delta_1(x)\right], \quad \nu_2(x) = \frac{1}{2}\left[\alpha_1(x) - \delta_1(x)\right]. \tag{35}$$

Integrating the first of equalities (35) from -1 to x, we obtain

$$\tau_2(x) = \frac{1}{2}\int_{-1}^{x} \left[\alpha_1(t) + \delta_1(t)\right] dt + \psi_2(-1), \ -1 \le x \le 0. \tag{36}$$

And solving system (31), (34), we get

$$\tau_3'(x) = \frac{1}{2}\left[\alpha_1(x) + \delta_2(x)\right], \quad \nu_3(x) = \frac{1}{2}\left[\alpha_1(x) - \delta_2(x)\right]. \tag{37}$$

Integrating the first of equalities (37) from 2 to x, we have

$$\tau_3(x) = \frac{1}{2}\int_{2}^{x} \left[\alpha_1(t) + \delta_2(t)\right] dt + \psi_1(2), \ 1 \le x \le 2. \tag{38}$$

Now, by differentiating the equation (21) with respect to y passing to the limit at $y \to 0$ in the resulting equation and in equation (22) ($i = 2$), we obtain

$$\nu_1''(x) - \mu_1(x) = \omega_{12}'(0), \quad \tau_1''(x) - \mu_1(x) = \omega_{21}(x) + \omega_{22}(0).$$

Eliminating the function $\mu_1(x)$ from these relations and integrating the resulting equation twice from 0 to x, we arrive at the relation

$$\nu_1(x) - \tau_1(x) = -\int_0^x (x-t)\left[\omega_{21}(t) + \omega_{22}(0)\right] dt + \frac{1}{2}\omega_{12}'(0)\, x^2 + k_1 x + k_2,$$

where $\omega_{12}'(0)$, k_1 and k_2 are unknown constants.

Excluding the function $\nu_1(x)$ from the last relation and from (32), we have

$$\tau_1'(x) + \tau_1(x) = \alpha_2(x) + \frac{1}{2}\omega_{12}'(0)\, x^2 + k_1 x + k_2, \ 0 \le x \le 1, \tag{39}$$

where

$$\alpha_2(x) = \alpha_1(x) + \int_0^x (x-t)\left[\omega_{21}(t) + \omega_{22}(0)\right] dt.$$

Solving equation (39) under the conditions (see (35), (36), (37), (38))

$$\tau_1(0) = \psi_2(-1) + \frac{1}{2}\int_{-1}^0 \left[\alpha_1(t) + \delta_1(t)\right] dt, \quad \tau_1'(0) = \frac{1}{2}\left[\alpha_1(0) + \delta_1(0)\right],$$

$$\tau_1'(1) = \frac{1}{2}\left[\alpha_1(1) + \delta_2(1)\right], \quad \tau_1(1) = \psi_1(2) - \frac{1}{2}\int_1^2 \left[\alpha_1(t) + \delta_2(t)\right] dt,$$

we find the function $\tau_1(x)$ as

$$\tau_1(x) = \int_0^x e^{t-x}\alpha_2(t)\, dt + \frac{\omega_{12}'(0)}{2}\left(x^2 - 2x + 2 - 2e^{-x}\right) + k_1\left(x - 1 + e^{-x}\right) + k_2\left(1 - e^{-x}\right) + k_3 e^{-x},$$

where

$$k_3 = \frac{1}{2}\int_{-1}^0 \left[\alpha_1(t) + \delta_1(t)\right] dt + \psi_2(-1),$$

$$k_2 = \frac{1}{2}\left[\alpha_1(0) + \delta_1(0)\right] - \alpha_2(0) + k_3,$$

$$k_1 = \frac{e-2}{2(e-3)}\left[\alpha_1(1) + \delta_2(1)\right] + \frac{e-2}{3-e}\alpha_2(1) + \frac{2}{3-e}\psi_1(2) +$$

$$+\frac{3e-4}{e(3-e)}k_2 + \frac{4-e}{e(3-e)}k_3 - \frac{1}{3-e}\int_0^1 e^t\alpha_2(t)\, dt - \frac{1}{3-e}\int_1^2 \left[\alpha_1(t) + \delta_2(t)\right] dt,$$

$$\omega'_{12}(0) = \frac{e}{2}\left[\alpha_1(1) + \delta_2(1)\right] - \alpha_2(1)\, e + \int_0^1 e^t\alpha_2(t)\, dt - k_2 + k_3 - k_1(e-1).$$

Thus, we have defined the functions $\nu_1(x)$, $\mu_1(x)$ and $u_2(x, y)$.

Now, let us consider the problem in the domain G_3. Passing to the limit at $y \to 0$ in equations (22) $(i = 2)$ and (22) $(i = 3)$, we find

$$\omega_{31}(x) = \omega_{21}(x), \ -1 \le x \le 0,$$

where it should be $\omega_{32}(0) = \omega_{22}(0)$.

Consider the following problem:

$$\begin{cases} u_{3xx} - u_{3yy} = \Omega_{31}(x) + \omega_{32}(y), \\ u_3(x, 0) = T_2(x), \ u_{3y}(x, 0) = N_2(x), \quad -2 \le x \le 1, \\ u_3(-1, y) = \varphi_2(y), \ u_{3x}(-1, y) = \varphi_4(y), \ u_3(0, y) = \tau_4(y), \ 0 \le y \le 1. \end{cases}$$

Here we used (4), (5) in the form $u_3(-1, y) = \phi_2(y)$, $u_{3x}(-1, y) = \phi_4(y)$.

We will seek a solution to this problem satisfying all conditions except the condition $u_{3x}(-1, y) = \varphi_4(y)$, in the form

$$u_3(x, y) = u_{31}(x, y) + u_{32}(x, y) + u_{33}(x, y), \tag{40}$$

where $u_{31}(x, y)$ is the solution of the problem

$$\begin{cases} u_{31xx} - u_{31yy} = 0, \\ u_{31}(x, 0) = T_2(x), \ u_{31y}(x, 0) = 0, \ -2 \le x \le 1, \\ u_{31}(-1, y) = \varphi_2(y), \ u_{31}(0, y) = \tau_4(y), 0 \le y \le 1, \end{cases} \tag{41}$$

$u_{32}(x, y)$ is the solution of the problem

$$\begin{cases} u_{32xx} - u_{32yy} = \omega_{32}(y), \\ u_{32}(x, 0) = 0, \ u_{32y}(x, 0) = N_2(x), \ -2 \le x \le 1, \\ u_{32}(-1, y) = 0, \ u_{32}(0, y) = 0, 0 \le y \le 1, \end{cases} \tag{42}$$

$u_{33}(x, y)$ is the solution of the problem

$$\begin{cases} u_{33xx} - u_{33yy} = \Omega_{31}(x), \\ u_{33}(x, 0) = 0, \ u_{33y}(x, 0) = 0, \ -2 \le x \le 1, \\ u_{33}(-1, y) = 0, \ u_{33}(0, y) = 0, 0 \le y \le 1. \end{cases} \tag{43}$$

Using the continuation method, we find solutions to problems (41), (42) and (43). They have the following forms

$$u_{31}(x, y) = \frac{1}{2}\left[T_2(x + y) + T_2(x - y)\right], \tag{44}$$

$$u_{32}(x, y) = \frac{1}{2}\int\limits_{x-y}^{x+y} N_2(t)\,dt - \int\limits_0^y (y - \eta)\,\omega_{32}(\eta)\,d\eta, \tag{45}$$

$$u_{33}(x, y) = -\frac{1}{2}\int\limits_0^y d\eta \int\limits_{x-y+\eta}^{x+y-\eta} \Omega_{31}(\xi)\,d\xi, \tag{46}$$

where

$$T_2(x) = \begin{cases} 2\varphi_2(-1-x) - \tau_2(-2-x), & -2 \le x \le -1, \\ \tau_2(x), & -1 \le x \le 0, \\ 2\tau_4(x) - \tau_2(-x), & 0 \le x \le 1, \end{cases}$$

$$\Omega_{31}(x) = \begin{cases} -\omega_{31}(-2-x), & -2 \le x \le -1, \\ \omega_{31}(x), & -1 \le x \le 0, \\ -\omega_{31}(-x), & 0 \le x \le 1, \end{cases}$$

$$N_2(x) = \begin{cases} -\nu_2(-2-x) + 2\int\limits_0^{-1-x} \omega_{32}(\eta)\, d\eta, & -2 \le x \le -1, \\ \nu_2(x), & -1 \le x \le 0, \\ -\nu_2(-x) + 2\int\limits_0^{x} \omega_{32}(\eta)\, d\eta, & 0 \le x \le 1. \end{cases}$$

Substituting (44), (45) and (46) into (40), we have

$$u_3(x, y) = \frac{1}{2}\left[T_2(x+y) + T_2(x-y)\right] + \frac{1}{2}\int\limits_{x-y}^{x+y} N_2(t)\, dt$$

$$-\frac{1}{2}\int\limits_0^y d\eta \int\limits_{x-y+\eta}^{x+y-\eta} \Omega_{31}(\xi)\, d\xi - \int\limits_0^y (y-\eta)\,\omega_{32}(\eta)\, d\eta.$$

Differentiating this solution with respect to x, we obtain

$$u_{3x}(x, y) = \frac{1}{2}\left[T_2'(x+y) + T_2'(x-y)\right] + \frac{1}{2}\left[N_2(x+y) - N_2(x-y)\right]$$

$$-\frac{1}{2}\int\limits_0^y \left[\Omega_{31}(x+y-\eta) - \Omega_{31}(x-y+\eta)\right] d\eta. \quad (47)$$

Passing to the limit in (47) at $x \to -1$ and considering the condition $u_{3x}(-1, y) = \varphi_4(y)$, we find

$$\omega_{32}(y) = \tau_2''(y-1) + \nu_2'(y-1) - \varphi_2''(y) - \varphi_4'(y) - \omega_{31}(y-1).$$

Similarly, from (47) using the conditions (15), (16), we have

$$\nu_4(y) = \tau_4'(y) + \beta_1(y), \quad (48)$$

where

$$\beta_1(y) = \tau_2'(-y) - \nu_2(-y) + \int\limits_0^y \omega_{31}(\eta - y)\, d\eta + \int\limits_0^y \omega_{32}(\eta)\, d\eta.$$

Now let us move to the domain G_4. Passing in equations (22) $(i = 4)$, (22) $(i = 2)$ to the limit at $y \to 0$ and considering (12) and (14) for $0 \le x \le 1$, we have $\omega_{41}(x) + \omega_{42}(0) = \omega_{21}(x) + \omega_{22}(0)$. Let's assume that $\omega_{42}(0) = \omega_{22}(0)$. Then, we have $\omega_{41}(x) = \omega_{21}(x)$.

Next, passing in equations (22) ($i = 4$), (22) ($i = 2$) to the limit at $x \to 1$ due to (18) and (20), we find

$$\omega_{12}(y) = \tau_5''(y) - \tau_5'(y) + \omega_{42}(y) + \omega_{41}(1) - \omega_{11}(1). \tag{49}$$

Passing in equations (22) ($i = 3$) and (21) to the limit at $x \to 0$ due to (15) and (17), we obtain

$$\omega_{12}(y) = \tau_4''(y) - \tau_4'(y) + \omega_{32}(y) + \omega_{31}(0) - \omega_{11}(0). \tag{50}$$

Eliminating function $\omega_{12}(y)$ from (49) and (50), we find

$$\omega_{42}(y) = \left[\tau_4''(y) - \tau_4'(y)\right] - \left[\tau_5''(y) - \tau_5'(y)\right] + \omega_{32}(y) + \omega_{31}(0) - \omega_{41}(1) - \omega_{11}(0) + \omega_{11}(1).$$

Consider the following problem:

$$\begin{cases} u_{4xx} - u_{4yy} = \Omega_{41}(x) + \omega_{42}(y), \\ u_4(x, 0) = T_3(x), \ u_{4y}(x, 0) = N_3(x), \quad 0 \le x \le 3, \\ u_4(2, y) = \varphi_1(y), \ u_4(1, y) = \tau_5(y), \ 0 \le y \le 1. \end{cases}$$

Here we used condition (3) in the form $u_4(2, y) = \phi_1(y)$.

We will seek a solution to the last problem in the form

$$u_4(x, y) = u_{41}(x, y) + u_{42}(x, y) + u_{43}(x, y), \tag{51}$$

where $u_{41}(x, y)$ is the solution of the problem

$$\begin{cases} u_{41xx} - u_{41yy} = 0, \\ u_{41}(x, 0) = T_3(x), \ u_{41y}(x, 0) = 0, \ 0 \le x \le 3, \\ u_{41}(2, y) = \varphi_1(y), \ u_{41}(1, y) = \tau_5(y), \ 0 \le y \le 1, \end{cases} \tag{52}$$

and $u_{42}(x, y)$ and $u_{43}(x, y)$ are the solution of the problems respectively

$$\begin{cases} u_{42xx} - u_{42yy} = \omega_{42}(y), \\ u_{42}(x, 0) = 0, \ u_{42y}(x, 0) = N_3(x), \ 0 \le x \le 3, \\ u_{42}(2, y) = 0, \ u_{42}(1, y) = 0, \ 0 \le y \le 1, \end{cases} \tag{53}$$

$$\begin{cases} u_{43xx} - u_{43yy} = \Omega_{41}(x), \\ u_{43}(x, 0) = 0, \ u_{43y}(x, 0) = 0, \ 0 \le x \le 3, \\ u_{43}(2, y) = 0, \ u_{43}(1, y) = 0, \ 0 \le y \le 1. \end{cases} \tag{54}$$

By using the continuation method, it is easy to see that the solutions to the problems (52), (53) and (54) have the forms

$$u_{41}(x, y) = \frac{1}{2}\left[T_3(x + y) + T_3(x - y)\right], \tag{55}$$

$$u_{42}(x, y) = \frac{1}{2}\int\limits_{x-y}^{x+y} N_3(t)\, dt - \int\limits_0^y (y - \eta)\, \omega_{42}(\eta)\, d\eta, \tag{56}$$

$$u_{43}(x, y) = -\frac{1}{2}\int\limits_0^y d\eta \int\limits_{x-y+\eta}^{x+y-\eta} \Omega_{41}(\xi)\, d\xi, \tag{57}$$

where

$$T_3(x) = \begin{cases} 2\varphi_1(x-2) - \tau_3(4-x), \ 2 \le x \le 3, \\ \tau_3(x), \ 1 \le x \le 2, \\ 2\tau_5(1-x) - \tau_3(2-x), \ 0 \le x \le 1, \end{cases}$$

$$\Omega_{41}(x) = \begin{cases} -\omega_{41}(4-x), \ 2 \le x \le 3, \\ \omega_{41}(x), \ 1 \le x \le 2, \\ -\omega_{41}(2-x), \ 0 \le x \le 1, \end{cases}$$

$$N_3(x) = \begin{cases} -\nu_3(4-x) + 2\int\limits_0^{x-2} \omega_{42}(\eta)\, d\eta, \ 2 \le x \le 3, \\ \nu_3(x), \ 1 \le x \le 2, \\ -\nu_3(2-x) + 2\int\limits_0^{1-x} \omega_{42}(\eta)\, d\eta, \ 0 \le x \le 1. \end{cases}$$

Substituting (55), (56) and (57) into (51), we have

$$u_4(x, y) = \frac{1}{2}\left[T_3(x+y) + T_3(x-y)\right] + \frac{1}{2}\int\limits_{x-y}^{x+y} N_3(t)\, dt - \frac{1}{2}\int\limits_0^y d\eta \int\limits_{x-y+\eta}^{x+y-\eta} \Omega_{41}(\xi)\, d\xi - \int\limits_0^y (y-\eta)\,\omega_{42}(\eta)\, d\eta.$$

Differentiating this solution with respect to x, we obtain

$$u_{41x}(x, y) = \frac{1}{2}\left[T_3'(x+y) + T_3'(x-y)\right] + \frac{1}{2}\left[N_3(x+y) - N_3(x-y)\right] - \frac{1}{2}\int\limits_0^y \Omega_{41}(x+y-\eta)\, d\eta + \frac{1}{2}\int\limits_0^y \Omega_{41}(x-y+\eta)\, d\eta. \quad (58)$$

Passing to the limit at $x \to 1$ in (58) after some calculations and transformations by virtue of (18), (19), we obtain the relation

$$\nu_5(y) = -\tau_4'(y) + \tau_4(y) - \tau_5(y) + \beta_2(y), \quad (59)$$

where

$$\beta_2(y) = \nu_1(0) - \tau_1(0) - \nu_1(1) + \tau_1(1) + \gamma_1(y),$$

$$\gamma_1(y) = \tau_3'(1+y) + \nu_3(1+y) - \int\limits_0^y \omega_{41}(1+y-\eta)\, d\eta +$$

$$+ \int\limits_0^y \omega_{32}(\eta)\, d\eta + \left[\omega_{31}(0) - \omega_{41}(1) + \tau_1''(1) - \nu_1(1) - \mu_1(0) + \nu_1(0) - \omega_{31}(0) - \omega_{32}(0)\right] y.$$

Finally, we consider the problem in the domain G_1. Passing to the limit in equation (21) at $y \to 0$, we find

$$\omega_{11}(x) = \tau''_1(x) - \nu_1(x) - \omega_{12}(0).$$

Next, we write down the solution to equation (21), satisfying conditions (12) for $0 \le x \le 1$, (15) and (18), differentiating this solution with respect to x after some calculations and transformations, we obtain

$$u_{1x}(x,\, y) = -\int_0^y \tau'_4(\eta)\, N(x,\, y;\, 0,\, \eta)\, d\eta + \int_0^y \tau'_5(\eta)\, N(x,\, y;\, 1,\, \eta)\, d\eta$$
$$+ \int_0^1 \tau'_1(\xi)\, N(x,\, y;\, \xi,\, 0)\, d\xi + \int_0^y [\omega_{11}(1) + \omega_{12}(\eta)]\, N(x,\, y;\, 1,\, \eta)\, d\eta$$
$$+ \int_0^y [\omega_{11}(0) + \omega_{12}(\eta)]\, N(x,\, y;\, 0,\, \eta)\, d\eta - \int_0^y d\eta \int_0^1 \omega'_{11}(\xi)\, N(x,\, y;\, \xi,\, \eta)\, d\xi, \quad (60)$$

where

$$\left.\begin{matrix} G(x,\, y;\, \xi,\, \eta) \\ N(x,\, y;\, \xi,\, \eta) \end{matrix}\right\} = \frac{1}{2\sqrt{\pi(y-\eta)}} \sum_{n=-\infty}^{+\infty} \left\{ \exp\left[-\frac{(x-\xi-2n)^2}{4(y-\eta)}\right] \mp \exp\left[-\frac{(x+\xi-2n)^2}{4(y-\eta)}\right] \right\}$$

are the Green's functions of the first and second boundary value problems for the heat equation.

Passing to the limits $x \to 0$ and $x \to 1$ in (60) by virtue of (48) and (59), we obtain two Abel's type integral equations for the unknowns $\tau''_4(y)$ and $\tau'_5(y)$. Applying the Abel inversion to these equations, we obtain a system of Volterra integral equations of the second kind for $\tau''_4(y)$ and $\tau'_5(y)$:

$$\tau''_4(y) + \int_0^y K_1(y,\eta)\, \tau''_4(\eta)\, d\eta + \int_0^y K_2(y,\eta)\, \tau'_5(\eta)\, d\eta = g_1(y), \quad (61)$$

$$\tau'_5(y) + \int_0^y K_3(y,\eta)\, \tau'_5(\eta)\, d\eta + \int_0^y K_4(y,\eta)\, \tau''_4(\eta)\, d\eta = g_2(y), \quad (62)$$

where $K_1(y,\eta)$, $K_2(y,\eta)$, $K_3(y,\eta)$, $K_4(y,\eta)$, $g_1(y)$, $g_2(y)$ are known functions, $K_1(y,\eta)$ and $K_3(y,\eta)$ have a weak singularity (1/2), and the remaining functions are continuous. Therefore, system $\{(61),(62)\}$ admits a unique solution in the class of continuous functions. Solving the system $\{(61),(62)\}$, we find functions $\tau''_4(y)$, $\tau'_5(y)$, and thus functions $\tau_4(y)$, $\tau_5(y)$, $\nu_4(y)$, $\nu_5(y)$. Then all the functions $u_3(x,y)$, $u_4(x,y)$ and $u_1(x,y)$, will be known. So, we have found a solution to the considered problem M in a unique way.

Conclusion

In this paper, we consider a new correct boundary value problem for a third-order parabolic-hyperbolic equation in a pentagonal domain consisting of three rectangles and one triangle. In the central rectangular domain the equation is parabolic, and in the two side rectangles and in the lower triangle it belongs to hyperbolic type. Straight lines $x = 0$, $x = 1$ and $y = 0$ are lines of change in the type of equation.

When constructing a solution in the lower characteristic triangle, writing a solution to equation (22) $(i = 2)$ that satisfies conditions (12), (13) and substituting this solution into conditions (4) and (5), we find the function $\omega_2(x)$.

Then, substituting this solution into conditions (6), we obtain the first functional relation between the unknown functions $T(x)$ and $N(x)$ for $-1 \le x \le 2$. Next, substituting this solution into condition (7), we obtain another relation between $\tau_2(x)$ and $\nu_2(x)$ for $-1 \le x \le 0$. From these two relations we find the functions $\tau_2(x)$ and $\nu_2(x)$ for $-1 \le x \le 0$. Similarly, satisfying condition (8), we find the functions $\tau_3(x)$ and $\nu_3(x)$ for $1 \le x \le 2$.

Then, differentiating equation (21) with respect to y, in the resulting equation and in equation (22) $(i = 2)$ assuming $y = 0$, we obtain two more relations between the unknown functions $\tau_1(x)$, $\nu_1(x)$ and $\mu_1(x)$ for $1 \le x \le 2$. Eliminating from these two relations and from the relation between the functions $T(x)$ and $N(x)$ for $0 \le x \le 1$ the function $\nu_1(x)$ and $\mu_1(x)$, we arrive at an ordinary differential equation of the second order with respect to $\tau_1(x)$, on the right side of which one unknown constant is involved. Solving this equation under the known three conditions, we find the function $\tau_1(x)$, and thus the functions $\nu_1(x)$, $u_2(x, y)$.

Further, by the method of continuation in the left and right rectangular regions using the written solutions, directing x to zero and to one, we obtain two relations between the unknown functions $\tau_4(y)$, $\nu_4(y)$ and $\tau_5(y)$, $\nu_5(y)$ respectively.

Next, passing to the limit in equation (21) at $y \to 0$, we find the unknown function $\omega_{11}(x)$. In the parabolic part of the rectangular region, writing the representation of the solution in terms of the known Green's function of the first boundary value problem and differentiating this solution with respect to x and assuming $x \to 0$ and $x \to 1$, we obtain two more relations between the unknown functions $\tau_4(y)$, $\nu_4(y)$ and $\tau_5(y)$, $\nu_5(y)$ respectively. Excluding the functions $\nu_4(y)$ and $\nu_5(y)$ from these four relations, we arrive at a system of Volterra integral equations of the second kind with respect to $\tau''_4(y)$ and $\tau'_5(y)$. The unique solvability of this system follows from the theory of integral equations. Solving this system, we find traces of the solution $\tau''_4(y)$ and $\tau'_5(y)$. Thus, we have proven the unique solvability of the considered problem.

Conflict of Interest

The author declares no conflict of interest.

References

1 Gelfand, I.M. (1959). Nekotorye voprosy analiza i differentsialnykh uravnenii [Some questions of analysis and differential equations]. *Uspekhi Matematicheskikh Nauk — Advances in Mathematical Sciences, 14*(3), 3–19 [in Russian].

2 Struchina, G.M. (1961). Zadacha o sopriazhenii dvukh uravnenii [The problem of conjugating two equations]. *Inzhenerno-fizicheskii zhurnal — Engineering and Physics Journal, 4*(11), 99–104 [in Russian].

3 Uflyand, Ya.S. (1964). K voprosu o rasprostranenii kolebanii v sostavnykh elektricheskikh liniiakh [On the question of the propagation of oscillations in composite electric lines]. *Inzhenerno-fizicheskii zhurnal — Engineering and Physics Journal, 4*(1), 89–92 [in Russian].

4 Dzhuraev, T.D., & Mamazhanov, M. (1986). Kraevye zadachi dlia odnogo klassa uravnenii chetvertogo poriadka smeshannogo tipa [Boundary value problems for one class of fourth-order equations of mixed type]. *Differentsialnye uravneniia — Differential equations, 22*(1), 25–31 [in Russian].

5 Dzhuraev, T.D., Sopuev, A., & Mamazhanov, M. (1986). *Kraevye zadachi dlia uravnenii parabolo-giperbolicheskogo tipa [Boundary value problems for equations of parabolic-hyperbolic type].* Tashkent: Izdatelstvo Fan [in Russian].

6 Dzhuraev, T.D., & Sopuev, A. (2000). *K teorii differentsialnykh uravnenii v chastnykh proizvodnykh chetvertogo poriadka [Boundary value problems for equations of parabolic-hyperbolic type].* Tashkent: Izdatelstvo Fan [in Russian].

7 Apakov, Yu.P., & Mamajonov, S.M. (2021). Solvability of a Boundary Value Problem for a Fourth Order Equation of Parabolic-Hyperbolic Type in a Pentagonal Domain. *Journal of Applied and Industrial Mathematics, 15*(4), 586–596. https://doi.org/10.1134/S1990478921040025

8 Apakov, Y.P., & Mamajonov, S.M. (2022). Boundary Value Problem for a Fourth-Order Equation of the Parabolic-Hyperbolic Type with Multiple Characteristics with Slopes Greater Than One. *Russian Mathematics, 66*(4), 1–11. https://doi.org/10.3103/S1066369X22040016

9 Mamajonov, M., & Shermatova, X.M. (2022). On a boundary value problem for a third-order equation of the parabolic-hyperbolic type in a triangular domain with three type change lines. *Journal of applied and industrial mathematics, 16*, 481–489. https://doi.org/10.1134/S1990478922030127

10 Mamajonov, S. (2018). Ob odnom klasse kraevykh zadach dlia uravneniia chetvertogo poriadka parabolo-giperbolicheskogo tipa v piatiugolnoi oblasti [On a class of boundary value problems for a fourth-order equation of parabolic-hyperbolic type in a pentagonal domain]. *Vestnik KRAUNTs. Fiziko-matematicheskie nauki — Bulletin of KRAUNTS. Physical and Mathematical Sciences, 4*(24), 40–49 [in Russian]. https://doi.org/10.18454/2079-6641-2018-24-4-40-49

11 Mamajonov, M. (2018). Ob odnoi kraevoi zadache dlia parabolo-giperbolicheskogo uravneniia chetvertogo poriadka v piatiugolnoi oblasti [On a boundary value problem for a parabolo-hyperbolic equation of the fourth order in a pentagonal domain]. *Vestnik KRAUNTs. Fiziko-matematicheskie nauki — Bulletin of KRAUNTS. Physical and Mathematical Sciences, 4*(24), 29–39 [in Russian]. https://doi.org/10.18454/2079-6641-2018-24-4-29-39

12 Takhirov, Zh.O. (1988). Kraevye zadachi dlia smeshannogo parabolo-giperbolicheskogo uravneniia s izvestnoi i neizvestnoi liniiami razdela [Boundary value problems for a mixed parabolic-hyperbolic equation with known and unknown dividing lines]. *Extended abstract of candidate's thesis* [in Russian].

13 Takhirov, Zh.O. (1999). K teorii nelokalnykh kraevykh zadach i zadach so svobodnoi granitsei dlia parabolicheskikh, giperbolicheskikh i smeshannykh parabolo-giperbolicheskikh uravnenii [On the theory of nonlocal boundary value problems and problems with a free boundary for parabolic, hyperbolic and mixed parabolic-hyperbolic equations]. *Extended abstract of Doctor's thesis* [in Russian].

14 Berdyshev, A.S. (2015). *Kraevye zadachi i ikh spektralnye svoistva dlia uravneniia smeshannogo parabolo-giperbolicheskogo i smeshanno-sostavnogo tipov [Boundary value problems and their spectral properties for equations of mixed parabolic-hyperbolic and mixed-composite types].* Almaty [in Russian].

15 Baltaeva, U.I. (2019). Kraevye zadachi dlia lineinykh nagruzhennykh differentsialnykh i integro-differentsialnykh uravnenii smeshannogo tipa tretego poriadka [Boundary value problems for linear loaded differential and integro-differential equations of mixed type of third order]. *Doctor's thesis.* Tashkent [in Russian].

16 Mamajonov, M., & Shermatova, Kh.M. (2022). Ob odnoi kraevoi zadache dlia uravneniia tretego poriadka parabolo-giperbolicheskogo tipa v treugolnoi oblasti s tremia liniiami izmeneniia tipa uravneniia [On a boundary value problem for a third-order equation of parabolic-hyperbolic type in a triangular domain with three lines of change in the type of equation]. *Sibirskii zhurnal indus-*

trialnoi matematiki — Siberian Journal of Industrial Mathematics, 25(3), 93–103 [in Russian].

17 Mamajonov, M., Shermatova, Kh.M., & Mukhtorova, T. (2022). On a boundary value problem for a parabolic-hyperbolic equation of the third-order, when the characteristic of the first order operator is parallel to the y ordinate axis. *International journal of social science and interdisciplinary research, 11*, 105–110.

18 Mamajonov, M., & Shermatova, H.M. (2022). Statement and study of a boundary value problem for a third-order equation of parabolic-hyperbolic type in a mixed pentagonal domain, when the slope of the characteristic of the operator the first order is greater than one. *International journal of research in commerce, IT, engineering and social sciences, 16*(5), 117–130.

19 Mamajonov, M., Suleymanov, M.M., Vokhobov, F.F. & Gafurova, M.A. (2022). Formulation and study of one boundary value problem for a third-order equation of a parabolic-hyperbolic type of the form in a concave hexagonal area with two lines of type change. *International Journal of Early Childhood Special Education*, 1980–1990.

20 Mamajonov, M., & Kharimova, Yu. (2022). On one boundary problem for one parabolic-hyperbolic equation of the third order in a quadrangular domain with two lines type changes. *Galaxy international interdisciplinary research journal, 10*(12), 68–77.

21 Mamajonov, M., Aroev, D.D., & Shermatova, G. (2022). Statement and investigation of one boundary problem for one parabolic-hyperbolic equation of the third order in a pentagonal domain with three lines of type change. *Galaxy international interdisciplinary research journal, 10*(12), 332–342.

22 Mamajonov, M., & Turdiboeva, M.M. (2022). On one boundary problem for a parabolic-hyperbolic equation of the third order, when characteristics of the first order operator parallel to the X-axis. *Galaxy international interdisciplinary research journal, 10*(12), 343–349.

23 Mamajonov, M. (2023). O nekotorykh kraevykh zadachakh dlia odnogo klassa uravnenii tretego poriadka parabolo-giperbolicheskogo tipa v treugolnoi oblasti s tremia liniiami izmeneniia tipa [On some boundary value problems for one class of third-order equations of parabolic-hyperbolic type in a triangular domain with three lines of change of type]. *Vestnik Oshskogo gosudarstvennogo universiteta. Matematika. Fizika. Tekhnika — Bulletin of Osh State University. Mathematics. Physics. Engineering, 1*, 119–131 [in Russian]. https://doi.org/10.52754/16948645_2023_1_121

24 Apakov, Y.P., & Mamajonov, S.M. (2024). Boundary Value Problem for Fourth Order Inhomogeneous Equation with Variable Coefficients. *Journal of Mathematical Sciences (United States), 284*(2), 153–165. https://doi.org/10.1007/s10958-024-07340-5

25 Apakov, Y.P., & Mamajonov, S.M. (2024). Boundary-Value Problem for the Fourth-Order Equation with Multiple Characteristics in a Rectangular Domain. *Journal of Mathematical Sciences (United States), 272*(2), 185–201. https://doi.org/10.1007/s10958-023-06409-x

26 Apakov, Y.P., & Mamajonov, S.M. (2024). Second Boundary Value Problem for a Fourth-Order Inhomogeneous Equation with Variable Coefficients. *Lobachevskii J Math, 45*(8), 3849–3859. https://doi.org/10.1134/S1995080224604466

27 Apakov, Y.P., & Mamazhonov, S.M. (2023). Boundary Value Problem for an Inhomogeneous Fourth-Order Equation with Lower-Order Terms. *Differential Equations, 59*(2), 188–198. https://doi.org/10.1134/S0012266123020040

28 Apakov, Y.P., & Mamajonov, S.M. (2023). Boundary Value Problem for an Inhomogeneous Fourth Order Equation with Constant Coefficients. *Cheliabinsk Physical and Mathematical Journal, 8*(2), 157–172.

29 Mamazhanov, M. (2024). O postanovke nekotorykh kraevykh zadach dlia parabolo-giperboli-

cheskikh uravnenii chetvertogo poriadka v smeshannoi piatiugolnoi oblasti [On some boundary value problems for one class of third-order equations of parabolic-hyperbolic type in a triangular domain with three lines of change of type]. *"Matematikaning zamonaviy masalalari: muammo va yechimlar" mavzusidagi respublika ilmiy-amaliy anjuman materiallari. Qo'qon – 2024*, 121–127 [in Russian].

*Author Information**

Mirza Mamajonov — Candidate of physical and mathematical sciences, Professor, Position, Kokand State Pedagogical Institute, 23 Turon street, Kokand, 150700, Uzbekistan; e-mail: *mirzamamajonov@gmail.com*; https://orcid.org/0009-0003-8413-0549

*The author's name is presented in the order: First, Middle and Last Names.

https://doi.org/10.31489/2025M1/133-142 *Research article*

Inverse boundary value problem for a linearized equations of longitudinal waves in rods

Y.T. Mehraliyev[1], R. Iskanderov[2], A.T. Ramazanova[3,*]

[1] *Baku State University, Baku, Azerbaijan;*
[2] *Azerbaijan University of Architecture and Construction, Baku, Azerbaijan;*
[3] *University Duisburg-Essen, Essen, Germany*
(E-mail: yashar_aze@mail.ru, r.iskanderov@gmail.com, aysel.ramazanova@uni-due.de)

In this article, a question regarding the solvability of an inverse boundary value problem for the linearized equation of longitudinal waves in rods with an integral condition of the first kind was considered. For the considered inverse boundary value problem, the definition of a classical solution was introduced. Using the Fourier method, the problem was reduced to solving a system of integral equations. The method of contraction mappings is applied to prove the existence and uniqueness of a solution to the system of integral equations. The problem is to deduce the existence and uniqueness of the classical solution for the original problem.

Keywords: Inverse boundary value problem, longitudinal wave equations, Fourier method, classical solution.

2020 Mathematics Subject Classification: 49k10, 49k20.

Introduction

At present, the theory of nonlocal problems is being intensively developed and represents an important branch of the theory of partial differential equations. Problems with nonlocal integral conditions are of great interest in this area. Conditions of this kind may appear in the mathematical modeling of phenomena related to plasma physics [1], heat propagation [2], moisture transfer in capillary-porous media [3], demography, and mathematical biology.

Inverse problems with an integral overdetermination condition for partial differential equations have been studied in many papers. Let us note the articles [4–6] and the references therein.

The study of various aspects of inverse problems of recovering the coefficients of partial differential equations, as well as the study of inverse problems by reducing them to variational formulations, is considered in the works of Kozhanov A.I. [6], Denisov A.M. [7], Ivanchov M.I. [8], and others.

Works are devoted to the study of nonlinear inverse problems for the linearized equation of longitudinal waves in rods. The questions of solvability of problems with nonlocal integral conditions for partial differential equations were studied in [5, 9–18].

1 Statement of the problem and its reduction to an equivalent problem

Let D_T be a Domain, $D_T = \{(x,t) : 0 < x < 1,\ 0 < t \le T\}$. Consider for linearized equation an inverse boundary value problem [6]

$$u_{tt}(x,t) + u_{ttxx}(x,t) - u_{xx}(x,t) = a(t)u(x,t) + f(x,t), \quad (x,t) \in D_T \tag{1}$$

[*]Corresponding author. *E-mail:* *aysel.ramazanova@uni-due.de*
Received: 6 May 2024; *Accepted:* 17 December 2024.

with the initial condition (IC)

$$u(x,0)=\varphi(x),\ u_t(x,T)=\psi(x)\quad (0\le x\le 1). \tag{2}$$

Neumann boundary condition (BC)

$$u_x(0,t)=0\quad (0\le t\le T), \tag{3}$$

integral condition (IgC)

$$\int\limits_0^1 u(x,t)dx=0\quad (0\le t\le T) \tag{4}$$

and additional conditions (AD)

$$u(0,t)=h(t)\quad (0\le t\le T), \tag{5}$$

$f(x,t)$, $\varphi(x)$, $\psi(x)$, and $h(t)$ are given functions, $u(x,t)$ and $a(t)$ are the required functions.

Definition. By the classical solution of the inverse boundary value problem (1)–(5) we mean the pair $\{u(x,t),\, a(t)\}$ of the functions $u(x,t)$, $a(t)$, where $u(x,t)\in \tilde{C}^{2,2}(\bar{D}_T)$, $a(t)\in C[0,T]$, and they satisfy the equations (1)–(5) in the ordinary sense, where

$$\tilde{C}^{2,2}(D_T)=\left\{u(x,t)\,:\,u(x,t)\in C^2(D_T),\ u_{ttxx}(x,t)\in C(D_T)\right\}.$$

To study problem (1)–(5), we first consider the following:

$$y''(t)=a(t)y(t)\quad (0\le t\le T), \tag{6}$$

$$y(0)\,=0\,,\ \ y'(T)\,=0, \tag{7}$$

where $a(t)\in C[0,T]$ are given functions, and $y=y(t)$ is the required function. Furthermore, by solving the problem (6), (7) we mean a function $y(t)$, that, together with all its derivatives in equation (6), is continuous on $[0,T]$ and satisfies conditions (6), (7) in the usual sense. The following lemma holds:

Lemma 1. [7] Let $a(t)\in C[0,T]$ be such that

$$\|a(t)\|_{C[0,T]}\le R=const.$$

Supplementarily, $\frac{1}{2}RT^2<1$. Then problem (6), (7) has only a trivial solution.

With the addition of the inverse boundary value problem (1)–(5), consider the following auxiliary inverse boundary value problem. We must define a pair $\{u(x,t),a(t)\}$ of functions $u(x,t)\in\tilde{C}^{2,2}(\bar{D}_T)$ and $a(t)\in C[0,T]$, from (2)-(3),

$$u_x(1,t)=0\quad (0\le t\le T), \tag{8}$$

$$h''(t)+u_{ttxx}(0,t)-u_{xx}(0,t)=a(t)h(t)+f(0,t)\quad (0\le t\le T). \tag{9}$$

Theorem 1. Let $\varphi(x)$, $\psi(x)\in C^1[0,1]$, $\varphi'(1)=0$, $\psi'(1)=0$, $h(t)\in C^2[0,T]$, $f(x,t)\in C(\bar{D}_T)$, $\int\limits_0^1 f(x,t)dx=0$ $(0\le t\le T)$, $h(t)\ne 0$ $(0\le t\le T)$ and

$$\int_0^1\varphi(x)dx=0,\qquad \int_0^1\psi(x)dx=0, \tag{10}$$

$$\varphi(0)=h(0),\quad \psi(0)=h'(T). \tag{11}$$

Then the following statements are true:

1. Each classical solution $\{u(x,t),a(t)\}$ of problem (1)–(5) is also a solution of problem (1)–(3), (8), (9).

2. Each solution $\{u(x,t), a(t)\}$ of problem (1)–(3), (8), (9) is a classical solution of the problem (1)–(5) if

$$\frac{1}{2}\,\|a(t)\|_{C[0,T]}\,T^2 < 1\,, \tag{12}$$

it is a classical solution (1)–(5).

Proof. Let $\{u(x,t), a(t)\}$ be a classical solution to problem (2)–(5). Integrating equation (2), we get:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx + u_{ttx}(1,t) - u_{ttx}(0,t) - (u_x(1,t) - u_x(0,t)) =$$

$$= a(t)\int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx \quad (0 \le t \le T). \tag{13}$$

Suppose it is the case that $\int\limits_0^1 f(x,t)dx = 0$ $(0 \le t \le T)$, taking into account (3), (4), we identify

$$u_{ttx}(1,t) - u_x(1,t) = 0 \quad (0 \le t \le T). \tag{14}$$

Due to (2), $\varphi'(1) = 0,\ \psi'(1) = 0$, therefore

$$u_x(1,0) = \varphi'(1) = 0, \quad u_{tx}(1,T) = \psi'(1) = 0. \tag{15}$$

Obviously, problem (14), (15) has only a trivial solution, $u_x(1,t) = 0$ $(0 \le t \le T)$, i.e. conditions (8) are satisfied.

Considering $h(t) \in C^2[0,T]$ and differentiating (5) twice, we obtain:

$$u_{tt}(0,t) = h(t) \quad (0 \le t \le T). \tag{16}$$

Further, from (2) we get:

$$\frac{d^2}{dt^2}u(0,t) + u_{ttxx}(0,t) - u_{xx}(0,t) = a(t)h(t) + f(0,t) \quad (0 \le t \le T). \tag{17}$$

From (17), regarding to (5) and (16), we obtained (9).

Now, assume that $\{u(x,t), a(t)\}$ is a solution to problem (1)–(3), (8), (9), and (12) is satisfied. Then from (13), taking into account (3) and (8), we deduce:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx - a(t)\int_0^1 u(x,t)dx = 0 \quad (0 \le t \le T). \tag{18}$$

As a corollary to (2) and (10), it is evident that

$$\int_0^1 u(x,0)dx = \int_0^1 \varphi(x)dx = 0, \quad \int_0^1 u_t(x,T)dx = \int_0^1 \psi(x)dx = 0. \tag{19}$$

Since, by virtue of Lemma 1, problem (18), (19) has only a trivial solution, then $\int\limits_0^1 u(x,t)dx = 0$ $(0 \le t \le T)$, i.e. condition (4) is satisfied.

Further, from (9) and (17):

$$\frac{d^2}{dt^2}(u(0,t) - h(t)) = a(t)(u(0,t) - h(t)) \ (0 \le t \le T). \tag{20}$$

From (2) and (11), we get:

$$u(0,0) - h(0) = \varphi(0) - h(0) = 0. \tag{21}$$

From (20) and (21), based on Lemma 1, we conclude that condition (5) is fulfilled. The theorem is proven.

2 The Existence and uniqueness of the classical solution of the inverse boundary value problem

Component $u(x,t)$ of solution $\{u(x,t), a(t)\}$ of problem (1)–(3), (8), (9) is studied in the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = \pi k), \tag{22}$$

$u_k(t) = m_k \int_0^1 u(x,t) \cos \lambda_k x dx \ \ (k = 0, 1, 2, ...)$, and

$$m_k = \begin{cases} 1, & k = 0, \\ 2, & k = 1, 2, \ldots \end{cases}$$

By applying the scheme of the Fourier method from (2), we get:

$$(1 - \lambda_k^2) u_k''(t) + \lambda_k^2 u_k(t) = F_k(t; u, a) \ \ (k = 0, 1, 2, \ldots ; \ 0 \le t \le T), \tag{23}$$

$$u_k(0) = \varphi_k, \ \ u_k'(T) = \psi_k \ \ (k = 0, 1, 2, \ldots), \tag{24}$$

$$F_k(t; u, a) = f_k(t) + a(t) u_k(t), \ \ f_k(t) = m_k \int_0^1 f(x,t) \cos \lambda_k x dx,$$

$$\varphi_k = m_k \int_0^1 \varphi(x) \cos \lambda_k x dx, \psi_k = m_k \int_0^1 \psi(x) \cos \lambda_k x dx \ \ (k = 0, 1, 2, \ldots).$$

Solving problem (1)–(6), we obtain:

$$u_0(t) = \varphi_0 + \int_0^T m(t) u_0(t) dt + \psi_0 t + \int_0^T G_0(t,\tau) F_0(\tau; u, a) d\tau, \tag{25}$$

$$u_k(t) = \frac{ch(\beta_k(T-t))}{ch(\beta_k T)} \varphi_k +$$

$$+ \frac{sh(\beta_k t)}{\beta_k ch(\beta_k T)} \psi_k - \frac{1}{(\lambda_k^2 - 1)\beta_k} \int_0^T G_k(t,\tau) F_k(\tau; u, a) d\tau \ \ (k = 1, 2, \ldots), \tag{26}$$

where

$$G_0(t,\tau) = \begin{cases} -t, & t \in [0, \tau], \\ -\tau, & t \in [\tau, T], \end{cases}$$

$$\beta_k^2 = \frac{\lambda_k^2}{\lambda_k^2 - 1} > 0,$$

$$G_k(t,\tau) = \begin{cases} -\frac{[sh(\beta_k(T+t-\tau)) - sh(\beta_k(T-(t+\tau)))]}{2ch(\beta_k T)}, & t \in [0, \tau], \\ -\frac{sh(\beta_k(T-(t+\tau))) - sh(\beta_k(T-(t-\tau)))}{2ch(\beta_k T)}, & t \in [\tau, T]. \end{cases}$$

After substituting the expression from (25), (26) into (22), to define a component $u(x,t)$ of the solution of problem (1)–(4), (8), (9), we obtain:

$$u(x,t) = \varphi_0 + \psi_0 t + \int_0^T G_0(t,\tau) F_0(\tau; u, a) d\tau + \sum_{k=1}^{\infty} \left\{ \frac{ch(\beta_k(T-t))}{ch(\beta_k T)} \varphi_k + \right.$$

$$\left. + \frac{sh(\beta_k t)}{\beta_k ch(\beta_k T)} \psi_k - \frac{1}{(\lambda_k^2 - 1)\beta_k} \int_0^T G_k(t,\tau) F_k(\tau; u, a) d\tau \right\} \cos \lambda_k x. \tag{27}$$

Now, from (8) including (22) and from (23) including (26), we get:

$$a(t) = [h(t)]^{-1} \left\{ h''(t) - f(0,t) - \sum_{k=1}^{\infty} \lambda_k^2 (u_k''(t) - u_k(t)) \right\}. \quad (28)$$

$$-\lambda_k^2 u_k''(t) + \lambda_k^2 u_k(t) = F_k(t;u,a) - u_k''(t) = \frac{\lambda_k^2}{\lambda_k^2 - 1} F_k(t;u,a) - \frac{\lambda_k^2}{\lambda_k^2 - 1} u_k(t) =$$

$$= \frac{\lambda_k^2}{\lambda_k^2 - 1} F_k(t;u,a) - \frac{\lambda_k^2}{\lambda_k^2 - 1} \left[\frac{ch(\beta_k(T-t))}{ch(\beta_k T)} \varphi_k + \right.$$

$$\left. + \frac{sh(\beta_k t)}{\beta_k ch(\beta_k T)} \psi_k - \frac{1}{(\lambda_k^2 - 1)\beta_k} \int_0^T G_k(t,\tau) F_k(\tau;u,a) d\tau \right] \quad (k = 1, 2, \dots). \quad (29)$$

Aimed at defining an equation for the second component $a(t)$ of the solution $\{u(x,t), a(t)\}$ of problem (1)–(3), (8), (9), we substitute expression (29) into (28):

$$a(t) = [h(t)]^{-1} \left\{ h''(t) - f(0,t) + \sum_{k=1}^{\infty} \left[\frac{\lambda_k^2}{\lambda_k^2 - 1} F_k(t;u,a) + \right. \right.$$

$$- \frac{\lambda_k^2}{\lambda_k^2 - 1} \left[\frac{ch(\beta_k(T-t))}{ch(\beta_k T)} \varphi_k + \frac{sh(\beta_k t)}{\beta_k ch(\beta_k T)} \psi_k - \right.$$

$$\left. \left. \left. - \frac{1}{(\lambda_k^2 - 1)\beta_k} \int_0^T G_k(t,\tau) F_k(\tau;u,a) d\tau \right] \right] \right\} \ (0 \le t \le T). \quad (30)$$

Thus, the solution of problem (1)–(3), (8), (9) is reduced to the solution of system (27), (30) with respect to unknown functions $u(x,t)$ and $a(t)$.

The subsequent lemma plays an essential role in the disquisition of the uniqueness question of the solution to problem (2)-(3), (8), (9).

Lemma 2. If $\{u(x,t), a(t)\}$ is any solution of problem (1)–(3), (8), (9), then the functions

$$u_k(t) = m_k \int_0^1 u(x,t) \cos \lambda_k x dx \quad (k = 0, 1, 2, \dots)$$

satisfy the system consisting of equations (25), (26) on $[0,T]$.

Proof. Let $\{u(x,t), a(t)\}$ be any solution of problem (1)–(3), (8), (9). Then multiplying both sides of equation (2) by the function $m_k \cos \lambda_k x$ $(k = 0, 1, 2, \dots)$, integrating the resulting equality over x from 0 to 1 and using $m_k \int_0^1 u_{tt}(x,t) \cos \lambda_k x dx = \frac{d^2}{dt^2} \left(m_k \int_0^1 u(x,t) \cos \lambda_k x dx \right) = u_k''(t)$ $(k = 0, 1, 2, ...)$,

$$m_k \int_0^1 u_{xx}(x,t) \cos \lambda_k x dx = -\lambda_k^2 \left(m_k \int_0^1 u(x,t) \cos \lambda_k x dx \right) = -\lambda_k^2 u_k(t) \quad (k = 0, 1, 2, \dots),$$

$$m_k \int_0^1 u_{ttxx}(x,t) \cos \lambda_k x dx = -\lambda_k^2 m_k \int_0^1 u_{tt}(x,t) \cos \lambda_k x dx =$$

$$= -\lambda_k^2 \frac{d^2}{dt^2} \left(m_k \int_0^1 u(x,t) \cos \lambda_k x dx \right) = -\lambda_k^2 u_k''(t) \quad (k = 0, 1, 2, \dots),$$

we obtain that Eq. (23) is satisfied.

Similarly, from (2) we obtain that condition (24) is fulfilled.

Thus, $u_k(t)$ $(k = 0, 1, 2, \dots)$ is a solution to problem (23), (24). Moreover, it directly follows that the functions $u_k(t)$ $(k = 0, 1, 2, \dots)$ satisfy the system (25), (26) on $[0,T]$. The lemma is proven.

Therefore, if $u_k(t)=2\int\limits_0^1 u(x,t)\cos\lambda_k x dx$ $(k=0,1,2,\dots)$ is a solution to system (25), (26), then the pair $\{u(x,t),a(t)\}$ of functions $u(x,t)=\sum\limits_{k=0}^{\infty}u_k(t)\cos\lambda_k x, a(t)$ and $a(t)$ is a solution to system (27), (30).

Corollary 1. Let system (27), (30) have a unique solution. Then problem (1)–(3), (8), (9) can not have more than one solution, i.e. if problem (1)–(3), (8), (9) has a solution, then it is unique.

To study the problem (1)–(3), (8), (9), we introduce two spaces.

By $B_{2,T}^{\alpha}$ [8], we denote the set of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x \ \ (\lambda_k=k\pi),$$

considered in D_T, where each function $u_k(t)$ $(k=0,1,\dots)$ is continuous on $[0,T]$ and

$$J(u)=\|u_0(t)\|_{C[0,T]}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^{\alpha}\|u_k(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}<+\infty,$$

where $\alpha\geq 0$. We define the norm in this set as follows:

$$\|u(x,t)\|_{B_{2,T}^{\alpha}}=J(u).$$

E_T^{α} denotes the space $B_{2,T}^{\alpha}\times C[0,T]$ with vector functions $z(x,t)=\{u(x,t),a(t)\}$ and norm

$$\|z(x,t)\|_{E_T^{\alpha}}=\|u(x,t)\|_{B_{2,T}^{\alpha}}+\|a(t)\|_{C[0,T]}.$$

$B_{2,T}^{\alpha}$ and E_T^{α} are Banach spaces.

In the space E_T^3, we define an operator:

$$\Phi(u,a,b)=\{\Phi_1(u,a),\Phi_2(u,a)\},$$

with

$$\Phi_1(u,a)=\tilde{u}(x,t)\equiv\sum_{k=0}^{\infty}\tilde{u}_k(t)\cos\lambda_k x, \quad \Phi_2(u,a,b)=\tilde{a}(t),$$

$\tilde{u}_0(t),\tilde{u}_k(t)$ $(k=1,2,\dots)$ and $\tilde{a}(t)$ are equal to the right-hand sides of (25), (26) and (30).

It is easy to see that

$$\frac{ch(\beta_k(T-t))}{ch(\beta_k T)}<1, \quad \frac{sh(\beta_k t)}{ch(\beta_k T)}<1, \quad \frac{sh(\beta_k(T+t-\tau))}{ch(\beta_k T)}<1 \ \ (t\in[0,\tau]),$$

$$\frac{sh(\beta_k(T-(t+\tau)))}{ch(\beta_k T)}<1, \quad \frac{sh(\beta_k(T-(t-\tau)))}{ch(\beta_k T)}<1 \ \ (t\in[\tau,T]),$$

$$\lambda_k^2-1>\frac{1}{2}\lambda_k^2, \quad 1<\beta_k=\frac{\lambda_k}{\sqrt{\lambda_k^2-1}}<\sqrt{2}, \quad \frac{1}{\sqrt{2}}<\frac{1}{\beta_k}<1.$$

Then, we have:

$$\|\tilde{u}_0(t)\|_{C[0,T]} \leq |\varphi_0| + T\,|\psi_0| + 2T\sqrt{T}\left(\int_0^t |f_0(\tau)|^2\,d\tau\right)^{\frac{1}{2}} + \ 2T^2\,\|a(t)\|_{\ C[0,T]}\,\|u_0(t)\|_{C[0,T]}\,, \tag{31}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq 2\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_k|)^2\right)^{\frac{1}{2}} + 2\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_k|)^2\right)^{\frac{1}{2}} +$$

$$+\,4\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\,|f_k(\tau)|)^2\right)^{\frac{1}{2}} + 4T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{32}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]}\Big\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]} +$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k\,\|f_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \ \|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} +\right.$$

$$+\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_k|)^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_k|)^2\right)^{\frac{1}{2}} + 2\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\,|f_k(\tau)|)^2\right)^{\frac{1}{2}} +$$

$$\left.+\ 2T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\Big\}. \tag{33}$$

Let us assume that the data of problem (1)–(3), (8), (9), satisfy the following conditions:
1) $\varphi(x) \in C^2[0,1],\ \varphi'''(x) \in L_2(0,1),\ \varphi'(0) = \varphi'(1) = 0;$
2) $\psi(x) \in C^2[0,1],\ \psi'''(x) \in L_2(0,1),\ \psi'(0) = \psi'(1) = 0;$
3) $f(x,t) \in C(D_T),\ f_x(x,t) \in L_2(D_T),\ f_x(0,t) = f_x(1,t) = 0\ \ (0 \leq t \leq T);$
4) $h(t) \in C^2[0,T],\ h(t) \neq 0\ \ (0 \leq t \leq T).$

Then from (31)–(33), we have:

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \leq A_1(T) + B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}\,, \tag{34}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \ A_2(T) + \ B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} +$$

$$+\,B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + C_2(T)\,\|u(x,t)\|_{B^3_{2,T}+D_2(T)\|a(t)\|_{C[0,T]}}\,, \tag{35}$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\,\|\psi(x)\|_{L_2(0,1)} + 2T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)} +$$

$$+2\left\|\varphi'''(x)\right\|_{L_2(0,1)} + 2\left\|\psi'''(x)\right\|_{L_2(0,1)} + 4\sqrt{T}\,\|f_x(x,t)\|_{L_2(D_T)}\,,$$

$$B_1(T) = 6T, \quad A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\Big\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]} +$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0.1)} + \left\|\varphi'''(x)\right\|_{L_2(0,1)} + \left\|\psi'''(x)\right\|_{L_2(0,1)} +\right.$$

$$\left.+\ 2\sqrt{T}\,\|f_x(x,t)\|_{L_2(D_T)}\right]\Big\},$$

$$B_2(T) = 2\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}(2T+1).$$

From inequalities (34), (35) we conclude:

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \le A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}\,, \tag{36}$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T).$$

Theorem 2. Let conditions 1–3 be satisfied and

$$B(T)(A(T)+2)^2 < 1. \tag{37}$$

Then, in $K = K_R$ $(\|\,z\,\|_{E^3_T} \le R = A(T)+2)$, in the space E^3_T, problem (1)–(3), (8), (9) has only one solution.

Proof. In the space E^3_T consider the equation

$$z = \Phi z, \tag{38}$$

where $z = \{u, a\}$ and components $\Phi_i(u,a)$ $(i = 1, 2)$ of operators $\Phi(u,a)$ are defined by the right-hand sides of equations (27) and (30).

Now, consider the operator $\Phi(u,a)$ in the ball $K = K_R$ of the space E^3_T. Analogously to (36), we obtain that for any $z, z_1, z_2 \in K_R$, the following estimates hold:

$$\|\Phi z\|_{E^3_T} \le A\,(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}}\,, \tag{39}$$

$$\|\Phi z_1 - \Phi z_s\|_{E^3_T} \le 2B(T)R\,(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}}). \tag{40}$$

As a result, the operator (38) has a unique solution for $K = K_R$.

Then, from estimates (39) and (40), allowing for (37), it follows that the operator Φ acts in the ball $K = K_R$ and is contractive.

Functions $u(x,t)$, as an element of space $B^3_{2,T}$, are continuous and have continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in $\bar{D}_T$.

From (23), it is evident that

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{2}\left\{\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \right.$$

$$\left. + \left\|\,\|a(t)u_x(x,t) + f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right\}.$$

Hence, it follows that $u_{tt}(x,t)$,$u_{ttx}(x,t)$,$u_{ttxx}(x,t)$ are continuous in $\bar{D}_T$.

It is easy to check that equation (1) and conditions (3), (4), (7), (9) are satisfied in the usual sense. Consequently, $\{u(x,t), a(t)\}$ is a solution to problem (1)–(3), (7)–(9). By the corollary of Lemma 1, it is unique in the ball $K = K_R$. The theorem has been proven.

In the proof of Theorem 1 and Theorem 2, the next Theorem plays, an essential role of unique and solvability of the problem (1)–(5).

Theorem 3. Let all the conditions of Theorem 2 and (10), (11) be satisfied, $\int_0^1 f(x,t)dx = 0$ $(0 \le t \le T)$, and

$$\frac{1}{2}(A(T)+2)\,T^2 < 1\,.$$

Then, problem (1)–(5) has the only classical solution in the ball $K = K_R$ $(||z||_{E^3_T} \le A(T)+2)$ from E^3_T.

Conclusion

An inverse boundary value problem for a linearized equations of longitudinal waves in rods with integral condition of the first kind are studied.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Samarskii, A.A. (1980). O nekotorykh problemakh teorii differentsialnykh uravnenii [On some problems of the theory of differential equations]. *Differentsialnye uravneniia — Differential Equations, 16*(11), 1925–1935 [in Russian].

2 Ionkin, N.I. (1977). Reshenie odnoi kraevoi zadachi teorii teploprovodnosti s neklassicheskim kraevym usloviem [Solutions of boundary value problem in heat conductions theory with nonlocal boundary conditions]. *Differentsialnye uravneniia — Differential Equations, 13*(2), 294–304 [in Russian].

3 Nakhushev, A.M. (1982). Ob odnom priblizhennom metode resheniia kraevykh zadach dlia differentsialnykh uravnenii i ego prilozheniia k dinamike pochvennoi vlagi i gruntovykh vod [An approximate method for solving boundary value problems for differential equations and its application to the dynamics of ground moisture and ground water]. *Differentsialnye uravneniia — Differential Equations, 18*(1), 72–81 [in Russian].

4 Kozhanov, A.I., & Pul'kina, L.S. (2006). On the solvability of boundary value problems with a nonlocal boundary condition of integral form for the multidimensional hyperbolic equations. *Differential Equations, 42*(9), 1233–1246. https://doi.org/10.1134/S0012266106090023

5 Kapustin, N.Yu., & Moiseev, E.I. (1997). O spektralnykh zadachakh so spektralnym parametrom v granichnom uslovii [On spectral problems with a spectral parameter in the boundary condition]. *Differentsialnye uravneniia — Differential Equations, 33*(1), 115–119 [in Russian].

6 Erofeev, V.I., Kazhaev, V.V., & Semerikova, N.P. (2002). *Volny v sterzhniakh: Dispersiia. Dissipatsiia. Nelineinost [Waves in Rods: Dispersion. Dissipation. Nonlinearity].* Moscow: Fizmatlit [in Russian].

7 Denisov, A.M. (1994). *Vvedenie v teoriiu obratnykh zadach [Introduction to Theory of Inverse Problem].* Moscow: MSU [in Russian].

8 Ivanchov, M.I. (2003). *Inverse Problems for Equation of Parabolic Type.* Ukraine: VNTL Publishers.

9 Megraliev, Ya.T. (2012). Obratnaia kraevaia zadacha dlia ellipticheskogo uravneniia vtorogo poriadka s dopolnitelnym integralnym usloviem [Inverse boundary value problem for second order elliptic equation with additional integral condition] *Vestnik Udmurtskogo universiteta. Matematika. Mekhanika. Kompiuternye nauki — Bulletin of Udmurt University. Mathematics. Mechanics. Computer sciences*, (1), 32–40 [in Russian].

10 Mehraliev, Ya.T. (2012). On an inverse boundary value problem for a second order elliptic equation with integral condition. *Visnyk of the Lviv Univ. Series Mech. Math.*, (77), 145–156.

11 Juraev, D.A. (2023). The Cauchy problem for matrix factorization of the Helmholtz equation in a multidimensional unbounded domain. *Boletim da Sociedade Paranaense de Matemática, 41*, 1–18. https://doi.org/10.5269/bspm.63779

12 Mehraliyev, Y.T., Allahverdiyeva, S., & Ramazanova, A.T. (2023). On one coefficient inverse boundary value problem for a linear pseudoparabolic equation of the fourth order. *AIMS Mathematics, 8*(2), 2622–2633. https://doi.org/10.3934/math.2023136

13 Mehraliyev, Y.T., Huntul, M.J., Ramazanova, A.T., Tamsir, M., & Emadifar, H. (2022). An inverse boundary value problem for transverse vibrations of a bar. *Boundary Value Problems, 2022,* 96. https://doi.org/10.1186/s13661-022-01679-x

14 Mehraliyev, Y.T., Ramazanova, A.T., & Huntul, M.J. (2022). An inverse boundary value problem for a two-dimensional pseudo-parabolic equation of third order. *Results in Applied Mathematics, 14*, 100274. https://doi.org/10.1016/j.rinam.2022.100274

15 Mehraliyev, Y.T., Sadikhzade, R., & Ramazanova, A.T. (2023). Two-dimensional inverse boundary value problem for a third-order pseudo-hyperbolic equation with an additional integral condition. *European Journal of Pure and Applied Mathematics, 16*(2), 670–686. https://doi.org/10.29020/nybg.ejpam.v16i2.4743

16 Juraev, D.A., Shokri, A., & Marian, D. (2022). Solution of the Ill-Posed Cauchy Problem for Systems of Elliptic Type of the First Order. *Fractal and Fractional, 6*(7), 358. https://doi.org/10.3390/fractalfract6070358

17 Mehraliyev, Y.T., Ramazanova, A.T., & Sevdimaliyev, Y.M. (2020). An inverse boundary value problem for the equation of flexural vibrations of a bar with an integral conditions of the first kind. *Journal of Mathematical Analysis, 11*(5), 1–12.

18 Farajov, A.S. (2022). On a Solvability of the Nonlinear Inverse Boundary Value Problem for the Boussinesq equation. *Advanced Mathematical Models & Applications, 7*(2), 241–248.

*Author Information**

Yashar Topush oglu Mehraliyev — Doctor of mathematical sciences, Professor, Head of Differential and Integral Equation Department, Faculty Mathematics, Baku State University, 33 Akademik Zahid Khalilov street, Baku, AZ1148, Azerbaijan; e-mail: *yashar_ aze@mail.ru*

Ramiz Iskanderov — Doctor of physical and mathematical sciences, Professor, Azerbaijan University of Architecture and Construction, 11 Ayna Sultanova street, Baku, AZ1073, Azerbaijan; e-mail: *r.iskanderov@gmail.com*; https://orcid.org/0000-0003-1730-9461

Aysel Telman kizi Ramazanova (*corresponding author*) — Doctor of mathematical sciences, Assistent of Professor, Departament of Mathematics, Nonlinear optimization, University of Duisburg Essen, 9 Thea-Leyman street, Essen, 45127, Germany; e-mail: *aysel.ramazanova@uni-due.de*; https://orcid.org/0000-0003-0166-6018

*The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 143–154

https://doi.org/10.31489/2025M1/143-154 *Research article*

Some methods for solving boundary value problems for polyharmonic equations

M.T. Sabirzhanov[1], B.D. Koshanov[2,*], N.M. Shynybayeva[2], P.Zh. Kozhobekova[1]

[1] *Osh State University, Osh, Kyrgyzstan;*
[2] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan*
(E-mail: smskg@bk.ru, koshanov@math.kz, shynybayeva001@mail.ru, pardaz@mail.ru)

This article consists of three sections. In the first section the concept of Vekua space is introduced, where for elliptic systems of the first order, the theorem on the representation of the solution of a homogeneous equation and the theorem on the continuity of the solution of an inhomogeneous equation are valid. In the second section the Vekua method for solving boundary value problems for a polyharmonic equation is described. In the third section the Otelbaev method describes the correct boundary value problems for a polyharmonic equation in a multidimensional sphere.

Keywords: first order elliptic system, polyharmonic equation, continuity of solution, boundary value problem, integral representations of solution.

2020 Mathematics Subject Classification: 31B30, 31B20, 31B10.

1 Vekua space for analytic functions

The question of the existence of solutions in the classical sense and of methods for finding these solutions has not lost interest. It is particularly interesting to consider equations of order higher than two using the method of potential theory.

Recently, the theory of boundary value problems for polyharmonic equations and elliptic systems has attracted the attention of mathematicians, due to their great theoretical and practical importance. For example, hydrodynamic and elasticity problems can be formulated using these equations.

The object of our research is polyharmonic equations. Vekua's method is applicable for fixed boundary conditions on the surface of the domain, and the equation itself can change, i.e. minor terms can be added to the main equation. Otelbaev's method is used for fixed equations, but the boundary conditions can vary.

The problem is that in these methods under what conditions both methods are applicable. This article is devoted to this problem of the applicability of the Vekua and Otelbaev methods.

In [1] I.N. Vekua proved that for a first-order elliptic system

$$Lu = \partial_{\overline{z}}u + a(z)u + b(z)\overline{u} = f, \quad z \in \Omega, \tag{1.1}$$

when $a, b, f \in L_p(\Omega),\ p > 2$, any solution from $W_2^1(\Omega)$ is continuous in Ω, and any solution of the corresponding homogeneous system is representable in the form

$$u(z) = \Phi(z)e^{\omega(z)}, \tag{1.2}$$

*Corresponding author. *E-mail: koshanov@math.kz*

This research was funded by the grant AP 19678182 of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

Received: 5 June 2024; *Accepted:* 10 December 2024.

where $\Phi(z)$ is analytic in Ω and $\omega(z)$ are continuous functions in Ω. Here Ω is any open set, $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, $i^2 = -1$.

It is not possible to transfer this result to the case $1 \le p \le 2$. Therefore, the problem naturally arises: What should be the Banach space B of a function that is continuously embedded in L_1, such that a) any solution of equation (1.1) from $L_{1,loc}$, when $a, b, f \in B$, is continuous in Ω and b) any solution of the corresponding homogeneous system can be represented in the form (1.2), where $\omega(z)$ is continuous in Ω?

This problem was first posed by M.O. Otelbaev in his works [2, 3] and in the same article gave a comprehensive answer to this problem. To solve this problem, the paper introduces the concept of Vekua spaces (V-spaces), namely: a Banach function space B is a Vekua space if for any $a(z)$, $b(z)$ and $f(z)$, the theorem on the continuity of the solution of an inhomogeneous equation and the theorem on the representation of solutions of a homogeneous equation are valid.

Let $\mathbb{C}$ be the complex plane of points $z = x+iy$, and let Q be the square $\{-\pi \le x \le \pi, -\pi \le y \le \pi\}$. Let us assume that a certain norm $\|\cdot\|_B$ is defined on the set of trigonometric polynomials $\mathcal{F}$. Let $B(Q)$ denote the Banach space obtained by completing $\mathcal{F}$ with respect to the $\|\cdot\|_B$ norm.

Let us give the exact definition of Vekua space (V-space). Throughout what follows we will assume that $B(Q)$ satisfies the following three properties:

1°. A multiplication operator is defined in the space $B(Q)$. The operator of multiplication by the characteristic function of any rectangle located in Q, and the operator of multiplication by any function $\psi \in C^\infty_\pi(Q)$ are bounded, where $C^\infty_\pi(Q)$ is the space of infinitely smooth periodic functions with period 2π in each variable x, y.

2°. If $f \in B(Q)$ and $a \in \mathbb{C}$, then $f(z+a), |f| \in B(Q)$ and $\|f(z+a)\|_B \le C\,\|f\|_B$, $\|\,|f|\,\|_B \le C\,\|f\|_B$. Here and below, C will denote, generally speaking, various positive constants.

3°. $B(Q)$ is continuously embedded in $L_1(Q)$ $(B(Q) \hookrightarrow L_1(Q))$.

Let $\mathbb{P}_1(Q)$ denote the completion of $\mathcal{F}$ with respect to norm

$$\langle f\rangle_{1,Q} = \sup_{z\in Q} \int\limits_Q P(z-\zeta)\,|f(\zeta)|\,dQ_\zeta,$$

where $P(\cdot)$ is periodic function, with period 2π in each variable, such that

$$P(z) = \begin{cases} |z|^{-1}, \text{ at } & |z| \le 1, \quad z \in Q, \\ 1, \quad\ \text{ at } & |z| \ge 1, \quad z \in Q. \end{cases} \tag{1.3}$$

We will denote the integral operator with kernel $P(z-\xi)$ by P. Let's introduce one more operator

$$Tf = \int\limits_Q T(z-\zeta)\,f(\zeta)\,dQ_\zeta,$$

where $T(\cdot)$ is a continuous function for $|z| > 0$, $z \in Q$, and 2π-periodic function for each variable, such that

$$T(z) = \begin{cases} C\,|z|^{-1} + K_1(z), \text{ at } & |z| \le 1, \quad z \in Q, \\ K_2(z), \quad\quad\quad\quad \text{ at } & |z| \ge 1, \quad z \in Q. \end{cases} \tag{1.4}$$

Here, $K_1(z)$ and $K_2(z)$ are continuous functions for $|z| > 0$, in addition, $|K_j(z)| \le C\,|z|^{-1+\varepsilon_0}$, $\varepsilon_0 > 0$, $j = 1, 2$.

Let us recall the definition of Lorentz spaces.

Let $1 \le p,\ q < \infty$. The completion of $\mathcal{F}$ by the norm

$$|f : \mathcal{L}(p,q)| = \left(\int_0^\infty \{[\mu(z \in Q : |f(z) \ge t|)]^{\frac{1}{p}} t\} \frac{dt}{t}\right)^{\frac{1}{q}},$$

where $\mu(\cdot)$ is the Lebesgue measure, will be called the Lorentz space $\mathcal{L}(p,q)$.

The main result of the work [2] is the following statement.

Theorem 1.1. A function space B with properties $1^\circ - 3^\circ$ is a Vekua space if and only if $B \hookrightarrow \mathbb{P}_1$.

This result implies that a symmetric space is a V space if and only if it is continuously embedded in the Lorentz space $\mathcal{L}(2;1)$.

Thus, we can say that the widest space to which Vekua's theory can be extended is $\mathbb{P}_1(\cdot)$, and among all symmetric spaces, this is $\mathcal{L}(2;1)$.

In the prove of the main result, we used information about the complete continuity of some integral operators, in particular the operators introduced in (1.3) and (1.4). Such statements play a very important role in Vekua's theory [1].

Theorem 1.2. Let $B^s_{p,\theta}$ be completion of $\mathcal{F}$ according to the Besov norm

$$\|f\|_{B^s_{p,\theta}(Q)} = \left(\|f\|^\theta_{W^{n,p}(Q)} + \int\limits_0^\infty \left(\frac{\omega_p^2(f^{(n)},t)}{t^2} \right)^\theta \right)^{1/\theta},$$

where $\omega_p^2(f^{(n)},t) = \sup\limits_{|h|\le t} \|\Delta_h^2 f\|_p$ is continuity modulus, $\Delta_h f = f(x+h) - f(x)$.

If $s \ge \frac{2}{p} - 1, \theta = 1$, or $s > \frac{2}{p} - 1$ and $p \ge 1$, then $B^s_{p,\theta}(Q)$ is V-space.

Remark 1.1. It can be shown that if the relations on s,p,θ specified in the theorem are violated, then $B^s_{p,\theta}(Q)$ is not a Vekua space.

Remark 1.2. We will say that $\varphi(\cdot) \in B_{loc}$ in a neighborhood of the point z_0 if $\psi(z)\varphi(z) \in B(Q)$ for $\psi(z) \in C_0^\infty(Q), \psi(z) = 1$ in the neighborhood of z_0.

The theory constructed in [2] is also applicable in local Vekua spaces.

Corollary of Theorem 1.2. Let $\overset{\circ}{W}{}^s_p(Q)$ be completion of $C_0^\infty(Q)$ by the norm

$$\|(-\Delta)^{s/2}u\|_{L_p(R^n)} = \left(\int\limits_{R^n} |F^{-1}|\xi|^s F u|^p dx \right)^{1/p},$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2$, F is Fourier transform. It can be shown that $\overset{\circ}{W}{}^s_p(Q)$ is a Vekua space if and only if $s > \frac{2}{p} - 1$. In particular, for $s = 0$, the space $L_p(Q)$ is a Vekua space if and only if $p > 2$.

2 On one Vekua method for solving boundary value problems for polyharmonic equations

Now we turn to the study of similar problems for polyharmonic equations in an arbitrary multidimensional domain.

In the monograph by I.N.Vekua [1], the calculus theory of the simplest problems for polyharmonic equations is given, namely: it is required to solve the equation

$$\Delta^m u + a_1 \Delta^{m-1} u + a_2 \Delta^{m-2} u + \cdots + a_m u = 0 \tag{2.1}$$

under boundary conditions

$$u|_s = \varphi_0,\ \Delta u|_s = \varphi_1,\ \Delta^2 u|_s = \varphi_2,\ \ldots,\ \Delta^{m-1} u|_s = \varphi_{m-1}, \tag{2.2}$$

where $\varphi_0, \ldots, \varphi_{m-1}$ are given functions on the boundary $S = \partial\Omega$ of a bounded domain $\Omega \subseteq \mathbb{R}^n$, $a_i \in \mathbb{R}$, $i = \overline{1,m}$.

The solution of this problem (2.1), (2.2) is decomposed into the solution of m Dirichlet problems for equations of the form

$$\Delta v - k_i^2 v = 0,\ i = 1, 2, \ldots, m,$$

where k_i^2 are the roots of the characteristic equation

$$p^m + a_1 p^{m-1} + a_2 p^{m-2} + \cdots + a_m = 0.$$

Similarly, the boundary value problem

$$\partial_n u|_s = \psi_0,\ \partial_n \Delta u|_s = \psi_1,\ \partial_n \Delta^2 u|_s = \psi_2,\ \ldots,\ \partial_n \Delta^{m-1} u|_s = \psi_{m-1}$$

reduces to m Neumann problems for equations (2.1). Here, $\partial_n = \frac{\partial}{\partial n}$ is outward normal to boundary $S = \partial\Omega$.

For the case of two independent variables I.N. Vekua gave a general theory of linear boundary value problems based on the methods of the theory of analytic functions and on the theory of singular integral equations with Cauchy kernels. The main works in this area are the monographs of I.N. Vekua [1], N.I. Muskhelishvili [4].

Let us briefly outline the idea of the method proposed by I.N. Vekua. Let us assume that the problem of finding a function $u(x),\ x = (x_1, ..., x_n)$, that satisfies the equation

$$\Delta^m u = f(x),\ x \in \Omega \tag{2.3}$$

and homogeneous boundary conditions

$$R_1(u) = 0,\ ...,\ R_m(u) = 0, \tag{2.4}$$

in a simply connected domain Ω, admits a solution for any function $f(x) \in L_p(\Omega),\ p \geq 1$, and the solution of this problem (2.3), (2.4) is represented in the form

$$u(x) = L_0 f = \int\limits_{\Omega} G(x, y) f(y) dy,\ dy = dy_1 ... dy_n. \tag{2.5}$$

It is important to note that for the case when Ω is a multidimensional ball and $R_k = \partial^{k-1}/\partial n^{k-1}$ (k-th outer normal to the boundary surface), function G is constructed explicitly (see, for example, [5–8]).

Considering now the problem of finding a solution to the more general equation

$$F(x, u, Du, ..., D^{2m} u) = 0,\ D^p u = \partial^p u / \partial x_1^{k_1} ... \partial x_n^{k_n},\ \sum_{j=1}^{n} k_j = p$$

with the same boundary conditions

$$R_1(u) = 0,\ ...,\ R_m(u) = 0,$$

we can look for its solution in the form (2.5). This will lead us for $f(x)$ to the functional equation $F(x, L_0 f, ..., L_{2m} f) = 0$ with operators $L_k f = D^k L_0 f,\ k = 0, 1, ..., 2m$.

The operators $L_k f$ are linear and completely continuous for $k \leq 2m - 1$. As for the operators $L_{2m} f$, their boundedness in L_p, $p > 1$, is proved by using Zygmund-Calderon equality [9], which generalizes of the well-known Riesz inequality for the singular operator with a Cauchy type kernel. In this way, the problem with unbounded operators $D^k u$ is reduced to the equivalent problem of studying the functional equation $F(x, L_0 f, ..., L_{2m} f) = 0$ with bounded operators $L_k f$. Using the basic principles of functional analysis, it is possible to prove the solvability of this equation for a very wide range of problems for linear and quasilinear differential equations of elliptic type. It should be noted that this method allows the study of boundary value problems with minimal assumptions regarding the coefficients of the equation and the domain. In addition, by using the embedding theorems of S.L. Sobolev [10] and using formula (2.5), it is possible to prove almost extremely accurate theorems on the nature of the smoothness of the generalized solution depending on the degree of smoothness of the coefficients.

3 Description of correct boundary value problems for polyharmonic equations in a ball

Let m be a natural number and in an n-dimensional ball $\Omega = \{x : |x| < r\}$ consider *Dirichlet problem for a polyharmonic equation*

$$\Delta^m u(x) = f(x), \quad x \in \Omega, \tag{3.1}$$

$$\frac{\partial^j u(x)}{\partial n_x^j} = \varphi_j(x), \ \ 0 \le j \le m-1, \ \ x \in \partial\Omega. \tag{3.2}$$

The classical solution $u(x) \in C^{2m}(\Omega) \cap C^{m-1}(\overline{\Omega})$ of the Dirichlet problem (3.1), (3.2) exists, is unique, and it represented using the Green's function $G_{2m,n}(x,y)$ in the following form [10]

$$u(x) = \int_\Omega G_{2m,n}(x,y) f(y) dy + \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}(x,y) \cdot \Delta_y^{m-1-j} \varphi(y) - \right.$$

$$\left. - \Delta_y^j G_{2m,n}(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} \varphi(y) \right] dS_y,$$

where $\frac{\partial}{\partial n_y}$ is outer normal to boundary $\partial\Omega$.

The Green's function of the Dirichlet problem (3.1), (3.2) is defined as follows

$$\Delta^m G_{2m,n}(x,y) = \delta(x-y), \quad x, y \in \Omega, \tag{3.3}$$

$$\frac{\partial^j G_{2m,n}(x,y)}{\partial n_x^j} = 0, \ \ x \in \partial\Omega, \ \ y \in \Omega, \ \ 0 \le j \le m-1, \tag{3.4}$$

where $\delta(x-y)$ is the Dirac delta function.

In further studies we will use the following notation

$$X^2 = X^2(x,y) = |x-y|^2, \quad Y^2 = Y^2(x,y) = \left|\frac{y}{r}\right|^2 \left|x - \frac{y}{|y|^2} r^2\right|^2,$$

$$Z^2 = Z^2(x,y) = \left(1 - \left|\frac{y}{r}\right|^2\right)\left(1 - \left|\frac{x}{r}\right|^2\right) r^2.$$

Theorem 3.1. [5] a) In the case of odd n, as well as for even n, if $2m < n$ the Green's function of the Dirichlet problem (3.3), (3.4) can be represented as

$$G_{2m,n}(x,y) = d_{2m,n} \Big[X^{2m-n} - Y^{2m-n} -$$

$$- \sum_{k=1}^{m-1} \frac{(-1)^k}{k!} (m - \frac{n}{2}) ... \Big(m - \frac{n}{2} - k + 1\Big) Y^{2m-n-2k} Z^{2k} \Big], \tag{3.5}$$

and

$$d_{2m,n} = \frac{1}{(m-1)!\,(2m-n)(2(m-1)-n)...(4-n)(2-n)} \cdot \frac{\Gamma(n/2)}{2^m \pi^{n/2}},$$

where $\Gamma(\cdot)$ is gamma function;

b) In the case of even n and $2m \geq n$, the Green's function of the Dirichlet problem (3.3), (3.4) can be represented as

$$G_{2m,n}(x,y) = d_{2m,n}\Big[X^{2m-n}\ln X - Y^{2m-n}\ln Y -$$

$$-\sum_{\nu=1}^{m-n/2}(-1)^{\nu}C_{\nu}^{m-n/2}\Big[\ln Y + \sum_{\mu=m-n/2+1-\nu}^{m-n/2}\frac{1}{2\mu}\Big]Z^{2\nu}Y^{2m-2\nu-n}+$$

$$+(-1)^{m-n/2}\sum_{\nu=1}^{n/2-1}\frac{2^{2m+2\nu-n}}{2\nu C_{\nu+n/2}^{m+\nu}}Z^{2(m+\nu)}Y^{-2\nu-n}\Big], \tag{3.6}$$

and

$$d_{2m,n} = \frac{(-1)^{n/2-1}}{\Gamma(m)\Gamma(m-n/2+1)\cdot 2^{2m-1}\pi^{n/2}}.$$

In this case, $\Omega = \{x \in R^n : |x| < r\}$ or Ω is simply connected domains homeomorphic to the ball. Let us choose the domain of definition of the maximal operator $\widehat{L}$

$$D(\widehat{L}) = W_2^{2m}(\Omega).$$

On the domain $D(\widehat{L})$ we define the operator $\widehat{L}$ by the formula

$$\widehat{L}u = \Delta^m u, \ \forall u \in D(\widehat{L}).$$

Recall that the domain of the maximal operator

$$R(\widehat{L}) = L_2(\Omega),$$

and $Ker\widehat{L}$ its kernel is not trivial.

The Dirichlet boundary value problem for the polyharmonic equation

$$L_0 u := \begin{cases} \Delta_x^m u(x) = f(x), \quad x \in \Omega = \{x : |x| < r\}, \\ \dfrac{\partial^j u(x)}{\partial n_x^j} = 0, \ \ 0 \leq j \leq m-1, \ \ x \in \partial\Omega, \end{cases}$$

has a unique solution $u(x)$ for any $f \in L_2(\Omega)$, which has an integral representation

$$L_0^{-1} f = u(x) = \int_{\Omega} G_{2m,n}^D(x,y) f(y) dy, \tag{3.7}$$

where $G_{2m,n}^D(x,y) \equiv G_{2m,n}(x,y)$ is Green's function of the Dirichlet problem from (3.5) or (3.6).

Note that the zero Dirichlet boundary conditions for a polyharmonic equation are equivalent to the following boundary conditions for the same equation.

Theorem 3.2. a) For any $f \in L_2(\Omega)$, the function $u(x)$, given by formula (3.7) with $m = 2p$, is a solution to the boundary value problem:

$$\Delta_x^m u(x) = f(x), \quad x \in \Omega,$$

$$u(x)\,|_{\partial\Omega} = 0, \ \ \frac{\partial}{\partial n_x}u(x)\Big|_{\partial\Omega} = 0, \ \ \Delta_x u(x)\,|_{\partial\Omega} = 0, \ \ \frac{\partial}{\partial n_x}\Delta_x u(x)\Big|_{\partial\Omega} = 0,$$

$$\dots\dots\dots\ \Delta_x^{p-1}u(x)\,|_{\partial\Omega}=0,\ \ \frac{\partial}{\partial n_x}\Delta_x^{p-1}u(x)\bigg|_{\partial\Omega}=0.$$

b) For any $f\in L_2(\Omega)$, the function $u(x)$, given by formula (3.7) with $m=2p+1$, is a solution to the boundary value problem:

$$\Delta_x^m u(x)=f(x),\quad x\in\Omega,$$

$$u(x)\,|_{\partial\Omega}=0,\ \ \frac{\partial}{\partial n_x}u(x)\bigg|_{\partial\Omega}=0,\ \ \Delta_x u(x)\,|_{\partial\Omega}=0,\ \ \frac{\partial}{\partial n_x}\Delta_x u(x)\bigg|_{\partial\Omega}=0,$$

$$\dots\dots\dots\ \frac{\partial}{\partial n_x}\Delta_x^{p-1}u(x)\bigg|_{\partial\Omega}=0,\ \ \Delta_x^p u(x)\,|_{\partial\Omega}=0.$$

Based on the representation of the solution (3.7) of the Dirichlet problem, we present other well-posed boundary value problems for an inhomogeneous polyharmonic equation. To do this, we apply Otelbaev's theorem [5] to describe correct restrictions of the maximal operator $\widehat{L}$.

Now we can describe the domain of the maximal operator $\widehat{L}$ in terms of the Green's function $G_{2m,n}$.

Lemma 3.1. [5] The domain of the maximal operator $\widehat{L}$ has the representation

$$D(\widehat{L})=\{w:\ w(x)=\int_\Omega G_{2m,n}(x,y)f(y)dy+$$

$$+\sum_{j=0}^{m-1}\int_{\partial\Omega}\left[\frac{\partial\Delta_y^j G_{2m,n}(x,y)}{\partial n_y}\cdot\Delta_y^{m-1-j}h(y)-\Delta_y^j G_{2m,n}(x,y)\cdot\frac{\partial\Delta_y^{m-1-j}h(y)}{\partial n_y}\right]dS_y,$$

$$\forall f\in L_2(\Omega),\ \forall h\in W_2^{2m}(\Omega).$$

In particular, if

$$\Delta_y^{m-1-j}h(y)|_{y\in\partial\Omega}=0,\ \ \frac{\partial}{\partial n_y}\Delta_y^{m-1-j}h(y)|_{y\in\partial\Omega}=0,\ \ j=0,...,m-1,$$

then we obtain that the $D(L_0)$ is domain of the operator L_0.

Now the next question arises: how to describe the domains of other possible correct restrictions of the maximal operator $\widehat{L}$?

Let K be an operator putting each function $f(x)\in L_2(\Omega)$ into correspondence to a unique function $h(x)\in W_2^{2m}(\Omega)$, such that $\|Kf\|_{L_2(\Omega)}\le C\|f\|_{L_2(\Omega)}$.

Using the chosen operator K, we construct the set

$$D_K=\{\,w(x)\in D(\widehat{L}):\ h=Kf\}.$$

On the set D_K we define the operator

$$\widehat{L}\Big|_{D_K}=L_K.$$

From Otelbaev's theorem [5] it follows that L_K is a correct restriction of the maximal operator $\widehat{L}$. In conclusion, we give another description of the operator L_K in terms of boundary conditions.

Theorem 3.3. [5] Let K be an arbitrary continuous operator acting from $L_2(\Omega)$ to $D(\widehat{L})$. Then the inhomogeneous operator equation $L_K w=f$ is equivalent to the following boundary value problem

a) for $m=2p$

$$\Delta_x^m w(x)=f(x),\quad x\in\Omega,$$

$$w\,|_{\partial\Omega}=K(\Delta_x^m w)|_{\partial\Omega},\ \ \frac{\partial}{\partial n_x}w\bigg|_{\partial\Omega}=\frac{\partial}{\partial n_x}K(\Delta_x^m w)\bigg|_{\partial\Omega},\dots\dots\dots$$

$$\Delta_x^{p-1}w\,|_{\partial\Omega} = \Delta_x^{p-1}K(\Delta_x^m w)|_{\partial\Omega},\ \frac{\partial}{\partial n_x}\Delta_x^{p-1}w\Big|_{\partial\Omega} = \frac{\partial}{\partial n_x}\Delta_x^{p-1}K(\Delta_x^m w)\Big|_{\partial\Omega};$$

b) for $m = 2p+1$

$$w\,|_{\partial\Omega} = K(\Delta_x^m w)|_{\partial\Omega},\ \frac{\partial}{\partial n_x}w\Big|_{\partial\Omega} = \frac{\partial}{\partial n_x}K(\Delta_x^m w)\Big|_{\partial\Omega}, \dots\dots\dots$$

$$\frac{\partial}{\partial n_x}\Delta_x^{p-1}w\Big|_{\partial\Omega} = \frac{\partial}{\partial n_x}\Delta_x^{p-1}(K\Delta_x^m w)\Big|_{\partial\Omega},\ \Delta_x^p w\,|_{\partial\Omega} = \Delta_x^p(K\Delta_x^m w)|_{\partial\Omega}.$$

In [11,12] the Fredholm property and index of the generalized Neumann problem containing powers of normal derivatives in the boundary conditions are investigated. The problems of solvability of various boundary value problems for differential-operator equations are studied in the works [13–19]. Applications of the Green function to problems in mechanics and physics can be found, in [20–24].

4 Example. General presentation of solutions of boundary value problems for biharmonic equations

As an example, we consider the following biharmonic equation

$$\Delta^2 u = f, \quad z = x+iy \in \Omega, \tag{4.1}$$

where f is a given function. This equation is often encountered in the study of two-dimensional problems of linear elasticity theory. Let us construct regular solutions of equation (4.1) in the two-dimensional region Ω of the plane of the complex variable $z = x = iy$. To find a particular solution $u_1(x,y)$ of equation (4.1), we adopt the notation $v = \Delta u_1$. The function u_1 will be a solution of equation (4.1) if $v(x,y)$ is a solution of the Poisson equation $\Delta v = f$.

The solution to this equation is given by the formula [25]

$$v \equiv 4\frac{\partial^2 u_1}{\partial z\partial \overline{z}} = \frac{1}{2\pi}\int_\Omega f(t)\log|t-z|d\xi d\eta, \quad t = \xi + i\eta.$$

Using the following obvious equality

$$\frac{\partial^2}{\partial z\partial\overline{z}}[(t-z)(\overline{t}-\overline{z})\log(t-z)(\overline{t}-\overline{z})] - 2 = 2\log|t-z|,$$

equation (4.1) can be written as

$$\frac{\partial^2}{\partial z\partial\overline{z}}[u_1 - \frac{1}{8\pi}\int_\Omega f(t)|t-z|^2\log|t-z|d\xi d\eta] = -\frac{1}{8\pi}\int_\Omega f(t)d\xi d\eta \equiv C = const.$$

Therefore,

$$u_1 = \frac{1}{8\pi}\int_\Omega f(t)|t-z|^2\log|t-z|d\xi d\eta + \Phi(z,\overline{z}),$$

where

$$\Phi(z,\overline{z}) = Cz\overline{z} + \varphi_1(z) + \varphi_2(\overline{z}),$$

and φ_1 and φ_2 are arbitrary analytic functions of the variables z and $\overline{z}$, respectively. Since Φ is a biharmonic function, the function

$$u_1 = \frac{1}{8\pi}\int_\Omega f(t)|t-z|^2\log|t-z|d\xi d\eta \tag{4.2}$$

can be taken as one of the particular solutions of equation (4.1).

If $u(x, y)$ is the desired solution of equation (4.1), then the function $w = u - u_1$ will be biharmonic, i.e.

$$\Delta^2 w = 0. \tag{4.3}$$

According to formula (85) from [25] the solution of equation (4.3) can be written in the form

$$w = \overline{z}\varphi(z) + z\overline{\varphi}(\overline{z}) + \psi(z) + \overline{\psi}(\overline{z}). \tag{4.4}$$

In the representation (4.4) of real biharmonic functions, the imaginary parts of the functions $\overline{z}\varphi(z)$ $\psi(z)$ are not present. Therefore, without loss of generality, we can assume that the analytic functions $\varphi(z)$ and $\psi(z)$ included in formula (4.4) at some point z_1 of the domain Ω satisfy the conditions

$$\varphi(z_1) = 0, \mathrm{Im}\varphi'(z_1) = 0 \tag{4.5}$$

and

$$\mathrm{Im}\psi'(z_1) = 0. \tag{4.6}$$

Thus we have proved the following theorem.

Theorem 4.1. a) To each pair of analytic functions $\varphi(z), \psi(z)$ formula (4.4) associates a well-defined biharmonic function $w(x, y)$. The converse statement is also true.

b) For each biharmonic function $w(x, y)$ there is a well-defined pair of analytic functions $\varphi(z)$, $\psi(z)$ satisfying conditions (4.5), (4.6) and $w(x, y)$ is represented by formula (4.4).

From this theorem we conclude that formula (4.4) gives a general representation of real biharmonic functions. Further, in view of the fact that

$$u = w + u_1,$$

on the basis of formulas (4.2) and (4.4) we arrive at the general complex representation of real solutions of equations (4.1):

$$u(x, y) = \frac{1}{8\pi} \int_\Omega f(t)|t - z|^2 \log|t - z| d\xi d\eta + \overline{z}\varphi(z) + z\overline{\varphi}(\overline{z}) + \psi(z) + \overline{\psi}(\overline{z}), \tag{4.7}$$

where $\varphi(z)$ and $\psi(z)$ are arbitrary analytic functions of a complex variable z satisfying conditions (4.5), (4.6). Formula (4.7) allows any linear problem for equation (4.1) to be reduced to the corresponding problem for biharmonic functions.

Conclusion

The studies carried out in this article are of significant importance in the theory of boundary value problems of linear and nonlinear partial differential equations, spectral theory, and the theory of numerical methods for approximate solutions of individual classes of boundary value problems for differential equations.

Thus, the object of our research was polyharmonic equations. Vekua's method is applicable for fixed boundary conditions on the surface of the domain, and the equation itself can change, i.e. minor terms can be added to the main equation. Otelbaev's method is applied for fixed equations, and the boundary conditions can change.

The problem is to determine the conditions under which both methods are applicable. This article is devoted to this problem and the applicability of the Vekua and Otelbaev methods. As an example, a biharmonic equation is given, which has an applied character in the theory of elasticity. A general complex representation of real solutions of the biharmonic equation is given in the form of formula (4.7).

Acknowledgments

This work was done with support of grant AP 19678182 of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

Author Contributions

N.M. Shynybayeva collected and analyzed data, and led manuscript preparation. P.Zh. Kozhobekova and M.T. Sabirzhanov assisted in data collection and analysis. B.D. Koshanov served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Vekua, I.N. (2014). *Generalized analytic functions.* Elsevier.

2 Otelbayev, M.O. (1979). K teorii obobshchennykh analiticheskikh funktsii Vekua [On the theory of generalized analytical functions of Vekua]. *Primenenenie metodov funktsionalnogo analiza i vychislitelnoi matematiki. Materialy shkoly-seminara – Application of methods of functional analysis and computational mathematics. School-seminar materials.* Novosibirsk, 80–99 [in Russian].

3 Kal'menov, T.S., Otelbaev, M., & Arepova, G.D. (2020). Bitsadze-Samarskii Boundary Conditions for an Elliptic-Parabolic Volume Potential with Smooth Matching. *Differential Equations, 56*(6), 740–755. https://doi.org/10.1134/S0012266120060075

4 Muskhelishvili, N.I., & Radok, J.R.M. (2008). *Singular integral equations: boundary problems of function theory and their application to mathematical physics.* Courier Corporation.

5 Koshanov, B.D. (2021). Green's functions and correct restrictions of the polyharmonic operator. *Journal of Mathematics, Mechanics and Computer Science, 109*(1), 34–54. https://doi.org/10.26577/JMMCS.2021.v109.i1.03

6 Koshanov, B.D. (2013). About solvability of boundary value problems for the nonhomogeneous polyharmonic equation in a ball. *Journal APAM, Advances in Pure and Applied Mathematics, 4*(4), 351–373. https://doi.org/10.1515/apam-2013-0022

7 Karachik, V. (2021). Dirichlet and Neumann boundary value problems for the polyharmonic equation in the unit ball. *Mathematics, 9*(16), 1907. https://doi.org/10.3390/math9161907

8 Karachik, V. (2023). Representation of the Green's Function of the Dirichlet Problem for the Polyharmonic Equation in the Ball. *Differential Equations, 59*, 1061–1074. https://doi.org/10.1134/S0012266123080050

9 Calderon, A.P., & Zygmund, A. (1952). On the existence of certain singular integrals. *Acta Math, 88*, 85–139. https://doi.org/10.1007/BF02392130

10 Sobolev, S.L. (1974). *Vvedenie v teoriiu kubaturnykh formul [Introduction to the theory of cubature formulas].* Moscow: Nauka [in Russian].

11 Koshanov, B.D., & Soldatov, A.P. (2018). On the Solvability of the Boundary Value Problems for the Elliptic Equation of High Order on a Plane. *Bulletin of the Karaganda University. Mathematics Series,* 3(91), 24–31. https://doi.org/10.31489/2018M3/24-30

12 Koshanov, B., & Soldatov, A. (2022). On Fredholm solvability and on the index of the generalized Neumann problem for an elliptic equation. *Complex Variables and Elliptic Equations, 12*(67), 2907–2923. https://doi.org/10.1080/17476933.2021.1958797

13 Kozhanov, A.I., Koshanov, B.D., Sultangazieva, Zh.B., Emir Kadyoglu, A.N., & Smatova, G.D. (2020). The spectral problem for nonclassical differential equations of the sixth order. *Bulletin of the Karaganda University. Mathematics Series,* 1(97), 79–86. https://doi.org/10.31489/2020M1/79-86

14 Koshanov, B.D., Kakharman, N., Segizbaeva, R.U., & Sultangazieva, Zh.B. (2022). Two theorems on estimates for solutions of one class of nonlinear equations in a finite-dimensional space. *Bulletin of the Karaganda University. Mathematics Series,* 3(107), 70–84. https://doi.org/10.31489/2022m3/70-84

15 Kanguzhin, B.E., & Koshanov, B.D. (2022). Uniqueness Criteria for Solving a Time Nonlocal Problem for a High-Order Differential Operator Equation $l(\cdot) - A$ with a Wave Operator with Displacement. *Simmetry, 14*(6), 1239. https://doi.org/10.3390/sym14061239

16 Kanguzhin, B.E., & Koshanov, B.D. (2023). Solution Uniqueness Criteria in a Time-Nonlocal Problem for the Operator Differential Equation with the Tricomi Operator. *Differential Equations, 59*(1), 1–12. https://doi.org/10.1134/S0012266123010019

17 Otelbaev, M., & Soldatov, A.P. (2021). Integral representations of vector functions based on the parametrix of first-order elliptic systems. *Computational Mathematics and Mathematical Physics, 61*, 964–973. https://doi.org/10.1134/S0965542521030143

18 Soldatov, A.P. (2022). On a Boundary Problem for a Fourth-Order Elliptic Equation on a Plane. *Computational Mathematics and Mathematical Physics, 62*, 599–607. https://doi.org/10.1134/S096554252204011X

19 Yuldashev, T.K., & Eshkuvatov, Y.F. (2022). On a Fredholm type integro-differential equations of fifth order with degenerate kernel. *Journal of Contemporary Applied Mathematics, 12*(1), 27–34.

20 Chung, F., & Zeng, J. (2023). Forest formulas of discrete Green's functions. *Journal of Graph Theory, 102*, 556–577. https://doi.org/10.1002/jgt.22887

21 Herrera, W.J., Vinck-Posada, H., & Paez, S.G. (2022). Green's functions in quantum mechanics courses. *American Journal of Physics, 90*, 763–769. https://doi.org/10.1119/5.0065733

22 Harkonen, V.J. (2022). Exact factorization of the many-body Green's function theory of electrons and nuclei. *Physical Review B, 106.* https://doi.org/10.1103/PhysRevB.106.205137

23 Dong, H., & Li, H. (2019). Optimal Estimates for the Conductivity Problem by Green's Function Method. *Archive for Rational Mechanics and Analysis, 231*, 1427–1453. https://doi.org/10.1007/s00205-018-1301-x

24 Grebenkov, D.S., & Traytak, S.D. (2019). Semi-analytical computation of Laplacian Green functions in three-dimensional domains with disconnected spherical boundaries. *Journal of Computational Physics, 379*, 91–117. https://doi.org/10.1016/j.jcp.2018.10.033

25 Bitsadze, A.V. (1984). *Osnovy teorii analiticheskikh funktsii komplesksnogo peremennogo [Fundamentals of the theory of complex variable analytic functions].* Moscow: Nauka [in Russian].

Author Information[*]

Muzaffar Takhirovich Sabirzhanov — Doctor's student, Osh State University, Osh, Kyrgyzstan; e-mail: *smskg@bk.ru*; https://orcid.org/0009-0008-2431-7033

[*]The author's name is presented in the order: First, Middle and Last Names.

Bakytbek Danebekovich Koshanov (*corresponding author*) — Doctor of physical and mathematical sciences, Professor, Chief researcher at Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan; e-mail: *koshanov@math.kz*; https://orcid.org/0000-0002-0784-5183

Nazym Myrzabekkyzy Shynybayeva — Junior Researcher, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan; e-mail: *shynybayeva001@mail.ru*

Pardaz Zhumabaevna Kozhobekova — Doctor's student, Osh State University, Osh, Kyrgyzstan; e-mail: *pardaz@mail.ru*; https://orcid.org/0009-0002-3143-3442

https://doi.org/10.31489/2025M1/155-169 *Research article*

Solving Volterra-Fredholm integral equations by non-polynomial spline function based on weighted residual methods

S.H. Salim[1], R.K. Saeed[2], K.H.F. Jwamer[3,*]

[1] *College of Basic Education, University of Sulaimani, Sulaymaniyah, Iraq;*
[2] *College of Science, Salahaddin University-Erbil, Erbil, Kurdistan Region, Iraq;*
[3] *College of Science, University of Sulaimani, Sulaymaniyah, Iraq*
(E-mail: sarfraz.salim@univsul.edu.iq, rostam.saeed@su.edu.krd, karwan.jwamer@univsul.edu.iq)

In this paper, a method that utilizes a non-polynomial spline function based on the weighted residual technique to approximate solutions for linear Volterra-Fredholm integral equations is presented. The approach begins with the selection of a series of knots along the integration interval. We then create a set of basis functions, defined as non-polynomial spline functions, between each pair of adjacent knots. The unknown function is expressed as a linear combination of these basis functions to approximate the solution of integral equations. The coefficients of the spline function are calculated by solving a system of linear equations derived from substituting the spline approximation into the integral equation while maintaining continuity and smoothness at the knots. Non-polynomial splines are beneficial for approximating functions with complex shapes and for solving integral equations with non-smooth kernels. However, the solution's accuracy significantly relies on the selection of knots, and the method may require extensive computational resources for large systems. To illustrate the effectiveness of the method, three examples are presented, implemented using Python version 3.9. The paper also addresses the error analysis theorem relevant to the proposed non-polynomial spline function.

Keywords: Volterra integral equation, Fredholm integral equation, non-polynomial spline, weighted residual methods.

2020 Mathematics Subject Classification: 45B05, 45B05, 41A15.

Introduction

Volterra-Fredholm integral equations play a crucial role in mathematical modeling in numerous scientific and engineering disciplines, such as physics, biology, and finance. These equations, which involve both Volterra and Fredholm integral operators, often arise in the analysis of intricate systems where time-dependent and spatially distributed processes are interrelated. However, despite their significance, solving Volterra-Fredholm integral equations presents challenges due to their complexity and the presence of mixed integral terms. For additional information, refer to [1–5].

The literature presents Volterra-Fredholm integral equations in the following form:

$$u(x) = f(x) + \lambda_1 \int_a^x K(x,t)u(t)dt + \lambda_2 \int_a^b L(x,t)u(t)dt, \tag{1}$$

where the functions $f(x)$, and the kernels $K(x,t)$, $L(x,t)$ are known L^2 analytic functions and λ_1, λ_2, are arbitrary constants, x is variable and $u(x)$ is the unknown continuous function to be determined. These integral equations allow physicists to formulate and solve problems where traditional differential equations may not be applicable or are too complex to solve directly. Volterra-Fredholm integral equations are used to describe the time evolution of quantum systems. For example, they can model

*Corresponding author. *E-mail: karwan.jwamer@univsul.edu.iq*
Received: 15 July 2024; *Accepted:* 25 November 2024.

scattering processes and interactions between particles. In the study of electromagnetic fields, these equations can help solve problems related to wave propagation, radiation, and diffraction, especially in complex media. They are employed in the analysis of many-body systems, where the interactions between particles can be described by integral equations. Also, they are used in modelling fluid flow, especially in non-linear and time-dependent situations, in systems with memory effects. These equations can be used to model the behaviour of materials that exhibit viscous and elastic characteristics, which is crucial in material science and engineering applications.

Traditional numerical methods have been developed and used for solving Volterra-Fredholm integral equation, including the use of Touchard Polynomials, spline function, rational interpolations, etc. [3, 6–19]. In addition, non-polynomial spline functions are used to solve integral equations and differential equations [20–31]. Recently, Salim et al. [32–35] used linear, quadratic, and cubic spline functions to solve equation (1).

In this study, the authors combine non-polynomial spline functions with weighted residual methods, which minimize the residuals of the integral equations in a weighted manner, to develop a robust and efficient technique for solving the equation (1).

The structure of this paper is outlined as follows: Section 1 provides an overview of the weighted residual method and its various types. Section 2 describes our methodology in detail. Section 3 focuses on error analysis. Section 4 presents several numerical examples to illustrate the effectiveness of our technique. Finally, some tentative conclusions are given.

1 *Weighted Residual Methods*

The weighted residual methods [2,36–38] is a mathematical technique commonly used in numerical analysis and finite element analysis to solve partial differential equations (PDEs). The idea behind the method is to represent the solution of the PDE as a linear combination of a set of basis functions, and then to find the coefficients of the basis functions by minimizing the residual error. The residual error is defined as the difference between the exact solution of the given problem and the approximate solution obtained using the basis functions. The minimization is performed using a set of weighting functions, which give greater weight to certain regions of the domain where the solution is expected to be more important. The weighted residual method can be used to solve a wide range of PDEs, including elliptic, parabolic, and hyperbolic equations, different types of ordinary differential equations and integral equations. It is a very flexible method that can handle boundary conditions. We present these methods by considering the following residues $E(x)$ or $E(\overline{C_j};x)$ depends on x as well as on the parameters $a_j,\ b_j,\ c_j,\ d_j,\ j = 0,\ 1,\ \ldots,\ n-1$. We define $E(x)$ as follows:

$$E(\overline{C_j};x) = \overline{u_n}(x) - f(x) - \lambda_1 \int_a^x K(x,t)\overline{u_n}(t)dt - \lambda_2 \int_a^b L(x,t)\overline{u_n}(t)dt, \quad x \in D = [a,b]\,, \tag{2}$$

for solving (1), where

$$\overline{u_n}\,(x) = \sum_{j=0}^{n-1} \overline{\alpha}_{ij}\phi_j(x),$$

for $i = 0,\ 1,\ 2,\ \ldots,\ n-1$ and D is a prescribed domain. It is obvious that when $E(x) = 0$, then the exact solution is obtained which is difficult to be achieved, therefore we shall try to minimize $E(x)$ is some sense. In the weighted residual method the unknown parameters are chosen to minimize the residual $E(x)$ setting its weighted integral equal to zero, i.e.

$$\int_D w_j E\,(x)\,dx = 0, \quad j = 0,1,2,\ldots,n-1, \tag{3}$$

where w_j is prescribed weighting function, the technique based on equation (3) is called weighted residual method. Different choices of w_j yield different methods with different approximate solutions. Below we discuss some of the weighted residual method.

1.1 Collocation Method (CM)

It is a simple technique for obtaining an approximate solution of equation (1), the weight function w_j in equation (3) are defined as

$$w_j = \delta\left(x - x_j\right), \tag{4}$$

where the fixed points $x_j \in D$, $j = 0, 1, \ldots, n-1$ are called collocation points. Here Dirac's delta function $\delta(x - x_j)$ is defined as

$$\delta\left(x - x_j\right) = \begin{cases} 1, & \text{if} \quad x_j = x, \\ 0, & \text{else.} \end{cases}$$

Inserting equation (4) in equation (3) gives

$$\int_D w_j E\left(x_j\right) dx = \int_D \delta\left(x - x_j\right) E\left(x_j\right) dx = \int_{x_j^-}^{x_j^+} \delta\left(x - x_j\right) E\left(x_j\right) dx =$$

$$= E\left(x_j\right) \int_{x_j^-}^{x_j^+} \delta\left(x - x_j\right) dx = E\left(x_j\right) = 0, \quad \text{for} \quad j = 0, 1, \ldots, n-1. \tag{5}$$

Equation (5) will provide us with n simultaneous equations in n unknowns to determine the parameters. Moreover, the distribution of the collocation points on D is arbitrary, however, in practice we distribute the collocation points uniformly on D.

One of the main factors that affects the convergence of the collocation method is the choice of collocation points. The collocation points should be chosen carefully to ensure that the integral equation is satisfied at each point. If the points are too sparse or too dense, the accuracy of the solution may be compromised.

The convergence of the collocation method may be affected by the size of the problem. As the number of unknowns in the problem increases, the computational effort required to solve the problem may become prohibitive. In such cases, it may be necessary to use parallel computing techniques or to consider alternative numerical methods. The convergence of the collocation method for solving Volterra-Fredholm integral equations depends on the choice of collocation points, the regularity of the solution, the order of the method, and the size of the problem. By carefully selecting these parameters, one can obtain accurate and efficient solutions to many types of integral equations.

1.2 Subdomain (Partition) Method (PM)

In this method the domain D is divided into $n+1$ non-overlapping subdomains D_j, $j = 0, 1, 2, \ldots, n$, with the weighting functions are taken as

$$\mathrm{w}_{\mathbf{j}} = \begin{cases} 1, & x \in D_j, \\ 0, & x \notin D_j, \end{cases} \qquad j = 0,\ 1, \ldots, n.$$

Hence equation (2) is satisfied in each of $(n+1)$ subdomain D_j, therefore equation (3) becomes

$$\int_{D_j} E\left(x\right) dx = 0, \qquad j = 0,\ 1, \ldots, n. \tag{6}$$

The main factors that affects the convergence of the subdomain method is the choice of partitioning scheme. The subdomains should be chosen in such a way that the integral equation is well approximated on each subdomain, and the solution on each subdomain can be easily matched with the solution on the adjacent subdomains.

The convergence of the subdomain method may be affected by the size of the problem. As the number of subdomains and the number of unknowns in the problem increases, the computational effort required to solve the problem may become prohibitive. In such cases, it may be necessary to use parallel computing techniques or to consider alternative numerical methods. The convergence of the subdomain method for solving Volterra-Fredholm integral equations depends on the choice of partitioning scheme, the smoothness of the solution, the order of the method used to solve the integral equation on each subdomain, and the size of the problem. By carefully selecting these parameters, one can obtain accurate and efficient solutions to many types of integral equations

1.3 Galerkin's Method (GM)

Galerkin method is the most important of the weighted residual method. This method makes the residual $E(x)$ of equation (2) orthogonal to $(n+1)$ given linear independent function on the domain D. In this method the weighting functions w_j are chosen to be

$$w_j\left(x\right)=\frac{\partial S_j(x)}{\partial \beta_j},\qquad j=0,1,\ldots,n,$$

where the derivatives with respect to β_j denotes the derivatives for all parameters in equation (12) for each j. Then equation (3) becomes

$$\int_D \frac{\partial S_j\left(x\right)}{\partial \beta_j}E\left(x\right)dx=0,\qquad j=0,1,\ldots,n. \tag{7}$$

Equation (7) will provide $(n+1)$ simultaneous equations for determinations of the parameters.

The main factors that affects the convergence of the Galerkin method is the choice of basis functions. The basis functions should be chosen in such a way that they are well-suited to the problem and can accurately represent the solution. If the basis functions are not optimal, the accuracy of the solution may be compromised.

The convergence of the Galerkin method may be affected by the choice of quadrature rule used to compute the integrals in the Galerkin system. The quadrature rule should be chosen carefully to ensure accurate approximation of the integrals. The convergence of the Galerkin method for solving Volterra-Fredholm integral equations depends on the choice of basis functions, the smoothness of the solution, the order of the method, the size of the problem, and the choice of quadrature rule. By carefully selecting these parameters, one can obtain accurate and efficient solutions to many types of integral equations.

1.4 Least Square Method (LM)

In this method the weighting function w_j is defined as

$$w_j=\frac{\partial E(x)}{\partial \beta_j},\qquad j=0,1,\ldots,n, \tag{8}$$

where $E(x)$ is given by equation (2). In this method, we take the square of the error on the domain D as follows:

$$J=\int_D \left[E\left(x\right)\right]^2dx.$$

Now, we compute the derivatives with respect to β_j, yields:

$$\frac{\partial J}{\partial \beta_j} = 2\int_D E\left(x\right)\frac{\partial E\left(x\right)}{\partial \beta_j}dx, \qquad j = 0, 1, \ldots, n. \tag{9}$$

It implies from equation (8) and equation (9) that

$$\frac{\partial J}{\partial \beta_j} = \int_D E\left(x\right)\frac{\partial E\left(x\right)}{\partial \beta_j}dx = 0, \qquad j = 0, 1, \ldots, n. \tag{10}$$

Therefore, J is stationary and the square of residual $E(x)$ attains its minimum.

The main factors that affect the convergence of the least-squares method is the choice of basis functions. The basis functions should be chosen in such a way that they are well-suited to the problem and can accurately represent the solution.

Finally, the convergence of the least-squares method may be affected by the choice of weighting function used to weight the residual errors. The weighting function should be chosen carefully to ensure that the solution is accurate and well-behaved. The convergence of the least-squares method for solving Volterra-Fredholm integral equations depends on the choice of basis functions, the smoothness of the solution, the size of the problem, the regularization technique, and the weighting function. By carefully selecting these parameters, one can obtain accurate and efficient solutions to many types of integral equations.

2 Description of the Method

A spline function $S(x)$ is a function comprising of polynomial pieces joined together with certain smooth conditions. We need to express $S(x)$ as follows:

$$S(x) = \begin{cases} S_0(x), & x \in [x_0, x_1], \\ S_1(x), & x \in [x_1, x_2], \\ \vdots & \\ S_{n-1}(x), & x \in [x_{n-1}, x_n]. \end{cases} \tag{11}$$

In this paper, we use the following non-polynomial spline function

$$S_j\left(x\right) = a_j \sin\left(x - x_j\right) + b_j \cos\left(x - x_j\right) + c_j\left(x - x_j\right) + d_j, \quad j = 0,\ 1,\ \ldots,\ n-1. \tag{12}$$

Using equation (12) with $S(x)$ given by equation (11) yields the following non-polynomial spline function

$$S(x) = \begin{cases} S_0(x) = a_0 \sin\left(x - x_0\right) + b_0 \cos\left(x - x_0\right) + c_0\left(x - x_0\right) + d_0, & x_0 \le x \le x_1, \\ S_1(x) = a_1 \sin\left(x - x_1\right) + b_1 \cos\left(x - x_1\right) + c_1\left(x - x_1\right) + d_1, & x_1 \le x \le x_2, \\ \vdots & \vdots \\ S_{n-1}(x) = a_{n-1} \sin\left(x - x_{n-1}\right) + b_{n-1} \cos\left(x - x_{n-1}\right) + c_{n-1}\left(x - x_{n-1}\right) + d_{n-1}, & x_{n-1} \le x \le x_n. \end{cases} \tag{13}$$

To solve equation (1) by non-polynomial spline function based on weighted residual method (13), using equations (2) and (3) for $x \in D = [x_j, x_{j+1}]$, we obtain:

$$\begin{aligned} E(\overline{C_j};x) &= S_j(x) - f(x) - \lambda_1 \int_a^x K(x,t)S_j(t)dt - \lambda_2 \int_a^b L(x,t)S_j(t)dt \\ &= a_j \sin(x-x_j) + b_j \cos(x-x_j) + c_j(x-x_j) + d_j - f(x) \\ &- \lambda_1 \int_a^x K(x,t)[a_j \sin(t-x_j) + b_j \cos(t-x_j) + c_j(t-x_j) + d_j]dt \\ &- \lambda_2 \int_a^x L(x,t)[a_j \sin(t-x_j) + b_j \cos(t-x_j) + c_j(t-x_j) + d_j]dt, \end{aligned} \tag{14}$$

where $t_j = x_j = x_0 + jh, \quad h = \dfrac{b-a}{n}, \quad j = 0, 1, \ldots, n.$

To find a_j, b_j, c_j and d_j, we use the four above methods.

3 Error Analysis

In this section the error analysis theorem for the proposed non-polynomial spline function is proved, where $u(x)$ is a sufficiently smooth function in $[a,b]$, and $S_j(x)$ is the non-polynomial spline given by equation (12), that interpolate $u(x)$ at n nodes x_j, $j = 0, 1, \ldots, n-1$ in $[a,b]$, such that $h = \dfrac{b-a}{n}$, $x_0 = a$, $x_j = x_0 + jh$ for $j = 0, 1, \ldots, n-1$.

Theorem 1. (Fundamental Theorem of Error Interpolation) [39]

Let f be a polynomial in $C^{n+1}[a,b]$, and let p be a polynomial of degree $\leq n$ that interpolate the function f at $n+1$ distinct points $x_0, x_1, \cdots, x_n \in [a,b]$. Then for each $x \in [a,b]$ there exists a point $c \in (a,b)$ such that

$$E_n(x) = f(x) - p_n(x) = \frac{1}{(n+1)!} f^{n+1}(c) \prod_{i=0}^{n} (x - x_i).$$

Theorem 2. Let

$$u_n(x) = \sum_{j=0}^{n-1} \alpha_j \varphi_j(x)$$

be the expansion of the exact solution $u(x)$ of equation (1). Also, let

$$S_i(x) \approx \overline{u_n}(x) = \sum_{j=0}^{n-1} \overline{\alpha}_{ij} \phi_j(x),$$

for $i = 0, 1, ..., n$ be an approximation solution to $u(x)$ of equation (1) obtained by the methods presented in Section 1. Then, there exist real numbers β_i and γ_i such that

$$\|u(x) - \overline{u_n}(x)\|_2 \leq \beta_i \frac{M_n(ih)^n}{n!} + \gamma_i \|\overline{C}_i - C\|_2, \tag{15}$$

where $\overline{C}_i = \left[\overline{\alpha}_{i0}, \overline{\alpha}_{i1}, ..., \overline{\alpha}_{i(n-1)}\right]$, $C = [\alpha_0, \alpha_1, ..., \alpha_{n-1}]$ and the norm is the Euclidean norm of vectors.

Proof. It is clear that

$$\|u(x) - \overline{u}_n(x)\|_2 \leq \|u(x) - u_n(x)\|_2 + \|u_n(x) - \overline{u}_n(x)\|_2. \tag{16}$$

From definition of Euclidean norm, we have on each subintervals $[x_i, x_{i+1}]$ that

$$\begin{aligned}\|u(x)-u_n(x)\|_2 &= \sqrt{\int_{x_i}^{x_{i+1}} |u(x)-u_n(x)|^2 dx} \\ &\le \sqrt{\int_{x_i}^{x_{i+1}} \left(\frac{M_n(ih)^n}{n!}\right)^2 dx} \quad \text{by} \quad \text{Theorem 1} \\ &= \sqrt{ih}\frac{M_n(ih)^n}{n!}. \end{aligned} \tag{17}$$

Also, we have

$$\begin{aligned}\|u_n(x)-\overline{u}_n(x)\|_2 &= \|\sum_{j=0}^{n-1}\alpha_j\varphi_j(x) - \sum_{j=0}^{n-1}\overline{\alpha}_{ij}\phi_j(x)\|_2 \\ &= \|\sum_{j=0}^{n-1}\left(\alpha_j\varphi_j(x)-\overline{\alpha}_{ij}\phi_j(x)\right)\|_2 \\ &\le \sqrt{\int_{x_i}^{x_{i+1}}\left[\sum_{j=0}^{n-1}\left(\alpha_j\varphi_j(t)-\overline{\alpha}_{ij}\phi_j(t)\right)\right]^2 dx} \\ &\le \sqrt{\int_{x_i}^{x_{i+1}}\sum_{j=0}^{n-1}|\alpha_j-\overline{\alpha}_{ij}|^2\sum_{j=0}^{n-1}|\varphi(x)_j-\phi_j(x)|^2 dx} \\ &\le \sqrt{\sum_{j=0}^{n-1}|\alpha_j-\overline{\alpha}_{ij}|^2}\sqrt{\int_{x_i}^{x_{i+1}}\sum_{j=0}^{n-1}|\varphi_j(x)-\phi_j(x)|^2 dx} \\ &\le 2M\left\|C-\overline{C}_i\right\|_2\sqrt{ihn}. \end{aligned} \tag{18}$$

Finally, substitute equation (17) and equation (18) in equation (16), we see that equation (15) is valid with $\alpha_i = \sqrt{ih}$ and $\gamma_i = 2M\sqrt{ihn}$.

4 Numerical Examples

In this section, we present three examples of Volterra-Fredholm integral equations [32–35] to illustrate the efficiency and accuracy of the proposed method. The computed errors e_i are defined by $e_i = |u_i - S_i|$, where u_i is the exact solution of equation (1) and S_i is an approximate solution of the same equation. Also we compute Least square error (LSE), which is defined by formula $\sum_{i=0}^{n}(u_i - S_i)^2$ and all computations are performed using the Python program.

Example 1. Consider the Volterra-Fredholm integral equation

$$u(x) = -\frac{x^2}{2} - \frac{7x}{2} + 2 + \int_0^x u(t)dt + \int_0^1 xu(t)dt$$

with the exact solution given by $u(x) = x + 2$.

Collocation method.

Form equation (14) for $j = 0$, we have

$$\begin{aligned}E\left(\overline{C}_0; x\right) &= \left(\frac{1}{2}-\frac{c_0}{2}\right)x^2 + \left(\frac{c_0}{2} - a_0 - 2d_0 + a_0\cos(1) - b_0\sin(1) + \frac{7}{2}\right)x + d_0 \\ &+ b_0\cos(x) + a_0\sin(x) - b_0\sin(x) + a_0(\cos(x)-1) - 2, \end{aligned} \tag{19}$$

and from equation (13) for $j = 0$, we have:

$$S_0(x) = a_0 \sin(x - x_0) + b_0 \cos(x - x_0) + c_0(x - x_0) + d_0. \tag{20}$$

For finding a_0, b_0, c_0 and d_0 in equation (20), we need four equations. To construct this four equations, the interval $[x_0, x_1]$ divided as follows:

$$s_l = x_0 + lh, \quad \text{where} \quad h = \frac{x_i - x_0}{3} \quad \text{and} \quad l = 0, 1, 2, 3.$$

Substituting this values of $s_0, \ldots, s_3$ in place of x in equation (19) when $j = 0$, we get the following equations:

$$\begin{aligned}
E\left(\overline{C}_0; 0\right) &= 0 \Longrightarrow b_0 + d_0 = 2,\\
E\left(\overline{C}_0; 0.0333\right) &= 0 \Longrightarrow \frac{29}{1800}c_0 - \frac{1}{30}a_0 + \frac{14}{15}d_0 + \frac{\cos(1)}{30}a_0 + \frac{\sin(1)}{30}b_0 + \frac{\sin(1)}{30}a_0\\
&\quad - \frac{\sin(1)}{30}b_0 - \frac{\sin(1)}{30}b_0 + (\frac{\cos(1)}{30} - 1)a_0 = \frac{3389}{1800},\\
E\left(\overline{C}_0; 0.0666\right) &= 0 \Longrightarrow \frac{7}{25}c_0 - \frac{1}{15}a_0 + \frac{13}{15}d_0 + \frac{\cos(1)}{15}a_0 + \frac{\sin(1)}{30}b_0 + \frac{\sin(1)}{15}a_0\\
&\quad - \frac{\sin(1)}{15}b_0 - \frac{\sin(1)}{15}b_0 + \left(\frac{\cos(1)}{15} - 1\right)a_0 = \frac{397}{225},\\
E\left(\overline{C}_0; 0.1\right) &= 0 \Longrightarrow \frac{9}{200}c_0 - \frac{1}{10}a_0 + \frac{4}{5}d_0 + \frac{\cos(1)}{10}a_0 + \frac{\sin(1)}{10}b_0 + \frac{\sin(1)}{10}a_0\\
&\quad - \frac{\sin(1)}{10}b_0 - \frac{\sin(1)}{10}b_0 + \left(\frac{\cos(1)}{10} - 1\right)a_0 = \frac{329}{200}.
\end{aligned}$$

Solving the above linear system, we get

$$\overline{C_0} = \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Hence

$$S_0\left(x\right) = 0\sin\left(x - x_0\right) \; + 0\cos\left(x - x_0\right) + \left(x - x_0\right) + 2 = x + 2.$$

In a similar manner, we get

$$S_s\left(x\right) = \left(x - x_s\right) + 2.s,$$

for s =1, 2, ..., 9.

Subdomain method.

For finding a_0, b_0, c_0 and d_0 in equation (20), we need Four equations. To construct this four equations, we divide the interval $[x_0, x_1]$ as follows:

$$s_l = x_0 + lh, \quad \text{where} \quad h = \frac{x_1 - x_0}{4} \quad \text{and} \quad l = 0, 1, 2, 3, 4.$$

Using equation (6) with $j = 0$, the following equations obtained:

$$\int_{s_0}^{s_1} E(\overline{C}_0; x)dx = \int_0^{0.025} E(\overline{C}_0; x)dx = 0,$$

$$\int_{s_1}^{s_2} E(\overline{C}_0;x)dx = \int_{0.025}^{0.05} E(\overline{C}_0;x)dx = 0,$$
$$\int_{s_2}^{s_3} E(\overline{C}_0;x)dx = \int_{0.05}^{0.075} E(\overline{C}_0;x)dx = 0,$$
$$\int_{s_3}^{s_4} E(\overline{C}_0;x)dx = \int_{0.075}^{0.1} E(\overline{C}_0;x)dx = 0.$$

Solving the above four equations, we get

$$\overline{C_0} = \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Hence

$$\begin{aligned} S_0(x) &= 0\sin(x-x_0) \ +0\cos(x-x_0) + (x-x_0) + 2 \\ &= (x-x_0) + 2. \end{aligned}$$

In a similar manner, we get

$$S_s(x) = (x - x_s) \ + 2.s,$$

for s =1, 2, ..., 9.

Galerkin method.

To find a_0, b_0, c_0 and d_0 in equation (20). First, we have to find weighted functions $w_j(x) = \frac{\partial S_0(x)}{\partial B_j}$, $j = 0, 1, 2, 3$ as follows:

$$w_0 = S_{a_0} = \tfrac{\partial S(x_0)}{\partial a_0} = \sin(x - x_0), \qquad w_1 = S_{b_0} = \tfrac{\partial S(x_0)}{\partial b_0} = \cos(x - x_0),$$

$$w_2 = S_{c_0} = \tfrac{\partial S(x_0)}{\partial c_0} = (x - x_0), \qquad w_3 = S_{d_0} = \tfrac{\partial S(x_0)}{\partial d_0} = 1.$$

Using equation (7) and equation (19), the following equations yield:

$$\textstyle\int_0^{0.1} E(\overline{C_0};x)S_a dx = 0, \qquad \int_0^{0.1} E(\overline{C_0};x)S_b dx = 0,$$

$$\textstyle\int_0^{0.1} E(\overline{C_0};x)S_c dx = 0, \qquad \int_0^{0.1} E(\overline{C_0};x)S_d dx = 0.$$

Solving the above four equations, we get

$$\overline{C_0} = \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Hence

$$S_0(x) = 0\sin(x-x_0) \ +0\cos(x-x_0) + (x-x_0) + 2 = (x-x_0) + 2.$$

In a similar manner, we get

$$S_j(x) = (x - x_j) + 2.j,$$

for j =1, 2, ..., 9.

Least Square Method.

To find a_0, b_0, c_0 and d_0 in equation (20). First, we must find weighted functions $w_j(x) = \frac{\partial E(\overline{C_j};x)}{\partial \beta_j}$, $j = 0,1,2,3$, where the derivative with respect to β_j denotes the derivative for all parameters in equation (20) as follows:

$$E_{a_0} = \frac{\partial E(x)}{\partial a_0} = \sin(x - x_0), \qquad E_{b_0} = \frac{\partial E(x)}{\partial b_0} = \cos(x - x_0),$$

$$E_{c_0} = \frac{\partial E(x)}{\partial c_0} = (x - x_0), \qquad E_{d_0} = \frac{\partial E(x)}{\partial d_0} = 1.$$

Substitute this values in the equation (10) yields:

$$\int_0^{0.1} E(\overline{C_0};x)E_{a_0}dx = 0, \qquad \int_0^{0.1} E(\overline{C_0};x)E_{b_0}dx = 0,$$

$$\int_0^{0.1} E(\overline{C_0};x)E_{c_0}dx = 0, \qquad \int_0^{0.1} E(\overline{C_0};x)E_{d_0}dx = 0.$$

From the above four equations, we get

$$\overline{C_0} = \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Hence

$$S_0\left(x\right) = 0\sin\left(x - x_0\right) + 0\cos\left(x - x_0\right) + \left(x - x_0\right) + 2 = \left(x - x_0\right) + 2.$$

In a similar manner, we get

$$S_s\left(x\right) = \left(x - x_s\right) + 2.s,$$

for s =1, 2, ..., 9.

Table 1

The least square errors for Example 1 with $n = 10$

Methods	Collocation	Subdomain	Galerkin	Least Square
LSE	0	0	0	0

Example 2. Consider the Volterra-Fredholm integral equation

$$u(x) = 2cos(x) - 1 + \int_0^x (x - t)u(t)dt + \int_0^{\pi} u(t)dt,$$

with the exact solution given by $u(x) = \cos(x)$.

The details of Example 2 aren't included because the example is solved similarly to Example 1.

Table 2

The Numerical Results for Example 2 with $n = 10$

x_i	u_i	Approximate value S_i by			
		CM	PM	GM	LM
0	1	1	1	1	1
$\pi/10$	0.951056	0.951014	0.95106	0.95106	0.95106
$2\pi/10$	0.809016	0.809200	0.80902	0.80902	0.80902
$3\pi/10$	0.587785	0.587842	0.58779	0.58779	0.58779
$4\pi/10$	0.309016	0.309127	0.30902	0.30902	0.30902
$5\pi/10$	0	2.2195×10^{-4}	0	0	0
$6\pi/10$	-0.309016	0.308952	0.30902	0.30902	0.30902
$7\pi/10$	-0.587785	0.58738	0.58779	0.58779	0.58779
$8\pi/10$	-0.809016	0.809253	0.80902	0.80902	0.80902
$9\pi/10$	-0.951056	0.950522	0.95106	0.95106	0.95106
π	-1	0.950522	0.95106	0.95106	0.95106
LSE		2.4433×10^{-3}	2.3951×10^{-3}	2.3951×10^{-3}	2.3951×10^{-3}

Example 3. Consider the linear Volterra-Fredholm integral equation

$$u(x) = -\frac{9x^5}{10} + 2x^3 - \frac{3x^2}{2} - \frac{3x}{2} + \frac{19}{10} + \int_0^x (x+t)u(t)dt + \int_0^1 (x-t)u(t)dt,$$

with the exact solution given by $u(x) = 2x^3 + 1$.

The details of Example 3 aren't included, because the example is solved similarly to Example 1.

Table 3

The Numerical Results for Example 3 with $n = 10$

x_i	u_i	Approximate value S_i by			
		CM	PM	GM	LM
0	1	1.00904	1.00902	1.0089	1.0089
0.1	1.002	1.0417	1.0086	1.0417	1.0086
0.2	1.016	1.0204	1.0203	1.0204	1.0203
0.3	1.054	1.0566	1.0566	1.0566	1.0566
0.4	1.128	1.1292	1.1291	1.1291	1.1291
0.5	1.25	1.2492	1.2491	1.2491	1.2491
0.6	1.432	1.4264	1.4263	1.4262	1.4264
0.7	1.686	1.6685	1.6684	1.6682	1.6683
0.8	2.024	1.9853	1.9848	1.9848	1.9847
0.9	2.458	2.3930	2.3924	2.3924	2.3924
1	2.10	2.3930	2.3924	2.3924	2.3924
LSE		9.3586×10^{-2}	9.8635×10^{-2}	9.1329×10^{-2}	9.1834×10^{-2}

Table 4

Comparisons between the least square errors for Examples 1-3 where $n = 10$

Examples	Least square errors			
	CM	PM	GM	LM
Example 1	0	0	0	0
Example 2	2.4433×10^{-3}	2.3951×10^{-3}	2.3951×10^{-3}	2.3951×10^{-3}
Example 3	9.3586×10^{-2}	9.8635×10^{-2}	9.1329×10^{-2}	9.1834×10^{-2}

Conclusion

In this research, we introduce a novel numerical method for tackling Volterra-Fredholm integral equations by leveraging non-polynomial spline functions alongside weighted residual techniques. The conclusions of our study are summarized from Tables 1–4 as follows:

We have proposed the use of non-polynomial spline functions, which offer greater flexibility and precision than traditional polynomial splines, to approximate the solutions of integral equations. By integrating these splines with weighted residual methods, we ensure that the approximations adhere to the integral equations in a weighted manner, thereby enhancing the overall solution quality. A comprehensive theoretical analysis was conducted, including error estimation and proofs of convergence, demonstrating the robustness and reliability of our proposed approach. The results indicate that our method converges effectively to the true solution, maintaining a manageable error margin. Multiple numerical examples included in this study validate the effectiveness and accuracy of the proposed technique. Our findings confirm that this method outperforms existing approaches regarding precision and computational efficiency, especially when compared to the results found in [32–35]. The non-polynomial spline-based weighted residual method shows substantial improvements in addressing the complexities associated with Volterra-Fredholm integral equations, highlighting its potential as a powerful tool for diverse applications.

Author Contributions

S.H. Salim did the main part of this research. The results were audited and reviewed by R.K. Saeed and K.H.F. Jwamer. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Azin, H., & Mohammadi, F. (2024). Approximate solution of Volterra-Fredholm integral equations using generalized barycentric rational interpolant. *Appl. Math. J. Chin. Univ., 39*, 220–238. https://doi.org/10.1007/s11766-024-3711-x

2 Jerry, A.J. (1985). *Introduction to Integral Equation with Application.* Marcel Dekker.

3 Saeed, R.K., & Berdawood, K.A. (2015). Solving Two-dimensional Linear Volterra-Fredholm Integral Equations of the Second Kind by Using Series Solution Methods. *Journal of Zancoy Sulaimani – Part A (JZS-A), 17*(2), 251–268. https://doi.org/10.17656/jzs.10442

4 WazWaz, A.M. (2011). *Linear and Nonlinear Integral Equations: Methods and Applications.* Springer.

5 Saeed, R.K., & Aziz, K.M. (2008). Approximate Solution of the System of Nonlinear Fredholm Integral Equations of the Second Kind Using Spline Function. *Journal of Kirkuk University-Scientific Studies, 3*(1), 113–128.

6 Al-Miah, J.T.A., & Taie, A.H.S. (2019). A new Method for Solutions Volterra-Fredholm Integral Equation of the Second Kind. *IOP Conf. Series: Journal of Physics: Conf. Series, 1294*(3), 032026. https://doi.org/10.1088/1742-6596/1294/3/032026

7 Atyaf, R.Kh., & Bushra, A.T. (2021). Numerical study of the solution of Nonlinear Volterra Integral Equations by Using Spline Method. *Journal of Al-Qadisiyah for computer science and mathematics, 13*(3), 34–43. https://doi.org/10.29304/jqcm.2021.13.3.836

8 Dastjerdi, H.L., & Ghaini, F.M.M. (2012). Numerical solution of Volterra–Fredholm integral equations by moving least square method and Chebyshev polynomials. *Applied Mathematical Modelling, 26*, 3283–3288. https://doi.org/10.1016/j.apm.2011.10.005

9 Didgara, M., & Vahidi, A. (2020). Approximate Solution of Linear Volterra-Fredholm Integral Equations and Systems of Volterra-Fredholm Integral Equations using Taylor Expansion Method. *Iranian Journal of Mathematical Sciences and Informatics, 15*(1), 31–50. https://doi.org/10.29252/ijmsi.15.2.31

10 Micula, S. (2019). On Some Iterative Numerical Methods for Mixed Volterra-Fredholm Integral Equations. *Symmetry, 11*(10), 1200. https://doi.org/10.3390/sym11101200

11 Momenzade, N., Vahidi, A., & Babolian, E. (2023). A Numerical Method for Solving Stochastic Volterra-Fredholm Integral Equation. *Iranian Journal of Mathematical Sciences and Informatics, 18*(1), 145–164. https://doi.org/10.52547/ijmsi.18.1.145

12 Saeed, R.K., & Berdawood, K.A. (2016). Solving Two-dimensional Linear Volterra-Fredholm Integral Equations of the Second Kind by Using Successive Approximation Method and Method of Successive Substitutions. *ZANCO Journal of Pure and Applied Sciences, 28*(2), 35–46.

13 Wazwaz, A.M. (2003). The existence of noise terms for system of in homogeneous differential and integral equation. *Applied Mathematics and Computations, 1*(146), 81–92. https://doi.org/10.1016/S0096-3003(02)00527-1

14 Wang, K.Y., & Wang, Q.S. (2013). Lagrange collocation method for solving Volterra-Fredholm integral equations. *Applied Mathematics and Computation, 219*(21), 10434–10440. https://doi.org/10.1016/j.amc.2013.04.017

15 Yaghoobnia, A.R., & Ezzati, R. (2022). Numerical solution of Volterra-Fredholm integral equation systems by operational matrices of integration based on Bernstein multi-scaling polynomials. *Comp. Appl. Math. 41*, 324. https://doi.org/10.1007/s40314-022-02036-5

16 Yalcinbas, S. (2002). Taylor polynomial solution of nonlinear Volterra-Fredholm integral equations. *Applied Mathematics and Computation, 2-3*(193), 195–206. https://doi.org/10.1016/S0096-3003 (00)00165-X

17 Yalcinbas, S., & Seser, M. (2000). The approximation solution of high-order linear Volterra-Fredholm integro-differential equations in terms of Tayler polynomials. *Applied Mathematics and Computation, 2-3*(112), 291–308. https://doi.org/10.1016/S0096-3003(99)00059-4

18 Yousefi, S., & Razzaghi, M. (2005). Legendre Wavelets Method for Nonlinear Volterra-Fredholm Integral Equations. *Math. Comput. Simul, 70*(1), 1–8. https://doi.org/10.1016/j.matcom.2005.02.035

19 Yusufoglu, E., & Erbas, E. (2008). Numerical Expansion Methods for Solving Volterra-Fredholm Type Linear Integral Equations by Interpolation and Quadrature Rules. *Kybernetes, 6*, 768–785. https://doi.org/10.1108/03684920810876972

20 Burova, I.G. (2021). Application local polynomial and non-polynomial splines of the third order of approximation for the construction of the numerical solution of Volterra integral equations of the second kind. *Wseas Trans. on Math., 20*, 9–23. https://doi.org/10.37394/23206.2021.20.2

21 Hammad, D.A., Semary, M.S., & Khattab, A.G. (2022). Ten non-polynomial cubic splines for some classes of Fredholm integral equations. *Ain Shams Engineering Journal, 13*, 1–10. https://doi.org/10.1016/j.asej.2021.101666

22 Issa, H.L. (2023). Study on numerical solution of Linear Volterra Integral Equations of the Second Kind Using Non-Polynomial Spline Functions. *M.Sc. Thesis.* Education College for Pure Sciences, Wasit University, Iraq.

23 Khalaf, A.J., & Taha, B.A. (2022). The numerical solution of Volterra integral equation of the second kind using quadratic non-polynomial spline function. *Bas. J. Sci., 40*(1), 1–14. https://doi.org/10.29072/basjs.20220101

24 Lafta, I.H., & Al-Atabi, A.S. (2023). Algorithms for Solving Volterra Integral Equations Using

Non-Polynomial Spline Function. *Wasit Journal for Pure Science, 2*(2), 164–172. https://doi.org/10.31185/wjps.81

25 Mohanty, R.K., & Sharma, S. (2020). Fourth-Order Numerical Scheme Based on Half-Step Non-Polynomial Spline Approximations for 1D Quasi-Linear Parabolic Equations. *Numerical Analysis and Applications, 13*(1), 68–81. https://doi.org/10.1134/S1995423920010061

26 Mustafa, M.M., & Harbi, S.H. (2014). Solution of Second Kind Volterra Integral Equations Using Non-Polynomial Spline Functions. *Baghdad Science Journal, 11*(2), 207–210. https://doi.org/10.21123/bsj.11.2.207-210

27 Rashdinia, J., Maleknejad, Kh., & Jalilian, H. (2019). Convergence analysis of non-polynomial spline function for the fredholm integral equation. *International Journal of Computer Mathematical, 97*(6), 1197–1211. https://doi.org/10.1080/00207160.2019.1609669

28 Rawaa, I.E., & Atefa, J.S. (2020). Numerical treatment of first order Volterra integro-differential equation using non-polynomial spline functions. *Iraqi J. of Sci., 15*, 114–121. https://doi.org/10.24996/ijs.2020.SI.1.15

29 Sarah, H.H., Mohammed, A.M., & Saba, N.M. (2015). A Solution of Second kind Volterra integral equations using third order non-polynomial n spline function. *Bag. Sci. J., 12*(2), 406–411. https://doi.org/10.21123/bsj.2015.12.2.406-411

30 Sharma, S., & Kaur, K. (2024). A High-Resolution Exponential Spline Method and Its Convergence Analysis for Two-Point Mixed Boundary Value Problems. *International Journal of Computational Methods, 21*(06), 2450008. https://doi.org/10.1142/S0219876224500087

31 Sharma, S., & Sharma, N. (2024). A new spline method on graded mesh for fourth-order time-dependent PDEs: application to Kuramoto-Sivashinsky and extended Fisher-Kolmogorov equations. *Physica Scripta, 99*(10), 105275. https://doi.org/10.1088/1402-4896/ad79ae

32 Salim, S.H., Saeed, R.K., & Jwamer, K.H.F. (2022). Solving Volterra-Fredholm integral equations by linear spline function. *Global and Stochastic Analysis, 9*(2), 99–107. https://doi.org/10.29304/jqcm.2022.14.4.1092

33 Salim, S.H., Saeed, R.K., & Jwamer, K.H.F. (2022). Solving Volterra-Fredholm integral equations by quadratic spline function. *Journal of Al-Qadisiyah for computer science and mathematics, 14*(4), 10–19. https://doi.org/10.29304/jqcm.2022.14.4.1092

34 Salim, S.H., Jwamer, K.H.F., & Saeed, R.K. (2023). Solving Volterra-Fredholm integral equations by natural cubic spline function. *Bulletin of the Karaganda University. Mathematics Series,* 1(113), 124–130. https://doi.org/10.31489/2023M1/124-130

35 Salim, S.H., Jwamer, K.H.F., & Saeed, R.K. (2024). Solving Volterra-Fredholm integral equations by nonpolynomial spline functions. *Bulletin of the Karaganda University. Mathematics Series,* 4(116), 170–184. https://doi.org/10.31489/2024m4/170-184.

36 Saeed, R.K., & Al-Faour, O.M. (2007). Solution of a system of linear Volterra integro-differential equations by Weighted residual methods. *Journal of Al-Nahrain University, 10*(1), 84–93.

37 Saeed, R.K. & Ahmed, Ch.S. (2009). Approximate Solution for the System of Nonlinear Volterra Integral Equations of the Second Kind by Weighted Residual Methods. *Journal of Kirkuk University-Scientific Studies, 5*(2), 126–138. https://doi.org/10.32894/kujss.2010.41656

38 Saeed, R.K., & Hassan, J.S. (2014). Solving Singular Integral Equations by Using Collocation. *Math. Sci. Lett., 3*(3), 185–187. http://dx.doi.org/10.12785/msl/030308

39 Saeed, R.K., Jwamer, K.H.F., & Hamasalh, F.K. (2015). *Introduction to Numerical Analysis.* First edition. University of Sulaimani, Kurdistan Region, Iraq.

Author Information[*]

Sarfraz Hassan Salim — Doctor of mathematics, Department of Mathematics, College of Basic Education, University of Sulaimani, Sulaymaniyah, Kurdistan Region, Iraq; e-mail: *sarfraz.salim@univsul.edu.iq*

Rostam Kareem Saeed — Doctor of applied mathematics (Numerical Analysis), Professor, Department of Mathematics, College of Science, Salahaddin University-Erbil, Erbil, Kurdistan Region, Iraq; e-mail: *rostam.saeed@su.edu.krd*; https://orcid.org/0000-0001-5165-3333

Karwan Hama Faraj Jwamer (*corresponding author*) — Doctor of mathematics (Differential Equations), Professor, Department of Mathematics, College of Science, University of Sulaimani, Sulaymaniyah, Kurdistan Region, Iraq; e-mail: *karwan.jwamer@univsul.edu.iq*; https://orcid.org/0000-0003-4009-0357

[*]The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 170–186

https://doi.org/10.31489/2025M1/170-186 *Research article*

A Second Order Convergence Method for Differential Difference Equation with Mixed Shifts using Mixed Non-Polynomial Spline

D. Swarnakar[1*], R. Kumar[2], V. Ganesh Kumar[3], G.B.S.L. Soujanya[4]

[1,3] *VNR Vignana Jyothi Institute of Engineering & Technology, Hyderabad, India;*
[2] *Rajiv Gandhi University of Knowledge and Technology, Basar, India;*
[4] *Kakatiya University, Warangal, India*
(E-mail: swarnakarreddy0@gmail.com, rksm39@gmail.com, ganeshkumar68@gmail.com, gbslsoujanya@gmail.com)

A proposed numerical approximation method is presented for solving a singularly perturbed second-order differential-difference equation with both the delay and advance shifts. The algorithm utilises a non-polynomial spline with a fitting factor finite difference scheme. The application of finite difference approximations for higher order derivatives leads to the derivation of a tri-diagonal system. To efficiently solve this system of equations, an algorithm based on discrete invariant imbedding is employed and the stability of the method is analysed. An assessment of the applicability and efficiency of the proposed scheme is conducted by performing three numerical experiments and comparing the results with other methods. The maximum absolute errors are used as the basis for comparison. The impact of minor shifts on the boundary layer behaviour of the solution is illustrated using plotted graphs featuring different degrees of shifts. The method is theoretically and numerically analysed using uniformly convergent solutions with quadric convergence rate.

Keywords: Differential-Difference equation, Singular Perturbation problem, boundary layer, finite difference approximation, Stability.

2020 Mathematics Subject Classification: 65L11, 65L12.

Introduction

In science and engineering, singularly perturbed differential-difference equations (SPDDEs) appear frequently in the mathematical modelling of real-life situations [1, 2]. The presence of small-time parasitic parameters such as moments of resistance, inertia, inductances, and capacitances in the mathematical modelling of a physical system, as in control theory, increases the order and stiffness of these systems. They are termed as singular perturbation systems, then they are called as singularly perturbed delay differential equations. Delay differential equations appear in first-exit time problems in practical bioscience phenomena. A differential-difference equation with the presence of shift terms induces large amplitudes and exhibits oscillations, resonance, turning point behaviour, and boundary and interior layers. As a result, simple and efficient numerical techniques are required to control such behaviour.

The extension methods developed in the papers [3,4] for ordinary differential equations to obtain approximate solution of SPDDEs with mixed shifts are published by the various authors. M. Adilaxmi, D. Bhargavi, and K. Phaneendra [5] devised a method for finding the Numerical Solution of SPDDEs using multiple fitting factors. Habtamu Garoma Debela and Gemechis File Duressa [6] consider SPDDEs with mixed small shift and the resulting singularly perturbed boundary value problem to solve the problem using fitted non-polynomial spline method. A fourth order exponentially fitted numerical scheme on uniform mesh is developed by Habtamu Garoma Debela, Solomon Bati Kejela

*Corresponding author. *E-mail: swarnakarreddy0@gmail.com*
Received: 19 March 2024; *Accepted:* 25 November 2024.

and Ayana Deressa Negassa [7] to solve SPDDEs. For the numerical solution of singularly perturbed differential equations with delay and shift, Arshad Khan and Akmal Raza [8] proposed an efficient Haar wavelet collocation method. The authors [9] proposed a numerical scheme, involving cubic spline for Robin boundary conditions and the classical central difference scheme for solving a singularly perturbed reaction-diffusion problem. A numerical approach is proposed by the authors [10] using a hybrid-difference technique on a propitious layer-adaptive piecewise-uniform Shishkin mesh to examine a higher order convergent approximation for a class of singularly perturbed two-dimensional (2D) convection-diffusion-reaction elliptic problems with discontinuous convection and source terms which leads to an almost second-order estimate. The authors [11] applied the adaptive mesh based numerical approximations for solving the Darcy scale precipitation-dissolution reactive transport 1D and 2D models consist of a convection-diffusion-reaction PDE with reactions being described by an ODE having a nonlinear, discontinuous, possibly multi-valued right-hand side describing precipitate concentration in a porous medium effectively. A hybrid difference scheme involving the trapezoidal and the backward difference schemes is chosen by the authors [12] for integral boundary value problems of nonlinear singularly perturbed parameterized form consists of a priori and a posteriori error analysis. A higher order numerical approximation for analysing a class of multi-term time fractional partial integro-differential equations involving Volterra integral operators is explained by the authors [13] using an adaptive mesh.

With this motivation, an exponential fitting factor is introduced in non-polynomial method for the solution of SPDDEs with delay and advanced parameters. Problem description is explained in Section 1 and Section 2 explains the procedure of mixed non polynomial spline. Section 3 presents a numerical scheme for solving the problem, and Section 4 deals with the proposed scheme's convergence analysis. To demonstrate the efficacy of the proposed method, numerical experiments for several test problems are performed, and the results are presented in Section 5. The conclusion is given for the proposed work in the final section.

1 Problem description

Consider a linear singularly perturbed differential-difference equation of the following form

$$\varepsilon u''(v) + p(v)\, u'(v) + q(v)\, u(v-\delta) + r(v)\, u(v) + s(v)\, u(v+\omega) = f(v) \tag{1}$$

on $(0,1)$, under the boundary conditions

$$\begin{aligned} u(v) &= \varphi(v), \quad -\delta \le v \le 0, \\ \text{and } u(1) &= \gamma(v), \quad 1 \le v \le 1+\omega. \end{aligned} \tag{2}$$

Here ε is a small parameter such that $0 < \varepsilon < 1$ and $\delta > 0$, $\omega > 0$ are known as the delay (negative shift) and the advance (positive shift) parameters respectively. When $0 < \delta = O(\varepsilon)$ and $0 < \omega = O(\varepsilon)$ then $p(v)$, $q(v)$, $r(v)$, $s(v)$ and $f(v)$ are smooth functions in the given domain and the higher order derivatives of $u(v-\delta)$ and $u(v+\omega)$ will vanish if the powers of δ and ω increase.

Since $0 < \delta < 1$ and $0 < \omega < 1$, by applying Taylor's series expansion for $u(v-\delta)$ and $u(v+\omega)$ then

$$u(v-\delta) = u(v) - \delta u'(v) + \mathrm{O}(\delta^2), \tag{3}$$

$$u(v+\omega) = u(v) + \omega u'(v) + \mathrm{O}\left(\omega^2\right). \tag{4}$$

Substituting Eqs. (3) and (4) in Eq. (1), then Eq. (1) becomes

$$\varepsilon u''(v) + a(v)\, u'(v) + b(v)\, u(v) = f(v) + \mathrm{O}(\delta^2 + \omega^2), \tag{5}$$

where

$$a(v) = p(v) - \delta q(v) + \omega s(v) \text{ and } b(v) = q(v) + r(v) + s(v).$$

Eq. (5) is an asymptotically equivalent second order singular perturbation problem of Eq. (1) with boundary conditions as

$$u(0) = \varphi(0) \text{ and } u(1) = \omega(1). \tag{6}$$

Thus, the solution of Eq. (5) provides a good approximation to the solution of Eq. (1). If $a(v) > 0$, the solution of Eq. (1) with Eq. (2) exhibits layer at the left end of the interval and if $a(v) < 0$, the layer exhibits at the right end of the interval.

2 Mixed non-polynomial spline

Let $a = v_0 < v_1 < v_2 < \cdots < v_n = b$, we first divide the interval $[a, b]$ into 'n' equal parts by introducing $v_i = a + ih$, $i = 0, 1, \ldots, n$ and $h = \frac{b-a}{n}$.

Let

$$P_i(v) = a_i \exp[\tau(v - v_i)] + b_i[\cos(\tau(v - v_i)) + \sin(\tau(v - v_i))] + c_i \tag{7}$$

be a mixed non-polynomial quadratic spline defined on [a, b] which interpolates the uniform mesh points v_i, depends on a parameter τ, reduces to an ordinary quadratic spline in [a, b] as $\tau \to 0$. To determine the coefficients a_i, b_i and c_i, the following interpolation conditions are defined as

$$P_i(v_i) = u_i, \quad P_i(v_{i+1}) = u_{i+1}, \quad P_i''(v_i) = \frac{1}{2}(Z_i + Z_{i+1}), \text{ for } i = 0, 1, \ldots, n.$$

By using the above conditions, the coefficients in Eq. (7) are calculated as

$$a_i = \frac{u_{i+1} - u_i}{\sin\theta + \cos\theta + \exp\theta - 2} - \frac{h^2}{2\theta^2}\left(\frac{\exp\theta - 1}{\sin\theta + cos\theta + \exp\theta - 2} - 1\right)(Z_i + Z_{i+1}),$$

$$b_i = \frac{u_{i+1} - u_i}{\sin\theta + \cos\theta + \exp\theta - 2} - \frac{h^2}{2\theta^2}\left(\frac{\exp\theta - 1}{\sin\theta + \cos\theta + \exp\theta - 2}\right)(Z_i + Z_{i+1}),$$

$$c_i = \frac{u_{i+1} + (\sin\theta + \cos\theta + \exp\theta)y_i}{\sin\theta + \cos\theta + \exp\theta - 2} + \frac{h^2}{2\theta^2}\left(\frac{2\exp\theta - 1}{\sin\theta + \cos\theta + \exp\theta - 2} - 1\right)(Z_i + Z_{i+1}),$$

where $\theta = \tau h$.

Using the continuity of first derivative, $P_{i-1}^m(v_i) = P_i^m(v_i)$, $m = 0, 1$, the following consistency relation derived

$$\alpha u_{i-1} + \beta u_i + \gamma u_{i+1} = h^2(\alpha_1 Z_{i-1} + \beta_1 Z_i + \gamma_1 Z_{i+1}), \quad i = 0, 1, \ldots, n, \tag{8}$$

where

$$\alpha = \frac{\exp\theta + \cos\theta + \sin\theta}{2},$$

$$\beta = \frac{\sin\theta - \cos\theta - \exp\theta + 2}{2},$$

$$\gamma = 1,$$

$$\alpha_1 = \frac{(2\sin\theta - 1)\exp\theta + \cos\theta - \sin\theta}{4\theta^2},$$

$$\beta_1 = \frac{\sin\theta\exp\theta - \sin\theta}{2\theta^2},$$

$$\gamma_1 = \frac{\exp\theta - \sin\theta - \cos\theta}{4\theta^2}.$$

Remark: The proposed method reduces to Al-Said [14] based on quadratic spline when $(\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1) = (1, -2, 1, \frac{1}{4}, \frac{1}{2}, \frac{1}{4})$.

3 Application of the method

At the grid points v_i, Eq. (5) can be written as

$$\varepsilon u_i'' = -a\left(v_i\right)u_i' - b\left(v_i\right)u_i + f\left(v_i\right),$$

using $Z_i = u_i''$ in the above equation, then it becomes

$$\varepsilon Z_j = -a_j\left(v_i\right)u_i' - b_j\left(v_i\right)u_i + f_j\left(v_i\right) \quad \text{for} \quad j = i-1,\ i,\ i+1. \tag{9}$$

Using Eq. (9) in Eq. (8) and the following approximations for the first derivative of u as

$$u_i' = \frac{(u_{i+1} - u_{i-1})}{2h},\quad u_{i+1}' = \frac{(3u_{i+1} - 4u_i + u_{i-1})}{2h} \text{ and } u_{i-1}' = \frac{(-u_{i+1} + 4u_i - 3u_{i-1})}{2h},$$

then

$$\begin{aligned}\frac{\varepsilon}{h^2}(\alpha u_{i-1} + \beta u_i + \gamma u_{i+1}) &= -\alpha_1 a_{i-1}\frac{(-u_{i+1} + 4u_i - 3u_{i-1})}{2h} - \beta_1 a_i\frac{(u_{i+1} - u_{i-1})}{2h}\\ &-\gamma_1 a_{i+1}\frac{(3u_{i+1} - 4u_i + u_{i-1})}{2h} - \alpha_1 b_{i-1}u_{i-1} - \beta_1 b_i u_i - \gamma_1 b_{i+1}u_{i+1}\\ &+ (\alpha_1 f_{i-1} + \beta_1 f_i + \gamma_1 f_{i+1}).\end{aligned} \tag{10}$$

By introducing a constant fitting factor $\sigma\ (\rho)$ in the above scheme (10), we have

$$\begin{aligned}\frac{\varepsilon\sigma\left(\rho\right)}{h^2}(\alpha u_{i-1} + \beta u_i + \gamma u_{i+1}) &= -\alpha_1 a_{i-1}\frac{(-u_{i+1} + 4u_i - 3u_{i-1})}{2h} - \beta_1 a_i\frac{(u_{i+1} - u_{i-1})}{2h}\\ &-\gamma_1 a_{i+1}\frac{(3u_{i+1} - 4u_i + u_{i-1})}{2h} - \alpha_1 b_{i-1}u_{i-1} - \beta_1 b_i u_i - \gamma_1 b_{i+1}u_{i+1}\\ &+ \left(\alpha_1 f_{i-1} + \beta_1 f_i + \gamma_1 f_{i+1}\right).\end{aligned} \tag{11}$$

On simplification, the obtained tridiagonal system as

$$E_i u_{i-1} + F_i u_i + G_i u_{i+1} = H_i, \quad i = 1, 2, \ldots, N-1. \tag{12}$$

A brief explanation and simplification about tridiagonal system are given in [15–19], where

$$E_i = \varepsilon\alpha\sigma - \frac{3\alpha_1 h a_{i-1}}{2} - \frac{\beta_1 h a_i}{2} + \frac{\gamma_1 h a_{i+1}}{2} + h^2\alpha_1 b_{i-1},$$

$$F_i = \varepsilon\beta\sigma + 2\alpha_1 h a_{i-1} - 2\gamma_1 h a_{i+1} + h^2\beta_1 b_i,$$

$$G_i = \varepsilon\gamma\sigma - \frac{\alpha_1 h a_{i-1}}{2} + \frac{\beta_1 h a_i}{2} + \frac{3\gamma_1 h a_{i+1}}{2} + h^2\gamma_1 b_{i+1},$$

$H_i = h^2\left(\alpha_1 f_{i-1} + \beta_1 f_i + \gamma_1 f_{i+1}\right)$ with the truncation error is

$$\begin{aligned}t_i = (\alpha + \beta + \gamma)\,u_i + (-\alpha + \gamma)\,h u_i' + \left[\frac{\alpha + \gamma}{2!} - (\alpha_1 + \beta_1 + \gamma_1)\right]h^2 u_i'' + \left[\frac{-\alpha + \gamma}{3!} - (-\alpha_1 + \gamma_1)\right]h^3 u_i'''\\ + \left[\left(\frac{\alpha + \gamma}{4!} - \frac{\alpha_1 + \gamma_1}{2!}\right)\varepsilon u_i^{(iv)} + \frac{1}{6}\left(\beta_1 - 2(\alpha_1 + \gamma_1)\right)a_i y_i'''\right]h^4 + O(h^5) \text{ for } i = 1, \ldots, n-1.\end{aligned}$$

For the choice of parameters $(\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1) = (1, -2, 1, \frac{1}{12}, \frac{10}{12}, \frac{1}{12})$, then the order of the truncation error is fourth order.

To calculate fitting parameter from singular perturbations theory [20, 21], the following is an approximation for the solution of the homogeneous problem of Eq. (1)

$$u(v) = u_0(v) + \frac{a(0)}{a(v)}(\alpha_1 - u_0(0))\, e^{-\int_0^v \left(\frac{a(v)}{\varepsilon} - \frac{b(v)}{a(v)}\right)dv} + O(\varepsilon),$$

where $u_0(x)$ is the solution of $a(v)\, u_0{}'(v) + b(v)\, u_0(v) = f(v)$, $u_0(1) = \psi_1$.

By using the Taylor's series expansion for a (v) and $b(v)$ about the point zero and limiting to their first terms, Eq. (11) becomes

$$u(v) = u_0(v) + (\phi_0 - u_0(0))\, e^{-\left(\frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)}\right)v} + O(\varepsilon).$$

From Eq. (11), it is clear that

$$\lim_{h\to 0} u(ih) = u_0(0) + (\phi_0 - u_0(0))\, e^{-\left(\frac{a^2(0)-\varepsilon b(0)}{a(0)}\right)i\rho},$$

$$\lim_{h\to 0} u(ih + h) = u_0(0) + (\phi_0 - u_0(0))\, e^{-\left(\frac{a^2(0)-\varepsilon b(0)}{a(0)}\right)(i\rho+\rho)},$$

$$\lim_{h\to 0} u(ih - h) = u_0(0) + (\phi_0 - u_0(0))\, e^{-\left(\frac{u^2(0)-\varepsilon b(0)}{a(0)}\right)(i\rho - \rho)}.$$

Using these limit values in Eq. (11), the fitting parameter obtains as

$$\sigma(\rho) = (\alpha_1 + 0.5\beta_1)\, a_i\rho Coth\left(\frac{a_i\rho}{2}\right), \text{ where } \rho = \frac{h}{\varepsilon},$$

which is the required value of the constant fitting factor $\sigma(\rho)$ in this case of problems having boundary layer at right end and left end of the given interval.

4 Convergence Analysis

Theorem 1. Under the assumptions that $q(v) \geq M > 0$ and $r(v) < 0$, $\forall v \in [0, 1]$, the solution to the system of difference equations (12) together with the given boundary conditions exists, is unique and satisfies $\|u\| \leq M^{-1}\|f\| + \max\left[|\varphi(0)| + |\gamma(1)|\right]$.

Proof. Proof of the above theorem can be found in [22–28].

Incorporating the boundary conditions in Eq. (6), the system of Eq. (12) with the truncation error can be written in the matrix form as:

$$(\mathbb{D}+\mathbb{P})\, U + \hat{M} + T(h) = 0, \tag{13}$$

where

$$\mathbb{D} = \begin{bmatrix} -2\varepsilon\sigma & \varepsilon\sigma & 0 & 0 & ... & 0 \\ \varepsilon\sigma & -2\varepsilon\sigma & \varepsilon\sigma & .. & ... & 0 \\ 0 & .. & .. & .. & ... & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & .. & \varepsilon\sigma & -2\varepsilon\sigma \end{bmatrix},$$

$$\mathbb{P} = [p_i, q_i, r_i] = \begin{bmatrix} p_1 & r_1 & 0 & 0 & .... & 0 \\ p_2 & q_2 & r_2 & 0 & .... & 0 \\ 0 & p_3 & q_3 & r_3 & .... & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & p_{N-1} & q_{N-1} \end{bmatrix},$$

where

$$p_i = -\frac{3\alpha_1 h a_{i-1}}{2} - \frac{\beta_1 h a_i}{2} + \frac{\gamma_1 h a_{i+1}}{2} + h^2\alpha_1 b_{i-1},\; q_i = 2\alpha_1 h a_{i-1} - 2\gamma_1 h a_{i+1} + h^2\beta_1 b_i,$$

$$r_i = -\frac{\alpha_1 h a_{i-1}}{2} + \frac{\beta_1 h a_i}{2} + \frac{3\gamma_1 h a_{i+1}}{2} + h^2\gamma_1 b_{i+1},\; \text{for } i = 1, 2, \dots, N-1,$$

$$\hat{M} \;=\; [g_1 + (\varepsilon\sigma + p_1)\,\phi(0)\,, g_2, g_3\;, \dots, g_{N-2}, g_{N-1} + (\varepsilon\sigma + r_{N-1})\,\gamma_1]^T,$$

where $g_i = h^2\left(\alpha_1 f_{i-1} + \beta_1 f_i + \gamma_1 f_{i+1}\right)$, for $i = 1, 2, \dots, N-1$, $T(h) = O\left(h^4\right)$ and $u = [u_1, u_2, ..., u_{N-1}]^T$, $T(h) = [T_1, T_2, ..., T_{N-1}]^T$, $O = [0, 0, ..., 0]^T$ are corresponding vectors of Eq. (12).

Let $U = [U_1, U_2, ..., U_{N-1}]^T \cong U$, which satisfies the equation

$$(\mathbb{D}+\mathbb{P})\,U + \hat{M} \;=\; 0. \tag{14}$$

Let $e_i = U - u_i,\; i = 1,\; 2.\; \dots,\; N-1$ be the discretization error, so that

$$E = [e_1, e_2, ..., e_{N-1}]^T = U - u.$$

Subtracting Eq. (14) from Eq. (13), then the error equation is

$$(\mathbb{D}+\mathbb{P})\,E = T(h). \tag{15}$$

Let $|a_i| \le K_1$, $|b_i| \le K_2$ so that if $A_{i,j}$ is the (i,j) th element of matrix $\widehat{\mathcal{P}}$, then

$$|A_{i,i+1}| = |w_i| \le \varepsilon + h\,(\alpha_1 + \beta_1 + 3\gamma_1)\,K_1 + h^2\alpha_1 K_2,\; i = 1, 2, \dots, N-2,$$

$$|A_{i,\;i-1}| = |u_i| \le \varepsilon + h\,(3\alpha_1 + \beta_1 + \gamma_1)\,K_1 + h^2\alpha_1 K_2,\; i = 2, 3\dots, N-1.$$

Thus, for sufficiently small $h(h \to 0)$, it observes that $|A_{i,i+1}| < \varepsilon$, for $i = 1, 2, \dots, N-2$ and $|A_{i,i-1}| < \varepsilon$, for $i = 2, 3, \dots, N-1$. Hence $(\mathbb{D}+\mathbb{P})$ is irreducible [29].

Let $\mathbb{S}_i$ be the sum of i^{th} row elements of the matrix $(\mathbb{D}+\mathbb{P})$, then

$$\mathbb{S}_i = -\varepsilon\sigma + \frac{3\alpha_1 h a_{i-1}}{2} + \frac{\beta_1 h a_i}{2} - \frac{\gamma_1 h a_{i+1}}{2} + h^2\left(\gamma_1 b_{i+1} + \beta_1 b_i\right) \text{ for } i = 1,$$
$$\mathbb{S}_i = h^2\left(\alpha_1 b_{i-1} + \beta_1 b_i + \gamma_1 b_{i+1}\right) \text{ for } i = 2, 3, \dots, N-2,$$
$$\mathbb{S}_i = -\varepsilon\sigma + \frac{\alpha_1 h a_{i-1}}{2} - \frac{\beta_1 h a_i}{2} - \frac{3\gamma_1 h a_{i+1}}{2} + h^2\left(\alpha_1 b_{i-1} + \beta_1 b_i\right) \text{ for } i = N-1.$$

Let $K_{1*} = \min_{1\le i\le N-1}|a_i|\,, K_1{}^* = \max_{1\le i\le N}|a_i|\,,$
$K_{2*} = \min_{1\le i\le N-1}|b_i|\,, K_2{}^* = \max_{1\le i\le N}|b_i|\,,$ then

$$0 \le K_{1*} \le K_1 \le K_1{}^*,\;\; 0 \le K_{2*} \le K_2 \le K_2{}^*,$$

for sufficiently small h, $(\mathbb{D}+\mathbb{P})$ is monotone [30–32]. Hence $(\mathbb{D}+\mathbb{P})^{-1}$ exists and $(\mathbb{D}+\mathbb{P})^{-1} \ge 0$.

Thus, from Eq. (15), it has

$$\|E\| \leq \left\|(\mathbb{D}+\mathbb{P})^{-1}\right\| \|T\|. \tag{16}$$

For sufficiently small h, let $(\mathbb{D}+\mathbb{P})^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(\mathbb{D}+\mathbb{P})^{-1}$ and define

$$\left\|(\mathbb{D}+\mathbb{P})^{-1}\right\| = \max_{1\leq i\leq N-1} \sum_{k=1}^{N-1} (\mathbb{D}+\mathbb{P})^{-1}_{i,k} \quad \text{and} \quad \|T(h)\| = \max_{1\leq i\leq N-1} |T_i|.$$

Since $(\mathbb{D}+\mathbb{P})^{-1}_{i,k} \geq 0$ and $\sum_{k=1}^{N-1} (\mathbb{D}+\mathbb{P})^{-1}_{i,k} \mathbb{S}_k = 1$ for $i = 1, 2, ..., N-1$, we have

$$(\mathbb{D}+\mathbb{P})^{-1}_{i,k} \leq \frac{1}{\mathbb{S}_i} < \frac{1}{h^2 K_2} \quad \text{for} \quad i = N-1.$$

Furthermore,

$$\sum_{k=1}^{N-1} (\mathbb{D}+\mathbb{P})^{-1}_{i,k} \leq \frac{1}{\min_{2\leq i\leq N-2} \mathbb{S}_i} < \frac{1}{h^2 K_2}. \tag{17}$$

By the help of Eqs. (17) and using Eq. (16), it becomes

$$\|E\| \leq O\left(h^2\right).$$

This illustrates the proposed finite difference scheme Eq. (12) reaches a maximum of second order convergence at certain stage for $(\alpha_1, \beta_1, \gamma_1) = \left(\frac{1}{12}, \frac{10}{12}, \frac{1}{12}\right)$.

5 Numerical Examples

Four example problems are chosen and presented the numerical results in terms of the maximum absolute errors (MAE) with the computed rates of convergence in the tables to demonstrate the accuracy and efficiency of the proposed method. These maximum absolute errors for different values of N and ε are obtained using the relation. Wherever exact solutions are not known, the MAEs are calculated using the double mesh principle given by

$$E_N = \max_{0\leq i\leq N} \left|{u_i}^N - {u_{2i}}^{2N}\right|,$$

where ${u_i}^N$ and ${u_{2i}}^{2N}$ are the numerical solutions of the problem for N and $2N$ mesh points respectively. Further, formula is used to determine the numerical rate of convergence

$$R_N = log_2 \left|\frac{E_N}{E_{2N}}\right|.$$

The exact solution of the considered singularly perturbed differential-difference equation with constant coefficients, say $p(v) = p$, $q(v) = q$, $r(v) = r$, $s(v) = s$, $f(v) = f$, $\phi(v) = \phi$, $\psi(v) = \psi$ in Eq. (1) and Eq. (2), then

$$\varepsilon u''(v) + pu'(v) + qu(v-\delta) + ru(v) + su(v+\eta) = f, \ 0 < v < 1,$$

with respect to the interval and boundary conditions $u(v) = \phi$, $-\delta \leq v \leq 0$ and $u(v) = \psi$, $1 \leq v \leq 1+\eta$ is given by

$$y(x) = C_1 e^{m_1(x)} + C_2 e^{m_2(x)} + \frac{f}{c},$$

where

$$C_1 = \frac{\left[e^{m_2}\left(f - c\phi\right) - f + \psi c\right]}{c\left(e^{m_1} - e^{m_2}\right)},$$

$$C_2 = \frac{\left[e^{m_1}\left(-f + c\phi\right) - f + \psi c\right]}{c\left(e^{m_1} - e^{m_2}\right)}.$$

$$m_1 = \frac{-\left(p - q\delta + s\eta\right) + \sqrt{\left(p - q\delta + s\eta\right)^2 - 4c\varepsilon}}{2\varepsilon}, m_2 = \frac{-\left(p - q\delta + s\eta\right) - \sqrt{\left(p - q\delta + s\eta\right)^2 - 4c\varepsilon}}{2\varepsilon},$$

with $c = q + r + s$.

Example 1. Consider a boundary value problem $\varepsilon u''(v) + u'(v) - u(v-\delta) + u(v) - u(v+\eta) = -1$, with boundary constraints $u(v) = 1;\ -\delta \le v \le 0,\ u(v) = 1,\ 1 \le v \le 1+\eta$.

Example 2. Consider a boundary value problem $\varepsilon u''(v) + 2.5u'(v) - 2exp(v)\,u(v-\delta) - u(v) - vu(v+\eta) = 1$, with boundary constraints $u(v) = 1;\ -\delta \le v \le 0,\ u(v) = 1,\ 1 \le v \le 1+\eta$.

Example 3. Consider a boundary value problem $\varepsilon u''(v) - \left(1 + exp\left(-v^2\right)\right)u'(v) - vu(v-\delta) - v^2u(v) - (1.5 - exp(-v))\,u(v+\eta) = 1$, with boundary constraints $u(v) = 1; -\delta \le v \le 0$, $u(v) = 1,\ 1 \le v \le 1+\eta$.

Example 4. Consider a boundary value problem $\varepsilon u''(v) - \left(1 + exp\left(v^2\right)\right)u'(v) - vu(v-\delta) + v^2u(v) - (1 - exp(-v))\,u(v+\eta) = 1$, with boundary constraints $u(v) = 1\ ;\ -\delta \le v \le 0,\ u(v) = -1$, $1 \le v \le 1+\eta$.

Table 1

The MAEs in Example 1 for various values of ε

$\varepsilon \downarrow \quad N \rightarrow$	2^3	2^4	2^5	2^6	2^7	2^8
Present method $\eta = \delta = 0.5\varepsilon$						
10^{-1}	2.405(-03)	6.367(-04)	1.615(-04)	4.053(-05)	1.014(-05)	2.553(-06)
	1.9174	1.9788	1.9948	1.9988	1.9898	
10^{-2}	9.363(-03)	4.724(-03)	1.765(-03)	5.110(-04)	1.331(-04)	3.365(-05)
	0.9870	1.4203	1.7882	1.9408	1.9834	
10^{-3}	9.438(-03)	5.180(-03)	2.724(-03)	1.397(-03)	6.805(-04)	2.679(-04)
	0.8655	0.9272	0.9633	1.0376	1.3449	
10^{-4}	9.441(-03)	5.183(-03)	2.725(-03)	1.398(-03)	7.087(-04)	3.568(-04)
	0.8651	0.9275	0.9628	0.9801	0.9900	
10^{-5}	9.442(-03)	5.183(-03)	2.725(-03)	1.398(-03)	7.088(-04)	3.568(-04)
	0.8334	0.9075	0.9516	0.9752	0.9874	
10^{-6}	9.442(-03)	5.183(-03)	2.725(-03)	1.398(-03)	7.088(-04)	3.568(-04)
	0.8653	0.9075	0.9516	0.9752	0.9874	
Results in [22]						
10^{-1}	3.658(-03)	9.595(-04)	2.409(-04)	6.759(-05)	1.776(-05)	1.232(-05)
10^{-2}	1.695(-02)	7.297(-03)	2.486(-03)	6.964(-04)	1.776(-04)	2.616(-05)
10^{-3}	2.0208(-02)	1.047(-02)	5.210(-03)	2.461(-03)	1.057(-03)	3.771(-04)
10^{-4}	2.052(-02)	1.079(-02)	5.520(-03)	2.769(-03)	1.363(-03)	6.539(-04)
10^{-5}	2.061(-02)	1.088(-02)	5.608(-03)	2.858(-03)	1.453(-03)	7.417(-04)
10^{-6}	1.951(-02)	9.783(-03)	4.513(-03)	1.762(-03)	3.577(-04)	3.729(-04)

Table 2

MAEs of Example 2 with various values of ε

$\varepsilon \downarrow \quad N \rightarrow$	10^1	10^2	10^3	10^4
Present method $\delta = 0.7\varepsilon, \eta = 0.5$				
10^{-1}	1.429(-02)	1.798(-04)	1.803(-06)	1.803(-08)
10^{-2}	2.692(-02)	1.852(-03)	2.111(-05)	2.111(-07)
10^{-3}	2.718(-02)	3.371(-03)	1.918(-04)	2.161(-06)
10^{-4}	2.721(-02)	3.375(-03)	3.461(-04)	1.925(-05)
Results in [22]				
10^{-1}	1.533(-02)	1.917(-04)	1.921(-06)	1.917(-08)
10^{-2}	2.817(-02)	1.865(-03)	2.024(-05)	2.026(-07)
10^{-3}	2.853(-02)	3.389(-03)	1.919(-04)	2.162(-06)
10^{-4}	2.857(-02)	3.395(-03)	3.463(-04)	1.925(-05)

Table 3

MAEs and rate of convergence of Example 2 with various values of ε

$\varepsilon \downarrow \quad N \rightarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
Present method $\eta = \delta = 0.5\varepsilon$						
2^{-3}	1.338(-03)	3.389(-04)	8.501(-05)	2.127(-05)	5.318(-06)	1.329(-06)
	1.9811	1.9951	1.9987	1.9988	2.000	
2^{-4}	2.827(-03)	7.339(-04)	1.852(-04)	4.642(-05)	1.161(-05)	2.904(-06)
	1.9456	1.9864	1.9962	1.9993	1.9992	
2^{-5}	5.401(-03)	1.511(-03)	3.898(-04)	9.822(-05)	2.460(-05)	6.154(-06)
	1.8377	1.9546	1.9886	1.9973	1.9990	
2^{-6}	8.369(-03)	2.842(-03)	7.859(-04)	2.019(-04)	5.084(-05)	1.273(-05)
	1.5581	1.8544	1.9607	1.9896	1.9977	
2^{-7}	9.770(-03)	4.381(-03)	1.461(-03)	4.014(-04)	1.029(-04)	2.590(-05)
	1.1570	1.5843	1.8638	1.9637	1.9902	
2^{-8}	9.930(-03)	5.106(-03)	2.244(-03)	7.411(-04)	2.029(-04)	5.200(-05)
	0.9596	1.1861	1.5983	1.8688	1.9641	
2^{-9}	9.946(-03)	5.183(-03)	2.613(-03)	1.136(-03)	3.733(-04)	1.020(-04)
	0.9191	0.9770	1.1965	1.6038	1.8703	
Results in [22]						
2^{-3}	1.378(-03)	3.486(-04)	8.742(-05)	2.187(-05)	5.469(-06)	1.367(-06)
2^{-4}	2.880(-03)	7.458(-04)	1.881(-04)	4.714(-05)	1.179(-05)	2.948(-06)
2^{-5}	5.477(-03)	1.526(-03)	3.930(-04)	9.902(-05)	2.480(-05)	6.204(-06)
2^{-6}	8.487(-03)	2.862(-03)	7.898(-04)	2.028(-04)	5.105(-05)	1.278(-05)
2^{-7}	9.922(-03)	4.413(-03)	1.466(-03)	4.024(-04)	1.031(-04)	2.596(-05)
2^{-8}	1.009(-02)	5.148(-03)	2.252(-03)	7.424(-04)	2.032(-04)	5.206(-05)
2^{-9}	1.011(-02)	5.228(-03)	2.624(-03)	1.138(-03)	3.736(-04)	1.021(-04)

Table 4

MAEs and rate of convergence of Example 3 with various values of ε

$\varepsilon\downarrow$ $N\rightarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
Present method $\eta=\delta=0.5\varepsilon$						
2^{-3}	3.025(-04)	7.465(-05)	1.899(-05)	4.693(-06)	1.183e-06	2.924e-07
	1.9943	1.9846	1.9940	2.0000	1.9998	
2^{-4}	6.723(-04)	1.674(-04)	4.258(-05)	1.034(-05)	2.609(-06)	6.524(-07)
	1.9979	1.9950	1.9962	1.9991	1.9996	
2^{-5}	1.524e-03	3.658e-04	9.013e-05	2.143e-05	5.612e-06	1.404e-06
	1.0934	1.9846	1.9200	1.9970	1.9989	
2^{-6}	2.230(-03)	7.967(-04)	3.222(-04)	1.852(-04)	1.174(-05)	2.940(-06)
	1.4849	1.3060	0.7988	0.979	1.9975	
2^{-7}	2.298(-03)	1.157(-03)	4.099(-04)	9.466(-05)	2.429(-05)	6.034e-06
	0.9899	1.4973	1.9933	1.9626	1.0984	
2^{-8}	2.294(-03)	1.181(-03)	5.905(-04)	2.085(-04)	4.798(-05)	1.232(-05)
	0.9578	1.0000	1.5046	1.9781	1.9584	
2^{-9}	2.298(-03)	1.177(-03)	5.998(-04)	2.985(-04)	1.059(-04)	2.506(-05)
	0.9652	0.9725	1.0060	1.5087	2.0000	
Results in [22]						
2^{-3}	8.434(-04)	2.112(-04)	5.284(-05)	1.321(-05)	3.303(-06)	8.260(-07)
2^{-4}	4.172(-03)	1.047(-03)	2.640(-04)	6.602(-05)	1.650(-05)	4.127(-06)
2^{-5}	1.858(-02)	4.743(-03)	1.190(-03)	2.980(-04)	7.452(-05)	1.864(-05)
2^{-6}	6.074(-02)	1.988(-02)	5.080(-03)	1.275(-03)	3.192(-04)	7.981(-05)
2^{-7}	1.111(-01)	6.451(-02)	2.061(-02)	5.270(-03)	1.323(-03)	3.311(-04)
2^{-8}	1.297(-01)	1.176(-01)	6.658(-02)	2.101(-02)	5.372(-03)	1.349(-03)
2^{-9}	1.310(-01)	1.372(-01)	1.212(-01)	6.766(-02)	2.122(-02)	5.425(-03)

Table 5

MAEs and rate of convergence of Example 4 with various values of ε

$\varepsilon\downarrow$ $N\rightarrow$	2^5	2^6	2^7	2^8	2^9
Present method $\eta=\delta=0.5\varepsilon$					
2^{-3}	1.826(-03)	4.086(-04)	1.002(-04)	2.543(-05)	6.246(-06)
	1.9982	1.9994	1.9989	2.0000	
2^{-4}	4.576(-03)	9.745(-04)	2.165(-04)	5.245(-05)	1.310(-05)
	1.9567	1.9988	1.9909	1.0099	
2^{-5}	9.233(-03)	2.431(-03)	4.959(-04)	1.104(-04)	2.678(-05)
	1.9359	1.9989	1.9846	1.9999	
2^{-6}	1.229(-02)	4.704(-03)	1.225(-03)	2.420(-04)	5.598(-05)
	1.3855	1.9411	2.0000	1.9994	
2^{-7}	1.275(-02)	6.203(-03)	2.374(-03)	6.166(-04)	1.264(-04)
	1.0394	1.3856	1.9430	1.9998	
2^{-8}	1.275(-02)	6.417(-03)	3.115(-03)	1.192(-03)	3.098(-04)
	0.8131	1.0426	1.3858	1.9439	
Results in [33]					
2^{-3}	8.354(-03)	2.013(-03)	4.986(-04)	1.249(-04)	3.121(-05)
2^{-4}	1.719(-02)	4.378(-03)	1.041(-03)	2.571(-04)	6.429(-05)
2^{-5}	2.517(-02)	8.889(-03)	2.238(-03)	5.290(-04)	1.303(-04)
2^{-6}	3.154(-02)	1.294(-02)	4.516(-03)	1.131(-03)	2.664(-04)
2^{-7}	4.478(-02)	1.622(-02)	6.559(-03)	2.276(-03)	5.686(-04)
2^{-8}	7.878(-02)	2.317(-02)	8.224(-03)	3.301(-03)	1.142(-03)

Table 6

MAE of Example 4 with $\varepsilon = 0.1$

$\delta \downarrow N \rightarrow$	2^3	2^5	2^7	2^9
Present method with $\eta = 0.5 * \varepsilon$				
0.00	3.889(-02)	2.576(-03)	1.305(-04)	8.065(-06)
0.05	3.850(-02)	2.527(-03)	1.286(-04)	7.948(-06)
0.09	3.818(-02)	2.489(-03)	1.272(-04)	7.853(-06)
Results in [33]				
0.00	9.109(-02)	1.112(-02)	6.382(-04)	4.004(-05)
0.05	9.047(-02)	1.095(-02)	6.306(-04)	3.950(-05)
0.09	8.996(-02)	1.082(-02)	6.244(-04)	3.906(-05)
$\eta \downarrow N \rightarrow$	2^3	2^5	2^7	2^9
Present Method $\delta = 0.5 * \varepsilon$				
0.00	3.835(-02)	2.502(-03)	1.277(-04)	7.888(-06)
0.05	3.850(-02)	2.527(-03)	1.286(-04)	7.948(-06)
0.09	3.862(-02)	2.548(-03)	1.294(-04)	7.995(-06)
Results in [33]				
0.00	9.604(-02)	1.116(-02)	6.458(-04)	3.924(-05)
0.05	9.621(-02)	1.124(-02)	6.494(-04)	3.952(-05)
0.09	9.634(-02)	1.131(-02)	6.522(-04)	3.970(-05)

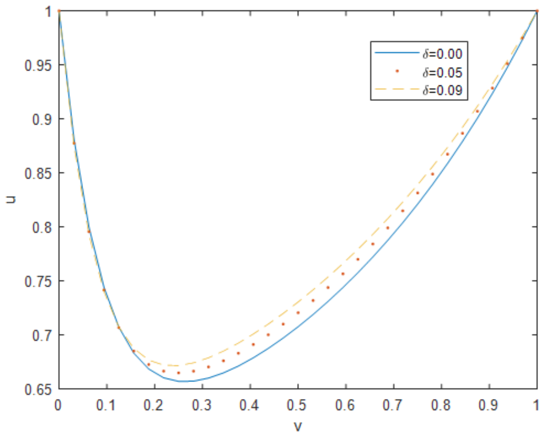

Figure 1. Numerical solution of Ex. 1 for various values of δ with N=2^5, $\varepsilon = 10^{-1}$ and $\eta = 0.05$

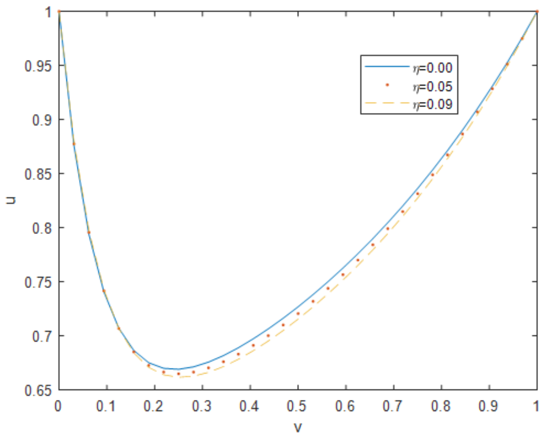

Figure 2. Numerical solution of Ex. 1 for various values of η with N=2^6, $\varepsilon = 10^{-2}$ and δ=0.05

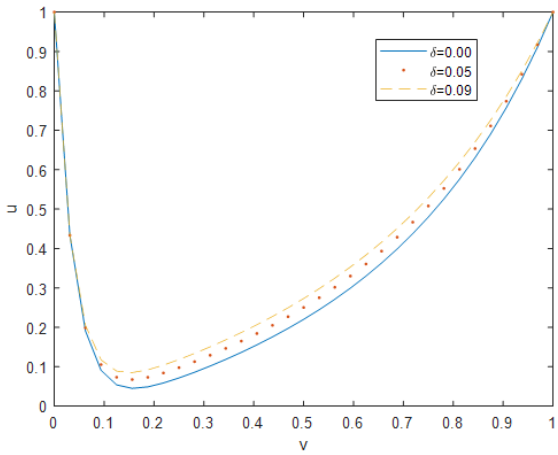

Figure 3. Numerical solution of Ex. 2 for various values of δ with N=2^5, $\varepsilon = 10^{-1}$ and η=0.05

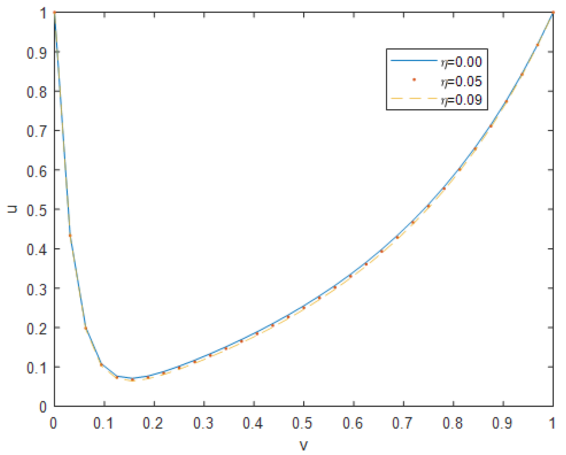

Figure 4. Numerical solution of Ex. 2 for various values of η with N=2^5, $\varepsilon = 10^{-1}$ and δ=0.05

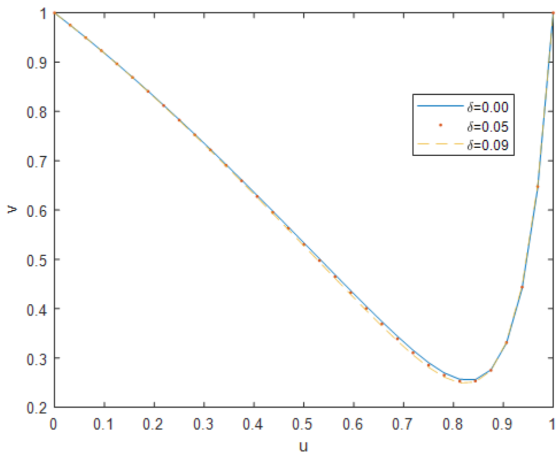

Figure 5. Numerical solution of Ex. 3 for various values of δ with N=2^5, $\varepsilon = 10^{-1}$ and η=0.05

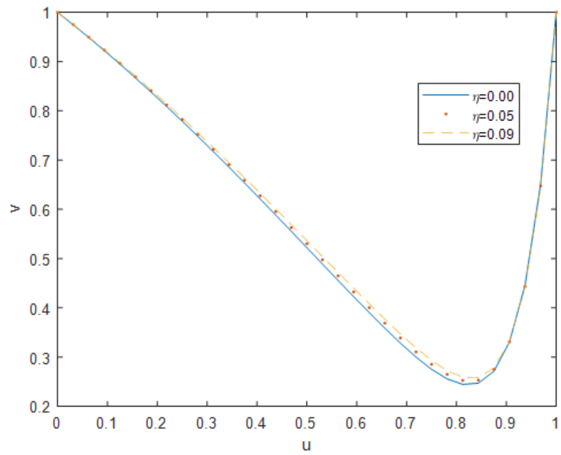

Figure 6. Numerical solution of Ex. 3 for various values of η with N=2^5, $\varepsilon = 10^{-1}$ and δ=0.05

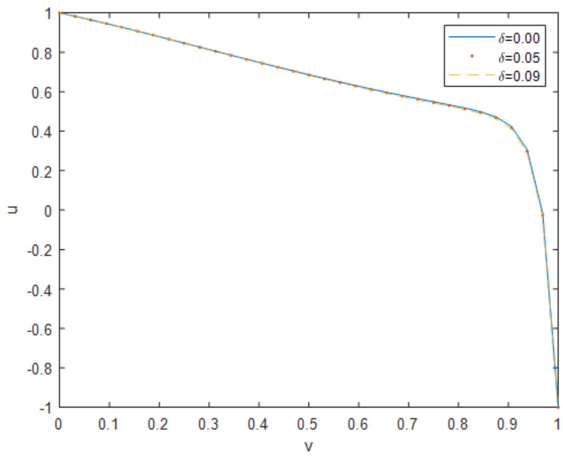

Figure 7. Numerical solution of Ex. 4 for various values of δ with N=2^5, $\varepsilon = 10^{-1}$ and η=0.05

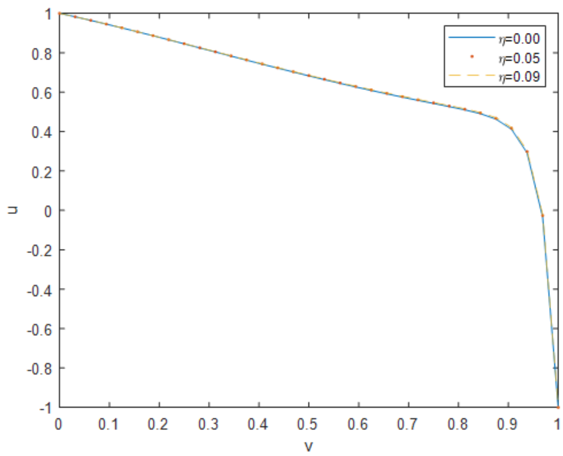

Figure 8. Numerical solution of Ex. 4 for various values of η with N=2^5, $\varepsilon = 10^{-1}$ and δ=0.05

Conclusion

For solving SPDDEs of second order with mixed shifts and boundary layers at the left or right end of the underlying interval using a non-polynomial cubic spline with fitting factor, a novel finite difference algorithm is recommended. To illustrate the accuracy and effectiveness of the approach, four example problems are tested for different values of N and with $\delta = \eta = 0.5\varepsilon$ and presented the numerical results in terms of maximum absolute errors and numerical rates of convergence. Using MATLAB, the MAEs in the solutions listed in comparison to the method given in [22] in Tables 1, 2, 3 and 4. Tables 5 and 6 give the MAEs in the solution of Example 4 to compare the method given in [33]. The detailed examination of the solution graphs plotted in Figs. 1, 2, 3, 4, 5, 6, 7 and 8 reveals that the mixed shifts have no significant impact on the boundary layer behaviour of the solution for problems with boundary layers at the left-end points of the given interval (Figs. 1, 2, 3, and 4), whereas these parameters affect the solution for problems with boundary layers at the right-end points of the given interval (Figs. 5, 6, 7, and 8). According to the results, the thickness of the layer increases as the delay parameter size increases and it decreases as the advance parameter size increases. The proposed method is simple and can be easily implemented on a computer.

Acknowledgments

We would like to express our sincere gratitude to all those who contributed to this research and its successful completion.

Author Contributions

Swarnakar Dornala selected the problem and analyzed it, and led the manuscript preparation. Kumar Ragula analyzed the problem and developed a method to solve it. Ganesh Kumar Vadla made a MATLAB programme to find the numerical solutions of the problem using the proposed method. BSL Soujanya G supervised the manuscript without spelling mistakes and grammatical errors. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Doolan, E.P., Miller, J.J.H., & Schilders, W.H.A. (1980). *Uniform Numerical Methods for Problems with Initial and Boundary Layers.* Boole Press, Dublin, Ireland.

2 Swamy, D.K., Phaneendra, K., & Reddy, Y.N. (2018). Accurate Numerical Methods for Singularly Perturbed Differential-Difference Equations with Mixed Shifts. *Khayyam Journal of Mathematics, 4*(2), 110–122. https://doi.org/10.22034/kjm.2018.57949

3 Priyadarshana, S., Sahu, S. R., & Mohapatra, J. (2022). Asymptotic and numerical methods for solving singularly perturbed differential difference equations with mixed shifts. *Iranian Journal of Numerical Analysis and Optimization, 1*(12), 55–72. https://doi.org/10.22067/ijnao.2021.70731.1038

4 Swamy, D.K., Phaneendra, K., & Reddy, Y.N. (2016). Solution of Singularly Perturbed Differential-Difference Equations with Mixed Shifts Using Galerkin Method with Exponential Fitting. *Chinese Journal of Mathematics, 2016*(10). https://doi.org/10.1155/2016/1935853

5 Adilaxmi, M., Bhargavi, D., & Phaneendra, K. (2019). Numerical Solution of Singularly Perturbed Differential-Difference Equations using Multiple Fitting Factors. *Communications in Mathematics and Applications, 10*(4), 681–691. https://doi.org/10.26713/cma.v10i4.1129

6 Debela, H.G., & Duressa, G.F. (2022). Robust Numerical Method for Singularly Perturbed Two-Parameter Differential-Difference Equations. *Applied Mathematics E-Notes, 22*, 200–209.

7 Debela, H.G., Solomon, B.K., & Negassa, A.D. (2020). Exponentially Fitted Numerical Method for Singularly Perturbed Differential-Difference Equations. *International Journal of Differential Equations, 2020*(13). https://doi.org/10.1155/2020/5768323

8 Khan, A., & Raza, A. (2022). Treatment of singularly perturbed differential equations with delay and shift using Haar wavelet collocation method. *Tamkang Journal of Mathematics, 53*(4), 303–322. https://doi.org/10.5556/j.tkjm.53.2022.3250

9 Das, P., & Natesan, S. (2012). Higher-order parameter uniform convergent schemes for robin type reaction-diffusion problems using adaptively generated grid. *International Journal of Computational Methods, 09*(04), 1250052. https://doi.org/10.1142/S0219876212500521

10 Shiromani, R., Shanthi, V., & Das, P. (2023). A higher order hybrid-numerical approximation for a class of singularly perturbed two-dimensional convection-diffusion elliptic problem with non-smooth convection and source terms. *Computers & Mathematics with Applications, 142*, 9–30. https://doi.org/10.1016/j.camwa.2023.04.004

11 Kumar, S., Das, P., & Kumar, K. (2024). Adaptive mesh based efficient approximations for Darcy scale precipitation- dissolution models in porous media. *International Journal for Numerical Methods in Fluids, 96*, 1415–1444. https://doi.org/10.1002/fld.5294

12 Kumar, S.K., Kumar, S., & Das, P. (2024). Second order a priori and a posteriori error estimation for integral boundary value problems of nonlinear singularly perturbed parameterized form. *Numerical Algorithms.* http://doi.org/10.1007/s11075-024-01918-5

13 Saini, S., Das, P., & Kumar, S. (2024). Parameter uniform higher order numerical treatment for singularly perturbed Robin type parabolic reaction diffusion multiple scale problems with large delay in time. *Applied Numerical Mathematics, 196*, 1–21. https://doi.org/10.1016/j.apnum.2023.10.003

14 Al-Said, E.A. (1997). Quadratic spline solution of two-point boundary value problems. *Journal of Natural Geometry, 12*, 125–134.

15 Kumar, S., Aishwarya, R., & Das, P. (2024). Impact of mixed boundary conditions and non-smooth data on layer originated non-premixed combustion problems: Higher order convergence analysis. *Studies in Applied Mathematics, 153*(4), 12763. https://doi.org/10.1111/sapm.12763

16 Saini, S., Das, P., & Kumar, S. (2023). Computational cost reduction for coupled system of multiple scale reaction diffusion problems with mixed type boundary conditions having boundary layers. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser A-Ma, 117*, 66. https://doi.org/10.1007/s13398-023-01397-8

17 Kumar, S., & Das, P. (2025). A uniformly convergent analysis for multiple scale parabolic singularly perturbed convection-diffusion coupled systems: Optimal accuracy with less computational time. *Applied Numerical Mathematics, 207*, 534–557. https://doi.org/10.1016/j.apnum.2024.09.020

18 Kumar, K., Chakravarthy, P.P., Das, P., & Ramos, H. (2021). A graded mesh refinement approach for boundary layer originated singularly perturbed time-delayed parabolic convection diffusion problems. *Mathematical Methods in the Applied Sciences, 44*(16), 12332–12350. https://doi.org/10.1002/mma.7358

19 Das, P. (2019). An a posteriori-based convergence analysis for a nonlinear singularly perturbed

system of delay differential equations on an adaptive mesh. *Numerical Algorithms, 81*, 465–487. https://doi.org/10.1007/s11075-018-0557-4

20 El'sgol'ts, L.E., & Norkin, S.B. (1973). *Introduction to the Theory and Application of Differential Equations with Deviating Arguments.* Academic Press.

21 O'Malley, R. E. (1974). *Introduction to Singular Perturbations.* Academic Press, New York.

22 Ranjan, R., & Prasad, H.S. (2021). A novel approach for the numerical approximation to the solution of singularly perturbed differential-difference equations with small shifts. *J. Appl. Math. Comput. 65*, 403–427. https://doi.org/10.1007/s12190-020-01397-6

23 Srivastava, H.M., Nain, A.K., Vats, R.K., & Das, P. (2023). A theoretical study of the fractional-order p-Laplacian nonlinear Hadamard type turbulent flow models having the Ulam-Hyers stability. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A–Mat., 117*, 160. https://doi.org/10.1007/s13398-023-01488-6

24 Das, P., & Rana, S. (2021). Theoretical prospects of fractional order weakly singular Volterra Integro differential equations and their approximations with convergence analysis. *Mathematical Methods in the Applied Sciences, 44*(11), 9419–9440. https://doi.org/10.1002/mma.7369

25 Das, P., Rana, S., & Ramos, H. (2022). On the approximate solutions of a class of fractional order nonlinear Volterra integro-differential initial value problems and boundary value problems of first kind and their convergence analysis. *Journal of Computational and Applied Mathematics, 404*, 113–116. https://doi.org/10.1016/j.cam.2020.113116

26 Das, P., Rana, S., & Ramos, H. (2019). A perturbation-based approach for solving fractional-order Volterra–Fredholm integro differential equations and its convergence analysis. *International Journal of Computer Mathematics, 97*(10), 1994–2014. https://doi.org/10.1080/00207160.2019.1673892

27 Das, P. (2015). Comparison of a priori and a posteriori meshes for singularly perturbed nonlinear parameterized problems. *Journal of Computational and Applied Mathematics, 290*, 16–25. https://doi.org/10.1016/j.cam.2015.04.034

28 Das, P. (2018). A higher order difference method for singularly perturbed parabolic partial differential equations. *Journal of Difference Equations and Applications, 24*(3), 452–477. https://doi.org/10.1080/10236198.2017.1420792

29 Varga, R.S. (1962). *Matrix Iterative Analysis.* Prentice-Hall, Englewood Cliffs, NJ.

30 Santra, S., Mohapatra, J., Das, P., & Choudhuri, D. (2023). Higher order approximations for fractional order integro-parabolic partial differential equations on an adaptive mesh with error analysis. *Computers & Mathematics with Applications, 150*, 87–101. https://doi.org/10.1016/j.camwa.2023.09.008

31 Choudhary, R., Singh, S., Das, P., & Kumar, D. (2024). A higher order stable numerical approximation for time-fractional non-linear Kuramoto-Sivashinsky equation based on quintic $\mathfrak{B}$-spline. *Mathematical Methods in the Applied Sciences, 47*(15), 11953–11975. https://doi.org/10.1002/mma.9778

32 Shakti, D., Mohapatra, J., Das, P., & Aguiar, J.V. (2022). A moving mesh refinement based optimal accurate uniformly convergent computational method for a parabolic system of boundary layer originated reaction-diffusion problems with arbitrary small diffusion terms. *Journal of Computational and Applied Mathematics, 404*, 113–167. https://doi.org/10.1016/j.cam.2020.113167

33 Ayalew, M., Kiltu, G.G, & Duressa, G.F. (2021). Fitted Numerical Scheme for Second Order Singularly Perturbed Differential-Difference Equations with Mixed Shifts. *Abstract and Applied Analysis, 2021*, Article ID 4573847. https://doi.org/10.1155/2021/4573847

Author Information[*]

Dornala Swarnakar (*corresponding author*) — Doctor of Philosophy in Mathematics, Assistant Professor, Department of Mathematics, VNR Vignana Jyothi Institute of Engineering & Technology, Hyderabad, India; e-mail: *swarnakarreddy0@gmail.com*; https://orcid.org/0000-0002-5655-7508

Ragula Kumar — Doctor of Philosophy in Mathematics, Assistant Professor, Department of Mathematics, Rajiv Gandhi University of Knowledge and Technology, Basar, India; e-mail: *rksm39@gmail.com*; https://orcid.org/0000-0003-3671-3590

Vadla Ganesh Kumar — Doctor of Philosophy in Mathematics, Assistant Professor, Department of Mathematics, VNR Vignana Jyothi Institute of Engineering & Technology, Hyderabad, India; *ganeshkumar68@gmail.com*; https://orcid.org/0000-0001-9821-3159

G.B.S.L. Soujanya — Doctor of Philosophy in Mathematics, Assistant Professor, Department of Mathematics, Kakatiya University, Warangal, India; e-mail: *gbslsoujanya@mail.com*; https://orcid.org/0000-0002-4522-7896

[*]The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 187–198

https://doi.org/10.31489/2025M1/187-198 *Research article*

Approximation of a singular boundary value problem for a linear differential equation

R. Uteshova[1,*], Y. Kokotova[2]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan*
(E-mail: r.uteshova@math.kz, kokotovae@mail.ru)

This paper addresses the approximation of a bounded (on the entire real axis) solution of a linear ordinary differential equation, where the matrix approaches zero as $t \to \mp\infty$ and the right-hand side is bounded with a weight. We construct regular two-point boundary value problems to approximate the original problem, assuming the matrix and the right-hand side, both weighted, are constant in the limit. An approximation estimate is provided. The relationship between the well-posedness of the singular boundary value problem and the well-posedness of an approximating regular problem is established.

Keywords: linear differential equation, bounded solution, singular boundary value problem, approximation, well-posedness, parameterization method.

2020 Mathematics Subject Classification: 34B05, 65L10.

Introduction

In many fields of applied mathematics, systems of ordinary differential equations that involve singularities or are defined over an infinite interval frequently occur. Numerous studies (see, for example, [1–8]) have explored the existence of bounded solutions for these types of problems and the approximation of these solutions.

In the present paper, we consider the differential equation

$$\frac{dx}{dt} = A(t)x + f(t), \quad x \in \mathbb{R}^n, \quad t \in (-\infty, \infty), \tag{1}$$

where the matrix function $A(t)$ is continuous on $\mathbb{R}$ and $\|A(t)\| := \max\limits_{j} \sum\limits_{k=1}^{n} |a_{jk}(t)| \leq \alpha(t)$. We assume that $\alpha(t) > 0$ is a continuous function such that

$$\int\limits_{-\infty}^{0} \alpha(t)dt = \infty, \quad \lim_{t\to-\infty} \alpha(t) = 0, \quad \int\limits_{0}^{\infty} \alpha(t)dt = \infty, \quad \lim_{t\to\infty} \alpha(t) = 0.$$

As is known (see, e.g. [9]), the above assumption implies that equation (1) has a bounded solution not for any function $f(t)$ continuous and bounded on the whole axis. For this reason, in [10] the existence and uniqueness of a bounded solution of equation (1) was investigated under the assumption that $f(t)$ is continuous and bounded with a weight.

We will use the following notation:

---

*Corresponding author. *E-mail: r.uteshova@math.kz*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant no. BR20281002).

Received: 16 July 2024; *Accepted:* 28 October 2024.

$\widetilde{C}(\mathbb{R},\mathbb{R}^n)$ is the space of continuous and bounded functions $x:\mathbb{R}\to\mathbb{R}^n$ equipped with the norm $\|x\|_1=\sup\limits_{t\in\mathbb{R}}\|x(t)\|$;

$\widetilde{C}_{1/\alpha}(\mathbb{R},\mathbb{R}^n)$ is the space of functions $f:\mathbb{R}\to\mathbb{R}^n$ that are continuous and bounded with the weight $1/\alpha(t)$, i.e. $f(t)/\alpha(t)\in\widetilde{C}(\mathbb{R},\mathbb{R}^n)$, equipped with the norm $\|f\|_\alpha=\sup\limits_{t\in\mathbb{R}}\|f(t)/\alpha(t)\|$.

PROBLEM 1 is the problem of finding a bounded on the whole axis solution of equation (1) with $f(t)\in\widetilde{C}_{1/\alpha}(\mathbb{R},\mathbb{R}^n)$.

We say that Problem 1 is well-posed with constant K if it has a unique solution $x(t)\in\widetilde{C}(\mathbb{R},\mathbb{R}^n)$ for any $f(t)\in\widetilde{C}_\alpha(\mathbb{R},\mathbb{R}^n)$, and

$$\|x\|_1\le K\|f\|_\alpha,$$

where K is a constant independent of $f(t)$.

In [10], Problem 1 was studied by the parameterization method [11] with nonuniform partition $\mathbb{R}=\bigcup\limits_{s=-\infty}^{\infty}[t_{s-1},t_s)$. For a fixed number $\theta>0$, the partition points $t_s\in\mathbb{R}$, $s\in\mathbb{Z}$, are determined as

$$t_0=0,\qquad \int\limits_{t_{s-1}}^{t_s}\alpha(t)dt=\theta.$$

Let $\widetilde{h}(\theta)$ denote a bilaterally infinite sequence of partition step sizes $h_s(\theta)=t_s-t_{s-1}$, $s\in\mathbb{Z}$, i.e. $\widetilde{h}(\theta)=(\ldots,h_s(\theta),h_{s+1}(\theta),\ldots)$. We will use the following spaces:

m_n is the space of bilaterally infinite sequences of $\lambda_s\in\mathbb{R}^n$ equipped with the norm

$$\|\lambda\|_2=\|(\ldots,\lambda_s,\lambda_{s+1},\ldots)\|_2=\sup_s\|\lambda_s\|,\quad s\in\mathbb{Z};$$

$L(m_n)$ is the space of bounded linear operators mapping m_n to itself, equipped with the induced norm;

$m_n(\widetilde{h}(\theta))$ is the space of bounded bilaterally infinite sequences of functions $x_s(t)$, each of which is continuous and bounded on its domain $[t_{s-1},t_s)$, equipped with the norm

$$\|x[t]\|_3=\|(\ldots,x_s(t),x_{s+1}(t),\ldots)\|_3=\sup_s\sup_{t\in[t_{s-1},t_s)}\|x_s(t)\|,\quad s\in\mathbb{Z}.$$

Well-posedness criteria for Problem 1 were obtained in [10] in terms of a bilaterally infinite block-diagonal matrix $Q_{\nu,\widetilde{h}(\theta)}:m_n\to m_n$ of the form

$$Q_{\nu,\widetilde{h}(\theta)}=\left\|\begin{array}{ccccccc}\ldots&\ldots&\ldots&\ldots&\ldots&\ldots&\ldots\\ \ldots&0&I+D_{\nu,s}(h_s(\theta))&-I&0&0&\ldots\\ \ldots&0&0&I+D_{\nu,s+1}(h_{s+1}(\theta))&-I&0&\ldots\\ \ldots&\ldots&\ldots&\ldots&\ldots&\ldots&\ldots\end{array}\right\|,$$

where $D_{\nu,s}(h_s(\theta))=\int\limits_{t_{s-1}}^{t_s}A(\tau_1)d\tau_1+\ldots+\int\limits_{t_{s-1}}^{t_s}A(\tau_1)\ldots\int\limits_{t_{s-1}}^{\tau_{\nu-1}}A(\tau_\nu)d\tau_\nu\ldots d\tau_1$, $s\in\mathbb{Z}$, and I is the identity matrix of order n.

1 Statement of the problem of approximation. A criterion for the well-posedness of Problem 1

In this paper we consider the issue of approximation of Problem 1 by regular two-point boundary value problems. For this purpose, we pose the following problem.

Problem 2. For a given $\varepsilon > 0$ find numbers T_1, $T_2 > 0$, real $n \times n$ matrices B, C, and vector $d \in R^n$, such that a solution $x_{T_1,T_2}(t)$ of the two-point boundary value problem

$$\frac{dx}{dt} = A(t)x + f(t), \qquad t \in (-T_1, T_2), \tag{2}$$

$$Bx(-T_1) + Cx(T_2) = d \tag{3}$$

satisfies the inequality

$$\max_{t\in[-T_1,T_2]} ||x_{T_1,T_2}(t) - x^*(t)|| < \varepsilon,$$

where $x^*(t)$ is a solution of Problem 1.

Problem 2 is considered under the following assumptions.

Assumption 1. $\lim\limits_{t\to\mp\infty} \frac{A(t)}{\alpha(t)} = A_{(\mp)}$, and Re $\xi_j^{\mp} \neq 0$, where $\xi_j^{\mp}$ are the eigenvalues of the matrices $A_{(\mp)}$, $j = 1, 2, \ldots, n$.

Assumption 2. $\lim\limits_{t\to\mp\infty} \frac{f(t)}{\alpha(t)} = f_{(\mp)}$.

We introduce the following functions:

$$\delta_1^-(T) := \sup_{t\in(-\infty,-T]} \left\| \frac{A(t)}{\alpha(t)} - A_{(-)} \right\|, \quad \delta_1^+(T) := \sup_{t\in[T,\infty)} \left\| \frac{A(t)}{\alpha(t)} - A_{(+)} \right\|,$$

$$\delta_2^-(T) := \sup_{t\in(-\infty,-T]} \left\| \frac{f(t)}{\alpha(t)} - f_{(-)} \right\|, \quad \delta_2^+(T) := \sup_{t\in[T,\infty)} \left\| \frac{f(t)}{\alpha(t)} - f_{(+)} \right\|.$$

Obviously, $\delta_r^{\mp}(T) \to 0$ as $T \to \infty$, $r = 1, 2$.

There exist nonsingular real $n \times n$ matrices $S_{(\mp)}$ that transform the matrices $A_{(\mp)}$ into the real Jordan canonical form [12]

$$\widetilde{A}_{(\mp)} = S_{(\mp)} A_{(\mp)} S_{(\mp)}^{-1} = \left\| \begin{matrix} A_{11}^{\mp} & 0 \\ 0 & A_{22}^{\mp} \end{matrix} \right\|, \tag{4}$$

where $A_{11}^{\mp}$ and $A_{22}^{\mp}$ consist of generalized Jordan blocks associated with the eigenvalues of $A_{(\mp)}$ that have negative and positive real parts, the numbers of which we denote by $n_1^{\mp}$ and $n_2^{\mp}$, respectively. We form the $n \times n$ matrices

$$P_1 = \left\| \begin{matrix} I_{n_1} & 0 \\ 0 & 0 \end{matrix} \right\|, \qquad P_2 = \left\| \begin{matrix} 0 & 0 \\ 0 & I_{n_2} \end{matrix} \right\|,$$

where I_{n_r} are the identity matrices of orders n_r, $r = 1, 2$.

The following statement establishes the interrelation between the well-posedness of Problem 1 and that of a two-point boundary value problem.

Theorem 1. Under Assumption 1, Problem 1 is well-posed if and only if:

(i) $n_1^- = n_1^+ = n_1$ and $n_2^- = n_2^+ = n_2$;

(ii) there exist T_0^1, $T_0^2 > 0$ such that for any $T_1 > T_0^1$, $T_2 > T_0^2$ the boundary value problem (2), (3) with $B = -P_1 S_{(-)}$ and $C = P_2 S_{(+)}$, is well-posed with a constant K_1 independent of T_1, T_2.

Proof. Necessity. Let Assumption 1 be fulfilled and let Problem 1 be well-posed. Then, by Theorem 3 [10], there exist $\theta_0 > 0$ such that the matrix $Q_{1,\widetilde{h}(\theta)}$ has an inverse for all $\theta \in (0, \theta_0]$, and the estimate $\|Q_{1,\widetilde{h}(\theta)}^{-1}\|_{L(m_n)} \leq \gamma/\theta$ holds, where γ is a constant independent of $\widetilde{h}(\theta)$. For a fixed $\theta > 0$ we choose T_1 and T_2, so that $t_{-N_1} = -T_1$ and $t_{N_2} = T_2$, and construct the matrix $Q_{1,\widetilde{h}(\theta)}$. In this matrix we then replace $A(t)$ by $\alpha(t)A_{(-)}$ in the block rows numbered $-N_1, -N_1 - 1, \ldots,$ and by $\alpha(t)A_{(+)}$ in the block rows numbered $N_2, N_2+1,\ldots$, and denote the resulting matrix by Q_{θ,T_1,T_2}. Assumption 1 implies that $\|Q_{1,\widetilde{h}(\theta)} - Q_{\theta,T_1,T_2}\|_{L(m_n)} \le \max\{\delta_1^-(T_1), \delta_1^+(T_2 - h_N(\theta))\}\theta$. Hence, by the theorem on small perturbations of boundedly invertible linear operators, if we choose T_0^1, T_0^2 satisfying $\gamma\max\{\delta_1^-(T_0^1), \delta_1^+(T_0^2 - h_N(\theta))\} \le 1/2$, we obtain that the matrix $Q_{\theta,T_1,T_2}: m_n \to m_n$ has an inverse for all $T_1 \ge T_0^1$ and $T_2 \ge T_0^2$, and the estimate

$$\|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \le \frac{\gamma_{T_1,T_2}}{\theta} \le \frac{2\gamma}{\theta}$$

holds. Here $\gamma_{T_1,T_2} = \dfrac{\gamma}{1-\gamma\max\{\delta_1^-(T_1), \delta_1^+(T_2)\}} \to \gamma$ as $T_1 \to \infty$, $T_2 \to \infty$.

We form a bilaterally infinite matrix $D = \mathrm{diag}(d_{ss})$, where $d_{ss} = S_{(-)}$ for $s = 0, -1, -2, \ldots$, and $d_{ss} = S_{(+)}$ for $s = 1, 2, \ldots$. The matrix $\widetilde{Q}_{\theta,T_1,T_2} = DQ_{\theta,T_1,T_2}D^{-1}$ has a bounded inverse and

$$\|\widetilde{Q}^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \le \|D^{-1}\|_{L(m_n)}\|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)}\|D\|_{L(m_n)} \le \zeta_1\gamma_{T_1,T_2}\zeta_2/\theta.$$

Here $\zeta_1 = \|D^{-1}\|_{L(m_n)} = \max\left(\|S^{-1}_{(-)}\|, \|S^{-1}_{(+)}\|\right)$ and $\zeta_2 = \|D\|_{L(m_n)} = \max\left(\|S_{(-)}\|, \|S_{(+)}\|\right)$. In the matrix $\widetilde{Q}_{\theta,T_1,T_2}$ the block rows numbered $s: s \le -N_1$, $s \ge N_2$, are of the form

$$\left\| \begin{array}{ccccccc} \ldots & 0 & I + \left\| \begin{array}{cc} A_{11}^{\mp} & 0 \\ 0 & A_{22}^{\mp} \end{array} \right\| \theta & -I & 0 & \ldots \end{array} \right\|.$$

Rearranging the blocks in $\widetilde{Q}_{\theta,T_1,T_2}$, we obtain the matrix

$$M_{\theta,T_1,T_2} = \left\| \begin{array}{ccccc} M_{11}(\theta) & 0 & 0 & 0 & 0 \\ 0 & M_{22}(\theta) & M_{23}(\theta) & 0 & 0 \\ M_{31}(\theta) & 0 & M_{33}(\theta) & 0 & M_{35}(\theta) \\ 0 & 0 & M_{43}(\theta) & M_{44}(\theta) & 0 \\ 0 & 0 & 0 & 0 & M_{55}(\theta) \end{array} \right\|.$$

The one-sided infinite matrices $M_{kk}(\theta)$, $k = 1, 2, 4, 5$, are of the form

$$M_{11}(\theta) = \left\| \begin{array}{ccccc} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & I_{n_1^-} + A_{11}^-\theta & -I_{n_1^-} & 0 \\ \ldots & 0 & 0 & I_{n_1^-} + A_{11}^-\theta & -I_{n_1^-} \end{array} \right\|,$$

$$M_{22}(\theta) = \left\| \begin{array}{cccccc} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & I_{n_2^-} + A_{22}^-\theta & -I_{n_2^-} & 0 \\ \ldots & 0 & 0 & I_{n_2^-} + A_{22}^-\theta & -I_{n_2^-} \\ \ldots & 0 & 0 & 0 & I_{n_2^-} + A_{22}^-\theta \end{array} \right\|,$$

$$M_{44}(\theta) = \left\| \begin{array}{ccccc} -I_{n_1^+} & 0 & 0 & 0 & \ldots \\ I_{n_1^+} + A_{11}^+\theta & -I_{n_1^+} & 0 & 0 & \ldots \\ 0 & I_{n_1^+} + A_{11}^+\theta & -I_{n_1^+} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \right\|,$$

$$M_{55}(\theta) = \left\| \begin{array}{ccccc} I_{n_2^+} + A_{22}^+\theta & -I_{n_2^+} & 0 & 0 & \ldots \\ 0 & I_{n_2^+} + A_{22}^+\theta & -I_{n_2^+} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \right\|.$$

The matrix $M_{33}(\theta)$ of dimension $[(N_1+N_2-1)n+n_1^-+n_2^+]\times(N_1+N_2)n$ is of the form

$$M_{33}(\theta)=\begin{Vmatrix} -P_1^{(-)} & 0 & 0 & \dots & 0 & 0 & 0\\ I+\widetilde{A}_{-N_1+1}(\theta) & -I & 0 & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & I+\widetilde{A}_{N_2-1}(\theta) & -I & \\ 0 & 0 & 0 & \dots & 0 & 0 & P_2^{(+)}(I+\widetilde{A}_{(+)}\theta)\end{Vmatrix},$$

where $P_1^{(-)}=(I_{n_1^-},0)$ is a matrix of dimension, $n_1^-\times n$, $P_2^{(+)}=(0,I_{n_2^+})$ is a matrix of dimension $n_2^+\times n$,

$$\widetilde{A}_p(\theta)=\begin{cases} S_{(-)}\int\limits_{t_{p-1}}^{t_p}A(t)dtS_{(-)}^{-1}, & p=-N_1+1,-N_1+2,\dots,1,0,\\ S_{(+)}\int\limits_{t_{p-1}}^{t_p}A(t)dtS_{(+)}^{-1}, & p=1,2,\dots,N_2-1.\end{cases}$$

In the block row of $M_{33}(\theta)$ corresponding to $p=0$, the term $-I$ is replaced by $-S_{(-)}S_{(+)}^{-1}$.

The off-diagonal nonzero blocks of the matrix M_{θ,T_1,T_2} satisfy the relations

$$\|M_{31}(\theta)\|=\|I_{n_1^-}+A_{11}^-\theta\|,\quad \|M_{23}(\theta)\|=1,\quad \|M_{43}(\theta)\|=\|I_{n_1^+}+A_{11}^+\theta\|,\quad \|M_{35}(\theta)\|=1.$$

Due to the invertibility of $\widetilde{Q}_{\theta,T_1,T_2}$, the matrix M_{θ,T_1,T_2} is also invertible, and its inverse satisfies the estimate

$$\|M_{\theta,T_1,T_2}^{-1}\|_{L(m_n)}=\|\widetilde{Q}_{\theta,T_1,T_2}^{-1}\|_{L(m_n)}\le\frac{\zeta_1\gamma_{T_1,T_2}\zeta_2}{\theta}=\frac{\widetilde{\gamma}_{T_1,T_2}}{\theta}.$$

Following the proof scheme in [13], we establish the invertibility of the matrices $M_{kk}(\theta)$, $k=\overline{1,5}$, and the estimates

$$\|[M_{kk}(\theta)]^{-1}\|\le\left[\max_{r=1,2}\left(\|S_{r,\mp}\|,\|S_{r,\mp}^{-1}\|\right)\right]^2\frac{2}{\xi\theta}=\frac{\beta}{\theta},\quad k=1,2,4,5, \tag{5}$$

$$\|[M_{33}(\theta)]^{-1}\|\le\frac{\widetilde{\gamma}_{T_1,T_2}}{\theta}. \tag{6}$$

Here $\xi=\min\left\{|\mathrm{Re}\ \xi_j^{\mp}|,j=1,2,\dots,n\right\}$ and $S_{r,\mp}$ $(r=1,2)$ are nonsingular complex matrices of order $n_r^{\mp}$ reducing $A_{rr}^{\mp}$ to Jordan form with the eigenvalues on the diagonal and $\xi/4$ or zeros on the superdiagonal.

Since the matrix $M_{33}(\theta)$ of dimension $[(N_1+N_2-1)n+n_1^-+n_2^+]\times(N_1+N_2)n$ is invertible, it follows that $n_1^-+n_2^+=n$. In view of the structure of the matrices $\widetilde{A}_{(\mp)}$, we also have $n_1^-+n_2^-=n_1^++n_2^+=n$. Hence, $n_1^-=n_1^+=n_1$, $n_2^-=n_2^+=n_2$.

By rearranging of terms in the matrix $M_{33}(\theta)$, we obtain the invertible matrix

$$N_{33}(\theta)=\begin{Vmatrix} -P_1 & 0 & 0 & \dots & 0 & 0 & P_2(I+\widetilde{A}_{(+)}\theta)\\ I+\widetilde{A}_{-N_1+1}(\theta) & -I & 0 & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 0 & I+\widetilde{A}_{N_2-1}(\theta) & -I\end{Vmatrix},$$

inverse of which, by (6), satisfies the estimate

$$\|[N_{33}(\theta)]^{-1}\|=\|[M_{33}(\theta)]^{-1}\|\le\frac{\widetilde{\gamma}_{T_1,T_2}}{\theta}\le\frac{2\widetilde{\gamma}}{\theta}.$$

Let D_{N_1,N_2} denote the block diagonal matrix consisting of blocks D numbered $s = -N_1, -N_1+1, \dots, N_2-2, N_2-1$. By premultiplying each but the first block row of $N_{33}(\theta)D_{N_1,N_2}$ with $S_{(-)}^{-1}$ or $S_{(+)}^{-1}$, respectively, we obtain the matrix $V_1(\theta)$. Its inverse satisfies the estimate

$$\|[V_1(\theta)]^{-1}\| \le \max(1,\zeta_1)\zeta_2\|[N_{33}(\theta)]^{-1}\| \le \frac{2\widetilde{\gamma}\max(1,\zeta_1)\zeta_2}{\theta} = \frac{\gamma_1}{\theta},$$

where γ_1 is independent of T_1 and T_2. Hence, by following the proof scheme of Theorem 3 in [13] and considering the specifics of our partitioning, it can be shown that for all $T_1 \ge T_0^1$ and $T_2 \ge T_0^2$, the two-point boundary value problem (2), (3) with $B = -P_1S_{(-)}$ and $C = P_2S_{(+)}$ is well-posed with constant K_1 independent of T_1 and T_2.

Sufficiency. Let conditions (i) and (ii) be fulfilled and let $\widetilde{Q}_1(\theta)$ denote the matrix $N_{33}(\theta)$ with the first block row scaled by $\theta > 0$. Then, adapting Theorem 3 in [13] to our partitioning, we obtain that for any $\varepsilon > 0$ there exists $\theta_1 = \theta_1(\varepsilon) > 0$ such that the matrix $\widetilde{Q}_1(\theta)$ is invertible for all $\theta \in (0,\theta_1]$, and

$$\|[\widetilde{Q}_1(\theta)]^{-1}\| \le \frac{(1+\varepsilon)\zeta_1\zeta_2K_1}{\theta} \le \frac{(1+\varepsilon)K_1}{\theta}. \tag{7}$$

The invertibility of $\widetilde{Q}_1(\theta)$ implies that of $M_{33}(\theta)$. Taking into account the bounded invertibility of the matrices $M_{kk}(\theta)$, $k = 1,2,4,5$, and the structure of the matrix M_{θ,T_1,T_2}, we obtain that the last one has a bounded inverse. Let us show that

$$\|M_{\theta,T_1,T_2}^{-1}\|_{L(m_n)} \le \frac{\widetilde{\gamma}}{\theta}, \tag{8}$$

where $\widetilde{\gamma}$ is constant independent of θ. To this end, we consider the equation

$$M_{\theta,T_1,T_2}\mu = b, \qquad \mu, b \in m_n, \tag{9}$$

which can be rewritten as the system

$$M_{11}(\theta)\mu^{(1)} = b^{(1)},$$

$$M_{22}(\theta)\mu^{(2)} + M_{23}(\theta)\mu^{(3)} = b^{(2)}, \tag{10}$$

$$M_{31}(\theta)\mu^{(1)} + M_{33}(\theta)\mu^{(3)} + M_{35}(\theta)\mu^{(5)} = b^{(3)}, \tag{11}$$

$$M_{43}(\theta)\mu^{(3)} + M_{44}(\theta)\mu^{(4)} = b^{(4)}, \tag{12}$$

$$M_{55}(\theta)\mu^{(5)} = b^{(5)}.$$

Here $\mu = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \mu^{(4)}, \mu^{(5)})$ and $b = (b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}, b^{(5)})$.

The bounded invertibility of $M_{11}(\theta)$, $M_{55}(\theta)$ and estimate (5) imply the existence of $\mu^{(1)} = [M_{11}(\theta)]^{-1}b^{(1)}$ and $\mu^{(5)} = [M_{55}(\theta)]^{-1}b^{(5)}$, as well as the estimates

$$\|\mu^{(1)}\| \le \frac{\beta}{\theta}\|b^{(1)}\|, \qquad \|\mu^{(5)}\| \le \frac{\beta}{\theta}\|b^{(5)}\|. \tag{13}$$

Let us now multiply by θ the first (from the bottom) block row in equation (10), the first (of dimension n_1) and the last (of dimension n_2) block rows in (11), and the first (from the top) block row in (12). We denote the matrices transformed in this way by $M_{22,\theta}$, $M_{23,\theta}$, $M_{31,\theta}$, $M_{33,\theta}$, $M_{35,\theta}$, $M_{43,\theta}$, $M_{44,\theta}$, the vectors by $b_\theta^{(2)}, b_\theta^{(3)}, b_\theta^{(4)}$ and the equations by $(10)'$, $(11)'$ and $(12)'$. Substituting the obtained sequences $\mu^{(1)}$ and $\mu^{(5)}$ into (11)$'$, we determine $\mu^{(3)}$. Taking into account $\|M_{33,\theta}^{-1}\| = \|[\widetilde{Q}_1(\theta)]^{-1}\|$ and estimate (7), we obtain

$$\|\mu^{(3)}\| = \|M_{33,\theta}^{-1}\left\{b_\theta^{(3)} - M_{31,\theta}[M_{11}(\theta)]^{-1}b^{(1)} - M_{35,\theta}[M_{55}(\theta)]^{-1}b^{(5)}\right\}\| \leq$$

$$\leq \frac{(1+\varepsilon)\widetilde{K}_1}{\theta}\left[\|b_\theta^{(3)}\| + (1+\zeta\theta)\beta\|b^{(1)}\| + \beta\|b^{(5)}\|\right] \leq \frac{(1+\varepsilon)\widetilde{K}_1}{\theta}[1+(2+\zeta\theta)\beta]\max_{k=1,3,5}\|b^{(k)}\|, \tag{14}$$

where $\zeta = [\max(\zeta_1, \zeta_2)]^2$. The one-sided infinite matrices $M_{22,\theta}$ and $M_{44,\theta}$ have bounded inverses, and

$$\|M_{22,\theta}^{-1}\| \leq \beta\frac{\xi}{2}\max\left(\frac{2}{\xi}, 1\right)\frac{1}{\theta}, \quad \|M_{44,\theta}^{-1}\| \leq \beta\frac{\xi}{2}\max\left(\frac{2}{\xi}, 1\right)\frac{1}{\theta}.$$

Substituting $\mu^{(3)}$ into (10) and (12), we determine $\mu^{(2)}$ and $\mu^{(4)}$ and obtain the estimates

$$\begin{aligned}\|\mu^{(2)}\| &\leq \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)(\|b_\theta^{(2)}\| + \theta\|\mu^{(3)}\|) \leq \\ &\leq \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)\{1 + (1+\varepsilon)\widetilde{K}_1[1+(2+\zeta\theta)\beta]\}\max_{k=1,2,3,5}\|b^{(k)}\|,\end{aligned} \tag{15}$$

$$\begin{aligned}\|\mu^{(4)}\| &\leq \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)(\|b_\theta^{(4)}\| + (1+\zeta\theta)\theta\|\mu^{(3)}\|) \leq \\ &\leq \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)\{1 + (1+\varepsilon)\widetilde{K}_1(1+\zeta\theta)[1+(2+\zeta\theta)\beta]\}\max_{k=2,3,4,5}\|b^{(k)}\|.\end{aligned} \tag{16}$$

Thus, for any $b \in m_n$ equation (9) has a unique solution $\mu \in m_n$, and, by (13)–(16), the estimate

$$\|\mu\|_2 \leq \frac{K}{\theta}\|b\|_2$$

holds, where

$$K = \max\{\beta,\ (1+\varepsilon)\widetilde{K}_1[1+(2+\zeta\theta)\beta],\ (\beta\xi/2)\max(2/\xi, 1)[1+(1+\varepsilon)\widetilde{K}_1](1+2\beta+\zeta\beta\theta)]\}.$$

Hence, for any $\varepsilon_1 > 0$ choosing $\theta_2 = \theta_2(\varepsilon_1) > 0$ small enough, we obtain that estimate (8) with $\widetilde{\gamma} = \widetilde{K} + \varepsilon_1 = (\beta\xi/2)\max(2/\xi, 1)[1 + \widetilde{K}_1(1+2\beta)] + \varepsilon_1$ is valid for all $\theta \in (0, \theta_2]$. Under condition (ii) the constant K_1 does not depend of T_1 and T_2, as well as the constant $\widetilde{\gamma} = \widetilde{K} + \varepsilon_1$. Thus, taking into account the estimates

$$\|\widetilde{Q}_{1,\theta} - \widetilde{Q}_{\theta,T_1,T_2}\|_{L(m_n)} \leq \delta_1(T_1, T_2 - h_N(\theta_2))\theta, \quad \|\widetilde{Q}_{\theta,T_1,T_2}^{-1}\|_{L(m_n)} = \|M_{\theta,T_1,T_2}^{-1}\|_{L(m_n)} \leq \frac{\widetilde{K}+\varepsilon_1}{\theta},$$

and choosing T_0^1 and T_0^2 such that $(\widetilde{K}+\varepsilon_1)\zeta\delta_1(T_0^1, T_0^2 - h_N(\theta_2)) \leq 1/2$, we obtain that $\widetilde{Q}_{1,\theta}$ is invertible and $\|\widetilde{Q}_{1,\theta}^{-1}\|_{L(m_n)} \leq 2\widetilde{\gamma}/\theta$. It follows then that

$$\|\widetilde{Q}_{1,\theta}\|_{L(m_n)} \leq \|D^{-1}\|_{L(m_n)}\|\widetilde{Q}_{1,\theta}^{-1}\|_{L(m_n)}\|D\|_{L(m_n)} \leq 2\zeta\widetilde{\gamma}/\theta.$$

Thus, by Theorem 3 in [10], Problem 1 is well-posed for $\nu = 1$. This finishes the proof.

Application of Theorem 1 allows one to obtain effective well-posedness criteria for Problem 1. But condition (ii) somewhat narrows the scope of application, since it becomes necessary to check the well-posedness of the two-point boundary value problem for all T_1 and T_2. However, if we repeat the proof of the sufficiency part of Theorem 1 setting $T_1^0 = T_0^1$, $T_2^0 = T_0^2$ and using the introduced numbers β, ξ, ζ, and then pass in the right part of the inequality to the limit, we establish the following statement.

Theorem 2. Let Assumption 1 hold and the following conditions be met:

(i) $n_1^- = n_1^+ = n_1$ and $n_2^- = n_2^+ = n_2$;

(ii) there exist $T_1^0, T_2^0 > 0$ such that the boundary value problem

$$\frac{dx}{dt} = A(t)x + f(t), \quad t \in (-T_1^0, T_2^0), \tag{17}$$

$$-P_1 S_{(-)} x(-T_1^0) + P_2 S_{(+)} x(T_2^0) = d \tag{18}$$

is well-posed with a constant K_1 satisfying the inequality $\widetilde{K}\zeta\delta_1(T_1^0, T_2^0) < 1$ with

$$\widetilde{K} = (\beta\xi/2)\max(2/\xi, 1)[1 + (1 + 2\beta)K_1\zeta].$$

Then Problem 1 is well-posed with the constant $K = \widetilde{K}\zeta/[1 - \widetilde{K}\zeta\delta_1(T_1^0, T_2^0)]$.

2 An approximating regular boundary value problem and the estimate for the approximation

The following theorem provides an approximating two-point boundary value problem and the estimate for the approximation.

Theorem 3. Under Assumptions 1 and 2, let Problem 1 be well-posed with constant K. Then for all $T_1 \geq T_0^1$ and $T_2 \geq T_0^2$, where $T_0^1, T_0^2 > 0$ are some constants determined by $K\max(\delta_1^-(T_0^1), \delta_1^+(T_0^2)) < 1$, the boundary value problem

$$\frac{dx}{dt} = A(t)x + f(t), \qquad t \in (-T_1, T_2), \tag{19}$$

$$P_1 S_{(-)} A_{(-)} x(-T_1) + P_2 S_{(+)} A_{(+)} x(T_2) = -P_1 S_{(-)} f_{(-)} - P_2 S_{(+)} f_{(+)} \tag{20}$$

has a unique solution $x_{T_1,T_2}(t)$, and

$$\max_{t\in[-T_1,T_2]} \|x_{T_1,T_2}(t) - x^*(t)\| \leq$$
$$\leq \frac{K}{1 - K\max(\delta_1^-(T_1), \delta_1^+(T_2))}[K\|f\|_\alpha \max(\delta_1^-(T_1), \delta_1^+(T_2)) + \max(\delta_2^-(T_1), \delta_2^+(T_2))], \tag{21}$$

where $x^*(t)$ is the solution of Problem 1.

Proof. We choose $\theta > 0$ and, applying the parameterization method, obtain that the solution $(\lambda^*, u^*(t)) \in m_n \times m_n(\widetilde{h}(\theta))$ of the boundary value problem with parameter (2)–(5) in [10] satisfies the equation

$$\left[I + \int\limits_{t_{s-1}}^{t_s} A(t)dt\right]\lambda_s^* + \lambda_{s+1}^* = -\int\limits_{t_{s-1}}^{t_s} f(t)dt - \int\limits_{t_{s-1}}^{t_s} A(t)u_s^*(t)dt, \quad s \in Z. \tag{22}$$

By Theorem 3 in [10], for any $\varepsilon > 0$ there exists $\overline{\theta} = \overline{\theta}(\varepsilon)$, such that the estimate $\|Q^{-1}_{1,\widetilde{h}(\theta)}\|_{L(m_n)} \leq \frac{(1+\varepsilon)K}{\theta}$ holds for all $\theta \in (0, \overline{\theta}]$, and, in addition,

$$\left\|\int\limits_{t_{s-1}}^{t_s} A(t)u_s^*(t)dt\right\| \leq c\theta^2, \quad s \in Z,$$

where $c = [1 + (1+\varepsilon)K]e^{\overline{\theta}}\|f\|_\alpha$, then the last term in (22) can be neglected for θ small enough. Let us separate the system (22) into three parts. Replacing $A(t), f(t)$ by $\alpha(t)A_{(-)}, \alpha(t)f_{(-)}$ for $s : s \le N_1$, and by $\alpha(t)A_{(+)}, \alpha(t)f_{(+)}$ for $s : s \ge N_2$, we obtain

$$(I + A_{(-)}\theta)\lambda_{r_1} - \lambda_{r_1+1} = -f_{(-)}\theta, \qquad r_1 = -N_1, -N_1 - 1, \dots, \tag{23}$$

$$\left[I + \int\limits_{t_{r_2-1}}^{t_{r_2}} A(t)dt\right]\lambda_{r_2} + \lambda_{r_2+1} = -\int\limits_{t_{r_2-1}}^{t_{r_2}} f(t)dt, \qquad r_2 = -N_1 + 1, \dots, N_2 - 1, \tag{24}$$

$$(I + A_{(+)}\theta)\lambda_{r_3} - \lambda_{r_3+1} = -f_{(+)}\theta, \qquad r_3 = N_2, N_2 + 1, \dots \tag{25}$$

We rewrite this system in the form

$$Q_{\theta,T_1,T_2}\lambda = -F_{\theta,T_1,T_2}. \tag{26}$$

If we choose $\varepsilon > 0$ to satisfy the inequality, then, by the theorem on small perturbations of boundedly invertible operators, it follows that the matrix Q_{θ,T_1,T_2} is invertible, and its inverse satisfies the estimate

$$\|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \le \frac{(1+\varepsilon)K}{[1 - (1+\varepsilon)K\max(\delta_1^-(T_1), \delta_1^+(T_2))]\theta}. \tag{27}$$

Hence, by Assumptions 1 and 2, we obtain the estimate for the difference between λ^* and the solution λ_{T_1,T_2} of equation (26):

$$\begin{aligned}
\|\lambda_{T_1,T_2} - \lambda^*\|_2 &\le \|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)}\|F_{\theta,T_1,T_2} - F_1(\widetilde{h}(\theta)) + [F_1(\widetilde{h}(\theta)) + Q_{\theta,T_1,T_2}\lambda^*]\|_2 = \\
&= \|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)}\|F_{\theta,T_1,T_2} - F_1(\widetilde{h}(\theta)) + [Q_{1,\widetilde{h}(\theta))}\lambda^* + G_1(u^*, \widetilde{h}(\theta)) - Q_{\theta,T_1,T_2}\lambda^*]\|_2 \le \\
&\le \frac{(1+\varepsilon)K[\max(\delta_2^-(T_1), \delta_2^+(T_2)) + K\|f\|_\alpha \max(\delta_1^-(T_1), \delta_1^+(T_2)) + c\theta]}{1 - (1+\varepsilon)K\max(\delta_1^-(T_1), \delta_1^+(T_2))}.
\end{aligned} \tag{28}$$

The components of λ_{T_1,T_2} numbered with $s : s \le N_1$ and $s \ge N_2$ satisfy equations (23) and (25), respectively. Hence, the corresponding components of the vector $\mu_{T_1,T_2} = D\lambda_{T_1,T_2}$ solve the equations

$$(I + \widetilde{A}_{(-)}\theta)\mu_{r_1} - \mu_{r_1+1} = -S_{(-)}f_{(-)}\theta, \qquad r_1 = -N_1, -N_1 - 1, \dots,$$

$$(I + \widetilde{A}_{(+)}\theta)\mu_{r_3} - \mu_{r_3+1} = -S_{(+)}f_{(+)}\theta, \qquad r_3 = N_2, N_2 + 1, \dots$$

Then, taking into account the decomposibility of the matrices $\widetilde{A}_{(-)}$ and $\widetilde{A}_{(+)}$, we obtain that $P_1^{(-)}\mu_{r_1}$ and $P_2^{(+)}\mu_{r_3}$ satisfy the equations

$$(I_{n_1^-} + A_{11}^-\theta)P_1^{(-)}\mu_{r_1} - P_1^{(-)}\mu_{r_1+1} = -P_1^{(-)}S_{(-)}f_{(-)}\theta, \tag{29}$$

$$(I_{n_2^+} + A_{22}^+\theta)P_2^{(+)}\mu_{r_3} - P_2^{(+)}\mu_{r_3+1} = -P_2^{(+)}S_{(+)}f_{(+)}\theta. \tag{30}$$

In the proof of Theorem 1 it was shown that the matrices $M_{11}(\theta)$ and $M_{55}(\theta)$ have bounded inverses. Thus, the one-sided infinite systems (29) and (30) have the unique solutions

$$P_1^{(-)}\mu_{-N_1+1} = P_1^{(-)}\mu_{-N_1+2} = \ldots = -[A_{11}^-]^{-1}P_1^{(-)}S_{(-)}f_{(-)},$$

$$P_2^{(+)}\mu_{N_2} = P_2^{(+)}\mu_{N_2+1} = \ldots = -[A_{22}^+]^{-1}P_2^{(+)}S_{(+)}f_{(+)}.$$

Returning to the variable λ, we obtain

$$A_{11}^{-}P_1^{(-)}S_{(-)}\lambda_{-N_1+1} = -P_1^{(-)}S_{(-)}f_{(-)}, \quad A_{22}^{+}P_2^{(+)}S_{(+)}\lambda_{N_2} = -P_2^{(+)}S_{(+)}f_{(+)}.$$

Then, in view of (4), we have

$$\begin{aligned} A_{11}^{-}P_1^{(-)}S_{(-)}\lambda_{-N_1+1} =& P_1^{(-)}\widetilde{A}_{(-)}S_{(-)}\lambda_{-N_1+1} = P_1^{(-)}S_{(-)}A_{(-)}S_{(-)}^{-1}S_{(-)}\lambda_{-N_1+1} \\ =& P_1^{(-)}S_{(-)}A_{(-)}\lambda_{-N_1+1} = -P_1^{(-)}S_{(-)}f_{(-)}, \\ P_2^{(+)}S_{(+)}A_{(+)}\lambda_{N_2} =& -P_2^{(+)}S_{(+)}f_{(+)}. \end{aligned}$$

These equations together with (24) constitute a closed system in parameters λ_{-N_1+1}, λ_{-N_1+2}, $\ldots$, λ_{N_2-1}, λ_{N_2}. If estimate (27) holds, the boundary value problem (17), (18) is well-posed for all $T_1 \geq T_0^1$, $T_2 \geq T_0^2$. Taking into account that (18) multiplied by $\left\| \begin{matrix} -A_{11}^{-} & 0 \\ 0 & A_{22}^{+} \end{matrix} \right\|$ yields the left-hand side of the boundary condition (20), we obtain that problem (19), (20) is well-posed for all $T_1 \geq T_0^1$, $T_2 \geq T_0^2$.

Let x_{T_1,T_2} be a solution of problem (19), (20), and let $[\lambda_{T_1,T_2}]_{N_1,N_2}$ be the vector composed of those components of $\lambda_{T_1,T_2} \in m_n$ that are numbered $s = -N_1+1, -N_1+2, \ldots, N_2-1, N_2$. Since

$$\max_{s} \sup_{t\in[t_{s-1},t_s)} \|x_{T_1,T_2} - [\lambda_{T_1,T_2}]_{N_1,N_2}\| \leq c_1\theta,$$

where c_1 is a constant independent of θ, we obtain, in view of (28), the following estimate:

$$\max_{t\in[-T_1,T_2]} \|x_{T_1,T_2}(t) - x^*(t)\| \leq \|[\lambda_{T_1,T_2}]_{N_1,N_2} - [\lambda^*]_{N_1,N_2}\| + (c+c_1)\theta \leq$$

$$\leq \frac{(1+\varepsilon)K[K\|f\|_\alpha \max(\delta_1^-(T_1),\delta_1^+(T_2)) + \max(\delta_2^-(T_1),\delta_2^+(T_2)) + c\theta]}{1-(1+\varepsilon)K\max(\delta_1^-(T_1),\delta_1^+(T_2))} + (c+c_1)\theta.$$

Passing to the limit as $\theta \to 0$, we obtain (21). Theorem 3 is proved.

Conclusion

By approximating Problem 1 with a two-point boundary value problem and utilizing well-known results, we developed an approximate method for finding the bounded solution. The form of matrices P_1 and P_2 indicates that the approximating problem involves separated boundary conditions. Theorem 2 allows one to establish the well-posedness of the singular boundary value problem (Problem 1) using the well-posedness constant K_1 of the two-point boundary value problem, the eigenvalues $\xi_j^{\mp}$ of the limit matrices $A_{\mp}$, and the nonsingular matrices $S_{(\mp)}$. This approach provides a robust framework for addressing similar singular problems.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant no. BR20281002).

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Fazio, R. (2002). A survey on free boundary identification of the truncated boundary in numerical BVPs on infinite intervals. *Journal of Computational and Applied Mathematics, 140*(1-2), 331–344. https://doi.org/10.1016/S0377-0427(01)00526-X

2 Ronto, M., & Samoilenko, A.M. (2000). *Numerical-Analytic Methods in the Theory of Boundary-Value Problems.* World Scientific. https://doi.org/10.1142/3962

3 Mitropolsky, Yu.A., Samoilenko, A.M., & Kulik, V.L. (2002). *Dichotomies and Stability in Nonautonomous Linear Systems.* Taylor & Francis Ltd, United Kingdom. https://doi.org/10.1201/9781482264890

4 Dzhumabaev, D.S. & Uteshova, R.Ye. (2018). Weighted limit solution of a nonlinear ordinary differential equation at a singular point and its property. *Ukrainian Mathematical Journal, 69*(12), 1997–2004. https://doi.org/10.1007/s11253-018-1483-2

5 Muratbekov, M.B., & Bayandiyev, Ye.N. (2022). Existence and smoothness of solutions of a singular differential equation of hyperbolic type. *Bulletin of the Karaganda University. Mathematics Series,* 3(107), 98–104. https://doi.org/10.31489/2022M3/98-104

6 Yesbayev, A.N., & Ospanov, M.N. (2021). The solvability conditions for the second order nonlinear differential equation with unbounded coefficients in $L_2(R)$. *Bulletin of the Karaganda University. Mathematics Series,* 1(101), 104–110. https://doi.org/10.31489/2021M1/104-110

7 Uteshova, R.Ye., & Kokotova, Ye.V. (2022). On bounded solutions of linear systems of differential equations with unbounded coefficients. *Bulletin of the Karaganda University. Mathematics Series,* 4(108), 107–116. https://doi.org/10.31489/2022M4/107-116

8 Assanova, A.T., & Uteshova, R.E. (2021). A singular boundary value problem for evolution equations of hyperbolic type. *Chaos, Solitons and Fractals, 143*, article number 110517. https://doi.org/10.1016/j.chaos.2020.110517

9 Daleckii, Ju.L., & Krein, M.G. (1974). *Stability of solutions of differential equations in Banach space.* American Mathematical Society.

10 Uteshova, R.E., & Kokotova, Y.V. (2023). On the well-posedness of a singular problem for linear differential equations. *Lobachevskii Journal of Mathematics, 44*(7), 2961–2969. https://doi.org/10.1134/S1995080223070430

11 Dzhumabaev, D.S. (1989). Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. *USSR Computational Mathematics and Mathematical Physics, 29*(1), 34–46. https://doi.org/10.1016/0041-5553(89)90038-4

12 Horn, R.A., & Johnson, C.R. (2013). *Matrix Analysis.* 2nd Edition, Cambridge University Press.

13 Dzhumabaev, D.S. (1990). Approximation of the bounded solution of an ordinary linear differential equation by solutions of two-point boundary-value problems. *USSR Computational Mathematics and Mathematical Physics, 30*(2), 34–45. https://doi.org/10.1016/0041-5553(90)90074-3

Author Information[*]

Roza Uteshova (*corresponding author*) — Candidate of sciences in physics and mathematics, Associate Professor, Institute of Mathematics and Mathematical Modeling, 125 Pushkin Street, Almaty, 050010, Kazakhstan; e-mail: *r.uteshova@math.kz*; https://orcid.org/0000-0002-8809-9310

Yelena Kokotova — Candidate of sciences in physics and mathematics, K. Zhubanov Aktobe Regional University, 34 Moldagulova Avenue, Aktobe, 030000, Kazakhstan; e-mail: *kokotovae@mail.ru*; https://orcid.org/0009-0007-9625-3334

[*]The author's name is presented in the order: First, Middle and Last Names.

Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 199–210

https://doi.org/10.31489/2025M1/199-210 *Research article*

Normal Jonsson theories and their Kaiser classes

A.R. Yeshkeyev, M.T. Kassymetova*, N.M. Mussina

Institute of Applied Mathematics, Karaganda Buketov University, Karaganda, Kazakhstan
(E-mail: aibat.kz@gmail.com, mairushaasd@mail.ru, mussinanazerke@gmail.com)

We present results concerning new notion connected with the study of Jonsson theories. The new notion is a Kaiser class of models for arbitrary Jonsson theories. All results are obtained within the framework of the normality of the considered Jonsson theory. Additionally, we describe the properties of lattices formed by perfect fragments of a fixed Jonsson theory and their relationship with the #-companion of these fragments. The results we obtained are the model-theoretic properties of the #-companion of a normal perfect Jonsson fragment. Furthermore, we establish necessary and sufficient conditions for a normal Jonsson theory to be perfect, expressed in terms of the lattices of existential formulas.

Keywords: Jonsson theory, semantic model, Jonsson set, almost Jonsson set, normality.

2020 Mathematics Subject Classification: 03C35, 03C48, 03C52, 03C65.

Introduction

This article investigates the model-theoretic properties of certain subsets within the semantic model of a fixed normal Jonsson theory. In particular, it considers regular almost Jonsson sets as such subsets. Previously, in [1] introduced the notions of normality for Jonsson theories and the concept of an almost Jonsson set. The notion of regularity is a natural requirement for definable sets, which are Jonsson sets. The formula that defines the Jonsson set for a more convenient study must satisfy certain well-established properties in a model theoretical sense. In this case, the axiomatic formulation of the regularity property is considered as a set defined by a formula that has Morley rank. Moreover, all this is determined within the framework of the study of Jonsson theories regarding the Morley rank, previously it was defined in the work [2]. In addition, it should be noted that the main results of this article are related to previous results from the works [1, 3, 4]. Since the Jonsson theories are not, generally speaking, complete theories, in the general case we do not have the opportunity to consider the Lindenbaum–Tarski Boolean algebra of formulas and its corresponding Stone space of types. Therefore, just as in [3, 4] we consider the lattice of existential formulas and the corresponding existential types for which the results are obtained in the framework of the study of almost Jonsson sets. The most important difference and innovation in this work from the previous studies in [3, 4] is the application of double factorization to the class of cosemanticness under consideration. The main novelty in this article is the transition from fragments obtained by closing Jonsson sets to fragments obtained by closing almost Jonsson sets. In the future, the class of models whose Kaiser hull forms the Jonsson theory will be called the Kaiser class of models of the considered Jonsson theory. Thus, almost Jonsson sets distinguish a special class of models among the class of all models of the considered Jonsson theory.

*Corresponding author. *E-mail: mairushaasd@mail.ru*

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23489523).

Received: 28 August 2024; *Accepted:* 15 December 2024.

1 Necessary information related to the study of the Jonsson theories

In order to understand the content and nuances of our next consideration we have to recall the main definitions and statements which are connected with definitions from [1]. Throughout this article, we will deal with a countable first-order language and all the considered theories will be countable, respectively.

The central concept of this article is the notion of a Jonsson theory.

Definition 1. [5; 80] A theory T is defined as a Jonsson theory if it satisfies the following conditions:
(i) T has infinite models;
(ii) T is an inductive theory (such that, $\forall\exists$-axiomatizable);
(iii) T has the joint embedding property (JEP);
(iv) T has the amalgam property (AP).

The technique reflecting its essence and application in the study of various concepts related to the Jonsson theories is described in the following works: [6–19].

Remark 1. We will always work within the framework of a pregeometry [20], which is defined by the closure operator cl on the set of all subsets of the semantic model of the given Jonsson theory.

The most important type of considered models in the framework of studying Jonsson theories is a class of its existentially closed models. Let us recall this important definition.

Definition 2. [5; 105] Let M be a structure and N be an extension of M. We say that M is existentially closed in N if for any tuple $\bar{a}$ from M and any quantifier-free formula $f(\bar{x}, \bar{y})$ in the language of M, the following holds: if $N \vdash (\exists \bar{y}) f(\bar{a}, \bar{y})$, then $M \vdash (\exists \bar{y}) f(\bar{a}, \bar{y})$.

One of the syntactic invariants of a Jonsson theory is its center.

Definition 3. [20] The center T^* of a Jonsson theory T is the elementary theory of its semantic model C, i.e., $T^* = Th(C)$.

Definition 4. [5; 156] Let T be an arbitrary theory of the language L. We say that T' is a model companion of T, if the following conditions hold:
(i) $T'_\forall = T_\forall$ (i.e. T and T' are mutually model-consistent, meaning any model of T can be embedded into a model of T' and vice versa),
(ii) T' is model complete.

The concept of a model companion is well-known and useful in the study of Jonsson theories. The following notion generalizes this concept.

Definition 5. [5; 158] Let T be an arbitrary theory. A #-companion $T^\#$ of T is a theory in the same signature that satisfies the following conditions:
(i) $(T^\#)_\forall = T_\forall$;
(ii) if $T_\forall = T'_\forall$, then $T^\# = (T')^\#$;
(iii) $T_{\forall\exists} \subseteq T^\#$.

The concept of model completeness is closely connected to the notion of mutual model consistence for the theory under consideration. It is evident that the model companion is a $T^\#$-companion of theory T.

Remark 2. When we say that $T'_\forall = T_\forall$, we mean precisely that:
(i) any model of theory T is a substructure of a model of T';
(ii) any model of theory T' is a substructure of a model of T.

It is clear that T' is a model companion of T if and only if T' is a model companion of $T_\forall$.

Proposition 1. [5; 159] Let T be an arbitrary theory.

(i) T has a model companion if and only if the class of existentially closed models of $T_\forall$ is an elementary class.

(ii) If a model companion of T exists, it is unique and is corresponds to the theory of existentially closed models of $T_\forall$.

In addition to the model companion, other companions are well-known in the study of Jonsson theories, such as the forcing companion, the existentially closed companion, and the Kaiser hull. Moreover, the Kaiser hull is closely connected to the companion that characterizes the class of existentially closed models of the theory being studied, as well as to the companion that defines the class of generic models when studying the forcing companion of the given theory.

It is well-known that the natural interpretations of the companion $T^\#$ include T^*, T^f, T^M, and T^l, where:

(i) T^* is the center of the Jonsson theory T;

(ii) T^f is the forcing companion of the Jonsson theory T;

(iii) T^M is the model companion of the theory T;

(iv) $T^e = Th(E_T)$, where E_T denotes the class of existentially closed models of the theory T.

It is important to note that if E_T represents the class of T-existentially closed models of an inductive theory T, then E_T is always non-empty [20].

A. Robinson introduced the concepts of finite forcing and the forcing #-companion in Model Theory [21]. In [21], it is demonstrated that a theory in a countable language that satisfies the Joint Embedding Property (JEP) has a forcing #-companion that is complete.

The following theorem establishes that any Jonsson theory T always has a forcing companion, which is a complete theory.

Theorem 1. [5; 162]

(i) $T^\#$ is a #-companion of T.

(ii) $T^\#$ is complete if and only if T satisfies JEP.

Consequently, the simultaneous existence of all interpretations of the #-companion is closely related to the existence of a model companion.

In the study of Jonsson theories, the class of perfect Jonsson theories holds a significant role.

Definition 6. [20] A Jonsson theory T is said to be perfect if every its semantic model is an ω^+-saturated model of T^*.

It turns out that the semantic model of the Jonsson theory under consideration is existentially closed.

Lemma 1. [20] The semantic model C_T of a Jonsson theory T is T-existentially closed.

The following theorem provides a criterion for perfectness:

Theorem 2. (Criterion of Perfectness). [20] Let T be an arbitrary Jonsson theory. Then the following conditions are equivalent:

(i) the theory T is perfect;

(ii) the theory T^* is a model companion of theory T.

In the case of a completely Jonsson theory, the concept of a companion coincides with the center of the theory for any type of companion.

Corollary 1. For a perfect Jonsson theory T, the #-companion coincides with the center of T.

Before studying the concepts of fragments of the theory under consideration, it is important to note that the definability of a set in models means that the set is a solution to some formula in the first-order language of the given signature.

Definition 7. [4] A set A is called A-definable if it can be defined by some formula in the language L. A set A is said to be Jonsson in the theory T if it satisfies the following conditions:

(i) A is a definable subset of M;

(ii) $cl(A)$ is a carrier of some existentially closed submodel M, where $cl(A)$ is the set of all A-definable elements $a \in A$ such that, for some formula $\varphi(x) \in L(A)$, it holds that $\varphi(M) = \{a\}$.

Let A be a Jonsson set in the theory T, and let N be an existentially closed submodel of a semantic model M of the considered Jonsson theory T, where $cl(A) = N$.

We denote by $Th_{\forall\exists}(N)$ the sets of all $\forall\exists$-sentences in the language that are true in the model N.

The following concept is simply a Jonsson theory obtained by closing a definable subset that is Jonsson.

Definition 8. [4] The fragment $Fr(X)$ of the Jonsson set X is a Jonsson theory obtained as a $\forall\exists$-sentences true in the model N which is a closure of this set, i.e. $Fr(X) = Th_{\forall\exists}(N)$.

Furthermore, we will use the denotation N^0 for the fragment $Fr(X)$, where N is $cl(X)$ and sometimes we can say instead of N^0 the Kaiser hull of the model N.

The following definition gives us important notion of Jonsson spectrum, the details of this one can extract from [1, 10, 17].

Definition 9. [22] Let K be a class of L-structures. The Jonsson spectrum of K, denoted $JSp(K)$, is defined as the set of all theories satisfying the following conditions:

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } \forall A \in K\ A \models T\}.$$

The following definition is an essential generalization of the concept of a Jonsson set.

Definition 10. [1] A set X is called an almost Jonsson if it satisfies the following conditions:

1) X is a definable subset of C_T, where C_T is the semantic model of theory T;

2) $cl(X) = M \in Mod(T)$.

Furthermore, $Th_{\forall\exists}(M) = M^0 = Fr(X)$ and $M^0 \in JSp(C_T)$, where $cl(X)$ is the definable closure of the set X in the frame of given above pregeometry [20].

This concept can be illustrated by an example of an arbitrary abelian group, which turns out to be the closure of some existential formula defining an almost Jonsson set of this abelian group.

The concept of normality for a Jonsson theory is defined for a class of theories where any fragment of their semantic models belongs to the Jonsson spectrum of the given Jonsson theory's semantic model.

The following definition identifies specific subsets within the semantic model of the given Jonsson theory.

Definition 11. [1] The Jonsson theory T is said to be normal if, for any $X \subseteq C_T$ such that $cl(X) = M \in Mod(T)$, $Fr(X) = M^0 \in JSp(C_T)$ and $C_{M^0} \preceq_{\exists_1} C_T$, where C_{M^0} is a semantic model of M^0.

An example of a normal theory is the universal theory of all unars. This theory is characterized by the fact that it has an empty list of axioms and is Jonsson.

Lemma 2. Let T be a normal Jonsson theory, K be a subset of E_T. Then $Th_{\forall\exists}(K)$ is a normal Jonsson theory.

Proof. Let $K \subseteq E_T$, T be a normal Jonsson theory. Since T is a normal Jonsson theory, it follows that T has JEP, then according to Theorem 1 $\forall A, B \in E_T \Rightarrow T^0(A) = T^0(B)$.

Theorem 3. [23; 353] Let L be a first-order language and T be a theory in L. Suppose that T has JEP, and let A, B be existentially closed models of T. Then each $\forall\exists$-sentence of the language L that is true in A will also be true in B.

Let T be a Jonsson theory, C_T be the semantic model of theory T, E_T be class of existentially closed models of theory T, $E_T \subset ModT$. And now let us define the concept of the Kaiser class of models for the arbitrary of the Jonsson theory. $K_T = \{A \in ModT / Th_{\forall\exists}(A)$ is a Jonsson theory$\}$. It is clear that $E_T \subseteq K_T$.

1) $K_T \neq \emptyset$ (if $T = T^* = Th(C_T)$ and T is perfect, then $K_T = E_T$).

2) $\forall M \subseteq K_T$, $Th_{\forall\exists}(M)$ is a Jonsson theory.

The Jonsson sets and almost Jonsson sets generate fragments in the case of normality of considered Jonsson theory from models of the Kaiser class of this theory, and these fragments' semantic models are existentially closed submodels of semantic model of considered Jonsson theory.

Definition 12. [20] The Jonsson theory T_1 is said to be cosemantic to the Jonsson theory T_2 (denoted by $T_1 \bowtie T_2$) if their semantic models are identical, i.e. $\mathcal{C}_{T_1} = \mathcal{C}_{T_2}$, where $\mathcal{C}_{T_i}$ represents the semantic models of T_i for $i = 1, 2$.

Definition 13. [20] Let T_1 and T_2 be arbitrary Jonsson theories. We say that T_1 and T_2 are Jonsson syntactically similar if there exists a bijection $f : E(T_1) \to E(T_2)$ satisfying the following conditions:

1) Isomorphism of lattices: for each $n < \omega$, the restriction of f to $E_n(T_1)$ is an isomorphism between the lattices $E_n(T_1)$ and $E_n(T_2)$.

2) Preservation of Existential Quantifiers: for every $\varphi \in E_{n+1}(T)$ and $n < \omega$, $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$.

3) Preservation of Equality: the bijection preserves equality formulas, i.e., $f(v_1 = v_2) = (v_1 = v_2)$.

The examples of syntactic similarities of two Jonsson theories are given in [18].

Definition 14. [20] The structure $\langle C, Aut(\mathcal{C}), Sub(C)\rangle$ is said to be the Jonsson semantic triple, where C denotes the domain of the semantic model $\mathcal{C}$ of the theory T, $Aut(\mathcal{C})$ represents the automorphism group of $\mathcal{C}$, and $Sub(C)$ is a class of all subsets of C that serve as domains of the corresponding existentially closed submodels of $\mathcal{C}$.

Definition 15. [20] Two Jonsson theories T_1 and T_2 are said to be Jonsson semantically similar if their Jonsson semantic triples are isomorphic as pure triples.

We introduce the following notation:

(i) the syntactic similarity of complete theories T_1 and T_2 is denoted by $T_1 \overset{S}{\bowtie} T_2$;

(ii) the semantic similarity of complete theories T_1 and T_2 is denoted by $T_1 \underset{S}{\bowtie} T_2$.

For Jonsson theories T_1 and T_2:

(i) the Jonsson syntactic similarity of theories T_1 and T_2 is written as $T_1 \overset{S}{\rtimes} T_2$;

(ii) the Jonsson semantic similarity of theories T_1 and T_2 is written as $T_1 \underset{S}{\rtimes} T_2$.

The following result, obtained by the first author of this paper, highlights the relationship between the Jonsson syntactic similarity of Jonsson theories and the Jonsson syntactic similarity of their centers.

Theorem 4. [20] Let T_1 and T_2 be $\exists$-complete perfect Jonsson theories. The following conditions are equivalent:

1) $T_1 \overset{S}{\rtimes} T_2$;

2) $T_1^* \overset{S}{\bowtie} T_2^*$, where T_1^* and T_2^* are the centers of T_1 and T_2, respectively.

The following results, concerning the case of two Jonsson theories, were obtained by Yeshkeyev A.R.:

Theorem 5. Let T_1 and T_2 be two Jonsson theories. If T_1 and T_2 are Jonsson syntactically similar, then they are also Jonsson semantically similar.

Thus, it follows that

$$T_1 \overset{S}{\rtimes} T_2 \Rightarrow T_1 \underset{S}{\rtimes} T_2$$

for any two Jonsson theories T_1 and T_2. This demonstrates that Jonsson syntactic similarity is a sufficient condition for Jonsson semantic similarity of theories.

There are also cases where this condition is necessary. In this paper, we explore specific classes of Jonsson theories for which these two relations are equivalent. We denote this equivalence by the following notation:

$$T_1 \overset{SS}{\rtimes} T_2.$$

Lemma 3. [18] Any two cosemantic Jonsson theories are also Jonsson semantically similar.

The proof follows from the definition of cosemantic Jonsson theories.

The definitions of Jonsson semantic and syntactic similarity relations were extended to classes of Jonsson theories in [18]:

Definition 16. [18] Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is syntactically similar to the class $[T]_2$, denoted by $[T]_1 \overset{S}{\rtimes} [T]_2$, if for every theory $\Delta \in [T]_1$ there exists a theory $\Delta' \in [T]_2$ such that Δ and Δ' are syntactically similar.

Definition 17. [18] The pure triple $\langle C, Aut(\mathcal{C}), \overline{E}_{[T]}\rangle$ is referred to the Jonsson semantic triple for the class $[T] \in JSp(A)/_{\bowtie}$, where C represents the semantic model of $[T]$, $AutC$ denotes the group of automorphisms of C, and $\overline{E}_{[T]}$ is the class of isomorphically images of all existentially closed models of $[T]$.

Definition 18. [18] Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, and $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is Jonsson semantically similar to class $[T]_2$, denoted by $[T]_1 \underset{S}{\rtimes} [T]_2$, if their semantic triples are isomorphic as pure triples.

2 *On fragments and its #-companions for models from K_T*

This paragraph is related to the description of the companions of fragments of Jonsson sets or almost Jonsson sets. Actually it is equivalent to the following fact: any considered set of this paragraph belongs to K_T, where T is considered normal Jonsson theory. The properties of the lattice of existential formulas were described quite well in works [20, 24–26]. Let's recall the main definitions.

Let a theory T be fixed as above.

Definition 19. [24; 843] Let $\varphi^T, \psi^T \in E_n(T)$ and assume that $\varphi^T \cap \psi^T = 0$.

(i) Complement: ψ^T is considered a complement of φ^T if their union equals the total element, i.e., $\varphi \cup \psi = 1$.

(ii) Pseudo-Complement: ψ^T is called a pseudo-complement of φ^T if, for every $\mu^T \in E_n(T)$, whenever $\varphi^T \cap \mu^T = 0$, it follows that $\mu^T \leq \psi^T$.

(iii) Weak Complement: ψ^T is weakly complementary to φ^T if, for every $\mu^T \in E_n(T)$, the condition $(\varphi^T \cup \mu^T) \cap \mu^T = 0$ implies $\mu^T = 0$.

Definition 20. [24; 843] Properties of φ^T and $E_n(T)$: φ^T is complemented if there exists another element that serves as its complement; φ^T is weakly complemented if there exists a weak complement for it; φ^T is pseudo-complemented if there exists a pseudo-complement for it; $E_n(T)$ is complemented if every $\varphi^T \in E_n(T)$ has a complement; $E_n(T)$ is weakly complemented if every $\varphi^T \in E_n(T)$ has a weak complement; $E_n(T)$ is pseudo-complemented if every $\varphi^T \in E_n(T)$ has a pseudo-complement.

We now turn our attention to the formulas that are preserved under extensions of models and submodels within the context of the theory T.

Definition 21. [20] A formula $\varphi(x_1, ..., x_n)$ is said to be preserved under extensions in $ModT$ if for any models $A \subset B$ of T, and any $a_1, ..., a_n \in A$, $A \models \varphi[a_1, ..., a_n] \Rightarrow \varphi[a_1, ..., a_n]$.

Definition 22. [20] A formula $\varphi(x_1, ..., x_n)$ is said to be preserved under submodels in $ModT$ if for any models $A \subset B$ of T, and any $a_1, ..., a_n \in A$, $B \models \varphi[a_1, ..., a_n] \Rightarrow A \models \varphi[a_1, ..., a_n]$.

We now define the concept of an invariant formula and examine the connection between the invariance of an existential formula and the complementarity of its class in $E(T)$.

Definition 23. [24; 844]. A formula φ is said to be invariant in $ModT$ if it is preserved both under extensions of models in $ModT$ and submodels in $ModT$.

Definition 24. [24; 843] A theory T is called positively model complete if it satisfies the following conditions:

(i) T is model complete, meaning every formula φ is equivalent to an existential formula within T.

(ii) Every existential formula in the language L of T is equivalent to a positive existential formula within T.

We present the required definitions and summarize established results that clarify the connection between model completeness, quantifier elimination in a theory T, and the structural properties of lattices of existential formulas $E_n([[M^0]])$.

Some results from [20] and classical model theory can be further refined within the context of studying fragments of Jonsson theories.

These refined results will serve as a basis for deriving the key conclusions in this section.

Theorem 6. [24; 846] An existential formula φ is invariant in $Mod(Th_{\forall\exists}(E_T))$, where E_T is the class of all existentially closed models of T, if and only if φ^T is weakly complemented in $E(T)$.

Theorem 7. [20]

(i) A theory T is model complete if and only if all formulas are preserved when passing to submodels in $Mod(T)$.

(ii) A theory T is model complete if and only if all formulas are preserved when extending to larger models in $Mod(T)$.

Theorem 8. [20]

(i) If T' is a model companion of a universal theory T, then T' is a model completion of T if and only if T' allows elimination of quantifiers.

(ii) If T is a model companion of T, then T' is a model completion of T if and only if T satisfies AP.

Theorem 9. [20] A theory T is a submodel complete if and only if T allows elimination of quantifiers.

Theorem 10. [24; 843] A theory T is called positively model complete if and only if every $\varphi^T \in E_n(T)$ has a positive existential complement.

Theorem 11. [24; 843] A theory T has a model companion if and only if $\varphi^T \in E_n(T)$ has a weakly complement.

Theorem 12. [24; 843] A theory T has a model companion if and only if $\varphi^T \in E_n(T)$, the lattice of existential formulas forms a Stone algebra.

Theorem 13. [24; 843] A theory T has a model companion if and only if every $\varphi^T \in E_n(T)$ has a weakly quantifier-free complement.

Theorem 14. [20] Let T be a Jonsson theory. Then the next conditions are equivalent:

1) the theory T is perfect;

2) the theory T has a model companion.

In [26], a connection between the completeness and model completeness of a Jonsson theory was established.

The following result represents a particular case of Lindstrom's theorem, which describes the relationship between model completeness and completeness.

Let T be a fixed, normal Jonsson theory. Within T, we consider N contained in K_T. Next, we examine the Jonsson spectrum $JSp(N)$, and proceed to perform a double factorization of $JSp(N)/_{\underset{\bowtie}{SS}}$.

Theorem 15. Let T be a normal Jonsson theory, $A \in C_T$, where A is an almost Jonsson theory, then $cl(A) = M \in K_T$. And let $K_T \neq \emptyset$, $N \subseteq K_T$. We consider a Jonsson theory $M^0 = Th_{\forall\exists}(M) \in JSp(N)$, and $[M^0] \in JSp(N)/_{\bowtie}$ and $[[M^0]] \in JSp(N)/_{\underset{\bowtie}{SS}}$, where M^0 is a Kaiser hull. Then the following conditions are equivalent:
(i) $[[M^0]]$ is complete;
(ii) $[[M^0]]$ is model complete.

In [27], a connection was demonstrated between the perfectness of a Jonsson theory and the properties of the lattice $E_n(T)$ [3]. Final point of proof of this fact follows from the definition of normality.

Theorem 16. Let T be a normal Jonsson theory. And let $K_T \neq \emptyset$, $N \subseteq K_T$, $[M^0] \in JSp(N)/_{\bowtie}$ and $[[M^0]] \in JSp(N)/_{\underset{\bowtie}{SS}}$. Then the following conditions are equivalent:
(i) $[[M^0]]$ is perfect;
(ii) $[[M^0]]^{\#}$ is normal Jonsson theory;
(iii) $E_n([[M^0]])$ is a Boolean algebra;
(iv) $[[M^0]]^*$ is model-complete.

Proof. It follows from results which connected the properties of lattice of existential formulas $E_n(T)$ and above results from Theorem 15.

Now, we study the model-theoretic properties of the #-companion of a normal perfect Jonsson fragment. Let T be a normal Jonsson theory in a countable language L, and let $A \in C_T$, where A is an almost Jonsson theory, then $cl(A) = M \in K_T$. And let $K_T \neq \emptyset$, $N \subseteq K_T$. We consider a Jonsson theory $M^0 = Th_{\forall\exists}(M) \in JSp(N)$, and $[M^0] \in JSp(N)_{\bowtie}$ and $[[M^0]] \in JSp(N)/_{\underset{\bowtie}{SS}}$. The distributive lattice $E_n([[M^0]])$ consists of equivalence classes of formulas defined as $\varphi^{[[M^0]]} = \{\psi \in E_n(L) | [[M^0]] \vdash \varphi \leftrightarrow \psi, \varphi \in E_n(L), K([[M^0]]) = \cup_{n<\omega} E_n([[M^0]])\}$.
We now focus on a fragment $[[M^0]]$ that is complete for existential sentences and satisfies the conditions described above.

Theorem 17. Let T be a normal Jonsson theory. And let $K_T \neq \emptyset$, $N \subseteq K_T$, $[M^0] \in JSp(N)_{\bowtie}$ and $[[M^0]] \in JSp(N)/_{\underset{\bowtie}{SS}}$. Let $[[M^0]]$ be a perfect fragment of a Jonsson set A, $[[M^0]]^{\#}$ be its #-companion. Then the following hold:
(i) $[[M^0]]^{\#}$ admits elimination of quantifiers if and only if every $\varphi \in E_n([[M^0]])$ has a quantifier-free complement;
(ii) $[[M^0]]^{\#}$ is positively model-complete if and only if every $\varphi \in E_n([[M^0]]$ has an existential complement.

Proof. (i) Given that $[[M^0]]$ is a fragment of a Jonsson set A and M' is a semantic model of $[[M^0]]$, we know $[[M^0]]$ is perfect, we have that $[[M^0]]^{\#} = Th(M')$, where the center of M' admits elimination of quantifiers. By Theorem 9, $[[M^0]]^{\#}$ is submodel complete, and therefore, is model complete, and by Theorem 16, the lattice $E_n([[M^0]])$ is a Boolean algebra, meaning every $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has a complement. Since $[[M^0]]^{\#}$ admits elimination of quantifiers, every class $\varphi \in E_n([[M^0]])$ must have a quantifier-free complement.

Conversely, assume that every class $\varphi \in E_n([[M^0]])$ has a quantifier-free complement. Then $E_n([[M^0]])$ is a Boolean algebra (by definition). By Theorem 16, this implies that $[[M^0]]^{\#}$ is model-complete. By part (ii) of Theorem 7, every formula in $[[M^0]]^{\#}$ is equivalent to some existential formula, i.e., the class of such formulas belongs to $E_n([[M^0]]^{\#})$. Since $[[M^0]]$ is complete existential sentences, we know $E_n([[M^0]]) = En([[M^0]]^{\#})$. Therefore, every $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has a quantifier-free complement, and $E_n([[M^0]]^{\#})$ consists of quantifier-free formulas. Hence, $[[M^0]]^{\#}$ admits elimination of quantifiers.

(ii) Assume that $[[M^0]]^{\#}$ is positively model-complete. By Definition 11, $[[M^0]]^{\#}$ is model-complete and for every existential formula φ there exists a positive existential formula ψ such that $[[M^0]]^{\#} \vdash \varphi \leftrightarrow \psi$. By Theorem 16, $E_n([[M^0]])$ is a Boolean algebra, so every $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has an existential complement. Since $[[M^0]]^{\#}$ ensures the existence of a positive existential formula ψ equivalent to φ, every there is a positive existential formula such that $[[M^0]]^{\#} \vdash \varphi \leftrightarrow \psi$, we obtain that every $\varphi \in E_n([[M^0]])$ has a positive existential complement. The necessary condition of part (ii) is proved.

To prove the sufficiency of part (ii): Assume that every $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has a positive existential complement. By Theorem 10, this implies that $[[M^0]]$ is positive model-complete. By the definition of positive model-completeness, $[[M^0]]$ is also model-complete. By Theorem 15, the fragment $[[M^0]]$ is complete. Since $[[M^0]]^{\#}$ is a central complement of the theory $[[M^0]]$, we have $[[M^0]] = [[M^0]]^{\#}$. Therefore, $[[M^0]]^{\#}$ is positively model-complete. The sufficiency is proven by using Theorem 10 (positive model-completeness of $[[M^0]]$), Theorem 15 (completeness of $[[M^0]]$), and the relationship between $[[M^0]]$ and $[[M^0]]^{\#}$. This completes the proof that $[[M^0]]^{\#}$ is positively model-complete.

In the following theorem, we establish necessary and sufficient conditions for the perfectness of a normal fragment $[[M^0]]$ within the framework of the class $[M^0]$, using the structure of the lattice of existential formulas $E_n([[M^0]])$.

Theorem 18. Let T be a normal Jonsson theory. And let $K_T \neq \emptyset$, $N \subseteq K_T$, $[M^0] \in JSp(N)_{\bowtie}$ and $[[M^0]] \in JSp(N)_{/\overset{SS}{\bowtie}}$, and let $[[M^0]]$ be a perfect fragment with $Fr^{\#}(A)$ as its #-companion. Then the following conditions are equivalent:

(i) $[[M^0]]$ is perfect;

(ii) $E_n([[M^0]])$ is weakly complemented;

(iii) $\varphi \in E_n([[M^0]])$ is a Stone algebra.

Proof. Let $[[M^0]]$ be a perfect fragment of a Jonsson set A, and $[[M^0]]^{\#}$ its #-companion. We prove the equivalence of the three conditions:

(i) $\Rightarrow$ (ii). Assume that a normal Jonsson theory $[[M^0]]$ is perfect. By Theorem 17, $[[M^0]]$ has a #-companion $[[M^0]]^{\#}$. By results in [24], we know $[[M^0]]^f = [[M^0]]^0$, where $[[M^0]]^0 = Th_{\forall\exists}(E_{[[M^0]]})$ is Kaiser's hull of $[[M^0]]$. Since $[[M^0]]^{\#}$ is model-complete (by the definition of a #-companion), part (ii) of Theorem 7 states that every formula of the language is preserved under submodels in $Mod[[M^0]]^{\#}$.

Consequently, every existential formula in the language is preserved under both submodels and extensions in $Mod[[M^0]]^{\#}$. By Definition 23, these formulas are invariant in $Mod[[M^0]]^{\#}$. From Theorem 6, it follows that every existential formula in $E_n([[M^0]])$ is weakly complemented. Thus, $E_n([[M^0]])$ is weakly complemented.

(ii) $\Rightarrow$ (i). Assume $E_n([[M^0]])$ is weakly complemented.

1. By Theorem 11, the fragment $[[M^0]]$ has a model companion.

2. By Theorem 17, if $[[M^0]]$ has a models companion, then $[[M^0]]$ is perfect. Thus, $[[M^0]]$ is perfect. Therefore, (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (iii). Assume $[[M^0]]$ is perfect.

1. By part (ii) of Theorem 10, the model companion of a normal Jonsson theory is its model completion.

2. From the perfectness of $[[M^0]]$, it follows by Theorem 12 that $E_n([[M^0]])$ forms a Stone algebra. Thus, $E_n([[M^0]])$ is a Stone algebra.

(iii) $\Rightarrow$ (i). Assume $E_n([[M^0]])$ is a Stone algebra.

1. By Theorem 12, if $E_n([[M^0]])$ is a Stone algebra, then $[[M^0]]$ has a #-companion $[[M^0]]^{\#}$.

2. By Theorem 16, if $[[M^0]]$ has a #-companion, then $[[M^0]]$ is perfect. Thus, $[[M^0]]$ is perfect.

Theorem 19. Let $[[M^0]]$ be a perfect normal Jonsson theory. And let $K_T \neq \emptyset$, $N \subseteq K_T$, $[M^0] \in JSp(N)_{\bowtie}$ and $[[M^0]] \in JSp(N)/_{\underset{\bowtie}{SS}}$, $[[M^0]]^{\#}$ be its #-companion. Then the following conditions are equivalent:

(i) $[[M^0]]^{\#}$ is a normal Jonsson theory;

(ii) every $\varphi \in E_n([[M^0]])$ has a weakly quantifier-free complement.

To prove the necessity we need the following statement.

If the model companion $[M^0]^m$ is defined, then a model companion $([M^0]_{\forall})^m$ is defined and

$$[M^0]^m = ([M^0]_{\forall})^m \tag{1}$$

(see [5]).

Proof. (i) ⇒ (ii). Assume $[[M^0]]^{\#}$ is a normal Jonsson theory. By results from [24], the fragment $[[M^0]]$ is perfect. From Theorem 18, a perfect fragment $[[M^0]]$ has a #-companion, which coincides with the theory $[[M^0]]^{\#}$. By part (ii) of Theorem 7, the #-companion $[[M^0]]^{\#}$ is a model completion of fragment $[[M^0]]$. Due to the mutual model consistency between the fragment $[[M^0]]$ and its universal theory $[[M^0]]_{\forall}$, we have the model completion of $[M^0]$ (the fragment's class) is also the model completion of $[M^0]_{\forall}$ (its universal consequences). By Theorem 13, it follows that every existential formula $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has a weakly quantifier-free complement.

(ii) ⇒ (i). Assume that every existential formula $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has a weakly quantifier-free complement. By definition, if every $\varphi^{[[M^0]]}$ has a weakly quantifier-free complement, then every $\varphi^{[[M^0]]} \in E_n([[M^0]])$ is weakly complemented, i.e. $E_n([[M^0]])$ is weakly complemented. By result from [24], it follows that the #-companion $[[M^0]]^{\#}$ is a normal Jonsson theory.

All necessary concepts and statements related to these notions that were not defined and were not noted in the text of this article can be extracted from the following list of sources.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Yeshkeyev, A.R, Ulbrikht, O.I, & Omarova, M.T. (2024). Double factorization of the Jonsson spectrum. *Bulletin of the Karaganda University. Mathematics Series*, 4(116), 185–196. https://doi.org/10.31489/2024M4/185-196

2 Yeshkeyev, A.R. (2016). Some properties of Morly rank over Jonsson sets. *Bulletin of the Karaganda University. Mathematics Series*, 4(84), 57–62.

3 Yeshkeyev, A.R. (2013). The structure of lattices of positive existential formulae of (Δ-PJ)-theories. *Science Asia*, 39(2), 19–24.

4 Yeshkeyev, A.R., Kasymetova, M.T., & Shamatayeva, N.K. (2018). Model-theoretic properties of the #-companion of a Jonsson set. *Eurasian mathematical journal, 9*(2), 68–81.

5 Barwise, J. (1982). *Teoriia modelei: spravochnaia kniga po matematicheskoi logike. Chast 1 [Model theory: Handbook of mathematical logic. Part 1].* Moscow: Izdatelstvo Nauka [in Russian].

6 Yeshkeyev, A.R., & Mussina, N.M. (2021). An algebra of the central types of the mutually model-consistent fragments. *Bulletin of the Karaganda University. Mathematics Series*, 4(101), 111–118. https://www.doi.org/10.31489/2021M1/111-118

7 Yeshkeyev, A.R., & Omarova, M.T. (2021). An essential base of the central types of the convex theory. *Bulletin of the Karaganda University. Mathematics Series*, 1(101), 119–126. https://www.doi.org/10.31489/2021M1/119-126

8 Yeshkeyev, A.R., Kassymetova, M.T., & Ulbrikht, O.I. (2021). Independence and simplicity in Jonsson theories with abstract geometry. *Bulletin of the Karaganda University. Mathematics Series*, 1(18), 433–455. https://www.doi.org/10.33048/semi.2021.18.030

9 Yeshkeyev, A.R., Tungushbayeva, I.O., & Kassymetova, M.T. (2022). Connection between the amalgam and joint embedding properties. *Bulletin of the Karaganda University. Mathematics Series*, 1(105), 127–135. https://www.doi.org/10.31489/2022M1/127-135

10 Yeshkeyev, A.R., Ulbrikht, O.I., & Omarova, M.T. (2023). The Number of Fragments of the Perfect Class of the Jonsson Spectrum. *Lobachevskii Journal of Mathematics, 43*(12), 3658–3673. https://www.doi.org/10.1134/S199508022215029X

11 Yeshkeyev, A.R., Yarullina, A.R., & Kassymetova, M.T. (2023). On Robinson spectrum of the semantic Jonsson quasivariety of unars. *Bulletin of the Karaganda University. Mathematics Series*, 2(110), 169–178. https://www.doi.org/10.31489/2023M2/169-178

12 Yeshkeyev, A.R., Ulbrikht, O.I., & Yarullina, A.R. (2022). Existentially positive Mustafin theories of S-acts over a group. *Bulletin of the Karaganda University. Mathematics Series*, 2(106), 172–185. https://www.doi.org/10.31489/2022M2/172-185

13 Yeshkeyev, A.R., Tungushbayeva, I.O., & Omarova, M.T. (2022). Forcing companions of Jonsson AP-theories. *Bulletin of the Karaganda University. Mathematics Series*, 3(107), 152–163. https://www.doi.org/10.31489/2022M3/152-163

14 Yeshkeyev, A.R., Tungushbayeva, I.O., & Amanbekov, S.M. (2022). Existentially prime Jonsson quasivarieties and their Jonsson spectra. *Bulletin of the Karaganda University. Mathematics Series*, 4(108), 117–124. https://doi.org/10.31489/2022M4/117-124

15 Yeshkeyev, A.R., Ulbrikht, O.I., & Issayeva, A.K. (2023). Algebraically prime and atomic sets. *Turkic World Mathematical Society (TWMS) Journal of Pure and Applied Mathematics, 14*(2), 232–245. https://www.doi.org/10.30546/2219-1259.14.2.2023.232

16 Yeshkeyev, A.R., Tungushbayeva, I.O., & Koshekova, A.K. (2024). The cosemanticness of Kaiser hulls of fixed classes of models. *Bulletin of the Karaganda University. Mathematics Series*, 1(113), 208–217. https://doi.org/10.31489/2024m1/208-217

17 Yeshkeyev, A.R., Ulbrikht, O.I., & Mussina, N.M. (2023). Similarities of Hybrids from Jonsson Spectrum and S-Acts. *Lobachevskii Journal of Mathematics, 44*(12), 5502–5518. https://doi.org/10.1134/s1995080223120399

18 Yeshkeyev, A.R., Ulbrikht, O.I., & Urken, G.A. (2023). Similarities of Jonsson spectra's classes. *Bulletin of the Karaganda University. Mathematics Series*, 4(112), 130–143. https://doi.org/10.31489/2023m4/130-143

19 Yeshkeyev, A.R., Tungushbayeva, I.O., & Ulbrikht, O.I. (2024). On some types of algebras on a Jonsson spectrum. *Siberian Electronic Methematical Reports, 21*(2), 866–881.

20 Yeshkeyev, A.R. (2024). *Theories and their models.* Karaganda, Kazakhstan: izd. KarU [in Russian].

21 Yeshkeyev, A.R. (2015). On Jonsson sets and some of their properties. *Bulletin of the Karaganda University. Mathematics Series*, 1(21), 99–100.

22 Yeshkeyev, A.R., & Ulbrikht, O.I. (2019). JSp-cosemanticness of R-modules. *Siberian Electronic*

Methematical Reports, 16, 1233–1244. https://doi.org/10.33048/semi.2019.16.084

23 Hodges, W. (1993). *Model Theory. Encyclopedia of Mathematics and its Applications.* Cambridge: Cambridge University Press.

24 Weispfenning, V. (1981). The model-theoretic signicance of complemented existential formulas. *The Journal of Symbolic Logic*, 46(4), 843–849.

25 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Mathematics Series*, 3(79), 33–39.

26 Yeshkeyev, A.R., & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Mathematics Series*, 3(79), 25–32.

27 Robinson, A. (1963). Introduction to model theory and to the metamathematics of algebra. *Amsterdam.*

*Author Information**

Aibat Rafhatuly Yeshkeyev — Doctor of physical and mathematical sciences, Professor, Professor Researcher, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *aibat.kz@gmail.com*; https://orcid.org/0000-0003-0149-6143

Maira Tekhnikovna Kassymetova (*corresponding author*) — PhD, Associate Professor of the Department of Algebra, Mathematical Logic and Geometry named after T.G. Mustafin, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *mairushaasd@mail.ru*; https://orcid.org/0000-0002-4659-0689

Nazerke Mukhtaramkyzy Mussina — PhD, Associate Professor of the Department of Algebra, Mathematical Logic and Geometry named after T.G. Mustafin, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *mussinanazerke@gmail.com*; https://orcid.org/0000-0002-4360-8767

*The author's name is presented in the order: First, Middle and Last Names.