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CONTENT

MATHEMATICS

MATHEMATICS

https://doi.org/10.31489/2024M4/4-17 *Research article*

Structural properties of the sets of positively curved Riemannian metrics on generalized Wallach spaces

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In the present paper sets related to invariant Riemannian metrics of positive sectional and (or) Ricci curvature on generalized Wallach spaces are considered. The problem arises in studying of the evolution of such metrics under the influence of the normalized Ricci flow. For invariant Riemannian metrics of the Wallach spaces which admit positive sectional curvature and belong to a given invariant surface of the normalized Ricci flow equation we establish that they form a set bounded by three connected and pairwise disjoint regular space curves such that each of them approaches two others asymptotically at infinity. Analogously, for all generalized Wallach spaces with coincided parameters the set of Riemannian metrics which belong to the invariant surface of the normalized Ricci flow and admit positive Ricci curvature is bounded by three space curves each consisting of exactly two connected components as regular curves. Mutual intersections and asymptotical behaviors of these components are studied as well. We also establish that curves corresponding to Kähler metrics of spaces under consideration form separatrices of saddles of a three-dimensional system of nonlinear autonomous ordinary differential equations obtained from the normalized Ricci flow equation.

Keywords: generalized Wallach space, Riemannian metric, Kähler metric, normalized Ricci flow, sectional curvature, Ricci curvature, dynamical system, singular point.

2020 Mathematics Subject Classification: 53C30, 53E20, 37C10.

Introduction

The paper is devoted to the study of structural properties of two important sets responsible for positivity of the sectional and the Ricci curvatures of invariant Riemannian metrics on the Wallach spaces and generalized Wallach spaces. The Wallach spaces

$$W_6 := \mathrm{SU}(3)/T_{\max}, \quad W_{12} := \mathrm{Sp}(3)\,/\,\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1), \quad W_{24} := F_4/\,\mathrm{Spin}(8) \qquad (1)$$

are well-known and admit invariant Riemannian metrics of positive sectional curvature as it was shown in [1]. As for generalized Wallach space, firstly, recall its definition and basic properties (see [2, 3]). Let G/H be a homogeneous almost effective compact space with a (compact) semisimple connected Lie group G and its closed subgroup H. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras of G

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and H. Then $[\cdot\,,\cdot]$ is a corresponding Lie bracket of $\mathfrak{g}$ whereas $B(\cdot\,,\cdot)$ is the Killing form of $\mathfrak{g}$. Note that $\langle\cdot\,,\cdot\rangle = -B(\cdot\,,\cdot)$ is a bi-invariant inner product on $\mathfrak{g}$. In this way invariant Riemannian metrics on G/H can be identified with $\operatorname{Ad}(H)$-invariant inner products on the orthogonal complement $\mathfrak{p}$ of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $\langle\cdot\,,\cdot\rangle$. Compact homogeneous spaces G/H whose isotropy representation admits a decomposition into a direct sum $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ of three $\operatorname{Ad}(H)$-invariant irreducible modules $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ satisfying $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for each $i \in \{1,2,3\}$ were called *generalized Wallach spaces* in the terminology of [3]. The main characteristic of these spaces is that every generalized Wallach space can be described by a triple of real parameters $a_i := A/d_i \in (0, 1/2]$, $i = 1, 2, 3$, where $d_i = \dim(\mathfrak{p}_i)$ and A is some important positive constant (see [2] for details). It should be also noted that not every triple $(a_1, a_2, a_3) \in (0, 1/2] \times (0, 1/2] \times (0, 1/2]$ corresponds to some generalized Wallach spaces. An interesting fact is the fact that the Wallach spaces (1) are partial cases $a_1 = a_2 = a_3 = a$ of generalized Wallach spaces with $a = 1/6$, $a = 1/8$ and $a = 1/9$ respectively (see [4]).

As noted above for a fixed bi-invariant inner product $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{g}$ of the Lie group G, any G-invariant Riemannian metric $\mathbf{g}$ on G/H can be determined by an $\operatorname{Ad}(H)$-invariant inner product

$$(\cdot,\cdot) = x_1\langle\cdot,\cdot\rangle|_{\mathfrak{p}_1} + x_2\langle\cdot,\cdot\rangle|_{\mathfrak{p}_2} + x_3\langle\cdot,\cdot\rangle|_{\mathfrak{p}_3}\,, \tag{2}$$

where x_1, x_2, x_3 are positive real numbers (a detailed exposition can be found in [2,3,5] and references therein). In [2] the explicit expressions $\operatorname{Ric}_{\mathbf{g}} = \mathbf{r_1}\,\operatorname{Id}|_{\mathfrak{p}_1} + \mathbf{r_2}\,\operatorname{Id}|_{\mathfrak{p}_2} + \mathbf{r_3}\,\operatorname{Id}|_{\mathfrak{p}_3}$ and $S_{\mathbf{g}} = d_1\mathbf{r_1} + d_2\mathbf{r_2} + d_3\mathbf{r_3}$ were derived for the Ricci tensor $\operatorname{Ric}_{\mathbf{g}}$ and the scalar curvature $S_{\mathbf{g}}$ of the metric (2) on generalized Wallach spaces, where

$$\mathbf{r_i} := \frac{1}{2x_i} + \frac{1}{2a_i}\left(\frac{x_i}{x_j x_k} - \frac{x_k}{x_i x_j} - \frac{x_j}{x_i x_k}\right) \tag{3}$$

are the principal Ricci curvatures, $\{i,j,k\} = \{1,2,3\}$.

Knowing $\operatorname{Ric}_{\mathbf{g}}$ and $S_{\mathbf{g}}$ allowed us to initiate in [6,7] the study of the normalized Ricci flow equation

$$\frac{\partial}{\partial t}\mathbf{g}(t) = -2\operatorname{Ric}_{\mathbf{g}} + 2\mathbf{g}(t)\frac{S_{\mathbf{g}}}{n} \tag{4}$$

introduced by R. Hamilton in [8] on generalized Wallach spaces. Since then studies related to this topic were continued in [9–14] concerning classifications of singular (equilibria) points of (4) being Einstein metrics and their bifurcations. The authors of [15–17] studied an interesting and quite complicated surface of bifurcations of (4) defined by a symmetric polynomial equation in three variables a_1, a_2, a_3 of degree 12. In the sequel authors of [4,18] considered the evolution of positively curved Riemannian metrics under the influence of (4) on an interesting class of generalized Wallach spaces with coincided parameters $a_1 = a_2 = a_3 := a \in (0, 1/2)$ generalizing some results of [19,20]. In this case (4) can be reduced to the following system of three autonomous ordinary differential equations (see [4]):

$$\frac{dx_i}{dt} = f_i(x_1, x_2, x_3) := \frac{x_i}{x_j} + \frac{x_i}{x_k} + 2a\left(\frac{x_j}{x_k} + \frac{x_k}{x_j} - 2\,\frac{x_i^2}{x_j x_k}\right) - 2 \tag{5}$$

with $\{i,j,k\} = \{1,2,3\}$.

In [4] it was proved that (4) deforms all generic metrics with positive sectional curvature into metrics with mixed sectional curvature on each Wallach space in (1) (Theorem 1 in [4]) and all generic metrics with positive Ricci curvature will be deformed into metrics with mixed Ricci curvature for W_{12} and W_{24} (see Theorem 2 in [4]), where given metric is said to be generic if $x_i \neq x_j \neq x_k \neq x_i$ for $i, j, k \in \{1,2,3\}$. According to Theorems 3 and 4 in [4] and Theorem 3 in [18] positiveness of the Ricci curvature will be preserved for all generic metrics at $a \in (1/6, 1/2)$ and for a special kind of metrics satisfying $x_k < x_i + x_j$ at $a = 1/6$ (the equalities $x_k = x_i + x_j$ correspond to Kähler metrics), whereas all positively curved metrics will be deformed into metrics with mixed Ricci curvature if $a \in (0, 1/6)$. In [4,18] we used the description $S := \big\{(x_1, x_2, x_3) \in (0, +\infty)^3 \mid \gamma_1 > 0,\ \gamma_2 > 0,\ \gamma_3 > 0\big\} \setminus \big\{(r, r, r) \in \mathbb{R}^3 \mid r > 0\big\}$ of the set of Riemannian metrics with positive sectional curvature on the Wallach spaces (1) given in [21], where

$$\gamma_i := (x_j - x_k)^2 + 2x_i(x_j + x_k) - 3x_i^2, \quad \{i, j, k\} = \{1, 2, 3\}. \tag{6}$$

Analogously, $R := \big\{(x_1, x_2, x_3) \in (0, +\infty)^3 \mid \lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 > 0\big\} \setminus \big\{(r, r, r) \in \mathbb{R}^3 \mid r > 0\big\}$ is the set of all Riemannian metrics of positive Ricci curvature on every generalized Wallach spaces with $a_1 = a_2 = a_3 := a \in (0, 1/2)$, where

$$\lambda_i := x_j x_k + a\left(x_i^2 - x_j^2 - x_k^2\right), \quad \{i, j, k\} = \{1, 2, 3\} \tag{7}$$

in accordance with (3).

The present paper is devoted to detailed proof of our observations in [4, 18] concerning structural properties of surfaces and curves obtained from (6) and (7). For each $i = 1, 2, 3$ introduce the surfaces (cones) $\Gamma_i := \{(x_1, x_2, x_3) \in (0, +\infty)^3 \mid \gamma_i = 0\}$ and $\Lambda_i := \{(x_1, x_2, x_3) \in (0, +\infty)^3 \mid \lambda_i = 0\}$.

Denote by Σ the surface defined by the equation $V = 1$, where $V := x_1 x_2 x_3$. Introduce also space curves $s_i := \Sigma \cap \Gamma_i$, $r_i := \Sigma \cap \Lambda_i$. The main result of this paper is contained in the following two theorems.

Theorem 1. The following assertions hold for all indices with $\{i, j, k\} = \{1, 2, 3\}$:

1 For each Wallach space in (1) the set of invariant Riemannian metrics (2) which belong to the invariant surface Σ of the differential system (5) and admit positive sectional curvature is bounded by the pairwise disjoint regular space curves s_1, s_2 and s_3 in Σ such that each s_k is connected and can be parameterized as

$$x_k = t^{-1}\alpha^{-2}, \quad x_i = t\alpha, \quad x_j = \alpha,$$

where

$$\alpha = \alpha(t) := \begin{cases} \sqrt[3]{\left(-t - 1 + 2\sqrt{t^2 - t + 1}\right) t^{-1}(t-1)^{-2}}, & \text{if} \quad t > 0,\ t \neq 1, \\ \sqrt[3]{6}/2, & \text{if} \quad t = 1, \end{cases}$$

and $\alpha(t) > 0$ for all $t > 0$.

2 Every invariant curve I_k of the differential system (5) given by the equations $x_i = x_j = p$, $x_k = p^{-2}$, $p > 0$, intersects the only border curve s_k at the unique point with coordinates $x_i = x_j = p_0$, $x_k = p_0^{-2}$ approaching at infinity the other two curves s_i and s_j as close as we like, where $p_0 = \sqrt[3]{6}/2$.

The results of Theorem 1 are illustrated in the left panel of Figure 1, where the curves s_1, s_2 and s_3 are depicted respectively in red, teal and blue colors, the invariant curves I_1, I_2, I_3 are all yellow colored.

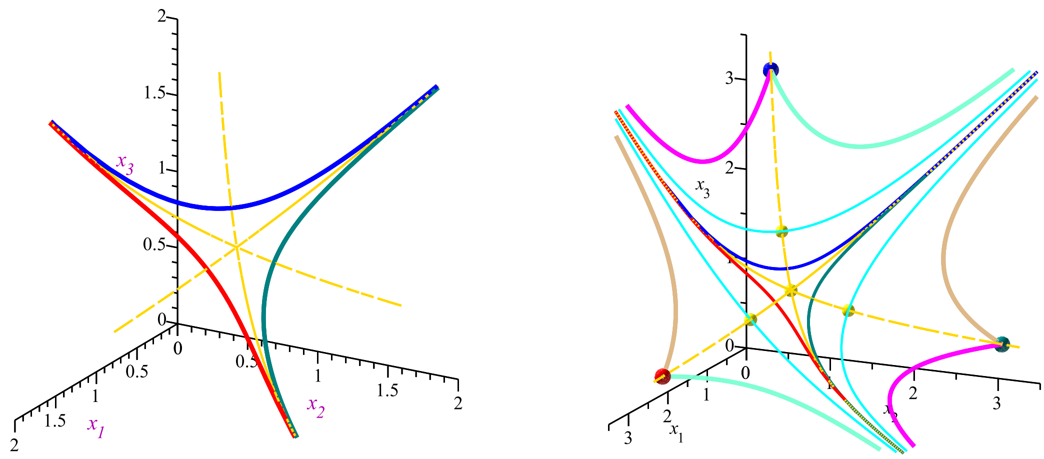

Figure 1. The curves s_1, s_2, s_3 (the left panel); the curves r_1, r_2, r_3, l_1, l_2, l_3 and singular points $\mathbf{o}_0, \mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3$ corresponding to $a = 1/6$ (the right panel)

Theorem 2. The following assertions hold for all indices with $\{i, j, k\} = \{1, 2, 3\}$:

1 For every generalized Wallach space with $a_1 = a_2 = a_3 = a \in (0, 1/2)$ the set of invariant Riemannian metrics (2) which belong to the invariant surface Σ of the differential system (5) and admit positive Ricci curvature is bounded by the space curves r_1, r_2 and r_3 in Σ such that each r_k consists of two regular connected components r_{ki} and r_{kj} parameterized by equations

$$x_k = t^{-1}\beta^{-2}, \quad x_i = t\beta, \quad x_j = \beta \tag{8}$$

and

$$x_k = t^{-1}\beta^{-2}, \quad x_j = t\beta, \quad x_i = \beta \tag{9}$$

respectively, where

$$\beta = \beta(t) := \sqrt[6]{(t^4 - a^{-1}t^3 + t^2)^{-1}} > 0,$$

$t \in (0, a]$.

2 Every pair of the curves r_i and r_j admits a unique common point P_{ij} with coordinates $x_i = x_j = a^{\frac{1}{3}}$, $x_k = a^{-\frac{2}{3}}$ which belong to the components r_{ij} and r_{ji}; In addition, every invariant curve I_k of the system (5) meets the components r_{ij} and r_{ji} of r_i and r_j exactly at the point P_{ij} approaching their another components r_{ik} and r_{jk} at infinity as close as we like.

3 For every $a \in (0, 1/2)$ all singular (equilibria) points of the differential system (5) belong to the set $\Sigma \cap R$.

4 Kähler metrics $x_k = x_i + x_j$ of generalized Wallach spaces with $a = 1/6$ form separatrices l_k of saddles of (5) in Σ which can be defined by parametric equations

$$x_k = t^{-1}\phi^{-2}, \quad x_i = t\phi, \quad x_j = \phi, \tag{10}$$

where $\phi = \phi(t) := \sqrt[3]{(t^2 + t)^{-1}}, \quad t > 0.$

The results of Theorem 2 are illustrated in the right panel of Figure 1 for the case $a = 1/6$, where the curves r_1, r_2 and r_3 are depicted respectively in magenta, aquamarine and burlywood colors, the curves l_1, l_2 and l_3 are depicted by cyan colored curves and yellow colored points correspond to singular points of (5).

It should be noted that we will consider only Riemannian metrics satisfying the unit volume condition $V := x_1x_2x_3 = 1$ (see [4,6]). In general, surfaces $V = c$, where $c > 0$, play the significant role for study (5) on generalized Wallach spaces. It is known that any set determined by the equation $V = c$ is invariant under (5), moreover $V = c$ is its first integral. Surfaces $V = c$ will also be unstable (or stable) manifolds of (5) and contain leading directions of motions of its trajectories (see [22]). Since the right hand sides of (5) are all homogeneous, namely $f_i(cx_1, cx_2, cx_3) = f_i(x_1, x_2, x_3)$ for any c, we can pass to a new differential system of the same form as the original one, but with $\widetilde{x}_1\widetilde{x}_2\widetilde{x}_3 = 1$. Actually this is reachable by replacings $x_i(t) = \widetilde{x}_i(\tau)\sqrt[3]{c}$ and $t = \tau\sqrt[3]{c}$. Therefore without loss of generality we assume that the invariant surface is given by $V \equiv 1$.

1 *Proofs of Theorems 1 and 2*

Observe that the expressions for γ_i and λ_i in (6) and (7) are symmetric under the permutations $i \to j \to k \to i$. Therefore it suffices to consider representatives only at fixed (i, j, k), where $\{i, j, k\} = \{1, 2, 3\}$.

1.1 *Proof of Theorem 1*

Proof. (1) *The curves s_1, s_2, s_3 are pairwise disjoint and form the boundary of the set $\Sigma \cap S$.* For each Wallach space in (1) the set S of Riemannian metrics (2) admitting positive sectional curvature is bounded by the pairwise disjoint cones Γ_1, Γ_2 and Γ_3 (these cones are depicted in the left panel of Figure 2 in red, teal and blue colors respectively). Although this fact was proved in [22] we repeat here the sketch of reasonings for convenience of the readers. Indeed the equation $\gamma_k = 0$ defines two connected components $x_k = 3^{-1}\left(x_i + x_j - 2\sqrt{x_i^2 - x_ix_j + x_j^2}\right)$ and $x_k = \Phi_k(x_i, x_j) := 3^{-1}\left(x_i + x_j + 2\sqrt{x_i^2 - x_ix_j + x_j^2}\right)$ of the cone Γ_k. Since the first of them gives $x_k < 0$ for all $x_i, x_j > 0$ then $\gamma_k > 0$ is equivalent to $0 < x_k < \Phi_k(x_i, x_j)$ meaning that S is bounded by the plane $x_k = 0$ and the positive part Γ_k of the cone $\gamma_k = 0$. By symmetry we have the same for Γ_i and Γ_j. Thus $\partial(S) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and hence $\partial(\Sigma \cap S) = s_1 \cup s_2 \cup s_3$.

Consider now the pair (i, j). The equations $\gamma_i = 0$ and $\gamma_j = 0$ defining the surfaces Γ_i and Γ_j can admit only the following two family of common solutions $x_i = x_j$, $x_k = 0$ and $x_i = x_k$, $x_j = 0$. But we need in positive solutions only. Hence $\Gamma_i \cap \Gamma_j = \emptyset$ for all positive x_1, x_2, x_3. By symmetry the same assertions hold for the pairs (i, k) and (j, k).

Parameterizations of the curves s_1, s_2 and s_3. Due to symmetry fix any unordered triple (i, j, k). The parametric representation $x_k = t^{-1}\alpha^{-2}$, $x_i = t\alpha$, $x_j = \alpha$ of the curve s_k can be obtained putting $x_k = x_i^{-1}x_j^{-1}$ in $\gamma_k = 0$. Then we have the following polynomial equation of degree 6 in two variables x_i and x_j: $x_i^2x_j^2(x_i - x_j)^2 + 2x_ix_j(x_i + x_j) - 3 = 0$.

Substituting $x_i = tx_j$, $x_j = \sqrt[3]{u}$ into the obtained equation and solving it with respect to u we find its two different roots $u_1 := \left(-t - 1 + 2\sqrt{t^2 - t + 1}\right)t^{-1}(t-1)^{-2}$, $u_2 := \left(-t - 1 - 2\sqrt{t^2 - t + 1}\right)t^{-1}(t-1)^{-2}$, where $t > 0$, $t \neq 1$, but the second of them, taken with the minus sign, gives only negative values of x_i and x_j.

Denote $\widetilde{\alpha}(t) = \sqrt[3]{u_1(t)} > 0$. Note that $\lim_{t\to 0+}\widetilde{\alpha}(t) = +\infty$ and $\lim_{t\to+\infty}\widetilde{\alpha}(t) = 0$. This predicts the behavior of the curve s_k for values $t \to 0+$ and $t \to +\infty$ of the parameter t: $\lim_{t\to+\infty} x_j(t) = 0$, $\lim_{t\to+\infty} x_i(t) = \lim_{t\to+\infty} x_k(t) = +\infty$ and $\lim_{t\to 0+} x_i(t) = 0$, $\lim_{t\to 0+} x_j(t) = \lim_{t\to 0+} x_k(t) = +\infty$.

Connectedness of the curves s_1, s_2 and s_3. Note also that $\lim_{t\to 1+}\widetilde{\alpha}(t) = \lim_{t\to 1-}\widetilde{\alpha}(t) = p_0 := \sqrt[3]{6}/2$. Hence assigning $\alpha(1) := p_0$ and

$$\alpha(t) := \begin{cases} \widetilde{\alpha}(t), & \text{if } t > 0,\ t \neq 1, \\ p_0, & \text{if } t = 1 \end{cases}$$

we define a continuous function $\alpha\colon G \to G$ on $G := (0, +\infty)$. Therefore in the standard topology of $\mathbb{R}^3$ the set (curve) $s_k = F(G) \subset G^3$ must be connected as a continuous image of the connected set G under a function $F\colon G \to G^3$ with continuous coordinate components $x_i, x_j, x_k\colon G \to G$ such that $x_i(t) = t\alpha(t)$, $x_j(t) = \alpha(t)$ and $x_k(t) = t^{-1}\alpha(t)^{-2}$.

Smoothness of the curves s_1, s_2 and s_3 can be proved using their parametric equations. But we prefer another way. Due to symmetry it suffices to prove smoothness of the curve $s_i = \Sigma \cap \Gamma_i$. Since Σ and Γ_i are smooth (regular) surfaces it remains to show that their intersection is transversal, in other words their gradient vectors $\nabla V = (x_2x_3, x_1x_3, x_1x_2) = \left(x_1^{-1}, x_2^{-1}, x_3^{-1}\right)$ and $\nabla\gamma_i = (\gamma_{i1}, \gamma_{i2}, \gamma_{i3})$ are linearly independent along s_i, where

$$\gamma_{ij} := \frac{\partial\gamma_i}{\partial x_j} = \begin{cases} x_i + x_j - x_k, & \text{if } j \neq i, \\ -3x_i + x_j + x_k, & \text{if } j = i, \end{cases}$$

for $i, j \in \{1, 2, 3\}$. Due to symmetry fix any i and suppose by contrary that $\nabla\gamma_i = c\,\nabla V$ for some real $c \neq 0$. This means that the equalities $\gamma_{ij} = cx_j^{-1}$ hold for $j \in 1, 2, 3$. Then for $j \neq i$ and $k \neq i$ we obtain equalities $(x_i + x_j - x_k)\,x_j = (x_i + x_k - x_j)\,x_k = c$ equivalent to $(x_j - x_k)(x_i + x_j + x_k) = 0$ which is impossible for $x_i \neq x_j \neq x_k \neq x_i$. Actually we proved the more strong fact that the normal vectors ∇V and $\nabla\gamma_i$ are linearly independent not only along s_i, but everywhere where the surfaces Σ and Γ_i are defined excepting points (x_1, x_2, x_3) with non positive or coincided components.

(2) *Intersections of s_1, s_2, s_3 with I_1, I_2, I_3.* Due to symmetry it suffices to take the invariant curve I_k of the system (5) defined as $x_i = x_j = p$, $x_k = p^{-2}$, $p > 0$. Consider the curve s_k. The question is whether I_k will cross the curve s_k or not. It suffices to answer this question for I_k and the surface Γ_k because existing of a point Z in $(0, +\infty)^3$ such that $Z \in I_k \cap \Gamma_k$ implies $Z \in I_k \subset \Sigma$ and hence $Z \in \Sigma \cap \Gamma_k = s_k$. Thus substituting $x_i = x_j = p$, $x_k = p^{-2}$ into the equation $\gamma_k = 0$ of Γ_k, we obtain the equation $\gamma_k = (4p^3 - 3)p^{-4} = 0$ which can admit the single root $p = p_0 = \sqrt[3]{6}/2$ providing the unique common point $x_i = x_j = p_0$, $x_k = p_0^{-2}$ of I_k with s_k.

Consider now any curve s_i such that $i \neq k$. Then we obtain an incompatible system of equations $x_i = x_j = p$, $x_k = p^{-2}$ and $\gamma_i = 0$ because of $\gamma_i = p^{-4} \neq 0$. Moreover, s_i asymptotically tends to I_k as $p \to +\infty$ according to $\lim_{p\to+\infty}\gamma_i = \lim_{p\to+\infty} p^{-4} = 0$. The same result holds for s_j by symmetry in the equation of I_k. Theorem 1 is proved.

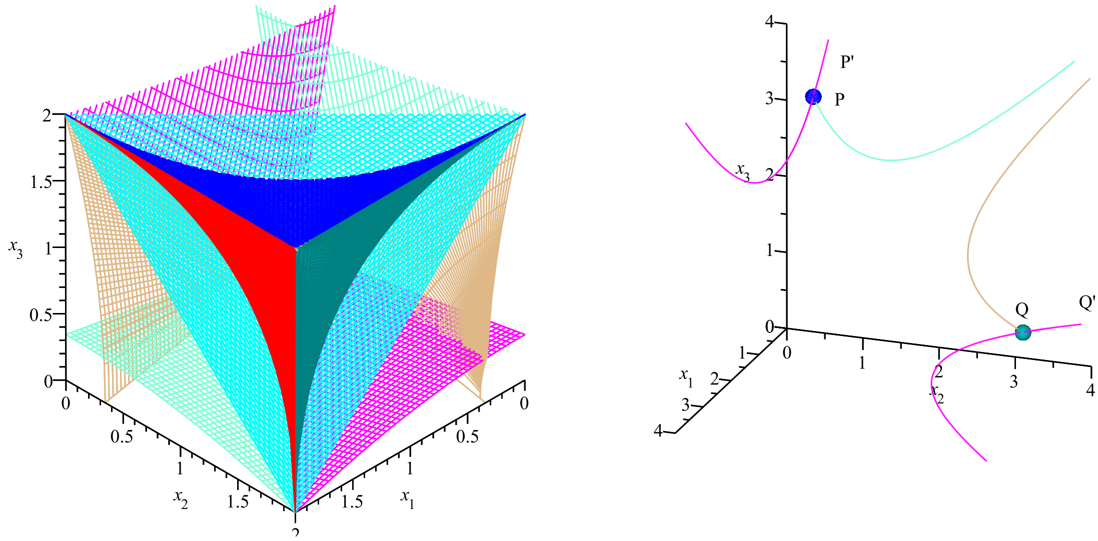

Figure 2. The cones $\Gamma_1, \Gamma_2, \Gamma_3$, $\Lambda_1, \Lambda_2, \Lambda_3$ and the planes $x_k = x_i + x_j$ for $\{i, j, k\} = \{1, 2, 3\}$ (the left panel); Crossing r_2 and r_3 by r_1 (the right panel)

1.2 Proof of Theorem 2

To prove Theorem 2 we need the following Lemma containing auxiliary results.

Lemma 1. For every generalized Wallach space with $a \in (0, 1/2)$ the set R is bounded by the conic surfaces Λ_1, Λ_2 and Λ_3. Each pair Λ_i and Λ_j has intersections along two different straight lines $x_i = x_j = u$, $x_k = 0$ and $x_i = x_j = av$, $x_k = v$, where $u, v > 0$.

The cones Λ_1, Λ_2 and Λ_3 are depicted in the left panel of Figure 2 in magenta, aquamarine and burlywood colors respectively.

Proof of Lemma 1. Consider the surface Λ_k. Since $D := x_i^2 - a^{-1}x_i x_j + x_j^2$ is symmetric with respect to x_i and x_j it can be considered as a quadratic polynomial in x_j without loss of generality. Then $D \le 0$ if $m x_i \le x_j \le M x_i$ and $D > 0$ if $0 < x_j < m x_i$ or $x_j > M x_i$, where

$$m = m(a) := \left(1 - \sqrt{1 - 4a^2}\right)(2a)^{-1}, \qquad M = M(a) := \left(1 + \sqrt{1 - 4a^2}\right)(2a)^{-1}. \qquad (11)$$

It is easy to see that $0 < m(a) < M(a)$ for all $a \in (0, 1/2)$.

Depending on the sign of D the inequality $\lambda_k > 0$ admits the positive solution $x_k > \sqrt{D}$ if $D > 0$ and any $x_k > 0$ can satisfy $\lambda_k > 0$ if $D \le 0$. This means that besides the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ the set R is bounded by two disjoint connected components Λ_{kj} and Λ_{ki} of the surface $\Lambda_k = \Lambda_{ki} \cup \Lambda_{kj}$ defined by the same equation $x_k = \Psi(x_i, x_j) := \sqrt{x_i^2 - a^{-1}x_i x_j + x_j^2}$ but on the different domains $\left\{(x_i, x_j) \in \mathbb{R}^2 \;\middle|\; x_i > 0,\, 0 < x_j < m x_i\right\}$ and $\left\{(x_i, x_j) \in \mathbb{R}^2 \;\middle|\; x_i > 0,\, x_j > M x_i\right\}$ respectively.

Due to symmetry the same properties hold for the surfaces Λ_i and Λ_j as well. Thus $\partial(R) = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$.

By the same reason it suffices to analyze only $\Lambda_i \cap \Lambda_j$. Assume that some triple (x_1, x_2, x_3) satisfies both of $\lambda_i = 0$ and $\lambda_j = 0$. Then $\lambda_i - \lambda_j = 0$ and $\lambda_i + \lambda_j = 0$ imply the system of equations $(x_i - x_j)(x_k - 2a(x_i + x_j)) = 0$ and $x_k(x_i + x_j - 2a x_k) = 0$. In what follows that the system of the equations $\lambda_i = 0$ and $\lambda_j = 0$ can admit only the following two different families of one-parametric solutions $x_i = x_j = u$, $x_k = 0$ and $x_i = x_j = av$, $x_k = v$ with parameters $u, v > 0$. Lemma 1 is proved.

Proof of Theorem 2. (1) Clearly $\partial(\Sigma \cap R) = r_1 \cup r_2 \cup r_3$ directly follows from $\partial(R) = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ proved in Lemma 1. Intersecting both of the connected components Λ_{ki} and Λ_{kj} of the cone Λ_k the surface Σ forms components $r_{ki} = \Sigma \cap \Lambda_{ki}$ and $r_{kj} = \Sigma \cap \Lambda_{kj}$ of the curve r_k such that $r_k = r_{ki} \cup r_{kj}$ and $r_{ki} \cap r_{kj} = \emptyset$.

Smoothness of the components of r_1*,* r_2 *and* r_3. Consider the curve r_k. We claim that the gradient vectors $\nabla V = \left(x_1^{-1}, x_2^{-1}, x_3^{-1}\right)$ and $\nabla\lambda_k = (\lambda_{k1}, \lambda_{k2}, \lambda_{k3})$ of the surfaces Σ and Λ_k are linearly independent for all positive x_1, x_2, x_3 such that $x_1 \neq x_2 \neq x_3 \neq x_1$, where

$$\lambda_{kj} := \frac{\partial\lambda_k}{\partial x_j} = \begin{cases} x_i - 2ax_j, & \text{if } \ j \neq k, \\ 2ax_k, & \text{if } \ j = k, \end{cases}$$

for $k, j \in \{1, 2, 3\}$. Indeed supposing $\nabla\lambda_k = c\nabla V$, where c is a nonzero real number, we obtain immediately an unreachable equality $(x_j - x_i)(x_j + x_i) = 0$. In what follows that each component r_{k1} and r_{k2} of the curve r_k is a smooth curve as a transversal intersection of two smooth surfaces.

Connectedness of the components of r_1, r_2, r_3. The variable x_k can be eliminated from the system of equations $x_i x_j x_k = 1$ and $\lambda_k = 0$ to obtain the equation

$$a x_i^2 x_j^2 \left(x_i^2 + x_j^2\right) - x_i^3 x_j^3 - a = 0$$

of the projection of the curve r_k onto the coordinate plane (x_i, x_j). By the same way as in Theorem 1 substituting $x_i = tx_j$, $x_j = \sqrt[3]{u}$ into the last equation and solving it with respect to u we obtain the parametric equation

$$x_k = t^{-1}\beta^{-2}, \quad x_i = t\beta, \quad x_j = \beta$$

of the curve r_k, where $\beta = \beta(t) := \left(t^4 - a^{-1}t^3 + t^2\right)^{-\frac{1}{6}} > 0$. It is easy to see that the numbers $m = m(a)$ and $M = M(a)$, $0 < m < M$, given in (11) are different real roots of the polynomial $t^2 - a^{-1}t + 1$ for all $a \in (0, 1/2)$. Therefore β can be rewritten in the form

$$\beta = \beta(t) := \left(t^2(t - m)(t - M)\right)^{-\frac{1}{6}}.$$

In what follows that the function $\beta(t)$ is well defined, continuous and positive valued for $t \in (0, m) \cup (M, +\infty)$. We conclude now that the components r_{ki} and r_{kj} of r_k are respectively continuous images of the connected sets $(0, m)$ and $(M, +\infty)$ under a vector-function with coordinates $x_i(t)$, $x_j(t)$ and $x_k(t)$. Therefore r_{ki} and r_{kj} are connected too.

Note that the components r_{ki} and r_{kj} are symmetric under the permutation $i \to j \to i$. Therefore we can parameterize them on the same interval but using different formulas (8) and (9) respectively. For simplicity we choose the interval $(0, m)$.

Intersections of r_1*,* r_2 *and* r_3. Consider the pair r_1 and r_2. By Lemma 1 the only common line of the surfaces Λ_1 and Λ_2 which consists of points with nonzero coordinates is the straight line $x_1 = x_2 = av$, $x_3 = v$, $v > 0$. This line intersects the surface Σ at a unique point, denote it P_{12}. Indeed substituting $x_1 = x_2 = av$, $x_3 = v$ into $x_1x_2x_3 = 1$ we get the unique value $v = v_0 := a^{-2/3}$. This yields coordinates $\left(a^{1/3}, a^{1/3}, a^{-2/3}\right)$ of P_{12}. Note that P_{12} (the point P in the right panel of Figure 2) is also the only intersection point of the curves r_1 and r_2 (their components r_{12} and r_{21}).

Now a value of t at which P_{12} is located in r_1 can be found from the parametric representation $x_1(t) = t^{-1}\beta(t)^{-2}$, $x_2(t) = t\beta(t)$, $x_3(t) = \beta(t)$ of r_{12}. The condition $x_1 = x_2$ implies an equation $t^{-1}\beta^{-2} = t\beta$ admitting the single root $t_0 = a$ for all $a \in (0, 1/2)$. Therefore the curve r_1 passes through P_{12} at $t = t_0$ only. It should be noted that the curves r_1 and r_2 leave extra pieces after crossing each other. In principle, we can preserve them, but it is advisable to remove them for greater clarity of pictures. Basing on the values of the limits $\lim_{t\to 0+} x_2(t) = 0$, $\lim_{t\to 0+} x_1(t) = \lim_{t\to 0+} x_3(t) = +\infty$

and $\lim_{t\to m-} x_1(t) = 0$, $\lim_{t\to m-} x_2(t) = \lim_{t\to m-} x_3(t) = +\infty$ we conclude that the tail PP' corresponds to values $t \in (a, m)$. Therefore the original interval of parametrization $(0, m)$ can be reduced to the interval $(0, a]$ shown in the text of Theorem 2.

By symmetry the analysis of the pairs $r_1 \cap r_3$ and $r_2 \cap r_3$ (points in teal and red color in the right panel of Figure 1) will be the same using permutations of the indices $\{i, j, k\} = \{1, 2, 3\}$. For example, the equations $x_1(t) = t^{-1}\beta(t)^{-2}$, $x_2(t) = \beta(t)$ and $x_3(t) = t\beta(t)$ define another connected component r_{13} of the curve r_1 (which intersects r_3) on the same interval $(0, a]$. Then coordinates $\left(a^{1/3}, a^{-2/3}, a^{1/3}\right)$ of the point P_{13} (in fact $\{P_{13}\} = r_{13} \cap r_{31}$) can be obtained at the same boundary value $t = a$ (the point Q in the right panel of Figure 2). Analogously at $t \in (a, m)$ we get the tail QQ' of r_{13}.

(2) *Intersections of r_1, r_2, r_3 with I_1, I_2, I_3.* Without loss of generality consider the invariant curve I_k. As in Theorem 1 it suffices to consider the surfaces Λ_i instead of the corresponding curves r_i. The curve I_k crosses both of the curves r_i and r_j (the components r_{ij} and r_{ji}) exactly at their common point P_{ij} because substituting $x_i = x_j = p$, $x_k = p^{-2}$ into $\lambda_i = 0$ and $\lambda_j = 0$ yields the equation

$$\lambda_i = \lambda_j = (p^3 - a)p^{-4} = 0$$

which admit a single root $p = a^{1/3}$ corresponding to P_{ij}. Therefore $I_k \cap r_{ij} \cap r_{ji} = \{P_{ij}\}$.

At the same time I_k approximates both of r_i and r_j (their components r_{ik} and r_{jk}) at infinity. Indeed

$$\lim_{p\to+\infty} \lambda_i = \lim_{p\to+\infty} \lambda_j = \lim_{p\to+\infty} (p^3 - a)p^{-4} = 0.$$

For the curve r_k we have $\lambda_k = (1 - 2a)p^2 + p^{-4} > 0$ under the same substitutions. Therefore I_k never cross r_k, moreover, $\lim_{p\to+\infty} \lambda_k = +\infty$.

(3) *Every singular point of* (5) *belongs to* $\Sigma \cap R$. As it follows from [6] the system of algebraic equations $f_i(x_1, x_2, x_3) = 0$ has the following four families of one-parametric solutions for every $a \in (0, 1/2) \setminus \{1/4\}$:

$$x_1 = x_2 = x_3 = \tau, \qquad x_i = \tau\kappa,\ x_j = x_k = \tau, \quad \tau > 0, \quad \{i, j, k\} = \{1, 2, 3\}, \tag{12}$$

where $\kappa := (1 - 2a)(2a)^{-1}$. At $a = 1/4$ these families merge to the unique family $x_1 = x_2 = x_3 = \tau$.

Substituting $x_i = \tau\kappa$ and $x_j = x_k = \tau$ into the expressions (7) for λ_1, λ_2 and λ_3 we obtain

$$\lambda_1 = \lambda_2 = \lambda_3 = (1 - 2a)(1 + 2a)(4a)^{-1}\,\tau^2 > 0,$$

because $a \in (0, 1/2)$. Obviously,

$$\lambda_1 = \lambda_2 = \lambda_3 = (1 - a)\,\tau^2 > 0$$

at $x_1 = x_2 = x_3 = \tau$. Therefore the straight lines (12) lye in the set R for all $a \in (0, 1/2)$ according to the definition of R. These lines cross the invariant surface Σ at the points (see also [22])

$$\mathbf{o}_0 := (1, 1, 1), \quad \mathbf{o}_1 := (q\kappa, q, q)\,, \quad \mathbf{o}_2 := (q, q\kappa, q)\,, \quad \mathbf{o}_3 := (q, q, q\kappa)\,,$$

being the singular points of the system (5) on Σ, where $q := \sqrt[3]{\kappa^{-1}}$ (obviously, the unique singular point $(1, 1, 1)$ will be obtained if $a = 1/4$). Thus we conclude that $\mathbf{o}_i \in \Sigma \cap R$ for every $a \in (0, 1/2)$ and $i \in \{0, 1, 2, 3\}$.

(4) *Invariancy of the curves l_1, l_2, l_3.* According to [6] the curves I_1, I_2 and I_3 are separatrices of the unique saddle point $\mathbf{o}_0$ (which has the *linear zero type*) of the system (5) if $a = 1/4$. For $a \in (0, 1/2) \setminus \{1/4\}$ the points $\mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3$ are all *hyperbolic type* saddles and $\mathbf{o}_0$ is a stable (respectively

unstable) *hyperbolic* node if $1/4 < a < 1/2$ (respectively if $0 < a < 1/4$). Additionally, each invariant curve I_k is one of two separatrices of the saddle $\mathbf{o}_k$ (see [22]), where $k = 1, 2, 3$. At $a = 1/6$ we have an opportunity to find analytically the second separatrice of each $\mathbf{o}_k$ different from I_k. Indeed it is easy to see that coordinates of $\mathbf{o}_k$ satisfy the system of equations

$$\begin{cases} x_k = x_i + x_j, \quad \{i, j, k\} = \{1, 2, 3\}, \\ x_i x_j x_k = 1, \end{cases} \tag{13}$$

where the equalities $x_k = x_i + x_j$ describe the set of Kähler metrics on a given generalized Wallach space G/H with $a_1 = a_2 = a_3 = a = 1/6$ (see also [4]). Therefore each saddle $\mathbf{o}_k$ belongs to the intersection l_k of the invariant surface Σ with the plane $x_k = x_i + x_j$ (the curves l_1, l_2, l_3 are depicted in the right panel of Figure 1 in cyan color for all indices $\{i, j, k\} = \{1, 2, 3\}$). Parametric equations (10) of the curves l_k can be obtained repeating similar procedures as in the case of the curves s_k and r_k.

It is easy to show that $l_i \cap l_j = \emptyset$ for $i \neq j$. Moreover, we claim that each of l_1, l_2, l_3 is also an invariant curve of the differential system (5). To show it consider the case $k = 3$ due to symmetry. Substitute the parametric representation $x_1 = \phi$, $x_2 = t\phi$, $x_3 = t^{-1}\phi^{-2}$ of the curve l_3 into f_1, f_2 and f_3 in (5), where

$$\phi = \phi(t) := \sqrt[3]{(t^2 + t)^{-1}}, \quad t > 0.$$

For $x_1 = \phi$, $x_2 = t\phi$ and $x_3 = t^{-1}\phi^{-2}$ the functions f_1, f_2, f_3 take the following forms

$$f_1 = -\frac{2}{9}\,\frac{(2t+1)(t-1)}{t(t+1)}, \qquad f_2 = \frac{2}{9}\,\frac{(t+2)(t-1)}{t+1}, \qquad f_3 = \frac{2}{9}\,\frac{(t-1)^2}{t}.$$

The value $t = 1$ providing $f_1 = f_2 = f_3 = 0$ gives a stationary trajectory, namely it is the singular point $\mathbf{o}_3 = (q, q, q\kappa)$ itself. Thus assume $t \neq 1$. The identities

$$\begin{aligned} \frac{dx_2}{dx_1} &= \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} \equiv \frac{(t\phi)'}{\phi'} = -\frac{(t+2)t}{2t+1} = \frac{f_2}{f_1}, \\ \frac{dx_3}{dx_1} &= \frac{\left(t^{-1}\phi^{-2}\right)'}{\phi'} = -\frac{t^2-1}{2t+1} = \frac{f_3}{f_1}, \\ \frac{dx_3}{dx_2} &= \frac{\left(t^{-1}\phi^{-2}\right)'}{(t\phi)'} = \frac{t^2-1}{t(t+2)} = \frac{f_3}{f_2} \end{aligned}$$

imply that l_3 is a trajectory of (5) for $t > 0$ and $t \neq 1$. Moreover, l_3 passes through the singular point $\mathbf{o}_3$. This means that l_3 is a separatrice of $\mathbf{o}_3$. Invariancy of the curves l_1 and l_2 respectively passing through $\mathbf{o}_1$ and $\mathbf{o}_2$ can be proved using the same idea. Theorem 2 is proved.

Remark 1. As it was noted in the proof of Theorem 2 the equations (8) define for $t \in (M, +\infty)$ the same curve as (9) for $t \in (0, m)$. In the case $t \in (M, +\infty)$ the tail removing procedure leads to the equation $t^{-1}\beta^{-2} = \beta$ equivalent to $at^2 - (a^2 + 1)t + a = 0$. Its first root $t = a$ corresponds to the point P_{ki} and the second root $t = 1/a$ gives the point P_{kj}. Obviously $0 < a < m < M < 1/a$ for all $a \in (0, 1/2)$. Therefore both components of each curve r_k can be parameterized by one formula, say (8), but using the different intervals $(0, a]$ and $[1/a, +\infty)$.

Remark 2. We proved that all singular points $\mathbf{o}_0, \mathbf{o}_1, \mathbf{o}_2$ and $\mathbf{o}_3$ of the normalized Ricci flow on generalized Wallach spaces with $a_1 = a_2 = a_3 = a$ belong to the set $\Sigma \cap R$ of metrics with positive Ricci curvature. Unfortunately a similar assertion does not hold for the set $\Sigma \cap S$. Lemma 3 in [22] shows that there exists a critical value $a = 3/14$ such that $\mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3 \in \Sigma S$ only if $a \in (3/14, 1/2)$ and the boundary cases $\mathbf{o}_i \in s_i$ $(i = 1, 2, 3)$ hold if $a = 3/14$. The only generalized Wallach spaces which admit metrics with positive sectional curvature are the Wallach spaces (1) which satisfy the condition $a \in (0, 3/14)$.

Remark 3. The case $a = 1/6$ is original, where Kähler metrics provide separatrices of saddles $\mathbf{o}_i$. For $a \neq 1/6$ it is a difficult problem to find similar separatrices analytically. Knowing all separatrices allows to predict the dynamics of the Ricci flow in more detail. To demonstrate the main idea consider an arbitrarily chosen singular point in the case $a = 1/6$. Without loss of generality take $\mathbf{o}_3 = \left(2^{-1/3}, 2^{-1/3}, 2^{2/3}\right)$ (the Kähler-Einstein metric) and observe that the curve l_3 defined by the equations $x_3 = x_1 + x_2$ and $x_1x_2x_3 = 1$ coincides with the unstable manifold W_3^u of $\mathbf{o}_3$ as it was shown in [22]. The stable manifold of $\mathbf{o}_3$ is $W_3^s := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_3 = p^{-2},\, x_1 = x_2 = p,\; 0 < p < 1\right\} \subset I_3$. It is clear now that controlling by W_i^s and W_i^u trajectories of (5) never can leave the domain bounded by the curves (13) because of no trajectory originated in that domain can intersect separatrices by the uniqueness of a solution of an initial value problem. This explains the fact proved in Theorem 4 in [4] that Riemannian metrics (2) on generalized Wallach spaces with $a = 1/6$ (on the Wallach space $\mathrm{SU}(3)/T_{\max}$ in particular) preserve the positivity of their Ricci curvature for $x_k < x_i + x_j$ $(\{i, j, k\} = \{1, 2, 3\})$. In Figure 3 the separatrices l_1, l_2, l_3 and some trajectories of (5) are depicted for illustrations.

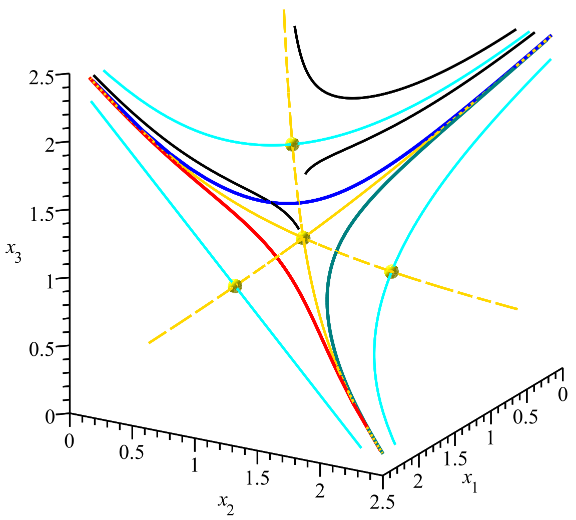

Figure 3. The case $a = 1/6$: the separatrices l_i (in cyan color), I_i (in yellow color) of the saddles $\mathbf{o}_i$ and some trajectories (in black color) of system (5), $i = 1, 2, 3$

2 *Additional remarks*

i) The well known fact that the positivity of the Ricci curvature follows from the positivity of the sectional curvature can be justified and illustrated via inclusion $S \subset R$, where S is depicted in Figure 2 as a set bounded by three cones in red, teal and blue colors, respectively R is bounded by six conic surfaces in magenta, aquamarine and burlywood colors.

To establish $S \subset R$ for all $a \in (0, 1/2)$ it suffices to show the inclusion $\partial(S) \subset R$. We will follow this opportunity since a direct attempt to establish $S \subset R$ leads to pairs of inequalities of the kind $\gamma_i > 0$ and $\lambda_i > 0$ whose analysis is much more complicated than to deal with the system consisting of one equation $\gamma_i = 0$ and one inequality $\lambda_i > 0$:

$$\begin{cases} (x_j - x_k)^2 + 2x_i(x_j + x_k) - 3x_i^2 = 0, \\ x_j x_k + a\left(x_i^2 - x_j^2 - x_k^2\right) > 0. \end{cases} \tag{14}$$

By symmetry fix any $i \in \{1,2,3\}$ and consider the component Γ_i of the boundary of S. Every point of the cone Γ_i belongs to some its generator line $x_i = \nu t,\ x_j = \mu t,\ x_k = t,\ t > 0$, where $\mu = 1 - \nu + 2\sqrt{\nu(\nu-1)} > 0,\ \nu > 1$ (see also [22]). Indeed generators satisfy the equation in (14) and the inequality in (14) takes the form $(X - Y)\,t^2 > 0$ with $X := \big(4a(\nu-1)+2\big)\sqrt{\nu(\nu-1)}$ and $Y := 4a\nu^2 + (1-6a)\nu + 2a - 1$.

Obviously $X > 0$ for all $a \in (0, 1/2)$ and $\nu > 1$. Since $Y = 0$ has roots $\nu_1 = \frac{2a-1}{4a} < 0$ and $\nu_2 = 1$ the inequality $Y > 0$ holds as well at $\nu > 1$. Thus $X - Y > 0$ is equivalent to $X^2 - Y^2 = (\nu-1)\,p(\nu) > 0$, where the quadratic polynomial $p(\nu) = 8a\nu^2 - (2a+3)(2a-1)\nu + (2a-1)^2$ admits two different negative roots $\nu_1 = \frac{2a-1}{16a}\left(2a+3+\sqrt{(2a-1)(2a-9)}\right)$ and $\nu_2 = \frac{2a-1}{16a}\left(2a+3-\sqrt{(2a-1)(2a-9)}\right)$ for every $a \in (0,1/2)$. It follows then $p(\nu) > 0$ and hence $X^2 - Y^2 > 0$ at $\nu > 1$ independently on $a \in (0,1/2)$. Therefore $\lambda_i > 0$ for any point of Γ_i which means that $\Gamma_i \subset R$. Since i was chosen arbitrarily we obtain $\partial(S) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \subset R$ and hence $S \subset R$ with the obvious consequence $\Sigma \cap S \subset \Sigma \cap R$.

ii) There are useful asymptotical representations for practical aims. For instance, at $t \to 0$ the expressions $x_1(t) = x_3(t) = t^{-1/3} + \mathcal{O}\big(t^{5/3}\big)$, $x_2(t) = t^{2/3} + \mathcal{O}\big(t^{8/3}\big)$ are valid for coordinates of points of the curve s_3 defined as a variety of solutions of the system

$$\begin{cases} (x_1 - x_2)^2 + 2x_3(x_1 + x_2) - 3x_3^2 = 0, \\ x_1 x_2 x_3 = 1. \end{cases} \tag{15}$$

For t tending to 0 the curve r_1 : $\begin{cases} x_2x_3 + a\left(x_1^2 - x_2^2 - x_3^2\right) = 0, \\ x_1x_2x_3 = 1 \end{cases}$ has a similar asymptotic $x_1(t) = t^{-1/3} + \mathcal{O}\big(t^{5/3}\big)$, $x_2(t) = t^{2/3} + \frac{t^{5/3}}{6a} + \mathcal{O}\big(t^{8/3}\big)$, $x_3(t) = t^{-1/3} + \frac{t^{2/3}}{6a} + \mathcal{O}\big(t^{5/3}\big)$ in accordance with the fact that s_3 and $r_{12} \subset r_1$ approach the same invariant curve I_2 at infinity.

iii) Often it is easier to deal with a planar analysis of the dynamics of the normalized Ricci flow. Choose the coordinate plane $x_3 = 0$ without loss of generality. Then the projection of the set $\Sigma \cap S$ of Riemannian metrics with positive sectional curvature onto the plane $x_3 = 0$ is bounded by the following plane curves s_1', s_2' and s_3' defined implicitly $3x_1^4x_2^2 - 2x_1^3x_2^3 - x_1^2x_2^4 - 2x_1^2x_2 + 2x_1x_2^2 - 1 = 0$, $3x_2^4x_1^2 - 2x_2^3x_1^3 - x_2^2x_1^4 - 2x_2^2x_1 + 2x_2x_1^2 - 1 = 0$ and $x_1^4x_2^2 - 2x_1^3x_2^3 + x_1^2x_2^4 + 2x_1^2x_2 + 2x_1x_2^2 - 3 = 0$.

For example the equation of s_3' can be obtained eliminating x_3 in the system (15).

Analogously, boundary curves of the projection of the set $\Sigma \cap R$ of Ricci positive metrics onto the plane $x_3 = 0$ have equations $ax_1^4x_2^2 - ax_1^2x_2^4 + x_1x_2^2 - a = 0$, $ax_2^4x_1^2 - ax_2^2x_1^4 + x_2x_1^2 - a = 0$ and $ax_1^4x_2^2 + ax_1^2x_2^4 - x_1^3x_2^3 - a = 0$.

Projections of the Kähler metrics $x_1 = x_2 + x_3$, $x_2 = x_1 + x_3$ and $x_3 = x_1 + x_2$ will be defined by $x_1x_2(x_1 - x_2) = 1$, $x_1x_2(x_2 - x_1) = 1$ and $x_1x_2(x_1 + x_2) = 1$ respectively.

We recommend to compare the pictures demonstrated in this paper with planar pictures depicted in the right panels of Figures 3, 6 and 7 obtained in [4] in the coordinate plane (x_1, x_2).

Author Contributions

The single author contributed to this work.

Conflict of Interest

The author declares no conflict of interest.

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The compact eighth-order of approximation difference schemes for fourth-order differential equation

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Local and nonlocal boundary value problems (LNBVPs) related to fourth-order differential equations (FODEs) were explored. To tackle these problems numerically, we introduce novel compact four-step difference schemes (DSs) that achieve eighth-order of approximation. These DSs are derived from a novel Taylor series expansion involving five points. The theoretical foundations of these DSs are validated through extensive numerical experiments, demonstrating their effectiveness and precision.

Keywords: Taylor's decomposition on five points (TDFP), LNBVPs, DSs, approximation, numerical experiment.

2020 Mathematics Subject Classification: 34B10, 35K10, 49K40.

Introduction

In applied sciences, achieving high precision in numerical algorithms is crucial, particularly when exact solutions are not feasible. Currently, a key focus is on developing and analyzing highly accurate DSs for ordinary and partial DEs with variable coefficients. Previous research has extensively explored the use of Taylor series expansions for constructing high-order compact finite DSs. For example, on two and three points Taylor's decomposition (TDs) has been used for approximate solutions of linear ordinary and partial DSs, as detailed in sources [1], [2], [3]. Further advancements include the use of three-step schemes with fourth-order of accuracy, derived from TDs on four points, for the numerical solution of several LNBVPs related to third-order DEs, as discussed in [4], [5], and [6]. These techniques have also been applied to third-order time-varying linear dynamical systems, as evidenced by the numerical analysis conducted on an up-converter in communication systems.

Recent studies [7] and [8] have expanded this work to include four-step DSs with fourth- and sixth-order accuracy, generated from TDFPs, specifically for linear ordinary DEs with boundary value problems (BVPs).

BVPs for ordinary DEs are fundamental in both theoretical and applied contexts, modeling a wide array of physical, biological, and chemical processes. Notable applications include Timoshenko's work on elasticity [9], Soedel's analysis of structural deformation [10], and Dulacska's research on soil settlement effects [11].

The literature on BVPs for higher-order DEs is extensive, including recent contributions [12], [13], and [14]. For a comprehensive overview of known results and additional references, see the monographs [15], [16], and paper [17].

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Nonlinear FODEs, often termed beam equations, have also been studied under various boundary conditions. Zill and Cullen [18] provide a clear discussion and physical interpretation of boundary conditions for linear beam equations, contrasting with other conditions like conjugate [19], focal [12], [20], and [21].

In this paper, we introduce new compact eighth-order finite DSs, derived from an innovative TDFPs, for solving FODEs with variable coefficients.

We consider FSDSs of eighth-order approximation for the numerical solutions of three types of BVPs

$$\begin{cases} u^{(4)}(s) + a(s)u(s) = F(s), 0 < s < \Upsilon, \\ u(0) = \varphi, u^{(1)}(0) = \eta, u(\Upsilon) = \omega, u^{(1)}(\Upsilon) = \varrho, \end{cases} \tag{1}$$

$$\begin{cases} u^{(4)}(s) + a(s)u(s) = F(s), 0 < s < \Upsilon, \\ u(0) = \varphi, u(0) = \eta, u(\Upsilon) = \omega, u^{(2)}(\Upsilon) = \varrho, \end{cases} \tag{2}$$

$$\begin{cases} u^{(4)}(s) + a(s)u(s) = F(s), 0 < s < \Upsilon, \\ u(0) = \varphi, u^{(3)}(0) = \eta, u(\Upsilon) = \omega, u^{(3)}(\Upsilon) = \varrho, \end{cases} \tag{3}$$

and of the nonlocal BVP

$$\begin{cases} u^{(4)}(s) + a(s)u(s) = F(s), 0 < s < \Upsilon, u(0) = u(\Upsilon) + \varphi, \\ u^{(1)}(0) = u^{(1)}(\Upsilon) + \eta, u^{(2)}(0) = u^{(2)}(\Upsilon) + \omega, u^{(3)}(0) = u^{(3)}(\Upsilon) + \varrho \end{cases} \tag{4}$$

for the FODEs. We introduce the uniform grid space

$$[0, \Upsilon]_h = \{y_k = kh, k = 0, 1, \cdots, N, Nh = \Upsilon\}.$$

The primary objective of this paper is to develop highly accurate four-step DSs for solving local and nonlocal FODEs. We introduce eighth-order accurate DSs generated by a new technique based on a five-point stencil: $y_{k\pm 2}$, $y_{k\pm 1}$, and y_k within the interval $[0, \Upsilon]_h$. The theoretical underpinnings of these schemes are corroborated by numerical experiments. The structure of the paper is as follows: Section 1 details the construction of the new technique using five points. Sections 2 through 5 explore local BVPs (1), (2), (3) and a nonlocal BVP (4).

1 *A new TDFPs*

The design of eighth order of approximation DSs for the numerical solutions of the LNBVPs (1), (2), (3), and (4) is based on the subsequent theorem on new TDFPs.

Theorem 1.1. Let $W(y)$ be a function defined on the interval $[0, \Upsilon]$ with a continuous twelfth derivative. Then the subsequent relation is satisfied:

$$h^{-4}(W(y_{k+2}) - 4W(y_{k+1}) + 6W(y_k) - 4W(y_{k-1}) + W(y_{k-2})) \tag{5}$$

$$= \frac{76}{105}W^{(4)}(y_k) + \frac{9}{70}(W^{(4)}(y_{k+1}) + W^{(4)}(y_{k-1})) + \frac{1}{105}(W^{(4)}(y_{k+2}) + W^{(4)}(y_{k-2}))$$

$$-\frac{97}{1680}h^4 W^{(8)}(y_k) + o(h^8).$$

Proof. By applying Taylor's formula, we obtain

$$h^{-4}(W(y_{k+2}) - 4W(y_{k+1}) + 6W(y_k) - 4W(y_{k-1}) + W(y_{k-2})) \tag{6}$$

$$= W^{(4)}(y_k) + W^{(6)}(y_k)\frac{1}{6}h^2 + W^{(8)}(y_k)\frac{1}{80}h^4 + W^{(10)}(y_k)\frac{17}{7!6}h^6 + o(h^8).$$

Applying the method of undetermined coefficients(MUCs), we will aim to find

$$h^{-4}(W(y_{k+2})-4W(y_{k+1})+6W(y_k)-4W(y_{k-1})+W(y_{k-2}))$$
$$-\alpha W(y_k)-\beta(W^{(4)}(y_{k+1})+W^{(4)}(y_{k-1}))-\gamma(W^{(4)}(y_{k+2})+W^{(4)}(y_{k-2}))-dh^4W^{(8)}(y_k)=o(h^8).$$

Utilizing Taylor's formula, we derive

$$\alpha W^{(4)}(y_k)+\beta(W^{(4)}(y_{k+1})+W^{(4)}(y_{k-1}))+\gamma(W^{(4)}(y_{k+2})+W^{(4)}(y_{k-2}))$$
$$=(\alpha+2\beta+2\gamma)W^{(4)}(y_k)+(\beta+4\gamma)W^{(6)}(y_k)h^2+(\frac{1}{12}\beta+\frac{4}{3}\gamma)W^{(8)}(y_k)h^4$$
$$+(\frac{1}{5!3}\beta+\frac{2^7}{6!}\gamma)W^{(10)}(y_k)h^6+o(h^8).$$

Using formula (6) and above formula , we get

$$h^{-4}(W(y_{k+2})-4W(y_{k+1})+6W(y_k)-4W(y_{k-1})+W(y_{k-2}))$$
$$-\alpha W^{(4)}(y_k)-\beta(W^{(4)}(y_{k+1})+W^{(4)}(y_{k-1}))-\gamma(W^{(4)}(y_{k+2})+W^{(4)}(y_{k-2}))-dh^4W^{(8)}(y_k)$$
$$=(1-\alpha-2\beta-2\gamma)W^{(4)}(y_k)+(\frac{1}{6}-\beta-4\gamma)W^{(6)}(y_k)h^2+(\frac{1}{80}-\frac{1}{12}\beta-\frac{4}{3}\gamma-d)W^{(8)}(y_k)h^4$$
$$+(\frac{17}{7!6}-\frac{1}{5!3}\beta-\frac{2^7}{6!}\gamma)W^{(10)}(y_k)h^6+o(h^8).$$

By setting the coefficient of the lowest power of h to zero, we derive the following system of algebraic equations(SAEs).

$$\begin{cases}\alpha+2\beta+2\gamma=1,\\ \beta+4\gamma=\frac{1}{6},\\ \frac{1}{12}\beta+\frac{4}{3}\gamma+d=\frac{1}{80},\\ \frac{1}{5!3}\beta+\frac{2^7}{6!}\gamma=\frac{17}{7!6}.\end{cases}$$

Upon resolving this SAEs, we find $\alpha=\dfrac{76}{105}$, $\beta=\dfrac{9}{70}$, $\gamma=\dfrac{1}{105}$, $d=-\dfrac{97}{1680}$. The relation (5) is obtained. Theorem 1.1 is established.

Theorem 1.2. Let $W(y)$ be a function defined on the interval $[0,\Upsilon]$ with a continuous fifth derivative. Then the subsequent relation holds:

$$W^{(1)}(y_k)=\frac{2}{3h}(W(y_{k+1})-W(y_{k-1}))-\frac{1}{12h}(W(y_{k+2})-W(y_{k-2}))+o\left(h^4\right). \tag{7}$$

Proof. By applying Taylor's formula, we obtain

$$W^{(1)}(y_k)=\beta(W(y_{k+1})-W(y_{k-1}))+\gamma(W(y_{k+2})-W(y_{k-2}))+o\left(h^4\right).$$

Utilizing Taylor's formula, we derive

$$(h^{-1}-(2\beta+4\gamma))W^{(1)}(y_k)h+(\frac{2}{3!}\beta+\frac{16}{3!}\gamma)W^{(3)}(y_k)h^3+(\beta+\gamma)o(h^5).$$

By setting the coefficient of the lowest power of h to zero, we derive the following SAEs.

$$\begin{cases}2\beta+4\gamma=h^{-1},\\ \frac{2}{3!}\beta+\frac{16}{3!}\gamma=0.\end{cases}$$

Upon resolving this SAEs, we find $\beta=\frac{2}{3}h^{-1}$, $\gamma=-\frac{1}{12}h^{-1}$. So, relation (7) is proved. Theorem 1.2 is established.

Theorem 1.3. Let $W(y)$ be a function defined on the interval $[0, \Upsilon]$ with a continuous sixth derivative. Then the subsequent relation holds:

$$W^{(2)}(y_k) = \frac{4}{3h^2}\left(W(y_{k+1}) + W(y_{k-1}) - 2W(y_k)\right) \tag{8}$$

$$-\frac{1}{12h^2}\left(W(y_{k+2}) + W(y_{k-2}) - 2W(y_k)\right) + o\left(h^4\right).$$

Proof. By applying Taylor's formula, we obtain

$$W^{(2)}(y_k) = \beta\left(W(y_{k+1}) + W(y_{k-1}) - 2W(y_k)\right) + \gamma\left(W(y_{k+2}) + W(y_{k-2}) - W(y_k)\right) + o\left(h^4\right).$$

Utilizing Taylor's formula, we derive

$$\left(h^{-2} - \left(\frac{2}{2!}\beta + \frac{8}{2!}\gamma\right)\right) W^{(2)}(y_k)\,h^2 + \left(\frac{2}{4!}\beta + \frac{32}{4!}\gamma\right) W^{(4)}(y_k)\,h^4 + (\beta+\gamma)o(h^6).$$

To obtain the SAEs, equate the coefficients of the smallest power of h in the above identity to 0.

$$\begin{cases} \frac{2}{2!}\beta + \frac{8}{2!}\gamma = h^{-2}, \\ \frac{2}{4!}\beta + \frac{32}{4!}\gamma = 0. \end{cases}$$

Upon resolving this SAEs, we find $\beta = \frac{4}{3}h^{-2}$, $\gamma = -\frac{1}{12}h^{-2}$. So, relation (8) is proved. Theorem 1.3 is established.

Theorem 1.4. Let $W(y)$ be a function defined on the interval $[0, \Upsilon]$ with a continuous seventh derivative. Then the subsequent relation holds:

$$W^{(3)}(y_k) = \frac{896}{159h^3}\left(W(y_{k+1}) - W(y_{k-1}) - 2W^{(1)}(y_k)\,h\right) \tag{9}$$

$$-\frac{419}{1272h^3}\left(W(y_{k+2}) - W(y_{k-2}) - 4W^{(1)}(y_k)\,h\right) + o\left(h^4\right).$$

Proof. Applying the MUCs, we will aim to find

$$W^{(3)}(y_k) = \beta\left(W(y_{k+1}) - W(y_{k-1}) - 2W^{(1)}(y_k)\,h\right) + \gamma\left(W(y_{k+2}) - W(y_{k-2}) - 4W^{(1)}(y_k)\,h\right)$$

$$+h^4\left(p\left(W^{(4)}(y_{k+1}) - W^{(4)}(y_{k-1})\right) + q\left(W^{(4)}(y_{k+2}) - W^{(4)}(y_{k-2})\right)\right) + o\left(h^8\right).$$

By applying Taylor's formula, we obtain

$$\left(h^{-3} - \left(\frac{2}{3!}\beta + \frac{16}{3!}\gamma\right)\right) W^{(3)}(y_k)\,h^3 + \left(\frac{2}{5!}\beta + \frac{64}{5!}\gamma\right) W^{(5)}(y_k)\,h^5 + (\beta+\gamma)\,o\left(h^7\right).$$

To obtain the SAEs, equate the coefficients of the smallest power of h in the above identity to 0.

$$\begin{cases} \frac{2}{3!}\beta + \frac{16}{3!}\gamma = h^{-3}, \\ \frac{2}{5!}\beta + \frac{64}{5!}\gamma = 0. \end{cases}$$

Upon resolving this SAEs, we find $\beta = \dfrac{896}{159}h^{-3}$, $\gamma = -\dfrac{419}{1272}h^{-3}$. So, relation (9) is proved. Theorem 1.4 is established.

2 Local BVP (1)

Let us consider BVP (1). For the application of TDFPs (5), we have to give the eighth order of approximation formulas for $W^{(1)}(0)$ and $W^{(1)}(\Upsilon)$.

Theorem 2.1. Let $W(y)$ be a function defined on the interval $[0,\Upsilon]$ with a continuous fifth derivative. Then the subsequent relations hold:

$$W^{(1)}(0) = h^{-1}\left\{\frac{2223}{518}\left(W(h)-W(0)\right)-\frac{3735}{1036}\left(W(2h)-W(0)\right)+\frac{1535}{777}\left(W(3h)-W(0)\right)\right.$$
$$\left.-\frac{45}{74}\left(W(4h)-W(0)\right)+\frac{243}{2590}\left(W(5h)-W(0)\right)-\frac{23}{3108}\left(W(6h)-W(0)\right)\right\}$$
$$-\frac{69h^3}{518}\left(W^{(4)}(h)-W^{(4)}(0)\right)+\frac{21h^3}{148}\left(W^{(4)}(2h)-W^{(4)}(0)\right)+o\left(h^8\right), \tag{10}$$

$$W^{(1)}(\Upsilon) = h^{-1}\left\{-\frac{2223}{518}\left(W(\Upsilon-h)-W(\Upsilon)\right)+\frac{3735}{1036}\left(W(\Upsilon-2h)-W(\Upsilon)\right)\right.$$
$$-\frac{243}{2590}\left(W(\Upsilon-5h)-W(\Upsilon)\right)+\frac{23}{3108}\left(W(\Upsilon-6h)-W(\Upsilon)\right)$$
$$-\frac{1535}{777}\left(W(\Upsilon-3h)-W(\Upsilon)\right)+\frac{45}{74}\left(W(\Upsilon-4h)-W(\Upsilon)\right)+o\left(h^8\right). \tag{11}$$

Proof. Applying the MUCs, we will aim to find

$$W^{(1)}(0) = \beta\left(W(h)-W(0)\right)+\gamma\left(W(2h)-W(0)\right)+d\left(W(3h)-W(0)\right)$$
$$+p\left(W(4h)-W(0)\right)+q\left(W(5h)-W(0)\right)+w\left(W(6h)-W(0)\right)$$
$$+h^4 m\left(W^{(4)}(h)-W^{(4)}(0)\right)+h^4 n\left(W^{(4)}(2h)-W^{(4)}(0)\right)+o\left(h^8\right).$$

By applying Taylor's formula, we obtain

$$W^{(1)}(0) = \beta\sum_{l=1}^{8}\frac{h^l}{l!}W^{(l)}(0)+\gamma\sum_{l=1}^{8}\frac{(2h)^l}{l!}W^{(l)}(0)+d\sum_{l=1}^{8}\frac{(3h)^l}{l!}W^{(l)}(0)$$
$$+p\sum_{l=1}^{8}\frac{(4h)^l}{l!}W^{(l)}(0)+q\sum_{l=1}^{8}\frac{(5h)^l}{l!}W^{(l)}(0)+w\sum_{n=1}^{8}\frac{(6h)^l}{l!}W^{(l)}(0)$$
$$+h^4 m\sum_{l=1}^{4}\frac{h^l}{l!}W^{(l+4)}(0)+h^4 n\sum_{l=1}^{4}\frac{(2h)^l}{l!}W^{(l+4)}(0)+o\left(h^8\right).$$

To obtain the SAEs, equate the coefficients of the smallest power of h in the above identity to 0.

$$\begin{cases}
3d+4p+\beta+2\gamma+5q+6w=h^{-1},\\
\frac{9}{2!}d+\frac{16}{2!}p+\frac{1}{2!}\beta+\frac{4}{2!}\gamma+\frac{25}{2!}q+\frac{36}{2!}w=0,\\
\frac{27}{3!}d+\frac{64}{3!}p+\frac{1}{3!}\beta+\frac{8}{3!}\gamma+\frac{125}{3!}q+\frac{216}{3!}w=0,\\
\frac{81}{4!}d+\frac{256}{4!}p+\frac{1}{4!}\beta+\frac{16}{4!}\gamma+\frac{625}{4!}q+\frac{1296}{4!}w=0,\\
\frac{243}{5!}d+\frac{1024}{5!}p+\frac{1}{5!}\beta+\frac{32}{5!}\gamma+\frac{3125}{5!}q+\frac{7776}{5!}w+m+2n=0,\\
\frac{729}{6!}d+\frac{4096}{6!}p+\frac{1}{6!}\beta+\frac{64}{6!}\gamma+\frac{15\,625}{6!}q+\frac{46\,656}{6!}w+\frac{1}{2!}m+\frac{4}{2!}n=0,\\
\frac{2187}{7!}d+\frac{16\,384}{7!}p+\frac{1}{7!}\beta+\frac{128}{7!}\gamma+\frac{78\,125}{7!}q+\frac{279\,936}{7!}w+\frac{1}{3!}m+\frac{8}{3!}n=0,\\
\frac{6561}{8!}d+\frac{65\,536}{8!}p+\frac{1}{8!}\beta+\frac{256}{8!}\gamma+\frac{390\,625}{8!}q+\frac{1679\,616}{8!}w+\frac{1}{4!}m+\frac{16}{4!}n=0,\\
\frac{19\,683}{9!}d+\frac{262\,144}{9!}p\frac{1}{9!}\beta+\frac{512}{9!}\gamma+\frac{1953\,125}{9!}q+\frac{10\,077\,696}{9!}w+\frac{1}{5!}m+\frac{32}{5!}n=0,\\
\frac{59\,049}{10!}d+\frac{1048\,576}{10!}p+\frac{1}{10!}\beta+\frac{1024}{10!}\gamma+\frac{9765\,625}{10!}q+\frac{60\,466\,176}{10!}w+\frac{1}{6!}m+\frac{64}{6!}n=0.
\end{cases}$$

Upon resolving this SAEs, we find $\beta = \dfrac{2223}{518h}$, $\gamma = -\dfrac{3735}{1036h}$, $d = \dfrac{1535}{777h}$, $p = -\dfrac{45}{74h}$, $q = \dfrac{243}{2590h}$, $w = -\dfrac{23}{3108h}$, $m = -\dfrac{69}{518h}$, $n = \dfrac{21}{148h}$. So, relation (10) is established. In a similar fashion, one can derive the relationship (11). Theorem 2.1 is established.

Now, we consider the application of Theorems 1.1–1.4 and Theorem 2.1 for the numerical solution of the BVP (1). Using the equation (1) and formulas (5), (7), (8), (9), (10), (11), and disregarding minor terms, we can present the eighth order of approximation DS

$$\begin{cases} h^{-4}(u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}) \\ + \left(\frac{76}{105} - \frac{97}{1680}h^4 a(x_k)\right) a(t_k)u_k + \frac{9}{70}a(x_{k+1})u_{k+1} \\ + a(x_{k-1})u_{k-1} + \frac{1}{105}\left(a(x_{k+2})u_{k+2} + a(x_{k-2})u_{k-2}\right) \\ = \left(\frac{76}{105} + \frac{97}{1680}h^4 a(x_k)\right) F(x_k) + \frac{9}{70}\left(F(x_{k+1}) + F(x_{k-1})\right) \\ + \frac{1}{105}\left(F(x_{k+2}) + F(x_{k-2})\right) - \frac{97}{1680}h^4 F^{(4)}(x_k), \\ \\ \left(-\frac{5543}{2590} + \frac{9}{1036}h^4 a(0)\right) u_0 + \left(\frac{2223}{518} + \frac{69}{518}h^4 a(h)\right) u_1 \\ - \left(\frac{3735}{1036} + \frac{21}{148}h^4 a(2h)\right) u_2 + \frac{1535}{777}u_3 - \frac{45}{74}u_4 + \frac{243}{2590}u_5 - \frac{23}{3108}u_6 \\ = h\eta + h^4\left[\frac{69}{518}\left(F(x_1) - F(x_0)\right) - \frac{21}{148}\left(F(x_2) - F(x_0)\right)\right], u_0 = \varphi, \\ \\ \left(\frac{5543}{2590} - \frac{9}{1036}h^4 a(0)\right) u_N - \left(\frac{2223}{518} + \frac{69}{518}h^4 a(h)\right) u_{N-1} \\ + \left(\frac{3735}{1036} + \frac{21}{148}h^4 a(2h)\right) u_{N-2} - \frac{1535}{777}u_{N-3} + \frac{45}{74}u_{N-4} - \frac{243}{2590}u_{N-5} \\ + \frac{23}{3108}u_{N-6} = h\rho - h^4\frac{69}{518}\left(F(x_{N-1}) - F(x_N)\right) \\ - h^4\frac{21}{148}\left(F(x_{N-2}) - F(x_N)\right), u_N = \omega \end{cases} \tag{12}$$

for the numerical solution of the BVP (1).

3 Local BVP (2)

Consider the BVP (2). For the application of TD's on five points (5), we have to give the eighth order of approximation formulas for $W^{(2)}(0)$ and $W^{(2)}(\Upsilon)$.

Theorem 3.1. Let $W(y)$ be a function defined on the interval $[0, \Upsilon]$ with a continuous tenth derivative. Then the subsequent relations hold:

$$W^{(2)}(0) - h^{-2}\left\{\frac{6937\,573}{3439\,828}W(0) - \frac{26\,121\,217}{5159\,742}W(h) + \frac{21\,060\,241}{5159\,742}W(2h)\right. \tag{13}$$

$$\left. -\frac{892\,879}{859\,957}W(3h) - \frac{26\,209}{10\,319\,484}W(4h) + \frac{24\,995}{5159\,742}W(5h)\right\} - h^2\frac{23\,426\,639}{206\,389\,680}W^{(4)}(0)$$

$$-h^2\left(\frac{12\,741\,989}{20\,638\,968}W^{(4)}(h) + \frac{5216\,939}{29\,484\,240}W^{(4)}(2h) - \frac{1324\,691}{103\,194\,840}W^{(4)}(3h)\right) = o(h^8),$$

$$W^{(2)}(\Upsilon) - h^{-2}\left\{\frac{6937\,573}{3439\,828}W(\Upsilon) - \frac{26\,121\,217}{5159\,742}W(\Upsilon - h)\right. \tag{14}$$

$$\left. +\frac{21\,060\,241}{5159\,742}W(\Upsilon - 2h) - \frac{892\,879}{859\,957}W(\Upsilon - 3h) - \frac{26\,209}{10\,319\,484}W(\Upsilon - 4h)\right\}$$

$$-h^{-2}\frac{24\,995}{5159\,742}W(\Upsilon - 5h) - h^2\left(\frac{23\,426\,639}{206\,389\,680}W^{(4)}(\Upsilon) + \frac{12\,741\,989}{20\,638\,968}W^{(4)}(\Upsilon - h)\right.$$

$$\left. +\frac{5216\,939}{29\,484\,240}W^{(4)}(\Upsilon - 2h) - \frac{1324\,691}{103\,194\,840}W^{(4)}(\Upsilon - 3h)\right) = o(h^8).$$

Proof. Applying the MUCs, we will aim to find

$$W^{(2)}(0)=\alpha W(0)+\beta W(h)+\gamma W(2h)+dW(3h)+pW(4h)+qW(5h)$$

$$+h^4mW^{(4)}(0)+h^4nW^{(4)}(h)+h^4fW^{(4)}(2h)+h^4wW^{(4)}(3h)+o\left(h^8\right).$$

By applying Taylor's formula, we obtain

$$W^{(2)}(0)=\alpha W(0)+\beta\sum_{l=0}^{9}\frac{h^l}{l!}W^{(l)}(0)+\gamma\sum_{l=0}^{9}\frac{(2h)^l}{l!}W^{(l)}(0)+d\sum_{l=0}^{9}\frac{(3h)^l}{l!}W^{(l)}(0)$$

$$+p\sum_{l=0}^{9}\frac{(4h)^l}{l!}W^{(l)}(0)+q\sum_{l=0}^{9}\frac{(5h)^l}{l!}W^{(l)}(0)+h^4mW^{(4)}(0)$$

$$+h^4n\sum_{l=0}^{5}\frac{h^l}{l!}W^{(l+4)}(0)+h^4f\sum_{l=0}^{5}\frac{(2h)^l}{l!}W^{(l+4)}(0)+h^4w\sum_{l=0}^{5}\frac{(3h)^l}{l!}W^{(l+4)}(0)+o\left(h^8\right).$$

To obtain the SAEs, equate the coefficients of the smallest power of h in the above identity to 0.

$$\begin{cases} d+p+\alpha+\beta+\gamma+q=0,\\ 3d+4p+\beta+2\gamma+5q=0,\\ \frac{9}{2!}d+\frac{16}{2!}p+\frac{1}{2!}\beta+\frac{4}{2!}\gamma+\frac{25}{2!}q=h^{-2},\\ \frac{27}{3!}d+\frac{64}{3!}p+\frac{1}{3!}\beta+\frac{8}{3!}\gamma+\frac{125}{3!}q=0,\\ \frac{81}{4!}d+\frac{256}{4!}p+\frac{1}{4!}\beta+\frac{16}{4!}\gamma+\frac{625}{4!}q+m+n+f+w=0,\\ \frac{243}{5!}d+\frac{1024}{5!}p+\frac{1}{5!}\beta+\frac{32}{5!}\gamma+\frac{3125}{5!}q+n+2f+3w=0,\\ \frac{729}{6!}d+\frac{4096}{6!}p+\frac{1}{6!}\beta+\frac{64}{6!}\gamma+\frac{15\,625}{6!}q+\frac{1}{2!}n+\frac{4}{2!}f+\frac{9}{2!}w=0,\\ \frac{2187}{7!}d+\frac{16\,384}{7!}p+\frac{1}{7!}\beta+\frac{128}{7!}\gamma+\frac{78\,125}{7!}q+\frac{1}{3!}n+\frac{8}{3!}f+\frac{27}{3!}w=0,\\ \frac{3^8}{8!}d+\frac{4^8}{8!}p+\frac{1}{8!}\beta+\frac{2^8}{8!}\gamma+\frac{5^8}{8!}q+\frac{1}{4!}n+\frac{16}{4!}f+\frac{81}{4!}w=0,\\ \frac{3^9}{9!}d+\frac{4^9}{9!}p+\frac{1}{9!}\beta+\frac{2^9}{9!}\gamma+\frac{5^9}{9!}q+\frac{1}{5!}n+\frac{32}{5!}f+\frac{243}{5!}w=0. \end{cases}$$

Upon resolving this SAEs, we find $\alpha=\frac{6937\,573}{3439\,828}, \beta=-\frac{26\,121\,217}{5159\,742}$, $\gamma=\frac{21\,060\,241}{5159\,742}$, $d=-\frac{892\,879}{859\,957}$, $p=-\frac{26\,209}{10\,319\,484}$, $q=\frac{24\,995}{5159\,742}$, $m=\frac{23\,426\,639}{206\,389\,680}$, $n=\frac{12\,741\,989}{20\,638\,968}$, $f=\frac{5216\,939}{29\,484\,240}$, $w=-\frac{1324\,691}{103\,194\,840}$. So, relation (13) is proved. In a similar fashion, one can derive the relationship (14). Theorem 3.1 is established.

Now, we consider the application of Theorems 1.1–1.4 and Theorem 3.1 for the numerical solution of the BVP (2). Using the equation (2) and formulas (5), (7), (8), (9), (13), (14), and disregarding minor terms, we can present the eighth order of approximation DS

$$\left\{\begin{array}{l}\left(1+\frac{1}{105}h^4a(x_{k-2})\right)u_{k-2}+\left(-4+\frac{9}{70}h^4a(x_{k-1})\right)u_{k-1}\\+\left(6+\left(\frac{76}{105}-\frac{97}{1680}h^4a(x_k)\right)a(x_k)h^4\right)u_k\\+\left(-4+\frac{9}{70}h^4a(x_{k+1})\right)u_{k+1}+\left(1+\frac{1}{105}h^4a(x_{k+2})\right)u_{k+2}\\=h^4\left[\left(\frac{76}{105}+\frac{97}{1680}h^4a(x_k)\right)F(x_k)+\frac{9}{70}\left(F(x_{k+1})+F(x_{k-1})\right)\right.\\\left.+\frac{1}{105}\left(F(x_{k+2})+F(x_{k-2})\right)-\frac{97}{1680}h^4F^{(4)}(x_k)\right],\ 2\le k\le N-2,\\\left(\frac{6937573}{3439828}-\frac{23426639}{206389680}h^4a(0)\right)u_0-\left(\frac{26121217}{5159742}+\frac{12741989}{20638968}h^4a(h)\right)u_1\\+\left(\frac{21060241}{5159742}-\frac{5216939}{29484240}h^4a(2h)\right)u_2\\-\left(\frac{892879}{859957}-\frac{1324691}{103194840}h^4a(3h)\right)u_3\\-\frac{26209}{10319484}u_4+\frac{24995}{5159742}u_5=h^2\eta-\frac{23426639}{206389680}f(0)\\-\left[\frac{12741989}{20638968}f(h)+\frac{5216939}{29484240}f(2h)-\frac{1324691}{103194840}f(3h)\right],u_0=\varphi,\\\left(\frac{6937573}{3439828}-\frac{23426639}{206389680}h^4a(\Upsilon)\right)u_N\\-\left(\frac{26121217}{5159742}+\frac{12741989}{20638968}h^4a(\Upsilon-h)\right)u_{N-1}\\+\left(\frac{21060241}{5159742}-\frac{5216939}{29484240}h^4a(\Upsilon-2h)\right)u_{N-2}\\\left(-\frac{892879}{859957}+\frac{1324691}{103194840}h^4a(\Upsilon-3h)\right)u_{N-3}-\frac{26209}{10319484}u_{N-4}\\+\frac{24995}{5159742}u_{N-5}=h^2\rho-h^4\left[\frac{23426639}{206389680}F(\Upsilon)+\frac{12741989}{20638968}F(\Upsilon-h)\right.\\\left.+\frac{5216939}{29484240}F(\Upsilon-2h)-\frac{1324691}{103194840}F(\Upsilon-3h)\right],u_N=\omega,\end{array}\right. \tag{15}$$

for the numerical solution of the BVP (2).

4 Local BVP (3)

Let us consider BVP (3). For the application of TD's on five points (5), we have to give the eighth order of approximation formulas for $W^{(3)}(0)$ and $W^{(3)}(\Upsilon)$.

Theorem 4.1. Let $W(y)$ be a function defined on the interval $[0,\Upsilon]$ with a continuous eleventh derivative. Then the subsequent relations hold:

$$W^{(3)}(0)-h^{-3}\left\{\frac{126630131}{4505760}\left(W(h)-W(0)\right)-\frac{78574591}{1501920}\left(W(2h)-W(0)\right)\right. \tag{16}$$

$$+\frac{45949355}{901152}\left(W(3h)-W(0)\right)-\frac{24699239}{901152}\left(W(4h)-W(0)\right)+\frac{1667173}{214560}\left(W(5h)-W(0)\right)$$

$$\left.-\frac{4609391}{4505760}\left(W(6h)-W(0)\right)+\frac{309293}{4505760}\left(W(7h)-W(0)\right)\right\}$$

$$+h\left\{-\frac{597497}{1501920}\left(W^{(4)}(h)-W^{(4)}(0)\right)+\frac{1528979}{500640}\left(W^{(4)}(2h)-W^{(4)}(0)\right)\right.$$

$$\left.-\frac{1173833}{500640}\left(W^{(4)}(3h)-W^{(4)}(0)\right)\right\}=o\left(h^8\right),$$

$$
\begin{aligned}
& W^{(3)}(\Upsilon)-h^{-3}\left\{-\frac{126\,630\,131}{4505\,760}(W(\Upsilon-h)-W(\Upsilon))\right. \\
& +\frac{78\,574\,591}{1501\,920}(W(\Upsilon-2h)-W(\Upsilon))-\frac{45\,949\,355}{901\,152}(W(\Upsilon-3h)-W(\Upsilon)) \\
& +\frac{24\,699\,239}{901\,152}(W(\Upsilon-4h)-W(\Upsilon))-\frac{1667\,173}{214\,560}(W(\Upsilon-5h)-W(\Upsilon)) \\
& \left.+\frac{4609\,391}{4505\,760}(W(\Upsilon-6h)-W(\Upsilon))-\frac{309\,293}{4505\,760}(W(\Upsilon-7h)-W(\Upsilon))\right\} \\
& +h\left\{\frac{597\,497}{1501\,920}\left(W^{(4)}(\Upsilon-h)-W^{(4)}(\Upsilon)\right)-\frac{1528\,979}{500\,640}\left(W^{(4)}(\Upsilon-2h)-W^{(4)}(\Upsilon)\right)\right. \\
& \left.+\frac{1173\,833}{500\,640}\left(W^{(4)}(\Upsilon-3h)-W^{(4)}(\Upsilon)\right)\right\}=o\left(h^{8}\right).
\end{aligned}
\tag{17}
$$

Proof. Applying the MUCs, we will aim to find

$$
\begin{aligned}
& W^{(3)}(0)-\beta(W(h)-W(0))+\gamma(W(2h)-W(0))+d(W(3h)-W(0)) \\
& +p(W(4h)-W(0))+q(W(5h)-W(0))+w(W(6h)-W(0))+f(W(7h)-W(0)) \\
& +h^{4}m\left(W^{(4)}(h)-W^{(4)}(0)\right)+h^{4}n(W^{(4)}(2h)-W^{(4)}(0))+h^{4}s\left(W^{(4)}(3h)-W^{(4)}(0)\right)=o\left(h^{8}\right).
\end{aligned}
$$

By applying Taylor's formula, we obtain

$$
\begin{aligned}
W^{(3)}(0)= & \beta\sum_{l=1}^{10}\frac{h^{l}}{l!}W^{(l)}(0)+\gamma\sum_{l=1}^{10}\frac{(2h)^{l}}{l!}W^{(l)}(0)+d\sum_{l=1}^{10}\frac{(3h)^{l}}{l!}W^{(l)}(0) \\
& +p\sum_{l=1}^{10}\frac{(4h)^{l}}{l!}W^{(l)}(0)+q\sum_{l=1}^{10}\frac{(5h)^{l}}{l!}W^{(l)}(0)+w\sum_{l=1}^{10}\frac{(6h)^{l}}{l!}W^{(l)}(0) \\
& +f\sum_{l=1}^{10}\frac{(7h)^{l}}{l!}W^{(l)}(0)+h^{4}m\sum_{l=1}^{5}\frac{h^{l}}{l!}W^{(4+l)}(0) \\
& +h^{4}n\sum_{l=1}^{5}\frac{(2h)^{l}}{l!}W^{(4+l)}(0)+h^{4}s\sum_{l=1}^{5}\frac{(3h)^{l}}{l!}W^{(4+l)}(0)+o\left(h^{8}\right).
\end{aligned}
$$

To obtain the SAEs, equate the coefficients of the smallest power of h in the above identity to 0.

$$
\left\{\begin{array}{l}
3d+4p+5q+6w+\beta+2\gamma+7f=0, \\
\frac{9}{2!}d+\frac{4^{2}}{2!}p+\frac{5^{2}}{2!}q+\frac{6^{2}}{2!}w+\frac{1}{2!}\beta+\frac{4}{2!}\gamma+\frac{7^{2}}{2!}f=0, \\
\frac{3^{3}}{3!}d+\frac{4^{3}}{3!}p+\frac{5^{3}}{3!}q+\frac{6^{3}}{3!}w+\frac{1}{3!}\beta+\frac{8}{3!}\gamma+\frac{7^{3}}{3!}f=h^{-3}, \\
\frac{3^{4}}{4!}d+\frac{4^{4}}{4!}p+\frac{5^{4}}{4!}q+\frac{6^{4}}{4!}w+\frac{1}{4!}\beta+\frac{2^{4}}{4!}\gamma+\frac{7^{4}}{4!}f=0, \\
\frac{3^{5}}{5!}d+\frac{4^{5}}{5!}p+\frac{5^{5}}{5!}q+\frac{6^{5}}{5!}w+\frac{1}{5!}\beta+\frac{2^{5}}{5!}\gamma+\frac{7^{5}}{5!}f+m+2n+3s=0, \\
\frac{3^{6}}{6!}d+\frac{4^{6}}{6!}p+\frac{5^{6}}{6!}q+\frac{6^{6}}{6!}w+\frac{1}{6!}\beta+\frac{2^{6}}{6!}\gamma+\frac{7^{6}}{6!}f+\frac{1}{2!}m+\frac{4}{2!}n+\frac{3^{2}}{2!}s=0, \\
\frac{3^{7}}{7!}d+\frac{4^{7}}{7!}p+\frac{5^{7}}{7!}q+\frac{6^{7}}{7!}w+\frac{1}{7!}\beta+\frac{2^{7}}{7!}\gamma+\frac{7^{7}}{7!}f+\frac{1}{3!}m+\frac{8}{3!}n+\frac{3^{3}}{3!}s=0, \\
\frac{3^{8}}{8!}d+\frac{4^{8}6}{8!}p+\frac{5^{8}}{8!}q+\frac{6^{8}}{8!}w+\frac{1}{8!}\beta+\frac{2^{8}}{8!}\gamma+\frac{7^{8}}{8!}f+\frac{1}{4!}m+\frac{16}{4!}n+\frac{3^{4}}{4!}s=0, \\
\frac{3^{9}}{9!}d+\frac{4^{9}}{9!}p+\frac{5^{9}}{9!}q+\frac{6^{9}}{9!}w+\frac{1}{9!}\beta+\frac{2^{9}}{9!}\gamma+\frac{7^{9}}{9!}f+\frac{1}{5!}m+\frac{32}{5!}n+\frac{3^{5}}{5!}s=0, \\
\frac{3^{10}}{10!}d+\frac{4^{10}}{10!}p+\frac{5^{10}}{10!}q+\frac{6^{10}}{10!}w+\frac{1}{10!}\beta+\frac{2^{10}}{10!}\gamma+\frac{7^{10}}{10!}f+\frac{1}{6!}m+\frac{64}{6!}n+\frac{3^{6}}{6!}s=0.
\end{array}\right.
$$

Upon resolving this SAEs, we find $\beta = \frac{126\,630\,131}{4505\,760h^3}$, $\gamma = -\frac{78\,574\,591}{1501\,920h^3}$, $d = \frac{45\,949\,355}{901\,152h^3}$, $p = -\frac{24\,699\,239}{901\,152h^3}$, $q = \frac{1667\,173}{214\,560h^3}$, $w = -\frac{4609\,391}{4505\,760h^3}$, $f = \frac{309\,293}{4505\,760h^3}$, $m = -\frac{597\,497}{1501\,920h^3}$, $n = \frac{1528\,979}{500\,640h^3}$, $s = -\frac{1173\,833}{500\,640h^3}$. So, relation (16) is proved. In a similar fashion, one can derive the relationship (17). Theorem 4.1 is established.

Now, we consider the application of Theorems 1.1–1.4 and Theorem 4.1 for the numerical solution of the BVP (3). Using the equation (3) and formulas (5), (7), (8), (9), (16), (17), and disregarding minor terms, we can present the eighth order of approximation DS

$$h^{-4}(u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}) + b_k u_k \tag{18}$$

$$+c_k u_{k+1} + d_k u_{k-1} + h_k u_{k+2} + g_k u_{k-2} = \varphi_k, 2 \le k \le N-2,$$

$$a_{1,0}u_0 + a_{1,1}u_1 + a_{1,2}u_2 + a_{1,3}u_3 + a_{1,4}u_4 + a_{1,5}u_5 + a_{1,6}u_6 = -a_{1,7} + \eta,$$

$$u_0 = \varphi, u_N = \omega,$$

$$a_{1,N}u_N + a_{1,N-1}u_{N-1} + a_{1,N-2}u_{N-2} + a_{1,N-3}u_{N-3}$$

$$+a_{1,N-4}u_{N-4} + a_{1,N-5}u_{N-5} + a_{1,N-6}u_{N-6} = -a_{1,N-7} + \rho,$$

where

$$b_k = \left(\frac{76}{105} - \frac{97}{1680}h^4 a(y_k)\right) a(y_k) - \frac{97}{1680}h^4 a^{(4)}(y_k) - \frac{697}{1680}h^4 a''(y_k) 6A_k^{20},$$

$$c_k = \frac{9}{70}a(y_{k+1}) - \frac{97}{1680}h^4\left[4a'(y_k)B_k^{30} + 6a''(y_k)B_k^{20} + 4a'''(y_k)B_k^{10}\right],$$

$$d_k = \frac{9}{70}a(y_{k-1}) - \frac{97}{1680}h^4\left[4a'(y_k)C_k^{30} + 6a''(y_k)C_k^{20} + 4a'''(y_k)C_k^{10}\right],$$

$$h_k = \frac{1}{105}a(y_{k+2}) - \frac{97}{1680}h^4\left[4a'(y_k)D_k^{30} + 6a''(y_k)D_k^{20} + 4a'''(y_k)D_k^{10}\right],$$

$$g_k = \frac{1}{105}a(y_{k-2}) - \frac{97}{1680}h^4\left[4a'(y_k)E_k^{30} + 6a''(y_k)E_k^{20} + 4a'''(y_k)E_k^{10}\right],$$

$$\varphi_k = \left(\frac{76}{105} + \frac{97}{1680}h^4 a(y_k)\right) F(y_k) + \frac{9}{70}\left(F(y_{k+1}) + F(y_{k-1})\right)$$

$$+\frac{1}{105}\left(F(y_{k+2}) + F(y_{k-2})\right) - \frac{97}{1680}h^4 F^{(4)}(y_k)$$

for the numerical solution of the BVP (3).

5 *The nonlocal BVP (4)*

Now, we consider the application of Theorems 1.1–1.4 and Theorems 2.1, 3.1, and 4.1 for the numerical solution of the nonlocal BVP (4). Using the equation (4) and formulas (5), (7), (8), (9), (10), (11), (13), (14), (16), (17), and disregarding minor terms, we can present the eighth order of approximation DS

$$\left\{\begin{array}{l}
h^{-4}(u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2})+\big(\frac{76}{105}\\
-\frac{97}{1680}h^4a(y_k)\big)\,a(y_k)u_k+\frac{9}{70}a(y_{k+1})u_{k+1}+a(y_{k-1})u_{k-1}\\
+\frac{1}{105}\,(a(y_{k+2})u_{k+2}+a(y_{k-2})u_{k-2})=\big(\frac{76}{105}+\frac{97}{1680}h^4a(y_k)\big)\,F(y_k)\\
+\frac{9}{70}\,(F(y_{k+1})+F(y_{k-1}))+\frac{1}{105}\,(F(y_{k+2})+F(y_{k-2}))\\
-\frac{97}{1680}h^4F^{(4)}(y_k),\\
u_0=u_N+\varphi,\\
h^{-1}\big\{\big(-\frac{5543}{2590}+\frac{9}{1036}h^4a\,(0)\big)\,u_0+\big(\frac{2223}{518}+\frac{69}{518}h^4a(h)\big)\,u_1\\
-\frac{45}{74}u_4-\big(\frac{3735}{1036}+\frac{21}{148}h^4a(2h)\big)\,u_2\\
+\frac{1535}{777}u_3+\frac{243}{2590}u_5-\frac{23}{3108}u_6\big\}\\
=h^{-1}\big\{\big(\frac{5543}{2590}-\frac{9}{1036}h^4a\,(0)\big)\,u_N-\big(\frac{2223}{518}+\frac{69}{518}h^4a(h)\big)\,u_{N-1}\\
+\big(\frac{3735}{1036}+\frac{21}{148}h^4a(2h)\big)\,u_{N-2}-\frac{1535}{777}u_{N-3}\\
+\frac{45}{74}u_{N-4}-\frac{243}{2590}u_{N-5}+\frac{23}{3108}u_{N-6}\big\}+\eta,\\
\\
h^{-2}\big\{\big(\frac{6937\,573}{3439\,828}-\frac{23\,426\,639}{206\,389\,680}h^4a\,(0)\big)\,u_0-\big(\frac{26\,121\,217}{5159\,742}\\
-+\frac{12\,741\,989}{20\,638\,968}h^4a(h)\big)\,u_1+\big(\frac{21\,060\,241}{5159\,742}-\frac{5216\,939}{29\,484\,240}h^4a(2h)\big)\,u_2\\
-\big(\frac{892\,879}{859\,957}-\frac{1324\,691}{103\,194\,840}h^4a(3h)\big)\,u_3-\frac{26\,209}{10\,319\,484}u_4\\
+\frac{24\,995}{5159\,742}u_5\big\}=h^{-2}\big\{\big(\frac{6937\,573}{3439\,828}-\frac{23\,426\,639}{206\,389\,680}h^4a\,(\Upsilon)\big)\,u_N\\
+\big(-\frac{26\,121\,217}{5159\,742}-\frac{12\,741\,989}{20\,638\,968}h^4a(\Upsilon-h)\big)\,u_{N-1}\\
+\big(\frac{21\,060\,241}{5159\,742}-\frac{5216\,939}{29\,484\,240}h^4a(\Upsilon-2h)\big)\,u_{N-2}\\
+\big(-\frac{892\,879}{859\,957}+\frac{1324\,691}{103\,194\,840}h^4a(\Upsilon-3h)\big)\,u_{N-3}\\
-\frac{26\,209}{10\,319\,484}u_{N-4}+\frac{24\,995}{5159\,742}u_{N-5}\big\}+\omega,\\
\\
h^{-3}\big\{\big(-\frac{17\,265\,457}{300\,384}+\frac{467\,941}{1501\,920}h^4a\,(0)\big)\,u_0\\
+\big(\frac{126\,630\,131}{4505\,760}+\frac{597\,497}{1501\,920}h^4a(h)\big)\,u_1-\big(\frac{4609\,391}{4505\,760}\\
+\frac{1528\,979}{500\,640}h^4a(2h)\big)\,u_2+\big(\frac{45\,949\,355}{901\,152}+\frac{1173\,833}{500\,640}h^4a(3h)\big)\,u_3\\
-\frac{24\,699\,239}{901\,152}u_4+\frac{1667\,173}{214\,560}u_5-\frac{4609\,391}{4505\,760}u_6+\frac{309\,293}{4505\,760}u_7\big\}\\
=\varrho+h^{-3}\big\{\big(\frac{27\,867\,473}{4505\,760}-\frac{467\,941}{1501\,920}h^4a\,(0)\big)\,u_N\\
-\big(\frac{126\,630\,131}{4505\,760}+\frac{597\,497}{1501\,920}h^4a(h)\big)\,u_{N-1}+\big(\frac{78\,574\,591}{1501\,920}+\\
\frac{1528\,979}{500\,640}h^4a(2h)\big)\,u_{N-2}+\big(-\frac{45\,949\,355}{901\,152}-\frac{1173\,833}{500\,640}h^4a(3h)\big)\,u_{N-3}\\
+\frac{24\,699\,239}{901\,152}u_{N-4}-\frac{1667\,173}{214\,560}u_{N-5}+\frac{4609\,391}{4505\,760}u_{N-6}-\frac{309\,293}{4505\,760}u_{N-7}\big\}
\end{array}\right. \tag{19}$$

for the numerical solution of the nonlocal BVP (4).

Now, for numerical analysis we consider the BVPs (1)–(4), for the simple case when $\Upsilon=1$, $a\,(y)=1,\varphi=\eta=\omega=\chi=0$, and

$$F(y)=\frac{y^8(1-y)^8}{8!}+\frac{1}{120}y^4\;(y-1)^4\left(130y^4-260y^3+182y^2-52y+5\right).$$

Then,

$$\mathbb{U}(y)=\frac{y^8(1-y)^8}{8!}$$

is the exact solution of these BVPs. For solving these problems, we use the eighth order of approximation DSs (12), (15), (18), and (19), respectively, with different values of h. The error is computed by

$$E_N=\max_{0\le k\le N}|u(y_k)-u_k|.$$

The error analysis shown in Table indicates that all DSs have correct convergence rates.

Table

Numerical Results

$h=\frac{1}{N}$	$N=40$	$N=80$	$N=160$
DS (12)	1.2225e-13	5.5447e-15	3.0095e-16
DS (15)	2.0641e-12	3.7546e-14	6.4559e-16
DS (18)	5.2435e-13	1.2208e-14	3.1736e-16
DS (19)	3.6094e-09	1.7743e-10	6.4050e-12

Conclusion

1. In this work, we examine LNBVPs for FODEs with variable coefficients. We develop and analyze finite DSs of eighth-order accuracy using a novel method based on five-point grids for addressing these problems. Our findings are validated through extensive numerical experiments.

2. Highly accurate four-step finite DSs for solving LNBVPs of the general FODE

$$u^{(4)}(s)+d(s)u^{(3)}(s)+c(s)u^{(2)}(s)+b(s)u^{(1)}(s)+a(s)u(s)=\Psi(s),0<s<\Upsilon$$

will be presented and investigated.

3. Highly accurate four-step finite DSs for solving LNBVPs for elliptic FODEs

$$u^{(4)}(s)+Au(s)=\Psi(s),0<s<\Upsilon$$

will be constructed and studied. Here A is a self-adjoint positive definite operator in a Hilbert space H. The stability of these DSs is ensured by the operator method discussed in reference [1].

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Punctual numberings for families of sets

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This work investigates the structure of punctual numberings for families of punctually enumerable sets with respect to primitive recursively reducibility. We say that a numbering of a certain family is primitive recursively reducible to another numeration of the same family if there exists a primitive recursively procedure (an algorithm not employing unbounded search) mapping the numbers of objects in the first numbering to the numbers of the same objects in the second numbering. This study was motivated by the work of Bazhenov, Mustafa, and Ospichev on punctual Rogers semilattices for families of primitive recursively enumerable functions. The concept of punctually enumerable sets was introduced in the paper, and it was proven that not all recursively enumerable sets are punctually enumerable, but in all m-degrees, recursively enumerable sets include punctually enumerable sets. For two-element families of punctual sets, it was demonstrated that punctual Rogers semilattices can be of at least three types: (1) one-element family, (2) isomorphic to the upper semilattice of recursively enumerable sets with respect to primitive recursively m-reducibility, (3) without the greatest element. It was also proven that the set of all punctually enumerable sets does not have a punctual numbering, and punctual families with a Friedberg numbering do not have the least numbering.

Keywords: primitive recursive functions, punctually enumerable sets, Rogers semilattice, quick functions, punctual numberings.

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Introduction

Theory of computable numberings is one of the actively developing areas in the computability theory. *Numbering* of a countable set S is any surjective mapping $\nu : \omega \to \mathcal{S}$ (Here and further as ω we denote the set of natural numbers). A numbering ν is called computable if the set

$$\{\langle n, x\rangle : n \in \omega,\ x \in \nu(n)\}$$

is computably enumerable (c.e.) set.

The set of all computable numberings for family $\mathcal{S}$ denotes as $\mathrm{Com}(\mathcal{S})$. Let ν and μ are numberings for family S. Numbering ν is *reducible* to μ if there is computable function f such that $\nu = \mu \circ f$ (denotes $\nu \leq \mu$). This reducibility induces a partially preordered set structure, which factor structure is called *Rogers semilattice for family* S and denoted as $\mathcal{R}(\mathcal{S})$.

There are several interesting results known about Rogers semilattice. For example, if $\mathcal{S}$ is a family of c.e. sets, then either $|\mathcal{R}(\mathcal{S})| = 1$ or $|\mathcal{R}(\mathcal{S})| = \infty$ [1]. In the case when $|\mathcal{R}(\mathcal{S})| = \infty$ the semilattice is not

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a lattice [2]. There is a universal numbering of all partial computable functions [3] and a computable numbering of all c.e. sets without repetitions [4] (these numberings are called friedberg numbering). For more information about the properties of the classical Rogers semilattice, refer to the following articles: [5–8].

In recent years, under the influence of the work [9], interest in primitive recursive (or punctual) properties of algebraic structures has increased. The next articles will help you to find more information about punctual structures [10–27]. In connection with this, studying the punctual properties of numberings is also relevant. Bazhenov, Mustafa, and Ospichev considered punctual numberings of families of functions in their article [28]. The authors established punctual reducibility between numberings, induced by primitive recursive functions, leading to the creation of upper semilattices of degrees known as Rogers pr-semilattices. They demonstrated that any infinite, uniformly primitive recursive family S induces an infinite Rogers pr-semilattice $\mathcal{R}$. It was proven that the semilattice $\mathcal{R}$ is downwards dense, with every nontrivial interval within $\mathcal{R}$ containing an infinite antichain. Additionally, every non-greatest element in $\mathcal{R}$ is a part of an infinite antichain. The authors showed that the Σ_1-fragment of the theory $Th(R)$ is decidable. Several examples were provided to emphasize the contrasts between the punctual framework and the classical theory of computable numberings. Notably, it was demonstrated that some infinite Rogers pr-semilattices $\mathcal{R}$ are lattices, while others are not. The authors obtained a series of results concerning special classes of punctual numberings, including Friedberg numberings and decidable numberings with primitive recursive numeration equivalence. This paper is a logical continuation of the article [28] and aims to investigate punctual numberings for families of sets.

In Chapter 2, we introduce punctual analogs of concepts standard in the theory of computable numberings and define the punctual Rogers semilattice for sets. Chapter 3 is devoted to the structural properties of c.e. degrees induced by the restriction of m-reducibility by primitive recursive functions (called prm-reducibility). Chapters 4 and 5 present some properties of the punctual Rogers semilattice for finite and infinite families, including its connection with the structure of c.e. prm-degrees.

We adhere to the notations and terminology adopted in [29, 30]. We denote by $\{p_e\}_{e\in\omega}$ the computable numbering of all primitive recursive functions. In this article we will consider restricted Church-Turing thesis for primitive recursive functions. We can define this thesis as follows: a function is primitive recursive if and only if it can be described by an algorithm that uses only bounded loops. More about restricted Church-Turing thesis you can find in the work [31].

1 Punctually enumerable sets and numberings

In the paper [28], the numbering ν of a family of primitive recursive functions is called "punctual" if the function $g_\nu(n, x) := (\nu(n))(x)$ is primitive recursive. It seems natural to attempt to extend this definition to a family of c.e. sets, but here we face some difficulties.

The thing is, such a definition of punctual numbering yields the same class of computable numberings for families of c.e. sets because any c.e. set can be represented as the range of a primitive recursive function, which means that a family can be enumerated in a punctual way. On the other hand, even with the presence of a punctual enumeration of a c.e. set, it is not always possible to use primitive recursive constructions, for example, due to the unbounded repetition of elements in the enumeration. In this regard, it makes sense to consider families of sets with stricter enumeration constraints than c.e. sets.

Definition 1. A set A is called *punctually enumerable*, if there is a primitive recursive function p, such that

1) $A = \text{range}(p)$, and
2) If $p(x) = p(y)$ for some $x < y$, then $\text{range}(p) = \{p(0), p(1), \ldots, p(x)\}$.

We will call the function p as a *quick* function for A.

Thus, the quick function p from the definition is injective for an infinite set A, and for a finite set it eventually enumerates all of its elements and then starts repeating them.

Definition 2. A numbering ν of a family $\mathcal{S}$ of punctually enumerable sets is called *punctual*, if there exists a primitive recursive function $g_\nu(n,x)$ such that $\lambda x.g_\nu(n,x)$ is a quick function for $\nu(n)$ for any n. The set of all punctual numberings for family $\mathcal{S}$ we will denote as $\mathrm{Com}_{pr}(\mathcal{S})$.

Reductions on numberings are defined analogously to [28].

Definition 3. We say that numbering ν is *punctually reducible* to numbering μ (denoted as $\nu \leq_{pr} \mu$), if there is primitive recursive function f such that $\nu = \mu \circ f$.

Numberings ν and μ are punctually equivalent and denote as $\nu \equiv_{pr} \mu$, if $\nu \leq_{pr} \mu$ and $\mu \leq_{pr} \nu$.

As in the computable case, the least upper bound of the numberings ν and μ is the numbering $\nu \oplus \mu$, which is defined as

$$(\nu \oplus \mu)(2x) = \nu(x),\ (\nu \oplus \mu)(2x+1) = \mu(x).$$

As punctual Rogers semilattice of the family of punctually enumerable sets $\mathcal{S}$, we will call partially ordered set $\mathcal{R}_{pr}(\mathcal{S}) = (\mathrm{Com}_{pr}(\mathcal{S})_{/\equiv_{pr}}, \leq_{pr}, \oplus)$.

The following theorem demonstrates the independence of the concepts of primitive recursive set (having primitive recursive characteristic function) and punctually enumerable.

Theorem 1. There exist sets A and B such that A is punctually enumerable but not primitive recursive, and B is primitive recursive but not punctually enumerable.

Proof. As a set A we can choose the set $K \oplus \omega$. For this set its quick function we can construct as follows: we fix a primitive recursive approximation of the creative set K, denoted K_i, which at each step enumerates at most one element. Then, we set $f(0) = 1$, and $f(x) = 2s$, where $s \in K_x \setminus K_{x-1}$, if such s exists. If there is no such s, then $f(x)$ is defined as the smallest odd number that has not been used before. It is clear that f is injective primitive recursive function and A is the range of f.

For set B we will construct its primitive recursive characteristic function ϕ such that for B there is no quick function. We fix a computable numbering of all injective primitive recursive functions with the following condition

$$i_e(x)[t]\downarrow = a \Rightarrow a < t.$$

We will define B as a infinite set. So, it is sufficient that there is no injective quick function for B.

At step s we will define $\phi(s)$ as follow: Assume that k is the cardinality of the set $\{t:\ \phi(t) = 1\ \&\ t < s\}$. If there is more that k elements $x \leq s$ such that $i_k(x)[s]\downarrow$, then define $\phi(s) = 1$. Otherwise, define $\phi(s) = 0$.

It is not hard to see that B is infinite set and any i_k can not enumerate B.

2 *The structure of prm-degrees*

In recent work [32] considered a many-one reductions for computable sets under primitive recursive functions, and have been proven that first-order theory of upper semilattice of degrees of computable sets with respect to primitive recursive many-one reducibility is hereditarily undecidable.

Definition 4. [32] The set A is *prm-reducible* to the set B (written as $A \leq_m^{pr} B$), if there exists a primitive recursive function f such that $A \leq_m B$ via f.

Remark 1. The computable m-degree contains infinitely many prm-degrees.

Theorem 2. For any c.e., but not computable set A, there is c.e. set B such that $A \equiv_m B$ and $A \not\leq_m^{pr} B$.

Proof. Let's define computable majorant for all primitive recursive functions:

$$f(0) = p_0(0) + 1,$$
$$f(x+1) = \max_{i,j \leq x+1} \{p_i(j), f(x)\} + 1.$$

Note that f is not primitive recursive but the set range(f) is primitive recursive.

Let $B = f(A) = \{f(x) : \ x \in A\}$.

It is clear, that $A \leq_m^f B$, since f is strongly increasing, then $x \in B \Leftrightarrow \exists n \leqslant x \, (n \in A \, \& \, f(n) = x)$, which means that reverse reducibility is correct.

Let's show that if $A \leq_m^{pr} B$, then A is computable. We fix primitive recursive function p_e, which reduces A to B, and also step s such that $A \restriction e = A_s \restriction e$ (here by $A \restriction e$ denotes the set $\{x : \ x \in A \, \& \, x \leq e\}$).

Let x be an arbitrary number. If $x \leqslant e$, then $x \in A \Leftrightarrow x \in A_s$. Otherwise, we check the following condition: $p_e(x) \in range(f)$? If it is not, then $x \notin A$. If it is, we effectively find z_o such that $f(z_0) = p_e(x)$.

Repeat for z_0 same procedure as we did for x, and, if $z_0 > e$, then we find number z_1 such that $f(z_1) = p_e(z_0)$ and so on. As a result, we receive sequence $(z_k)_k$. Since $f(z_0) = p_e(x) < f(x)$, by definition of f, then $z_0 < x$, consequently, the sequence $(z_k)_k$ decreases and we find k, such that $p_e(z_k) \notin range(f)$ or $z_k \leqslant e$. Then $x \in A \Leftrightarrow p_e(x) = f(z_0) \Leftrightarrow z_0 \in A \Leftrightarrow p_e(z_0) = f(z_1) \Leftrightarrow \cdots \Leftrightarrow p_e(z_{k-1}) = f(z_k) \Leftrightarrow z_k \in A$. If $p_e(z_k) \notin range(f)$, then $z_k \notin A$, otherwise $z_k \leqslant e$ and $z_k \in A \Leftrightarrow z_k \in A_s$.

Thus, we can effectively define that x belongs to A or not.

Corollary 1. Every non-computable c.e. m-degree contains infinitely many prm-degrees.

Proof. Let A_0 be non-computable c.e. set. By using the previous theorem, we will build c.e. set A_1 such that $A_0 \equiv_m A_1$ and $A_0 \not\leq_m^{pr} A_1$, for A_1 similarly build A_2, and for A_2 build A_3 and so on. All sets A_n, $n \in \omega$ are m-equivalent, and for $i < j$ set A_i m-reduces to A_j by $f^{j-i}(x)$ (($j-i$)-th composition of function f from the previous theorem), consequently, $A_i \not\leq_m^{pr} A_j$. Here, note that $B = f(A) \leq_m^{pr} A$. (Proof is similar).

3 Punctual semilattice of two-element families

In the work [28] it was shown that the punctual Rogers semilattice of a finite family of functions always has exactly one element. However, it turns out that this is not the case for families of sets.

In this chapter we assume, that $\mathcal{S} = \{A, B\}$, where A, B are different punctually enumerable sets.

Note that in this case $\mathcal{R}_{pr}(\mathcal{S}) \neq \emptyset$, since the function

$$\alpha(n)(x) = \begin{cases} f(x), & \text{for } n = 2k, \\ g(x), & \text{for } n = 2k+1, \end{cases}$$

where f and g are quick functions for A and B respectively, gives the punctual numbering of the family $\mathcal{S}$.

Proposition 1. Let $\mathcal{S}$ is punctual two-element family such that A or B is finite then $|\mathcal{R}_{pr}(\mathcal{S})| = 1$.

Proof. Let $|A| = N \leqslant |B|$; f and g are quick functions of the sets A and B, respectively.

Let $\nu, \mu \in \mathrm{Com}_{pr}(\mathcal{S})$ are arbitrary and k_ν, k_μ their quick functions. Let's show that $\nu \equiv_{pr} \mu$.

Fix numbers a and b such that $\mu(a) = A$ and $\mu(b) = B$. Then $\nu \leqslant_{pr} \mu$ by primitive recursive function h, which defines as:

$$h(n) = \begin{cases} a, & \text{if } |k_\nu(n,\cdot) \restriction N| \leqslant N, \\ b, & \text{otherwise.} \end{cases}$$

Really, $|k_\nu(n,\cdot) \restriction N| \leqslant N$ means that quick function k_ν on N-th argument starts to repeat the values, consequently, $|\nu(n)| \leqslant N$ and that's why $\nu(n) = A = \mu(a)$.

Reverse reducibility is proved similarly.

Proposition 2. Let $\mathcal{S}$ be a two-element family such that A and B are infinite, $A \cap B$ is finite and one of the sets is primitive recursive, then $|\mathcal{R}_{pr}(\mathcal{S})| = 1$.

Proof. Let $|A \cap B| = N$ and A is primitive recursive.

Let's take two numberings ν and μ of the family $\mathcal{S}$ and fix numbers a and b such that $\mu(a) = A$ and $\mu(b) = B$.

We define

$$h(n) = \begin{cases} a, & \text{if } \exists x \in (k_\nu(n,\cdot) \restriction N) \cap (A \setminus B), \\ b, & \text{otherwise.} \end{cases}$$

Note, that $|k_\nu(n,\cdot) \restriction N| > |A \cap B| \Rightarrow \exists x \in (k_\nu(n,\cdot) \restriction N) \setminus (A \cap B)$. We can check that x belongs to A by primitive recursive procedure, and $\nu(n) = A \Leftrightarrow x \in A$. Consequently, $\nu \leqslant_{pr} \mu$ by function h. It is clear that reverse reducibility is true, then $\nu \equiv_{pr} \mu$.

Theorem 3. There exists family $\mathcal{S} = \{A, B\}$, where $|A \cap B| < \infty$, such that there is no universal numbering for $\mathcal{S}$.

Proof. We will build the sets A, B and numbering α_e for family $\mathcal{S} = \{A, B\}$, satisfying the following requirements:

$$\mathcal{P}_{e,i}: \ \pi_e \in \mathrm{Com}_{pr}(\mathcal{S}) \to \alpha_e \not\leqslant_{pr} \pi_e \text{ by function } p_i,$$

where π_e is computable numbering of all primitive recursive numberings, p_i is computable numbering of all primitive recursive functions. Let k_e be primitive recursive quick function for numbering π_e.

Strategy for $\mathcal{P}_{e,i}$:

1) Pick $w_{e,i}$ – the least number, that we do not use before.
2) Wait until $p_i(w_{e,i}) \downarrow$ and $k_e(p_i(w_{e,i}), 0) \downarrow$ on the step t. While we are waiting, list to $\alpha_e(w_{e,i})$ new numbers.
3) We perform one of the following cases:

Case 1: If $k_e(p_i(w_{e,i}), 0) \in B$, then all elements that we listed to $\alpha_e(w_{e,i})$ until this step, we add to A. Also, we add to $\alpha(w_{e,i})$ all elements from A. After this, we add to $\alpha_e(w_{e,i})$ all elements that we add to A.

Case 2: If $k_e(p_i(w_{e,i}), 0) \in A$, then all elements that we listed to $\alpha_e(w_{e,i})$ until this step, we add to B. Also, we add to $\alpha(w_{e,i})$ all elements from B. After this, we add to $\alpha_e(w_{e,i})$ all elements that we add to B.

Case 3: If $k_e(p_i(w_{e,i}), 0) \notin A \cup B$, then $k_e(p_i(w_{e,i}), 0) \in B$ and return to the Case 1.

Construction. Fix effective linear order of requirements:

$$\mathcal{P}_{0,0} < \mathcal{P}_{1,0} < \mathcal{P}_{0,1} < \mathcal{P}_{2,0} < \mathcal{P}_{1,1} < \mathcal{P}_{0,2} < \ldots$$

On step s of the construction we visit the first s strategies from the list. At every step, fresh numbers are selected and thrown into the sets A or B.

Let $\pi_e \in \mathrm{Com}_{pr}(\mathcal{S})$, by this we build α_e and let there is primitive recursive function p_i, such that $\alpha_e \leq_{pr} \pi_e$ by function p_i. So the following should be performed:

$$\forall x[\alpha_e(x) = \pi_e(p_i(x))].$$

Let's check the witness $w_{e,i}$ in $\mathcal{P}_{e,i}$. By construction, while $p_i(w_{e,i})$ defines on step t, in $\alpha_e(w_{e,i})$ we add new numbers. Suppose that on stage 3 the number $k_e(p_i(w_{e,i}), 0)$ is in the set B, since $A \cap B = \emptyset$, the following performed $\pi_e(p_i(w_{e,i})) = B$. But, by construction $\alpha_e(w_{w,i}) = A$. This contradicts to reducibility α_e to π_e.

For case, when the number $k_e(p_i(w_{e,i}), 0)$ is in the set A, we do the same action. If number $k_e(p_i(w_{e,i}), 0) \notin A \cup B$, then the action is performed as in the first case.

Proposition 3. Let $\mathcal{S}$ be two-elemented family, such that $A \subset B$, then $|\mathcal{R}_{pr}(\mathcal{S})| = \infty$ with universal numbering.

Proof. Since the sets A and B are punctually enumerable, then there are functions p, q – quick functions for A and B, respectively.

We will build the numberings for family $\mathcal{S}$ as follows:

Let W be arbitrary c.e. set, which is not empty and not ω, then numbering ν defines as:

$$\nu_W(x) = \begin{cases} A, \text{if } x \notin W, \\ B, \text{if } x \in W. \end{cases}$$

$W = \cup_s W_s$.

Quick function $\lambda x.h(x, y)$ of the numbering ν will add elements as:

On step 0. $h_\nu(x, 0) = p(0)$.

On step s.

1) $h_\nu(x, s) = p(s)$, if $x \notin W_s$,

2) $h_\nu(x, s) = q(\mu_{z \leq s+1}[q(z) \notin h_\nu(x, \cdot) \upharpoonright s])$, if $x \in W_s$.

It is easy to check, that for an arbitrary W_i, which is not empty and not ω, we can decide that $\nu_{W_i} \in \mathrm{Com}_{pr}(\mathcal{S})$.

W_i reduces to W_j by primitive recursive function if and only if ν_{W_i} reduces by primitive recursive function to ν_{W_j}.

Now, let α be an arbitrary punctual numbering of the family $\mathcal{S}$. Then we can find c.e. set W_i such that $\alpha = \nu_{W_i}$. Since α is numbering of the family $\mathcal{S}$, and W_i is not empty, then there is an element $b \in B \setminus A$, then we can define the function $\varphi(x) = \mu_z[h_\alpha(x, z) = b]$, which is range of c.e. set W_i. Which means that $\mathcal{R}_{pr}(\mathcal{S})$ is isomorphic to $\mathcal{L}^0_{prm}$.

It is known, that the set $K_0 = \{\langle x, y\rangle | x \in W_y\}$ is universal in $\mathcal{L}^0_{prm}$. Consequently, ν_{K_0} is universal punctual numbering of the family $\mathcal{S}$.

Note that $\mathcal{R}_{pr}(\mathcal{S})$ is isomorphic to the upper semilattice of all c.e. sets under *pr*-many-one reducibility. By [32] we can say that first-order theory of $\mathcal{R}_{pr}(\mathcal{S})$ is undecidable.

Proposition 4. There exists the family $\mathcal{S} = \{A, B\}$ such that $|A \cap B| = \infty$ and $|\mathcal{R}_{pr}(\mathcal{S})| = \infty$ without universal numbering.

Proof. Let $A = \omega \setminus \{0\}$ and $B = \omega \setminus \{1\}$. And let μ be an arbitrary numbering of the family $\mathcal{S}$. We will build the numbering $\nu \in \mathrm{Com}_{pr}(\mathcal{S})$ so that $\nu \not\leq_{pr} \mu$.

We will construct quick function q_ν for ν as follows:

$$q_\nu(w, y) = \begin{cases} y + 2, & \{0, 1\} \cap h_\mu(p_e(w) \downarrow, \cdot) \upharpoonright y = \emptyset; \\ 0, & h_\mu(p_e(w), y) = 1; \\ 1, & h_\mu(p_e(w), y) = 0; \\ y + 1, & \{0, 1\} \cap h_\mu(p_e(w) \downarrow, \cdot) \upharpoonright y \neq \emptyset, \end{cases}$$

where h is quick function of the numbering μ and p_e is function that does not reduce ν to μ.

Let numbering ν reduces to numbering μ by primitive recursive function p, then there is w such that $\nu(w) = 0$ and $\mu(p(w)) = 1$ or $\nu(w) = 1$ and $\mu(p(w)) = 0$. Contradiction.

Since μ is an arbitrary numbering from the $\mathrm{Com}_{pr}(\mathcal{S})$, then for any numbering from $\mathrm{Com}_{pr}(\mathcal{S})$ we can construct ν, which does not reduce to μ, which means that there is no universal numbering in this family.

4 Punctual semilattice of the infinite families

Theorem 4. There is no punctual numbering for the family of all punctually enumerable sets.

Proof. Suppose that there is the numbering ν for the family of all punctually enumerable sets. Let $g(x, y)$ be a quick function for ν. We construct punctually enumerable set A such that $\nu(e) \neq A$ for all $e \in \omega$; thus we will come to the contradiction.

Let $q(0) = p(0, 0) + 1$ and $q(n + 1) = \max(\{p(n + 1, z) : \ z \leq n + 1\} \cup \{q(n)\}) + 1$.

It is clear that q is primitive recursive increasing function. Assume $A = \mathrm{range}(q)$.

By contradiction, assume $\nu(n) = A$ for some $n \in \omega$. Since, A is infinite, the function $\lambda y.g(n, y)$ is injective. It is clear that the set $\{p(n, z) : \ z \leq n\}$ has $n+1$ different elements, hence in $\{p(n, z) : \ z \leq n\}$ there is a number greater than $q(n)$. Contradiction.

Corollary 2. There is no punctual numbering for the family of all primitive recursive punctually enumerable sets.

The proof of corollary is the same as the proof of the theorem.

Definition 5. The numbering $\nu \in \mathrm{Com}_{pr}(\mathcal{S})$ is called *friedberg*, if it is injective.

Proposition 5. If the infinite family $\mathcal{S}$ has friedberg numbering, then
1) $\mathcal{R}_{pr}(\mathcal{S})$ does not have the least element,
2) $|\mathcal{R}_{pr}(\mathcal{S})| = \infty$.

Proof. 1) Let ν be friedberg numbering for the infinite family $\mathcal{S}$. Suppose that α is the least numbering of the family $\mathcal{S}$. Then $\alpha \leq_{pr} \nu$ by primitive recursive function g, which means that $alpha$ is punctually decidable, since $\forall n, m\, \alpha(m) = \alpha(n) \Leftrightarrow \nu(g(m)) = \nu(g(n)) \Leftrightarrow g(m) = g(n)$. Consequently ([28], Proposition 3.1(ii)), there is *spd*-numbering $\mu \equiv_{pr} \alpha$. By using the construction from the Theorem 4.1 of the same paper, we can construct the numbering $\mu_0 <_{pr} \mu$, which contradicts to choice of α.

2) Let ν be friedberg numbering of the infinite family $\mathcal{S}$. Consider $\mu = \nu \circ f$, where f is a primitive recursive bijective function such that f^{-1} is not primitive recursive (existence of such function is shown in [33]). It is clear, that $\mu \leq_{pr} \nu$ and μ friedberg: $\mu(m) = \mu(n) \Leftrightarrow \nu(f(m)) = \nu(f(n)) \Leftrightarrow f(m) = f(n) \Leftrightarrow m = n$. Wherein, $\nu = \mu \circ f^{-1}$ and f^{-1} is not primitive recursive, which means that $\nu \nleq_{pr} \mu$. Thus $\mu <_{pr} \nu$. Continuing the process, you can build an endless-waning chain of friedberg numberings, from where we get required.

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Author Contributions

A. Askarbekkyzy proved Theorem 3. R. Bagaviev worked on Proposition 5. V. Isakov proved Proposition 1. B. Kalmurzayev set the direction of the research and participated in discussions of all the evidence and propositions in this article. D. Nurlanbek proved Proposition 3. F. Rakymzhankyzy proved Theorem 4. A. Slobozhanin proved Theorem 1. All authors participated in discussions of the definitions that were introduced in this article. All authors participated in the revision of the manuscript and approved the final submission.

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2024M4/41-56 *Research article*

On solution of non-linear $\mathbb{FDE}$ under tempered $\Psi-$Caputo derivative for the first-order and three-point boundary conditions

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In this article, the existence and uniqueness of solutions for non-linear fractional differential equation with Tempered $\Psi-$Caputo derivative with three-point boundary conditions were studied. The existence and uniqueness of the solution were proved by applying the Banach contraction mapping principle and Schaefer's fixed point theorem.

Keywords: fractional differential equations, tempered $\Psi-$Caputo derivative, nonlinear analysis, Schaefer's fixed point theorem; Banach contraction.

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Introduction

Fractional calculus is a strong tool of mathematical analysis that studies derivatives and integrals of a fractional order. Fractional differential equations ($\mathbb{FDE}$s) are used in many fields of engineering and sciences such as physics, mechanics, chemistry, viscoelasticity, electro chemistry, porous media, electromagnetic, for more details see the books [1–3] and applicable papers [4–10].

One of the useful generalizations of a fractional derivative and an integral is associated with a dependent function [11]. Mali *et al.* developed well the theory of tempered fractional integrals and derivatives of a function with respect to another function [12]. This theory combines the tempered fractional calculus with the ψ-fractional calculus, both of which have found applications in topics including continuous time random walks. In [13], Benchohra *et al.*, by means of the Banach fixed point theorem and the nonlinear alternative of Leray-Schauder type, proved the existence of solutions for the first order boundary value problem ($\mathbb{BVP}$) for a $\mathbb{FDE}$

$$\mathcal{D}^{\eta}_{C}\varkappa(\mathrm{w}) = \mathfrak{h}(\mathrm{w}, \varkappa(\mathrm{w})), \qquad \mathrm{w}, \eta \in \Omega := (0, 1), \tag{1}$$

under condition $p\varkappa(0) + q\varkappa(1) = \mathfrak{h}_{\circ}$. In 1996, authors proved existence and uniqueness of problem (1), for $\mathrm{w} \in \Omega$, where $\mathfrak{h} : [0, \top] \times \mathbb{R} \to \mathbb{R}$, $0 < \top < +\infty$ is a given continuous function [14]. Also, the authors in [15] by the Banach contraction principle and Schauder's fixed point theorem investigated the existence of solutions for problem (1) with integral conditions $\varkappa(0) + p\int_0^{\top} \varkappa(\grave{\zeta})\,\mathrm{d}\grave{\zeta} = \varkappa(\top)$. Recently,

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authors have presented very valuable works on the ability of fractional derivatives and fractional q-derivatives with Caputo sense [16–23]. Salim *et al.* concerned some existence and uniqueness results for a class of problems for nonlinear Caputo tempered implicit $\mathbb{FDE}$s

$$\begin{cases} \mathcal{D}^{\eta,\psi}_{C;\kappa_1,\xi}\varkappa(\mathrm{w}) = \mathfrak{h}\left(\mathrm{w}, \varkappa(\mathrm{w}), \mathcal{D}^{\eta,\psi}_{C;\kappa_1,\xi}\varkappa(\mathrm{w})\right), & \mathrm{w} \in [\kappa_1, \kappa_2],\ \eta \in \Omega, \\ p_1\varkappa(\kappa_1) + p_2\varkappa(\kappa_2) = p_3\varkappa(\eta) + q, \end{cases} \tag{2}$$

in b-Metric spaces with three-point boundary conditions, where $\mathfrak{h} \in C(\Omega \times \mathbb{R}^2)$, $\kappa_1 < \eta < \kappa_2 < +\infty$ and $p_i, i = 1, 2, 3,\ q$ are real constants [24]. For more instance, consider [25–27].

Motivated by the studies [28–33], we characterize an alteration of the $\Psi-$Caputo derivative, the Tempered $\Psi-$Caputo derivative and consider the Cauchy problem for $\mathbb{FDE}$s with this type of a fractional derivative. This derivative incorporates as uncommon cases the Tempered Caputo [30]. In this manner, we study the following ($\mathbb{BVP}$) for a $\mathbb{FDE}$ with the tempered $\Psi-$Caputo fractional derivative type

$$\begin{cases} \mathcal{D}^{\eta,\lambda,\psi}_{C}\varkappa(\mathrm{w}) = \mathfrak{h}(\mathrm{w}, \varkappa(\mathrm{w})), & \mathrm{w} \in \bar{\mathrm{J}} = [0, \top],\ \eta \in \Omega, \\ p_1\varkappa(0) + p_2\varkappa(\top_\circ) + p_3\varkappa(\top) = q, \end{cases} \tag{3}$$

where $\mathfrak{h} : \bar{\mathrm{J}} \times \mathbb{R}^2$ is a continuous function, $\mathcal{D}^{\eta,\lambda,\psi}_{C}$ is a Tempered $\Psi-$Caputo fractional derivative, increase function Ψ is a continuously differentiable on $[0, \infty)$ with $\Psi(0) = 0$, $\Psi'(\mathrm{w}) > 0$, for each $\mathrm{w} \in (0, \infty)$, $\lim_{\mathrm{w}\to\infty} \Psi(\mathrm{w}) = \infty$, $p_i\ (i = 1, 2, 3)$ are real constants with $\acute{q} = p_1 + p_2e^{-\lambda\Psi(\top_\circ)} + p_3e^{-\lambda\Psi(\top)}$, $\acute{q} \neq 0$, $0 < \top_\circ < \top$.

In Section 2, we give a result, based on Banach (Theorem 2) and Schaefer's (Theorem 3) fixed point theorems. In Section 2.2, a case is given that illustrates the application of our primary comes about. These comes about can be considered as a commitment to this developing field.

1 Preliminaries

In this section we present definitions and theorems from fractional calculus theory which are used in this paper. Let $\Psi \in C^n[\top_1, \top_2]$ be an increasing differentiable function for all $\top_1 \leq \mathrm{w} \leq \top_2$. The tempered Ψ-fractional integral of an order $n - 1 < \eta < n\ (n \in \mathbb{N})$ is present by

$$\mathcal{I}^{\eta,\lambda,\Psi}_{\top_1}\varkappa(\mathrm{w}) = \int_{\top_1}^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1}\frac{e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})}\Psi'(\grave{\zeta})}{\Gamma(\eta)}\varkappa(\grave{\zeta})\,\mathrm{d}\grave{\zeta}, \qquad \lambda \geqslant 0,$$

where $\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}) = \Psi(\mathrm{w}) - \Psi(\mathrm{v})$. Now, let $\Psi'(\mathrm{w}) \neq 0$ for all $\mathrm{w} \in [\top_1, \top_2]$. The tempered $\Psi-$Caputo fractional derivative of an order η is defined as

$$\mathcal{D}^{\eta,\lambda,\Psi}_{C;\top_1}\varkappa(\mathrm{w}) = \int_{\top_1}^{\mathrm{w}} \frac{e^{-\lambda\Psi(\mathrm{w})}\Psi'(\mathrm{w})}{\Gamma(n-\eta)}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{n-\eta-1}\varkappa^{[n]}_{\lambda,\Psi}(\grave{\zeta})\,\mathrm{d}\grave{\zeta}, \qquad \lambda \geqslant 0,$$

where $\varkappa^{[n]}_{\lambda,\Psi}(\mathrm{w}) = \left[\frac{1}{\Psi'(\mathrm{w})}\frac{\mathrm{d}}{\mathrm{dw}}\right]^n\left(e^{\lambda\Psi(\mathrm{w})}\varkappa(\mathrm{w})\right)$. By employing the above assumptions the next theorem is satisfied.

Theorem 1. Let $\Psi \in C^n[\top_1, \top_2]$. Then the following holds (I) $\mathcal{D}^{\eta,\lambda,\Psi}_{C;\top_1}\left[\mathcal{I}^{\eta,\lambda,\Psi}_{\top_1}\varkappa(\mathrm{w})\right] = \varkappa(\mathrm{w})$; (II) $\mathcal{I}^{\eta,\lambda,\Psi}_{\top_1}\left[\mathcal{D}^{\eta,\lambda,\Psi}_{C;\top_1}(\varkappa(\mathrm{w}))\right] = \varkappa(\mathrm{w}) - e^{-\lambda\Psi(\mathrm{w})}\sum\limits_{k=0}^{n-1}\mathfrak{h}_k\left[\tilde{\Psi}_{\top_1}(\mathrm{w})\right]^k$ where

$$\mathfrak{h}_k = \frac{\varkappa^{[k]}_{\lambda,\Psi}(\top_1)}{k!} = \frac{1}{k!}\left[\frac{1}{\Psi'(\mathrm{w})}\frac{\mathrm{d}}{\mathrm{dw}}\right]\left(e^{\lambda\Psi(\mathrm{w})}\varkappa(\mathrm{w})\right)\Big|_{\mathrm{w}=\top_1}.$$

2 Main results

In this section, we consider 𝔹𝕍ℙ (3). We consider the norm $|\varkappa|_\infty := \sup\left\{\varkappa(\mathrm{w}) : \mathrm{w} \in \bar{\mathrm{J}}\right\}$ on space $C(\bar{\mathrm{J}})$.

2.1 Existence of solution

Let us start by defining what we mean by a solution of 𝔹𝕍ℙ (3).

Definition 1. A continuous function $\varkappa : \bar{\mathrm{J}} \to \mathbb{R}$ is a solution of the 𝔹𝕍ℙ (3), if $\mathcal{D}_C^{\eta,\lambda,\psi}\varkappa(\mathrm{w})$ exists for all $\mathrm{w} \in \bar{\mathrm{J}}$, continuous on $\bar{\mathrm{J}}$, and $\varkappa(\mathrm{w})$ fulfils equality (3) for all $\mathrm{w} \in \bar{\mathrm{J}}$.

Lemma 1. Let the function $\mathfrak{h} \in C(\bar{\mathrm{J}} \times \mathbb{R})$ be bounded. Then the function $\varkappa(\mathrm{w})$ is a solution of the 𝔹𝕍ℙ (3) defined on the interval $\bar{\mathrm{J}}$ iff it is a solution of the following equation

$$\varkappa(\mathrm{w}) = \tfrac{q}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})} + \int_0^{\top} \mathbb{G}(\mathrm{w}, \grave{\zeta})\tfrac{\Psi'(\grave{\zeta})}{\Gamma(\eta)}\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}$$

with $\mathbb{G}(\mathrm{w},\mathrm{v}) = \mathbb{G}_1(\mathrm{w},\mathrm{v})$, whenever $0 \le \mathrm{w} \le \top_\circ$, and $\mathbb{G}(\mathrm{w},\mathrm{v}) = \mathbb{G}_2(\mathrm{w},\mathrm{v})$, whenever $\top_\circ < \mathrm{w} \le \top$, where

$$\mathbb{G}_1(\mathrm{w},\mathrm{v}) = \begin{cases} -\frac{p_2 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}}(\tilde{\Psi}_{\mathrm{v}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top_\circ)} & \\ \quad -\frac{p_3 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & 0 \le \mathrm{v} \le \mathrm{w}, \\ -\frac{p_2 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}}(\tilde{\Psi}_{\mathrm{v}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top_\circ)} - \frac{p_3 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \mathrm{w} < \mathrm{v} \le \top, \\ -\frac{p_3 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \top_\circ < \mathrm{v} \le \top, \end{cases}$$

$$\mathbb{G}_2(\mathrm{w},\mathrm{v}) = \begin{cases} -\frac{p_2}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} & \\ \quad -\frac{p_3}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & 0 \le \mathrm{v} \le \top_\circ, \\ -\frac{p_3}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & \top_\circ < \mathrm{v} \le \mathrm{w}, \\ -\frac{p_3}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})}(\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \mathrm{w} < \mathrm{v} \le \top. \end{cases}$$

Proof. By performing the integral $\mathcal{I}_0^{\eta\lambda,\Psi}$ to both of Equation (3) and applying assertion (2) of Theorem 1, we get $\varkappa(\mathrm{w}) = c_0 e^{-\lambda\Psi(\mathrm{w})} + \frac{1}{\Gamma(\eta)}\int_0^{\mathrm{w}}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}$. Using condition (3) we have

$$\begin{aligned} c_0 = \tfrac{q}{\acute{q}} - \tfrac{p_2}{\acute{q}}&\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}\frac{e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}}{\Gamma(\eta)}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\ &- \int_0^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}\frac{p_3 e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}}{\acute{q}\Gamma(\eta)}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}, \end{aligned}$$

then the unique solution of (3) is given by the formula

$$\begin{aligned} \varkappa(\mathrm{w}) = \tfrac{q}{\acute{q}}e^{-\lambda\Psi(\mathrm{w})} &- \tfrac{p_2 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}\Gamma(\eta)}\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\ &- \tfrac{p_3}{\acute{q}\Gamma(\alpha)}e^{-\lambda\Psi(\mathrm{w})}\int_0^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \\ &+ \int_0^{\mathrm{w}}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1}\frac{e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})}}{\Gamma(\eta)}\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}. \end{aligned} \tag{4}$$

Let $0 \le \mathrm{w} \le \top_\circ$. Then (4) can be rewritten

$$\begin{aligned}
\varkappa(\mathrm{w}) = {} & \tfrac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} - \tfrac{p_2 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}\Gamma(\eta)} \Big\{ \int_0^{\mathrm{w}} (\tilde{\Psi}_{\xi}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta} \\
& + \int_{\mathrm{w}}^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta} \Big\} \\
& - \tfrac{p_3}{\acute{q}\Gamma(\eta)} e^{-\lambda\Psi(\mathrm{w})} \Big\{ \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta} \\
& + \int_{\mathrm{w}}^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta} \\
& + \int_{\top_\circ}^{\top} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta} \Big\} \\
& + \tfrac{1}{\Gamma(\eta)} \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})} \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta}.
\end{aligned}$$

Here grouping the like terms, and then simplifying, we get the new function as follows

$$\mathbb{G}_1(\mathrm{w}, \mathrm{v}) = \begin{cases} -\frac{p_2}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} (\tilde{\Psi}_{\mathrm{v}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top_\circ)} & \\ \quad -\frac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} (\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & 0 \le \mathrm{v} \le \mathrm{w}, \\ -\frac{p_2}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} (\tilde{\Psi}_{\mathrm{v}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top_\circ)} - \frac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} (\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \mathrm{w} < \mathrm{v} \le \top, \\ -\frac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} (\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \top_\circ < \mathrm{v} \le \top, \end{cases}$$

using this equality, relation (4) may be written as an integral equation,

$$\varkappa(\mathrm{w}) = \tfrac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} + \tfrac{1}{\Gamma(\eta)} \int_0^{\top_\circ} \mathbb{G}_1(\mathrm{w}, \grave{\zeta}) \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta},$$

for the case $\mathrm{w} \in [\top_\circ, \top]$ we can write equality (4) in the form

$$\begin{aligned}
\varkappa(\mathrm{w}) = {} & \tfrac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} - \tfrac{p_2 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}\Gamma(\eta)} \int_0^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta} \\
& - \tfrac{p_3 e^{-\lambda\Psi(\mathrm{w})}}{\acute{q}\Gamma(\eta)} \Big\{ \int_0^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta} \\
& + \int_{\top_\circ}^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta} \\
& + \int_{\mathrm{w}}^{\top} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta} \Big\} \\
& + \tfrac{1}{\Gamma(\eta)} \Big\{ \int_0^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})} \Psi'(\eta) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta} \Big\} \\
& + \int_{\top_\circ}^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})} \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta}.
\end{aligned}$$

Here we introduce the new function

$$\mathbb{G}_2(\mathrm{w}, \mathrm{v}) = \begin{cases} -\frac{p_2}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} (\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} & \\ -\frac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} (\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & 0 \le \mathrm{v} \le \top_\circ, \\ -\frac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} (\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)} + (\tilde{\Psi}_{\mathrm{v}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\mathrm{w})}, & \top_\circ < \mathrm{v} \le \mathrm{w}, \\ -\frac{p_3}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} (\tilde{\Psi}_{\mathrm{v}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\mathrm{v}}(\top)}, & \mathrm{w} < \mathrm{v} \le \top. \end{cases}$$

Hence for the case $\mathrm{w} \in [\top_\circ, \top]$, we can write (4) in the form $\varkappa(\mathrm{w}) = \frac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} + \frac{1}{\Gamma(\eta)} \int_{\top_\circ}^{\top} \mathbb{G}_2(\mathrm{w}, \grave{\zeta}) \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta}$. So, we conclude that the solution of $\mathbb{BVP}$ (3) has the form $\varkappa(\mathrm{w}) = \frac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} + \int_0^{\top} \mathbb{G}(\mathrm{w}, \grave{\zeta}) \frac{\Psi'(\grave{\zeta})}{\Gamma(\eta)} \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta}$. The proof is completed.

Theorem 2. Assume that

(H_1) There exists a constant $k > 0$ such that $|\mathfrak{h}(\mathrm{w}, \mathrm{v}_1) - \mathfrak{h}(\mathrm{w}, \mathrm{v}_2)| \leqslant k|\mathrm{v}_1 - \mathrm{v}_2|$, for all $\mathrm{w} \in \bar{\mathrm{J}}$, and for each $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{R}$.

If $k/\lambda^\eta \left(|p_2/\acute{q}| + |p_3/\acute{q}| + 1\right) < 1$, then the $\mathbb{BVP}$ (3) has a unique solution on $\bar{\mathrm{J}}$.

Proof. We transform the problem (3) into a fixed point problem considering the operator $\mathcal{O} : C(\bar{\mathrm{J}}) \to C(\bar{\mathrm{J}})$ defined by

$$\mathcal{O}(\varkappa)(\mathrm{w}) = \frac{q}{\acute{q}} e^{-\lambda\Psi(\mathrm{w})} + \int_0^{\top} \mathbb{G}(\mathrm{w}, \grave{\zeta}) \Psi'(\grave{\zeta}) \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta}))\, \mathrm{d}\grave{\zeta}. \tag{5}$$

It isn't troublesome to see that, a fixed point $\mathcal{O}$ is a solution of (3). We might utilize the Banach contraction principle to demonstrate that $\mathcal{O}$ characterized by (3) includes a fixed point and $\mathcal{O}$ is a contraction.

Case 1: Let $\mathrm{w} \in \bar{\mathrm{J}}$, so we have

$$\begin{aligned}
|\mathcal{O}(\varkappa_1)(\mathrm{w}) - \mathcal{O}(\varkappa_2)(\mathrm{w})| &= \left| \int_0^{\top} \mathbb{G}_1(\mathrm{w}, \grave{\zeta}) \tfrac{\Psi'(\grave{\zeta})}{\Gamma(\eta)} \left[\mathfrak{h}(\grave{\zeta}, \varkappa_1(\grave{\zeta})) - \mathfrak{h}(\grave{\zeta}, \varkappa_2(\grave{\zeta})) \right] \mathrm{d}\grave{\zeta} \right| \\
&\leq \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \int_0^{\top} |\mathbb{G}_1(\mathrm{w}, \grave{\zeta})| \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \\
&\leq \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \Bigg(\left| \tfrac{p_2}{\acute{q}} \right| e^{-\lambda\Psi(\mathrm{w})} \Bigg\{ \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \\
&\quad + \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \Bigg\} + \left| \tfrac{p_3}{\acute{q}} \right| e^{-\lambda\Psi(\mathrm{w})} \Bigg\{ \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \\
&\quad + \int_{\mathrm{w}}^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} + \int_{\top_\circ}^{\top} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \Bigg\} \\
&\quad + \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \Bigg) \\
&\leq \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \Bigg\{ \left| \tfrac{p_2}{\acute{q}} \right| e^{-\lambda\Psi(\mathrm{w})} \int_0^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \\
&\quad + \left| \tfrac{p_3}{\acute{q}} \right| e^{-\lambda\Psi(\mathrm{w})} \int_0^{\top} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} + \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1} e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})} \Psi'(\grave{\zeta})\, \mathrm{d}\grave{\zeta} \Bigg\} \\
&\leqslant \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \Bigg\{ \left| \tfrac{p_2}{\acute{q}} \right| \int_0^{\Psi(\top_\circ)} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} + \left| \tfrac{p_3}{\acute{q}} \right| \int_0^{\Psi(\top)} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} + \int_0^{\Psi(\mathrm{w})} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} \Bigg\} \\
&\leqslant \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \Bigg\{ \left| \tfrac{p_2}{\acute{q}} \right| \int_0^{\infty} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} + \left| \tfrac{p_3}{\acute{q}} \right| \int_0^{\infty} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} + \int_0^{\infty} \grave{\zeta}^{\eta-1} e^{-\lambda\grave{\zeta}}\, \mathrm{d}\grave{\zeta} \Bigg\} \\
&\leqslant \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \Bigg\{ \left| \tfrac{p_2}{\acute{q}} \right| \tfrac{\Gamma(\eta)}{\lambda^\eta} + \left| \tfrac{p_3}{\acute{q}} \right| \tfrac{\Gamma(\eta)}{\lambda^\eta} + \tfrac{\Gamma(\eta)}{\lambda^\eta} \Bigg\} \leqslant \tfrac{k}{\lambda^\eta} \left(\left| \tfrac{p_2}{\acute{q}} \right| + \left| \tfrac{p_3}{\acute{q}} \right| + 1 \right) |\varkappa_1 - \varkappa_2|.
\end{aligned}$$

Case 2: Let $\mathrm{w} \in [\top_\circ, \top]$, so we have

$$
\begin{aligned}
|\mathcal{O}(\varkappa_1)(\mathrm{w}) - \mathcal{O}(\varkappa_2)(\mathrm{w})| & \left| \tfrac{1}{\Gamma(\eta)} \int_0^{\top} \mathbb{G}_2(\mathrm{w}, \grave{\zeta}) \Psi'(\grave{\zeta}) \left[\mathfrak{h}(\grave{\zeta}, \varkappa)_1(\grave{\zeta})) - \mathfrak{h}(\grave{\zeta}, \varkappa_2(\grave{\zeta})) \right] \mathrm{d}\grave{\zeta} \right| \\
& \leq \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \int_0^{\top} |\mathbb{G}_2(\mathrm{w}, \grave{\zeta})| \Psi'(\grave{\zeta}) \, \mathrm{d}\grave{\zeta} \\
& \leq \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \left(\left| \tfrac{p_2}{\acute{q}} \right| e^{-\lambda \Psi(\mathrm{w})} \left\{ \int_0^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda \tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta}) \, \mathrm{d}\grave{\zeta} \right\} \right. \\
& \quad + \left| \tfrac{p_3}{\acute{q}} \right| e^{-\lambda \Psi(\mathrm{w})} \left\{ \int_0^{\top} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda \tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta}) \, \mathrm{d}\grave{\zeta} \right. \\
& \quad \left. + \int_{\top_\circ}^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda \tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta}) \, \mathrm{d}\grave{\zeta} \quad + \int_{\mathrm{w}}^{\top} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda \tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta}) \, \mathrm{d}\grave{\zeta} \right\} \\
& \quad \left. + \int_0^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1} e^{-\lambda \tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})} \Psi'(\grave{\zeta}) \, \mathrm{d}\grave{\zeta} + \int_{\top_\circ}^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1} e^{-\lambda \tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})} \Psi'(\grave{\zeta}) \, \mathrm{d}\grave{\zeta} \right) \\
& \leqslant \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \left\{ \left| \tfrac{p_2}{\acute{q}} \right| \int_0^{\top_\circ} (\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1} e^{-\lambda \tilde{\Psi}_{\grave{\zeta}}(\top_\circ)} \Psi'(\grave{\zeta}) \, \mathrm{d}\grave{\zeta} \right. \\
& \quad \left. + \left| \tfrac{p_3}{\acute{q}} \right| \int_0^{\top} (\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1} e^{-\lambda \tilde{\Psi}_{\grave{\zeta}}(\top)} \Psi'(\grave{\zeta}) \, \mathrm{d}\grave{\zeta} + \int_0^{\mathrm{w}} (\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}))^{\eta-1} e^{-\lambda \tilde{\Psi}_{\grave{\zeta}}(\mathrm{w})} \Psi'(\grave{\zeta}) \, \mathrm{d}\grave{\zeta} \right\} \\
& \leqslant \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \left\{ \left| \tfrac{p_2}{\acute{q}} \right| \int_0^{\Psi(\top_\circ)} \grave{\zeta}^{\eta-1} e^{-\lambda \grave{\zeta}} \, \mathrm{d}\grave{\zeta} + \left| \tfrac{p_3}{\acute{q}} \right| \int_0^{\Psi(\top)} \grave{\zeta}^{\eta-1} e^{-\lambda \grave{\zeta}} \, \mathrm{d}\grave{\zeta} + \int_0^{\Psi(\mathrm{w})} \grave{\zeta}^{\eta-1} e^{-\lambda \grave{\zeta}} \, \mathrm{d}\grave{\zeta} \right. \\
& \leqslant \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \left\{ \left| \tfrac{p_2}{\acute{q}} \right| \int_0^{\infty} \grave{\zeta}^{\eta-1} e^{-\lambda \grave{\zeta}} \, \mathrm{d}\grave{\zeta} + \left| \tfrac{p_3}{\acute{q}} \right| \int_0^{\infty} \grave{\zeta}^{\eta-1} e^{-\lambda \grave{\zeta}} \, \mathrm{d}\grave{\zeta} + \int_0^{\infty} \grave{\zeta}^{\eta-1} e^{-\lambda \grave{\zeta}} \, \mathrm{d}\grave{\zeta} \right\} \\
& \leqslant \tfrac{k|\varkappa_1 - \varkappa_2|}{\Gamma(\eta)} \left\{ \left| \tfrac{p_2}{\acute{q}} \right| \tfrac{\Gamma(\eta)}{\lambda^\eta} + \left| \tfrac{p_3}{\acute{q}} \right| \tfrac{\Gamma(\eta)}{\lambda^\eta} + \tfrac{\Gamma(\eta)}{\lambda^\eta} \right\} \leqslant \tfrac{k}{\lambda^\eta} \left(\left| \tfrac{p_2}{\acute{q}} \right| + \left| \tfrac{p_3}{\acute{q}} \right| + 1 \right) |\varkappa_1 - \varkappa_2|.
\end{aligned}
$$

Thus, for all $\mathrm{w} \in \bar{\mathrm{J}}$, $|\mathcal{O}(\varkappa_1)(\mathrm{w}) - \mathcal{O}(\varkappa_2)(\mathrm{w})| \leqslant \frac{k}{\lambda^\eta}\left(\left|\frac{p_2}{\acute{q}}\right| + \left|\frac{p_3}{\acute{q}}\right| + 1\right)|\varkappa_1 - \varkappa_2|$. Consequently by (3), $\mathcal{O}$ is a contraction. As a consequence of Banach fixed point theorem we deduce that $\mathcal{O}$ has a fixed point which is a solution of the problem (3).

The second result is based on Schaefer's fixed point.

Theorem 3. Assume that

(H_2) The function $\mathfrak{h} : \bar{\mathrm{J}} \times \mathbb{R} \to \mathbb{R}$ is continuous;

(H_3) There exists $\breve{\Delta} > 0$ such that $|\mathfrak{h}(\mathrm{w}, \varkappa)| \leq \breve{\Delta}$ for each $\mathrm{w} \in \bar{\mathrm{J}}$ and all $\varkappa \in \mathbb{R}$.

Then the $\mathbb{BVP}$ (3) has at least one solution on $\bar{\mathrm{J}}$.

Proof. We shall use Schaefer's fixed point theorem to prove that $\mathcal{O}$ defined by (5) has a fixed point. The proof will be given in several steps.

Step 1: $\mathcal{O}$ is continuous. Let $\{\varkappa_n\}$ be a sequence such that $\varkappa_n \to \varkappa$ in $C(\bar{\mathrm{J}})$. Then for each $\mathrm{w} \in \bar{\mathrm{J}}$,

$$
\begin{aligned}
|\mathcal{O}(\varkappa_n)(\mathrm{w}) - \mathcal{O}(\varkappa)(\mathrm{w})| &= \int_0^{\top} \mathbb{G}(\mathrm{w}, \grave{\zeta}) \tfrac{\Psi'(\grave{\zeta})}{\Gamma(\eta)} \mathfrak{h}(\grave{\zeta}, \varkappa_n(\grave{\zeta})) - \mathfrak{h}(\grave{\zeta}, \varkappa(\grave{\zeta})) \, \mathrm{d}\grave{\zeta} \\
&\leqslant \tfrac{1}{\Gamma(\eta)} \sup_{\mathrm{w} \in \bar{\mathrm{J}}} \left| \mathfrak{h}(\mathrm{w}, \varkappa_n(\mathrm{w})) - \mathfrak{h}(\mathrm{w}, \varkappa(\mathrm{w})) \right| \int_0^{T} \mathbb{G}(\mathrm{w}, \grave{\zeta}) \Psi'(\grave{\zeta}) \, \mathrm{d}\grave{\zeta} \\
&\leqslant \tfrac{1}{\lambda^\eta} \left| \tfrac{p_2}{\acute{q}} + \tfrac{p_3}{\acute{q}} + 1 \right| |\mathfrak{h}(\mathrm{w}, \varkappa_n(\mathrm{w})) - \mathfrak{h}(\mathrm{w}, \varkappa(\mathrm{w}))|_\infty.
\end{aligned}
$$

Since $\mathfrak{h}$ is a continuous function, we have

$$
|\mathcal{O}(\varkappa_n)(\mathrm{w}) - \mathcal{O}(\varkappa)(\mathrm{w})| \leqslant \tfrac{1}{\lambda^\eta} \left| \tfrac{p_2}{\acute{q}} + \tfrac{p_3}{\acute{q}} + 1 \right| |\mathfrak{h}(\mathrm{w}, \varkappa_n(\mathrm{w})) - \mathfrak{h}(\mathrm{w}, \varkappa(\mathrm{w}))|_\infty \to 0,
$$

as $n \longrightarrow \infty$.

Step 2: $\mathcal{O}$ maps bounded sets into bounded sets in $C(\bar{\mathrm{J}})$. Indeed, it is enough to show that for any $r>0$ there exists a positive constant l such that for each $\varkappa \in B_r = \{\varkappa \in C(\bar{\mathrm{J}}) : |\varkappa|_\infty \leqslant r\}$, we have $|\mathcal{O}(\varkappa)|_\infty \leqslant l$. By ($H_3$) we have for each $\mathrm{w} \in \bar{\mathrm{J}}$,

$$|\mathcal{O}(\varkappa)(\mathrm{w})| \leqslant \left|\tfrac{q}{\acute{q}}\right| + \tfrac{1}{\Gamma(\eta)}\left|\int_0^\top \mathbb{G}(\mathrm{w},\grave{\zeta})\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\right|$$
$$\leqslant \left|\tfrac{q}{\acute{q}}\right| + \tfrac{\breve{\Delta}}{\Gamma(\eta)}\int_0^\top \mathbb{G}(\mathrm{w},\grave{\zeta})\Psi'(\grave{\zeta}))\,\mathrm{d}\grave{\zeta} \leqslant \left|\tfrac{q}{\acute{q}}\right| + \tfrac{\breve{\Delta}}{\lambda}^{\eta}\left(\left|\tfrac{p_2}{\acute{q}}\right| + \left|\tfrac{p_3}{\acute{q}}\right| + 1\right) := l.$$

Step 3: Here we prove that the operator $\mathcal{O}$ maps bounded sets into equicontinuous sets from $C(\bar{\mathrm{J}})$. Let $\mathrm{w}_1, \mathrm{w}_2 \in \bar{\mathrm{J}}$, B_r be a bounded set in $C(\bar{\mathrm{J}})$. As in Step 2 we assume that $\varkappa \in B_r$ and $K_\Psi = \lambda \max\left\{\Psi'(\mathrm{w})e^{-\lambda\Psi(\mathrm{w})} : \mathrm{w} \in \bar{\mathrm{J}}\right\}$. Then the mean value theorem implies that $\left|e^{-\lambda\Psi(\mathrm{w}_2)} - e^{-\lambda\Psi(\mathrm{w}_1)}\right| \leqslant K_\Psi|\mathrm{w}_2 - \mathrm{w}_1|$.

Case 1: Let $\mathrm{w}_1, \mathrm{w}_2 \in \bar{\mathrm{J}}$. Then

$$\begin{aligned}
|\mathcal{O}(\varkappa)(\mathrm{w}_2) - \mathcal{O}(\varkappa)(\mathrm{w}_1)| &= \Big|\tfrac{q}{\acute{q}}\left(e^{-\lambda\Psi(\mathrm{w}_2)} - e^{-\lambda\Psi(\mathrm{w}_1)}\right)\\
&\quad + \int_0^\top \left(\mathbb{G}_1(\mathrm{w}_2,\grave{\zeta}) - \mathbb{G}_1(\mathrm{w}_1,\grave{\zeta})\right)\tfrac{\Psi'(\grave{\zeta})}{\Gamma(\eta)}\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\Big|\\
&\leqslant \left|\tfrac{q}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)} - e^{-\lambda\Psi(\mathrm{w}_1)}\right) + \int_0^\top \left(\mathbb{G}_1(\mathrm{w}_2,\grave{\zeta}) - \mathbb{G}_1(\mathrm{w}_1,\grave{\zeta})\right)\tfrac{\breve{\Delta}\Psi'(\grave{\zeta})}{\Gamma(\eta)}\,\mathrm{d}\grave{\zeta}\\
&\leqslant \left|\tfrac{q}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)} - e^{-\lambda\Psi(\mathrm{w}_1)}\right)\\
&\quad + \tfrac{\breve{\Delta}}{\Gamma(\eta)}\Big\{\Big(\int_0^{\mathrm{w}_2}\left|\tfrac{p_2}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&\quad + \int_0^{\mathrm{w}_2}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda(\tilde{\Psi}_{\grave{\zeta}}(\top))}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&\quad + \int_0^{\mathrm{w}_2} e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&\quad + \int_{\mathrm{w}_2}^{\top_\circ}\left|\tfrac{p_2}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&\quad + \int_{\mathrm{w}_2}^{\top_\circ}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\Big)\\
&\quad - \int_0^{\mathrm{w}_1}\mathrm{w}\tfrac{p_2}{\acute{q}}\mathrm{w}e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&\quad - \int_0^{\mathrm{w}_1}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(s)\,\mathrm{d}\grave{\zeta}\\
&\quad - \int_0^{\mathrm{w}_1} e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&\quad - \int_{\mathrm{w}_1}^{\top_\circ}\left|\tfrac{p_2}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&\quad - \int_{\mathrm{w}_1}^{\top_\circ}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\Big\}\\
&\leqslant \left|\tfrac{q}{\acute{q}}\right|K_\Psi|\mathrm{w}_2 - \mathrm{w}_1| + \tfrac{\breve{\Delta}}{\Gamma(\eta)}\Big\{\left|\tfrac{p_2}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)} - e^{-\lambda\Psi(\mathrm{w}_1)}\right)\\
&\quad \cdot \int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}
\end{aligned}$$

$$
\begin{aligned}
&+\left|\tfrac{p_3}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)}-e^{-\lambda\Psi(\mathrm{w}_1)}\right)\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\int_0^{\mathrm{w}_2}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}-\int_0^{\mathrm{w}_1}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\Bigg\}\\
\leqslant{}& K_\Psi|\mathrm{w}_2-\mathrm{w}_1|\Bigg\{\left|\tfrac{q}{\acute{q}}\right|+\tfrac{\check{\Delta}}{\Gamma(\eta)}\left|\tfrac{p_2}{\acute{q}}\right|\int_0^{\Psi(\top_\circ)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}\\
&+\tfrac{M}{\Gamma(\eta)}\left|\tfrac{p_3}{\acute{q}}\right|\int_{\tilde{\Psi}_{\top_\circ}(\top)}^{\Psi(\top_\circ)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}\Bigg\}+\int_{\Psi(\mathrm{w}_1)}^{\Psi(\mathrm{w}_2)}\grave{\zeta}^{\eta-1}e^{-\lambda\eta}\,\mathrm{d}\grave{\zeta}.
\end{aligned}
$$

Case 2: Let $\mathrm{w}_1\in[0,\top_\circ]$, $\mathrm{w}_2\in[\top_\circ,\top]$. Then

$$
\begin{aligned}
|\mathcal{O}(\varkappa)(\mathrm{w}_2)-\mathcal{O}(\varkappa)(\mathrm{w}_1)|={}&\Bigg|\tfrac{q}{\acute{q}}\left(e^{-\lambda\Psi(\mathrm{w}_2)}-e^{-\lambda\Psi(\mathrm{w}_1)}\right)\\
&+\tfrac{1}{\Gamma(\eta)}\int_0^{\top}\left(\mathbb{G}_2(\mathrm{w}_2,\grave{\zeta})-\mathbb{G}_1(\mathrm{w}_1,\grave{\zeta})\right)\Psi'(\grave{\zeta})\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\Bigg|\\
\leqslant{}&\left|\tfrac{q}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)}-e^{-\lambda\Psi(\mathrm{w}_1)}\right)\\
&+\tfrac{\check{\Delta}}{\Gamma(\eta)}\Bigg\{\left|\tfrac{p_2}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}\int_{\top_\circ}^{\mathrm{w}_2}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\int_{\top_\circ}^{\mathrm{w}_2}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}\int_{\mathrm{w}_2}^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\left|\tfrac{p_2}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}\int_0^{\mathrm{w}_1}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}\int_0^{\mathrm{w}_1}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\int_0^{\mathrm{w}_1}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\left|\tfrac{p_2}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}\int_{\mathrm{w}_1}^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}\int_{\mathrm{w}_1}^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&-\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}\int_{\top_\circ}^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
\leqslant{}& K_\Psi|\mathrm{w}_2-\mathrm{w}_1|\Bigg\{\left|\tfrac{q}{\acute{q}}\right|+\tfrac{\check{\Delta}}{\Gamma(\lambda)}\left|\tfrac{p_2}{\acute{q}}\right|\int_0^{\top_\circ}(\tilde{\Psi}_{\grave{\zeta}}(\top_\circ))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top_\circ)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\\
&+\tfrac{\check{\Delta}}{\Gamma(\lambda)}\left|\tfrac{p_3}{\acute{q}}\right|\int_0^{\top}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\Bigg\}
\end{aligned}
$$

$$+\int_0^{\mathrm{w}_2}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}-\int_0^{\mathrm{w}_1}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}$$
$$\leqslant K_\Psi|\mathrm{w}_2-\mathrm{w}_1|\Big\{\left|\tfrac{q}{\acute{q}}\right|+\tfrac{\check{\Delta}}{\Gamma(\eta)}\left|\tfrac{p_2}{\acute{q}}\right|\int_0^{\Psi(\top_\circ)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}$$
$$+\tfrac{\check{\Delta}}{\Gamma(\alpha)}\left|\tfrac{p_3}{\acute{q}}\right|\int_{\tilde{\Psi}_{\grave{\zeta}}(\top)}^{\Psi(\top_\circ)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}\Big\}+\int_{\Psi(\mathrm{w}_1)}^{\Psi(\mathrm{w}_2)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}.$$

Case 3: Let $\mathrm{w}_1,\mathrm{w}_2\in[\top_\circ,\top]$. Then

$$|\mathcal{O}(\varkappa)(\mathrm{w}_2)-\mathcal{O}(\varkappa)(\mathrm{w}_1)|=\Big|\tfrac{q}{\acute{q}}\left(e^{-\lambda\Psi(\mathrm{w}_2)}-e^{-\lambda\Psi(\mathrm{w}_1)}\right)$$
$$+\int_0^{\top}\mathbb{G}_2(\mathrm{w}_2,\grave{\zeta})-\mathbb{G}_2(\mathrm{w}_1,\grave{\zeta})\tfrac{\Psi'(\grave{\zeta})}{\Gamma(\eta)}\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\Big|$$
$$\leqslant\left|\tfrac{q}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)}-e^{-\lambda\Psi(\mathrm{w}_1)}\right)$$
$$+\tfrac{\check{\Delta}}{\Gamma(\eta)}\Big\{\int_{\top_\circ}^{\mathrm{w}_2}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}$$
$$+\int_{\top_\circ}^{\mathrm{w}_2}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}$$
$$+\int_{\mathrm{w}_2}^{\top}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}$$
$$-\int_{\top_\circ}^{\mathrm{w}_1}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}$$
$$-\int_{\tau}^{\mathrm{w}_1}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}$$
$$-\int_{\mathrm{w}_1}^{\top}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\Big\}$$
$$\leqslant\left|\tfrac{q}{\acute{q}}\right|\left(e^{-\lambda\Psi(\mathrm{w}_2)}-e^{-\lambda\Psi(\mathrm{w}_1)}\right)$$
$$+\tfrac{\check{\Delta}}{\Gamma(\eta)}\Big\{\int_{\top_\circ}^{\top}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_2)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}$$
$$-\int_{\top_\circ}^{\top}\left|\tfrac{p_3}{\acute{q}}\right|e^{-\lambda\Psi(\mathrm{w}_1)}(\tilde{\Psi}_{\grave{\zeta}}(\top))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\top)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}$$
$$+\int_{\top_\circ}^{\mathrm{w}_2}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}-\int_{\top_\circ}^{\mathrm{w}_1}(\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_2))^{\eta-1}e^{-\lambda\tilde{\Psi}_{\grave{\zeta}}(\mathrm{w}_1)}\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta}\Big\}$$
$$\leqslant K_\Psi|\mathrm{w}_2-\mathrm{w}_1|\Big\{\left|\tfrac{q}{\acute{q}}\right|+\tfrac{\check{\Delta}}{\Gamma(\eta)}\left|\tfrac{p_3}{\acute{q}}\right|\int_0^{\tilde{\Psi}_{\top_\circ}(\top)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}\Big\}+\int_{\tilde{\Psi}_{\top_\circ}(\mathrm{w}_2)}^{\tilde{\Psi}_{\top_\circ}(\mathrm{w}_2)}\grave{\zeta}^{\eta-1}e^{-\lambda\grave{\zeta}}\,\mathrm{d}\grave{\zeta}.$$

The right hand side of the above inequalities for all Cases 1–3 tend for zero by $\mathrm{w}_2\longrightarrow\mathrm{w}_1$. From this due to Arzelà-Ascoli theorem and Steps 1–3 follows that the mapping $\mathcal{O}:C(\bar{\mathrm{J}})\to\mathcal{O}(\bar{\mathrm{J}})$ is continuous.

Step 4: Here we prove the necessary prior bounds. Indeed we show that the set $\Upsilon=\big\{\varkappa\in C([0,\mathbb{R}]):\varkappa=\mu\mathcal{O}(\varkappa)$ for some $\mu\in\Omega\big\}$, is bounded. Suppose that $\varkappa=\mu\mathcal{O}(\varkappa)$ for some $0<\mu<1$. Then for each $\mathrm{w}\in\bar{\mathrm{J}}$ we can write

$$\varkappa(\mathrm{w})=\mu\Big\{\tfrac{q}{\acute{q}}+\int_0^{\top}\mathbb{G}(\mathrm{w},\grave{\zeta})\tfrac{\Psi'(s)}{\Gamma(\alpha)}\mathfrak{h}(\grave{\zeta},\varkappa(\grave{\zeta}))\,\mathrm{d}\grave{\zeta}\Big\}.$$

This fact in combination with (H_3) shows that for each $\mathrm{w} \in \bar{\mathrm{J}}$,

$$|\mathcal{O}(\varkappa)(\mathrm{w})| \leqslant \left|\frac{q}{\acute{q}}\right| + \frac{\check{\Delta}}{\Gamma(\eta)} \int_0^{\top} \mathbb{G}(\mathrm{w}, \grave{\zeta})\Psi'(\grave{\zeta})\,\mathrm{d}\grave{\zeta} \leqslant \left|\frac{q}{\acute{q}}\right| + \frac{\check{\Delta}}{\Gamma(\eta)}\left(\left|\frac{p_2}{\acute{q}}\right| + \left|\frac{p_3}{\acute{q}}\right| + 1\right),$$

for each $\mathrm{w} \in \bar{\mathrm{J}}$. Thus the set Υ is bounded and $\mathcal{O}$ has a fixed point by Schaefer's fixed point theorem, that is a solution of problem.

2.2 An illustrative example

In this section we give an example to illustrate the usefulness of our main results. Let us consider $\Psi(\mathrm{w}) = \ln(\mathrm{w}+1)$, and the following $\mathbb{FBVP}$,

$$\mathcal{D}_C^{\eta,\lambda,\Psi}\varkappa(\mathrm{w}) = \frac{e^{-2\mathrm{w}}|\varkappa(\mathrm{w})|}{24|1+\varkappa(\mathrm{w})|}, \qquad \mathrm{w} \in \bar{\mathrm{J}} = [0,1]\,,\ \eta \in (0,1]\,, \tag{6}$$

with

$$\varkappa(0) + \tfrac{3}{2}\varkappa\left(\tfrac{1}{2}\right) + 2\,\varkappa(1) = \tfrac{1}{2}. \tag{7}$$

Put $\mathfrak{h}(\mathrm{w}, \varkappa(\mathrm{w})) = \frac{e^{-2\mathrm{w}}\varkappa(\mathrm{w})}{24(\mathrm{w}+1)}$, $(\mathrm{w}, \varkappa) \in \bar{\mathrm{J}} \times [0, +\infty)$. Let $\varkappa_1, \varkappa_2 \in [0, +\infty)$ and $\mathrm{w} \in \bar{\mathrm{J}}$. Then we have

$$|\mathfrak{h}(\mathrm{w}, \varkappa_1) - \mathfrak{h}(\mathrm{w}, \varkappa_2)| = \frac{e^{-2\mathrm{w}}}{24}\left|\frac{\varkappa_1}{2\varkappa_1+1} - \frac{\varkappa_2}{2\varkappa_2+1}\right| = \frac{e^{-2\mathrm{w}}}{24}\frac{|\varkappa_1-\varkappa_2|}{(\varkappa_1+1)(\varkappa_2+1)} \leqslant \frac{e^{-2\mathrm{w}}}{24}|\varkappa_1 - \varkappa_2| \leqslant \frac{1}{24}|\varkappa_1 - \varkappa_2|.$$

Hence the condition (H_1) holds with $k = \frac{1}{24}$. We shall check that condition (7) is satisfied for appropriate values of $\eta \in]0,1[$ with $p_1 = 1$, $p_2 = \frac{3}{2}$, $p_3 = 2$, $\top = 1$, $\top_\circ = \frac{1}{2}$ and

$$\acute{q} = p_1 + p_2 e^{-\lambda\Psi(\top_\circ)} + p_3 e^{-\lambda\Psi(\top)} = 1 + \tfrac{3}{2}e^{-\lambda\Psi(1/2)} + 2e^{-\lambda\Psi(1)}. \tag{8}$$

Then by Theorem 2 the problem (6)-(7) has a unique solution on $\bar{\mathrm{J}}$ for values of η and λ satisfying condition (H_1). For example

- If $\lambda = 1$ and for all $\eta \in (0,1)$ then thanks to Eq. (8), we have $\acute{q} = 3$ and

$$\frac{k}{\lambda^\eta}\left(\left|\frac{p_2}{\acute{q}}\right| + \left|\frac{p_3}{\acute{q}}\right| + 1\right) \simeq 0.09027 < 1.$$

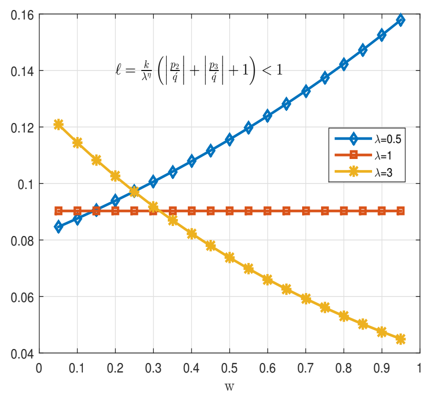

Figure 1. 2D-graph of $\ell < 1$ for the $\mathbb{BVP}$ (6) whenever $\lambda = 0.5, 1, 3$, $\eta, \mathrm{w} \in \Omega$

Fig. 1 shows 2D-graph of ℓ for the $\mathbb{BVP}$ (6) whenever $\lambda = 0.5, 1, 3$ and $\eta, \mathrm{w} \in \Omega$.

- If $\lambda = 3$ and $\eta \in \Omega$, we have

$$\frac{k}{\lambda^{\eta}}\left(\left|\frac{p_2}{\acute{q}}\right| + \left|\frac{p_3}{\acute{q}}\right| + 1\right) = \frac{1}{24\times 3^{\eta}}\left(\left|\frac{1}{2}\right| + \left|\frac{2}{3}\right| + 1\right) < 1. \tag{9}$$

Table 1 presents numerical values of ℓ for the 𝔹𝕍ℙ (6) whenever $\lambda = 0.5, 1, 3$ and $\eta, \mathrm{w} \in \Omega$. Fig. 1 shows 2D-graph of ℓ for the 𝔹𝕍ℙ (6) whenever $\lambda = 0.5, 1, 3$ and $\eta, \mathrm{w} \in \Omega$.

Table 1

Obtained results of $\ell < 1$ in (9) when $\lambda = 0.5, 1, 3$ and $\eta, \mathrm{w} \in \Omega$

w	ℓ		
	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 3$
0.05	0.08463	0.09028	0.12091
0.10	0.08761	0.09028	0.11444
0.15	0.09070	0.09028	0.10833
0.20	0.09390	0.09028	0.10254
0.25	0.09721	0.09028	0.09706
⋮	⋮	⋮	⋮
0.80	0.14232	0.09028	0.05304
0.85	0.14734	0.09028	0.05021
0.90	0.15254	0.09028	0.04752
0.95	0.15792	0.09028	0.04498

2.3 Data comparison

At present, we consider $\lambda = 3$, three values for $\eta = 0.7, 0.8, 0.9$ and four cases for $\Psi(\mathrm{w})$ as $\Psi_1(\mathrm{w}) = 2^{\mathrm{w}}$; $\Psi_2(\mathrm{w}) = \mathrm{w}$ (Caputo derivative); $\Psi_3(\mathrm{w}) = \ln \mathrm{w}$ (Caputo–Hadamard derivative); $\Psi_4(\mathrm{w}) = \sqrt{\mathrm{w}}$ (Katugampola derivative); for the 𝔹𝕍ℙ (6). Tables 2, 3 and 4 show the numerical results for these cases. One can see illustrative results in the Figs. 2, 3 and 4. Therefore, these results guarantee that for all of three different cases by terms of the order η and four standard fractional derivatives Ψ, the 𝔹𝕍ℙ admits at least a solution on $\bar{J}$.

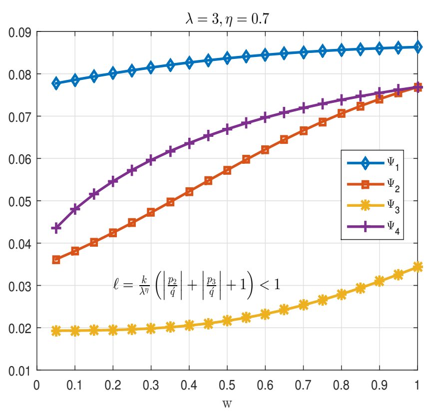

Figure 2. 2D plot of ℓ in 𝔹𝕍ℙ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.7$ for $\mathrm{w} \in \Omega$

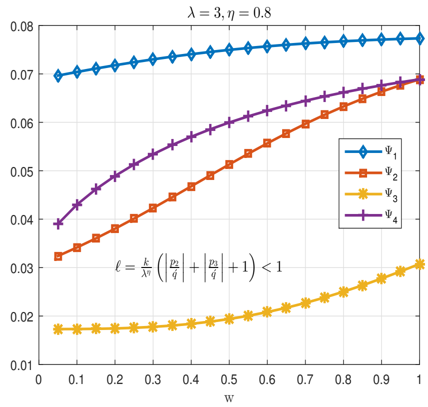

Figure 3. 2D plot of ℓ in $\mathbb{BVP}$ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.8$ for $\mathrm{w} \in \Omega$

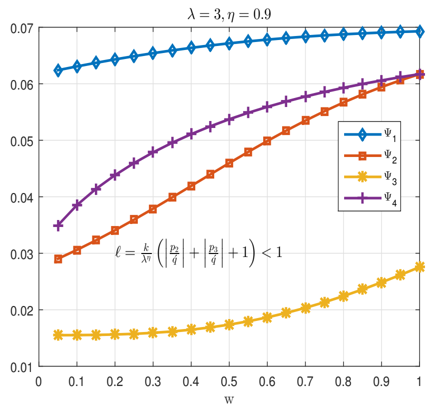

Figure 4. 2D plot of ℓ in $\mathbb{BVP}$ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.9$ for $\mathrm{w} \in \Omega$

Table 2

Obtained results of $\ell < 1$ in $\mathbb{BVP}$ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.7$ for $\mathrm{w} \in \Omega$

w	$\Psi_1(\mathrm{w}) = 2^{\mathrm{w}}$		$\Psi_2(\mathrm{w}) = \mathrm{w}$		$\Psi_3(\mathrm{w}) = \ln \mathrm{w}$		$\Psi_4(\mathrm{w}) = \sqrt{\mathrm{w}}$	
	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ
0.05	1.157	0.078	4.012	0.036	28001.000	0.019	2.790	0.044
0.10	1.140	0.079	3.593	0.038	3501.000	0.019	2.355	0.048
0.15	1.125	0.079	3.232	0.040	1038.037	0.019	2.095	0.052
0.20	1.112	0.080	2.921	0.042	438.500	0.019	1.915	0.055
0.25	1.099	0.081	2.653	0.045	225.000	0.020	1.781	0.057
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.70	1.027	0.085	1.429	0.067	11.204	0.025	1.284	0.072
0.75	1.023	0.085	1.369	0.069	9.296	0.027	1.260	0.073
0.80	1.019	0.086	1.318	0.071	7.836	0.028	1.239	0.074
0.85	1.016	0.086	1.273	0.072	6.699	0.029	1.220	0.075
0.90	1.013	0.086	1.235	0.074	5.801	0.031	1.203	0.075
0.95	1.011	0.086	1.202	0.076	5.082	0.033	1.188	0.076
1.00	1.009	0.086	1.174	0.077	4.500	0.034	1.174	0.077

Table 3

Obtained results of $\ell < 1$ in $\mathbb{BVP}$ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.8$ for $\mathrm{w} \in \Omega$

w	$\Psi_1(\mathrm{w}) = 2^{\mathrm{w}}$		$\Psi_2(\mathrm{w}) = \mathrm{w}$		$\Psi_3(\mathrm{w}) = \ln \mathrm{w}$		$\Psi_4(\mathrm{w}) = \sqrt{\mathrm{w}}$	
	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ
0.05	1.157	0.070	4.012	0.032	28001.000	0.017	2.790	0.039
0.10	1.140	0.070	3.593	0.034	3501.000	0.017	2.355	0.043
0.15	1.125	0.071	3.232	0.036	1038.037	0.017	2.095	0.046
0.20	1.112	0.072	2.921	0.038	438.500	0.017	1.915	0.049
0.25	1.099	0.072	2.653	0.040	225.000	0.018	1.781	0.051
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.75	1.023	0.077	1.369	0.062	9.296	0.024	1.260	0.065
0.80	1.019	0.077	1.318	0.063	7.836	0.025	1.239	0.066
0.85	1.016	0.077	1.273	0.065	6.699	0.026	1.220	0.067
0.90	1.013	0.077	1.235	0.066	5.801	0.028	1.203	0.068
0.95	1.011	0.077	1.202	0.068	5.082	0.029	1.188	0.068
1.00	1.009	0.077	1.174	0.069	4.500	0.031	1.174	0.069

Table 4

Obtained results of $\ell < 1$ in $\mathbb{BVP}$ (6) when $\lambda = 3$, $\Psi(\mathrm{w}) \in \{2^{\mathrm{w}}, \mathrm{w}, \ln \mathrm{w}, \sqrt{\mathrm{w}}\}$ and $\eta = 0.9$ for $\mathrm{w} \in \Omega$

w	$\Psi_1(\mathrm{w}) = 2^{\mathrm{w}}$		$\Psi_2(\mathrm{w}) = \mathrm{w}$		$\Psi_3(\mathrm{w}) = \ln \mathrm{w}$		$\Psi_4(\mathrm{w}) = \sqrt{\mathrm{w}}$	
	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ	$\acute{q}$	ℓ
0.05	1.157	0.062	4.012	0.029	28001.000	0.016	2.790	0.035
0.10	1.140	0.063	3.593	0.031	3501.000	0.016	2.355	0.039
0.15	1.125	0.064	3.232	0.032	1038.037	0.016	2.095	0.041
0.20	1.112	0.064	2.921	0.034	438.500	0.016	1.915	0.044
0.25	1.099	0.065	2.653	0.036	225.000	0.016	1.781	0.046
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.75	1.023	0.069	1.369	0.055	9.296	0.021	1.260	0.059
0.80	1.019	0.069	1.318	0.057	7.836	0.022	1.239	0.059
0.85	1.016	0.069	1.273	0.058	6.699	0.024	1.220	0.060
0.90	1.013	0.069	1.235	0.059	5.801	0.025	1.203	0.061
0.95	1.011	0.069	1.202	0.061	5.082	0.026	1.188	0.061
1.00	1.009	0.069	1.174	0.062	4.500	0.028	1.174	0.062

Conclusion

This paper contains a new fractional mathematical model of a $\mathbb{BVP}$ consisting of the Tempered $\Psi-$Caputo derivative in the framework of the generalized sequential G-operators. We turned to the investigation of the qualitative behaviors of its solutions including existence and uniqueness. To confirm the existence criterion, we used the Banach contraction mapping principle and Schaefer's fixed point theorem. Comparison of data obtained by choosing several types of fractional derivatives is of great importance.

Author Contributions

K. Bensassa: Actualization, formal analysis, methodology, initial draft, validation, investigation and was a major contributor in writing the manuscript. M. Benbachir: Methodology, actualization, validation, investigation, formal analysis and initial draft. M.E. Samei: Validation, actualization, formal analysis, methodology, investigation, simulation, initial draft, software and was a major contributor in writing the manuscript. S. Salahshour: Methodology, actualization, validation, investigation, formal analysis and initial draft. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare that they have no competing interests.

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Singularly perturbed integro-differential equations with degenerate Hammerstein's kernel

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Singularly perturbed integro-differential equations with degenerate kernels are considered. It is shown that in the linear case these problems are always uniquely solvable with continuous coefficients, while nonlinear problems either have no real solutions at all or have several of them. For linear problems, the results of Bobojanova are refined; in particular, necessary and sufficient conditions are given for the existence of a finite limit of their solutions as the small parameter tends to zero and sufficient conditions under which the passage to the limit to the solution of the degenerate equation is possible.

Keywords: singularly perturbed, Hammerstein's equation, degenerate kernel, Fredholm's equations, analytic function, Laurent's series, passage to the limit, the Maple program.

2020 Mathematics Subject Classification: 34E20, 45J05.

Introduction

Many applied problems lead to nonlinear Hammerstein's equations of the form

$$\varepsilon\frac{dy}{dt}=\int\limits_0^1 K\left(t,s\right)f\left(s,y\left(s,\varepsilon\right)\right)ds,\; y\left(0,\varepsilon\right)=y^0.$$

In the general case, it is impossible to obtain its solution in explicit form. However, if $K\left(t,s\right)$ is represented as a sum of products of functions with separated variables, then the study of this equation can be reduced to an algebraic system of equations. We will not consider the general case, but will show how this issue can be solved for a singularly perturbed equation of the form

$$\begin{gathered}\varepsilon\tfrac{dy(t)}{dt}=\int\limits_0^1 a_1\left(t\right)b_1\left(s\right)f\left(y\left(s,\varepsilon\right),s\right)\,ds+\\ +\int\limits_0^1 a_2\left(t\right)b_2\left(s\right)f\left(y\left(s,\varepsilon\right),s\right)ds,\quad y\left(0,\varepsilon\right)=y^0.\end{gathered}\tag{1}$$

Here $f\left(y,s\right)$ is a known continuous nonlinear function, $a_j\left(t\right),b_j\left(t\right)$ are known continuous functions on the segment $[0,1]$, $y=y\left(t,\varepsilon\right)$ is an unknown scalar function, $\varepsilon>0$ is a small parameter (the segment $[0,1]$ is taken to simplify the calculations; instead, you can take any segment $[0,T]$). Linear version of this problem:

$$\begin{gathered}\varepsilon\tfrac{dy}{dt}=\int\limits_0^1 a_1\left(t\right)b_1\left(s\right)y\left(s,\varepsilon\right)\,ds+\int\limits_0^1 a_2\left(t\right)b_2\left(s\right)y\left(s,\varepsilon\right)\,ds+\\ +h\left(t\right),\; y\left(0,\varepsilon\right)=y^0\end{gathered}\tag{2}$$

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was considered in [1]. Before examining the nonlinear equation (1), we present the results of this work. For a complete understanding, let us recall the scheme for solving equation (2) indicated in [1].

1 Linear singularly perturbed Fredholm's equations

Integrating (2) over t, assuming that it has a continuous solution, we obtain the equivalent problem

$$\varepsilon y(t,\varepsilon)=\int_0^t a_1(\theta)\,d\theta\int_0^1 b_1(s)\,y(s,\varepsilon)\,ds+$$

$$+\int_0^t a_2(\theta)\,d\theta\int_0^1 b_2(s)\,y(s,\varepsilon)\,ds+\int_0^t h(\theta)\,d\theta+\varepsilon y^0.$$

Using the notation $\int_0^t a_j(\theta)\,d\theta=q_j(t)\,,\ \int_0^t h(\theta)\,d\theta+\varepsilon y^0=h_1(t,\varepsilon)\,,$ we reduce the last equation to the integral equation

$$\varepsilon y(t,\varepsilon)=q_1(t)\int_0^1 b_1(s)\,y(s,\varepsilon)\ ds+q_2(t)\int_0^1 b_2(s)\,y(s,\varepsilon)\ ds+h_1(t,\varepsilon) \tag{3}$$

with a degenerate kernel and solve it using a well-known scheme (see, for example, [2]). Enter constants

$$w_1=\int_0^1 b_1(s)\,y(s,\varepsilon)\ ds,\ \ w_2=\int_0^1 b_2(s)\,y(s,\varepsilon)\,ds. \tag{4}$$

Then the solution to equation (3) will be written in the form

$$y(t,\varepsilon)=\frac{1}{\varepsilon}\left(q_1(t)w_1+q_2(t)\,w_2+h_1(t,\varepsilon)\right). \tag{5}$$

Substituting this into (4), we obtain a system of algebraic equations

$$\begin{cases}\varepsilon w_1=\int_0^1 b_1(s)\left(\left(q_1(s)\,w_1+q_2(s)\,w_2\right)\right)\ ds+\int_0^1 b_1(s)\,h_1(s,\varepsilon)\,ds,\\[2ex] \varepsilon w_2=\int_0^1 b_2(s)\left(\left(q_1(s)\,w_1+q_2(s)\,w_2\right)\right)\ ds+\int_0^1 b_2(s)\,h_1(s,\varepsilon)\,ds\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\varepsilon w_1=c_{11}w_1+c_{12}w_2+H_1(\varepsilon)\,,\\[1ex] \varepsilon w_2=c_{21}w_1+c_{22}w_2+H_2(\varepsilon)\,,\end{cases} \tag{6}$$

relative to the unknown constants w_1 and w_2. Here it is indicated:

$$c_{ij}=\int_0^1 b_i(s)\,q_j(s)\,ds,\ \ H_j(\varepsilon)=\int_0^1 b_j(s)\,h_1(s,\varepsilon)\,ds,\ \ i,\ j=1,2.$$

Let $\sigma(C)=\{\lambda_1,\lambda_2\}$ be the spectrum of the matrix $C=(c_{ij})$ (λ_1, λ_2 may coincide). Let's reduce the matrix C to normal form in the space $\mathbb{C}^2$ (see, for example, [3]). The following cases of normal forms of a matrix are possible:

$$1)\ \ J_1=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\,(\lambda_1\neq\lambda_2)\,,$$

$$2)\ J_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} (\lambda_1 = \lambda_2 = \lambda),$$

$$3)\ J_3 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} (\lambda_1 = \lambda_2 = \lambda),$$

two of which are diagonal, and one have a Jordan's structure. There exists (see, for example, [4]) a transformation matrix $T = T_j$ such that $T^{-1}CT = J_j$, $j = 1, 2, 3$. But then the same matrix T leads to the matrix

$$\varepsilon I - C \equiv \begin{pmatrix} \varepsilon - c_{11} & -c_{12} \\ -c_{21} & \varepsilon - c_{22} \end{pmatrix}$$

of the normal form, i.e. $T^{-1}(\varepsilon I - C)\, T$ will take one of the following forms:

$$1)\ J_1(\varepsilon) = \begin{pmatrix} \varepsilon - \lambda_1 & 0 \\ 0 & \varepsilon - \lambda_2 \end{pmatrix} (\lambda_1 \neq \lambda_2),$$

$$2)\ J_2(\varepsilon) = \begin{pmatrix} \varepsilon - \lambda & 0 \\ 0 & \varepsilon - \lambda \end{pmatrix} (\lambda_1 = \lambda_2 = \lambda),$$

$$3)\ J_3(\varepsilon) = \begin{pmatrix} \varepsilon - \lambda & 1 \\ 0 & \varepsilon - \lambda \end{pmatrix} (\lambda_1 = \lambda_2 = \lambda).$$

In this case, the solution of the system (6) will be written in one of the following forms:

$$w = w(\varepsilon) = \left(T J_j^{-1}(\varepsilon)\, T^{-1}\right) H(\varepsilon),\ j = 1, 2, 3. \tag{7}$$

Let us first assume that $\det C \neq 0$. Then the eigenvalues $\lambda_j \neq 0$. We have in the case $j = 1$:

$$w = \left[T \begin{pmatrix} (\varepsilon - \lambda_1)^{-1} & 0 \\ 0 & (\varepsilon - \lambda_2)^{-1} \end{pmatrix} T^{-1}\right] H(\varepsilon). \tag{8}$$

Since $(\varepsilon - \lambda_j)^{-1} = -\frac{1}{\lambda_j}\frac{1}{1-\frac{\varepsilon}{\lambda_j}} = -\frac{1}{\lambda_j}\sum\limits_{k=0}^{\infty}\left(\frac{\varepsilon}{\lambda_j}\right)^k$ is the analytic function with respect to ε, and the inhomogeneity $H(\varepsilon) = \{h_1(\varepsilon), h_2(\varepsilon)\}$ linearly depends on ε, then $w(\varepsilon)$ is an analytic function with respect to ε, and the solution (5) of the problem (2) will have a first-order pole with respect to ε.

In the case $j = 2$ expression (7) for w has the form

$$w = \left(T J_2^{-1}(\varepsilon)\, T^{-1}\right) H(\varepsilon) = T \begin{pmatrix} \frac{1}{\varepsilon-\lambda} & 0 \\ 0 & \frac{1}{\varepsilon-\lambda} \end{pmatrix} T^{-1} H(\varepsilon) =$$

$$= -\frac{1}{\lambda} T \begin{pmatrix} \frac{1}{1-\frac{\varepsilon}{\lambda}} & 0 \\ 0 & \frac{1}{1-\frac{\varepsilon}{\lambda}} \end{pmatrix} T^{-1} H(\varepsilon),$$

i.e. the vector $w = w(\varepsilon)$ is again an analytic function with respect to ε, and therefore the solution (5) of the problem (2) will have a pole of first order with respect to ε.

In the case $j = 3$ the vector w :

$$w = \left(T J_3^{-1}(\varepsilon)\, T^{-1}\right) H(\varepsilon) = T \begin{pmatrix} \varepsilon - \lambda & 1 \\ 0 & \varepsilon - \lambda \end{pmatrix}^{-1} T^{-1} H(\varepsilon) =$$

$$= T \begin{bmatrix} \frac{1}{\varepsilon-\lambda} & -\frac{1}{(\varepsilon-\lambda)^2} \\ 0 & \frac{1}{\varepsilon-\lambda} \end{bmatrix} T^{-1} H(\varepsilon) = T \begin{bmatrix} -\frac{1}{\lambda}\frac{1}{1-\frac{\varepsilon}{\lambda}} & -\frac{1}{\lambda^2}\frac{1}{\left(1-\frac{\varepsilon}{\lambda}\right)^2} \\ 0 & -\frac{1}{\lambda}\frac{1}{1-\frac{\varepsilon}{\lambda}} \end{bmatrix} T^{-1} H(\varepsilon)$$

is again an analytic function with respect to ε, and therefore the solution (5) of the problem (2) will have a pole of first order with respect to ε.

Let $\det C = 0$. Three cases have to be considered here:

$$a)\ \lambda_1 = \lambda_2 = 0, \quad b)\ \lambda_1 = 0,\ \lambda_2 \neq 0, \quad c)\ \lambda_1 \neq 0,\ \lambda_2 = 0.$$

In the case a), expression (8) for w takes the form

$$w = \left(T \begin{pmatrix} \frac{1}{\varepsilon} & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix} T^{-1}\right) H(\varepsilon) = \frac{1}{\varepsilon} H(\varepsilon)$$

if $C = 0$, and the form

$$w = \left(T \begin{pmatrix} \frac{1}{\varepsilon} & -\frac{1}{\varepsilon^2} \\ 0 & \frac{1}{\varepsilon} \end{pmatrix} T^{-1}\right) H(\varepsilon) = \frac{1}{\varepsilon^2} T \begin{pmatrix} \varepsilon & -1 \\ 0 & \varepsilon \end{pmatrix} H(\varepsilon)$$

if the matrix C is similar to a Jordan's cell $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case the solution (5) of the problem (2) will have a second-order pole with respect to ε and a third-order pole with respect to ε, if the matrix C is similar to a jordan's cell $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

In the case b), the expression (7) takes the form

$$w = T \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & (\varepsilon - \lambda_2)^{-1} \end{pmatrix} T^{-1} H(\varepsilon) = \frac{1}{\varepsilon} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{\lambda_2} \frac{\varepsilon}{1 - \frac{\varepsilon}{\lambda_2}} \end{bmatrix} H(\varepsilon),$$

therefore the solution (5) of the problem (2) will have a second-order pole with respect to ε. In the case c), we also obtain that the solution (5) of the problem (2) has a second-order pole with respect to ε.

Let us write the results obtained in the form of a theorem.

Theorem 1. Let the functions $a_j(t), b_j(t), h(t)$ in the equation (2) be continuous on the segment $[0,1]$. Then the following statements are true.

1. If $\det C \neq 0$, then the solution $y(t,\varepsilon)$ of the problem (2) exists in the class $C^1[0,1]$, is unique in this class and is represented as a Laurent's series $y(t,\varepsilon) = \sum_{k=-1}^{\infty} \varepsilon^k y_k(t)$.

2. If $\det C = 0$ and $\sigma(C) = \{\lambda_1, \lambda_2\}$, then the following statements hold:

a) when $\lambda_1 = \lambda_2 = 0$ the solution $y(t,\varepsilon)$ of the problem (2) exists in the class $C^1[0,1]$, is unique in this class and is represented as a Laurent's series $y(t,\varepsilon) = \sum_{k=-2}^{\infty} \varepsilon^k y_k(t)$, if $C = 0$, and in the form of Laurent's series $y(t,\varepsilon) = \sum_{k=-3}^{\infty} \varepsilon^k y_k(t)$, if C is similar to a Jordan cell $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

b) for $\lambda_1 = 0,\ \lambda_2 \neq 0$ or $\lambda_1 \neq 0,\ \lambda_2 = 0$ the solution $y(t,\varepsilon)$ of the problem (2) exists in the class $C^1[0,1]$, is unique in this class and is represented as a Laurent's series $y(t,\varepsilon) = \sum_{k=-2}^{\infty} \varepsilon^k y_k(t)$.

From this theorem it follows that in the general case the solution $y(t,\varepsilon)$ tends to infinity as $t > 0$ and $\varepsilon \to +0$. Only in exceptional cases $y(t,\varepsilon)$ may tend to a finite limit. For example, if $\det C \neq 0$, then for the existence of a finite limit it is necessary to require that $y_{-1}(t) \equiv 0$. This condition must be expressed through the initial data of the problem (2). This was done in [1], but it is quite cumbersome and we do not present it. In the case of one term in (2), i.e. in the case $a_2(t) \equiv 0$ or $b_2(t) \equiv 0$ condition $y_{-1}(t) \equiv 0$ becomes more visible. Let's show it.

Noting $a_1(t) = a(t)$, $b_1(t) = b(t)$, we rewrite equation (2) in the form

$$\varepsilon\frac{dy}{dt} = \int\limits_0^1 a(t)\, b(s)\, y(s,\varepsilon)\; ds + h(t)\,,\; y(0,\varepsilon) = y^0. \tag{9}$$

Applying the procedure described above to (9), we obtain the following solution:

$$y(t,\varepsilon) = \frac{1}{\varepsilon}\left[\frac{\int\limits_0^t a(x)dx \int\limits_0^1 b(s)\left(\int\limits_0^s h(\theta)d\theta + \varepsilon y^0\right)ds}{\varepsilon - \int\limits_0^1 b(s)\left(\int\limits_0^s a(x)dx\right)ds} + \int\limits_0^t h(\theta)\,d\theta + y^0\varepsilon\right]. \tag{10}$$

Summing up the expression in square brackets, we write the solution in the form

$$y(t,\varepsilon) = \frac{\varepsilon^{-1}}{\varepsilon - \int\limits_0^1 b(s)\left(\int\limits_0^s a(x)dx\right)ds}\left[-\varepsilon y^0 \int\limits_0^1 b(s)\left(\int\limits_0^s a(x)\,dx\right)ds + \right.$$

$$+\varepsilon^2 y^0 + \int\limits_0^t a(s)\,ds \int\limits_0^1 b(s)\left(\int\limits_0^s h(\theta)\,d\theta + \varepsilon y^0\right)ds -$$

$$\left. - \int\limits_0^t h(s)\,ds \int\limits_0^1 b(s)\left(\int\limits_0^s a(x)\,dx\right)ds + \varepsilon\int\limits_0^t h(s)\,ds\right].$$

The free term on ε in the square bracket does not allow one to go to the final limit as $\varepsilon \to +0$, therefore it must be removed. Let's calculate it:

$$\left(\int\limits_0^t a(s)\,ds\right)\left(\int\limits_0^1 b(s)\left(\int\limits_0^s h(\theta)\,d\theta\right)ds\right) - \left(\int\limits_0^t h(s)\,ds\right)\left(\int\limits_0^1 b(s)\left(\int\limits_0^s a(x)\,dx\right)ds\right).$$

This means that if for any $t \in [0,1]$ the condition

$$\left(\int\limits_0^t a(s)\,ds\right)\left(\int\limits_0^1 b(s)\left(\int\limits_0^s h(\theta)\,d\theta\right)ds\right) \equiv$$

$$\equiv \left(\int\limits_0^t h(s)\,ds\right)\left(\int\limits_0^1 b(s)\left(\int\limits_0^s a(x)\,dx\right)ds\right), \tag{$*$}$$

is satisfied, then there is a finite limit $y(t,\varepsilon) \to \bar{y}(t)$ as $\varepsilon \to +0$. This condition is necessary and sufficient for the existence of a finite limit $\lim\limits_{\varepsilon\to+0} y(t,\varepsilon) = \bar{y}(t)$.

Note that the condition $(*)$ is automatically satisfied if $a(t) \equiv h(t)$. It is curious that in this case the limit $\bar{y}(t)$ will coincide with the solution of the equation $\int\limits_0^1 b(s)\,\bar{y}(s)\; ds + 1 = 0$ degenerate with respect to (9). Let us prove this.

Let $h(t) \equiv a(t)$. Then the condition $(*)$ is satisfied and the solution of the problem (2) will be written in the form

$$y(t,\varepsilon) = -\frac{-y^0\left(\int\limits_0^1 b(s)\left(\int\limits_0^s a(x)dx\right)ds\right) + \varepsilon y^0 + \left(\int\limits_0^t a(s)ds\right)\left(\int\limits_0^1 b(s)y^0 ds\right) + \left(\int\limits_0^t h(s)ds\right)}{\left(-\varepsilon + \int\limits_0^1 b(s)\left(\int\limits_0^s a(x)\; dx\right)ds\right)} =$$

$$= \frac{y^0\left(\int\limits_0^1 b(s)\left(\int\limits_0^s a(x)dx\right)\; ds\right) - \varepsilon y^0 - \left(\int\limits_0^t a(s)ds\right)\left(\int\limits_0^1 b(s)y^0\; ds\right) - \left(\int\limits_0^t a(s)ds\right)}{\left(-\varepsilon + \int\limits_0^1 b(s)\left(\int\limits_0^s a(x)dx\right)ds\right)}.$$

Passing here to the limit when $\varepsilon \to +0$, we obtain

$$\bar{y}\,(t) = \frac{1}{\int\limits_0^1 b(s)\left(\int\limits_0^s a(x)\ dx\right)\ ds}\left[y^0\ \left(\int\limits_0^1 b\,(s)\left(\int\limits_0^s a\,(x)\ dx\right)\ ds\right) - \right.$$
$$\left. - \left(\int\limits_0^t a\,(s)\ ds\right)\left(\int\limits_0^1 b\,(s)\,y^0\ ds\right) - \left(\int\limits_0^t a\,(s)\ ds\right)\right]. \tag{11}$$

Let us show that $\bar{y}\,(t)$ is the solution to the degenerate equation

$$\int\limits_0^1 b\,(s)\cdot \bar{y}\,(s)\,ds + 1 = 0. \tag{12}$$

Substituting (11) into the left side of the equation (12), we will have

$$\int\limits_0^1 \frac{1}{\int\limits_0^1 b(s)\left(\int\limits_0^s a(x)dx\right)\ ds}\left[b(s)\left(-\left(\int\limits_0^s a\,(s)\,ds\right)y^0\left(\int\limits_0^1 b\,(s)\,ds\right) + \right.\right.$$
$$\left.\left. + y^0\left(\int\limits_0^1 b\,(s)\left(\int\limits_0^s a\,(x)\,dx\right)ds\right) - \left(\int\limits_0^s a\,(s)\,ds\right)\right)\right] + 1.$$

We must show that

$$\int\limits_0^1 b\,(s)\left(-\left(\int\limits_0^s a\,(x)\ dx\right)y^0\left(\int\limits_0^1 b\,(s)\,ds\right) + \right.$$
$$+\,y^0\left(\int\limits_0^1 b\,(s)\left(\int\limits_0^s a\,(x)\ dx\right)ds\right) - \overline{\overline{\int\limits_0^s a\,(x)\,dx}}\Bigg)ds -$$
$$-\int\limits_0^1 b\,(s)\,\overline{\overline{\left(\int\limits_0^s a\,(x)\ dx\right)}}ds\ \equiv 0.$$

Removing the terms underlined above and then canceling both sides by y^0, we arrive at the identity

$$\int\limits_0^1 b\,(s)\left(-\left(\int\limits_0^s a\,(x)\,dx\right)\left(\int\limits_0^1 b\,(s)\,ds\right) + \left(\int\limits_0^1 b\,(s)\left(\int\limits_0^s a\,(x)\,dx\right)\ ds\right)\right)\ ds \equiv 0.$$

The proof of this identity for arbitrary functions $a\,(t)$ and $b\,(t)$, continuous on an interval $[0,1]$, is problematic. However, in the case of polynomials $a\,(t)$ and $b\,(t)$, it can be proved by induction on the powers of the polynomials.

The following results were obtained.

Theorem 2. Let the functions $a\,(t)\,, b\,(t)\,, h\,(t)$ in equation (9) be continuous on the segment $[0,1]\,.$ Then:

1) equation (9) has a unique solution $y\,(t,\varepsilon) \in C^1\,[0,1]$ in the form (10), which for arbitrary $a\,(t)$ and $h\,(t) \in C\,[0,1]$ has a first-order pole with respect to ε if $C = \int\limits_0^1 b\,(s)\cdot\int\limits_0^s a\,(\theta)\,d\theta \neq 0$, and a second-order pole with respect to ε if $C = 0$;

2) in order for $y(t,\varepsilon)$ to be analytical in ε (for sufficiently small $\varepsilon > 0$), it is necessary and sufficient that the identity $(*)$ holds;

3) for $a(t) \equiv h(t)$ *, the exact solution $y(t,\varepsilon)$ of the equation (2) uniformly (for $t \in [0,1]$) tends to the solution (11) of the degenerate equation (12) when $\varepsilon \to +0$.

Remark 1. In work [1] statement 3) of this theorem was not given. Here it is proved for the first time.

Remark 2. It follows from Theorems 1 and 2 *that there is no boundary layer* in the solutions of problem (2).

Let's look at examples.

Example 1. Consider the problem

$$\varepsilon \frac{dy}{d\,t} = 5\,t^2 \int\limits_0^1 \left(4s^2 - 5s\right) y(s,\varepsilon)\,ds + 2t - 1,\ y(0,\varepsilon) = y^0. \tag{13}$$

Substituting $a(t) = 5t^2, b(t) = 4t^2 - 5t,\ h(t) = 2t - 1$ into formula (10), we find a solution to this problem in the form

$$y(t,\varepsilon) = y^0 - \frac{t}{\varepsilon} + \frac{t^2}{\varepsilon} - \frac{1}{4}\frac{t^3\left(70y^0\varepsilon - 13\right)}{\varepsilon\,(9\varepsilon + 5)}.$$

The condition $(*)$ that has the form $\frac{13}{36}t^3 \equiv -\frac{5}{9}t^2 + \frac{5}{9}t$, is not satisfied, and therefore the solution to problem (13) has a first-order pole in ε.

Example 2. Now consider the problem

$$\varepsilon \frac{dy}{d\,t} = 3(t-1)^2 \int\limits_0^1 \left(2s - \frac{6}{5}\right) y(s,\varepsilon)\ ds + 5\,t + 1,\ y(0,\varepsilon) = y^0.$$

Here: $a(t) = 3(t-1)^2$, $b(t) = 2t - \frac{6}{5}$, $h(t) = 5\,t + 1$, $C = \int\limits_0^1 b(s) \cdot \left(\int\limits_0^s a(x)\,dx\right) ds = 0$ and the condition $(*)$ is not met. Calculating the solution using formula (10), we obtain the following solution:

$$y(t,\varepsilon) = \frac{y^0}{60} \cdot \frac{-12t^3 + 36t^2 + 60\varepsilon - 36t}{\varepsilon} + \frac{150\varepsilon t^2 + 19t^3 + 60\varepsilon t - 57t^2 + 57t}{60\varepsilon^2}.$$

It can be seen that the solution has a pole of second order in ε.

Example 3. Consider another problem

$$\varepsilon \frac{dy}{dt} = \left(2 - 5\,t^2\right) \int\limits_0^1 s^3 y(s,\varepsilon)\,ds + \left(2 - 5t^2\right), y(0) = y^0. \tag{14}$$

Here $a(t) \equiv h(t) = \left(2 - 5t^2\right)$, it means that the condition $(*)$ is fulfilled in an obvious way and therefore there is a finite limit $\lim\limits_{\varepsilon \to +0} y(t,\varepsilon) = \bar{y}(t)$. Let's make sure of this. Solving problem (14) using the above method, we obtain the following solution:

$$y(t,\varepsilon) = \frac{1}{4}\frac{-175y^0t^3 - 700t^3 + 420\varepsilon y^0 + 210y^0t - 68y^0 + 840t}{-17 + 105\,\varepsilon}.$$

* In this case the identity $(*)$ is obvious.

We see that the solution is analytic with respect to ε for sufficiently small values $\varepsilon > 0$, and there is a uniform passage to the limit

$$y(t,\varepsilon) \to \bar{y}(t) = \frac{175}{68} y^0 t^3 + \frac{175}{17} t^3 - \frac{105}{34} t y^0 + y^0 - \frac{210}{17} t \ \ (\varepsilon \to +0).$$

Substituting $\bar{y}(t)$ into the right side of the degenerate equation $0 = \int\limits_0^1 s^3 \times \times \bar{y}(s)\, ds + 1$, we have

$$\int\limits_0^1 s^3 \left(\frac{175}{68} y^0 s^3 + \frac{175}{17} s^3 - \frac{105}{34} s y^0 + y^0 - \frac{210}{17} s \right) ds + 1 \equiv 0.$$

Thus, the function $\bar{y}(t)$ is the solution of a degenerate equation, which is consistent with statement 3) of Theorem 2.

2 Nonlinear singularly perturbed Hammerstein equations

Let's move on to studying the nonlinear equation (1). In the works known to us [5–7] more general linear, nonlinear differential and integro-differential equations are considered and systems are than in our work. However, they are devoted to the construction of asymptotic solutions and the study phenomena of initial and boundary jumps. Assuming that there is a continuous solution of this equation, integrating it by t over the segment $[0,t]$, we obtain the integral equation

$$\begin{gathered} \varepsilon y(t,\varepsilon) = q_1(t) \int\limits_0^1 b_1(s) f(y(s,\varepsilon), s)\, ds + \\ + q_2(t) \int\limits_0^1 b_2(s) f(y(s,\varepsilon), s)\, ds + \varepsilon y^0, \end{gathered} \tag{3*}$$

where the notations $q_j(t) = \int\limits_0^t a_j(\theta)\, d\theta$, $j = 1,2$, are introduced. Let us introduce constants

$$w_1 = \int\limits_0^1 b_1(s) f(y(s,\varepsilon), s)\ ds, \ \ w_2 = \int\limits_0^1 b_2(s) f(y(s,\varepsilon), s)\ ds. \tag{15}$$

Then the solution of the equation (3*) will be written in the form

$$y(t,\varepsilon) = \frac{1}{\varepsilon} \left(q_1(t) w_1 + q_2(t) w_2 + \varepsilon y^0 \right). \tag{16}$$

Substituting (16) into (15), we obtain an algebraic system of equations

$$\begin{aligned} w_1 &= \int\limits_0^1 b_1(s) f\left(\tfrac{1}{\varepsilon} \left(q_1(s) w_1 + q_2(s) w_2 + \varepsilon y^0 \right), s \right) ds, \\ w_2 &= \int\limits_0^1 b_2(s) f\left(\tfrac{1}{\varepsilon} \left(q_1(s) w_1 + q_2(s) w_2 + \varepsilon y^0 \right), s \right) ds. \end{aligned} \tag{17}$$

If the function $f(y,s)$ is known, then (17) is a nonlinear algebraic system of equations, the solvability of which relative tow_1 and w_2 is not guaranteed by anything. Therefore, it is unlikely that in the general case it will be possible to formulate the conditions for the solvability of the system (17) in terms of the initial data. In a specific case, when all the functions included in equation (1) are given, nothing can be also said about solvability. In this case, difficulties arise in calculating the integrals included in (17). Let's try to solve system (17) using the Maple program. We present the corresponding algorithm.

Restart:

Set the initial data

$$f := f\left(z,t\right);\ q_1 := q_1\left(t\right);\ q_2 := q_2\left(t\right);\ b_1 := b_1\left(t\right);\ b_2 := b_2\left(t\right).$$

We write system (17) for given data

$$w_1 = \int\limits_0^1 b_1\left(s\right) f\left(\tfrac{1}{\varepsilon}\left(q_1\left(s\right) w_1 + q_2\left(s\right) w_2 \ + \varepsilon y^0\right), s\right) ds,$$

$$w_2 = \int\limits_0^1 b_2\left(s\right) f\left(\tfrac{1}{\varepsilon}\left(q_1\left(s\right) w_1 + q_2\left(s\right) w_2 \ + \varepsilon y^0\right), s\right) ds.$$

A system of algebraic equations is obtained. We solve it using the *solve* operator. If we manage to find the constants $w_1 = w_1^0,\ w_2 = w_2^0$, then the solution of the equation (1) is obtained as follows:

$$y\left(t,\varepsilon\right) = \frac{1}{\varepsilon}\left(q_1\left(t\right) w_1 + q_2\left(t\right) w_2 \ + \varepsilon y^0\right);$$

$$subs\left(\left\{c_1 = c_1^0, c_2 = c_2^0\right\}, y\left(t,\varepsilon\right)\right).$$

Let us demonstrate the implementation of this procedure using specific examples.

Example 4. Solve the Cauchy's problem

$$\varepsilon \frac{d}{d\,t}\, y\left(t,\varepsilon\right) = 3t^2 \int\limits_0^1 s y^2\left(s,\varepsilon\right) ds,\ \ y\left(0,\varepsilon\right) = y^0. \tag{18}$$

Here: $q_1\left(t\right) = \frac{t^3}{3}$, $q_2\left(t\right) = 0$, $b_1\left(t\right) = t$, $b_2\left(t\right) = 0$. Applying the algorithm described above, we obtain the following solution to problem (18):

$$y\left(t,\varepsilon\right) = t^3\left(4\varepsilon - \frac{8}{5} y^0 \pm \frac{2}{5}\sqrt{100\varepsilon^2 - 80\varepsilon y^0 - 9\left(y^0\right)^2}\right) + y^0.$$

From this it is clear that for sufficiently small $\varepsilon > 0$ and $y^0 \neq 0$ equation (18) has no real solutions and only for $y^0 = 0$ it has two real solutions $y\left(t,\varepsilon\right) = t^3\left(4\varepsilon \pm 4\varepsilon\right)$, uniformly tending to zero as $\varepsilon \to +0$.

Example 5. Now consider the problem

$$\varepsilon \frac{d}{d\,t}\, y\left(t,\varepsilon\right) = 2t \int\limits_0^1 s y^3\left(s\right)\ ds,\ y\left(0,\varepsilon\right) = m. \tag{19}$$

Here, instead of quadratic nonlinearity, we took cubic nonlinearity $f\left(y\right) = y^3$. Using the Maple program algorithm described above, we find that problem (19) has only one real solution $y\left(t\right) = \frac{t^2}{\varepsilon} w + m$, where the constant w has the form

$$w = \left[\frac{1}{3}\left(-10m^3 - 144\varepsilon m + 6\sqrt{3m^6 + 72\varepsilon m^4 + 672\varepsilon^2 m^2 - 384\varepsilon^3}\right)^{1/3} - \right.$$

$$-\frac{3\left(\frac{2}{9}m^2-\frac{8}{3}\varepsilon\right)}{\left(-10m^3-144\varepsilon m+6\sqrt{3m^6+72\varepsilon m^4+672\varepsilon^2m^2-384\varepsilon^3}\right)^{1/3}}-\frac{4}{3}m\Bigg]\cdot\varepsilon.$$

When $\varepsilon\to+0$ the solution $y\left(t,\varepsilon\right)$ has a finite limit

$$\bar{y}\left(t\right)=t^2\left(\frac{1}{3\left(6\sqrt{3}\left|m\right|^3-10m^3\right)^{1/3}}-\frac{2m^2}{3\left(6\sqrt{3}\left|m\right|^3-10m^3\right)^{1/3}}-\frac{4}{3}m\right)+m.$$

For different signs of the initial condition m, the solution tends to different limits.

Remark 3. The results of studies for linear singularly perturbed problems are presented in the works [8–24].

Conclusion

The properties of nonlinear singularly perturbed problems of type (1) differ significantly from the properties of linear problems of type (2); linear problems are always uniquely solvable in the class $C^1\left[0,1\right]$ with continuous initial data, and nonlinear problems may not have real solutions at all or have several of them.

Author Contributions

M.A. Bobodzhanova collected and analyzed data, implemented the program on Maple, B.T. Kalimbetov assisted in collecting and analyzing data, supervised the preparation of the manuscript. V.F. Safonov was the main executor of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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A modified Jacobi elliptic functions method for optical soliton solutions of a conformable nonlinear Schrödinger equation

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In this paper, we study precise and exact traveling wave solutions of the conformable differential nonlinear Schrödinger equation. Then, we transform the given equation into an integer order differential equation by utilizing the wave transformation and the characteristics of the conformable derivative. To extract optical soliton solutions, we divide the wave profile into amplitude and phase components. Further, we introduce a new extension of a modified Jacobi elliptic functions method to the conformable differential nonlinear Schrödinger equation with group velocity dispersion and coefficients of second-order spatiotemporal dispersion.

Keywords: Non-linear Schrödinger equation, Conformable fractional derivative, Modified Jacobi elliptic functions method, Extracting optical solitons-solutions.

2020 Mathematics Subject Classification: 26A33, 34A25, 35R11.

Introduction

Fractional partial and ordinary differential equations (FPDEs and FODEs) are a type of differential equation that involve fractional derivatives. They have been extensively used in many areas of science and engineering, including physics, biology, and finance. One important aspect of fractional calculus is the ability to model complex systems with memory, where the behavior of the system depends on past history [1–8]. The conformable fractional sense is a new approach to fractional calculus that has gained significant attention in recent years. It provides a more accurate representation of non-local effects and has been used to model various physical and biological systems. Conformable PDEs are a type of FPDEs that utilize the conformable fractional derivative, and they have been used to provide a more realistic representation of a wide range of real-world phenomena, such as diffusion and wave propagation. As such, the study of FPDEs and FODEs in the conformable fractional sense is an active and exciting area of research with broad applications [9, 10].

The behavior of conformable PDEs has gained significant attention in recent years due to their wide-ranging applications in various domains, including physics, biology, and engineering. However, understanding the behavior of solutions to conformable PDEs is a complex task, making it challenging to determine accurate answers. To overcome this challenge, several approaches have been suggested to discover analytical solutions for conformable PDEs. These approaches include integral transform methods, numerical methods, and special function techniques, among others. Each of these methods

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has its strengths and weaknesses, making it essential to select the appropriate approach based on the specific characteristics of the conformable PDE being studied. A comprehensive reference list of these approaches can be found in [11], which can aid researchers in selecting the most appropriate approach to solve a particular conformable PDE.

In recent years, finding accurate traveling wave solutions of various nonlinear conformable PDEs has been the primary focus of many researchers. Numerous methods have been proposed to tackle this challenge and provide more general solutions to nonlinear PDEs. These methods include the Kudryashov method [12], the improved tan-expansion method [13], the Sine-Gordon Expansion method [14], the exponential rational function method [15], the sub-equation method [16], the tanh method [16], the auxiliary equation method [17], the Exp-function Method [18], the Jacobi elliptic function expansion method [19], the extended direct algebraic method [20], the first integral method [21] and the improved Bernoulli subequation function method [22]. Among these methods, the modified Jacobi elliptic functions approach [23] is the most crucial method for solitary wave solutions in optics, which has been widely used to give more general solutions to nonlinear PDEs. These approaches have enabled researchers to better understand the behavior of conformable PDEs and their solutions, leading to more accurate predictions and improved modeling of various physical systems.

The focus of this paper is on obtaining an accurate solution to a 1D conformable differential nonlinear Schrödinger equation (CNLSE). To accomplish this, the paper proposes using a second-order nonlinear ODE with a sixth-degree nonlinear component, which extends the elliptic equation. Additionally, the paper aims to develop solutions to the CNLSE using Jacobi's elliptic functions (JEFs). In doing so, optical solitons and other solutions can be observed in the limiting situation of the modulus of ellipticity. The remainder of the paper is dedicated to utilizing a modified auxiliary equation approach to identify all soliton solutions in terms of JEFs and providing soliton solutions with suitable limiting values of the modulus of ellipticity. Ultimately, the results of this paper will contribute to advancing the understanding and application of the CNLSE. However, this paper is structured as follows:

- Section 1 presents the modified auxiliary method (modified Jacobi elliptic functions (MJEFs) method) for obtaining solitary solutions of CNLSE, including a demonstration that some solutions from a previous paper are particular to our model.
- Section 2 discusses the obtained results and their novelty compared to previous methods.
- Section 2 presents the study's results.

1 Mathematical analysis

In this paper, we apply a MJEFs approach to obtain exact wave solutions for CNLSE having group velocity dispersion GVD and second order spatiotemporal dispersion coefficients provided $\overline{\omega} = 1$ and $b_2 = 0$ (see [24]). The governing model is written as follows:

$$\begin{cases} i\frac{\partial q}{\partial x} + i\rho\frac{\partial^{\overline{\omega}} q}{\partial t^{\overline{\omega}}} + +\beta\frac{\partial^{2\overline{\omega}} q}{\partial t^{2\overline{\omega}}} + \gamma\frac{\partial^2 q}{\partial t^2} + b_2|q|^2 q = 0, \\ t > 0, \overline{\omega} > 1, \end{cases} \tag{1}$$

where $q(x,t)$ denotes the macroscopic complex-valued wave profile, x and t are, respectively, the spatial and temporal variables. The numbers β and γ denote the coefficients of the GVD and spatial dispersion, respectively. Whereas, ρ is proportional to the group speed ratio and b_2 is nonzero real-valued constant coefficient which coefficient constitute the nonlinearity component. For extracting optical solitons-solutions, the wave profile is split into amplitude and phase components as

$$q(x,t) = u(\xi)e^{i\psi}, \tag{2}$$

where

$$\xi = x - \nu \frac{t^{\varpi}}{\varpi}, \tag{3}$$

ν being a real constant and $u(\xi)$ the amplitude components of the wave profiles. The phase factor is

$$\psi = -cx + \omega \frac{t^{\varpi}}{\varpi} + \theta_0, \tag{4}$$

where c is the frequency of the solitons and ω is the wave number and θ_0 is the phase constant. We reduce NLPDE (1) into one-dimensional ODE; if we take the necessary of (2) with (3) for (1), we get the following expressions:

$$\begin{cases} q_x = u'e^{i\psi} - icue^{i\psi}, \\ q_{xx} = u''e^{i\psi} - 2icu'e^{i\psi} - c^2ue^{i\psi}, \\ \frac{\partial^{\varpi} q}{\partial t^{\varpi}} = -\nu u'e^{i\psi} + i\omega ue^{i\psi}, \\ \frac{\partial^{2\varpi} q}{\partial t^{2\varpi}} = \nu^2 u''e^{i\psi} - 2i\nu\omega u'e^{i\psi} - \omega^2 ue^{i\psi}, |q|^2 q = u^3 e^{i\psi}. \end{cases} \tag{5}$$

By using (2), (4) and (5) in (1), CNLSE (1) turns into an ODE that we decompose into real and imaginary parts. The imaginary part yields a relation which is constraint between the soliton parameters as

$$\nu = \tfrac{1-2c\gamma}{\rho+2\omega\beta}. \tag{6}$$

The real part of the equation in (6) is

$$(c - \rho\omega - \beta\omega^2 - \gamma c^2)u + (\beta\nu^2 + \gamma)u'' + b_2 u^3 = 0. \tag{7}$$

The balance rule detailed in [25] gives $N = 1$. Solitons emerges from the limiting process are presenting in the next section.

1.1 Solitons-solutions

Applying the modified auxiliary equation method to the CNLSE (1) and using the balance rule of [25] (when $N = 1$),we get to write the solution of (7) as follows:

$$u(\xi) = \textstyle\sum_{i=0}^{N=1} a_i F^i(\xi) = a_0 + a_1 F(\xi), \tag{8}$$

where, a_0, a_1 are arbitrary constants such that $a_1 \neq 0$ and $F(\xi)$ is a Jacobian elliptic function (see [23]), when $F(\xi)$ satisfying the following:

$$(F'(\xi))^2 = A_2 F^2(\xi) + A_4 F^4(\xi) + A_6 F^6(\xi), \tag{9}$$

where A_2, A_4 and A_6 are arbitrary constants determined by Jacobi elliptic functions JEFs method [23]. Substituting (8) and the derivative of (9) in (7), while collecting all terms with the same power and setting them to zero, we have the following of algebraic equations:

$$\begin{cases} 3(\beta\nu^2 + \gamma)A_6 a_1 = 0, \\ 2(\beta\nu^2 + \gamma)A_4 + b_2 a_1^2 = 0, \\ 3b_2 a_0 a_1^2 = 0, \\ (\beta v^2 + \gamma)A_2 + 3b_2 a_0^2 + (c - \rho\omega - \beta\omega^2 - \gamma c^2) = 0, \\ b_2 a_0^3 + (c - \rho\omega - \beta\omega^2 - \gamma c^2)a_0 = 0. \end{cases} \tag{10}$$

Form the first equation of system (10), we take $A_6 = 0$ according to [23], we deduce existence of one modulus ($0 \leq k_1 \leq 1$ and $k_2 = 0$, see [23]).

Solving algebraic equation (10) by using any computer software (Matlab, Maple, Wolfram, Mathematica, ...) yields three cases of solutions and according with [23] as follow:

$$a_0 = 0, a_1^2 = -2(\beta\nu^2 + \gamma)\frac{A_4}{b_2} > 0, c = \frac{1}{2\gamma}(1 + \sqrt{4\beta\gamma\nu^2 A_2 - 4\beta\gamma\omega^2 + 4\beta\gamma A_2 - 4\gamma\omega\rho + 1}) \geq 0, A_6 = 0,$$

with

$$4\beta\gamma\nu^2 A_2 - 4\beta\gamma\omega^2 + 4\beta\gamma A_2 - 4\gamma\omega\rho + 1 \geq 0.$$

Case 1: When $A_2 = -(1 + k_1^2)$ and $A_4 = k_1^2 > 0$ with $(\beta\nu^2 + \gamma)\frac{A_4}{b_2} < 0$, we can acquire the following new complex Jacobi sine function solution for equation (1):

$$q_1(x,t,k_1) = \pm\sqrt{-2(\beta\nu^2 + \gamma)\frac{k_1^2}{b_2}}e^{i\left(\frac{-1}{2\gamma}\left(1\pm\sqrt{-4\beta\gamma\nu^2(1+k_1^2)-4\beta\gamma\omega^2-4\beta\gamma(1+k_1^2)-4\gamma\omega\rho+1}\right)x+\omega\frac{t^{\varpi}}{\varpi}+\theta_0\right)} sn(x-\nu\frac{t^{\varpi}}{\varpi}, k_1). \tag{11}$$

Case 2: When $A_2 = 2k_1^2 - 1$ and $A_4 = -k_1^2 < 0$ with $(\beta\nu^2 + \gamma)\frac{A_4}{b_2} > 0$, we can acquire the following new complex Jacobi cosine function solution for equation (1):

$$q_2(x,t,k_1) = \pm\sqrt{2(\beta\nu^2 + \gamma)\frac{k_1^2}{b_2}}e^{i\left(\frac{-1}{2\gamma}\left(1\pm\sqrt{4\beta\gamma\nu^2(2k_1^2-1)-4\beta\gamma\omega^2+4\gamma(2k_1^2-1)-4\gamma\omega\rho+1}\right)x+\omega\frac{t^{\varpi}}{\varpi}+\theta_0\right)} cn(x-\nu\frac{t^{\varpi}}{\varpi}, k_1). \tag{12}$$

Case 3: When $A_2 = 2 - k_1^2$ and $A_4 = -1 < 0$ with $(\beta\nu^2 + \gamma)\frac{A_4}{b_2} > 0$, we can acquire the following new complex Jacobi function solution of the third kind for equation (1):

$$q_3(x,t,k_1) = \pm\sqrt{\frac{2(\beta\nu^2 + \gamma)}{b_2}}e^{i\left(\frac{-1}{2\gamma}\left(1\pm\sqrt{4\beta\gamma\nu^2(2-k_1^2)-4\beta\gamma\omega^2+4\gamma(2-k_1^2)-4\gamma\omega\rho+1}\right)x+\omega\frac{t^{\varpi}}{\varpi}+\theta_0\right)} dn(x-\nu\frac{t^{\varpi}}{\varpi}, k_1). \tag{13}$$

1.2 Particular cases

When $k_1 \to 0$, the JEFs (11)-(12)-(13) degenerate to the triangular functions, that is,

$$sn\xi \to \sin\xi, cn\xi \to \cos\xi, dn\xi \to 1. \tag{14}$$

When $k_1 \to 0$, the JEFs (11)-(12)-(13) degenerate to the hyperbolic functions, that is,

$$sn\xi \to \tan\xi, cn\xi \to sech\xi, dn\xi \to sech\xi. \tag{15}$$

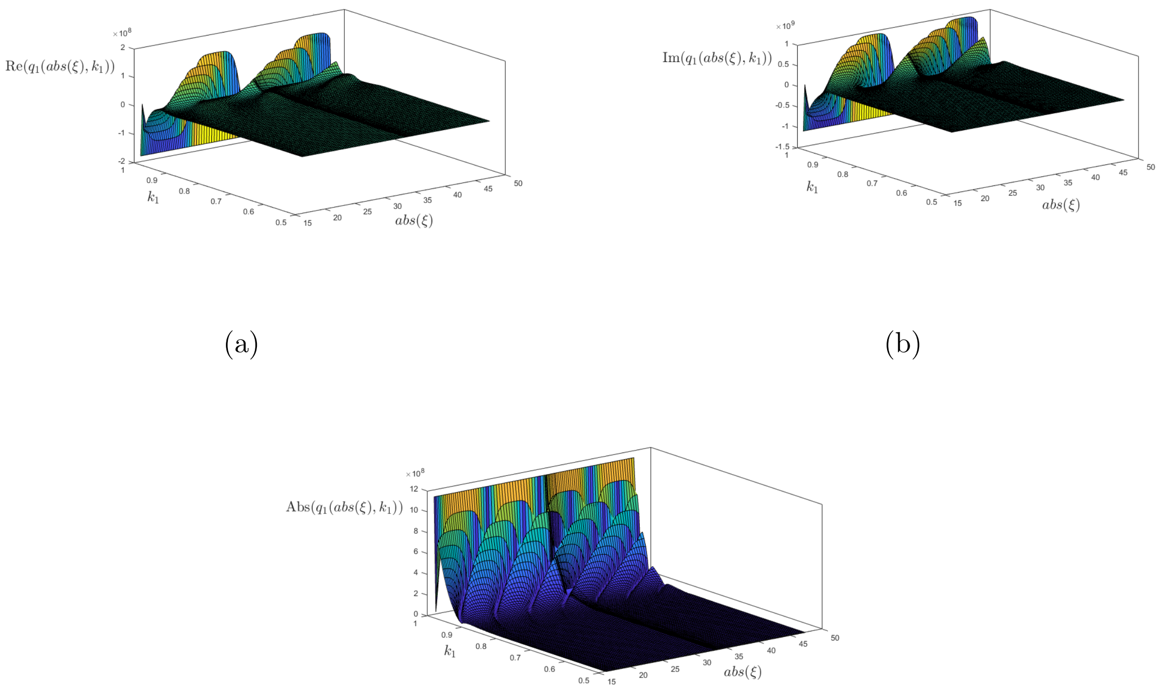
In [11], several specific solutions from (11)-(12)-(13) with (14)-(15) are described.

2 Physical interpretation and discussion

In a specific example of the constants when setting the variables $\gamma = 1$, $\beta = 3$ and $\varpi = 0.5$, it would be extremely helpful if we had real figures that visually illustrated some of the new solutions to equation 1 that were achieved, corresponding to case 1 (Fig. 1), case 2 (Fig. 2) and case 3 (Fig. 3). In this study, key characteristics of the modified JEFs approach were employed to provide a physical explanation for several complex and Jacobi elliptic solutions that were derived for an equation 1. The modified JEFs technique is more broad than the classical methods (such a tanh method, sin-cos method, simplest equation method [26] and the expansion method [18]) because it can discover additional types of analytical solutions that cannot be found by using Bäckland method [11] as an example. In order to acquire additional analytical answers, a better knowledge of engineering and physical challenges, and new physical predictions, the process described in [27] will help.

In section (1.1), we demonstrate that the JEFs solutions to (1) only have one k_1 modulus $0 \leq k_1 \leq 1$ according to [23]. As far as we are aware, this is the first place in the literature where the new solutions (11)-(12)-(13) of (1) may have been found.

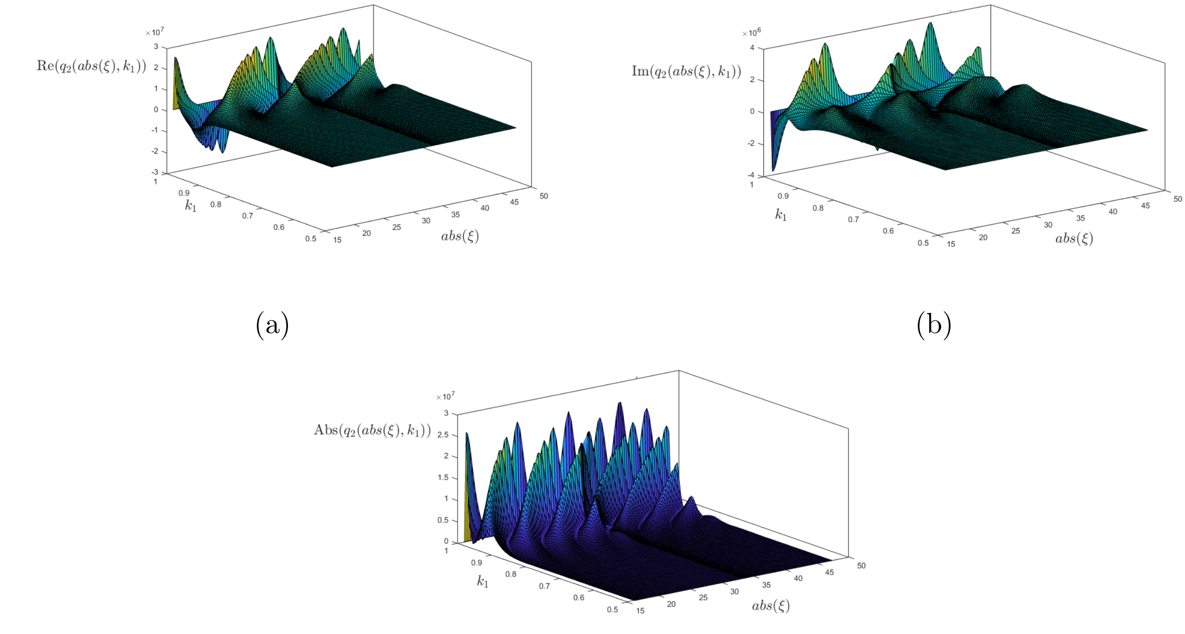
In regards to figures, surfaces have been plotted by taking into account the appropriate values for the parameters. When we verify every analytical solution produced in this study using the modified JEFs approach, we find that Figures 1, 2, and 3 have three-dimensional surfaces. As a result, it may be claimed that they are physically plausible because nearly every figure demonstrates similar wave behaviors given the appropriate parameter values.



(a) (b)

(c)

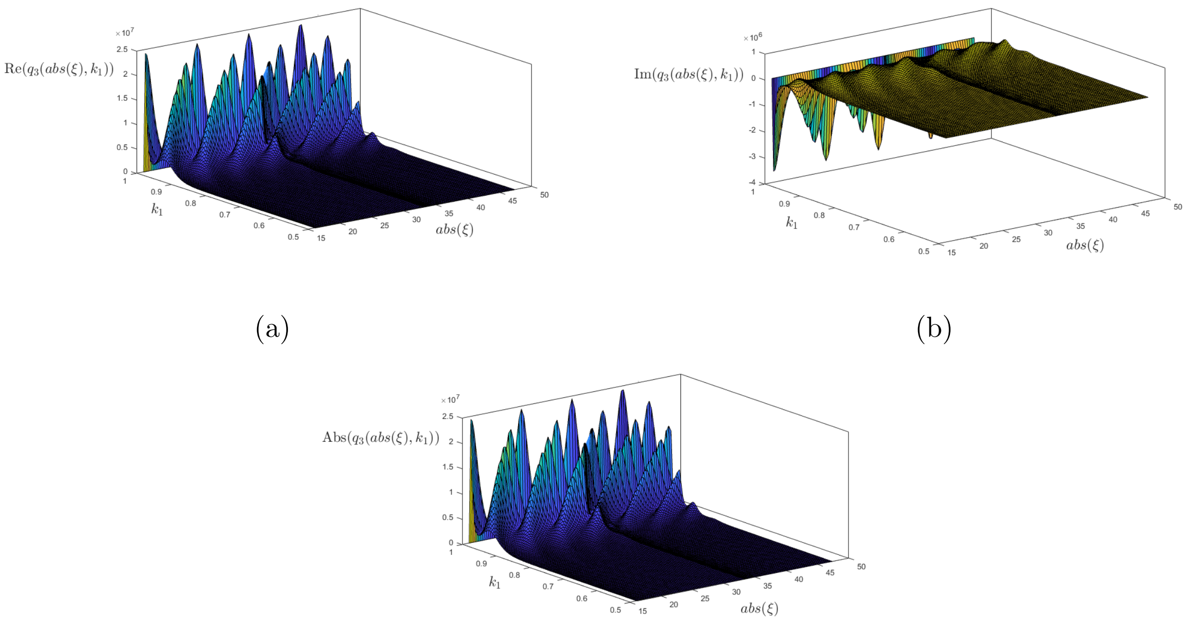
Figure 1. (a) Real, (b) imaginary and (c) absolute plots in 3D sketches of equation (11), respectively, when $v = 2$, $b_2 = -2$, $\omega = 5$, $\theta_0 = 2$ and $\rho = 2$



(a) (b)

(c)

Figure 2. (a) Real, (b) imaginary and (c) absolute plots in 3D sketches of equation (12), respectively, when $v = 2$, $b_2 = -2$, $\omega = 5$, $\theta_0 = 2$ and $\rho = 2$



(a) (b)

(c)

Figure 3. (a) Real, (b) imaginary and (c) absolute plots in 3D sketches of equation (13), respectively, when $v = 2$, $b_2 = -2$, $\omega = 5$, $\theta_0 = 2$ and $\rho = 2$

Conclusion

The current work employs the MJEFs approach to address the CNLSE, incorporating fractional derivatives featuring second-order spatiotemporal and GVD coefficients via wave transformation and conformable derivatives. A diverse array of optical solitons-solutions are constructed for the governing equation. Figures 1, 2, and 3 present a viewpoint of the resulting solitons solutions with respect to distinct parameters. Our novel MJEFs technique generates a new set of solutions (with one modulus) that are exclusively presented in this work. These unrestricted parameter solutions hold significant importance in elucidating various physical interpretations. The outcomes demonstrate the capability of our approach to be used in a variety of CPDEs and offer numerous precise solutions for CPDEs.

Author Contributions

Aicha Boussaha collected and analyzed data, and led manuscript preparation. B. Semmar and N. Djeddi assisted in data collection and analysis. M. Al-Smadi and S. Al-Omari served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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Fixed point results in C^*-algebra valued fuzzy metric space with applications to boundary value problem and control theory

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In this paper, we derive some new fixed point results in C^*-algebra valued fuzzy metric space with the help of subadditive altering distance function with respect to a t-norm. Our results generalize some existing fixed point results in the literature. A common fixed point result is also derived for a pair of mappings on complete C^*-algebra valued fuzzy metric space. The results are supported by suitable examples along with the graphical demonstration of the used conditions. As application, we establish the solvability of a second order boundary value problem. Moreover, the results are also applied in control theory to study the possibility of optimally controlling the solution of an ordinary differential equation in dynamic programming.

Keywords: C^*-algebra valued metric space, fuzzy metric space, fixed point, boundary value problem, control theory.

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Introduction

The concept of fuzzy metric was introduced by Kramosil and Michalek [1] in 1975 and the study of fixed points in fuzzy metric space was given by Grabiec [2] in 1988. Fixed point theory has emerging applications in various domains including applied analysis, physics, mechanics, medical science etc. During recent years, several researchers ([3–10]) have done the study of fixed point theory by introducing different types of mappings as well as considering different spaces along with various applications.

In 1984, Khan et al. [11] introduced the concept of altering distance function between two points and again in 2011, Shen et al. [12] defined the same by introducing a new condition and derived many fixed point results in fuzzy metric space. After that Roldán-López-de-Hierro et al. [3] established some results on common fixed point theorems for weakly compatible mappings in fuzzy metric spaces with new contractive conditions. In 2018, Shoaib et al. [13] derived some fixed point results in dislocated complete b-metric space and gave some examples as well as applications relating the results to common fixed points for multivalued mappings. Using the altering distance function, Patir et al. [5,6,8] derived some fixed point results using different types of mappings and gave examples as well as applications to boundary value problem and integral equations.

The concept of C^*-algebra valued metric space was given by Ma et al. [14] by replacing the set of non negative real numbers with a (unital) C^*-algebra. In 2020, Madadi et al. [15] introduced the concept of C^*-algebra valued fuzzy metric space and derived some topological properties of the same. After that in 2021, Khaofong et al. [16] gave a new definition of C^*-algebra valued fuzzy metric space by replacing $[0,1]$ by $[0_{\mathbb{A}}, 1_{\mathbb{A}}]$, where $0_{\mathbb{A}}$ and $1_{\mathbb{A}}$ are the zero element and the unit element of an algebra $\mathbb{A}$ respectively in the sense of George and Veeramani [17], and established some results by introducing C^*-algebra valued contraction mapping with application to integral equations.

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Motivated by these, in this paper, we establish some fixed point results for complete C^*-algebra valued fuzzy metric space using subadditive altering distance function with respect to some t-norm. We also derive a common fixed point result for a pair of mappings on complete C^*-algebra valued fuzzy metric space. Some of our results extend the works of Shoaib et al. [13] and Patir et al. [5, 6, 8, 18] in the setting of C^*-algebra valued fuzzy metric space. In the third section we give an application of our established result to second order boundary value problem. In view of the vast application of control theory in present times in different technological fields viz., spacecraft control, robot technology, smart fluid technology, etc., the section four of our paper is devoted to the study of control theory via our derived result. Here we apply our results to study the possibility of optimally controlling the solution of an ordinary differential equation in dynamic programming.

1 Preliminaries

Throughout the paper, $\mathbb{A}$ denotes a unital C^*-algebra with unity $1_{\mathbb{A}}$. A complex algebra $\mathbb{A}$ is called a complex *-algebra if there is an involution $* : \mathbb{A} \to \mathbb{A}$ defined on it by $u \to u^*$, where u^* is called the adjoint of u and having the properties that for all $u, v \in \mathbb{A}$, $(\lambda u + v)^* = \bar{\lambda} u^* + v^*$, $(uv)^* = v^* u^*$ and $(u^*)^* = u$, where $\bar{\lambda}$ denotes the conjugate of $\lambda \in \mathbb{C}$. A complete unital *-algebra is called a Banach *-algebra with the norm satisfying $||u^*|| = ||u||$ for all $u \in \mathbb{A}$. Moreover, a Banach *-algebra is a C^*-algebra if $||u^* u|| = ||u||^2$ for all $u \in \mathbb{A}$.

An element $\xi \in \mathbb{A}$ is called a positive element of $\mathbb{A}$ and denoted by $0_{\mathbb{A}} \preccurlyeq \xi$ ($0_{\mathbb{A}}$ being the zero element of $\mathbb{A}$) if $\xi \in \mathbb{A}_h$ and $\sigma(\xi) \subset [0, \infty)$, where $\mathbb{A}_h = \{\xi \in \mathbb{A} : \xi^* = \xi\}$ and $\sigma(\xi)$ is the spectrum of ξ. A partial ordering on $\mathbb{A}$ is defined by $\xi \preccurlyeq \eta$ (or, $\eta \succcurlyeq \xi$) if and only if $0_{\mathbb{A}} \preccurlyeq \eta - \xi$ (or, $\eta - \xi \succcurlyeq 0_{\mathbb{A}}$). When $\xi - \eta$ is positive and non-zero, we call $\xi - \eta$ as strictly positive and denote it by $\xi - \eta \succ 0_{\mathbb{A}}$ (or, $\xi \succ \eta$). The set $\{\xi \in \mathbb{A} : 0_{\mathbb{A}} \preccurlyeq \xi\}$ is denoted by $\mathbb{A}^+$ and we denote $(\xi^* \xi)^{\frac{1}{2}}$ as $|\xi|$ and for invertible η, $\xi \eta^{-1}$ as $\frac{\xi}{\eta}$. Let $\mathbb{A}'$ be the set $\{\xi \in \mathbb{A}^+ : \xi\eta = \eta\xi$ for all $\eta \in \mathbb{A}\}$. Moreover, $[0_{\mathbb{A}}, 1_{\mathbb{A}}]$ denotes the set $\{\xi \in \mathbb{A} : 0_{\mathbb{A}} \preccurlyeq \xi \preccurlyeq 1_{\mathbb{A}}\}$.

Definition 1. [14] Let X be a nonempty set and $\mathbb{A}$ be a C^*-algebra. Suppose that a mapping $d : X \times X \to \mathbb{A}^+$ satisfies:

(i) $d(\xi, \eta) = 0_{\mathbb{A}}$ if and only if $\xi = \eta$ for all $\xi, \eta \in X$,

(ii) $d(\xi, \eta) = d(\eta, \xi)$ for all $\xi, \eta \in X$,

(iii) $d(\xi, \eta) \preccurlyeq d(\xi, \zeta) + d(\zeta, \eta)$ for all $\xi, \eta, \zeta \in X$.

Then d is called a C^*-algebra valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

Example 1. [19] Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{M}_2(\mathbb{R})$, the set of all bounded linear operators on the Hilbert space $\mathbb{R}^2$. Define $d : X \times X \to \mathbb{A}^+$ by

$$d(\xi, \eta) = \begin{bmatrix} |\xi - \eta| & 0 \\ 0 & 2|\xi - \eta| \end{bmatrix},$$

where $\xi, \eta \in X$. Then, $(X, \mathbb{A}, d)$ is a C^*-algebra valued metric space.

Lemma 1. [20, 21] Suppose that $\mathbb{A}$ is a unital C^*-algebra with unit element $1_{\mathbb{A}}$.

(i) For any $\xi \in \mathbb{A}^+$, $\xi \preccurlyeq 1_{\mathbb{A}}$ if and only if $||\xi|| \leq 1$.

(ii) If $u \in \mathbb{A}^+$ with $||u|| \leq \frac{1}{2}$, then $1_{\mathbb{A}} - u$ is invertible and $||u(1_{\mathbb{A}} - u)^{-1}|| < 1$.

(iii) Suppose that $u, v \in \mathbb{A}$ with $0_{\mathbb{A}} \preccurlyeq u, v$ and $uv = vu$ then $0_{\mathbb{A}} \preccurlyeq uv$.

(iv) Suppose that $\tilde{\mathbb{A}} = \{u \in \mathbb{A} : uv = vu$ for all $v \in \mathbb{A}\}$. Let $u \in \tilde{\mathbb{A}}$, if $v, w \in \tilde{\mathbb{A}}$ with $0_{\mathbb{A}} \preccurlyeq w \preccurlyeq v$ and $1_{\mathbb{A}} - u$ is a positive element in $\tilde{\mathbb{A}}$ then $(1_{\mathbb{A}} - u)^{-1} w \preccurlyeq (1_{\mathbb{A}} - u)^{-1} v$.

Lemma 2. [22, 23] Let $\mathbb{A}$ be a C^*-algebra with unit element $1_{\mathbb{A}}$ and let $u, v \in \mathbb{A}$.

(i) If u is self-adjoint, then $u \preccurlyeq ||u|| 1_{\mathbb{A}}$.

(ii) If $0_{\mathbb{A}} \preccurlyeq u \preccurlyeq v$, then $||u|| \leq ||v||$.
(iii) If $u \in \mathbb{A}$, then $1_{\mathbb{A}} + uu^*$ is invertible in $\mathbb{A}$.
(iv) If $u \in \mathbb{A}^+$, then $u = \xi^*\xi$ for some $\xi \in \mathbb{A}$.

Madadi et al. [15] defined the triangular norm or t-norm as follows:

Definition 2. Let $\mathbb{A}$ be a C^*-algebra with unit element $1_{\mathbb{A}}$. A mapping $\tau : \mathbb{A}^+ \times \mathbb{A}^+ \to \mathbb{A}^+$ is called a t-norm if
(i) $\tau(a, 1_{\mathbb{A}}) = a$ for all $a \in \mathbb{A}^+$,
(ii) $\tau(a, b) = \tau(b, a)$ for all $a, b \in \mathbb{A}^+$,
(iii) $a \preccurlyeq a'$, $b \preccurlyeq b' \implies \tau(a, b) \preccurlyeq \tau(a', b')$ for all $a, b, c, d \in \mathbb{A}^+$,
(iv) $\tau(a, \tau(b, c)) = \tau(\tau(a, b), c)$ for all $a, b, c \in \mathbb{A}^+$.

Definition 3. [16] Let $\mathbb{A}$ be a C^*-algebra with unit element $1_{\mathbb{A}}$. For an arbitrary set X, let τ be a continuous t-norm on $\mathbb{A}^+$ and $M_{\mathbb{A}}$ be a fuzzy set from $X \times X \times (0, \infty) \to [0_{\mathbb{A}}, 1_{\mathbb{A}}]$. Then $(X, M_{\mathbb{A}}, \tau)$ is called a C^*-algebra valued fuzzy metric space, if it satisfies the following conditions, for each $\xi, \eta, \rho \in X$ and $t, s > 0$,
(i) $M_{\mathbb{A}}(\xi, \eta, t) \succ 0_{\mathbb{A}}$,
(ii) $M_{\mathbb{A}}(\xi, \eta, t) = 1_{\mathbb{A}}$ if and only if $\xi = \eta$ for all $t > 0$,
(iii) $M_{\mathbb{A}}(\xi, \eta, t) = M_{\mathbb{A}}(\eta, \xi, t)$,
(iv) $\tau(M_{\mathbb{A}}(\xi, \eta, s), M_{\mathbb{A}}(\eta, \rho, t)) \preccurlyeq M_{\mathbb{A}}(\xi, \rho, s + t)$,
(v) $M_{\mathbb{A}}(\xi, \eta) : (0, \infty) \to [0_{\mathbb{A}}, 1_{\mathbb{A}}]$ is continuous.

As in [12], we define the altering distance function in C^*-algebra valued fuzzy metric space as follows.

Definition 4. Let $(X, M_{\mathbb{A}}, \tau)$ be a C^*-algebra fuzzy metric space with unit element $1_{\mathbb{A}}$. Let $\phi : \mathbb{A}^+ \to \mathbb{A}^+$ be a mapping. Then ϕ is called an altering distance function if
(i) ϕ is strictly decreasing and left continuous,
(ii) $\phi(k) = 0_{\mathbb{A}}$ if and only if $k = 1_{\mathbb{A}}$, i.e., $\lim_{k \to 1_{\mathbb{A}}^-} \phi(k) = 0_{\mathbb{A}}$.

Using the subadditivity condition with respect to a t-norm τ, we give the following definition of subadditive altering distance function with respect to τ.

Definition 5. Let $(X, M_{\mathbb{A}}, \tau)$ be a C^*-algebra valued fuzzy metric space. An altering distance function ϕ is said to be subadditive with respect to the t-norm τ if $\phi(\tau(a, b)) \preccurlyeq \phi(a) + \phi(b)$, $a, b \in \{M_{\mathbb{A}}(\xi, \eta, t) : \xi, \eta \in X, t > 0\}$.

In the same line as Grabiec [2; Lemma 4], we can prove the following lemma in the setting of C*-algebra valued fuzzy metric space.

Lemma 3. Let $(X, M_{\mathbb{A}}, \tau)$ is a C^*-algebra valued fuzzy metric space. Then $M_{\mathbb{A}}(\xi, \eta, t) \preccurlyeq M_{\mathbb{A}}(\xi, \eta, kt)$, where $k \in \mathbb{N}$, $\xi, \eta \in X$ and $t > 0$.

Proof. Let $t, s > 0$ with $t < s$. Suppose that for all $\xi, \eta \in X$, $M_{\mathbb{A}}(\xi, \eta, t) \succ M_{\mathbb{A}}(\xi, \eta, s)$. Now, by condition (iv) of Definition 3,

$$\begin{aligned} \tau(M_{\mathbb{A}}(\xi, \eta, t), M_{\mathbb{A}}(\eta, \eta, s - t)) &\preccurlyeq M_{\mathbb{A}}(\xi, \eta, s) \\ &\prec M_{\mathbb{A}}(\xi, \eta, t), \\ \tau(M_{\mathbb{A}}(\xi, \eta, t), 1_{\mathbb{A}}) &\prec M_{\mathbb{A}}(\xi, \eta, t), \\ M_{\mathbb{A}}(\xi, \eta, t) &\prec M_{\mathbb{A}}(\xi, \eta, t), \end{aligned}$$

which is a contradiction. So, $M_{\mathbb{A}}(\xi, \eta, t) \preccurlyeq M_{\mathbb{A}}(\xi, \eta, s)$ when $t < s$.

Thus, $M_{\mathbb{A}}(\xi, \eta, t)$ is non-decreasing with respect to t for all $\xi, \eta \in X$ and hence the lemma easily follows.

Following the definition of Cauchy sequence in fuzzy metric space by George and Veeramani [17], the Cauchy sequence in C^*-algebra valued fuzzy metric space can be defined in a similar way.

Definition 6. Let $(X, M_{\mathbb{A}}, \tau)$ be a C^*-algebra valued fuzzy metric space. A sequence $\{\xi_n\}$ in X is said to be a Cauchy sequence if for all $\epsilon_{\mathbb{A}} \in (0_{\mathbb{A}}, 1_{\mathbb{A}})$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$, $M_{\mathbb{A}}(\xi_m, \xi_n, t) \succcurlyeq 1_{\mathbb{A}} - \epsilon_{\mathbb{A}}$ or equivalently, $\lim_{m,n\to\infty} M_{\mathbb{A}}(\xi_m, \xi_n, t) = 1_{\mathbb{A}}$.

The sequence $\{\xi_n\}$ is said to be convergent to ξ, if $\lim_{n\to\infty} M_{\mathbb{A}}(\xi_n, \xi, t) = 1_{\mathbb{A}}$. If every Cauchy sequence in $(X, M_{\mathbb{A}}, \tau)$ is convergent, then $(X, M_{\mathbb{A}}, \tau)$ is called a complete C^*-algebra valued fuzzy metric space.

2 Main Results

In this section, we derive some fixed point results considering a subadditive altering distance function with respect to a t-norm.

Theorem 1. Let $(X, M_{\mathbf{A}}, \tau)$ be a complete C^*-algebra valued fuzzy metric space. Let ϕ be a subadditive altering distance function with respect to the t-norm τ and let T be a self mapping on X such that

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi, T\eta, t)) &\preccurlyeq a_1^*\phi(M_{\mathbb{A}}(\xi, T\xi, t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\eta, T\eta, t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi, T\eta, 2t))a_3 \\ &+ a_4^*\phi(M_{\mathbb{A}}(\eta, T\xi, t))a_4 + a_5^*\phi(M_{\mathbb{A}}(\xi, \eta, t))a_5 \\ &+ a_6^*(\phi(M_{\mathbb{A}}(\xi, T\xi, t)) + \phi(M_{\mathbb{A}}(\eta, T\eta, t)))\frac{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi, T\xi, t)))}{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi, \eta, t)))}a_6, \qquad (1)\end{aligned}$$

where $a_i \in \mathbb{A}'$ for $i = 1, ..., 6$ with $\sum_{i=1}^{6} ||a_i||^2 + ||a_3||^2 + ||a_6||^2 < 1$. Then T has a unique fixed point in X.

Proof. For $\xi_0 \in X$, we consider the Picard sequence $\xi_{n+1} = T\xi_n$, $n \in \mathbb{N} \cup \{0\}$. Now,

$$\begin{aligned}&\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \\ &\preccurlyeq a_1^*\phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_{j-1}, t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\xi_j, T\xi_j, t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_j, 2t))a_3 \\ &+ a_4^*\phi(M_{\mathbb{A}}(\xi_j, T\xi_{j-1}, t))a_4 + a_5^*\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t))a_5 \\ &+ a_6^*(\phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_{j-1}, t)) + \phi(M_{\mathbb{A}}(\xi_j, T\xi_j, t)))\frac{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_{j-1}, t)))}{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)))}a_6 \\ &\preccurlyeq ||a_1||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t))) + ||a_2||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) + ||a_3||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_{j+1}, 2t)) \\ &+ ||a_5||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_6||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_6||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)). \qquad (2)\end{aligned}$$

Using the property of t-norm and the altering distance function ϕ, we have

$$\begin{aligned}M_{\mathbb{A}}(\xi_{j-1}, \xi_{j+1}, 2t) &\succcurlyeq \tau(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t), M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)), \\ \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_{j+1}, 2t)) &\preccurlyeq \phi(\tau(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t), M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))) \\ &\preccurlyeq \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)).\end{aligned}$$

So, from (2), we get

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) &\preccurlyeq ||a_1||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_2||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) + ||a_3||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) \\ &+ ||a_3||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) + ||a_5||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_6||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) \\ &+ ||a_6||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)).\end{aligned}$$

Then the above equation becomes

$$\begin{aligned}(1-||a_2||^2-||a_3||^2-||a_6||^2)1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) &\preccurlyeq (||a_1||^2+||a_3||^2+||a_5||^2+||a_6||^2)1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)),\\ \phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) &\preccurlyeq \frac{(||a_1||^2+||a_3||^2+||a_5||^2+||a_6||^2)}{(1-(||a_2||^2+||a_3||^2+||a_6||^2))}1_{\mathbb{A}}\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)),\\ \phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) &\preccurlyeq \gamma\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t))\\ &\preccurlyeq \gamma^j\phi(M_{\mathbb{A}}(\xi_0,\xi_1,t)), \end{aligned} \tag{3}$$

where $\gamma=\frac{(||a_1||^2+||a_3||^2+||a_5||^2+||a_6||^2)}{(1-(||a_2||^2+||a_3||^2+||a_6||^2))}1_{\mathbb{A}}$. Taking norm on both sides of the equation (3), we get

$$||\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))|| \preccurlyeq ||\gamma||^j||\phi(M_{\mathbb{A}}(\xi_0,\xi_1,t))||.$$

Taking the limit as $j\to\infty$, since $||\gamma||<1$, from the above equation, we get

$$\begin{aligned}\lim_{j\to\infty}\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) &= 0_{\mathbb{A}},\\ \lim_{j\to\infty}M_{\mathbb{A}}(\xi_j,\xi_{j+1},t) &= 1_{\mathbb{A}}.\end{aligned} \tag{4}$$

Next we show that $\{\xi_j\}$ is a Cauchy sequence. If not, then there exists $0_{\mathbb{A}}\succ\epsilon_{\mathbb{A}}\succ 1_{\mathbb{A}}$, for which we can find two subsequence $\{\xi_{r(j)}\}$ and $\{\xi_{s(j)}\}$ of $\{\xi_j\}$ with $r(j)>s(j)>j$, $j\in\mathbb{N}\cup\{0\}$ such that

$$M_{\mathbb{A}}(\xi_{r(j)},\xi_{s(j)},t)\preccurlyeq 1_{\mathbb{A}}-\epsilon_{\mathbb{A}}. \tag{5}$$

Now, without loss of generality, we can choose $r(j)$ as the smallest positive integer satisfying $r(j)>s(j)$ in (5). Then,

$$M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)},t)\succ 1_{\mathbb{A}}-\epsilon_A. \tag{6}$$

Now,

$$\begin{aligned}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t) &\succcurlyeq \tau(M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)},\frac{t}{2}),M_{\mathbb{A}}(\xi_{s(j)},\xi_{s(j)-1},\frac{t}{2})),\ j\in\mathbb{N}\\ &\succcurlyeq \tau(1_{\mathbb{A}}-\epsilon_{\mathbb{A}},M_{\mathbb{A}}(\xi_{s(j)},\xi_{s(j)-1},\frac{t}{2}))\ (by\ (6)),\\ \lim_{j\to\infty}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t) &\succcurlyeq \tau(1_{\mathbb{A}}-\epsilon_A,1_{\mathbb{A}})=1_{\mathbb{A}}-\epsilon_{\mathbb{A}}\ (by\ (4)),\\ \lim_{j\to\infty}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t) &\succcurlyeq 1_{\mathbb{A}}-\epsilon_{\mathbb{A}}.\end{aligned} \tag{7}$$

Again, from (5),

$$\begin{aligned}1_{\mathbb{A}}-\epsilon_{\mathbb{A}} &\succcurlyeq M_{\mathbb{A}}(\xi_{r(j)},\xi_{s(j)},4t)\\ &\succcurlyeq \tau(M_{\mathbb{A}}(\xi_{r(j)},\xi_{r(j)-1},2t),\tau(M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t),M_{\mathbb{A}}(\xi_{s(j)},\xi_{s(j)-1},t)))\\ &\succcurlyeq \tau(1_{\mathbb{A}},\tau(\lim_{j\to\infty}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t),1_{\mathbb{A}}))\ (by\ (4)),\\ 1_{\mathbb{A}}-\epsilon_{\mathbb{A}} &\succcurlyeq \lim_{j\to\infty}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t).\end{aligned} \tag{8}$$

Hence, from (7) and (8), we get

$$\lim_{j\to\infty}M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t)=1_{\mathbb{A}}-\epsilon_{\mathbb{A}}.$$

By (5),

$$
\begin{aligned}
\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) &\preccurlyeq \phi(M_{\mathbb{A}}(\xi_{r(j)},\xi_{s(j)},t))\\
&\preccurlyeq a_1^*\phi(M_{\mathbb{A}}(\xi_{r(j)-1},T\xi_{r(j)-1},t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\xi_{s(j)-1},T\xi_{s(j)-1},t))a_2\\
&+ a_3^*\phi(M_{\mathbb{A}}(\xi_{r(j)-1},T\xi_{s(j)-1},2t))a_3 + a_4^*\phi(M_{\mathbb{A}}(T\xi_{s(j)-1},T\xi_{r(j)-1},t))a_4\\
&+ a_5^*\phi(M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t))a_5 + a_6^*(\phi(M_{\mathbb{A}}(\xi_{r(j)-1},T\xi_{r(j)-1},t))\\
&+ \phi(M_{\mathbb{A}}(\xi_{s(j)-1},T\xi_{s(j)-1},t)))\frac{(1_{\mathbb{A}}+\phi(M_{\mathbb{A}}(\xi_{r(j)-1},T\xi_{r(j)-1},t)))}{(1_{\mathbb{A}}+\phi(M_{\mathbb{A}}(\xi_{r(j)-1},\xi_{s(j)-1},t)))}a_6.
\end{aligned}
$$

By taking the limit as $j\to\infty$ the above expression becomes

$$
\begin{aligned}
\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) &\preccurlyeq ||a_1||^2 1_{\mathbb{A}}\phi(1_{\mathbb{A}}) + ||a_2||^2 1_{\mathbb{A}}\phi(1_{\mathbb{A}}) + ||a_3||^2 1_{\mathbb{A}}\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) + ||a_4||^2 1_{\mathbb{A}}\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}})\\
&\quad + ||a_5||^2 1_{\mathbb{A}}\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) + ||a_6||^2(\phi(1_{\mathbb{A}})+\phi(1_{\mathbb{A}}))\frac{(1_{\mathbb{A}}+\phi(1_{\mathbb{A}}))}{(1_{\mathbb{A}}+\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}))},\\
\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) &\preccurlyeq ||a_3||^2\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) + ||a_4||^2\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) + ||a_5||^2\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}),\\
\phi(1_{\mathbb{A}}-\epsilon_{\mathbb{A}}) &\preccurlyeq 0_{\mathbb{A}} \implies 1_{\mathbb{A}}-\epsilon_{\mathbb{A}} = 1_{\mathbb{A}} \implies \epsilon_{\mathbb{A}} = 0_{\mathbb{A}},
\end{aligned}
$$

which is a contradiction. Therefore, $\{\xi_j\}$ is a Cauchy sequence. Then there exists a point z in X such that $\xi_n\to z$. Now,

$$
\begin{aligned}
\phi(M_{\mathbb{A}}(T\xi_n,Tz,t)) &\preccurlyeq a_1^*\phi(M_{\mathbb{A}}(\xi_n,T\xi_n,t))a_1 + a_2^*\phi(M_{\mathbb{A}}(z,Tz,t))a_2\\
&\quad + a_3^*\phi(M_{\mathbb{A}}(\xi_n,Tz,2t))a_3 + a_4^*\phi(M_{\mathbb{A}}(z,T\xi_n,t))a_4 + a_5^*\phi(M_{\mathbb{A}}(\xi_n,z,t))a_5\\
&\quad + a_6^*(\phi(M_{\mathbb{A}}(\xi_n,T\xi_n,t)) + \phi(M_{\mathbb{A}}(z,Tz,t)))\frac{(1_{\mathbb{A}}+\phi(M_{\mathbb{A}}(\xi_n,T\xi_n,t)))}{(1_{\mathbb{A}}+\phi(M_{\mathbb{A}}(\xi_n,z,t)))}a_6.
\end{aligned}
$$

Taking the limit as $n\to\infty$ and by Lemma 3, from the above equation, we get

$$
\begin{aligned}
&\phi(M_{\mathbb{A}}(z,Tz,t)) \preccurlyeq ||a_2||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(z,Tz,t)) + ||a_3||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(z,Tz,t)) + ||a_6||^2 1_{\mathbb{A}}\phi(M_{\mathbb{A}}(z,Tz,t)),\\
&(1-||a_2||^2-||a_3||^2-||a_6||^2)1_{\mathbb{A}}\phi(M_{\mathbb{A}}(z,Tz,t)) \preccurlyeq 0_{\mathbb{A}}.
\end{aligned}
$$

Since the left hand side of the above expression is positive and $a_2,a_3,a_6\in\mathbb{A}'$, using Lemma 1, we get

$$
\phi(M_{\mathbb{A}}(z,Tz,t)) = 0_{\mathbb{A}} \implies M_{\mathbb{A}}(z,Tz,t) = 1_{\mathbb{A}} \implies z = Tz.
$$

Uniqueness of the fixed point can be proved easily by (1).

Remark 1. The above theorem generalizes the results given by [13] and [5] if we consider C^*-algebra valued fuzzy metric space in place of b-metric space and fuzzy metric space respectively.

We present the following example to demonstrate the above theorem.

Example 2. Let $X=\mathbb{A}=[0,1]$ and $d(\xi,\eta)=|\xi-\eta|$ for all $\xi,\eta\in X$. Let $M_{\mathbb{A}}$ be a fuzzy set from $X^2\times(0,\infty)$ to $[0,1]$ such that $M_{\mathbb{A}}(\xi,\eta,t)=\frac{1}{1+d(\xi,\eta)}$. Then $(X,M_{\mathbb{A}},\tau)$ is a complete C^*-algebra valued fuzzy metric space with respect to the t-norm, $\tau(a,b)=\min\{a,b\}$, $a,b\in[0,1]$. Let $T:X\to X$ be defined by $T(\xi)=\frac{\xi}{7}$ for all $\xi\in X$ and $\phi(\lambda)=1-\lambda$, $\lambda\in[0,1]$. Let $a_i=\frac{1}{3}$ for $i=1,...,6$.
Now,

$$
\phi(M_{\mathbb{A}}(T\xi,T\eta,t)) = \phi\left(\frac{1}{1+|T\xi-T\eta|}\right) = 1-\frac{1}{1+|\frac{\xi}{7}-\frac{\eta}{7}|}. \tag{9}
$$

Again,

$$a_1^*\phi(M_{\mathbb{A}}(\xi,T\xi,t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\eta,T\eta,t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi,T\eta,2t))a_3 + a_4^*\phi(M_{\mathbb{A}}(\eta,T\xi,t))a_4$$
$$+ a_5^*\phi(M_{\mathbb{A}}(\xi,\eta,t))a_5 + a_6^*(\phi(M_{\mathbb{A}}(\xi,T\xi,t)) + \phi(M_{\mathbb{A}}(\eta,T\eta,t)))\frac{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi,T\xi,t)))}{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi,\eta,t)))}a_6$$
$$= \frac{1}{9}\left[5 - \left\{\frac{1}{1+|\xi-\frac{\xi}{7}|} + \frac{1}{1+|\eta-\frac{\eta}{7}|} + \frac{1}{1+|\xi-\frac{\eta}{7}|} + \frac{1}{1+|\eta-\frac{\xi}{7}|} + \frac{1}{1+|\xi-\eta|}\right\}\right]$$
$$+ \frac{1}{9}\left(2 - \frac{1}{1+|\xi-T\xi|} - \frac{1}{1+|\eta-T\eta|}\right)\frac{\left(2-\frac{1}{1+|\xi-T\xi|}\right)}{\left(2-\frac{1}{1+|\xi-\eta|}\right)}. \tag{10}$$

We represent the equations (9) and (10) in the following figure.

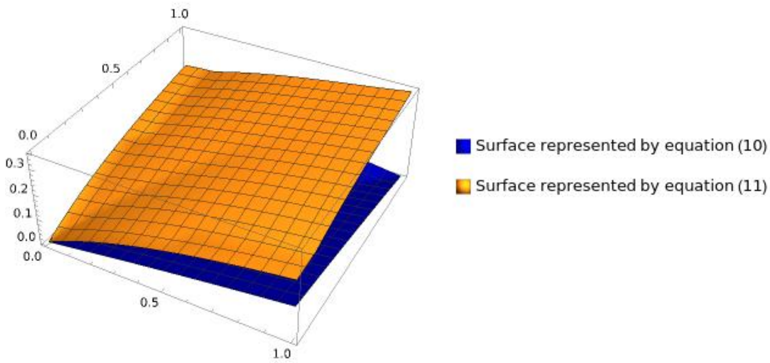

Figure 1. Demonstration of the condition of Theorem 1 by mapping T

In Figure 1, the yellow surface represents the equation (10) and the blue surface represents the equation (9), where the values of ξ, η are between 0 and 1. Clearly, for all values of ξ, η, the value of (10) is greater than the value of (9). Hence, the condition of Theorem 1 is satisfied. Clearly, 0 is the fixed point of T here.

Example 3. Let $X = \{(1,1),(2,1),(2,7)\} \subseteq \mathbb{R}^2$ and $\mathbb{A}$, $M_{\mathbb{A}}$, τ and ϕ be as in Example 2. Let $T : X \to X$ be defined by $T(1,1) = T(2,1) = (1,1)$ and $T(2,7) = (2,1)$ and $a_1 = a_2 = \sqrt{\frac{15}{100}}$, $a_4 = a_5 = \sqrt{\frac{22}{100}}$ and $a_3 = a_6 = 0$. Then for $\xi = (2,1)$ and $\eta = (2,7)$,

$$\phi(M_{\mathbb{A}}(T\xi,T\eta,t)) = 1 - \frac{1}{1+d((1,1),(2,1))} = 1 - \frac{1}{2} = \frac{1}{2}$$

and

$$a_1^*\phi(M_{\mathbb{A}}(\xi,T\xi,t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\eta,T\eta,t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi,T\eta,2t))a_3 + a_4^*\phi(M_{\mathbb{A}}(\eta,T\xi,t))a_4$$
$$+ a_5^*\phi(M_{\mathbb{A}}(\xi,\eta,t))a_5 + a_6^*(\phi(M_{\mathbb{A}}(\xi,T\xi,t)) + \phi(M_{\mathbb{A}}(\eta,T\eta,t)))\frac{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi,T\xi,t)))}{(1_{\mathbb{A}} + \phi(M_{\mathbb{A}}(\xi,\eta,t)))}a_6$$
$$= 0.15(1 - \frac{1}{2} + 1 - \frac{1}{7}) + 0.22(1 - \frac{1}{1+\sqrt{37}} + 1 - \frac{1}{6}) = 0.5734 > \frac{1}{2}.$$

Therefore, the condition of Theorem 1 is satisfied. Clearly, $(1,1)$ is the fixed point of T in X.

In the following result, we use minimum and maximum conditions to prove the existence of fixed point. We note that for $a_i \in [0_{\mathbb{A}}, 1_{\mathbb{A}}]$, $i = 1, 2, ..., n$, $n \in \mathbb{N}$, $MIN(a_i)$ denotes an element a_k, $1 \le k \le n$ such that $a_k \preccurlyeq a_i$ for each i, $1 \le i \le n$. Similarly, $MAX(a_i)$ denotes an element a_k, $1 \le k \le n$ such that $a_k \succcurlyeq a_i$ for each i, $1 \le i \le n$.

Theorem 2. Let $(X, M_{\mathbf{A}}, \tau)$ be a complete C^*-algebra valued fuzzy metric space. Let ϕ be a subadditive altering distance function with respect to the t-norm τ and let T be a self mapping on X such that

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi, T\eta, t)) \preccurlyeq\ & a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi, \eta, t)), \phi(M_{\mathbb{A}}(\xi, T\xi, t)), \phi(M_{\mathbb{A}}(\xi, T\eta, 2t)), \phi(M_{\mathbb{A}}(\eta, T\xi, t)),\\ & \phi(M_{\mathbb{A}}(\eta, T\eta, t))\}a_1 + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi, \eta, t)), \phi(M_{\mathbb{A}}(\xi, T\xi, t)),\\ & \phi(M_{\mathbb{A}}(\xi, T\eta, 2t)), \phi(M_{\mathbb{A}}(\eta, T\xi, t)), \phi(M_{\mathbb{A}}(\eta, T\eta, t))\}a_2, \end{aligned} \tag{11}$$

where $a_1, a_2 \in \mathbb{A}'$ with $||a_1||^2 + 2||a_2||^2 < 1$. Then T has a unique fixed point in X.

Proof. For $\xi_0 \in X$, let $\xi_{n+1} = T\xi_n$, $n \in \mathbb{N} \cup \{0\}$. Now,

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq\ & a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_{j-1}, t)), \phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_j, 2t)),\\ & \phi(M_{\mathbb{A}}(\xi_j, T\xi_{j-1}, t)), \phi(M_{\mathbb{A}}(\xi_j, T\xi_j, t))\}a_1 + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)),\\ & \phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_{j-1}, t)), \phi(M_{\mathbb{A}}(\xi_{j-1}, T\xi_j, 2t)), \phi(M_{\mathbb{A}}(\xi_j, T\xi_{j-1}, t)), \phi(M_{\mathbb{A}}(\xi_j, T\xi_j, t))\}a_2\\ \preccurlyeq\ & ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_{j+1}, 2t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\}\\ & + ||a_2||^2 1_{\mathbb{A}} MAX\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_{j+1}, 2t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\}\\ \preccurlyeq\ & ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)), \phi(\tau(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t), M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)))\}\\ & + ||a_2||^2 1_{\mathbb{A}} MAX\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)), \phi(\tau(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t), M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)))\}.\end{aligned}$$

Again, $\phi(a) \preccurlyeq \phi(\tau(a, b))$ and $\phi(b) \preccurlyeq \phi(\tau(a, b))$. So, $MIN\{\phi(a), \phi(b), \phi(\tau(a, b))\} = MIN\{\phi(a), \phi(b)\}$ for all $a, b \in [0_{\mathbb{A}}, 1_{\mathbb{A}}]$. Hence,

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq\ & ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\}\\ & + ||a_2||^2 1_{\mathbb{A}} \phi(\tau(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t), M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)))\\ \preccurlyeq\ & ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\}\\ & + ||a_2||^2 1_{\mathbb{A}} (\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))). \end{aligned} \tag{12}$$

If $MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\} = \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t))$, then

$$\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq ||a_1||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_2||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_2||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)),$$

$$\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq \frac{(||a_1||^2 + ||a_2||^2)}{(1 - ||a_2||^2)} 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) = \gamma_1 \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)).$$

Again, if $MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)), \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))\} = \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t))$, then from (12), we get

$$\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq ||a_1||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) + ||a_2||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) + ||a_2||^2 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)),$$

$$\phi(M_{\mathbb{A}}(\xi_j, \xi_{j+1}, t)) \preccurlyeq \Big(\frac{||a_2||^2}{1 - ||a_1||^2 - ||a_2||^2}\Big) 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)) = \gamma_2 \phi(M_{\mathbb{A}}(\xi_{j-1}, \xi_j, t)),$$

where $\gamma_1 = \big(\frac{||a_1||^2 + ||a_2||^2}{1 - ||a_2||^2}\big) 1_{\mathbb{A}}$ and $\gamma_2 = \big(\frac{||a_2||^2}{1 - ||a_1||^2 - ||a_2||^2}\big) 1_{\mathbb{A}}$ are positive elements in $\mathbb{A}$ and strictly less than $1_{\mathbb{A}}$. Proceeding as in Theorem 1, we can easily show that the sequence $\{\xi_n\}$ is a Cauchy sequence.

Let $\xi_n \to z$. Now,

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi_n, Tz, t)) \preccurlyeq\ & a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_n, z, t)), \phi(M_{\mathbb{A}}(\xi_n, T\xi_n, t)), \phi(M_{\mathbb{A}}(\xi_n, Tz, 2t)), \phi(M_{\mathbb{A}}(z, T\xi_n, t)), \\ & \phi(M_{\mathbb{A}}(z, Tz, t))\}a_1 + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_n, z, t)), \phi(M_{\mathbb{A}}(\xi_n, T\xi_n, t)), \phi(M_{\mathbb{A}}(\xi_n, Tz, 2t)), \\ & \phi(M_{\mathbb{A}}(z, T\xi_n, t)), \phi(M_{\mathbb{A}}(z, Tz, t))\}a_2 \\ & \preccurlyeq a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_n, z, t)), \phi(M_{\mathbb{A}}(\xi_n, \xi_{n+1}, t)), \phi(M_{\mathbb{A}}(\xi_n, Tz, 2t)), \phi(M_{\mathbb{A}}(z, \xi_{n+1}, t)), \\ & \phi(M_{\mathbb{A}}(z, Tz, t))\}a_1 + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_n, z, t)), \phi(M_{\mathbb{A}}(\xi_n, \xi_{n+1}, t)), \phi(M_{\mathbb{A}}(\xi_n, Tz, 2t)), \\ & \phi(M_{\mathbb{A}}(z, \xi_{n+1}, t)), \phi(M_{\mathbb{A}}(z, Tz, t))\}a_2.\end{aligned}$$

By taking the limit as $n \to \infty$, the above equation becomes

$$\begin{aligned}\phi(M_{\mathbb{A}}(z, Tz, t)) &\preccurlyeq ||a_1||^2 \phi(M_{\mathbb{A}}(z, Tz, t)) + ||a_2||^2 \phi(M_{\mathbb{A}}(z, Tz, t)), \\ (1 - ||a_1||^2 - ||a_2||^2) 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(z, Tz, t)) &\preccurlyeq 0_{\mathbb{A}},\end{aligned}$$

which gives $z = Tz$. Clearly, by using (11), the fixed point is unique.

Remark 2. The above theorem can be taken as a generalization of Theorem 2.11 of [5] and Theorem 2.1 of [24] in the setting of C^*-algebra valued fuzzy metric space.

It may be noted here that the mapping we have considered is not necessarily continuous, which can be seen from the following example.

Example 4. We consider $(X, M_{\mathbb{A}}, \tau)$ and ϕ as in Example 3. Let $T : X \to X$ be defined by

$$T(\xi) = \begin{cases} \frac{1}{6} \ if \ \xi \in [0, \frac{1}{2}) \\ \frac{1}{12} \ if \xi \in [\frac{1}{2}, 1]. \end{cases}$$

Let $a_1 = 0$ and $a_2 = \frac{7}{10}$. Now, three cases will arise:

Case 1. If $\xi, \eta \in [0, \frac{1}{2})$, then $\phi(M_{\mathbb{A}}(T\xi, T\eta, t)) = 1 - \frac{1}{1+d(\frac{1}{6}, \frac{1}{6})} = 0$. So, condition (11) is trivially true.

Case 2. If $\xi, \eta \in [\frac{1}{2}, 1]$, this is similar to Case 1.

Case 3. If $\xi \in [0, \frac{1}{2})$ and $\eta \in [\frac{1}{2}, 1]$, then

$$\phi(M_{\mathbb{A}}(T\xi, T\eta, t)) = 1 - \frac{1}{1 + |\frac{1}{6} - \frac{1}{12}|} = \frac{1}{13} \tag{13}$$

and

$$\begin{aligned}& a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi, \eta, t)), \phi(M_{\mathbb{A}}(\xi, T\xi, t)), \phi(M_{\mathbb{A}}(\xi, T\eta, 2t)), \phi(M_{\mathbb{A}}(\eta, T\xi, t)), \phi(M_{\mathbb{A}}(\eta, T\eta, t))\}a_1 \\ & + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi, \eta, t)), \phi(M_{\mathbb{A}}(\xi, T\xi, t)), \phi(M_{\mathbb{A}}(\xi, T\eta, 2t)), \phi(M_{\mathbb{A}}(\eta, T\xi, t)), \phi(M_{\mathbb{A}}(\eta, T\eta, t))\}a_2 \\ & = \frac{49}{100} MAX\left\{1 - \frac{1}{1 + |\xi - \frac{1}{6}|}, 1 - \frac{1}{1 + |\eta - \frac{1}{12}|}, 1 - \frac{1}{1 + |\xi - \frac{1}{12}|}, 1 - \frac{1}{1 + |\eta - \frac{1}{6}|}, 1 - \frac{1}{1 + |\xi - \eta|}\right\}.\end{aligned} \tag{14}$$

Figure 2 describes equations (13) and (14). Here, the yellow surface represents the equation (14) and the blue surface represents the equation (13). From Figure 2, it is clear that for all $\xi \in [0, \frac{1}{2})$ and $\eta \in [\frac{1}{2}, 1]$, the value of (14) is greater than the value of (13). Thus, the condition of Theorem 2 is satisfied. Clearly, $\frac{1}{6}$ is a fixed point of T.

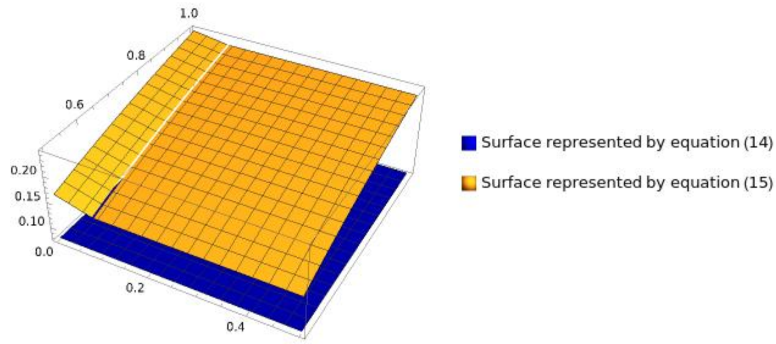

Figure 2. Demonstration of the condition of Theorem 2 by mapping T

Next we derive the following common fixed point theorem.

Theorem 3. Let $(X, M_{\mathbf{A}}, \tau)$ be a complete C^*-algebra valued fuzzy metric space. Let ϕ be a subadditive altering distance function with respect to the t-norm τ and let $T, S : X \to X$ be such that

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi, S\eta, t)) \preccurlyeq a_1^* MIN\{&\phi(M_{\mathbb{A}}(\xi, \eta, t)), \phi(M_{\mathbb{A}}(\xi, T\xi, t)), \phi(M_{\mathbb{A}}(\xi, S\eta, 2t)), \phi(M_{\mathbb{A}}(\eta, T\xi, 2t)),\\ &\phi(M_{\mathbb{A}}(\eta, S\eta, t))\} a_1 + a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi, \eta, t)), \phi(M_{\mathbb{A}}(\xi, T\xi, t)), \phi(M_{\mathbb{A}}(\xi, S\eta, 2t)),\\ &\phi(M_{\mathbb{A}}(\eta, T\xi, 2t)), \phi(M_{\mathbb{A}}(\eta, S\eta, t))\} a_2,\end{aligned}$$

where $a_1, a_2 \in \mathbb{A}'$ with $||a_1||^2 + 2||a_2||^2 < 1$. Then T and S have a unique common fixed point.

Proof. For $\xi_0 \in X$, let $\xi_{2i+1} = T\xi_{2i}$ and $\xi_{2i+2} = S\xi_{2i+1}$, $i \in \mathbb{N} \cup \{0\}$. Now,

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t)) &= \phi(M_{\mathbb{A}}(T\xi_{2i}, S\xi_{2i+1}, t))\\ &\preccurlyeq a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, T\xi_{2i}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, S\xi_{2i+1}, 2t)),\\ &\phi(M_{\mathbb{A}}(\xi_{2i+1}, T\xi_{2i}, 2t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, S\xi_{2i+1}, t))\} a_1\\ &+ a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, T\xi_{2i}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, S\xi_{2i+1}, 2t)),\\ &\phi(M_{\mathbb{A}}(\xi_{2i+1}, T\xi_{2i}, 2t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, S\xi_{2i+1}, t))\} a_2\\ &= ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+2}, 2t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}\\ &+ ||a_2||^2 1_{\mathbb{A}} MAX\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+2}, 2t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}\\ &\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(\tau(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))),\\ &\phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}\\ &+ ||a_2||^2 1_{\mathbb{A}} MAX\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(\tau(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))),\\ &\phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}.\end{aligned}$$

Since $MIN\{\phi(a), \phi(b), \phi(\tau(a, b))\} = MIN\{\phi(a), \phi(b)\}$ for all $a, b \in [0_{\mathbb{A}}, 1_{\mathbb{A}}]$, we have

$$\begin{aligned}\phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t)) &\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}\\ &+ ||a_2||^2 1_{\mathbb{A}} \phi(\tau(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t)))\\ &\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)), \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))\}\\ &+ ||a_2||^2 1_{\mathbb{A}} (\phi(M_{\mathbb{A}}(\xi_{2i}, \xi_{2i+1}, t)) + \phi(M_{\mathbb{A}}(\xi_{2i+1}, \xi_{2i+2}, t))). \qquad (15)\end{aligned}$$

Similarly,

$$
\begin{aligned}
\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t)) &= \phi(M_{\mathbb{A}}(S\xi_{2i+1},T\xi_{2i+2},t))\\
&\preccurlyeq a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+1},S\xi_{2i+1},t)),\phi(M_{\mathbb{A}}(\xi_{2i+1},T\xi_{2i+2},2t)),\\
&\phi(M_{\mathbb{A}}(\xi_{2i+2},S\xi_{2i+1},2t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},T\xi_{2i+2},t))\}a_1\\
&+ a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+1},S\xi_{2i+1},t)),\phi(M_{\mathbb{A}}(\xi_{2i+1},T\xi_{2i+2},2t)),\\
&\phi(M_{\mathbb{A}}(\xi_{2i+2},S\xi_{2i+1},2t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},T\xi_{2i+2},t))\}a_2\\
&\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(\tau(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))),\\
&\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))\} + ||a_2||^2 1_{\mathbb{A}} MAX\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(\tau(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\\
&\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))),\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))\}\\
&\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))\}\\
&+ ||a_2||^2 1_{\mathbb{A}} \phi(\tau(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t)))\\
&\preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)),\phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))\}\\
&+ ||a_2||^2 1_{\mathbb{A}} (\phi(M_{\mathbb{A}}(\xi_{2i+1},\xi_{2i+2},t)) + \phi(M_{\mathbb{A}}(\xi_{2i+2},\xi_{2i+3},t))). \qquad (16)
\end{aligned}
$$

Putting $j = 2i+1$, $i = 0, 1, 2, ...$, from (15) and (16), we get

$$
\begin{aligned}
\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) \preccurlyeq ||a_1||^2 1_{\mathbb{A}} MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)),\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))\}\\
+ ||a_2||^2 1_{\mathbb{A}} (\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)) + \phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))). \qquad (17)
\end{aligned}
$$

If $\min\{\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)),\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))\} = \phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t))$, then from (17), we get

$$
\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) \preccurlyeq \big(\frac{||a_1||^2 + ||a_2||^2}{1 - ||a_2||^2}\big) 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)). \qquad (18)
$$

Again, if $MIN\{\phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)),\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))\} = \phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t))$, then

$$
\phi(M_{\mathbb{A}}(\xi_j,\xi_{j+1},t)) \preccurlyeq \big(\frac{||a_2||^2}{1 - ||a_1||^2 - ||a_2||^2}\big) 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(\xi_{j-1},\xi_j,t)). \qquad (19)
$$

Proceeding as Theorem 1, from (18) and (19) we can easily show that $\{\xi_n\}$ is a Cauchy sequence and let $\lim_{n\to\infty} \xi_n = z$. Then,

$$
\begin{aligned}
\phi(M_{\mathbb{A}}(z,Sz,t)) &\preccurlyeq \phi(M_{\mathbb{A}}(z,\xi_{2n+1},t)) + \phi(M_{\mathbb{A}}(\xi_{2n+1},Sz,t))\\
&= \phi(M_{\mathbb{A}}(z,\xi_{2n+1},t)) + \phi(M_{\mathbb{A}}(T\xi_{2n},Sz,t))\\
&\preccurlyeq \phi(M_{\mathbb{A}}(z,\xi_{2n+1},t)) + a_1^* MIN\{\phi(M_{\mathbb{A}}(\xi_{2n},z,t)),\phi(M_{\mathbb{A}}(\xi_{2n},T\xi_{2n},t)),\\
&\phi(M_{\mathbb{A}}(\xi_{2n},Sz,2t)),\phi(M_{\mathbb{A}}(z,T\xi_{2n},2t)),\phi(M_{\mathbb{A}}(z,Sz,t))\}a_1\\
&+ a_2^* MAX\{\phi(M_{\mathbb{A}}(\xi_{2n},z,t)),\phi(M_{\mathbb{A}}(\xi_{2n},T\xi_{2n},t)),\\
&\phi(M_{\mathbb{A}}(\xi_{2n},Sz,2t)),\phi(M_{\mathbb{A}}(z,T\xi_{2n},2t)),\phi(M_{\mathbb{A}}(z,Sz,t))\}a_2.
\end{aligned}
$$

Taking the limit as $n \to \infty$ and by Lemma 3, from the above equation, we get

$$
\begin{aligned}
\phi(M_{\mathbb{A}}(z,Sz,t)) &\preccurlyeq ||a_1||^2 \phi(M_{\mathbb{A}}(z,Sz,t)) + ||a_2||^2 \phi(M_{\mathbb{A}}(z,Sz,t)),\\
(1 - ||a_1||^2 - ||a_2||^2) 1_{\mathbb{A}} \phi(M_{\mathbb{A}}(z,Sz,t)) &\preccurlyeq 0_{\mathbb{A}},
\end{aligned}
$$

which gives $z = Sz$. Similarly, we can show that z is also a fixed point of T.

Theorem 4. Let $(X, M_{\mathbf{A}}, \tau)$ be a complete C^*-algebra valued fuzzy metric space. Let ϕ be a subadditive altering distance function with respect to the t-norm τ and let T, S be two self mappings on X such that

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi, S\eta, t)) \preccurlyeq\ & a_1^*\phi(M_{\mathbb{A}}(\xi, T\xi, t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\eta, S\eta, t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi, S\eta, t))a_3 \\ & + a_4^*\phi(M_{\mathbb{A}}(\eta, T\xi, t))a_4 + a_5^*\phi(M_{\mathbb{A}}(\xi, \eta, t))a_5,\end{aligned}$$

where $a_i \in \mathbb{A}'$ for $i = 1$ to 5 with $\sum_{i=1}^{5} ||a_i||^2 < 1$. Then T and S have a unique common fixed point.

The proof is similar to Theorem 3.
For $T = S$, the above theorem reduces to the following fixed point theorem.

Theorem 5. Let $(X, M_{\mathbb{A}}, \tau)$ be a complete C^*-algebra valued fuzzy metric space. Let ϕ be a subadditive altering distance function with respect to the t-norm τ and let T be a self mapping on X such that

$$\begin{aligned}\phi(M_{\mathbb{A}}(T\xi, T\eta, t)) \preccurlyeq\ & a_1^*\phi(M_{\mathbb{A}}(\xi, T\xi, t))a_1 + a_2^*\phi(M_{\mathbb{A}}(\eta, T\eta, t))a_2 + a_3^*\phi(M_{\mathbb{A}}(\xi, T\eta, t))a_3 \\ & + a_4^*\phi(M_{\mathbb{A}}(\eta, T\xi, t))a_4 + a_5^*\phi(M_{\mathbb{A}}(\xi, \eta, t))a_5,\end{aligned}$$

where $a_i \in \mathbb{A}'$ for $i = 1$ to 5 with $||a_1||^2 + ||a_2||^2 + 2||a_3||^2 + ||a_5||^2 < 1$. Then T has a unique fixed point.

3 Application to boundary value problem

We consider the following boundary value problem:

$$x^2y'' + xy' - y = f(t, y(t)),\ 0 < x < 1,\ t \in I = [0, 1], \tag{20}$$

(where f is a function from $I \times \mathbb{R}$ to $\mathbb{R}$), with the boundary conditions: $y(x)$ is bounded as $x \to 0$ and $y(1) = 0$. This boundary value problem is equivalent to the integral equation:

$$u(t) = \int_0^1 G(t, s)f(s, u(s))ds,\ 0 < t, s < 1,$$

where

$$G(t, s) = \begin{cases} \frac{t}{2}(1 - \frac{1}{s^2}), & s > t \\ \frac{1}{2}(t - \frac{1}{t}), & s < t \end{cases}$$

is the Green's function.

Let $C(I, \mathbb{R})$ denote the set of all continuous functions $f : I \to \mathbb{R}$ such that for $x, y \in C(I, \mathbb{R})$, $|x(t) - y(t)| < k$ for some $k > 0$ and for all $t \in I$.

Theorem 6. For the above problem (20), we consider f as a continuous function from $I \times \mathbb{R} \to \mathbb{R}$ satisfying the following condition:

$$f(s, u(s)) - f(s, v(s)) \leq \frac{1}{9}|u(s) - v(s)|,\ \text{for all } u, v \in C(I, \mathbb{R}),\ s \in I.$$

Then the problem (20) has a unique solution.

Proof. Let $T : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ be defined by $Tu(t) = \int_0^1 G(t, s)f(s, u(s))ds,\ u \in C(I, \mathbb{R})$. Let $\mathbb{A} = [0, 1]$ with the usual norm on $\mathbb{R}$. Let $X = C(I, \mathbb{R})$ with $d(x, y) = \sup_{t \in I} |x(t) - y(t)|$, $x, y \in C(I, \mathbb{R})$. Here $0_{\mathbb{A}} = 0$ and $1_{\mathbb{A}} = 1$. We consider $M_{\mathbb{A}} : X \times X \times (0, \infty) \to [0, 1]$ given by $M_{\mathbb{A}}(x, y, t) = 1 - \frac{d(x,y)}{k}$, $x, y \in X$, $t > 0$. Then $(X, M_{\mathbb{A}}, \tau)$ is a complete C^*-algebra valued fuzzy metric space with respect to the t-norm $\tau(x,y) = \max\{x+y-1,0\}$, $x,y \in [0,1]$. Also let $\phi(t) = 1-t$, $t \in [0,1]$ be the subadditive altering distance function. Now, for $u,v \in X$ and $t_1 > 0$,

$$\begin{aligned}
\phi(M_{\mathbb{A}}(Tu,Tv,t_1)) = \frac{d(Tu,Tv)}{k} &= \frac{1}{k}\sup_{t\in I}|Tu(t)-Tv(t)| \\
&= \frac{1}{k}\sup_{t\in I}\left|\int_0^1 G(t,s)f(s,u(s))ds - \int_0^1 G(t,s)f(s,v(s))ds\right| \\
&= \frac{1}{k}\sup_{t\in I}\left|\int_0^1 G(t,s)(f(s,u(s)) - f(s,v(s)))ds\right| \\
&\le \frac{1}{k}\sup_{t\in I}\left|\int_0^1 G(t,s)\frac{1}{9}|u(s)-v(s)|ds\right| \\
&\le \frac{1}{9}\frac{d(u,v)}{k}\sup_{t\in I}\left|\int_0^1 G(t,s)ds\right| \\
&= \frac{1}{9}\frac{d(u,v)}{k}\sup_{t\in I}\left|\int_0^t \frac{1}{2}(t-\frac{1}{t})ds + \int_t^1 \frac{t}{2}(1-\frac{1}{s^2})ds\right| \\
&= \frac{1}{9}\frac{d(u,v)}{k}\sup_{t\in I}|t-1| = \frac{1}{3}\phi(M_{\mathbb{A}}(u,v,t_1))\frac{1}{3},
\end{aligned}$$

where $a_i = 0$ for $i = 1$ to 4 and $a_5 = \frac{1}{3}$. Then all the conditions of Theorem 5 are satisfied. Hence the boundary value problem has a unique solution.

4 Application to control theory

In [25], Pathak et al. and in [26] Rhoades et al. investigated the possibility of optimally controlling the solution of ordinary differential equation via dynamic programming. Inspired by their work, we give an application to solve such ordinary differential equations in control theory using C^*-algebra valued metric space.

Let K be a compact subset of $\mathbb{R}^n$ with the Euclidean distance which we denote here by $|.|$. Let $T_a : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping such that $T_a(\xi) = f(\xi,a)$ for each $a \in K$ and for all $\xi \in \mathbb{R}^n$, where $f : \mathbb{R}^n \times K \to \mathbb{R}^n$ is a bounded continuous function such that

$$|f(\xi,a)| \le C \text{ for some } C > 0 \tag{21}$$

and for $t_1 > 0$, $\xi,\eta \in \mathbb{R}^n$,

$$\begin{aligned}
\frac{t_1}{t_1+|f(\xi,a)-f(\eta,a)|} \le a_1^*\frac{t_1}{t+|\xi-f(\xi,a)|}a_1 + a_2^*\frac{t_1}{t_1+|\eta-f(\eta,a)|}a_2 + a_3^*\frac{t_1}{t_1+|\xi-f(\eta,a)|}a_3 \\
+ a_4^*\frac{t_1}{t_1+|\eta-f(\xi,a)|}a_4 + a_5^*\frac{t_1}{t_1+|\xi-\eta|}a_5,
\end{aligned} \tag{22}$$

where $||a_1||^2 + ||a_2||^2 + 2||a_3||^2 + ||a_5||^2 < 1$. For $X = \mathbb{R}^n$ and $\mathbb{A} = \mathbb{R}$, $\tau(\xi,\eta) = \min\{\xi,\eta\}$, $\xi,\eta \in \mathbb{R}^+$, we define $M_{\mathbb{A}}(\xi,\eta,t_1) = \frac{t_1}{t_1+|\xi-\eta|}$. We take ϕ as the identity mapping on $\mathbb{A}^+$.

Now, we study the possibility of optimally controlling the solution $\xi(.)$ of the ordinary differential equation:

$$\begin{cases} \xi'(s) = f(\xi(s),\alpha(s)), & t < s < T, \\ \xi(t) = \xi, \end{cases} \tag{23}$$

where $\xi \in \mathbb{R}^n$ is a given initial point, taken by $\xi(.)$ at the initial time $t \geq 0$, and $T > 0$ is a fixed terminal time and $\xi'(s) = \frac{d\xi(s)}{ds}$. Here $\alpha(.)$ is a control function which is some appropriate scheme for adjusting parameters from the compact set K as time progresses thereby affecting the dynamics of the system modelled by (23). We assume that

$$K' = \{\alpha : [0, T] \to K, \alpha(.) \text{ is measurable}\}$$

denotes the set of admissible controls. Since $T_a(\xi) = f(\xi, a)$ for all $\xi \in \mathbb{R}^n$, $a \in K$, from (21) and (22) we have

$$\frac{t_1}{t_1 + |T_a(\xi) - T_a(\eta)|} \leq a_1^* \frac{t_1}{t + |\xi - T_a(\xi)|} a_1 + a_2^* \frac{t_1}{t_1 + |\eta - T_a(\eta)|} a_2 + a_3^* \frac{t_1}{t_1 + |\xi - T_a(\eta)|} a_3 + a_4^* \frac{t_1}{t_1 + |\eta - T_a(\xi)|} a_4 + a_5^* \frac{t_1}{t_1 + |\xi - \eta|} a_5$$

for all $\xi, \eta \in \mathbb{R}^n$, $t_1 > 0$ and $a \in K$. Now, applying Theorem 5, we deduce that for each control $\alpha(.) \in K'$, the ordinary differential equation (23) has a unique continuous solution $\xi = \xi^{\alpha(.)}(.)$, existing on the time interval $[t, T]$. Solving the ordinary differential equation for almost everywhere time $t < s < T$, we say that $\xi(.)$ is the response of that system to the control $\alpha(.)$, and $\xi(s)$ is the state of the system at a particular time s.

To find a function $\alpha^*(.)$ which can control the system, the following cost criterion is introduced for each admissible control $\alpha(.) \in K'$ (refer to [27]).

$$\Omega_{\xi,t}(\alpha(.)) = \int_t^T p(\xi(s), \alpha(s))ds + q(\xi(T)), \tag{24}$$

where $\xi = \xi^{\alpha(.)}(.)$ is a solution of (23) and $p : \mathbb{R}^n \times K \to \mathbb{R}$, $q : \mathbb{R}^n \to \mathbb{R}$ are given functions, where p is the running cost per unit time and q is the terminal cost. Suppose that,

$$\begin{cases} \max\{|P_a(\xi)|, |q(\xi)|\} & \leq C \text{ for some } C > 0 \\ \frac{t_1}{t_1+|P_a(\xi)-P_a(\eta)|} & \leq a_1^* \frac{t_1}{t+|\xi-P_a(\xi)|} a_1 + a_2^* \frac{t_1}{t_1+|\eta-P_a(\eta)|} a_2 + a_3^* \frac{t_1}{t_1+|\xi-P_a(\eta)|} a_3 \\ & +a_4^* \frac{t_1}{t_1+|\eta-P_a(\xi)|} a_4 + a_5^* \frac{t_1}{t_1+|\xi-\eta|} a_5 \\ \frac{t_1}{t_1+|q(\xi)-q(\eta)|} & \leq a_1^* \frac{t_1}{t+|\xi-q(\xi)|} a_1 + a_2^* \frac{t_1}{t_1+|\eta-q(\eta)|} a_2 + a_3^* \frac{t_1}{t_1+|\xi-q(\eta)|} a_3 \\ & +a_4^* \frac{t_1}{t_1+|\eta-q(\xi)|} a_4 + a_5^* \frac{t_1}{t_1+|\xi-\eta|} a_5, \ for\ all\ \xi, \eta \in \mathbb{R}^n, \ a \in K, \end{cases}$$

where $P_a : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping such that $P_a(\xi) = p(\xi, a)$ for all $\xi \in \mathbb{R}^n$. For given $\xi \in \mathbb{R}^n$ and $0 < t < T$, we are to find if possible a control $\alpha^*(.)$ which minimizes the cost functional (24) among all other admissible controls.

For the solution of the above problem we now apply the dynamic programming as described in [27], where the value function $u(\xi, t)$ is defined by

$$u(\xi, t) = \inf_{\alpha(.) \in K'} \Omega_{\xi,t}(\alpha(.)) \ \xi \in \mathbb{R}^n, \ 0 \leq t \leq T.$$

Here $u(\xi, t)$ is the least cost for the position ξ at time t.

For fixed $\xi \in \mathbb{R}^n$ and $0 \leq t \leq T$, proceeding as in [27; 554], the following theorem gives the optimality conditions:

Theorem 7. For each $\zeta > 0$ small enough such that $t + \zeta \leq T$,

$$u(\xi, t) = \inf_{\alpha(.) \in K'} \left\{ \int_t^{t+\zeta} p(\xi(s), \alpha(s))ds + u(\xi(t + \zeta), t + \zeta) \right\},$$

where $\xi = \xi^{\alpha(.)}$ solves the ODE (23) for the control $\alpha(.)$.

Proof. The proof follows from Theorem 5 and [27; 554].

Conclusions and Future Works

In this paper, we have obtained some fixed point and common fixed point results for some generalized mappings in C^*-algebra valued fuzzy metric space. Moreover, the results are applied to boundary value problem and control theory. Some open problems concerning our results are as follows.

In Theorems 1, 2 and 3, we have considered complete C^*-algebra valued fuzzy metric space. The investigation of the existence of fixed point via our defined contractive conditions in case of incomplete C^*-algebra valued fuzzy metric space is a problem of further study.

In [28] and [29], the authors obtained some important results in fuzzy bipolar metric space. The analogous study in case of bipolar C^*-algebra valued fuzzy metric space for the mappings defined in this paper is a scope for future research.

In 2024, Gnanaprakasam et al. [30] applied fixed point techniques to discuss solvability of fractional integro-differential equation in orthogonal complete metric space. In this regard, we can extend our study to investigate solvability of fractional integro-differential equation.

Further investigation can be done considering coupled fixed point, best proximity point, coupled best proximity point, etc., using our mappings in the setting of C^*-algebra valued fuzzy metric space. The works done in this paper thus open up a wide scope of investigation in C^*-algebra valued fuzzy metric space considering various emerging applications.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Recurrence free decomposition formulas for the Lauricella special functions

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Expansion formulas associated with the multidimensional Lauricella hypergeometric functions are well-established and extensively utilized. However, the recurrence relations inherit in these formulas add extra complexities to their use. A thorough analysis of the characteristics of these expansion formulas shows that they can be simplified and converted into a more convenient form. This paper presents new recurrence free decomposition formulas, which are employed to solve boundary value problems.

Keywords: Appell Functions, Lauricella Functions, Recurrence Decomposition Formula, Recurrence Free Decomposition Formula.

2020 Mathematics Subject Classification: 33C65, 35J25, 35J70.

Introduction

The theory of multidimensional hypergeometric functions has gained significant interest because of its capability to solve numerous applied problems involving partial differential equations (for details, see [1]; also the references quoted in to [2] and [3]). As shown in work [4], for instance, hypergeometric special functions with many arguments can be widely used to estimate the energy absorbed by the non ferromagnetic conducting sphere located inside an internal magnetic field. In addition to their using in solving partial deferential equations, hypergeometric series of several variables are utilized into different quantum physical problems and also in quantum chemical applications [2,5]. Inter alia, the second order degenerate differential equations in partial derivatives of elliptic-parabolic types, which are particularly widespread in studying gas dynamics problems may be solved by means of diverse multidimensional Gaussian series. Interesting examples consist of the studying problem of the adiabatic plane parallel to the liquid or gas flow without any vortex. Also the problem of the flow of supersonic current from a container with smooth walls and several other technical issues of gas-liquid flow may arise in various applications [6, 7].

It is very essential to highlight that Riemann's and Green's special functions, as well as the fundamental solutions with singularity of the second order degenerate differential equations with partial derivatives may be also expressed by multidimensional Gaussian series. When we research problems with boundary values for similar differential equations in partial derivatives, we need to expand hypergeometric special functions of several variables into more simpler types of special functions, like Gauss or Appell functions.

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The known operator method of Burchnall and Chaundy [8] has been very widely used by scientists to receive formulas for expanding of hypergeometric functions of two independent arguments, the known operator method of Burchnall and Chaundy [8] has been very widely used by scientists to receive formulas for expanding of hypergeometric Gaussian series of two variables, expressing through the use of simple Gauss' hypergeometric series of one variable.

Based on the fundamental work of Burchnall and Chaundy [8], Hasanov and Srivastava [9, 10] introduced formulas which extend the capabilities Burchnall-Chaundy operator, this leads to another expansion formulas for various hypergeometric series of three variables. They also established recurrent formulas for higher-dimensional hypergeometric functions. Nonetheless, the recurrence introduces potential complications when applying these decomposition formulas.

In this study, we develop novel decomposition formulas for all four multiple Lauricella's hypergeometric functions, providing they are independent of recurrence.

1 *The expansions of Appell's two-variable functions*

The decomposition of a hypergeometric series with many arguments into several simpler components is one of the main problems of the special functions theory. Such a decomposition is valuable because it enables the simplification of complex calculations, reduces the dimensionality of the problem, and facilitates the development of new identities and relationships between special functions.

In 1940, Burchnall and Chaundy [8] introduce the operators

$$\nabla(h)=\frac{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)}{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)},\quad \Delta(h)=\frac{\Gamma(\delta_1+h)\,\Gamma(\delta_2+h)}{\Gamma(h)\,\Gamma(\delta_1+\delta_2+h)}, \tag{1}$$

where $\delta_1=x\dfrac{\partial}{\partial x}$ and $\delta_2=y\dfrac{\partial}{\partial y}$, through which they penned

$$F_2\left(a,b,b';c,c';x,y\right)=\nabla(a)F(a,b;c;x)F(a,b';c';y), \tag{2}$$

$$F_3\left(a,a',b,b';c;x,y\right)=\Delta(c)F(a,b;c;x)F(a',b';c;y),$$

$$F_1\left(a,b,b';c;x,y\right)=\nabla(a)\Delta(c)F(a,b;c;x)F(a,b';c;y),$$

$$F_4\left(a,b;c,c';x,y\right)=\nabla(a)\nabla(b)F(a,b;c;x)F(a,b;c';y),$$

thus decomposing Appell's functions using operators Δ and ∇; they also obtained transformations of Appell's functions including

$$F_1\left(a,b,b';c;x,y\right)=\nabla(a)F_3\left(a,a,b,b';c;x,y\right),$$

$$F_1\left(a,b,b';c;x,y\right)=\Delta(c)F_2\left(a,b,b';c,c;x,y\right),$$

$$F_4\left(a,b;c,c';x,y\right)=\nabla(b)F_2\left(a,b,b;c,c';x,y\right),$$

and some others.

These symbolic representations are utilized to derive numerous expansions of Appell's functions either as products of ordinary hypergeometric functions or conversely. For instance, employing Gauss' formula [11; 73],

$$F(a,b;c;x)\equiv F\left[\begin{matrix}a,b;\\ c;\end{matrix}\ x\right]=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!},\ c\neq 0,-1,-2,...$$

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ c\neq 0,-1,-2,...,\ Re(c-a-b)>0 \tag{3}$$

we symbolically express

$$\nabla(h) = \sum_{r=0}^{\infty} \frac{(-\delta)_r(-\delta')_r}{(h)_r r!}.$$

Now, by virtue of Poole's formula [12; 26]

$$(-\delta)_r f(r) = (-1)^r x^r \frac{d^r f(r)}{dx^r},$$

we obtain

$$(-\delta)_r F(a,b;c;x) = (-1)^r \frac{(a)_r(b)_r}{(c)_r} x^r F(a+r,b+r;c+r;x)$$

and therefore (2) indicates the decomposition formula [8]

$$F_2\left(a,b,b';c,c';x,y\right) = \sum_{r=0}^{\infty} \frac{(a)_r\,(b)_r\,(b')_r}{r!\,(c)_r\,(c')_r} \times \\ \times x^r y^r F\left(a+r,b+r;c+r;x\right) F\left(a+r,b'+r;c'+r;y\right). \tag{4}$$

Through the inversion of (2) in the following form

$$F(a,b;c;x)F(a,b';c';y) = \Delta(a) F_2\left(a,b,b';c,c';x,y\right)$$

and an associated expansion of $\Delta(a)$, which is related to (4),

$$F\left(a,b;c;x\right)F\left(a,b';c';y\right) = \sum_{r=0}^{\infty}(-1)^r \frac{(a)_r\,(b)_r\,(b')_r}{r!\,(c)_r\,(c')_r} \times \\ \times x^r y^r F_2\left(a+r,b+r,b'+r;c+r,c'+r;x,y\right)$$

is obtained.

These expansions can be established through coefficient comparison of corresponding powers of x and y.

Applying their way, Burchnall and Chaundy enacted 15 couples of expansions that binds Appell's two-variables functions to one-variables ordinary hypergeometric functions, along with many additional expansion formulas involving hypergeometric series of many variables and confluent hypergeometric series of Humbert.

The introduced method is applicable to functions with two arguments, relies on symbolic operators that are mutually inverse, as detailed in subsequent literature [8].

2 Decomposition formulas for multiple Lauricella hypergeometric functions

To extend the operators $\nabla(h)$ and $\Delta(h)$ introduced in (1), Hasanov and Srivastava [9,10] proposed new operators

$$\tilde{\nabla}_{x_1;x_2,\ldots,x_n}(h) = \frac{\Gamma(h)\,\Gamma(\delta_1+\ldots+\delta_n+h)}{\Gamma(\delta_1+h)\,\Gamma(\delta_2+\ldots+\delta_n+h)},$$

$$\tilde{\Delta}_{x_1;x_2,\ldots,x_n}(h) = \frac{\Gamma(\delta_1+h)\,\Gamma(\delta_2+\ldots+\delta_n+h)}{\Gamma(h)\,\Gamma(\delta_1+\ldots+\delta_n+h)},$$

where $\delta_k = x_k \dfrac{\partial}{\partial x_k}$ $(k=\overline{1,n})$, through which they successfully derived decomposition formulas for the entire class of multiple Gauss series.

Based on the ideas presented in [8], Hasanov and Srivastava [9] demonstrated that the recurrence formulas [10] hold for all $n \in N\backslash\{1\}$.

$$F_A^{(n)}(a,\mathbf{b};\mathbf{c};\mathbf{x}) = \sum_{|\mathbf{k}'|=0}^{\infty} \frac{(a)_{|\mathbf{k}'|}(b_1)_{|\mathbf{k}'|}}{(c_1)_{|\mathbf{k}'|}} x_1^{|\mathbf{k}'|} \prod_{j=2}^{n} \frac{(b_j)_{k_j}}{k_j!\,(c_j)_{k_j}} x_j^{k_j} \times$$
$$\times F\left(a+|\mathbf{k}'|, b_1+|\mathbf{k}'|; c_1+|\mathbf{k}'|; x_1\right) F_A^{(n-1)}\left(a+|\mathbf{k}'|, \mathbf{b}'+\mathbf{k}'; \mathbf{c}'+\mathbf{k}'; \mathbf{x}'\right), \tag{5}$$

$$F_B^{(n)}(\mathbf{a},\mathbf{b};c;\mathbf{x}) = \sum_{|\mathbf{k}'|=0}^{\infty} \frac{(-1)^{|\mathbf{k}'|}(a_1)_{|\mathbf{k}'|}(b_1)_{|\mathbf{k}'|}}{(c-1+|\mathbf{k}'|)_{|\mathbf{k}'|}(c)_{2|\mathbf{k}'|}} x_1^{|\mathbf{k}'|} \prod_{j=2}^{n} \frac{(a_j)_{k_j}(b_j)_{k_j}}{k_j!} x_j^{k_j} \times$$
$$\times F\left(a_1+|\mathbf{k}'|, b_1+|\mathbf{k}'|; c+2|\mathbf{k}'|; x_1\right) F_B^{(n-1)}\left(\mathbf{a}'+\mathbf{k}', \mathbf{b}'+\mathbf{k}'; c+2|\mathbf{k}'|; \mathbf{x}'\right), \tag{6}$$

$$F_C^{(n)}(a,b;\mathbf{c};\mathbf{x}) = \sum_{|\mathbf{k}'|+|\mathbf{l}'|=0}^{\infty} \frac{\left[(a)_{|\mathbf{k}'|}\right]^2 (b)_{2|\mathbf{k}'|+|\mathbf{l}'|}}{(a)_{|\mathbf{k}'|}(c_1)_{|\mathbf{k}'|+|\mathbf{l}'|}} x_1^{|\mathbf{k}'|+|\mathbf{l}'|} \prod_{j=2}^{n} \frac{x_j^{k_j+l_j}}{k_j! l_j!\,(c_j)_{k_j+l_j}} \times$$
$$\times F\left(a+|\mathbf{k}'|+|\mathbf{l}'|, b+2|\mathbf{k}'|+|\mathbf{l}'|; c_1+|\mathbf{k}'|+|\mathbf{l}'|; x_1\right),$$
$$F_C^{(n-1)}\left(a+|\mathbf{k}'|+|\mathbf{l}'|, b+2|\mathbf{k}'|+|\mathbf{l}'|; \mathbf{c}'+\mathbf{k}'; \mathbf{x}'\right), \tag{7}$$

$$F_D^{(n)}(a,\mathbf{b};c;\mathbf{x}) = \sum_{|\mathbf{k}'|+|\mathbf{l}'|=0}^{\infty} \frac{(-1)^{|\mathbf{k}'|}(a)_{2|\mathbf{k}'|+|\mathbf{l}'|}(b_1)_{|\mathbf{k}'|+|\mathbf{l}'|}(c)_{2|\mathbf{k}'|}}{(c-1+|\mathbf{k}'|)_{|\mathbf{k}'|}\left[(c)_{2|\mathbf{k}'|+|\mathbf{l}'|}\right]^2} x_1^{|\mathbf{k}'|+|\mathbf{l}'|} \times$$
$$\times \prod_{j=2}^{n} \frac{(b_j)_{k_j+l_j}}{k_j! l_j!} x_j^{k_j+l_j} F\left(a+2|\mathbf{k}'|+|\mathbf{l}'|, b_1+|\mathbf{k}'|+|\mathbf{l}'|; c+2|\mathbf{k}'|+|\mathbf{l}'|; x_1\right),$$
$$F_D^{(n-1)}\left(a+2|\mathbf{k}'|+|\mathbf{l}'|, \mathbf{b}'+\mathbf{k}'+\mathbf{l}'; c+2|\mathbf{k}'|+|\mathbf{l}'|; \mathbf{x}'\right), \tag{8}$$

where

$|\mathbf{k}'| := k_2+...+k_n$, $k_2 \geq 0,...,k_n \geq 0$; $|\mathbf{l}'| := l_2+...+l_n$, $l_2 \geq 0,...,l_n \geq 0$; $\mathbf{x}' := (x_2,...,x_n)$; $\mathbf{a}'+\mathbf{a}' := (a_2+k_2,...,a_n+k_n)$ and so on.

Certain properties of the Lauricella $F_A^{(n)}$ function have been studied previously, differentiation formulas, limit formulas, new integral representations and several decomposition formulas have been derived [13]. Nevertheless, the recurrence that presents in formulas (5)–(8) may introduce additional complexities when applying these expansions. Further investigation into the properties of Lauricella functions has shown that these recurrence formulas can be simplified into more manageable forms.

3 New recurrence free decomposition formulas for the Lauricella hypergeometric functions

Until the presentation the main results, let's determine some necessary notations

$$A(k) = A(k,n) = \sum_{i=2}^{k+1}\sum_{j=i}^{n} m_{i,j},\ A(0)=0;\ B(k) \equiv B(k,n) = \sum_{i=2}^{k} m_{i,k} + \sum_{i=k+1}^{n} m_{k+1,i},$$

$$|\mathbf{m}_n| := \sum_{i=2}^{n}\sum_{j=i}^{n} m_{i,j},\ \ M_n! := \prod_{i=2}^{n}\prod_{j=i}^{n} m_{i,j}!,$$

$$C(k) \equiv C(k,n) = \sum_{i=2}^{k+1}\sum_{j=i}^{n} p_{i,j},\ C(0)=0;\ D(k) \equiv D(k,n) = \sum_{i=2}^{k} m_{i,k} + \sum_{i=k+1}^{n} p_{k+1,i},$$

$$|\mathbf{p}_n| := \sum_{i=2}^{n}\sum_{j=i}^{n} p_{i,j}, \quad P_n! := \prod_{i=2}^{n}\prod_{j=i}^{n} m_{i,j}!,$$

where $k, n \in \mathrm{N}$, $k \leq n$; $m_{i,j} \in \mathrm{N} \cap \{0\}$ $(2 \leq i \leq j \leq n)$; if we interpret the $\sum\limits_{i=2}^{s}$ as zero when $s = 1$, for instance, our notations $A(0) = B(1) = C(0) = D(1) = 0$ are adopted.

Theorem 1. The following expansion formulas hold at $n \in \mathrm{N}$

$$\begin{aligned} F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(a)_{A(n)}}{M_n!} \prod_{k=1}^{n} \frac{(b_k)_{B(k)}}{(c_k)_{B(k)}} \times \\ \times \prod_{k=1}^{n} x_k^{B(k)} F\left[\begin{matrix} a + A(k), b_k + B(k); \\ c_k + B(k); \end{matrix} \; x_k\right], \end{aligned} \tag{9}$$

$$\begin{aligned} F_B^{(n)}(\mathbf{a}, \mathbf{b}; c; \mathbf{x}) = \sum_{|\mathbf{m}_n|=0}^{\infty} \frac{(-1)^{A(n)}}{(c)_{2A(n)} M_n!} \times \\ \times \prod_{k=1}^{n} \frac{(a_k)_{B(k)} (b_k)_{B(k)}}{(c - 1 + A(k) - A(k-1))_{A(k)-A(k-1)}} \times \\ \times \prod_{k=1}^{n} x_k^{B(k)} F\left[\begin{matrix} a_k + B(k), b_k + B(k); \\ c + 2A(k); \end{matrix} \; x_k\right], \end{aligned} \tag{10}$$

$$\begin{aligned} F_C^{(n)}(a, b; \mathbf{c}; \mathbf{x}) = \sum_{|\mathbf{m}_n|+|\mathbf{p}_n|=0}^{\infty} \frac{\left[(a)_{A(n)+C(n)}\right]^2 (b)_{2A(n)+C(n)}}{M_n! P_n!} \times \\ \times \prod_{k=1}^{n} \frac{x_k^{B(k)+D(k)}}{(c_k)_{B(k)+D(k)} (a + A(k-1) + C(k-1))_{A(k)-A(k-1)}} \times \\ \times \prod_{k=1}^{n} F\left[\begin{matrix} a + A(k) + C(k), b + 2A(k) + C(k); \\ c_k + B(k) + D(k); \end{matrix} \; x_k\right], \end{aligned} \tag{11}$$

$$\begin{aligned} F_D^{(n)}(a, \mathbf{b}; c; \mathbf{x}) = \sum_{|\mathbf{m}_n|+|\mathbf{p}_n|=0}^{\infty} \frac{(-1)^{A(n)} (a)_{2A(n)+C(n)}}{M_n! P_n! \left[(c)_{2A(n)+C(n)}\right]^2} \times \\ \times \prod_{k=1}^{n} \frac{(c + 2A(k-1) + C(k-1))_{2A(k)-2A(k-1)} (b_k)_{B(k)+D(k)}}{(c + A(k) + A(k-1) + C(k-1))_{A(k)-A(k-1)}} \times \\ \times \prod_{k=1}^{n} x_k^{B(k)+D(k)} F\left[\begin{matrix} a + 2A(k) + C(k), b_k + B(k) + D(k); \\ c_k + 2A(k) + C(k); \end{matrix} \; x_k\right]. \end{aligned} \tag{12}$$

Proof. Equality (9) is proved with the help of the mathematical induction method. Three new equalities (10)–(12) are also proved by mathematical induction.

Corollary 1. Let $a, b_1, \ldots, b_n$ be real numbers with a, c_k, $c_k - b_k \neq 0, -1, -2, \ldots$ and $a > |\mathbf{b}|$. Then the ensuing limit formulas valid at $n \in \mathrm{N}$

$$\lim_{\mathbf{x} \to 0} \left\{ \mathbf{x}^{-\mathbf{b}} F_A^{(n)}\left(a, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{\mathbf{x}}\right) \right\} = \frac{\Gamma(a - |\mathbf{b}|)}{\Gamma(a)} \prod_{k=1}^{n} \frac{\Gamma(c_k)}{\Gamma(c_k - b_k)}; \tag{13}$$

$$\lim_{\mathbf{x}\to 0}\left\{\mathbf{x}^{-\mathbf{b}}F_B^{(n)}\left(\mathbf{a};\mathbf{b};c;1-\frac{1}{\mathbf{x}}\right)\right\}=\frac{\Gamma(c)}{\Gamma(c-|\mathbf{b}|)}\prod_{k=1}^{n}\frac{\Gamma(a_k-b_k)}{\Gamma(a_k)}, \tag{14}$$

where

$$\mathbf{x}^{-\mathbf{b}}:=x_1^{-b_1}...x_n^{-b_n};\ \frac{1}{\mathbf{x}}:=\left(\frac{1}{x_1},...,\frac{1}{x_n}\right).$$

Proof. Due to the above decomposition formula (9) we get next formula

$$\begin{aligned}F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};1-\frac{1}{\mathbf{x}}\right)=\sum_{|\mathbf{m}_n|=0}^{\infty}\frac{(a)_{A(n,n)}}{M_n!}\prod_{k=1}^{n}\frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}}\times\\ \times\prod_{k=1}^{n}\left(1-\frac{1}{x_k}\right)^{B(k,n)}F\left[\begin{matrix}a+A(k,n),b_k+B(k,n);\\ c_k+B(k,n);\end{matrix}\ 1-\frac{1}{x_k}\right].\end{aligned} \tag{15}$$

Now applying the well-known Boltz's formula

$$F(a,b;c;z)=(1-z)^{-b}F\left(c-a,b;c;\frac{z}{z-1}\right)$$

for each hypergeometric function within sum (15), we obtain

$$\begin{aligned}F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};1-\frac{1}{\mathbf{x}}\right)=\mathbf{x}^{\mathbf{b}}\sum_{|\mathbf{m}_n|=0}^{\infty}\frac{(a)_{A(n,n)}}{M_n!}\prod_{k=1}^{n}\frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}}(x_k-1)^{B(k,n)}\times\\ \times\prod_{k=1}^{n}F\left[\begin{matrix}c_k-a+B(k,n)-A(k,n),b_k+B(k,n);\\ c_k+B(k,n);\end{matrix}\ 1-x_k\right].\end{aligned}$$

Utilizing the property parity of the sum

$$\sum_{k=1}^{n}B(k)=2\sum_{k=2}^{n}\sum_{i=2}^{k}m_{i,k}=2\sum_{k=1}^{n-1}\sum_{i=k+1}^{n}m_{k+1,i},$$

we calculate the limit

$$\begin{aligned}\lim_{\mathbf{x}\to 0}\left\{\mathbf{x}^{-\mathbf{b}}F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};1-\frac{1}{\mathbf{x}}\right)\right\}=\sum_{|\mathbf{m}_n|=0}^{\infty}\frac{(a)_{A(n,n)}}{M_n!}\prod_{k=1}^{n}\frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}}\times\\ \times\prod_{k=1}^{n}F\left[\begin{matrix}c_k-a+B(k,n)-A(k,n),b_k+B(k,n);\\ c_k+B(k,n);\end{matrix}\ 1\right]\end{aligned}$$

and utilizing identity (3) to transform the hypergeometric Gauss series in the final summation, by virtue of the previously received equality [14]

$$\begin{aligned}\sum_{|\mathbf{m}_n|=0}^{\infty}\frac{(a)_{A(n,n)}}{M_n!}\prod_{k=1}^{n}\frac{(b_k)_{B(k,n)}(a-b_k)_{A(k,n)-B(k,n)}}{(a)_{A(k,n)}}=\\ =\frac{\Gamma(a-|\mathbf{b}|)}{\Gamma(a)}\prod_{k=1}^{n}\frac{\Gamma(a)}{\Gamma(a-b_k)},\end{aligned}$$

we obtain equality (13). Equality (14) is proved analogously to the proof of (13).

4 Applications of the recurrence free decomposition formulas

Two dimensional case. In case $n = 2$, the formula (9) was known since 1940 in the work [8] (see the expansion (4)) and it was effectively used in studying problems with boundary values for the differential equation of elliptic type with two singular coefficients

$$u_{xx} + u_{yy} + \frac{2\alpha}{x} u_x + \frac{2\beta}{y} u_y = 0, \;\; [2\alpha, 2\beta \in (0,1)]$$

in the works [15, 16].

Three dimensional case. A following decomposition formula

$$\begin{aligned} &F_A^{(3)}\left(a, b_1, b_2, b_3; c_1, c_2, c_3; x_1, x_2, x_3\right) = \\ &= \sum_{i,j,k=0}^{\infty} \frac{(a)_{i+j+k}(b_1)_{j+k}(b_2)_{i+k}(b_3)_{i+j}}{i!j!k!(c_1)_{j+k}(c_2)_{i+k}(c_3)_{i+j}} \times \\ &\times x_1^{j+k} F\left(a + j + k, b_1 + j + k; c_1 + j + k; x_1\right) \times \\ &\times x_2^{i+k} F\left(a + i + j + k, b_2 + i + k; c_2 + i + k; x_2\right) \times \\ &\times x_3^{i+j} F\left(a + i + j + k, b_3 + i + j; c_3 + i + j; x_3\right) \end{aligned}$$

is used in solving various problems with boundary values for the three dimensional differential equation of elliptic type with the three singular coefficients

$$u_{xx} + u_{yy} + u_{yy} + \frac{2\alpha}{x} u_x + \frac{2\beta}{y} u_y + \frac{2\gamma}{z} u_z = 0, \;\; 0 < 2\alpha, 2\beta, 2\gamma < 1$$

in the works [17–19].

Four dimensional case. Sixteen fundamental solutions were constructed for degenerate elliptic type equation with four variables [20]

$$y^m z^k t^l u_{xx} + x^n z^k t^l u_{yy} + x^n y^m t^l u_{zz} + x^n y^m z^k u_{tt} = 0, \quad m, n, k, l \equiv const > 0, \tag{16}$$

by means of following recurrence free expansion formula for the hypergeometric Lauricella's series of four independent variables

$$\begin{aligned} &F_A^{(4)}\left(a; b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; x_1, x_2, x_3, x_4\right) = \\ &= \sum_{\substack{m_2, m_3, m_4, \\ i,j,k=0}}^{\infty} \frac{(a)_{m_2+m_3+m_4+i+j+k}(b_1)_{m_2+m_3+m_4}(b_2)_{m_2+i+j}(b_3)_{m_3+i+k}(b_4)_{m_4+j+k}}{(c_1)_{m_2+m_3+m_4}(c_2)_{m_2+i+j}(c_3)_{m_3+i+k}(c_4)_{m_4+j+k} m_2! m_3! m_4! i! j! k!} \times \\ &\times x_1^{m_2+m_3+m_4} x_2^{m_2+i+j} x_3^{m_3+i+k} x_4^{m_4+j+k} \\ &\times F\left(a + m_2 + m_3 + m_4, b_1 + m_2 + m_3 + m_4; c + m_2 + m_3 + m_4; x_1\right) \\ &\times F\left(a + m_2 + m_3 + m_4 + i + j, b_2 + m_2 + i + j; c_2 + m_2 + i + j; x_2\right) \\ &\times F\left(a + m_2 + m_3 + m_4 + i + j + k, b_3 + m_3 + i + k; c_3 + m_3 + i + k; x_3\right) \\ &\times F\left(a + m_2 + m_3 + m_4 + i + j + k, b_4 + m_4 + j + k; c_4 + m_4 + i + k; x_4\right). \end{aligned}$$

Using the obtained fundamental solutions, several boundary value problems were solved in both finite and infinite domains. For equation (16) in an infinite domain, Neumann, Dirichlet, and several mixed boundary value problems were solved [21, 22]. In a finite domain the Holmgren's problem analogue was solved [23].

Multidimensional case. It is known that all fundamental solutions of the elliptic type differential equation of many variables with singular coefficients

$$L_{\alpha,\lambda}^{(m,n)}(u) \equiv \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} + \sum_{j=1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} = 0, \ 0 < 2\alpha_j < 1, j = \overline{1,n} \tag{17}$$

in the domain $R_m^{n+} := \{(x_1, ..., x_m) : x_1 > 0, ..., x_n > 0\}$ $(m \geq 2, \ 1 \leq n \leq m)$ are expressed by the Lauricella hypergeometric function $F_A^{(n)}$ in the forms

$$\begin{gathered} q_k(x;\xi) = \gamma_k r^{-2\beta_k} \prod_{i=1}^{n} x_i^{2\alpha_i} \prod_{i=1}^{k} (x_i\xi_i)^{1-2\alpha_i} \times \\ \times F_A^{(n)} \left[\begin{matrix} \beta_k, 1-\alpha_1, ..., 1-\alpha_k, \alpha_{k+1}, ..., \alpha_n; \\ 2-2\alpha_1, ..., 2-2\alpha_k, 2\alpha_{k+1}, ..., 2\alpha_n; \end{matrix} \ \sigma \right], \ \ k = \overline{0, \, n}, \end{gathered} \tag{18}$$

where

$$\beta_k = \frac{m-2}{2} + k - \sum_{i=1}^{k} \alpha_i + \sum_{i=k+1}^{n} \alpha_i, \ \ k = \overline{0, \, n};$$

$$\gamma_k = 2^{2\beta_k - m} \frac{\Gamma(\beta_k)}{\pi^{m/2}} \prod_{i=1}^{k} \frac{\Gamma(1-\alpha_i)}{\Gamma(2-2\alpha_i)} \prod_{i=k+1}^{n} \frac{\Gamma(\alpha_i)}{\Gamma(2\alpha_i)}, k = \overline{0, \, n};$$

$$\xi = (\xi_1, ..., \xi_m) : \xi_1 > 0, ..., \xi_n > 0; \ \ \sigma = (\sigma_1, ..., \sigma_n), \ \ \sigma_j = 1 - \frac{r_j^2}{r^2},$$

$$r^2 = \sum_{i=1}^{m} (x_i - \xi_i)^2, \ r_j^2 = (x_j + \xi_j)^2 + \sum_{i=1, \, i \neq j}^{m} (x_i - \xi_i)^2, \ \ j = \overline{1, n}.$$

The singularity of fundamental solutions. By means of the expansion formula (9), it can be shown that the received fundamental solutions (18) have their singularity at $r = 0$. Indeed, it is easy to rewrite a fundamental solution $q_k(x;\xi)$ in the form

$$q_k(x;\xi) = \frac{1}{r^{m-2}} \, \tilde{q}_k(x;\xi), \ \ m > 2,$$

where

$$\tilde{q}_k(x;\xi) = \gamma_k \mathbf{X}^{-|\mathbf{b}|} \prod_{i=1}^{k} \frac{x_i \xi_i^{1-2\alpha_i}}{r_i^{2-2\alpha_i}} \prod_{i=k+1}^{n} \left(\frac{x_i}{r_i} \right)^{2\alpha_i} F_A^{(n)} \left(\beta_k, \mathbf{b}; \mathbf{c}; 1 - \frac{1}{\mathbf{X}} \right),$$

$$\mathbf{X} := \left(\frac{r^2}{r_1^2}, ..., \frac{r^2}{r_n^2} \right), \ \mathbf{X}^{-|\mathbf{b}|} := \prod_{i=1}^{k} \left(\frac{r}{r_i} \right)^{2-2\alpha_i} \prod_{i=k+1}^{n} \left(\frac{r}{r_i} \right)^{2\alpha_i},$$

$$|\mathbf{b}| := k - \sum_{i=1}^{k} \alpha_i + \sum_{i=k+1}^{n} \alpha_i, \ \ \mathbf{b} := (1-\alpha_1, ..., 1-\alpha_k, \alpha_{k+1}, ..., \alpha_n),$$

$$\mathbf{c} := (2-2\alpha_1, ..., 2-2\alpha_k, 2\alpha_{k+1}, ..., 2\alpha_n), \ \ k = \overline{0, \, n}.$$

Now using limit relation (13), we see that the function $\tilde{q}_k(x;\xi)$ is a limited expression at $x \to \xi$:

$$\lim_{r \to 0} \tilde{q}_k(x;\xi) = \frac{1}{4\pi^{m/2}} \Gamma\left(\frac{m-2}{2} \right).$$

So, the constructed fundamental solutions of the differential equation (17) have a singularity of the order $m-2$ when $r \to 0$.

Introduce the following notation:

$$S_p = \{x : x_1 > 0, ..., x_{p-1} > 0,\ x_p = 0,\ x_{p+1} > 0, ..., x_n > 0,\ -\infty < x_{n+1} < +\infty, ..., -\infty < x_m < +\infty\},$$

$$X_p^2 := 1 + x_1^2 + ... + x_{p-1}^2 + x_{p+1}^2 + ... + x_m^2,\ p = \overline{1,n}.$$

Dirichlet-Neumann problem $\left(D^k N^{n-k}\right)^\infty$ *in unbounded domains.* Find a regular solution $u_k(x)$ of equation (17) from the function class $C\left(\overline{\Omega}\right) \cap C^2(\Omega)$, satisfying conditions

$$u_k(x)|_{x_p=0} = \tau_p(\tilde{x}_p),\ \ p = \overline{1,k}, \tag{19}$$

$$\left.\left(x_p^{2\alpha_p} \frac{\partial u_k(x)}{\partial x_p}\right)\right|_{x_p=0} = \nu_p(\tilde{x}_p),\ \ p = \overline{k+1,n}, \tag{20}$$

and

$$\lim_{R\to\infty} u_k(x) = 0,\ m > 2,\ k = \overline{0,n} \tag{21}$$

(if $m = 2$, then the boundedless of the desired solution at infinity is required as well), where $\tau_p(\tilde{x}_p)$ and $\nu_p(\tilde{x}_p)$ are defined functions in the following form:

$$\tau_p(\tilde{x}_p) = \frac{\tilde{\tau}_p(\tilde{x}_p)}{X_p^{\varepsilon_p}},\quad \tilde{\tau}_p(\tilde{x}_p) \in C\left(\overline{S_p}\right),\quad \varepsilon_p > 0,\ p = \overline{1,k},$$

and

$$\nu_p(\tilde{x}_p) = \frac{\tilde{\nu}_p(\tilde{x}_p)}{X_p^{1-2\alpha_p+\varepsilon_p}},\quad \tilde{\nu}_p(\tilde{x}_p) \in C\left(\overline{S}_p\right),\quad \varepsilon_p > 0,\ p = \overline{k+1,n}.$$

The functions $\tau_p(\tilde{x})\ (p = \overline{1,k})$ satisfy the coordination conditions on the initial k lateral faces S_p of the domain and at the origin:

$$\tau_1|_{x_2=0} = \tau_2|_{x_1=0},\ \tau_2|_{x_3=0} = \tau_3|_{x_2=0},\ ....,\ \tau_{k-1}|_{x_k=0} = \tau_k|_{x_{k-1}=0};$$

$$\tau_1(0,0,...,0) = \tau_2(0,0,...,0) = ... = \tau_k(0,0,...,0).$$

The vector $\tilde{x}_p$ occurring in the problem setting is obtained from a vector x by excluding its pth component:

$$\tilde{x}_p := (x_1, ..., x_{p-1}, x_{p+1}, ..., x_m),\ p = \overline{1,n}.$$

The problem's unique solution $\left(D^k N^{n-k}\right)^\infty$ is represented in the next form

$$\begin{aligned} u_k(\xi) = \sum_{p=1}^{k} \int_{S_p} \tau_p(\tilde{x}_p)\, \tilde{x}_p^{(2\alpha)} \left.\left(x_p^{2\alpha_p} \frac{\partial q_k(x,\xi)}{\partial x_p}\right)\right|_{x_p=0} dS_p - \\ - \sum_{p=k+1}^{n} \int_{S_p} \nu_p(\tilde{x}_p)\, \tilde{x}_p^{(2\alpha)}\, q_k(x,\xi)|_{x_p=0} dS_p. \end{aligned} \tag{22}$$

In (22), we use the notation

$$\int_{S_p} ...dS_p := \underbrace{\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty}}_{m-n} \underbrace{\int_0^{+\infty} ... \int_0^{+\infty}}_{n-1} ...dx_1...dx_{p-1}dx_{p+1}...dx_n dx_{n+1}...dx_m.$$

By direct calculation, we establish that the function $u_k(\xi)$, defined in (22), is a solution to the equation (17). Using the decomposition formula (9) and limit relation (13) we can prove that the function $u_k(\xi)$ satisfies the conditions (19)–(21) of the problem $\left(D^k N^{n-k}\right)^\infty$ (for details, see [24]).

Other applications of the expansion formula (9) for the multiple Lauricella special function $F_A^{(n)}$ are found in [25].

We do not yet know any applications of the decomposition formulas (10), (11) and (12) for the well-known Lauricella's hypergeometric series $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$, respectively.

Conclusion

In this paper, recurrence free decomposition formulas for the four Lauricella functions were presented. The obtained formulas were proved using the mathematical induction. These expansions can be demonstrated by comparing the coefficients of equal powers of the variables $x_1, ..., x_n$ on both sides. Formulas (21) and (22) indicate a reciprocity property of the hypergeometric Lauricella functions F_A and F_B, as these functions exhibit reciprocal values in the limit. Do the F_C and F_D functions have similar properties? One of these decomposition formulas for the Lauricella's series F_A is often used in studying problems with boundary values for partial differential equations of various types.

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Author Contributions

T.G. Ergashev collected and analyzed data, led manuscript preparation and supervised the research process. A.R. Ryskan analyzed data, led manuscript preparation, served as the principal investigator of the research grant. N.N. Yuldashev collected and analyzed data.

All authors participated in the revision of the manuscript and approved the final submission.

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Conflict of Interest

The authors declare no conflict of interest.

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Pseudospectra of the direct sum of linear operators in ultrametric Banach spaces

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In this paper, a characterization of essential pseudospectra of bounded linear operators on ultrametric Banach spaces over a spherically complete field was given and the notions of pseudospectra and condition pseudospectra of the direct sum of linear operators on ultrametric Banach spaces were introduced. In particular, we proved that the pseudospectra of the direct sum of bounded linear operators associated with various ε are nested sets and that the intersection of all the pseudospectra of bounded linear operators is the spectrum of the direct sum of bounded linear operators in the direct sum of ultrametric Banach spaces. In addition, many results were proved about them and examples were given.

Keywords: Ultrametric Banach spaces, pseudospectrum, condition pseudospectrum, direct sum of operators, linear operator pencils.

2020 Mathematics Subject Classification: 47A10; 47S10.

1 Introduction and preliminaries

In the classical theory, Trefethen and Embree [1] studied the pseudospectra of bounded linear operators on complex Banach spaces. Recently, Otkun Çevik and Ismailov [2] studied some spectral properties of the direct sum of operators in the direct sum of Hilbert spaces. Ismailov and Ipek Al introduced and studied the pseudospectra of the direct sum of operators and they established some of its properties, for more details, see [3].

In ultaremetric operator theory, the authors [4] extended and studied the concept of pseudospectra of linear operators on ultrametric Banach spaces. The condition pseudospectra of bounded linear operators on ultrametric Banach spaces were extended and studied by Ammar et al. [5]. Recently, El Amrani et al. [6] studied the notion of bounded linear operator pencils on non-Archimedean Banach spaces. The concepts of pseudospectra and condition pseudospectra of ultrametric matrices were studied by El Amrani et al. [7].

In this paper, we will extend and study the pseudospectra and the condition pseudospectra of the direct sum of bounded linear operators on ultrametric Banach spaces.

Throughout this paper, F is an ultrametric Banach space over an ultrametric complete valued field $\mathbb{K}$ with a non-trivial valuation $|\cdot|, \mathcal{L}(F)$ denotes the set of all bounded linear operators on F and $F^* = \mathcal{L}(F, \mathbb{K})$ is the dual space of F. If $S \in \mathcal{L}(F)$, $N(S)$ and $R(S)$ denote the kernel and the range of S respectively, see [8]. Recall that, an unbounded linear operator $S : D(S) \subseteq F \to F$ is called closed, if for each $(x_n)_{n\in\mathbb{N}} \subset D(S)$ such that $\|x_n - x\| \to 0$ and $\|Sx_n - y\| \to 0$ as $n \to \infty$ for some $x \in F$ and $y \in F$, then $x \in D(S)$ and $y = Sx$. The collection of all closed linear operators on F is denoted by $\mathcal{C}(F)$. If $S \in \mathcal{L}(F)$ and B is an unbounded linear operator on F, then $S + B$ is closed if and only if B is closed [8]. We begin with the following preliminaries.

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Definition 1. [8] Let F be a vector space over $\mathbb{K}$. A non-negative real valued function $\|\cdot\| : F \to \mathbb{R}_+$ is an ultrametric norm if:
(i) for all $x \in F$, $\|x\| = 0$ if and only if $x = 0$,
(ii) for each $x \in F$ and $\lambda \in \mathbb{K}$, $\|\lambda x\| = |\lambda|\|x\|$,
(iii) for any $x, y \in F$, $\|x + y\| \leq \max(\|x\|, \|y\|)$.

Definition 2. [8] An ultrametric normed space is a pair $(F, \|\cdot\|)$ where F is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is an ultrametric norm on F.

Definition 3. [8] An ultrametric Banach space is a vector space endowed with an ultrametric norm which is complete.

Proposition 1. [8] The direct sum of two ultrametric Banach spaces is an ultrametric Banach space.

Definition 4. [8] An ultrametric Banach space F is said to be a free Banach space if there exists a family $(x_i)_{i\in I}$ of elements of F indexed by a set I such that each element $x \in F$ can be written uniquely like a pointwise convergent series defined by $x = \sum_{i\in I} \lambda_i x_i$ and $\|x\| = \sup_{i\in I} |\lambda_i|\|x_i\|$.

The family $(x_i)_{i\in I}$ is then called an orthogonal basis for F. If, for all $i \in I, \|x_i\| = 1$, then $(x_i)_{i\in I}$ is called an orthonormal basis of F.

Definition 5. [8] Let F be an ultrametric Banach space over $\mathbb{K}$ and let $A \in \mathcal{L}(F)$. The resolvent set $\rho(A)$ of A on F is defined by

$$\rho(A) = \{\lambda \in \mathbb{K} : (A - \lambda I)^{-1} \in \mathcal{L}(F)\}.$$

The spectrum $\sigma(A)$ of A on F is given by $\mathbb{K}\backslash\rho(A)$.

Example 1. [8] Let F be an ultrametric free Banach space with an orthogonal basis $(e_i)_{i\in\mathbb{N}}$. Consider A on F defined by for all $n \in \mathbb{N}, Ae_n = \lambda_n e_n$ whose domain is

$$D(A) = \{x = (x_n)_{n\in\mathbb{N}} \in F : \lim_{n\to\infty} |\lambda_n||x_n|\|e_n\| = 0\}.$$

If $x \in D(A)$, then one can see that

$$Ax = \sum_{n=0}^{\infty} \lambda_n x_n e_n.$$

Proposition 2. [8] Consider the diagonal operator A given above. Then

$$\rho(A) = \{\lambda \in \mathbb{K} : \lambda \neq \lambda_n \text{ for all } n \in \mathbb{N}\}.$$

Proposition 3. [8] The diagonal operator $A : D(A) \subset F \to F$ given above is closed.

Definition 6. [8] Let $A \in \mathcal{L}(F)$. A is called an operator of finite rank, if $R(A)$ is a finite-dimensional subspace of F.

Definition 7. [8] Let F be an ultrametric Banach space and let $A \in \mathcal{L}(F)$. A is said to be completely continuous, if there exists a sequence of finite rank linear operators $(A_n)_{n\in\mathbb{N}}$ such that $\|A_n - A\| \to 0$ as $n \to \infty$.

The collection of all completely continuous linear operators on F is denoted by $\mathcal{C}_c(F)$.
Ingleton [9] proved the following theorem.

Theorem 1. [9] Suppose that $\mathbb{K}$ is spherically complete. Let F be an ultrametric Banach space over $\mathbb{K}$. For all $x \in F\backslash\{0\}$, there exists $x^* \in F^*$ such that $x^*(x) = 1$ and $\|x^*\| = \|x\|^{-1}$.

From Lemma 4.11 and Lemma 4.13 of [10], we have:

Lemma 1. Let F be an ultrametric normed space over a spherically complete field $\mathbb{K}$. If $f_1^*, \cdots, f_n^*$ are linearly independent vectors in F^*, then there are vectors $f_1, \cdots, f_n$ in F such that

$$f_j^*(f_k) = \delta_{j,k} = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases} \quad 1 \leq j, k \leq n. \tag{1}$$

Moreover, if $f_1, \cdots, f_n$ are linearly independent vectors in F, then there are vectors $f_1^*, \cdots, f_n^*$ in F^* such that (1) holds.

Definition 8. [11] We say that $A \in \mathcal{L}(F)$ has an index when both $\alpha(A) = \dim N(A)$ and $\beta(A) = \dim \Big(F/R(A)\Big)$ are finite. In this case, the index of the linear operator A is defined by $ind(A) = \alpha(A) - \beta(A)$.

Definition 9. [11] Let $A \in \mathcal{L}(F)$. A is said to be an upper semi-Fredholm operator, if

$$\alpha(A) \text{ is finite and } R(A) \text{ is closed.}$$

The set of all upper semi-Fredholm operators on F is denoted by $\Phi_+(F)$.

Definition 10. [11] Let $A \in \mathcal{L}(F)$. A is said to be a lower semi-Fredholm operator, if

$$\beta(A) \ \text{ is finite.}$$

The set of all lower semi-Fredholm operators on F is denoted by $\Phi_-(F)$.

The set of all Fredholm operators on F is defined by

$$\Phi(F) = \Phi_+(F) \cap \Phi_-(F).$$

Lemma 2. [12] Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. If $S \in \Phi(F)$ and $C \in \mathcal{C}_c(F)$, then $S + C \in \Phi(F)$.

Lemma 3. [5] Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. If $S \in \Phi(F)$, then for all $C \in \mathcal{C}_c(F)$, $S + C \in \Phi(F)$ and $ind(S + C) = ind(S)$.

Definition 11. [4] Let F be an ultrametric Banach space over $\mathbb{K}$, let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S)$ of a bounded linear operator S on F is defined by

$$\sigma_\varepsilon(S) = \sigma(S) \cup \{\lambda \in \mathbb{K} : \|(S - \lambda I)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudoresolvent $\rho_\varepsilon(S)$ of a bounded linear operator S on F is defined by

$$\rho_\varepsilon(S) = \rho(S) \cap \{\lambda \in \mathbb{K} : \|(S - \lambda I)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(S - \lambda I)^{-1}\| = \infty$ if and only if $\lambda \in \sigma(S)$.

Theorem 2. [4] Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ and let $S \in \mathcal{L}(F)$. Then

$$\sigma_\varepsilon(S) = \bigcup_{D \in \mathcal{L}(F) : \|D\| < \varepsilon} \sigma(S + D).$$

Theorem 3. [5] Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ and let $S \in \mathcal{L}(F)$. Then

$$\sigma_e(S) = \bigcap_{K \in \mathcal{C}_c(F)} \sigma(S + K).$$

Definition 12. [8] Let F be an ultrametric Banach space over $\mathbb{K}$ and let $S, B \in \mathcal{L}(F)$. The resolvent set $\rho(S, B)$ of a bounded linear operator pencil (S, B) of the form $S - \lambda B$ on F is defined by

$$\rho(S, B) = \{\lambda \in \mathbb{K} : (S - \lambda B)^{-1} \in \mathcal{L}(F)\}.$$

The spectrum $\sigma(S, B)$ of a bounded linear operator pencil (S, B) of the form $S - \lambda B$ on F is given by $\mathbb{K} \backslash \rho(S, B)$.

Definition 13. [6] Let F be an ultrametric Banach space over $\mathbb{K}$, let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S, B)$ of a bounded linear operator pencil (S, B) of the form $S - \lambda B$ on F is defined by

$$\sigma_\varepsilon(S, B) = \sigma(S, B) \cup \{\lambda \in \mathbb{K} : \|(S - \lambda B)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudoresolvent $\rho_\varepsilon(S, B)$ of a bounded linear operator pencil (S, B) of the form $S - \lambda B$ on F is defined by

$$\rho_\varepsilon(S, B) = \rho(S, B) \cap \{\lambda \in \mathbb{K} : \|(S - \lambda B)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(S - \lambda B)^{-1}\| = \infty$ if and only if $\lambda \in \sigma(S, B)$.

Proposition 4. [13] Let F be an ultrametric Banach space over $\mathbb{K}$, let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$, we have:

(i) $\sigma(S, B) = \bigcap\limits_{\varepsilon > 0} \sigma_\varepsilon(S, B)$.

(ii) For any ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$, $\sigma(S, B) \subset \sigma_{\varepsilon_1}(S, B) \subset \sigma_{\varepsilon_2}(S, B)$.

Theorem 4. [13] Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|F\| \subseteq |\mathbb{K}|$, let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then

$$\sigma_\varepsilon(S, B) = \bigcup_{C \in \mathcal{L}(F) : \|C\| < \varepsilon} \sigma(S + C, B).$$

Now, we characterize the essential pseudospectra of bounded linear operator pencils in ultrametric Banach spaces over a spherically complete field $\mathbb{K}$.

Definition 14. [14] Let F be an ultrametric Banach space over $\mathbb{K}$, let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. The essential pseudospectrum $\sigma_{e,\varepsilon}(S, B)$ of a bounded linear operator pencil (S, B) of the form $S - \lambda B$ on F is defined by

$$\sigma_{e,\varepsilon}(S, B) = \mathbb{K} \backslash \{\lambda \in \mathbb{K} : S + C - \lambda B \in \Phi_0(F) \ \text{ for all } C \in \mathcal{L}(F) \text{ such that } \|C\| < \varepsilon\},$$

where $\Phi_0(F)$ is the set of all unbounded Fredholm operators on F of index 0.

We continue by recalling the following statements.

Theorem 5. [14] Let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then,

$$\sigma_{e,\varepsilon}(S, B) = \bigcup_{C \in \mathcal{L}(F) : \|C\| < \varepsilon} \sigma_e(S + C, B).$$

Theorem 6. [13] Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. Let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then,

$$\sigma_{e,\varepsilon}(S, B) = \sigma_{e,\varepsilon}(S + K, B) \quad \text{for all } \ K \in \mathcal{C}_c(F).$$

Remark 1. [13] From Theorem 6, it follows that the essential pseudospectrum of bounded linear operator pencils is invariant under perturbation of all completely continuous linear operators on ultrametric Banach spaces over a spherically complete field $\mathbb{K}$.

Theorem 7. [15] Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. Let $S, B \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then

$$\sigma_e(S,B) = \bigcap_{K \in \mathcal{C}_c(F)} \sigma(S+K,B).$$

From Example 1 of [16], we conclude the following example.

Example 2. Let F be an ultrametric free Banach space over $\mathbb{K}$ with an orthogonal basis $(e_i)_{i\in\mathbb{N}}$. Let $S \in \mathcal{C}(F)$, $B \in \mathcal{L}(F)$ be two diagonal operators such that B is invertible defined by for all $i \in \mathbb{N}, Se_i = \lambda_i e_i$ and $Be_i = \mu_i e_i$ where for all $i \in \mathbb{N}, \lambda_i, \mu_i \in \mathbb{K}$ such that $\lim\limits_{i\to\infty} |\lambda_i| = \infty$ and $\sup\limits_{i\in\mathbb{N}} |\mu_i|$ is finite, then

$$\sigma(S,B) = \{\lambda_i \mu_i^{-1}, i \in \mathbb{N}\}$$

and for all $\lambda \in \rho(S,B)$, we have

$$\begin{aligned} \|(S-\lambda B)^{-1}\| &= \sup_{i\in\mathbb{N}} \frac{\|(S-\lambda B)^{-1}e_i\|}{\|e_i\|} \\ &= \sup_{i\in\mathbb{N}} \left|\frac{1}{\lambda_i - \lambda\mu_i}\right| \\ &= \frac{1}{\inf\limits_{i\in\mathbb{N}} |\lambda_i - \lambda\mu_i|}. \end{aligned}$$

Thus

$$\left\{\lambda \in \mathbb{K} : \|(A-\lambda B)^{-1}\| > \frac{1}{\varepsilon}\right\} = \left\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}} |\lambda_i - \lambda\mu_i| < \varepsilon\right\}.$$

Hence

$$\sigma_\varepsilon(S,B) = \{\lambda_i\mu_i^{-1}, i \in \mathbb{N}\} \cup \left\{\lambda \in \mathbb{K} : \inf_{i\in\mathbb{N}} |\lambda_i - \lambda\mu_i| < \varepsilon\right\}.$$

For more details on pseudospectra and condition pseudospectra of linear operators on ultrametric Banach spaces, we refer to [4, 5, 7].

2 *Main Results*

We begin with the following theorem.

Theorem 8. Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then $\lambda \notin \bigcap\limits_{K\in\mathcal{C}_c(F)} \sigma_\varepsilon(S+K)$ if and only if for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, $S+C-\lambda I \in \Phi(F)$ and $ind(S+C-\lambda I) = 0$.

Proof. Let $\lambda \notin \bigcap\limits_{K\in\mathcal{C}_c(F)} \sigma_\varepsilon(S+K)$, then there exists $K \in \mathcal{C}_c(F)$ such that $\lambda \notin \sigma_\varepsilon(S+K)$. By Theorem 2, there is $K \in \mathcal{C}_c(F)$ such that for all $C \in \mathcal{L}(F)$ with $\|C\| < \varepsilon$, $\lambda \in \rho(S+K+C)$, hence for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$,

$$S+K+C-\lambda I \in \Phi(F)$$

and

$$ind(S+K+C-\lambda I) = 0.$$

The operator $S+C-\lambda I$ can be written in the form

$$S+C-\lambda I = S+C+K-\lambda I - K.$$

Since $K \in \mathcal{C}_c(F)$, by Lemmas 2 and 3, we have for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$,

$$S + C - \lambda I \in \Phi(F)$$

and

$$ind(S + C - \lambda I) = 0.$$

Conversely, let $\lambda \in \mathbb{K}$ and for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, we have $S + C - \lambda I \in \Phi(F)$ and $ind(S + C - \lambda I) = 0$. Put $\alpha(S + C - \lambda I) = \beta(S + C - \lambda I) = n$. Let $\{x_1, \cdots, x_n\}$ being the basis for $N(S + C - \lambda I)$ and $\{y_1^*, \cdots, y_n^*\}$ being the basis for $R(S + C - \lambda I)^{\perp}$. By Lemma 1, there are functionals $x_1^*, \cdots, x_n^*$ in F^* $\big(F^*$ is the dual space of $F\big)$ and elements $y_1, \cdots, y_n$ in F such that

$$x_j^*(x_k) = \delta_{j,k} \text{ and } y_j^*(y_k) = \delta_{j,k},\ 1 \leq j,\ k \leq n,$$

where $\delta_{j,k} = 0$, if $j \neq k$ and $\delta_{j,k} = 1$, if $j = k$. Consider the operator K defined on F by

$$\begin{aligned} K:\ & F \to F \\ & x \mapsto \sum_{i=1}^{n} x_i^*(x) y_i. \end{aligned}$$

It is easy to see that K is a linear operator and $D(K) = F$. In fact, for all $x \in F$,

$$\begin{aligned} \|Kx\| &= \|\sum_{i=1}^{n} x_i^*(x) y_i\| \\ &\leq \max_{1 \leq i \leq n} \|x_i^*(x) y_i\| \\ &\leq \max_{1 \leq i \leq n} (\|x_i^*\| \|y_i\|) \|x\|. \end{aligned}$$

Moreover, $R(K)$ is contained in a finite-dimensional subspace of F. So, K is a finite rank operator, then K is completely continuous. We show that for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, we have

$$N(S + C - \lambda I) \cap N(K) = \{0\} \tag{2}$$

and

$$R(S + C - \lambda I) \cap R(K) = \{0\}. \tag{3}$$

Let $x \in N(S + C - \lambda I) \cap N(K)$, hence $x \in N(S + C - \lambda I)$ and $x \in N(K)$. If $x \in N(S + C - \lambda I)$, then

$$x = \sum_{i=1}^{n} \alpha_i x_i \text{ with } \alpha_1, \cdots, \alpha_n \in \mathbb{K}.$$

Then for all $1 \leq j \leq n$, $x_j^*(x) = \sum_{i=1}^{n} \alpha_i \delta_{i,j} = \alpha_j$. If $x \in N(K)$, hence $Kx = 0$, so

$$\sum_{j=1}^{n} x_j^*(x) y_j = 0.$$

Therefore, we have for all $1 \leq j \leq n$, $x_j^*(x) = 0$. Hence $x = 0$. Consequently,

$$N(S + C - \lambda I) \cap N(K) = \{0\}.$$

Let $y \in R(S+C-\lambda I) \cap R(K)$, then $y \in R(S+C-\lambda I)$ and $y \in R(K)$. Let $y \in R(K)$, we have

$$y = \sum_{i=1}^{n} \alpha_i y_i \text{ with } \alpha_1, \cdots, \alpha_n \in \mathbb{K}.$$

Then for all $1 \leq j \leq n, y_j^*(y) = \sum_{i=1}^{n} \alpha_i \delta_{i,j} = \alpha_j$. Moreover, if $y \in R(S+C-\lambda I)$, hence for all $1 \leq j \leq n$, $y_j^*(y) = 0$. Thus $y = 0$. Therefore,

$$R(S+C-\lambda I) \cap R(K) = \{0\}.$$

Since K is a compact operator. By Lemmas 2 and 3, $S+C-\lambda I+K \in \Phi(F)$ and $ind(S+C+K-\lambda I) = 0$. Thus

$$\alpha(S+C+K-\lambda I) = \beta(S+C+K-\lambda I). \tag{4}$$

If $x \in N(S+C+K-\lambda I)$, then $(S+C-\lambda I)x = -Kx$ in $R(S+C-\lambda I) \cap R(K)$. It follows from (3) that $(S+C-\lambda I)x = -Kx = 0$, hence $x \in N(S+C-\lambda I) \cap N(K)$ and from (2), we have $x = 0$. Thus $\alpha(S+K+C-\lambda I) = 0$, it follows from (4), $R(S+C+K-\lambda I) = X$. Consequently, $S-\lambda I+K+C$ is invertible and from Theorem 2, we conclude that $\lambda \notin \bigcap_{K \in \mathcal{C}_c(F)} \sigma_\varepsilon(S+K)$.

Let $(X_i)_{1 \leq i \leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$. The space $X = \oplus_{i=1}^{n} X_i$ endowed by for all $i \in \{1, \cdots, n\}$, $x_i \in X_i$, $\|x_1 \oplus x_2 \oplus \cdots \oplus x_n\| = \max_{i \in \{1,\cdots,n\}} \|x_i\|$ is an ultrametric Banach space over $\mathbb{K}$ [8]. One can see that if for all $i \in \{1, \cdots, n\}$, $A_i \in \mathcal{L}(X_i)$, then $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \in \mathcal{L}(X)$. We introduce the following definition.

Definition 15. Let $(X_i)_{1 \leq i \leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$ and let $A_i \in \mathcal{L}(X_i)$. The spectrum $\sigma(A)$ of A on $\oplus_{i=1}^{n} X_i$ is given by

$$\sigma(A) = \{\lambda \in \mathbb{K} : A - \lambda I \text{ is not invertible in } \mathcal{L}(\oplus_{i=1}^{n} X_i)\},$$

where I denotes the identity operator of $\oplus_{i=1}^{n} X_i$ and $A = \oplus_{i=1}^{n} A_i$. The resolvent set of A on $\oplus_{i=1}^{n} X_i$ is defined by

$$\rho(A) = \{\lambda \in \mathbb{K} : (A - \lambda I)^{-1} \in \mathcal{L}(\oplus_{i=1}^{n} X_i)\}.$$

For $i = 2$, we have the following proposition.

Proposition 5. Let X, Y be two ultrametric Banach spaces over $\mathbb{K}$. Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$. The spectrum of $A \oplus B \in \mathcal{L}(X \oplus Y)$ is given by

$$\sigma(A \oplus B) = \sigma(A) \cup \sigma(B).$$

Proof. Let $\lambda \in \sigma(A \oplus B)$, then $(A \oplus B) - (I_X \oplus I_Y)$ is not invertible, hence $A - \lambda I_X$ is not invertible in $\mathcal{L}(X)$ or $B - \lambda I_Y$ is not invertible in $\mathcal{L}(Y)$, thus $\lambda \in \sigma(A) \cup \sigma(B)$. Hence $\sigma(A \oplus B) \subseteq \sigma(A) \cup \sigma(B)$. Similarly, we obtain that $\sigma(A) \cup \sigma(B) \subseteq \sigma(A \oplus B)$. Consequently, $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$.

More generally, one can see that.

Proposition 6. Let $(X_i)_{1 \leq i \leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$ and let $A_i \in \mathcal{L}(X_i)$. Set $A = \oplus_{i=1}^{n} A_i \in \mathcal{L}(\oplus_{i=1}^{n} X_i)$. Then

$$\sigma(A) = \bigcup_{i=1}^{n} \sigma(A_i)$$

and

$$\rho(A) = \bigcap_{i=1}^{n} \rho(A_i).$$

Now, we define the pseudospectrum of A where $A = \oplus_{i=1}^{n} A_i$ and for all $i \in \{1, \cdots, n\}, A_i \in \mathcal{L}(X_i)$ on the ultrametric Banach space $\oplus_{i=1}^{n} X_i$. We have the following definition.

Definition 16. Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(\oplus_{i=1}^{n} A_i)$ of $\oplus_{i=1}^{n} A_i$ on $\oplus_{i=1}^{n} X_i$ is given by

$$\sigma_\varepsilon(\oplus_{i=1}^{n} A_i) = \sigma(\oplus_{i=1}^{n} A_i) \cup \{\lambda \in \bigcap_{i=1}^{n} \rho(A_i) : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon^{-1}\}.$$

Remark 2. One can see that $\sigma_\varepsilon(\oplus_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} \sigma_\varepsilon(A_i)$.

Proposition 7. Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, then

(i) $\sigma(\oplus_{i=1}^{n} A_i) = \bigcap_{\varepsilon>0} \sigma_\varepsilon(\oplus_{i=1}^{n} A_i)$.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma(\oplus_{i=1}^{n} A_i) \subset \sigma_{\varepsilon_1}(\oplus_{i=1}^{n} A_i) \subset \sigma_{\varepsilon_2}(\oplus_{i=1}^{n} A_i)$.

Proof. (i) By Definition 16, for each $\varepsilon > 0$, $\sigma(\oplus_{i=1}^{n} A_i) \subset \sigma_\varepsilon(\oplus_{i=1}^{n} A_i)$, then $\sigma(\oplus_{i=1}^{n} A_i) \subset \bigcap_{\varepsilon>0} \sigma_\varepsilon(\oplus_{i=1}^{n} A_i)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \sigma_\varepsilon(\oplus_{i=1}^{n} A_i)$, since

$$\bigcap_{\varepsilon>0} \sigma_\varepsilon(\oplus_{i=1}^{n} A_i) = \sigma(\oplus_{i=1}^{n} A_i) \cup \bigcap_{\varepsilon>0} \{\lambda \in \bigcap_{i=1}^{n} \rho(A_i) : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon^{-1}\}$$

and $\bigcap_{\varepsilon>0} \{\lambda \in \bigcap_{i=1}^{n} \rho(A_i) : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon^{-1}\} = \emptyset$ because of for all $i \in \{1, \cdots, n\}$, $(A_i - \lambda I)^{-1}$ are bounded linear operators. Thus $\lambda \in \sigma(\oplus_{i=1}^{n} A_i)$.

(ii) For $0 < \varepsilon_1 < \varepsilon_2$. Let $\lambda \in \sigma_{\varepsilon_1}(\oplus_{i=1}^{n} A_i)$, consequently, $\sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$, hence $\lambda \in \sigma_{\varepsilon_2}(\oplus_{i=1}^{n} A_i)$.

Let $A \in \mathcal{L}(X)$, set $r(A) = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}}$. We have the following lemmas.

Lemma 4. Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, then $r(\oplus_{i=1}^{n} A_i) = \sup_{i\in\{1,\cdots,n\}} r(A_i)$. Furthermore $\sup_{\lambda\in\sigma(\oplus_{i=1}^{n} A_i)} |\lambda| \leq \sup_{\lambda\in\sigma_\varepsilon(\oplus_{i=1}^{n} A_i)} |\lambda|$.

Proof. Since for all $k \in \mathbb{N}$, $(A_1 \oplus \cdots \oplus A_n)^k = A_1^k \oplus \cdots \oplus A_n^k$. Thus

$$\begin{aligned} r(\oplus_{i=1}^{n} A_i) &= \lim_{k\to\infty} \|(A_1 \oplus \cdots \oplus A_n)^k\|^{\frac{1}{k}} \\ &= \lim_{k\to\infty} \|A_1^k \oplus \cdots \oplus A_n^k\|^{\frac{1}{k}} \\ &= \lim_{k\to\infty} \sup_{i\in\{1,\cdots,n\}} \|A_i^k\|^{\frac{1}{k}} \\ &= \sup_{i\in\{1,\cdots,n\}} \lim_{k\to\infty} \|A_i^k\|^{\frac{1}{k}} \\ &= \sup_{i\in\{1,\cdots,n\}} r(A_i). \end{aligned}$$

Since $\sigma(A) \subseteq \sigma_\varepsilon(A)$, then $\sup_{\lambda\in\sigma(\oplus_{i=1}^{n} A_i)} |\lambda| \leq \sup_{\lambda\in\sigma_\varepsilon(\oplus_{i=1}^{n} A_i)} |\lambda|$.

Set $r_\varepsilon(\oplus_{i=1}^n A_i) = \sup\limits_{\lambda\in\sigma_\varepsilon(\oplus_{i=1}^n A_i)} |\lambda|$, we have the following:

Lemma 5. Let $(X_i)_{1\le i\le n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, then $r_\varepsilon(\oplus_{i=1}^n A_i) = \sup\limits_{i\in\{1,\cdots,n\}} r_\varepsilon(A_i)$.

Proof. From Remark 2, $\sigma_\varepsilon(\oplus_{i=1}^n A_i) = \bigcup\limits_{i=1}^n \sigma_\varepsilon(A_i)$. One can see that $r_\varepsilon(\oplus_{i=1}^n A_i) = \sup\limits_{i\in\{1,\cdots,n\}} r_\varepsilon(A_i)$.

We have the following examples.

Example 3. Consider $(A_k)_{1\le k\le n}$ defined on $\mathbb{K}^2$ by

$$A_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \mu_k \end{pmatrix},$$

where $\lambda_k, \mu_k \in \mathbb{K}$ for all $k \in \{1, \cdots, n\}$ and $n \in \mathbb{N}$ is fixed. Then $\sigma(\oplus_{k=1}^n A_k) = \bigcup\limits_{k=1}^n \{\lambda_k, \mu_k\}$ and

$$\sigma_\varepsilon(\oplus_{k=1}^n A_k) = \bigcup_{k=1}^n \{\lambda_k, \mu_k\} \cup \{\lambda \in \mathbb{K} : \sup_{1\le k\le n} \|(\lambda I - A_k)^{-1}\| > \frac{1}{\varepsilon}\}.$$

Example 4. Let F be an ultrametric free Banach space over $\mathbb{K}$ with an orthogonal basis $(e_m)_{m\in\mathbb{N}}$. Let $(A_k)_{1\le k\le n}$ be defined on F by for all $x \in F$ and for each $k \in \{1, \cdots, n\}$, $A_k x = \lambda_k x$. Set $A = \oplus_{k=1}^n A_k$. One can see that

$$\sigma(A) = \bigcup_{k=1}^n \{\lambda_k\}$$

and for all $k \in \{1, \cdots, n\}$ and for each $\lambda \in \rho(A_k)$, $\|(\lambda - A_k)^{-1}\| = \frac{1}{|\lambda-\lambda_k|}$. Hence $\sigma_\varepsilon(A_k) = \{\lambda_k\} \cup B(\lambda_k, \varepsilon)$. Consequently,

$$\sigma_\varepsilon(A) = \bigcup_{k=1}^n \{\lambda_k\} \cup \bigcup_{k=1}^n B(\lambda_k, \varepsilon).$$

We introduce the following definition.

Definition 17. Let $(X_i)_{1\le i\le n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. The condition pseudospectrum $\Lambda_\varepsilon(\oplus_{i=1}^n A_i)$ of $\oplus_{i=1}^n A_i$ on $\oplus_{i=1}^n X_i$ is defined by

$$\Lambda_\varepsilon(\oplus_{i=1}^n A_i) = \sigma(\oplus_{i=1}^n A_i) \cup \{\lambda \in \mathbb{K} : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon^{-1}\},$$

with the convention $\sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| = \infty$ if $\lambda \in \sigma(\oplus_{i=1}^n A_i)$.

Remark 3. It is easy to see that $\bigcup\limits_{i=1}^n \Lambda_\varepsilon(A_i) \subset \Lambda_\varepsilon(\oplus_{i=1}^n A_i)$.

Proposition 8. Let $(X_i)_{1\le i\le n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, then

(i) $\sigma(\oplus_{i=1}^n A_i) = \bigcap\limits_{\varepsilon>0} \Lambda_\varepsilon(\oplus_{i=1}^n A_i)$.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma(\oplus_{i=1}^n A_i) \subset \Lambda_{\varepsilon_1}(\oplus_{i=1}^n A_i) \subset \Lambda_{\varepsilon_2}(\oplus_{i=1}^n A_i)$.

Proof. (i) By Definition 17, for each $\varepsilon > 0$, $\sigma(\oplus_{i=1}^{n} A_i) \subset \Lambda_\varepsilon(\oplus_{i=1}^{n} A_i)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \Lambda_\varepsilon(\oplus_{i=1}^{n} A_i)$ and $\lambda \notin \sigma(\oplus_{i=1}^{n} A_i)$. Using $\lim_{\varepsilon \to \to 0} \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| = \infty$, we get a contradiction.

(ii) For $0 < \varepsilon_1 < \varepsilon_2$. If $\lambda \in \Lambda_{\varepsilon_1}(\oplus_{i=1}^{n} A_i)$, thus for all $\sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)^{-1}\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$, then $\lambda \in \Lambda_{\varepsilon_2}(\oplus_{i=1}^{n} A_i)$.

Let $A \in \mathcal{L}(X)$, set $r(A) = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}}$. We have the following lemmas.

Lemma 6. Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, then $\sup_{\lambda\in\sigma(\oplus_{i=1}^{n} A_i)} |\lambda| \leq \sup_{\lambda\in\Lambda_\varepsilon(\oplus_{i=1}^{n} A_i)} |\lambda|$.

Proof. Since $\sigma(A) \subseteq \Lambda_\varepsilon(A)$, then $\sup_{\lambda\in\sigma(\oplus_{i=1}^{n} A_i)} |\lambda| \leq \sup_{\lambda\in\Lambda_\varepsilon(\oplus_{i=1}^{n} A_i)} |\lambda|$.

We introduce a new definition of the condition pseudospectrum of $\oplus_{i=1}^{n} A_i$ as follows.

Definition 18. Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$ and let $A_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. The condition pseudospectrum $\Lambda'_\varepsilon(\oplus_{i=1}^{n} A_i)$ of $\oplus_{i=1}^{n} A_i$ on $\oplus_{i=1}^{n} X_i$ is

$$\Lambda'_\varepsilon(\oplus_{i=1}^{n} A_i) = \sigma(\oplus_{i=1}^{n} A_i) \cup \{\lambda \in \mathbb{K} : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda I)\| \|(A_i - \lambda I)^{-1}\| > \varepsilon^{-1}\}.$$

Remark 4. (i) It is easy to see that $\Lambda'_\varepsilon(\oplus_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} \Lambda'_\varepsilon(A_i)$.

(ii) $\sigma(\oplus_{i=1}^{n} A_i) = \bigcap_{\varepsilon>0} \Lambda'_\varepsilon(\oplus_{i=1}^{n} A_i)$.

(iii) If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma(\oplus_{i=1}^{n} A_i) \subset \Lambda'_{\varepsilon_1}(\oplus_{i=1}^{n} A_i) \subset \Lambda'_{\varepsilon_2}(\oplus_{i=1}^{n} A_i)$.

(iv) For all $\varepsilon > 0$, we have $\sup_{\lambda\in\sigma(\oplus_{i=1}^{n} A_i)} |\lambda| \leq \sup_{\lambda\in\Lambda'_\varepsilon(\oplus_{i=1}^{n} A_i)} |\lambda|$.

(v) The condition pseudospectum $\Lambda'_\varepsilon(\oplus_{i=1}^{n} A_i)$ of $\oplus_{i=1}^{n} A_i$ gives nice properties than $\Lambda_\varepsilon(\oplus_{i=1}^{n} A_i)$.

We finish with the following example.

Example 5. Consider $(A_k)_{1\leq k\leq n}$ defined on $\mathbb{K}^2$ by

$$A_k = \begin{pmatrix} 0 & \lambda_k \\ \lambda_k & 0 \end{pmatrix}$$

for all $k \in \{1, \cdots, n\}$ and $n \in \mathbb{N}$ is fixed. Then $\sigma(\oplus_{k=1}^{n} A_k) = \bigcup_{k=1}^{n} \{-\lambda_k, \lambda_k\}$ and

$$\Lambda_\varepsilon(\oplus_{k=1}^{n} A_k) = \bigcup_{k=1}^{n} \{-\lambda_k, \lambda_k\} \cup \{\lambda \in \mathbb{K} : \sup_{1\leq k\leq n} \|\lambda I - A_k\| \sup_{1\leq k\leq n} \|(\lambda I - A_k)^{-1}\| > \frac{1}{\varepsilon}\}$$

where for all $k \in \{1, \cdots, n\}$, $\|\lambda - A_k\| = \max\{|\lambda|, |\lambda_k|\}$ and for all $\lambda \in \rho(A_k)$, $\|(\lambda I - A_k)^{-1}\| = \max\left\{\frac{|\lambda|}{|\lambda^2-\lambda_k^2|}, \frac{|\lambda_k|}{|\lambda^2-\lambda_k^2|}\right\}$.

Conflict of Interest

The author declares that there are no conflict of interest.

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https://doi.org/10.31489/2024M4/119-126 *Research article*

Model-theoretic properties of J-non-multidimensional theories

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The issues of utilizing the central type to analyze the theoretical and model properties of the idea of heredity were examined in this research, taking into account both theories and the Jonsson spectrum. Finding solutions to issues related to the enriching language for the fixed Jonsson theory is associated with the problems of heredity of Jonsson theory. Another feature of Jonsson theories was described in the presented article. That is, the conclusion concerning J-non-multidimensional theories was presented in this study. The connection between J-P-stable theories and J-non-multidimensional theories was also characterired. In addition, the main result in the article was considered for the class of semantic pairs.

Keywords: Jonsson theory, semantic model, perfect Jonsson theory, hereditary Jonsson theory, Jonsson spectrum, permissible enrichment, central type, existentially closed model, J-stable theory, semantic pairs, existentially finite cover property, J-non-multidimensional theory.

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Introduction

Although the Model Theory is the youngest science in terms of its development, it has spread its roots and is growing in all directions. It can be seen from the new concepts and findings that are being studied and defined year after year, and from the new methods of researching these findings. Model theorists use various limiting conditions to obtain results concerning incomplete theories. One of the relevant directions in this sense is studying of Jonsson theories. The number and variety of methodologies under development [1–7] indicates the tremendous expansion of the apparatus for analyzing Jonsson theories in recent years.

The difficulty of defining the notion of heredity in Jonsson theory remains unresolved. This problem's relevance is supported by the following significant counterexample: the elementary theory of an algebraically closed field loses its Jonsson character when it is enhanced with a unary predicate. Accordingly, one key model-theoretic challenge for characterizing the hereditary Jonsson theories is the study of model-theoretic features of central types in predicate enrichment.

We will further study the J-non-multidimensional theories. The study of non-multidimensional theories in general begins with the work of S. Shelah [8]. The theory T called non-multidimensional, if there is a bound to the size of families of pairwise orthogonal types. A. Pillay developed a classification of models for ω-stable non-multidimensional theories [9]. And T. Mustafin and T. Nurmagambetov obtained the main results of non-multidimensional theories for superstable theories [10].

The purpose of the article is to show the connection between J-non-multidimensional theories and J-P-stable theories. In general, the scope of study of P-stable theory is wide. For the first time, French mathematician B. Poizat began to study in [11], i.e. he found the conditions for the completeness of elementary pairs. E. Bouscaren [12, 13] further argued that a different class of stable

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theories should be used to address these questions. She showed that elementary pairs theories are complete for stable and superstable theories. In the work of D. Laskar [14] it is said that those theories should be uncountable categorical theories. And in work of T. Nurtazin [15] on the proof for the class of uncountable categorical theories. In [16] T. Mustafin introduces the concept of T^*-stability, which generalizes the well-known fact of λ-stability. In a separate case, the concept of P-stability related to the concept of elementary pairs is studied. In [10; 88–100] obtained a description of P-stability for any superstable theories. E^*-stable theories were introduced by E. A. Palyutin in [17]. The discontinuity requirement, which states that stability, P-stability, and other significant independent conditions are satisfied in the trivial situation, distinguishes this idea from T^*-stability. The types in P-sets of P-stable theories constructed by T. Nurmagambetov and B. Poizat for types in the context of P-models are defined [18], in addition to the definition of types in stable theories. As of right now, A.R. Yeshkeyev's work [19] has yielded several innovations in P-stability for the Jonsson theories, or perfect Jonsson theories.

This paper consists of two sections. In Section 1, we give some basic information on Jonsson theories. In Section 2, we present our results obtained for cosemanticness classes of Jonsson spectrum in permissible enrichment, so-called J-non-multidimensional theories.

1 Basic information concerning Jonsson theories

To set the stage for the major result, let us define several terms and results associated with Jonsson theories that are well known.

Definition 1. [20] A theory $\mathbb{T}$ is called a Jonsson theory, if
1. $\mathbb{T}$ has at least one infinite mode;
2. $\mathbb{T}$ is an inductive theory;
3. $\mathbb{T}$ has the joint embedding property (JEP);
4. $\mathbb{T}$ has the amalgam property (AP).

The main properties and theorems related to Jonsson theory can be found in work [20].

Any inductive theory has a nonempty class of existential closed models, and the class of Jonsson theories is a subset of inductive theories. Consequently, by including more Jonsson theory properties, the description of the class of existential closed models above can be strengthened. One of these characteristics is the power saturation of the semantic model. Such theories are called perfect Jonsson theories. Let's become acquainted with the features of these theories.

Definition 2. [20; 162] A Jonsson theory $\mathbb{T}$ is called a perfect theory, if its semantic model is saturated.

As the examples below (Fig.) make abundantly evident, any Jonsson theory can be perfect; not all Jonssons can be perfect.

One of the striking examples of such a phenomenon is the example of the theory of fields of a fixed characteristic. In this example, the interpretation of an one-place predicate is realized by an existentially closed submodel of the semantic model of this theory. It is well known that an algebraic closed field with the same characteristic will serve as this theory's semantic model. The notion of a hereditary Jonsson theory was defined with the knowledge of such cases. Hereditary Jonsson theories are those in which the qualities of jonssonness are retained, that is, the enriched theory stays Jonsson despite any permissible enrichment of the theory's language. The term "admissibility of enrichment" refers to the preservation of the definability of any new language type with respect to the stability of the enriched theory, where stability is considered in relation to the enrichment framework. The idea of a Jonsson theory's central type was established in order to research hereditary hypotheses.

Concepts of "hereditary" and "permissible enrichment" belongs to professor Yeshkeyev A.R.

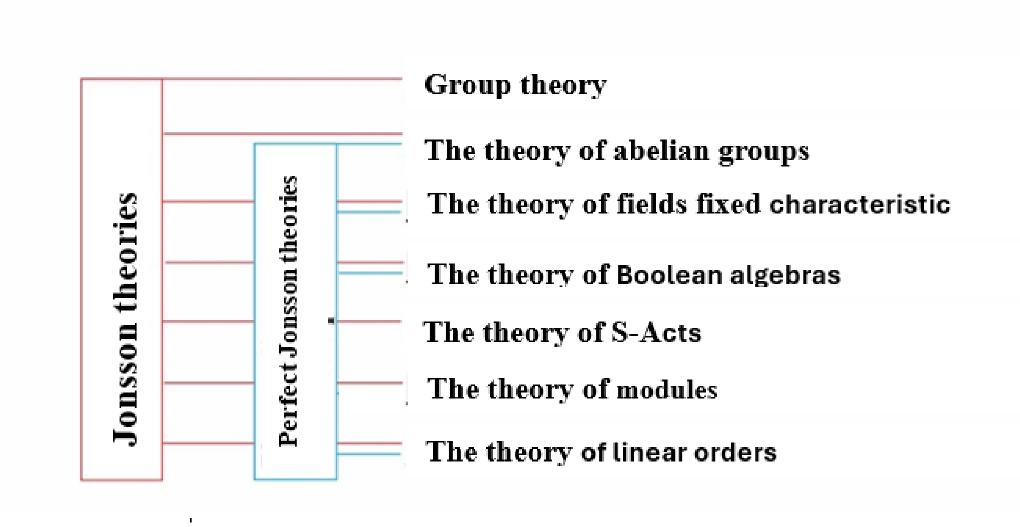

Figure. Examples of Jonsson theories

Definition 3. Let $\overline{\mathbb{T}}$ be an enrichment of Jonsson theory $\mathbb{T}$, $\Gamma = \{P\} \cup \{c\}$, P be some unary predicate symbol and c be a constant symbol. $\overline{\mathbb{T}}$ is called permissible, if any Δ-type in this enrichment is definable within $\mathbb{T}_\Gamma$-stability, where Δ is a subset of L_σ, a Δ-type means that any formula of this type belongs to Δ.

The following definition shows only the necessity of the concept, and the sufficient condition has not yet been defined. Let's give the definition.

Definition 4. A Jonsson theory is called hereditary, if with any of its permissible enrichments, any expansion of it in this enrichment will be Jonsson theory.

The subsequent discussion will solely focus on permissible enrichments, specifically examining those for the Jonsson spectrum that comprise solely of hereditary Jonsson theories [21–26].

Then the $JSp(A) = \{\mathbb{T}|\ A \in Mod\mathbb{T},\ \mathbb{T} - \text{Jonsson theory of the signature } \sigma\}$ is the Jonsson spectrum of the model A.

Definition 5. If two models A, B have the same semantic model, then we will state that they are cosemantic among themselves. Symbolically $A \bowtie B$.

The relation $\bowtie$ is the equivalence relation between models, which generalizes the concept of elementary equivalence. Therefore we consider the factor set $JSp(A)/_{\bowtie}$ for the model A.

The task of describing the properties of heredity of Jonsson theory is significantly complicated by the transition of the theory to a spectrum. We set ourselves the task of studying the cosemanticity classes of a fixed Jonsson spectrum with respect to all the above-mentioned arising questions, their consequences and all possible combinations. In this case, the cosemanticity classes from the considered Jonsson spectrum will, as a rule, be convex, that is, the theories from this class will be convex. Accordingly, the central types will reflect the essence of the permissible enrichment of the considered hereditary Jonsson theory along with the convexity of the obtained theories.

Definition 6. [12] A class $JSp(A)/_{\bowtie}$ is called perfect (hereinafter, $PJSp(A)/_{\bowtie}$), if each class $[\mathbb{T}] \in JSp(A)/_{\bowtie}$ is perfect, $[\mathbb{T}]$ is called perfect, if $C_{[\mathbb{T}]}$ is a saturated model.

Note that for perfect Jonsson theories $\mathbb{T} \in [\mathbb{T}]$ their central types are equal.

The study in article heavily relies on the central type. The law of first-order logic, which states that a constant can be substituted by a variable when the constant was not belong of the theory's language prior to enrichment, is used in its construction. Following this substitution, the formal expression that was formerly a sentence becomes a formula. The central type is a complete 1-type that results, if this formal expression represents complete theory.

The concept of "central type" was first introduced in 2008 by professor A.R. Yeshkeyev. We enter the central type according to the following algorithm:

1. We enrich the σ signature, i.e. $\sigma_\Gamma = \sigma \bigcup \Gamma$, where $\Gamma = \{P\} \bigcup \{c\}$.

2. We write the enriched theory accordingly $\overline{\mathbb{T}} = Th_{\forall\exists}\left(\mathcal{C}, c_a\right)_{a \in C} \bigcup Th_{\forall\exists}(E_{\mathbb{T}}) \bigcup \{P\left(c\right)\} \bigcup \{"P \subseteq "\}$, in the language of the signature σ_Γ, the interpretation of the symbol P is an existentially closed submodel, as expressed by the infinite set of sentences $\{"P \subseteq "\}$.

3. The solution to the equation $P\left(C\right) = M \in E_{\mathbb{T}}$ in the signature $\bar{\sigma}$ is the interpretation of the symbol P.

4. We take into account all complements in the $\bar{\sigma}$ signature of the $\mathbb{T}$ theory. The center $\bar{\mathbb{T}}^*$ of the theory $\bar{\mathbb{T}}$ is one of the complements of the theory $\bar{\mathbb{T}}$ exhibited since the theory $\mathbb{T}$ is a Jonsson theory.

5. We restrict the $\bar{\sigma}$ signature to the signature $\sigma \cup \{P\}$.

6. The constant c is not included in the new signature because of constraints.

7. We substitute any variable, such x, for the constant symbol in accordance with the law of first-order logic.

8. Accordingly, we designate the theory $\bar{\mathbb{T}}^*$ by p^c, which is a complete type 1 in the admissible enrichment.

This enrichment is denoted by $\odot$.

The concept of the Jonsson set, which is a definable set with the aid of an existential formula and whose definable closure defines some existentially closed submodel of the semantic model under consideration, is useful for manipulating the properties of elements and subsets of a semantic model.

Definition 7. [20]. In the theory $\mathbb{T}$, a set X is referred to as a Jonsson set, if it meets the following criteria:

1) X is a definable subset of $C_{\mathbb{T}}$, where $C_{\mathbb{T}}$ is a semantic model of the theory $\mathbb{T}$;

2) $dcl(X)$ is a universe of existentially closed submodel C_T, where $dcl(X)$ is definable closure of X.

2 *The connection between J-P-stable theory and J-non-multidimensional theory*

From the main result in paper [10], for superstable theories, the notions of P-stability and P-superstability and non-multidimensional theories for complete theories coincide.

Theorem 1. [10; 90] Let T be a superstable theory. Then the following conditions are equivalent:

1) the theory T is non-multidimensional;

2) the theory T is P-superstable;

3) the theory T is P-stable.

Now a stable theory is superstable iff every type does not fork over a finite set. A generalization of stability for Jonsson theories is proved in the work [19].

We'll also talk about the idea that "type p does not fork over" in relation to Theorem 8 from [5].

Definition 8. Let p be complete $\exists$-type over A, A is a Jonsson subset of C. Then p is J-stationary over A, if

1) p does not fork over A;

2) p has a unique consistent extension that does not fork over A.

Definition 9. 1) A is a Jonsson subset of C, if $p(\bar{x}_1), q(\bar{x}_2)$ are complete $\exists$-types over A. If and only if $p(\bar{x}_1) \cup q(\bar{x}_2)$ is a $\exists$-complete type (over A), then p is J-weakly orthogonal to q.

2) For each p_1 and p_2, let us consider two $\exists$-complete or J-stationary types, respectively. If A is the universe of a $\exists_1$-saturated model, then p_1 is J-orthogonal to p_2, and q_1 is weakly J-orthogonal to q_2, where q_1, q_2 are any J-nonforking extensions of p_1 and p_2 over A, respectively.

Definition 10. Given a $C_{\mathbb{T}}$ semantic model and a Jonsson theory $\mathbb{T}$, let A be a Jonsson subset of the model. If p is orthogonal to any complete $\exists$-type over A, then p is considered J-multidimensional and a $\exists$-complete type. If $\mathbb{T}$ has a J -multidimensional type, then it is called a J-multidimensional theory. In the absence of this, the theory $\mathbb{T}$ is called the J-non-multidimensional theory or the J-restricted dimension theory.

Definition 11. A class $[\mathbb{T}]$ is called J-non-multidimensional, if every theory in this class does not have a J-multidimensional type.

We consider the class of semantic pairs as the main result [27].

Definition 12. [27; 188]. An existentially closed pair $(C_{\mathbb{T}}, M)$ is a semantic pair, if the following conditions hold:

1) M is $|\mathbb{T}|^+$-$\exists$-saturated (it means that it is $|\mathbb{T}|^+$-saturated restricted up to existential types);

2) for any tuple $\bar{a} \in C$ each its $\exists$-type in sense of $\mathbb{T}$ over $M \cup \{\bar{a}\}$ is satisfiable in C.

Let $[\nabla]$ be $\exists$-complete and J-λ-stable class of Jonsson theories, $\mathcal{C}_{[\nabla]}$ be a semantic model of the theory $[\nabla]$, $\overline{[\nabla]} = [\nabla]$ in the enrichment of $\odot$, $\overline{[\nabla]}^*$ is the center of the $\overline{[\nabla]}$, $p, q \in S(\overline{[\nabla]}^*)$, $\nabla' = Th_{\forall\exists}(\mathcal{C}, \mathcal{M})$.

Theorem 2. [27; 189]. $(\mathcal{C}_{[\nabla]}, M_1)$ and $(\mathcal{C}_{[\nabla]}, M_2)$ are two semantic pairs, $\bar{a}$ and $\bar{b}$ tuples taken from each of them, M_1, $M_2 \in E_{[\nabla]}$. Then $(\mathcal{C}_{[\nabla]}, M_1) \equiv_{\forall\exists} (\mathcal{C}_{[\nabla]}, M_2)$, if their central types are equivalent by the fundamental order $\overline{\nabla}^*$.

Definition 13. [27; 187] Let T be the Jonsson L-theory and $f(\bar{x}, \bar{y})$ be an $\exists$ formula of L language. If for any arbitrary large n exists $\bar{a}^0, \ldots, \bar{a}^{n-1}$ in some existentially closed model of T and $\bar{a}^0, \ldots, \bar{a}^{n-1}$ satisfies $\neg(\exists \bar{x}) \bigwedge_{k<n} f(\bar{x}, \bar{a}^k)$ and for any $l < n$ $\neg(\exists \bar{x}) \bigwedge_{k<n} f(\bar{x}, \bar{a}^k)$, then $f(\bar{x}, \bar{y})$ is said to have e.f.c.p. (existentially finite cover property).

Theorem 3. [27; 189]. Let $[\nabla]$ be a hereditary, $\exists$-complete perfect, and J-λ-stable class of Jonsson theories. Then the following conditions are equivalent:

1) $\overline{[\nabla]}^*$ does not have e.f.c.p.

2) Any $|\mathbb{T}|^+$-saturated model from ∇' is a semantic pair.

3) Two tuples $\bar{a}$ and $\bar{b}$ from the models of $\overline{[\nabla]}^*$ have the same type if and only if their central types in sense of $\overline{[\nabla]}^*$ over $\mathcal{M}$ are equivalent by fundamental order $\overline{[\nabla]}^*$.

4) Two tuples $\bar{a}$ and $\bar{b}$ from models of ∇' and that are in $\mathcal{C}_{[\nabla]} \setminus M$ have the same central types in the sense of $\overline{[\nabla]}$ if and only if they have the same central types in the sense of $\overline{[\nabla]}^*$.

Theorem 4. Let $[\nabla]$ be a hereditary, $\exists$-complete perfect, and J-λ-stable class of Jonsson theories. If $\overline{[\nabla]}^*$ does not have e.f.c.p. and λ-stable class, then $[\nabla]'$ is J-λ-stable and does not have e.f.c.p.

Let be $K = \{(C, M) | M \preceq_{\exists_1} C, (C, M)\text{is semantic pair}\}$, $JSp(K) = \{\Delta | \Delta$ is Jonsson theory, $\Delta = Th_{\forall\exists}(\mathcal{C}, \mathcal{M})$, where $(\mathcal{C}, \mathcal{M}) \in K\}$, let $[\Delta] \in JSp(K)/_{\bowtie}$. Let $[\Delta]$ be a $\exists$-complete and J-λ-stable class of Jonsson theories, the class $\overline{[\Delta]}$ be $[\Delta]$ in an permissible enrichment $\odot$, let $\overline{[\Delta]}^*$ be the center of the class $\overline{[\Delta]}$.

Theorem 5. Let $\mathbb{T}$ be a perfect, J-λ-stable $\exists$-complete Jonsson theory, K be the class of J-beautiful pairs of T. Let $[\Delta] \in JSpK/_{\bowtie}$ be a complete for $\exists$-sentences. Then the following conditions are equivalent:

1) the class $\overline{[\Delta]}^*$ is non-multidimensional (in the classical sense);

2) the class $\overline{[\Delta]}$ is J -non-multidimensional.

Proof. Let us prove implication 2)$\Rightarrow$ 1). Suppose $\overline{[\Delta]}$ is a J-non-multidimensional class. That is, not every theory in this class is J-multidimensional. $p \perp A$, where A is the φ-Jonsson set of the semantic model $C_{[\Delta]}$, is a J-multidimensional orthogonal $\exists$-type, if the theory is not J-multidimensional. $[\Delta]$ is a

J-stable and $\exists$ -complete class according to the theorem's condition, which is equivalent to the Morley rank condition. Furthermore, a measure of forking is Morley rank. Furthermore, the Lindenbaum theorem states that each theory in the class $\overline{[\Delta]}$ can be extended to the maximum, or to complete theories, because the theories in this class are incomplete.

The proof of 1)$\Rightarrow$ 2) is trivial.

Theorem 6. If the class $[\Delta]$ is J-P-λ-stable, then it is J-non-multidimensional and does not have e.f.c.p.

Proof. If $[\Delta]$ is J-non-multidimensional, then every elementary extension of a semantic pair is a semantic pair. Indeed, let (C, M) be a semantic pair, that is, for each dimension the cardinality of M is at least $|\mathbb{T}|^+$. Then, by Theorems 3 and 4, $[\Delta]$ does not have e.f.c.p.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2024M4/127-137 *Research article*

Geometric properties of the Minkowski operator

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This article is about Minkowski difference of sets, which is one of the Minkowski operators. The necessary and sufficient conditions for the existence of the Minkowski difference of given regular polygons in the plane were derived. The method of finding the Minkowski difference of given regular tetrahedrons in the Euclidean space $\mathbb{R}^3$ was explained. At the end of the article, the obtained results were summarized and a geometric method for finding the Minkowski difference of the convex set M and compact set N given in $\mathbb{R}^n$ was shown. The theory of foliations was applied to find the Minkowski difference of sets. New geometric concepts such as "dense embedding" and "completely dense embedding" were introduced. An important geometric property of the Minkowski operator was introduced and proved as a theorem.

Keywords: Minkowski sum, Minkowski difference, orthogonal projection, foliation, dense embedding in a foliation.

2020 Mathematics Subject Classification: 52B11, 53C12.

Introduction

Not all operations on sets may have a geometric meaning. For sets with elements of any kind, we can perform operations such as union, intersection, and difference.

So, the above operations do not necessarily mean geometrically in some cases. The Minkowski sum and difference on the sets were introduced precisely for the purpose of solving geometric problems, and these operations depend on the nature of the elements that make up the sets. That is why Minkowski operations are not performed for the sets given in the above example.

Definitions and some properties of Minkowski operators are presented in works [1, 2]. Among the known scientific works, the Minkowski difference was first used in [3] to solve the problem of pursuit in differential games under the name "geometric difference". Later, in other works such as [4, 5], various properties of this "geometric difference" were studied, and with their help, the conditions for solving the problem of chasing were eased. Also, many geometric properties of Minkowski difference and sum are presented in [6–9]. To date, several scientific researches have been conducted to find algorithms for calculating the Minkowski sum. Y. Yan, D.S. Chirikjian, A. Baram, E. Fogel, D. Halperin, M. Hemmer, S. Morr, O. Eduard, M. Sharir, A. Kaul, M.A. O'Connor, V. Srinivasan, S. Das, S.D. Ranjan, S. Sarvottamananda, W. Cox, L. While, M. Reynolds and other scientists obtained fundamental results on the calculation of the Minkowski sum of polygons in the plane [10–15].

Finding the Minkowski difference of sets is more complicated than finding their Minkowski sum. There are also not many works on finding the Minkowski difference of given sets [16, 17]. Several properties and calculation methods of the Minkowski difference are presented in the works of specialists such as L.A. Tuan, L. Yang, H. Zhang, J.B. Jeannin, N. Ozay, Y.T. Feng, Y. Tan, Y. Zhang, W. Qilin [18–21]. However, so far, the conditions for the Minkowski difference of an arbitrary given set to be empty or non-empty have not been obtained.

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The theory of foliation is one of the developing branches of modern geometry, and it has applications to many areas of geometry [22–28]. In summarizing the obtained results in this article, the foliation theory was also used. Through new geometrical concepts, an efficient method for finding the Minkowski difference of given compact sets in $\mathbb{R}^n$ has been created.

This article presents important geometric properties of the Minkowski operator and geometric ways to find the Minkowski difference of some sets using these properties. In this article, we solved the following problems:

1) a new geometric method and exact formula for finding the Minkowski difference of given regular polygons in the plane $\mathbb{R}^2$;
2) finding the Minkowski difference of two given regular tetrahedrons in the Euclidean space $\mathbb{R}^3$;
3) a new geometric property for finding the Minkowski difference of arbitrary sets;
4) applying foliation theory to finding the Minkowski difference.

1 Research Methodology

Definition 1. Let the sets A and B be non-empty sets of the n dimensional Euclidean space $\mathbb{R}^n$. Their Minkowski sum is the set of points formed by adding each point of set A to each point of set B, i.e.

$$A + B = \{c \in \mathbb{R}^n : c = a + b,\ a \in A,\ b \in B\}.$$

Using this introduced operation, the Minkowski difference of two sets is defined as follows.

Definition 2. Let the sets A and B be non-empty sets of the n dimensional Euclidean space $\mathbb{R}^n$. The following set is called their Minkowski difference:

$$D = A \overset{*}{-} B = \{d \in \mathbb{R}^n : d + B \subset A\}.$$

Definition 3. The Minkowski operators of a multi-valued mapping $G : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ are the operators $A_G : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ and $B_G : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ given by the formulas

$$A_G S = \bigcup_{x \in S} (x + G(x)),$$

$$B_G S = \mathbb{R}^n \backslash (A_G(\mathbb{R}^n \backslash S)),$$

for any set S.

If, in particular, we take the multi-valued mapping G to be constant $G(x) = G_0$ for all $x \in S$, the Minkowski operators correspond to Minkowski sum and difference, respectively:

$$A_G S = S + G_0, \qquad B_G S = S \overset{*}{-} (-G_0).$$

Minkowski sum and Minkowski difference have been used to obtain sufficient conditions for ending the game in differential games [3–5]. Today, the approximate calculation of Minkowski sum and difference takes an important place in solving practical problems with the help of differential games. At the same time, it is one of the most important issues to evaluate the Minkowski difference from below and above in theoretical studies.

Minkowski operator were first applied to the study of differential games in the works of L.S. Pontryagin [3, 4]. He called this operator geometric difference and marked it as ($\overset{*}{-}$). In [17], a necessary and sufficient condition for the Minkowski difference of two squares to be non-empty was obtained. Formulas for calculating Minkowski differences are also presented in these works.

2 Minkowski Difference of Regular Polygons

On the Euclidean plane $\mathbb{R}^2$, let regular n-sided polygons P^A and P^B be given by vertices $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$, respectively. Using these points, we can express vectors corresponding to the sides of regular polygons P^A and P^B:

$$\overrightarrow{A_1A_2} = \vec{a}_1, \overrightarrow{A_2A_3} = \vec{a}_2, ..., \overrightarrow{A_nA_1} = \vec{a}_n,$$
$$\overrightarrow{B_1B_2} = \vec{b}_1, \overrightarrow{B_2B_3} = \vec{b}_2, ..., \overrightarrow{B_nB_1} = \vec{b}_n.$$

Theorem 1. In order for the Minkowski difference $P^A \overset{*}{-} P^B$ of regular polygons P^A and P^B given on the Euclidean plane $\mathbb{R}^2$ to be non-empty, the following relation is necessary and sufficient:

$$\frac{|\vec{a}_1|}{2\tan\frac{\pi}{n}} \geq \frac{\left|\vec{b}_1\right|}{2\sin\frac{\pi}{n}} \cdot \cos\left(\frac{\pi}{n} - \alpha_i\right). \tag{1}$$

Here $\alpha_i = \min\limits_{i=\overline{1,n}} \left\{ \arccos\left(\frac{\langle \vec{a}_1, \vec{b}_i \rangle}{|\vec{a}_1|\left|\vec{b}_i\right|}\right)\right\}$ is the smallest angle between vectors $\vec{a}_1$ and $\vec{b}_i, i = \overline{1,n}$.

Proof. Since P^A is a regular polygon, the centers of the circumcircle and incircles of this polygon are at the same point. Let's denote this point as O^A. In the same way, we mark the center of circumcircle and incircles of the polygon P^B as O^B. $P^A \overset{*}{-} P^B \neq \varnothing$ means that the set P^B can be nested inside the set P^A. For this, we move the set P^B parallel until the point O^B falls on the point O^A, that is, we move the set P^B parallel along the vector $\overrightarrow{O^BO^A}$. There can be two cases.

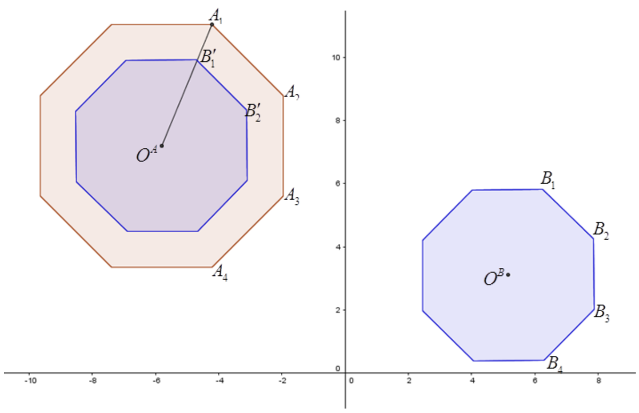

Figure 1. The Minkowski difference of regular polygons with parallel sides

In the first case, it can be $\vec{a}_1 \uparrow\uparrow \vec{b}_1$, $\vec{a}_2 \uparrow\uparrow \vec{b}_2, ..., \vec{a}_n \uparrow\uparrow \vec{b}_n$ (Fig. 1). In such a situation, the images of points $B'_1, B'_2, ..., B'_n$ formed by parallel displacement of points $B_1, B_2, ..., B_n$ along vector $\overrightarrow{O^BO^A}$ will be located on straight lines O^AA_i, $i = \overline{1,n}$. In order for the points $B'_1, B'_2, ..., B'_n$ to belong to the regular polygon P^A (here, the points inside the polygon are also considered to belong to the polygon), it is necessary and sufficient to satisfy the relation

$$\left|O^AA_i\right| \geq \left|O^AB'_i\right|, i = \overline{1,n}. \tag{2}$$

The length of the segments $O^AB'_i$, $i = \overline{1,n}$ is equal to the radius of the circumcircle of the P^B polygon, i.e

$$\left|O^AB'_i\right| = \frac{\left|\vec{b}_1\right|}{2\sin\frac{\pi}{n}}, i = \overline{1,n}. \tag{3}$$

The length of the segment $O^A A_i$, $i=\overline{1,n}$ is equal to the radius of the circumcircle of polygon P^A, but if we express it by the radius of the incircle of the polygon P^A, it will be in the form of

$$\left|O^A A_i'\right| = \frac{|\vec{a}_1|}{2\tan\frac{\pi}{n}} \cdot \frac{1}{\cos\frac{\pi}{n}},\ i=\overline{1,n}. \tag{4}$$

Since $\vec{a}_1 \uparrow\uparrow \vec{b}_1$, follows that $\alpha_i = \min\limits_{i=\overline{1,n}} \left\{\arccos\left(\frac{\langle\vec{a}_1,\vec{b}_i\rangle}{|\vec{a}_1||\vec{b}_i|}\right)\right\} = 0$. From this we can write equation(4) as

$$\left|O^A A_i'\right| = \frac{|\vec{a}_1|}{2\tan\frac{\pi}{n}} \cdot \frac{1}{\cos\left(\frac{\pi}{n}-\alpha_i\right)},\ i=\overline{1,n}. \tag{5}$$

If we put equations (5) and (3) to relation (2), condition (1) is obtained.

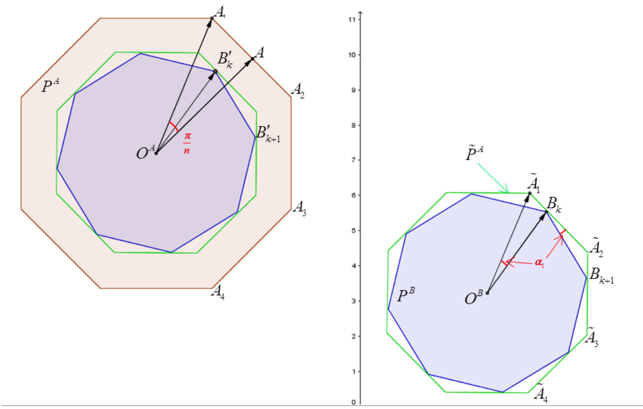

Figure 2. The Minkowski difference of regular polygons with corresponding sides not parallel

In the second case, relations $\vec{a}_i \nparallel \vec{b}_j;\ \ i,\ j=\overline{1,n}$ are appropriate, that is, none of the sides of the polygons P^A and P^B are parallel to each other (Fig. 2). In studying this situation, we must first determine the smallest angle between the vectors $\vec{a}_1$ and $\vec{b}_i$, $i=\overline{1,4}$ and we denote this angle as α_i and calculate it as follows

$$\alpha_i = \min_{i=\overline{1,n}} \left\{\arccos\left(\frac{\left\langle\vec{a}_1,\vec{b}_i\right\rangle}{|\vec{a}_1|\left|\vec{b}_i\right|}\right)\right\}.$$

Suppose this angle is the angle between the vector $\overrightarrow{A_1A_2}$ and the vector $\overrightarrow{B_kB_{k+1}}$, $k=\overline{1,n}(B_{n+1}=B_1)$. In that case, we construct the vector $\overrightarrow{O^AA}$, whose beginning is at the point O^A, and whose end is at the point A, the middle of the segment A_1A_2. This vector forms an angle $\frac{\pi}{n}-\alpha_i$, $i=\overline{1,n}$ with the vector $\overrightarrow{O^AB_k'}$, whose beginning is at point O^A and whose end is at point B_k'. In order for the points to belong to the regular polygon P^A, it is necessary and sufficient that the length of the orthogonal projection of the vector $\overrightarrow{O^AB_k'}$ onto the vector $\overrightarrow{O^AA}$ is not greater than the length of the vector $\overrightarrow{O^AA}$ (Fig. 3), i.e

$$\left|\overrightarrow{O^AA}\right| \geq \left|\overrightarrow{O^AB_k'}\right| \cdot \cos\left(\frac{\pi}{n}-\alpha_i\right). \tag{6}$$

The length of the vector $\overrightarrow{O^AA}$ is equal to the radius of the incircle of the regular polygon P^A,

$$\left|\overrightarrow{O^AA}\right| = \frac{|\vec{a}_1|}{2\tan\frac{\pi}{n}}. \tag{7}$$

The length of the vector $\overrightarrow{O^A B'_k}$ is equal to the radius of the circumcircle of the regular polygon P^B,

$$\left|\overrightarrow{O^A B'_k}\right| = \frac{\left|\vec{b}_1\right|}{2\sin\frac{\pi}{n}}. \tag{8}$$

If we put equations (8) and (7) to relation (6), condition (1) is obtained. This completes the proof.

3 Minkowski Difference of Regular Tetrahedrons

We know that a polyhedron is called a regular polyhedron, if all its faces are congruent regular polygons and all dihedral angles are also congruent. Since at least three edges of the polyhedron pass through each vertex, the sum of all plane angles at that end is less than 2π. A regular tetrahedron is a pyramid with all faces consisting of equilateral triangles, and it has 4 vertices, 4 faces and 6 edges. The spheres drawn inside and outside a regular tetrahedron have their centers at the same point. To define a tetrahedron in a three-dimensional Euclidean space, it is enough to give the coordinates of its vertices.

Let's say that the points corresponding to the vertices of the tetrahedron T^A are given by $A_i = \left\{\alpha_i^1, \alpha_i^2, \alpha_i^3\right\}$, $i = \overline{1,4}$ coordinates, and the points corresponding to the vertices of the tetrahedron T^B are given by $B_i = \left\{\beta_i^1, \beta_i^2, \beta_i^3\right\}$, $i = \overline{1,4}$ coordinates. Then the coordinate of the center of the circumsphere and insphere of the tetrahedron T^A is in the form

$$O^A = \{a_1, a_2, a_3\}, \quad a_j = \frac{1}{4}\sum_{i=1}^{4} \alpha_i^j, \quad j = \overline{1,3}.$$

Similarly, the coordinate of the center of the circumsphere and insphere of the tetrahedron T^B is also in the form

$$O^B = \{b_1, b_2, b_3\}, \quad b_j = \frac{1}{4}\sum_{i=1}^{4} \beta_i^j, \quad j = \overline{1,3}.$$

We denote the vectors starting at point O^A and ending at the points where the medians of the faces of the tetrahedron T^A intersect as $\vec{r}_i^A$, $i = \overline{1,4}$ and the coordinates of these vectors are in the form

$$\vec{r}_i^A = \frac{1}{3}\{a_1 - \alpha_i^1, a_2 - \alpha_i^2, a_3 - \alpha_i^3\}, \quad i = \overline{1,4}.$$

The lengths of these vectors are the same and equal to the radius of the insphere of the tetrahedron T^A, i.e.

$$\left|\vec{r}_i^A\right| = \frac{\sqrt{6}}{12}\left|\vec{a}_1\right|, \;\; i = \overline{1,4}.$$

Where $\vec{a}_1 = \overrightarrow{A_1 A_2}$ and represents the vector corresponding to the edge of the tetrahedron T^A.

Let's denote the vectors starting at O^B and ending at points B_i, $i = \overline{1,4}$ as $\vec{R}_i^B$, $i = \overline{1,4}$ respectively, and the coordinates of these vectors are in the form

$$\vec{R}_i^B = -\{b_1 - \beta_i^1, b_2 - \beta_i^2, b_3 - \beta_i^3\}, \quad i = \overline{1,4}.$$

The lengths of these vectors are equal to the radius of the circumsphere of the tetrahedron T^B:

$$\left|\vec{R}_i^B\right| = \frac{\sqrt{6}}{4}\left|\vec{b}_1\right|, \;\; i = \overline{1,4},$$

where $\vec{b}_1 = \overrightarrow{B_1 B_2}$ and represents the vector corresponding to the edge of the tetrahedron T^B. By α we denote the smallest angle between $\vec{r}_i^A$, $i = \overline{1,4}$ vectors and $\vec{R}_i^B$, $i = \overline{1,4}$ vectors.

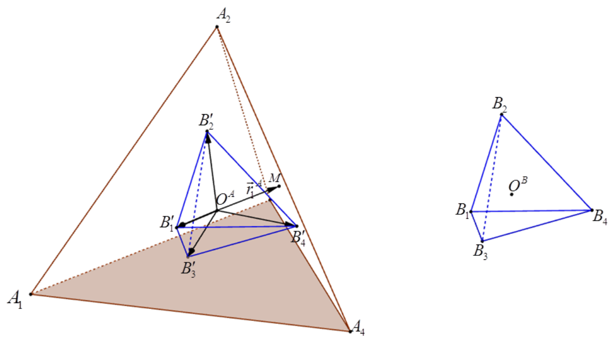

Figure 3. Minkowski difference of tetrahedrons

Theorem 2. In order for the Minkowski difference $T^A \overset{*}{-} T^B$ of regular tetrahedrons T^A and T^B given in Euclidean space $\mathbb{R}^3$ to be non-empty, the following relation is necessary and sufficient:

$$|\vec{a}_1| \geq 3|\vec{b}_1| \cos\alpha. \tag{9}$$

Proof. To calculate the difference $T^A \overset{*}{-} T^B$, we move the tetrahedron T^B parallel to the vector $\overrightarrow{O^B O^A}$. Let us denote the images of points $B_i = \left\{\beta_i^1, \beta_i^2, \beta_i^3\right\}$, $i = \overline{1,4}$ in this parallel displacement as B_i', $i = \overline{1,4}$ respectively (Fig. 3). In order for the difference $T^A \overset{*}{-} T^B$ not to be empty, these points must lie inside the tetrahedron T^A or at most on its faces.

Let the points B_i', $i = \overline{1,4}$ lie on the faces of the tetrahedron T^A. The radius of the insphere of the tetrahedron T^A drawn from the point O^A to the face formed by the vertices $A_2 = \left\{\alpha_2^1, \alpha_2^2, \alpha_2^3\right\}$, $A_3 = \left\{\alpha_3^1, \alpha_3^2, \alpha_3^3\right\}$, $A_4 = \left\{\alpha_4^1, \alpha_4^2, \alpha_4^3\right\}$ of the tetrahedron T^A falls on the point where the medians of the triangle $\triangle\ A_2A_3A_4$ intersect and is perpendicular to this face. Let's designate the vector corresponding to this radius as $\vec{r}_1^A$, its coordinate will be in the form

$$\vec{r}_1^A = \frac{1}{3}\{a_1 - \alpha_1^1, a_2 - \alpha_1^2, a_3 - \alpha_1^3\}.$$

The length of the orthogonal projection of all vectors starting from O^A and ending at points lying on the face $A_2A_3A_4$ onto the vector $\vec{r}_1^A$ is equal to $\left|\vec{r}_1^A\right|$. Hence, if any point B_i', $i = \overline{1,4}$ belongs to face $A_2A_3A_4$, equality

$$proj_{\vec{r}_1^A} \overrightarrow{O^A B_i'} = \left|\vec{r}_1^A\right|, \quad i = \overline{1,4} \tag{10}$$

holds. Points B_i', $i = \overline{1,4}$ can also be located inside the tetrahedron T^A, so we generalize equation (10) and write it in the form

$$proj_{\vec{r}_1^A} \overrightarrow{O^A B_i'} \leq \left|\vec{r}_1^A\right|, \quad i = \overline{1,4}. \tag{11}$$

We can write the same relation for other faces of the tetrahedron T^A:

$$\begin{aligned} proj_{\vec{r}_2^A} \overrightarrow{O^A B_i'} &\leq \left|\vec{r}_2^A\right|, \quad i = \overline{1,4}, \\ proj_{\vec{r}_3^A} \overrightarrow{O^A B_i'} &\leq \left|\vec{r}_3^A\right|, \quad i = \overline{1,4}, \\ proj_{\vec{r}_4^A} \overrightarrow{O^A B_i'} &\leq \left|\vec{r}_4^A\right|, \quad i = \overline{1,4}. \end{aligned} \tag{12}$$

Summarizing relations (11) and (12), we can write as follows

$$proj_{\vec{r}_j^A} \overrightarrow{O^A B_i'} \leq \left|\vec{r}_j^A\right|, \quad i = \overline{1,4}, \quad j = \overline{1,4}. \tag{13}$$

We know that the lengths of vectors $\overrightarrow{O^A B_i'}$ are the same and equal to the radius of the circumsphere of the tetrahedron T^B. $\vec{r}_j^A$ vectors have the same length and are equal to the radius of the insphere of the tetrahedron T. Based on these, we write relation (13) in form (9), were α is the smallest of the angles between vectors $\vec{r}_j^A, j = \overline{1,4}$ and vectors $\overrightarrow{O^A B_i'}$, $i = \overline{1,4}$. Because the cosine of a smaller angle is greater than the cosine of a larger angle. This means that if relation (9) holds for the smallest angle, it holds for the rest of the angles as well. Therefore, (9) is considered a necessary and sufficient condition for the relation $T^A \overset{*}{-} T^B$ not to be empty.

During the proof of the theorem, we derived the algorithm for finding the Minkowski difference of two tetrahedrons given by their vertices in the Euclidean space $\mathbb{R}^3$. According to it, the following should be done in sequence:

1) Let's say that the points corresponding to the vertices of the tetrahedron T^A are given by $A_i = \left\{\alpha_i^1, \alpha_i^2, \alpha_i^3\right\}$, $i = \overline{1,4}$ coordinates, and the points corresponding to the vertices of the tetrahedron T^B are given by $B_i = \left\{\beta_i^1, \beta_i^2, \beta_i^3\right\}$, $i = \overline{1,4}$ coordinates. First of all we determine the Minkowski difference of tetrahedrons T^A and T^B is not empty. For this we check relation (10) according to the above theorem. The numbers $|\vec{a}_1|$ and $\left|\vec{b}_1\right|$ in relation (10) are lengths of vectors $\overrightarrow{A_1 A_2}$ and $\overrightarrow{B_1 B_2}$ respectively, and they are founded by following equality:

$$|\vec{a}_1| = \left|\overrightarrow{A_1 A_2}\right| \sqrt{\left(\alpha_2^1 - \alpha_1^1\right)^2 + \left(\alpha_2^2 - \alpha_1^2\right)^2 + \left(\alpha_2^3 - \alpha_1^3\right)^2},$$

$$\left|\vec{b}_1\right| = \left|\overrightarrow{B_1 B_2}\right| \sqrt{\left(\beta_2^1 - \beta_1^1\right)^2 + \left(\beta_2^2 - \beta_1^2\right)^2 + \left(\beta_2^3 - \beta_1^3\right)^2}.$$

2) Suppose that as a result of the check, equality $|\vec{a}_1| = 3|\vec{b}_1| \cos\alpha$ is satisfied. This means that difference $T^A \overset{*}{-} T^B$ consists only one point and this point is in the form $O^A - O^B$.

3) Suppose that as a result of the check, relation $|\vec{a}_1| > 3|\vec{b}_1| \cos\alpha$ is satisfied. In this case to calculate the difference $T^A \overset{*}{-} T^B$, we construct a tetrahedron $\tilde{T}^B$ such that, the edges are parallel to the edges of the tetrahedron T^A, and the vertices of the tetrahedron T^B lie on the faces of this tetrahedron. Such a tetrahedron is unique, the center of the insphere of this is at point O^B and the radius is equal to $\frac{\sqrt{6}}{4}\left|\vec{b}_1\right| \cdot \cos\alpha$. If we designate the vertices of the tetrahedron $\tilde{T}^B$ as $\tilde{B}_i$, $i = \overline{1,4}$ the directions of the vectors $\overrightarrow{O^B \tilde{B}_i}$, $i = \overline{1,4}$ are the same as the directions of the vectors $\overrightarrow{O^A A_i}$, $i = \overline{1,4}$, and their lengths are equal to the radius of the circumsphere of the tetrahedron $\tilde{T}^B$. Since the radius of the circumsphere of the regular tetrahedron is three times longer than the radius of its insphere, the lengths of the vectors $\overrightarrow{O^B \tilde{B}_i}$, $i = \overline{1,4}$ are equal to the number $\frac{3\sqrt{6}}{4}\left|\vec{b}_1\right| \cdot \cos\alpha$. The coordinates of the vectors $\overrightarrow{O^A A_i}$, $i = \overline{1,4}$ are as follows:

$$\overrightarrow{O^A A_i} = \left\{\alpha_i^1 - a_1, \alpha_i^2 - a_2, \alpha_i^3 - a_3\right\}, \quad i = \overline{1,4}.$$

From these we can find the coordinates of vectors $\overrightarrow{O^B \tilde{B}_i}, i = \overline{1,4}$:

$$\overrightarrow{O^B \tilde{B}_i} = M \cdot \overrightarrow{O^A A_i}, \quad i = \overline{1,4}, \quad M = \frac{3\left|\vec{b}_1\right| \cdot \cos\alpha}{|\vec{a}_1|}.$$

Using these vectors, we find the points $\tilde{B}_i$, $i = \overline{1,4}$ the vertices of the tetrahedron $\tilde{T}^B$:

$$\tilde{B}_i = M\left\{\alpha_i^1 - a_1 + b_1, \alpha_i^2 - a_2 + b_2, \alpha_i^3 - a_3 + b_3\right\}, \quad i = \overline{1,4}.$$

4) We find the vertices of the tetrahedron formed as a result of difference $T^A \overset{*}{-} T^B$ by subtracting the corresponding coordinates of the points found from the coordinates of points $A_i = \left\{\alpha_i^1, \alpha_i^2, \alpha_i^3\right\}$, $i = \overline{1,4}$:

$$A_i - \tilde{B}_i.$$

4 Generalization of the results

In this section, we summarize the results obtained above [23–26]. Let, we are given convex set M and compact set N in $\mathbb{R}^n$. We denote by $\partial M_0 = L_0$ the boundary of a convex compact set $M = M_0$. $M_\alpha,\ \partial M_\alpha = L_\alpha,\ \alpha \in A$ are chosen in such a way that: 1) $\bigcup\limits_{\alpha\in A} L_\alpha = M$; 2) $M_\alpha \overset{*}{-} M_\beta \neq \varnothing$ for arbitrary $\alpha, \beta \in A$ and $\alpha \leq \beta$. Based on I. Tamura [23], we call $F = \{L_\alpha : L_\alpha = \partial M_\alpha,\ \alpha \in A\}$ a foliation and $L_\alpha,\ \alpha \in A$ a leaves of the foliation. Let $\partial \left(M_\alpha \overset{*}{-} M_\beta\right) \in F$ be for arbitrary $\alpha, \beta \in A$.

Theorem 3. If the condition $N \subset M_\alpha$ is satisfied for the convex compact sets M, M_α and compact set N given in $\mathbb{R}^n$, the equality

$$M \overset{*}{-} N = \left(M \overset{*}{-} M_\alpha\right) + \left(M_\alpha \overset{*}{-} N\right)$$

holds.

Proof. Let be $z \in M \overset{*}{-} N$, then we show that there are elements $z_1 \in M \overset{*}{-} M_\alpha$ and $z_2 \in M_\alpha \overset{*}{-} N$ such that $z = z_1 + z_2$. We can write the relation $z + N \subset M$ using the definition of the Minkowski difference for the element $z \in M \overset{*}{-} N$. Therefore, for any $c \in N$, there is an element $a \in M$ such that the equality $z + c = a$ holds. From this we get the expression

$$c = a - z \in N. \tag{14}$$

By condition, since $N \subset M_\alpha$, relation $M_\alpha \overset{*}{-} N \neq \varnothing$ is valid. Let $z_2 \in M_\alpha \overset{*}{-} N$. It follows that $z_2 + N \subset M_\alpha$. This relation holds for all elements of the set N. Hence, according to (14), we can write the relation

$$z_2 + a - z \in M_\alpha. \tag{15}$$

According to the condition, $M \overset{*}{-} M_\alpha \neq \varnothing$. Let $z_1 \in M \overset{*}{-} M_\alpha$. Then, $z_1 + M_\alpha \subset M$ is appropriate. Since this relation holds for all elements of the set M_α, it also holds for the element $z_2 + a - z$ in expression (15)

$$z_2 + z_1 + a - z \in M.$$

Since $a \in M,\ z_1 + z_2 - z = 0$ and hence, the equality $z_1 + z_2 = z$ holds true.

Now, let $z \in \left(M \overset{*}{-} M_\alpha\right) + \left(M_\alpha \overset{*}{-} N\right)$, then there are elements $z_1 \in M \overset{*}{-} M_\alpha$ and $z_2 \in M_\alpha \overset{*}{-} N$ such that $z_1 + z_2 = z$. According to the definition of Minkowski difference from relation $z_1 \in M \overset{*}{-} M_\alpha$, we can write relation $z_1 + M_\alpha \subset M$, similarly, we get the expression $z_2 + N \subset M_\alpha$ from the relation $z_2 \in M_\alpha \overset{*}{-} N$. From these two expressions we get $z_1 + z_2 + N \subset M$, which leads to $z_1 + z_2 \subset M \overset{*}{-} N$. The theorem is proved.

Definition 4. A compact set N is said to be embedded in a foliation F, if such a leaf $L_\alpha = \partial M_\alpha$, $\alpha \in A$ and an element $z \in \mathbb{R}^n$ are found for which the relation $z + N \subset M_\alpha$ holds.

Definition 5. A compact set N is said to be densely embedded in a foliation F, if $z + N \subset M_{\alpha_0}$ and the index α_0 is the smallest among the numbers $\alpha \in A$ for which the relation $z + N \subset M_\alpha$ holds.

It is easy to understand from this definition that if the compact set N is densely embedded in foliation F, the dimension of the geometric difference $M_\alpha \overset{*}{-} N$ is smaller than the dimension of the space $\mathbb{R}^n$.

Definition 6. A compact set N is said to be completely densely embedded in a foliation F, if Minkowski difference $M_\alpha \overset{*}{-} N = \{a\}$ consists of a single point.

Theorem 4. If compact set N completely densely embedded in a foliation F, then the equality

$$M \overset{*}{-} N = \left(M \overset{*}{-} M_\alpha\right) + a$$

holds.

Using the concept of "complete dense embedding", we can write the following results for cases where the "subtrahend" set in the theorem 1 and theorem 2, above is an arbitrary compact set N.

Theorem 5. For polygons P^A and P^B in the Euclidean plane $\mathbb{R}^2$, condition (1) holds. If compact set N is completely dense embedded in set P^B, then the equality $P^A \overset{*}{-} N = P^A \overset{*}{-} P^B$ holds.

Theorem 6. For tetrahedrons T^A and T^B in the Euclidean space $\mathbb{R}^3$, condition (9) holds. If compact set N is completely dense embedded in set T^B, then the equality $T^A \overset{*}{-} N = T^A \overset{*}{-} T^B$ holds.

Conclusion

The Minkowski difference is actually useful as a research and conceptual tool. But, unfortunately, it is well known that there are serious difficulties in finding the Minkowski difference for given arbitrary forms of sets. This is the main obstacle for using the Minkowski difference in various practical applications. The results of the analysis of the work done by experts so far on finding the Minkowski difference and sum have shown that the Minkowski sum of sets is sufficiently studied, but there is a lack of data and literature on the Minkowski difference and its calculation.

Above, we introduced new methods for finding Minkowski differences of regular polygons given by vertices in the plane $\mathbb{R}^2$, regular tetrahedron given by vertices in space $\mathbb{R}^3$. Taking these results, we came to the conclusion that the form of the Minkowski difference of these sets will be similar to the "minuend" set.

But we cannot state this conclusion for the Minkowski difference of n-dimensional cubes in $\mathbb{R}^n$. Because the Minkowski difference of two cubes can also be a rectangular parallelepiped edges of which are parallel to the edges of the "minuend" cube. At the end of the article, we stated a theorem that helps to calculate the Minkowski difference of arbitrary convex compact sets in $\mathbb{R}^n$ using the elements of the theory of foliation.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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https://doi.org/10.31489/2024M4/138-148 *Research article*

Uniform asymptotic expansion of the solution for the initial value problem with a piecewise constant argument

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The article is devoted to the study of a singularly perturbed initial problem for a linear differential equation with a piecewise constant argument second-order for a small parameter. This paper is considered the asymptotic expansion of the solution to the Cauchy problem for singularly perturbed differential equations with piecewise-constant argument. The initial value problem for first order linear differential equations with piecewise-constant argument was obtained that determined the regular members. The Cauchy problems for linear nonhomogeneous differential equations with a constant coefficient were obtained, which determined the boundary layer terms. An asymptotic estimate for the remainder term of the solution of the Cauchy problem was obtained. Using the remainder term, we construct a uniform asymptotic solution with accuracy $O(\varepsilon^{N+1})$ on the $\theta_i \le t \le \theta_{i+1}, i = \overline{0, p}$ segment of the singularly perturbed Cauchy problem with a piecewise constant argument.

Keywords: singular perturbation, asymptotics, small parameter, boundary layer part, piecewise constant argument.

2020 Mathematics Subject Classification: 34D15, 34E10, 34K26.

Introduction

The singularly perturbed differential equations arise in various fields of chemical kinetics, mathematical biology fluid dynamics and in a variety models for control theory. These problems depend on a small positive parameter such that the solution varies rapidly in some domains and varies slowly in other domains. Asymptotics of the solution of singularly perturbed initial and boundary value problems with the phenomenon of an initial jump for ordinary differential equations with smooth coefficients in the general formulation were studied by K.A. Kasymov [1] and others. Asymptotics of the solution of the first boundary value problem for the linear ordinary differential equation of the second order with piecewise smooth coefficients and a small parameter at the highest derivative was first studied in the work of V. G. Sushko [2]. In particular, in the work of Kasymov [3] the case is considered when the highest coefficient of a degenerate equation has discontinuities of the first kind at points $t = t_i$, $i = \overline{1, n}$ and it is proved that the desired solution of the boundary value problem has initial jumps at these points. However, the case when the coefficients of the of a linear differential equation depend on a piecewise constant variable has not been investigated by them and others. The initial and boundary value problems considered in the studies [4–11] are equivalent to the Cauchy problem with the initial jump for differential and integro-differential equations in the stable case. Methods of solving nonlocal problems for hyperbolic equations with piecewise constant argument of the generalized type are given in papers [12–14]. A mathematical model including a piecewise constant argument was first considered by Busenberg and Cooke in 1982. Systematic studies of theoretical and practical problems involving

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piecewise constant arguments were initiated in the early 1980s. Since then, differential equations with piecewise constant arguments have attracted great attention from researchers in mathematics, biology, engineering, and other fields. They constructed a first-order linear equation to investigate vertically transmitted diseases. Following this work, using the method of reduction to discrete equations, many authors have analyzed various types of differential equations with piecewise constant arguments. A system of differential equations with piecewise constant argument of generalized type was introduced in [15, 16]. Asymptotic estimations of the solution to a singularly perturbed equation with piecewise constant argument were published [17, 18].

1 *Statement of the problem*

We consider the initial value problem for linear differential equations with a piecewise constant argument of a small parameter

$$\varepsilon y''(t) + A(t)y'(t) + B(t)y(t) + C(t)y(\beta(t)) = F(t), \tag{1}$$

$$y(0,\varepsilon) = d_0,\ y'(0,\varepsilon) = d_1, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, d_0, d_1 are known constants. The piecewise constant argument is determined with the function $\beta(t) = \theta_i$, if $t \in [\theta_i, \theta_{i+1}),\ i = \overline{1,p},\ 0 < \theta_1 < \theta_2 < ... < \theta p < T$.

Let us assume that the following conditions are satisfied:

C1) $A(t), B(t), C(t), F(t) \in C[0,T]$;

C2) $A(t) > 0,\ 0 \le t \le T$.

Theorem 1. Suppose that conditions (C1)-(C2) are fulfilled. Then, for the solution of the initial problem (1), (2) and its derivatives in the interval $0 \le t \le T$ for $\varepsilon > 0$, the following limit transitions are valid

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = \overline{y}(t), \quad 0 \le t \le T, \tag{3}$$

$$\lim_{\varepsilon\to 0} y'(t,\varepsilon) = \overline{y'}(t), \quad 0 < t \le T,$$

where $\overline{y}^{(q)}(t),\ q = 0,1$ is the solution to the following initial problem:
if $t \in [0;\theta_1)$

$$\begin{cases} A(t)\overline{y}'(t) + B(t)\overline{y}(t) = F(t) - C(t) - d_0, \\ \overline{y}(0) = d_0, \end{cases}$$

and if $t \in [\theta_i;\theta_{i+1}),\ i = \overline{1,p}$

$$\begin{cases} A(t)\overline{y}'(t) + B(t)\overline{y}(t) = F(t) - C(t)\overline{y}(\theta_i), \\ \overline{y}(\theta_i) = \overline{y}(\theta_i). \end{cases}$$

The convergence (3) can be nonuniform near several points, that is to say, that multi-layers emerge. These layers occur on the neighborhoods of $t = 0$ and $t = \theta_i,\ i = \overline{1,p}$.

For example, we take $A(t) = 1, B(t) = 0, C(t) = -3, \beta(t) = \left[\frac{t}{2}\right], F(t) = 1$ and $t \in [2n, 2n+2)$, $n = 0,1,2,\ d_0 = 1,\ d_1 = 3$. Then the graph of the solution is shown in Figure.

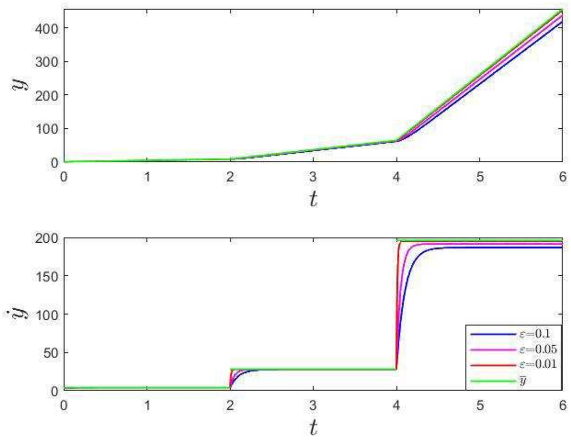

Figure. The blue, pink and red lines are graphs of solutions of example with initial values $d_0 = 1, d_1 = 3$ with values of $\varepsilon : 0.1, 0.05, 0.01$ respectively. The green line is the solution of unperturbed problem.

The derivative of solution of unperturbed problem is a discontinuity of the first kind.

2 Uniform asymptotic expansion of the solution for the initial problem

In the interval $\theta_i \le t \le \theta_{i+1},\ i = \overline{0,p}$, we look for a uniform asymptotic expansion of the solution to the initial problem (1), (2) in the following form

$$y(t,\varepsilon) = y_\varepsilon(t) + \varepsilon w_\varepsilon^{(i)}(\tau_i), \quad \tau_i = \frac{t-\theta_i}{\varepsilon}, \tag{4}$$

where

$$y_\varepsilon(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + ... + \varepsilon^k y_k(t) + ... \tag{5}$$

$$w_\varepsilon^{(i)}(\tau_i) = w_0^{(i)}(\tau_i) + \varepsilon w_1^{(i)}(\tau_i) + \varepsilon^2 w_2^{(i)}(\tau_i) + ... + \varepsilon^k w_k^{(i)}(\tau_i). \tag{6}$$

(5) is called the regular part of the asymptotics, and (6) is called the boundary-layer part of the asymptotics.

If we substitute expression (4) into equation (1), we obtain the following equality

$$\varepsilon \left(y_\varepsilon''(t) + \frac{\varepsilon}{\varepsilon^2}\ddot{w}_\varepsilon^{(i)}(\tau_i)\right) + A(t)\left(y_\varepsilon'(t) + \frac{\varepsilon}{\varepsilon}\dot{w}\varepsilon^{(i)}(\tau_i)\right) + \tag{7}$$

$$+B(t)\left(y_\varepsilon(t) + \varepsilon\dot{w}_\varepsilon^{(i)}(\tau_i)\right) + C(t)(y_\varepsilon(\theta_i) + \varepsilon w_\varepsilon^{(i)}(0)) = F(t).$$

From equation (7) we select equations that depend on the variables t and τ separately:

$$\varepsilon y_\varepsilon''(t) + A(t)y_\varepsilon'(t) + B(t)y_\varepsilon(t) + C(t)y_\varepsilon(\theta_i) = F(t), \tag{8}$$

$$\ddot{w}_\varepsilon^{(i)}(\tau_i) + A(\theta_i + \varepsilon\tau_i)\dot{w}_\varepsilon^{(i)}(\tau_i) + \varepsilon B(\theta_i + \varepsilon\tau_i)w_\varepsilon^{(i)}(\tau_i) + \varepsilon C(\theta_i + \varepsilon\tau_i)w_\varepsilon^{(i)}(0) = 0. \tag{9}$$

We substitute expression (5) into equation (8)

$$\varepsilon\left(y_0'(t)+\varepsilon y_1'(t)+...+\varepsilon^k y_k'(t)+...\right)+A(t)\left(y_0'(t)+\varepsilon y_1''(t)+...+\varepsilon^k y_k''(t)+...\right)+ \tag{10}$$

$$+B(t)\left(y_0(t)+\varepsilon y_1(t)+...+\varepsilon^k y_k(t)+...\right)+$$

$$+C(t)\left(y_0(\theta_i)+\varepsilon y_1(\theta_i)+...+\varepsilon^k y_k(\theta_i)+...\right)=F(t).$$

Equating the coefficients for a small parameter of the same degree in both sides of equation (10), we obtain a sequence of equations defining the $y_k(t),\ k=0,1,...$ functions

$$\varepsilon^0: A(t)y_0'(t)+B(t)y_0(t)+C(t)y_0(\theta_i)=F(t),$$

$$\varepsilon^1: A(t)y_1'(t)+B(t)y_1(t)+C(t)y_1(\theta_i)=-y_0''(t),$$

$$\varepsilon^k: A(t)y_k'(t)+B(t)y_k(t)+C(t)y_k(\theta_i)=-y_{k-1}''(t).$$

We classify the functions $A(\theta_i+\varepsilon\tau_i)$, $B(\theta_i+\varepsilon\tau_i)$, $i=\overline{0,p}$ into a Taylor series in the neighborhood of the point θ_i by degree ε

$$A(\theta_i+\varepsilon\tau_i)=A(\theta_i)+\frac{\varepsilon\tau_i}{1!}A'(\theta_i)+\frac{(\varepsilon\tau_i)^2}{2!}A''(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}A^{(k)}(\theta_i)+... \tag{11}$$

$$B(\theta_i+\varepsilon\tau_i)=B(\theta_i)+\frac{\varepsilon\tau_i}{1!}B'(\theta_i)+\frac{(\varepsilon\tau_i)^2}{2!}B''(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}B^{(k)}(\theta_i)+...$$

$$C(\theta_i+\varepsilon\tau_i)=C(\theta_i)+\frac{\varepsilon\tau_i}{1!}C'(\theta_i)+\frac{(\varepsilon\tau_i)^2}{2!}C''(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}C^{(k)}(\theta_i)+...$$

Substituting formulas (6), (11) into equation (9), we obtain the following expression

$$\ddot{w}_0^{(i)}(\tau_i)+\varepsilon\ddot{w}_1^{(i)}(\tau_i)+...+\varepsilon^k\ddot{w}_k^{(i)}(\tau_i)+...+\Big(A(\theta_i)+\frac{\varepsilon\tau_i}{1!}A'(\theta_i)+ \tag{12}$$

$$+\frac{(\varepsilon\tau_i)^2}{2!}A''(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}A^{(k)}(\theta_i)+...\Big)(\ddot{w}_0^{(i)}(\tau_i)+\varepsilon\ddot{w}_1^{(i)}(\tau_i)+...+\varepsilon^k\ddot{w}_k^{(i)}(\tau_i)+...)+$$

$$+\varepsilon\left(B(\theta_i)+\frac{\varepsilon\tau_i}{1!}B'(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}B^{(k)}(\theta_i)+...\right)(\ddot{w}_0^{(i)}(\tau_i)+\varepsilon\ddot{w}_1^{(i)}(\tau_i)+...+\varepsilon^k\ddot{w}_k^{(i)}(\tau_i)+...)+$$

$$+\varepsilon\left(C(\theta_i)+\frac{\varepsilon\tau_i}{1!}C'(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}C^{(k)}(\theta_i)+...\right)(\ddot{w}_0^{(i)}(\tau_i)+\varepsilon\ddot{w}_1^{(i)}(\tau_i)+...+\varepsilon^k\ddot{w}_k^{(i)}(\tau_i)+...)=0.$$

By equalizing the coefficients for a small parameter of the same degree in both sides of the equation (12), we obtain a sequence of equations defining the $w_k^{(i)}(\tau_i)$, $k=0,1,...$ functions

$$\varepsilon^0:\quad \ddot{w}_0^{(i)}(\tau_i)+A(\theta_i)\dot{w}_0^{(i)}(\tau_i)=0, \tag{13}$$

$$\varepsilon^1: \quad \ddot{w}_1^{(i)}(\tau_i) + A(\theta_i)\dot{w}_1^{(i)}(\tau_i) = \Phi_1(\tau_i), \tag{14}$$

where

$$\Phi_1(\tau_i) = -\tau_i A'(\theta_i)\dot{w}_0^{(i)}(\tau_i) - B(\theta_i)w_0^{(i)}(\theta_i) - C(\theta_i)w_0^{(i)}(0),$$

$$\varepsilon^k: \quad \ddot{w}_k^{(i)}(\tau_i) + A(\theta_i)\dot{w}_k^{(i)}(\tau_i) = \Phi_k(\tau_i), \tag{15}$$

where

$$\Phi_k(\tau_i) = -\sum_{m=1}^{k} \frac{(\tau_i)^m}{m!} A^{(m)}(\theta_i)\dot{w}_{k-m}^{(i)}(\tau_i) - \tag{16}$$

$$-\sum_{l=1}^{k} \frac{(\tau_i)^{l-1}}{(l-1)!} \left(B^{(l-1)}(\theta_i)w_{k-l}^{(i)}(\tau_i) + C^{(l-1)}(\theta_i)w_{k-l}^{(i)}(0) \right).$$

Consider the interval $t \in [0, \theta_1)$. Applying conditions (2) to the uniform asymptotic expansion of solution (4), equating the coefficients in front of the small parameter ε of the same degree, we determine the following conditions:

$$y_0(0) + \varepsilon y_1(0) + ... + \varepsilon^k y_k(0) + ...+$$

$$+\varepsilon \left(w_0^{(0)}(0) + \varepsilon w_1^{(0)}(0) + ... + \varepsilon^k w_k^{(0)}(0) + ... \right) = d_0,$$

$$y_0'(0) + \varepsilon y_1'(0) + ... + \varepsilon^k y_k'(0) + ...+$$

$$+\varepsilon \left(\dot{w}_0^{(0)}(0) + \varepsilon \dot{w}_1^{(0)}(0) + ... + \varepsilon^k \dot{w}_k^{(0)}(0) + ... \right) = d_1.$$

$$\varepsilon^0: y_0(0) = d_0, \;\; \dot{w}_0^{(0)}(0) = d_1 - y_0'(0),$$

$$\varepsilon^1: y_1(0) = -w_0^{(0)}(0), \;\; \dot{w}_1^{(0)}(0) = -y_1'(0),$$

$$\varepsilon^k: y_k(0) = -w_{k-1}^{(0)}0, \;\; \dot{w}_k^{(0)}(0) = -y_k'(0).$$

To determine the $w_k^{(i)}(\tau_i)$, $k = 0, 1, ...$ functions of the boundary layer, one more condition is necessary, since the order of equations (13)-(15) is equal to two. If we integrate equation (13) over $[\tau_i, \infty)$ and take into account condition $w_0^{(i)}(\infty) = 0$, $\dot{w}_0^{(i)}(\infty) = 0$, we determine the following expression

$$w_0^{(i)}(\tau_i) = -\frac{\dot{w}_0^{(i)}(\tau_i)}{A(\theta_i)}. \tag{17}$$

We substitute $\tau_0 = 0$ into equation (17):

$$w_0^{(0)}(0) = -\frac{d_1 - y_0'(0)}{A(0)}.$$

We continue this process and find the following conditions

$$w_k^{(0)}(0) = -\frac{1}{A(0)}\left(-y_k'(0) + \int_0^\infty \Phi_k(s)ds\right). \tag{18}$$

Let us determine the initial conditions for the interval $\theta_i \le t \le \theta_{i+1}$, $i = \overline{1,p}$:

$$y_0(\theta_i) + \varepsilon y_1(\theta_i) + ... + \varepsilon^k y_k(\theta_i) + ... + \varepsilon\left(w_0^{(i)}(0) + \varepsilon w_1^{(i)}(0) + ... + \varepsilon^k w_k^{(i)}(0) + ...\right) = y(\theta_0),$$

$$y_0'(\theta_i) + \varepsilon y_1'(\theta_i) + ... + \varepsilon^k y_k'(\theta_i) + ... + \varepsilon\left(\dot{w}_0^{(i)}(0) + \varepsilon \dot{w}_1^{(i)}(0) + ... + \varepsilon^k \dot{w}_k^{(i)}(0) + ...\right) = y'(\theta_i).$$

$$\varepsilon^0 : y_0(\theta_i) = y(\theta_i), \;\; \dot{w}_0^{(0)}(0) = y'(\theta_i) - y_0'(\theta_i),$$

$$\varepsilon^1 : y_1(\theta_i) = -w_0^{(i)}(0), \;\; \dot{w}_1^{(i)}(0) = -y_1'(\theta_i),$$

$$\varepsilon^k : y_k(\theta_i) = -w_{k-1}^{(i)}(0), \;\; \dot{w}_k^{(0)}(0) = -y_k'(\theta_i).$$

To determine the functions $w_k^{(i)}(\tau_0)$, $k = 0, 1, ...$ of the boundary layer, one more condition is necessary, since the order of equations (13)–(15) is equal to two. If we integrate equation (13) over $[\tau_i, \infty)$, $i = 1, 2, 3, ...$ and take into account the conditions $w_0^{(i)}(\infty) = 0$, $\dot{w}_0^{(i)}(\infty) = 0$, we determine the following expression:

$$w_0^{(i)}(\tau_i) = -\frac{1}{A(\theta_i)}\dot{w}_0^{(i)}(\tau_i). \tag{19}$$

If we substitute $\tau_i = \theta_i$, $i = 1, 2, 3, ...$ into equation (19),

$$w_0^{(i)}(0) = -\frac{1}{A(\theta_i)}(y'(\theta_i) - y_0'(\theta_i)).$$

Continuing this process, we obtain the following conditions

$$w_k^{(i)}(0) = -\frac{1}{A(\theta_i)}\left(-y_k'(\theta_i) + \int_{\theta_i}^\infty \Phi_k(s)ds\right), \;\; i = 1, 2, 3, ...$$

Problems defining regular terms for the interval $t \in [0, \theta_1)$

$$\begin{cases} A(t)y_0'(t) + B(t)y_0(t) + C(t)y_0(0) = F(t), \\ y_0(0) = d_0. \end{cases} \tag{20}$$

From the initial calculation (20), the $y_0(t)$ term of the regular part of the asymptotics is determined uniquely:

$$\begin{cases} A(t)y_1'(t) + B(t)y_1(t) + C(t)y_1(0) = -y_0''(t), \\ y_1(0) = -w_0^{(0)}(0). \end{cases} \tag{21}$$

From the initial calculation (21), the $y_1(t)$ term of the regular part of the asymptotics is determined uniquely:

$$\begin{cases} A(t)y_k'(t) + B(t)y_k(t) + C(t)y_k(0) = -y_{k-1}''(t), \\ y_k(0) = -w_{k-1}^{(0)}(0). \end{cases} \tag{22}$$

From the original description (22) the term $y_k(t)$ of the regular part of the asymptotics is determined uniquely.

Problems defining boundary-layer members for the interval $t \in [0, \theta_1)$

$$\ddot{w}_0^{(0)}(\tau_0) + A(0)\dot{w}_0^{(0)}(\tau_0) = 0, \tag{23}$$

$$w_0^{(0)}(0) = -\frac{d_1 - y_0'(0)}{A(0)},$$

$$\dot{w}_0^{(0)}(0) = d_1 - y_0'(0).$$

From the initial calculation (18) and (23), the zeroth approximation $w_0^{(0)}(\tau_0)$ of the boundary-layer part of the asymptotics is uniquely determined:

$$\ddot{w}_k^{(0)}(\tau_0) + A(0)\dot{w}_k^{(0)}(\tau_0) = \Phi_k(\tau_0), \tag{24}$$
$$w_k^{(0)}(0) = -\frac{1}{A(0)}\left(-y_k'(0) + \int_0^\infty \Phi_k(s)ds\right),$$
$$\dot{w}_k^{(0)}(0) = -y_k'(0),$$

where the function $\Phi_k(\tau_0)$ is determined by formula (16). From the initial calculation (24) the $w_k^{(0)}(\tau_0)$ $k-$th approximation of the boundary layer part of the asymptotics is uniquely determined.

Problems defining regular terms for the interval $t \in [\theta_i, \theta_{i+1})$, $i = 1, 2, ...$

$$\begin{cases} A(t)y_0'(t) + B(t)y_0(t) + C(t)y_0(\theta_i) = F(t), \\ y_0(\theta_i) = y(\theta_i), \end{cases}$$

$$\begin{cases} A(t)y_k'(t) + B(t)y_k(t) + C(t)y_k(\theta_i) = -y_{k-1}''(t), \\ y_k(0) = -w_{k-1}^{(0)}(\theta_i). \end{cases}$$

Problems of determining boundary-layer elements for the interval $t \in [\theta_i, \theta_{i+1})$, $i = 1, 2, ...$

$$\ddot{w}_0^{(i)}(\tau_i) + A(0)\dot{w}_0^{(i)}(\tau_i) = 0, \ i = \overline{1,p}$$

$$w_0^{(i)}(0) = -\frac{1}{A(\theta_i)}\left(y'(\theta_i) - y_0'(\theta_i)\right),$$

$$\dot{w}_0^{(i)}(0) = y'(\theta_i) - y_0'(\theta_i).$$

$$\ddot{w}_k^{(i)}(\tau_i) + A(\theta_i)\dot{w}_k^{(i)}(\tau_i) = \Phi_k(\tau_i), \ i = \overline{1,p}$$
$$w_k^{(i)}(0) = -\frac{1}{A(\theta_i)}\left(-y_k'(\theta_i) + \int_{\theta_i}^\infty \Phi_k(s)ds\right),$$
$$\dot{w}_k^{(i)}(0) = -y_k'(\theta_i).$$

3 Justification of the asymptotic behavior of the solution to the initial problem

Theorem 2. Let conditions (C1), (C2) be satisfied. Then, for a sufficiently small value of the small parameter ε (1), the initial problem (2) has a solution $y(t, \varepsilon)$ on the interval $\theta \leq t \leq \theta_{i+1}$, $i = \overline{0,p}$, which is unique and is expressed as

$$y(t, \varepsilon) = y_N(t, \varepsilon) + R_N(t, \varepsilon),$$

where the function $y_N(t, \varepsilon)$ is defined by the formula

$$y_N(t, \varepsilon) = \sum_{k=0}^N \varepsilon^k y_k(t) + \varepsilon \sum_{k=0}^N \varepsilon^k w_k^{(i)}(\tau_i), \ \ \tau_i = \frac{t - \theta_i}{\varepsilon}, \ \ \theta_i \leq t \leq \theta_{i+1}, \ i = \overline{0,p} \tag{25}$$

and the following estimates are suitable for the remainder term $R_N(t,\varepsilon)$

$$|R_N^{(q)}(t,\varepsilon)| \leq C\varepsilon^{N+1}, \quad q=0,1, \;\; \theta_i \leq t \leq \theta_{i+1}, \quad i=\overline{0,p}, \tag{26}$$

where $C>0$ is a quantity independent of ε.

Proof. We obtain the independent sum of series (25) from (4). If we substitute function (25) into equation (1), we obtain an equation

$$\varepsilon y_N''(t,\varepsilon) + A(t)y_N'(t,\varepsilon) + B(t)y_N(t,\varepsilon) + C(t)y_N'(\beta(t)) + D(t)y_N(\beta(t)) = F(t) + O\left(\varepsilon^{N+1}\right). \tag{27}$$

That is, the function $y_N(t,\varepsilon)$ satisfies the equation with accuracy $O\left(\varepsilon^{N+1}\right)$.

Satisfying the function (25) with conditions (2), we define the following conditions

$$y_N(0,\varepsilon) = d_0 + O\left(\varepsilon^{N+1}\right), \quad y_N'(0,\varepsilon) = d_1. \tag{28}$$

Let us introduce the difference between the exact solution and the approximate solution in the following form

$$R_N(t,\varepsilon) = y(t,\varepsilon) - y_N(t,\varepsilon) \Rightarrow y(t,\varepsilon) = y_N(t,\varepsilon) - R_N(t,\varepsilon). \tag{29}$$

The function $R_N(t,\varepsilon)$ is called the remainder term of the asymptotics.

Substituting formula (29) into equation (1), taking into account that the function $y_N(t,\varepsilon)$ satisfies equation (27) and conditions (28), we obtain the following equation defining the remainder term $R_N(t,\varepsilon)$

$$\varepsilon R_N''(t,\varepsilon) + A(t)R_N'(t,\varepsilon) + B(t)R_N(t,\varepsilon) + C(t)R_N'(\beta(t)) + D(t)R_N(\beta(t)) = O\left(\varepsilon^{N+1}\right), \tag{30}$$

$$R_N(0,\varepsilon) = O\left(\varepsilon^{N+1}\right), \;\; R_N'(0,\varepsilon) = 0.$$

Since the type of problem (30) is the same as the type of problem (1), (2), to solve problem (26) we use the solution estimate (1), (2)

$$|y^{(q)}(t,\varepsilon)| \leq C\left(|d_0|(1+\max_{\theta_i\leq t\leq\theta_{i+1}}|C(t)|) + \varepsilon|d_1| + \max_{\theta_i\leq t\leq\theta_{i+1}}|F(t)|\right) +$$

$$+C\varepsilon^{1-q}\exp^{-\gamma\frac{t-\theta_i}{\varepsilon}}\left(|d_0|(1+\max_{\theta_i\leq t\leq\theta_{i+1}}|C(t)|) + |d_1| + \max_{\theta_i\leq t\leq\theta_{i+1}}|F(t)|\right), \;\; q=0,1, \;\; i=\overline{0,p+1}.$$

Then we obtain the following estimate for the solution

$$|R_N(t,\varepsilon)| \leq C\varepsilon^{N+1} + C\varepsilon^{N+2}\exp^{-\gamma\frac{t-\theta_i}{\varepsilon}} \leq C\varepsilon^{N+1}, \; \theta_i\leq t\leq\theta_{i+1}, \; i=\overline{0,p+1}$$

$$|R_N'(t,\varepsilon)| \leq C\varepsilon^{N+1} + C\varepsilon^{N+1}\exp^{-\gamma\frac{t-\theta_i}{\varepsilon}} \leq C\varepsilon^{N+1}, \; \theta_i\leq t\leq\theta_{i+1}, \; i=\overline{0,p+1}.$$

The following conclusion follows: the function $y_N(t,\varepsilon)$ is called an asymptotic solution obtained with an accuracy of $O\left(\varepsilon^{N+1}\right)$. From Theorem 2 it is clear that for the solution of the perturbed problem there is a uniform limit transition and it has a discontinuity of the 1st kind. Theorem 2 is proved.

Conclusion

In this paper, we considered the asymptotic expansion of the solution to the Cauchy problem for a singularly perturbed initial value problem for a linear differential equation with a piecewise constant second-order argument in a small parameter. We have obtained the initial problem for first-order linear differential equations with piecewise constant argument that determine the regular terms. Cauchy problems were also obtained for linear nonhomogeneous differential equations with a constant coefficient, which determine the terms of the boundary layer. Using an estimate for the solution to the initial problem, we obtained an asymptotic estimate for the remainder term for the solution to the Cauchy problem. And using the remainder term, we constructed a uniform asymptotic solution with an accuracy of $O(\varepsilon^{N+1})$ on $\theta_i \leq t \leq \theta_{i+1}$, $i = \overline{0,p}$ segment of a singularly perturbed Cauchy problem with piecewise constant argument.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The author declares no conflict of interest.

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Conditions for maximal regularity of solutions to fourth-order differential equations

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This article investigates a fourth-order differential equation defined in a Hilbert space, with an unbounded intermediate coefficient and potential. The key distinction from previous research lies in the fact that the intermediate term of the equation does not obey to the differential operator formed by its extreme terms. The study establishes that the generalized solution to the equation is maximally regular, if the intermediate coefficient satisfies an additional condition of slow oscillation. A corresponding coercive estimate is obtained, with the constant explicitly expressed in terms of the coefficients' conditions. Fourth-order differential equations appear in various models describing transverse vibrations of homogeneous beams or plates, viscous flows, bending waves, and etc. Boundary value problems for such equations have been addressed in numerous works, and the results obtained have been extended to cases with smooth variable coefficients. The smoothness conditions imposed on the coefficients in this study are necessary for the existence of the adjoint operator. One notable feature of the results is that the constraints only apply to the coefficients themselves; no conditions are placed on their derivatives. Secondly, the coefficient of the lowest order in the equation may be zero, moreover, it may not be unbounded from below.

Keywords: fourth-order differential equation, unbounded coefficient, solution, existence, uniqueness, smoothness, operator, separability, regularity, coercive estimate.

2020 Mathematics Subject Classification: 34A30, 34C11.

1 Introduction. Formulation of the problem

Fourth-order differential equations describe various physical phenomena, such as transversal oscillations of homogeneous beams or plates, viscoelastic and inelastic flows, bending waves, and other [1, 2]. The issues of existence and uniqueness of solutions to boundary value problems posed for linear and nonlinear fourth-order differential equations have been studied extensively in the literature [3–5]. In the case of an infinite domain, the Cauchy problem for a fourth-order waves equation is considered in [6]. However, in these works, the coefficients of the equations are either constant or assumed to be bounded functions. Additionally, when investigating nonlinear equations, excessively strict restrictions are imposed on the coefficients to ensure the uniqueness of solutions [3–5]. In light of both theoretical and practical needs, there is a growing relevance in studying the solvability of fourth-order differential equations with variable coefficients and relaxing constraints on these coefficients. This concern is particularly pertinent to differential equations with independently growing coefficients that are given in an infinite domain.

Consider the following fourth-order differential equation defined on the real line:

$$L_0 y = y^{(4)} + p(x)\, y^{(3)} + q(x)\, y = F(x), \tag{1}$$

where $x \in \mathbb{R} = (-\infty, \infty)$, $p(x) > 0$, $p(x) \in C_{loc}^{(3)}(\mathbb{R})$, $q(x)$ is a continuous function, and $F(x) \in L_2(\mathbb{R})$.

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Let L denote the closure in the $L_2(\mathbb{R})$ norm of the operator

$$L_0 y = y^{(4)} + p(x)\, y^{(3)} + q(x)\, y$$

defined on the set $C_0^{(4)}(\mathbb{R})$ of continuously differentiable up to the fourth order functions with compact support. A solution to equation (1) is an element $y \in D(L)$ satisfying the equality $Ly = F$.

Our goal is to establish conditions sufficient for the fulfillment of the inequality

$$\left\| y^{(4)} \right\|_2 + \left\| p y^{(3)} \right\|_2 + \|(1+|q|)y\|_2 \le C\left(\|F\|_2 + \|y\|_2\right), \tag{2}$$

for a solution y, where $\|\cdot\|_2$ denotes the norm of the $L_2(\mathbb{R})$ space. Inequality (2) is referred to as a coercive estimate or an estimate of maximal regularity of the solution.

The equation (1) has been primarily studied in the case of $p = 0$ [7]. In addition, if $q \ge \delta > 0$, then (1) is a unique solvable. And if the oscillation of q satisfies certain additional conditions, then the inequality (2) is satisfied for a solution of (1). However, when $p(x)$ is a non-zero, rapidly growing function, the method of [7] is inapplicable. This is because the operator $p\frac{d^3}{dx^3}$ may not obey $\frac{d^4}{dx^4} + q(x)\, E$ (E is the identity operator). For the sake of completeness, we provide statements about the existence and uniqueness of solutions with proofs.

The aforementioned problem of unique and coercive solvability has been addressed in [8, 9] for second-order differential equations with rapidly growing intermediate coefficients, and in [10] for third-order differential equations. In [11], the authors developed an effective method for investigating the spectrum of a degenerate symmetric fourth-order differential operator. We build upon the ideas of the last four works. Unique and coercive solvability of various types of singular differential equations with intermediate coefficients is studied in [12–15].

In what follows, by C we will denote positive constants, which may have, in general, different values in the different places.

2 On an auxiliary binomial differential equation

Let us consider the operator $l_0 y = y^{(4)} + p(x)\, y^{(3)}$, $D(l_0) = C_0^{(4)}(\mathbb{R})$. We denote its closure in $L_2(\mathbb{R})$ by l.

Lemma 1. Suppose the function $p(x) \in C_{loc}^{(3)}(\mathbb{R})$ such that

$$p(x) \ge \varepsilon > 0. \tag{3}$$

Then, for any $y \in C_0^{(4)}(\mathbb{R})$, the following estimate holds

$$\left\| \sqrt{p} y^{(3)} \right\|_2 \le \left\| \frac{l_0 y}{\sqrt{p}} \right\|_2. \tag{4}$$

Proof. Let $y \in C_0^{(4)}(\mathbb{R})$. We concider the scalar product $A = \left(l_0 y, y^{(3)}\right)$. Since y is a function with compact support, the following equalities hold:

$$A = \int_{-\infty}^{\infty} y^{(4)}(x) y^{(3)}(x)\, dx + \int_{-\infty}^{\infty} p(x) \left[y^{(3)}(x)\right]^2 dx = \int_{-\infty}^{\infty} p(x) \left[y^{(3)}(x)\right]^2 dx. \tag{5}$$

On the other hand, using condition (3) and the Holder inequality, we obtain:

$$A \le \left(\int_{-\infty}^{\infty} |l_0 y|^2 \frac{1}{p(x)}\, dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} p(x) \left|y^{(3)}\right|^2 dx\right)^{\frac{1}{2}}.$$

From this and (5), inequality (4) follows. The right-hand side of (4) is bounded under the condition (3).

Let $\rho(t)$ and $v(t) \neq 0$ be given continuous functions, and k is a natural number. We introduce the following notations:

$$\alpha_{\rho,v,k} = \sup_{x>0} \left(\int_0^x \rho^2(t)dt \right)^{\frac{1}{2}} \left(\int_x^\infty t^{2(k-1)2} v^{-2}(t)\, dt \right)^{\frac{1}{2}},$$

$$\beta_{\rho,v,k} = \sup_{x<0} \left(\int_x^0 \rho^2(s)ds \right)^{\frac{1}{2}} \left(\int_{-\infty}^x s^{2(k-1)} v^{-2}(s)\, ds \right)^{\frac{1}{2}},$$

$$\gamma_{\rho,v,k} = \max\left(\alpha_{\rho,v,k}, \beta_{\rho,v,k}\right).$$

Lemma 2. [11] If functions $\rho(t)$ and $v(t)$ satisfy the relation

$$\gamma_{\rho,v,k} < \infty \, (k \in \mathbb{N}),$$

then for each $f \in C_0^{(k)}(\mathbb{R})$ the following inequality holds:

$$\|\rho f\|_2 \leq \frac{2}{(k-1)!} \gamma_{\rho,v,k} \left\| v f^{(k)} \right\|_2.$$

Lemma 3. Suppose the function $p(x)$ satisfies condition (3) and $\gamma_{1,\sqrt{p},3} < \infty$. Then the operator l is invertible, and for each $y \in D(l)$, the inequality holds

$$\|y\|_2 + \left\| \sqrt{p} y^{(3)} \right\|_2 \leq C \|ly\|_2. \tag{6}$$

Proof. Let $y \in C_0^{(3)}(\mathbb{R})$. According to the condition $\gamma_{1,\sqrt{p},3} < \infty$, Lemma 2, and estimate (4), we obtain the following inequalities:

$$\|y\|_2 \leq C \left\| \sqrt{p} y^{(3)} \right\|_2 \leq C \left\| \frac{l_0 y}{\sqrt{p}} \right\|_2.$$

By (3), we have

$$\left\| \sqrt{p} y^{(3)} \right\|_2 \leq \sqrt{\varepsilon} \|l_0 y\|_2 \tag{7}$$

and

$$\|y\|_2 \leq C \sqrt{\varepsilon} \|l_0 y\|_2. \tag{8}$$

Since $D(l_0) = C_0^{(3)}(\mathbb{R})$ and l is the closure of the operator l_0, from (7) and (8), the inequalities

$$\left\| \sqrt{p} y^{(3)} \right\|_2 \leq \sqrt{\varepsilon} \|ly\|_2$$

and

$$\|y\|_2 \leq C \sqrt{\varepsilon} \|ly\|_2$$

follow for each $y \in D(l)$, respectively. Combining them yields (6).

Consider the equation

$$ly = y^{(4)} + p(x) y^{(3)} = f(x). \tag{9}$$

An element $y \in D(l)$ satisfying $ly = f$ is called a solution to (9).

Lemma 4. Suppose that the conditions of Lemma 3 hold for $p(x)$. Then the solution to equation (9) is unique.

Proof. If y and z are two solutions to equation (9), then by definition, $y, z \in D(l)$ and $ly = f$, $lz = f$. For $v = y - z$, we have $lv = 0$. Then, by inequality (6), $\|v\| = 0$, i.e., $y = z$.

Lemma 5. Suppose that the conditions of Lemma 3 hold for $p(x)$. Then, for any $f(x) \in L_2(\mathbb{R})$, a solution to equation (9) exists.

Proof. According to Lemma 3, the operator l is invertible. It suffices to show that its range $R(l)$ coincides with the entire space $L_2(\mathbb{R})$. By Lemma 3, $y^{(3)} \in L_2(\mathbb{R})$, if $y \in D(l)$. Let $y^{(3)} = z$ and $\Theta z = z' + p(x) z$. Then $z \in L_2(\mathbb{R})$, and equation (9) takes the form:

$$\Theta z = z' + pz = f \in L_2(\mathbb{R}).$$

The equality $R(l) = R(\Theta)$ holds. Indeed,

$$R(\Theta) = \{v \in L_2(\mathbb{R}) : \exists z \in D(\Theta),\ \Theta z = v\} =$$

$$= \{v \in L_2(\mathbb{R}) : \exists y \in D(l),\ ly = v\} = R(l).$$

According to (6), $R(\Theta)$ is a closed set. It suffices to demonstrate that $R(\Theta) = L_2(\mathbb{R})$. Let us assume the opposite. Suppose that $R(\Theta) \neq L_2(\mathbb{R})$. Then there exists a non-zero element $w \in L_2(\mathbb{R})$, which is orthogonal to the set $R(\Theta)$: $(w, \Theta z) = 0, z \in D(\Theta)$. Since $(w, \Theta z) = (\Theta^* w, z)$, and the set $D(\Theta)$ is dense in $L_2(\mathbb{R})$, the function $w \in D(\Theta^*)$ satisfies the following homogeneous equation:

$$\Theta^* w = w - w' = 0.$$

Therefore, as $p(x)$ is continuous, it follows that $w' \in L_{2,loc}(\mathbb{R})$, then $w \in W^1_{2,loc}(\mathbb{R})$. Consequently, the function $w(x)$ is continuous, and

$$|w(x)| = |c|\, e^{\int_a^x p(t)\,dt},\ \forall x \in \mathbb{R}.$$

Hence, $|w(x)| \geq |c|$ for $x \geq a$, we obtain $w \notin L_2(\mathbb{R})$. This leads to a contradiction, demonstrating that $R(\Theta) = L_2(\mathbb{R})$.

3 Conditions for the separability of a binomial operator

Let $\lambda \in \mathbb{R}_+ = [0, +\infty)$. Consider the following differential operator $\Theta_{0\lambda} z = z' + (p + \lambda) z$, $D(\Theta_{0\lambda}) = C_0^{(1)}(\mathbb{R})$. Its closure in the space $L_2(\mathbb{R})$ we denote by Θ_λ.

Definition 1. It is said that the operator Θ_λ is separable in the space $L_2(\mathbb{R})$, if for any $z \in D(\Theta_\lambda)$, the following inequality holds:

$$\left\|z'\right\|_2 + \|pz\|_2 + \lambda \|z\|_2 \leq C(\|\Theta_\lambda z\|_2 + \|z\|_2). \tag{10}$$

It is evident that the operator Θ_λ is separable in the space $L_2(\mathbb{R})$, if and only if there exists $\mu \in \mathbb{R}$ such that the operator $\Theta_{\lambda+\mu} = \Theta_\lambda + \mu E$ is separable in this space.

Lemma 6. Let the coefficient p satisfy the conditions of Lemma 3 and the following relation:

$$\sup_{x, \eta \in \mathbb{R}, |x - \eta| \leq 1} \frac{p(x)}{p(\eta)} < \infty. \tag{11}$$

Then, the operator Θ_λ is separable in $L_2(\mathbb{R})$.

Proof. Let us observe that the conditions of Lemma 3 remain valid for the function p, and $\lambda\geq 0$. According to Lemma 4 and Lemma 5, the inverse operator $\Theta_\lambda^{-1}(\lambda\geq 0)$ exists and is continuous. We will now demonstrate that the operator Θ_λ is separable for at least one $\lambda\geq 0$.

Let $\Delta_j = [j, j+1)\,,\ \Omega_j = \left(j-\frac{1}{2}, j+\frac{3}{2}\right)(j\in Z)$. We choose the functions $\varphi_j(x)\ (j\in Z)$ from $C_0^\infty(\Omega_j)\,(j\in Z)$, satisfying the following conditions:

a) $0\leq\varphi_j(x)\leq 1,\ \varphi_j(x)=1\ \forall x\in\Delta_j,\ \sup\limits_{x\in\Omega_j}\max\limits_{j\in Z}\left|\varphi_j'(x)\right|\leq M.$

Then

$$\acute{\Delta}_j\subset\Omega_j\subset\Delta_{j-1}\cup\Delta_j\cup\Delta_{j+1}, \Delta_j\cap\Delta_k=\emptyset\,(j\neq k)\,,$$

$$\Omega_j\cap\Omega_m=\emptyset\ \ (|j-m|\geq 2)\,,\ \sum_{j=-\infty}^{\infty}\varphi_j(x)\,\chi_{\Delta_j}(x)=1.$$

Here χ_{Δ_j} is a characteristic function of Δ_j. Recall that the sequence $\{\varphi_j(x)\}_{j=-\infty}^{\infty}$, satisfying conditions a), exists [7].

Let $p_j(x)\,(j\in Z)$ be the extension to the entire $\mathbb{R}$ of the restriction in $\Omega_j\,(j\in Z)$ of the function $p(x)$ such that

$$\frac{1}{2}\inf_{z\in\Omega_j}p(z)\leq p_j(x)\leq 2\sup_{z\in\Omega_j}p(z),\ \ x\in\mathbb{R}. \tag{12}$$

According to condition (11), such an extension exists [7]. Let

$$\widetilde{\theta}_{j,\lambda}z=z'+(p_j+\lambda)\,z,\ \ z\in C_0^{(1)}(\mathbb{R})\,.$$

Denote the closure of the operator $\widetilde{\theta}_{j,\lambda}$ in the space $L_2(\mathbb{R})$ as $\theta_{j,\lambda}$. By Lemma 3, for any $z\in D(\theta_{j,\lambda})\,,$ we have

$$\left\|\sqrt{p_j+\lambda}z\right\|_2\leq\left\|\sqrt{\frac{1}{p_j+\lambda}}\theta_{j,\lambda}z\right\|_2.$$

Then,

$$\|z\|_2\leq\frac{1}{\mathop{inf}\limits_{x\in\mathbb{R}}(p_j(x)+\lambda)}\left\|\theta_{j,\lambda}z\right\|_2\ \ (j\in Z)\,. \tag{13}$$

In particular, based on (3) and (12), we obtain

$$\|z\|_2\leq\frac{2}{\varepsilon+2\lambda}\left\|\theta_{j,\lambda}z\right\|_2\ \ \ (j\in Z). \tag{14}$$

Therefore, the operator $\theta_{j,\lambda}$ is invertible. Due to Lemma 5, the operator $\theta_{j,\lambda}^{-1}\,(j\in Z)$ is continuous. Let $f\in C_0^{(1)}(\mathbb{R})$. Consider the following operators M_λ and B_λ:

$$M_\lambda f=\sum_j\varphi_j\theta_{j,\lambda}^{-1}\left(\chi_{\Delta_j}f\right),\ B_\lambda f=\sum_j\varphi_j'\theta_{j,\lambda}^{-1}\left(\chi_{\Delta_j}f\right).$$

Since f is a function with compact support, the number of terms in the sums on the right-hand side of the last equalities is finite. By our choice, for $z\in\Omega_j$, the equality $\Theta_\lambda z=\theta_{j,\lambda}z,\ z\in D(\Theta_\lambda)$, holds. Considering this and the properties of the function $\varphi_j\in C_0^\infty(\Omega_j)\,,$ we can easily demonstrate the equality

$$\Theta_\lambda(M_\lambda f)=(B_\lambda+E)\,f. \tag{15}$$

Note that the multiplicity of the intersection of intervals $\Omega_j\,(j\in Z)$ is at most two. Therefore, the following inequalities hold:

$$\|B_\lambda f\|_2^2=\sum_{j=-\infty}^{\infty}\int_{\Delta_j}|B_\lambda f|^2\,dx\leq\sum_{j=-\infty}^{\infty}\int_{\Delta_j}\left[\sum_{k=j-1}^{j+1}\left|\varphi'_k(x)\right|\left|\theta_{k,\lambda}^{-1}\left(\chi_{\Delta_k}f\right)\right|\right]^2dx\leq$$

$$\leq3\sum_{j=-\infty}^{\infty}\int_{\Delta_j}\sum_{k=j-1}^{j+1}\left|\varphi'_k\left(x\right)\right|^2\left|\theta_{k,\lambda}^{-1}\left(\chi_{\Delta_k}f\right)\right|^2dx\leq3M^2\sum_{j=-\infty}^{\infty}\int_{\Delta_j}\sum_{k=j-1}^{j+1}\left|\theta_{k,\lambda}^{-1}\left(\chi_{\Delta_k}f\right)\right|^2dx=$$

$$=3M^2\sum_{j=-\infty}^{\infty}\left\|\theta_{j,\lambda}^{-1}\left(\chi_{\Delta_j}f\right)\right\|_2^2.$$

According to inequality (14), we have

$$\|B_\lambda f\|_2^2\leq3M^2\left(\frac{2}{\varepsilon+2\lambda}\right)^2\|f\|_2^2.$$

Therefore, if we denote $\lambda_0=\sqrt{3}M\theta^{-1}-0,5\varepsilon$, then for $\lambda\geq\lambda_0$, we have

$$\|B_\lambda\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}\leq\mu\;\;(0<\mu<1)\,,$$

where $\|\cdot\|=\|\cdot\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}$ is the operator norm. By the well-known Banach theorem on small perturbations of a linear operator, for $\lambda\geq\lambda_0$, the operator $E+B_\lambda$ is invertible, and its inverse $(E+B_\lambda)^{-1}$ is bounded. The following inequalities are easily proven:

$$\frac{1}{1+\mu}\leq\left\|(E+B_\lambda)^{-1}\right\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}\leq\frac{1}{1-\mu}\;\;(\lambda\geq\lambda_0).\tag{16}$$

By (15), we obtain the following operator equality

$$\Theta_\lambda^{-1}=M_\lambda\left(E+B_\lambda\right)^{-1},\;\lambda\geq\lambda_0.\tag{17}$$

Let us estimate the norm $\left\|(p+\lambda)\Theta_\lambda^{-1}\right\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}$. By (16) and (17),

$$\left\|(p+\lambda)\,\Theta_\lambda^{-1}\right\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}\leq\frac{1}{1-\mu}\left\|(p+\lambda)\,M_\lambda\right\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}.$$

But

$$\|(p+\lambda)\,M_\lambda f\|_2^2=\sum_{j=-\infty}^{\infty}\int_{\Delta_j}\left(p\left(x\right)+\lambda\right)^2\left|\sum_{k=j-1}^{j+1}\varphi_k\left(x\right)\theta_{k,\lambda}^{-1}\left(\chi_{\Delta_k}f\right)\right|^2dx\leq$$

$$\leq3\sum_{j=-\infty}^{\infty}\int_{\Delta_j}(p(x)+\lambda)^2\left[\left|\varphi_{j-1}\theta_{j-1,\lambda}^{-1}\chi_{\Delta_{j-1}}f(x)\right|^2+\left|\varphi_j\theta_{j,\lambda}^{-1}\chi_{\Delta_j}f(x)\right|^2\right]dx+$$

$$+3\sum_{j=-\infty}^{\infty}\int_{\Delta_j}(p(x)+\lambda)^2\left|\varphi_{j+1}\theta_{j+1,\lambda}^{-1}\left(\chi_{\Delta_{j+1}}f\right)\right|^2dx\leq$$

$$\leq3\left(\sup_{x\in\Omega_j}p\left(x\right)+\lambda\right)^2\sum_{j=-\infty}^{\infty}\int_{\mathbb{R}}\left|\varphi_j\left(x\right)\theta_{j,\lambda}^{-1}\left(\chi_{\Delta_j}f\right)\right|^2dx.$$

According to inequality (13), property a) of the sequence $\{\varphi_j(x)\}_{j=-\infty}^{\infty}$ and condition (11), we obtain

$$\|(p+\lambda)\,M_\lambda f\|_2^2 \leq 3\left(\sup_{x\in\Omega_j} p\,(x)+\lambda\right)^2 \sum_{j=-\infty}^{\infty}\int_R \left|\theta_{j,\lambda}^{-1}\left(\chi_{\Delta_j} f\right)\right|^2 dx \leq$$

$$\leq 3\left(\sup_{x\in\Omega_j} p\,(x)+\lambda\right)^2 \frac{1}{\left(\inf\limits_{x\in R} p_j\,(x)+\lambda\right)^2} \sum_{j=-\infty}^{\infty}\int_R \left|\left(\chi_{\Delta_j} f\right)\right|^2 dx \leq$$

$$\leq 12\left(\frac{\sup\limits_{t\in\Omega_j} p\,(t)+\lambda}{\sup\limits_{t\in\Omega_j} p\,(t)+\lambda}\right)^2 \int_{\mathbb{R}} \left(\sum_{j=-\infty}^{\infty}\chi_{\Delta_j}^2\right) f^2\,(x)\,dx \leq 12\left(\sup_{x\in\Omega_j}\frac{p\,(x)}{p\,(t)}+1\right)^2 \|f\|_2^2\,.$$

So

$$\|(p+\lambda)M_\lambda f\|_2^2 \leq 12(K+1)^2\,\|f\|_2^2\ \ (\lambda\geq\lambda_0)\,,\ K=\sup_{x\in\Omega_j}\frac{p\,(x)}{p\,(t)}. \tag{18}$$

For $z\in D\,(\Theta_\lambda)\,,\ \Theta_\lambda z=f,\ \lambda\geq\lambda_0$, we have $z=\Theta_\lambda^{-1}f$. Therefore, according to (17), (18) and (16),

$$\|(p+\lambda)\,z\|_2=\left\|(p+\lambda)\,M_\lambda\,(E+B_\lambda)^{-1}\,f\right\|_2\leq$$

$$\leq 2\sqrt{3}\,(K+1)\left\|(E+B_\lambda)^{-1}\,f\right\|_2 \leq 2\sqrt{3}\,(K+1)\,\frac{1}{1-\mu}\,\|f\|_2\,.$$

Furthermore

$$\left\|z'\right\|_2=\|f-(p+\lambda)\,z\|_2\leq\left[2\sqrt{3}\,(K+1)\,\frac{1}{1-\mu}+1\right]\|f\|_2\,.$$

Consequently,

$$\left\|z'\right\|_2+\|pz\|_2+\|\lambda z\|_2\leq(6\sqrt{3}(K+1)\frac{1}{1-\mu}+1)\,\|f\|_2\,.$$

So, we have proven the inequality (10), and lemma.

From this lemma, taking into account the notation $(l+\lambda E)\,y=y^{(4)}+(p+\lambda)\,y^{(3)},\ y^{(3)}=z$, and Lemma 3, we come to the following conclusion.

Lemma 7. Let the function p satisfy the conditions of Lemma 3 and the relation (11). Then, the operator $l+\lambda E(\ \lambda\geq 0)$ is boundedly invertible in $L_2\,(\mathbb{R})$. Moreover, for any $y\in D(l+\lambda E)$, the following inequality holds:

$$\left\|y^{(4)}\right\|_2+\left\|(p+\lambda)y^{(3)}\right\|_2+\|y\|_2\leq C\,\|(l+\lambda E)\,y\|_2\,.$$

Remark 1. The condition (3), which was used in the proofs of Lemmas 3, 6, and 7, can be replaced with the condition $p\,(x)\geq 1$. Indeed, if we denote $x=\varepsilon^{-1}t\ (\ t>0)\ ,\ \widehat{y}\,(t)=y\left(\varepsilon^{-1}t\right)$ and $\widehat{p}\,(t)=p\left(\varepsilon^{-1}t\right)$. The operator $ly=y^{(4)}+p\,(x)\,y^{(3)}$ is transformed into

$$\varepsilon^4\widehat{l}\widehat{y}\,(t)=\widehat{y}^{(4)}\,(t)+\varepsilon^{-1}\widehat{p}\,(t)\,\widehat{y}^{(3)}\,(t)\,,$$

where $\varepsilon^{-1}\widehat{p}\,(t)\geq 1$.

4 Main result and its proof

Theorem 1. Assume that $p(x)$ satisfies conditions (3), $\gamma_{1,\sqrt{p},3} < \infty$ and $\gamma_{q,p,3} < \infty$. Then for any $f \in L_2(\mathbb{R})$ there exists a solution to equation (1) and it is unique. If, in addition, the relation (11) holds, then the solution y satisfies the following maximal regularity estimate

$$\left\| y^{(4)} \right\|_2 + \left\| p y^{(3)} \right\|_2 + \|(1+|q|)y\|_2 \le C \|f\|_2 . \tag{19}$$

Proof. In equation (1), we introduce a new variable t using the formula $x = \frac{t}{a}$. Let us denote:

$$\widetilde{y}(t) = y\left(a^{-1}t\right),\ \widetilde{p}(t) = p\left(a^{-1}t\right),\ \widetilde{q}(t) = q\left(a^{-1}t\right),\ \widetilde{F}(t) = a^{-4}F\left(a^{-1}t\right) \quad (t \in \mathbb{R}).$$

Then, equation (1) takes the form:

$$\widetilde{L}_{0a}\widetilde{y} = \widetilde{y}^{(4)}(t) + a^{-1}\widetilde{p}(t)\,\widetilde{y}^{(3)}(t) + a^{-4}\widetilde{q}(t)\,\widetilde{y}(t) = \widetilde{F}(t). \tag{20}$$

Let l_a be the closure of the differential operator

$$l_{0a}\widetilde{y} = \widetilde{y}^{(4)}(t) + a^{-1}\widetilde{p}(t)\,\widetilde{y}^{(3)}(t),\ \widetilde{y} \in C_0^{(4)}(\mathbb{R}),$$

in the space $L_2(\mathbb{R})$. It can be easily verified that $\gamma_{1,\sqrt{a^{-1}\widetilde{p}},3} = a^3\gamma_{1,\sqrt{p},3} < \infty$. By Lemma 3, the operator l_a is continuously invertible. Moreover, by Lemma 6, for each $\widetilde{y} \in D(l_a)$, we have

$$\left\| \widetilde{y}^{(4)}(t) \right\|_2 + \left\| a^{-1}\left(\widetilde{p}(t) + \lambda\right)\widetilde{y}^{(3)}(t) \right\|_2 + \|\widetilde{y}\|_2 \le C_a \|l_a\widetilde{y}\|_2 . \tag{21}$$

Further, $\gamma_{a^{-4}\widetilde{q},a^{-1}\widetilde{p},3} = \frac{1}{\sqrt{a}}\gamma_{q,p,3}$. Consequently, by Lemma 1, we obtain $\left\| a^{-4}\widetilde{q}\widetilde{y} \right\|_2 \le \frac{2}{\sqrt{a}}\gamma_{q,p,3}C_a \|l_a\widetilde{y}\|_2$. If we choose the parameter a such that $a \ge \max\left(\frac{4C_a^2}{\nu^2}\gamma_{q,p,3}^2,\ 1\right)$ $(0 < \nu < 1)$, then the following inequality holds:

$$\left\| a^{-4}\widetilde{q}\widetilde{y} \right\|_2 \le \nu \|l_a\widetilde{y}\|_2 ,\ 0 < \nu < 1. \tag{22}$$

Then, by the theorem on small perturbations, the closure $\widetilde{L}_a$ in $L_2(\mathbb{R})$ of the operator $\widetilde{L}_{0a}\widetilde{y} = l_a\widetilde{y} + a^{-4}\widetilde{q}(t)\,\widetilde{y}(t)$ is invertible, and its inverse $\widetilde{L}_a^{-1}$ is continuous. So, for each right-hand side $\widetilde{F}(t) \in L_2(\mathbb{R})$, the solution $\widetilde{y}$ of the equation (20) exists and is unique. Furthermore, by (22),

$$\left\| \widetilde{l}_a\widetilde{y} \right\|_2 \le \frac{1}{(1-\nu)} \left\| \widetilde{L}_a\widetilde{y} \right\|_2 .$$

In accordance with (21), we have

$$\left\| \widetilde{y}^{(4)}(t) \right\|_2 + \left\| a^{-1}\widetilde{p}(t)\,\widetilde{y}^{(3)}(t) \right\|_2 + \left\| a^{-4}\widetilde{q}\widetilde{y} \right\|_2 \le \left[C_a + \frac{1}{1-\nu} \right] \left\| \widetilde{L}_a\widetilde{y} \right\|_2 .$$

Returning by the substitution $x = \frac{1}{a}t$ to the variable x in this inequality, we obtain the estimate

$$\left\| y^{(4)} \right\|_2 + \left\| p y^{(3)} \right\|_2 + \|qy\|_2 \le C \|F\|_2 .$$

From here, the inequality (19) easily follows.

Conclusion

The qualitative properties of a fourth-order differential equation with unlimited intermediate and minor coefficients are studied in the work. For a wide class of coefficients the correctness of equation is proved and a maximal regularity estimate of the solution in the norm of the Hilbert space is obtained.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Advances in the generalized Cesàro polynomials

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Cesàro polynomials have been extended in various ways and applied in diverse areas. In this paper, we aim to introduce a multivariable and multiparameter generalization of Cesàro polynomials. Then we explore several generating functions, an addition formula, a differential-recurrence relation, a multiple integral formula for this extended Cesàro polynomial, as well as a multiple integral formula whose kernel is this extended Cesàro polynomial. Also we present several bilinear and bilateral generating functions for this extended Cesàro polynomial, two of whose examples are demonstrated.

Keywords: Cesàro polynomials, generating function, recurrence relation, hypergeometric function, integral representation.

2020 Mathematics Subject Classification: 33C45, 33C70.

Introduction

The generalized Cesàro polynomials $g_n^{(s)}(\lambda, x)$ are defined by [1]

$$g_n^{(s)}(\lambda, x) = \binom{s+n}{n} \, {}_2F_1 \left[\begin{matrix} -n, & \lambda; \\ & -s-n; \end{matrix} \; x \right], \tag{1}$$

where

$$g_n^{(s)}(x) := g_n^{(s)}(1, x), \;\; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \tag{2}$$

are the Cesàro polynomials [2–7]. Here ${}_2F_1$ denotes the hypergeometric function (or Gaussian hypergeometric function) [8]:

$${}_2F_1\left(a, b; c; x\right) = F\left(a, b; c; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k,$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol.

The generalized Cesàro polynomials $g_n^{(s)}(\lambda, x)$ in (1) have the following generating function [9]:

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x) t^n = (1-t)^{-s-1}(1-xt)^{-\lambda}. \tag{3}$$

Recall the following double series manipulations: Let f, $g : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ be functions and $p \in \mathbb{N}$. Then

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} f(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(k, n + pk), \tag{4}$$

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$$\sum_{k=0}^{n}\sum_{j=0}^{[k/p]} g\left(k,j\right)=\sum_{j=0}^{[n/p]}\sum_{k=0}^{n-pj} g\left(k+pj,j\right), \tag{5}$$

where $[\lambda]$ denotes the integer part of $\lambda\in\mathbb{R}$.

Cesàro polynomials have been generalized in various ways and used in diverse areas [1–7], [10; 62]. For example, Malik [11] has introduced Cesàro polynomials in two and three variables and has given their generating functions. In this paper, we provide a multivariable and multiparameter generalization of Cesàro polynomials. Then we investigate several generating functions, an addition formula, a differential-recurrence relation, a multiple integral formula for this extended Cesàro polynomial, as well as a multiple integral formula whose kernel is this extended Cesàro polynomial. Also we explore several bilinear and bilateral generating functions for this extended Cesàro polynomial, two examples of which are considered.

1 *Multivariable and multiparameter Cesàro polynomials*

In this section, we define a multivariable and multiparameter extension of the generalized Cesàro polynomials $g_n^{(s)}(\lambda,x)$ in (1) and obtain their generating functions. Also, we derive several properties for these polynomials.

Definition 1. Let $m\in\mathbb{N}$; $n\in\mathbb{N}_0$; $s\in\mathbb{C}\setminus\mathbb{N}_0$; λ_j, $x_j\in\mathbb{C}$ $(j=1,\ldots,m)$. Then an m variable and m parameter extension of the generalized Cesàro polynomials is defined by

$$\begin{aligned} &g_n^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m):\\ &=\sum_{r_1+\cdots+r_m=n}\binom{s+n}{n}\frac{(-n)_{\delta_m}}{(-s-n)_{\delta_m}}\prod_{j=1}^{m}\frac{(\lambda_j)_{r_j}\,x_j^{r_j}}{(r_j)!}, \end{aligned} \tag{6}$$

where

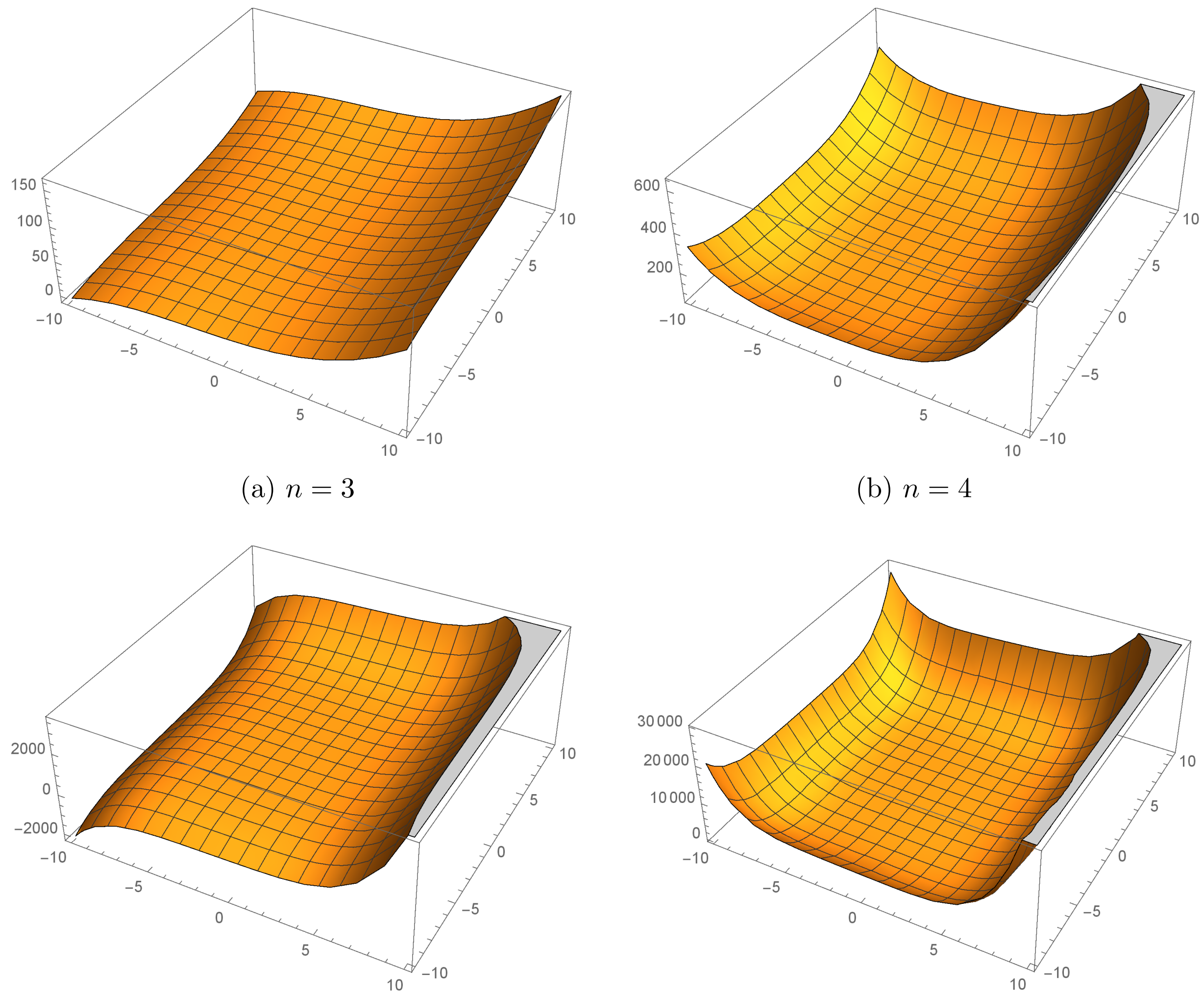
$$\delta_m:=r_1+\cdots+r_m. \tag{7}$$

The summation notation $\sum\limits_{r_1+\cdots+r_m=n}$ in (6) represents the following m-ple series:

$$\sum_{r_1+\cdots+r_m=n}=\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\cdots\sum_{r_m=0}^{n-r_1-\cdots-r_{m-1}}. \tag{8}$$

Figure demonstrates the surfaces of the generalized Cesàro polynomials $g_n^{(s)}(\lambda_1,\lambda_2,x_1,x_2)$ in two variables for some parameter values. We should remark that the special case of $m=1$ in (6) immediately reduces to the generalized Cesàro polynomials $g_n^{(s)}(\lambda,x)$ in (1). Also if we take $\lambda_j=1$ $(j=1,\ldots,m)$ in (6), we get the following multivariable generalization of the Cesàro polynomials $g_n^{(s)}(x)$ in (2):

$$g_n^{(s)}(x_1,\ldots,x_m):=\sum_{r_1+\cdots+r_m=n}\binom{s+n}{n}\frac{(-n)_{\delta_m}}{(-s-n)_{\delta_m}}\prod_{j=1}^{m}x_j^{r_j}.$$



(a) $n = 3$ (b) $n = 4$

(c) $n = 5$ (d) $n = 6$

Figure. Surfaces of the generalized Cesàro polynomials $g_n^{(s)}(\lambda_1, \lambda_2, x_1, x_2)$ in two variables for the parameter values $s = 4$, $\lambda_1 = 1/10$, $\lambda_2 = 1/20$ and $n = 3, 4, 5, 6$

In the study of special functions, a theoretical relationship to the unification of generating functions is critical. Several researchers have made strides in this approach [12–14].

The following theorems present two generating function relations for the multivariable-multiparameter Cesàro polynomials in (6).

Theorem 1. The multivariable-multiparameter generalized Cesàro polynomials in (6) are generated by the following function:

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m)\, t^n = (1-t)^{-s-1} \prod_{j=1}^{m} (1 - x_j t)^{-\lambda_j}, \tag{9}$$

where $|t| < \min\left\{ |x_1|^{-1}, \ldots, |x_m|^{-1}, 1 \right\}$ and $m \in \mathbb{N}$.

Proof. Let $\mathcal{L}_1$ be the left member of (9).

Replacing the $g_n^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)$ with (6) and (8), we get

$$\mathcal{L}_1=\sum_{n=0}^{\infty}\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\cdots\sum_{r_m=0}^{n-r_1-\cdots-r_{m-1}}\binom{s+n}{n}\times\frac{(-n)_{\delta_m}}{(-s-n)_{\delta_m}}\prod_{j=1}^{m}\frac{(\lambda_j)_{r_j}\,x_j^{r_j}}{(r_j)!}\,t^n. \tag{10}$$

Employing the case $p=1$ of (4) in the first double sums in (10) gives

$$\mathcal{L}_1=\sum_{r_1=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r_2=0}^{n}\sum_{r_3=0}^{n-r_2}\cdots\sum_{r_m=0}^{n-r_2-\cdots-r_{m-1}}\binom{s+n+r_1}{n+r_1}\times\frac{(-n-r_1)_{\delta_m}}{(-s-n-r_1)_{\delta_m}}\prod_{j=1}^{m}\frac{(\lambda_j)_{r_j}\,x_j^{r_j}}{(r_j)!}\,t^{n+r_1}. \tag{11}$$

Applying the same procedure as in getting (11) to the 2nd and 3rd double sums (11), and repeating the similar process, we find

$$\mathcal{L}_1=\sum_{n=0}^{\infty}\sum_{r_1=0}^{\infty}\cdots\sum_{r_m=0}^{\infty}\binom{s+n+\delta_m}{n+\delta_m}\frac{(-n-\delta_m)_{\delta_m}}{(-s-n-\delta_m)_{\delta_m}}\prod_{j=1}^{m}\frac{(\lambda_j)_{r_j}\,x_j^{r_j}}{(r_j)!}\,t^{n+\delta_m}, \tag{12}$$

where δ_m is given in (7).

Consider the following easily-derivable identity:

$$(-m-p)_p=(-1)^p\,\frac{(m+p)!}{m!}\qquad (m,\,p\in\mathbb{N}_0)\,. \tag{13}$$

Employing (13) in (12) offers

$$\begin{aligned}\mathcal{L}_1&=\sum_{n=0}^{\infty}\binom{s+n}{n}t^n\sum_{r_1=0}^{\infty}\ldots\sum_{r_m=0}^{\infty}\prod_{j=1}^{m}\frac{(\lambda_j)_{r_j}\,x_j^{r_j}}{(r_j)!}\,t^{\delta_m}\\&=\sum_{n=0}^{\infty}(s+1)_n\,\frac{t^n}{n!}\sum_{r_1=0}^{\infty}\frac{(\lambda_1)_{r_1}\,(x_1t)^{r_1}}{r_{1!}}\cdots\sum_{r_m=0}^{\infty}\frac{(\lambda_m)_{r_m}\,(x_mt)^{r_m}}{r_{m!}}.\end{aligned} \tag{14}$$

Using the generalized binomial theorem

$$(1-z)^{-\alpha}=\sum_{n=0}^{\infty}(\alpha)_n\,\frac{z^n}{n!}\qquad (|z|<1,\,\alpha\in\mathbb{C})$$

in each sum of the 2nd equality in (14), we arrive at the right member of (9).

Remark 1. The case $m=1$ of the generating function relation (9) reduces to the generating function relation (3).

Theorem 2. The multivariable-multiparameter generalized Cesàro polynomials in (6) are generated by the following generating function:

$$\sum_{n=0}^{\infty}\binom{n+k}{n}g_{n+k}^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)\,t^n=(1-t)^{-s-k-1}\times\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j}g_k^{(s)}\left(\lambda_1,\ldots,\lambda_m;\frac{x_1(1-t)}{1-x_1t},\cdots,\frac{x_m(1-t)}{1-x_mt}\right), \tag{15}$$

where $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $|t| < \min\left\{|x_1|^{-1}, \ldots, |x_m|^{-1}, 1\right\}$.

Proof. Replacing t by $t+u$ in (9) gives

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)(t+u)^n = (1-t-u)^{-s-1}\prod_{j=1}^{m}(1-x_jt-x_ju)^{-\lambda_j},$$

which, upon using binomial theorem, yields

$$\begin{aligned}\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)\sum_{k=0}^{n}\binom{n}{k}t^{n-k}u^k = (1-t)^{-s-1}\\ \times\left(1-\frac{u}{1-t}\right)^{-s-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j}\left(1-\frac{x_ju}{1-x_jt}\right)^{-\lambda_j}.\end{aligned} \tag{16}$$

Using (9) on the right member of (16), with the aid of the case $p=1$ of (4), offers

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k}{k}g_{n+k}^{(s)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)t^n\,u^k\\ =\;& (1-t)^{-s-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j}\\ &\times\sum_{k=0}^{\infty}g_k^{(s)}\left(\lambda_1,\ldots,\lambda_m;\frac{x_1(1-t)}{1-x_1t},\ldots,\frac{x_m(1-t)}{1-x_mt}\right)\left(\frac{u}{1-t}\right)^k,\end{aligned}$$

which, upon equating the coefficients of u^k on both sides, yields the desired identity (15).

Theorem 3. The following identity holds true:

$$\begin{aligned}&g_n^{(s_1+s_2+1)}(\lambda_1+\mu_1,\ldots,\lambda_m+\mu_m;x_1,\ldots,x_m)\\ &\quad=\sum_{k=0}^{n}g_{n-k}^{(s_1)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)g_k^{(s_2)}(\mu_1,\ldots,\mu_m;x_1,\ldots,x_m).\end{aligned} \tag{17}$$

Proof. From (9), we find

$$\begin{aligned}&\sum_{n=0}^{\infty}g_n^{(s_1+s_2+1)}(\lambda_1+\mu_1,\ldots,\lambda_m+\mu_m;x_1,\ldots,x_m)t^n\\ &=(1-t)^{-s_1-s_2-2}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j-\mu_j}\\ &=\sum_{n=0}^{\infty}g_n^{(s_1)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)t^n\sum_{k=0}^{\infty}g_k^{(s_2)}(\mu_1,\ldots,\mu_m,x_1,\ldots,x_m)t^k\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}g_{n-k}^{(s_1)}(\lambda_1,\ldots,\lambda_m;x_1,\ldots,x_m)g_k^{(s_2)}(\mu_1,\ldots,\mu_m;x_1,\ldots,x_m)\,t^n.\end{aligned}$$

Matching the coefficients of the first and last members yields the desired identity (17).

Theorem 4. The following differential-recurrence relation holds true:

$$\begin{aligned} &\frac{\partial}{\partial x_{j_0}} g_{n+1}^{(s)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m) \\ &= \lambda_{j_0} g_n^{(s)}(\lambda_1, \ldots, \lambda_{j_0}+1, \lambda_{j_0+1}, \lambda_m; x_1, \ldots, x_m), \end{aligned} \tag{18}$$

where $1 \le j_0 \le m$.

Proof. We will prove, when $j_0 = 1$. By symmetry, it will be easy to interpret the result into the general $1 \le j_0 \le m$.

Differentiating both sides of (9) with respect to x_1, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\partial}{\partial x_1} g_n^{(s)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m)\, t^{n-1} \\ &= \lambda_1 (1-t)^{-s-1} \left[(1 - x_1 t)^{-\lambda_1 - 1} \prod_{j=2}^{m} (1 - x_j t)^{-\lambda_j} \right], \end{aligned}$$

which, upon using (9), yields

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\partial}{\partial x_1} g_{n+1}^{(s)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m)\, t^{n} \\ &= \lambda_1\, g_n^{(s)}(\lambda_1 + 1, \lambda_2, \ldots, \lambda_m; x_1, \ldots, x_m)\, t^n. \end{aligned} \tag{19}$$

Equating the coefficients of t^n on both sides of (19) leads to the identity (18) when $j_0 = 1$.

Integrating both sides of (6) with respect to each of the variables x_j $(j = 1, \ldots, m)$ from 0 to 1 gives the result in the following theorem.

Theorem 5. Let $m \in \mathbb{N}$; $n \in \mathbb{N}_0$; $s \in \mathbb{C} \setminus \mathbb{N}_0$; $\lambda_j \in \mathbb{C}$ $(j = 1, \ldots, m)$. Then

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 g_n^{(s)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m)\, dx_1 \cdots dx_m \\ &= \sum_{r_1 + \cdots + r_m = n} \binom{s+n}{n} \frac{(-n)_{\delta_m}}{(-s-n)_{\delta_m}} \prod_{j=1}^{m} \frac{(\lambda_j)_{r_j}}{(r_j+1)!}, \end{aligned}$$

where δ_m is the same as in (7).

The following theorem provides an integral representation of the multivariable-multiparameter generalized Cesàro polynomials.

Theorem 6. Let $m \in \mathbb{N}$; $n \in \mathbb{N}_0$; $s \in \mathbb{C} \setminus \mathbb{N}_0$; λ_j, $x_j \in \mathbb{C}$ $(j = 1, \ldots, m)$. Also let $\Re(s+1) > 0$, $\Re(\lambda_j) > 0$ $(j = 1, \ldots, m)$. Then

$$\begin{aligned} &g_n^{(s)}(\lambda_1, \ldots, \lambda_m; x_1, \ldots, x_m) \\ &= \frac{1}{n!\,\Gamma(s+1) \prod\limits_{j=1}^{m} \Gamma(\lambda_j)} \int_0^{\infty} \cdots \int_0^{\infty} e^{-(u + u_1 + \cdots + u_m)} \\ &\times \left(u + \sum_{j=1}^{m} u_j x_j \right)^n u^s u_1^{\lambda_1 - 1} \cdots u_m^{\lambda_m - 1} du du_1 \cdots du_m. \end{aligned} \tag{20}$$

Proof. Recall that the well-known identity as

$$c^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-ct} t^{v-1} dt \quad (\Re(c) > 0, \, \Re(\nu) > 0) . \tag{21}$$

Using (21) in each factor of the right member of (9), under the restrictions in Theorem 1, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} g_n^{(s)}(\lambda_1, \ldots, \lambda_m, x_1, \ldots, x_m) t^n \\
&= \frac{1}{\Gamma(s+1)} \int_0^\infty e^{-(1-t)u} u^s du \frac{1}{\Gamma(\lambda_1)} \int_0^\infty e^{-(1-x_1 t)u_1} u_1^{\lambda_1 - 1} du_1 \\
&\times \cdots \frac{1}{\Gamma(\lambda_m)} \int_0^\infty e^{-(1-x_m t)u_m} u_m^{\lambda_m - 1} du_m \\
&= \frac{1}{\Gamma(s+1)\Gamma(\lambda_1)\cdots\Gamma(\lambda_m)} \int_0^\infty \cdots \int_0^\infty e^{-(u+u_1+\cdots+u_m)} u^s u_1^{\lambda_1-1} \cdots u_m^{\lambda_m-1} \\
&\times \sum_{n=0}^{\infty} \frac{(u + u_1 x_1 + \cdots + u_m x_m)^n}{n!} du du_1 \cdots du_m \, t^n.
\end{aligned}$$

Equating the coefficients of t^n on the first and last members of the last resulting identity yields the desired integral representation (20).

2 Miscellaneous generating function relations

Now, we obtain new substantial families of bilinear and bilateral generating function relations for the multivariable-multiparameter generalized Cesàro polynomials in (6).

Throughout this section, let $m, \, p, \, q, \, r \in \mathbb{N}; \; l \in \mathbb{N}_0; \;\; \mu, \, \nu \in \mathbb{C}; \; a_k \in \mathbb{C} \setminus \{0\} \; (k \in \mathbb{N}_0)$. Also let

$$\Omega_\mu : \mathbb{C}^r \longrightarrow \mathbb{C} \setminus \{0\}$$

be a bounded function.

Theorem 7. Let

$$\Lambda_{\mu,\nu}(y_1, \ldots, y_r; \eta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, \ldots, y_r) \, \eta^k$$

and

$$\begin{aligned}
&\Theta_{n,p}^{\mu,\nu} \left(\lambda_1, \ldots, \lambda_m, x_1, \ldots, x_m; y_1, \ldots, y_r; \xi\right) \\
&:= \sum_{k=0}^{[n/p]} a_k g_{n-pk}^{(s)} \left(\lambda_1, \ldots, \lambda_m, x_1, \ldots, x_m\right) \Omega_{\mu+\nu k}(y_1, \ldots, y_r) \, \xi^k.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\nu} \left(\lambda_1, \ldots, \lambda_m, x_1, \ldots, x_m; y_1, \ldots, y_r; \tfrac{\eta}{t^p}\right) t^n \\
&= (1-t)^{-s-1} \prod_{j=1}^{m} (1 - x_j t)^{-\lambda_j} \Lambda_{\mu,\nu}(y_1, \ldots, y_r; \eta).
\end{aligned} \tag{22}$$

Proof. Let $\mathcal{L}_2$ be the left member of (22). Then we have

$$\mathcal{L}_2 = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k g_{n-pk}^{(s)} (\lambda_1, \dots, \lambda_m, x_1, \dots, x_m) \, \Omega_{\mu+\nu k}(y_1, \dots, y_r) \eta^k t^{n-pk}.$$

Using (4), we obtain

$$\begin{aligned} \mathcal{L}_2 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \, g_n^{(s)} (\lambda_1, \dots, \lambda_m, x_1, \dots, x_m) \, \Omega_{\mu+\nu k}(y_1, \dots, y_r) \eta^k t^n \\ &= \sum_{n=0}^{\infty} g_n^{(s)} (\lambda_1, \dots, \lambda_m, x_1, \dots, x_m) \, t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, \dots, y_r) \eta^k \\ &= (1-t)^{-s-1} \prod_{j=1}^{m} (1 - x_j t)^{-\lambda_j} \Lambda_{\mu,\nu}(y_1, \dots, y_r; \eta), \end{aligned}$$

which is the right member of (22).

Theorem 8. Let

$$N_{n,l,q}^{\mu,p}(y_1, \dots, y_r; z) := \sum_{k=0}^{[n/q]} \binom{l+n}{n-qk} a_k \Omega_{\mu+pk}(y_1, \dots, y_r) z^k. \tag{23}$$

Also let

$$\begin{aligned} &\Lambda_{l,q}^{\mu,p} \left[\lambda_1, \dots, \lambda_m, x_1, \dots, x_m; y_1, \dots, y_r; t\right] \\ &\quad := \sum_{n=0}^{\infty} a_n g_{l+qn}^{(s)}(\lambda_1, \dots, \lambda_m, x_1, \dots, x_m) \Omega_{\mu+pn}(y_1, \dots, y_r) \, t^n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=0}^{\infty} g_{l+n}^{(s)}(\lambda_1, \dots, \lambda_m, x_1, \dots, x_m) N_{n,l,q}^{\mu,p}(y_1, \dots, y_r; z) t^n \\ &= (1-t)^{-s-l-1} \prod_{j=1}^{m} (1 - x_j t)^{-\lambda_j} \\ &\times \Lambda_{l,q}^{\mu,p} \left[\lambda_1, \dots, \lambda_m, \frac{x_1(1-t)}{1-x_1 t}, \dots, \frac{x_m(1-t)}{1-x_m t}; y_1, \dots, y_r; z\Big(\frac{t}{1-t}\Big)^q\right]. \end{aligned} \tag{24}$$

Proof. Let $\mathcal{L}_3$ be the left member of (24). Using (23), we have

$$\mathcal{L}_3 = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/q]} g_{l+n}^{(s)}(\lambda_1, \dots, \lambda_m, x_1, \dots, x_m) \binom{l+n}{n-qk} a_k \Omega_{\mu+pk}(y_1, \dots, y_r) z^k \, t^n.$$

Employing (4), in view of the result in Theorem 2, we may write

$$\begin{aligned} \mathcal{L}_3 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g_{l+n+qk}^{(s)}(\lambda_1, \dots, \lambda_m, x_1, \dots, x_m) \binom{l+n+qk}{n} \\ &\quad \times a_k \Omega_{\mu+pk}(y_1, \dots, y_r) z^k t^{n+qk} \end{aligned}$$

$$
\begin{aligned}
&= (1-t)^{-s-l-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j} \\
&\quad\times\sum_{k=0}^{\infty}a_k g_{l+qk}^{(s)}\left(\lambda_1,\ldots,\lambda_m,\frac{x_1(1-t)}{1-x_1t},\ldots,\frac{x_m(1-t)}{1-x_mt}\right) \\
&\quad\times\Omega_{\mu+pk}(y_1,\ldots,y_r)\frac{z^kt^{qk}}{(1-t)^{qk}} \\
&= (1-t)^{-s-l-1}\prod_{j=1}^{m}(1-x_jt)^{-\lambda_j} \\
&\quad\times\Lambda_{l,q}^{\mu,p}\left[\lambda_1,\ldots,\lambda_m,\frac{x_1(1-t)}{1-x_1t},\ldots,\frac{x_m(1-t)}{1-x_mt};y_1,\ldots,y_r;z\left(\frac{t}{1-t}\right)^q\right],
\end{aligned}
$$

which is the right member of (24).

Theorem 9. Let

$$
\begin{aligned}
&\Lambda_{\mu,\nu}^{n,p}(\lambda_1+\beta_1,\ldots,\lambda_m+\beta_m,x_1,\ldots,x_m;y_1,\ldots,y_r;z) \\
&:=\sum_{k=0}^{[n/p]}a_k g_{n-pk}^{(s_1+s_2+1)}(\lambda_1+\beta_1,\ldots,\lambda_m+\beta_m,x_1,\ldots,x_m)\Omega_{\mu+\nu k}(y_1,\ldots,y_r)z^k.
\end{aligned}
$$

Then, for $n\in\mathbb{N}_0$, we have

$$
\begin{aligned}
&\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l g_{n-k}^{(s_1)}(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m)g_{k-pl}^{(s_2)}(\beta_1,\ldots,\beta_m,x_1,\ldots,x_m) \\
&\quad\times\Omega_{\mu+\nu l}(y_1,\ldots,y_r)z^l \\
&\quad=\Lambda_{\mu,\nu}^{n,p}(\lambda_1+\beta_1,\ldots,\lambda_m+\beta_m,x_1,\ldots,x_m;y_1,\ldots,y_r;z).
\end{aligned}
\tag{25}
$$

Proof. Let $\mathcal{L}_4$ be the left member of (25). Using (5) and then using addition formula (17) for the multivariable-multiparameter generalized Cesàro polynomials, we get

$$
\begin{aligned}
\mathcal{L}_4 &= \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl}a_l g_{n-k-pl}^{(s_1)}(\lambda_1,\ldots,\lambda_m,x_1,\ldots,x_m)g_k^{(s_2)}(\beta_1,\ldots,\beta_m,x_1,\ldots,x_m) \\
&\quad\times\Omega_{\mu+\nu l}(y_1,\ldots,y_r)z^l \\
&= \sum_{l=0}^{[n/p]}a_l g_{n-pl}^{(s_1+s_2+1)}(\lambda_1+\beta_1,\ldots,\lambda_m+\beta_m,x_1,\ldots,x_m)\Omega_{\mu+\nu l}(y_1,\ldots,y_r)z^l \\
&= \Lambda_{\mu,\nu}^{n,p}(\lambda_1+\beta_1,\ldots,\lambda_m+\beta_m,x_1,\ldots,x_m;y_1,\ldots,y_r;z),
\end{aligned}
$$

which is the right member of (25).

3 Concluding remarks and examples

We proposed an extension of Cesàro polynomials to several variables and parameters. Then we investigated several generating functions, an addition formula, a differential-recurrence relation, a multiple integral formula for this extended Cesàro polynomial, as well as a multiple integral formula kernel of which is this extended Cesàro polynomial. Also we explored several bilinear and bilateral generating functions for this extended Cesàro polynomial, two examples of which are demonstrated in Examples 1 and 2.

Since the multivariable function $\Omega_{\mu+\nu k}(y_1, \ldots, y_r)$ is very general, we may deduce a number of particular formulas from the results in Sections 1 and 2. We just use Theorem 7 to present the following two examples.

Example 1. The Bessel function $J_\mu(x)$ are generated by (see, e.g., [15; p. 141])

$$\left(1 - \frac{2t}{x}\right)^{-\mu/2} J_\mu\left(\sqrt{x^2 - 2xt}\right) = \sum_{n=0}^{\infty} J_{\mu+n}(x) \frac{t^n}{n!}. \tag{26}$$

If we take $r = 1$, $a_k = \frac{1}{k!}$, $\nu = 1$ and substitute the Bessel function for $\Omega_{\mu+\nu k}$ in Theorem 7, using the relation (26), we can obtain the following result providing a class of bilateral generating function relation for the multivariable generalized Cesàro polynomials and the Bessel functions:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} g_{n-pk}^{(s)}(\lambda_1, \ldots, \lambda_m, x_1, \ldots, x_m) J_{\mu+k}(y) \eta^k t^{n-pk}$$
$$= \left(1 - \frac{2\eta}{y}\right)^{-\mu/2} J_\mu\left(\sqrt{y^2 - 2y\eta}\right) (1-t)^{-s-1} \prod_{j=1}^{m} (1 - x_j t)^{-\lambda_j}.$$

Example 2. Taking $r = m$, $a_k = 1$, $\mu = 0$, $\nu = 1$ and substituting the multivariable-multiparameter generalized Cesàro polynomials for $\Omega_{\mu+\nu k}$ in Theorem 7, and using the generating relation (9), we may get the following class of bilinear generating functions for the multivariable-multiparameter generalized Cesàro polynomials:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} g_{n-pk}^{(s)}(\lambda_1, \ldots, \lambda_m, x_1, \ldots, x_m)$$
$$\times g_k^{(s)}(\beta_1, \ldots, \beta_m, y_1, \ldots, y_m) \eta^k t^{n-pk}$$
$$= [(1-t)(1-\eta)]^{-s-1} \prod_{j=1}^{m} (1 - x_j t)^{-\lambda_j} (1 - y_j \eta)^{-\beta_j},$$

where each variable, each parameter, and each index can be suitably restricted so that this formula is meaningful.

Obviously, many other particular cases of Theorem 7 can be provided. Further, the results in the other theorems in Sections 1 and 2 can reduce to yield a variety of identities about the extended Cesàro polynomials (6) and their simpler ones.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Solving Volterra-Fredholm integral equations by non-polynomial spline functions

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It depends on our information, non-polynomial spline functions have not been applied for solving Volterra-Fredholm integral equations of the second kind yet. In this paper, we want to use such functions for finding approximation solutions of Volterra-Fredholm integral equations. In our approach, the coefficients of the non-polynomial spline were found by solving a system of linear equations. Then, these functions were utilized to reduce the fredholm integral equations to the solution of algebraic equations. Analysis of convergences investigated. Finally, three examples were presented to show the effectiveness of the method. This was done with the help of a computer program that used the Python code program version 3.9.

Keywords: Volterra integral equation, Fredholm integral equation, non-polynomial spline function.

2020 Mathematics Subject Classification: 45B05, 45B05, 41A15.

Introduction

Integral equations are a fundamental class of equations in mathematical analysis that involve an unknown function under an integral sign. They arise naturally in various fields such as physics, engineering, and applied mathematics, where they model phenomena ranging from heat conduction to quantum mechanics [1–3].

Volterra-Fredholm integral equations are vital in numerous scientific and engineering disciplines, such as mechanics, electrical engineering, and physics. These equations are instrumental in modeling intricate phenomena involving integration, which is essential for comprehending and addressing issues in these areas. To approximate solutions for these equations, numerical methods utilizing non-polynomial spline functions have been developed. This technique serves as an effective means for resolving integral equations lacking analytical solutions, delivering precise numerical answers for a range of significant problems. In recent years, there has been a growing interest in using non-polynomial splines to find numerical solutions for integral equations and other types of equations, as evidenced by the increasing number of published articles on the topic. Non-polynomial spline functions are a type of interpolation function that can be used to approximate the solution of integral equations. Non-polynomial spline functions are particularly useful because they can provide good approximation properties and can often handle irregularities in the solution or the kernel function better than polynomial-based methods. Numerous researchers utilize non-polynomial splines to solve Volterra and Fredholm integral equations [4–16].

Recently, the interplay of Volterra-Fredholm integral equation using many numerical techniques has been investigated [17–24]. For the first time, Salim, et al [25–27] used linear, quadratic and cubic

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spline function for solving the following linear Volterra-Fredholm integral equation of the second kind,

$$u(x) = f(x) + \lambda_1 \int_a^x K(x,t)u(t)dt + \lambda_2 \int_a^b L(x,t)u(t)dt, \tag{1}$$

where the functions $f(x)$, and the kernels $K(x,t)$ and $L(x,t)$ are known L^2 analytic functions and λ_1, λ_2 are arbitrary constants, x is variable and $u(x)$ is the unknown continuous function to be determined.

In this paper, we introduce a novel non-polynomial spline function to obtain the numerical solution of equation (1) for the first time.

The structure of this paper is as follows: Section 2 introduces our method for solving equation (1). Section 3 details our methodology, while Section 4 focuses on the convergence analysis. Section 5 presents several numerical examples to demonstrate the effectiveness of our technique. Finally, Section 6 offers some tentative conclusions.

1 *Non-polynomial spline function*

We describe the non-polynomial spline for solving equation (1) in similar manner of [11] The numerical scheme has been developed on the domain of integration $\omega = [a,b]$ with partitions

$$a = x_0 < x_1 < \cdots < x_n = b,$$

where $x_i = x_0 + ih$, $i = 0, \cdots, n$ and $h = \dfrac{b-a}{n}$. Let $S_i(x)$ be the interpolating non-polynomial spline function which interpolate y at x_i defined by [11–13]

$$S_i(x) = a_i + b_i(x - x_i) + c_i \sin\tau(x - x_i) + d_i \cos\tau(x - x_i), \tag{2}$$

where a_i, b_i, c_i and d_i are real numbers and τ is an arbitrary parameter. We denote the following relations

$$S_i(x_i,\tau) = y_i, \quad S_i(x_{i+1},\tau) = y_{i+1}, \quad S_i''(x_k,\tau) = M_i, \quad S_i''(x_{i+1},\tau) = M_{i+1}. \tag{3}$$

Using equation (2) and equation (3) we have the following expressions

$$a_i = y_i + \frac{M_i}{\tau^2}, \quad b_i = \frac{y_{i+1} - y_i}{h} + \frac{M_{i+1} - M_i}{\tau\theta}, \quad c_i = \frac{M_i \cos\theta - M_{i+1}}{\tau^2 \sin\theta},$$

$$d_i = \frac{-M_i}{\tau^2}. \tag{4}$$

With the continuity of first derivatives of $S_{i-1}(x)$ and $S_i(x)$ at $x = x_i$, $i = 1, 2, \cdots, n-1$, we obtain the following consistency relation,

$$\alpha M_{i-1} + 2\beta M_i + \alpha M_{i+1} = \frac{1}{h}(y_{i+1} - 2y_i + y_{i-1}), \tag{5}$$

where $\alpha = (\frac{1}{\theta^2})(\theta\csc\theta - 1)$, $\quad \beta = (\frac{1}{\theta^2})(1 - \theta\cot\theta)$.

Using the finite difference operator $E = e^{hD}$ in the above consistency relation where D is differential operator and by expanding in powers of hD, the error for relation equation (5) can be expressed as follows:

$$Error = (2\alpha + 2\beta - 1)(M_i - y_i'') + D^2h^2\left(\alpha - \frac{1}{12}\right)(M_i - y_i'') +$$

$$+ D^4h^4\left(\frac{\alpha}{12} - \frac{1}{360}\right)(M_i - y_i'') + O(h^6). \tag{6}$$

The consistency relation equation (5) for the above equation leads to the equation $2\alpha + 2\beta = 1$, which may also be expressed as $\tan\left(\frac{\theta}{2}\right) = \left(\dfrac{\theta}{2}\right)$. This equation has a zero root and an infinitely many of non-zero roots, the smallest positive root being $\theta = 8.98881$.

We use this θ as an optimal value in the convergence analysis and numerical computation. In this case, we have

$$\left|M_i - y_i''\right| \le k_2 h^2, \quad k_2 = 0.22 \max\left|y_i^4\right|,$$

provided that $2\alpha + 2\beta = 1$.

If we let $\alpha = \dfrac{1}{12}$ and $\beta = \dfrac{5}{12}$, the second term in the error equation (6) is also zero and the method can be modified and has a precision of the order $O(h^4)$ to calculate vector M:

$$M_{i+1} + 10M_i + M_{i-1} = \frac{12}{h}(y_{i+1} - 2y_i + y_{i-1}),$$

$$\left|M_i - y_i''\right| \le k_2 h^4, \quad k_2 = \frac{1}{240} \max\left|y_i^6\right|. \tag{7}$$

Provided that $2\alpha + 2\beta = 1$. If we let, $\alpha = \dfrac{1}{12}$ and $\beta = \dfrac{5}{12}$ the second term in the error equation (6) is also zero and the system (5) with natural cubic Spline initial condition $M_0 = M_n = 0$ is strictly diagonally dominant and has a unique solution to obtain $M_1, M_2, \cdots, M_{n-1}$.

From equation (7), we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 1 & 10 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 1 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 1 & 10 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ \vdots \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix} = \frac{12}{h^2} \begin{bmatrix} 0 \\ y_0 - 2y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ \vdots \\ \vdots \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \\ 0 \end{bmatrix}. \tag{8}$$

The above matrix form can be expressed as follows:

$$WM = \frac{12}{h^2} JY \quad \Rightarrow \quad M = \frac{12}{h^2} W^{-1} JY, \tag{9}$$

where $Y = (y_0, y_1, y_2, \cdots, y_{n-1}, y_n)^T$ and $M = (M_0, M_1, M_2, \cdots, M_{n-1}, M_n)^T$.

2 Methodology

In this section, we present numerical scheme to approximate equation (1). From equation (2) and equation (4) we have

$$U_i = y_i + \frac{M_i}{\tau^2} + \left(\frac{y_{i+1}}{h} + \frac{M_{i+1}}{h}\right)(x - x_i) - \left(\frac{y_i}{h} + \frac{M_i}{h}\right)(x - x_i) +$$

$$+ \frac{M_i \cos\theta}{\tau^2 \sin\theta} \sin\tau(x - x_i) - \frac{M_{i+1}\cos\theta}{\tau^2 \sin\theta} \sin\tau(x - x_i) - \frac{M_i}{\tau^2} \cos\tau(x - x_i). \tag{10}$$

By replacing equation (10) in equation (1) and using the collocation method, we have

$$U_i = f(x_i) + \int_a^x K(x_i,t)u(t)dt + \int_a^b L(x_i,t)u(t)dt$$

$$= f(x_i) + \sum_{j=0}^{i}\int_{x_j}^{x_{j+1}} K(x_i,t)u_j(t)dt + \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} L(x_i,t)u_j(t)dt$$

$$= f(x_i) + \sum_{j=0}^{i} u_{j(t)}\int_{x_j}^{x_{j+1}} K(x_i,t)dt + \sum_{j=0}^{n-1} u_{j(t)}\int_{x_j}^{x_{j+1}} L(x_i,t)dt$$

$$= f(x_i) + \sum_{j=0}^{i}(y_j + \frac{M_j}{\tau^2})\int_{x_j}^{x_{j+1}} K(x_i,t)dt + \sum_{j=0}^{n-1}(y_j + \frac{M_j}{\tau^2})\int_{x_j}^{x_{j+1}} L(x_i,t)dt$$

$$+ \sum_{j=0}^{i}\left(\frac{y_{j+1}}{h} + \frac{M_{j+1}}{\tau\theta}\right)\int_{x_j}^{x_{j+1}} K(x_i,t)(t-t_j)dt + \sum_{j=0}^{n-1}\left(\frac{y_{j+1}}{h} + \frac{M_{j+1}}{\tau\theta}\right)\int_{x_j}^{x_{j+1}} L(x_i,t)(t-t_j)dt$$

$$- \sum_{j=0}^{i}\left(\frac{y_j}{h} + \frac{M_{j+1}}{\tau\theta}\right)\int_{x_j}^{x_j} K(x_i,t)(t-t_j)dt - \sum_{j=0}^{n-1}\left(\frac{y_j}{h} + \frac{M_{j+1}}{\tau\theta}\right)\int_{x_j}^{x_{j+1}} L(x_i,t)(t-t_j)dt$$

$$+ \sum_{j=0}^{i}\frac{M_j\cos\theta}{\tau^2\sin\theta}\int_{x_j}^{x_{j+1}} K(x_i,t)\sin\tau(t-t_j)dt + \sum_{j=0}^{n-1}\frac{M_j\cos\theta}{\tau^2\sin\theta}\int_{x_j}^{x_{j+1}} L(x_i,t)\sin\tau(t-t_j)dt$$

$$+ \sum_{j=0}^{i}\frac{M_{j+1}\cos\theta}{\tau^2\sin\theta}\int_{x_j}^{x_{j+1}} K(x_i,t)\sin\tau(t-t_j)dt + \sum_{j=0}^{n-1}\frac{M_{j+1}\cos\theta}{\tau^2\sin\theta}\int_{x_j}^{x_{j+1}} L(x_i,t)\sin\tau(t-t_j)dt$$

$$+ \sum_{j=0}^{i}\frac{M_j}{\tau^2}\int_{x_j}^{x_{j+1}} K(x_i,t)\cos\tau(t-t_j)dt + \sum_{j=0}^{n-1}\frac{M_j}{\tau^2}\int_{x_j}^{x_{j+1}} L(x_i,t)\cos\tau(t-t_j)dt$$

$$= f(x_i) + \sum_{j=0}^{i}(y_j + \frac{M_j}{\tau^2})a_{ij} + \sum_{j=0}^{n-1}(y_j + \frac{M_j}{\tau^2})a^*_{ij} + \sum_{j=0}^{i}\left(\frac{y_{j+1}}{h} + \frac{M_{j+1}}{\tau\theta}\right)b_{ij+1}$$

$$+ \sum_{j=0}^{n-1}\left(\frac{y_{j+1}}{h} + \frac{M_{j+1}}{\tau\theta}\right)b^*_{ij+1} - \sum_{j=0}^{i}\left(\frac{y_j}{h} + \frac{M_j}{\tau\theta}\right)c_{ij} - \sum_{j=0}^{n-1}(\frac{y_j}{h} + \frac{M_j}{\tau\theta})c^*_{ij} + \sum_{j=0}^{i}\frac{M_j\cos\theta}{\tau^2\sin\theta}d_{ij}$$

$$+ \sum_{j=0}^{n-1}\frac{M_j\cos\theta}{\tau^2\sin\theta}d^*_{ij} + \sum_{j=0}^{i}\frac{M_{j+1}}{\tau^2 sin\theta}e_{ij+1} + \sum_{j=0}^{n-1}\frac{M_{j+1}}{\tau^2\sin\theta}e^*_{ij+1} + \sum_{j=0}^{i}\frac{M_j}{\tau^2}p_{ij} + \sum_{j=0}^{n-1}\frac{M_j}{\tau^2}p^*_{ij}$$

$$= f(i) + \sum_{j=0}^{i}(y_j + \frac{M_j}{\tau^2})a_{ij} + \sum_{j=0}^{n}(y_j + \frac{M_j}{\tau^2})a^*_{ij} + (-y_n - \frac{M_n}{\tau^2}a^*_{in})$$

$$- (\frac{y_0}{h} + \frac{M_0}{\tau\theta})b_{i0} + \sum_{j=0}^{i}(\frac{y_j}{h} + \frac{M_j}{\tau\theta})b_{ij} + \left(\frac{y_{i+1}}{h} + \frac{M_{i+1}}{\tau\theta}\right)b_{i,i+1} - (\frac{y_0}{h} + \frac{M_0}{\tau\theta})b^*_{i0}$$

$$+ \sum_{j=0}^{n}\left(\frac{y_j}{h} + \frac{M_j}{\tau\theta}\right)b^*_{ij} - \sum_{j=0}^{i}(\frac{y_j}{h} + \frac{M_j}{\tau\theta})c_{ij} - \sum_{j=0}^{n}\left(\frac{y_j}{h} + \frac{M_j}{\tau\theta}\right)c^*_{ij} + \left(-\left(\frac{y_n}{h} + \frac{M_n}{\tau\theta}\right)\right)c^*_{in}$$

$$+ \sum_{j=0}^{i}\frac{M_j\cos\theta}{\tau^2\sin\theta}d_{ij} + \sum_{j=0}^{n-1}\frac{M_j\cos\theta}{\tau^2\sin\theta}d^*_{ij} - \frac{M_n\cos\theta}{\tau^2\sin\theta}d^*_{in} - M_o e_{io} + \sum_{j=0}^{i}\frac{M_j}{\tau^2\sin\theta}e_{ij}$$

$$+ M_{i+1}e_{i,i+1} - M_o e^*_{io} + \sum_{j=0}^{n}\frac{M_j}{\tau^2\sin\theta}e^*_{ij} + \sum_{j=0}^{i}\frac{M_j}{\tau^2}p_{ij} + \sum_{j=0}^{n}\frac{M_j}{\tau^2}p^*_{ij} - \frac{M_n}{\tau^2}p_{in}$$

$$
\begin{aligned}
&= f(i) + \sum_{j=0}^{i}(y_j + \frac{M_j}{\tau^2})a_{ij} + \sum_{j=0}^{n}(y_j + \frac{M_j}{\tau^2})a^*_{ij} + \sum_{j=0}^{i}\left(\frac{y_j}{h} + \frac{M_j}{\tau\theta}\right) b_{ij} \\
&+ \sum_{j=0}^{n}(\frac{y_j}{h} + \frac{M_j}{\tau\theta})b^*_{ij} - \sum_{j=0}^{i}(\frac{y_j}{h} + \frac{M_j}{\tau\theta})c_{ij} - \sum_{j=0}^{n}\left(\frac{y_j}{h} + \frac{M_j}{\tau\theta}\right) c^*_{ij} + \sum_{j=0}^{i}\frac{M_j\cos\theta}{\tau^2\sin\theta}d_{ij} \\
&+ \sum_{j=0}^{n-1}\frac{M_j\cos\theta}{\tau^2\sin\theta}d^*_{ij} + \sum_{j=0}^{i}\frac{M_j}{\tau^2\sin\theta}e_{ij} + \sum_{j=0}^{n}\frac{M_j}{\tau^2\sin\theta}e^*_{ij} + \sum_{j=0}^{i}\frac{M_j}{\tau^2}p_{ij} \\
&+ \sum_{j=0}^{n}\frac{M_j}{\tau^2}p^*_{ij} + O(h^4), \quad i = 0, 1, 2, \cdots, n. \qquad (11)
\end{aligned}
$$

After finding the above integration by quadrature rules, we assuming $a^*_{in} = b_{i0} = b^*_{i0} = b_{i,i+1} = c^*_{in} = d^*_{in} = e_{i0} = e_{i,i+1} = e^*_{i0} = p^*_{in} = 0$. Now, if we suppose $a_{ij} = A$, $b_{ij+1} = B$, $c_{ij} = C$, $d_{ij} = D$, $e_{ij+1} = E$ and $p_{ij+1} = P$. Also, $a^*_{ij} = A^*$, $b^*_{ij+1} = B^*$, $c^*_{ij} = C^*$, $d^*_{ij} = D^*$, $e^*_{ij+1} = E^*$, $p^*_{ij+1} = P^*$,

$$
\begin{aligned}
a_{ij} &= \int_{x_j}^{x_{j+1}} K(x_i,t)dt, & a^*_{ij} &= \int_{x_j}^{x_{j+1}} L(x_i,t)dt, \\
b_{ij+1} &= \int_{x_j}^{x_{j+1}} K(x_i,t)(t-t_j)dt, & b^*_{ij+1} &= \int_{x_j}^{x_{j+1}} L(x_i,t)(t-t_j)dt, \\
c_{ij} &= \int_{x_j}^{x_{j+1}} K(x_i,t)(t-t_j)dt, & c^*_{ij} &= \int_{x_j}^{x_{j+1}} L(x_i,t)(t-t_j)dt, \\
d_{ij} &= \int_{x_j}^{x_{j+1}} K(x_i,t)sin\tau(t-t_j)dt, & d^*_{ij} &= \int_{x_j}^{x_{j+1}} L(x_i,t)\sin\tau(t-t_j)dt, \\
e_{ij+1} &= \int_{x_j}^{x_{j+1}} K(x_i,t)\sin\tau(t-t_j)dt, & e^*_{ij+1} &= \int_{x_j}^{x_{j+1}} L(x_i,t)\sin\tau(t-t_j)dt, \\
p_{ij} &= \int_{x_j}^{x_{j+1}} K(x_i,t)\cos\tau(t-t_j)dt, & p^*_{ij} &= \int_{x_j}^{x_{j+1}} L(x_i,t)\cos\tau(t-t_j)dt,
\end{aligned}
$$

$$
\hat{M} \approx M = (M_0, M_1, M_2, \cdots, M_{n-1}, M_n)^T, \hat{u} \approx U = (U_0, U_1, U_2, \cdots, U_{n-1}, U_n)^T,
$$
$$
F = (f_0, f_1, f_2, \cdots, f_{n-1}, f_n)^T,
$$

we have

$$
\begin{aligned}
\hat{U} = F + [(A+\hat{A}) + \frac{1}{h}(B+\hat{B}) - \frac{1}{h}(C+\hat{C})]\hat{U} + \frac{1}{\tau^2}[(A+\hat{A}) + \frac{1}{h}(B+\hat{B}) \\
-\frac{1}{h}(C+\hat{C}) + \cot\theta D - \csc\theta E - P]\hat{M}.
\end{aligned}
$$

Using equation (9) we have

$$
\begin{aligned}
[I - [(A+\hat{A}) + \frac{1}{h}(B+\hat{B}) - \frac{1}{h}(C+\hat{C})] - \frac{12}{\theta^2}[(A+\hat{A}) + \frac{1}{h}(B+\hat{B}) \\
-\frac{1}{h}(C+\hat{C}) + \cot\theta D - csc\theta E - P]Z]\hat{U} = F. \qquad (12)
\end{aligned}
$$

Let

$$
H_1 = [(A+\hat{A}) + \frac{1}{h}(B+\hat{B}) - \frac{1}{h}(C+\hat{C})],
$$

and

$$H_2 = [(A + \hat{A}) + \frac{1}{h}(B + \hat{B}) - \frac{1}{h}(C + \hat{C}) + \cot\theta D - \csc\theta E - P]Z],$$

where $Z = W^{-1}J$. This implies that

$$[I - (H_1 + H_2 Z)]\hat{U} = F.$$

If we suppose $W^{-1} = (u_{ij}),\ 1 \leq i, j \leq n+1$, then

$$Z = \begin{bmatrix} u_{1,2} & u_{1,3} - 2u_{1,2} & z_{1,3} & \cdots & z_{1,n-1} & u_{1,n-1} - 2u_{1,n} & u_{1,n} \\ u_{2,2} & u_{2,3} - 2u_{2,2} & z_{1,3} & \cdots & z_{2,n-1} & u_{2,n-1} - 2u_{2,n} & u_{2,n} \\ u_{3,2} & u_{3,3} - 2u_{3,2} & z_{1,3} & \cdots & z_{3,n-1} & u_{3,n-1} - 2u_{3,n} & u_{3,n} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{n+1,2} & u_{n+1,3} - 2u_{n+1,2} & z_{n+1,3} & \cdots & z_{n+1,n-1} & u_{1,n-1} - 2u_{n+1,n} & u_{n+1,n} \end{bmatrix},$$

where $z_{i,j} = u_{i,j-1} - 2u_{i,j} + u_{i,j+1}$ for $3 \leq j \leq n-1$ and $1 \leq i \leq n+1$.

Finally we can approximate the exact solution y by the non-polynomial Spline function $\hat{U}$ such that $\hat{U} = \hat{U}_i$ on $[x_i, x_{i+1}]$, $i = 0, 1, 2, \cdots, n-1$, where

$$\hat{U}(x) = \hat{y}_i + \frac{\hat{M}_i}{\tau^2} + \left(\frac{\hat{y}_{i+1}}{h} + \frac{\hat{M}_{i+1}}{\tau\theta}\right)(x - x_i) - \left(\frac{\hat{y}_i}{h} + \frac{\hat{M}_i}{\tau\theta}\right)(x - x_i)$$

$$+ \left(\frac{\hat{M}_i \cos\theta}{\tau^2 \sin\theta}\right)\sin\tau(x - x_i) - \left(\frac{\hat{M}_{i+1}}{\tau^2 \sin\theta}\right)\sin\tau(x - x_i) - \left(\frac{\hat{M}_i}{\tau^2}\right)\cos\tau(x - x_i). \quad (13)$$

3 Analysis of convergence

Lemma 1. [28] Let A be a $n \times n$ matrix with $\|A\|_\infty < 1$, then the matrix $(I - A)$ is invertable. Moreover $\left\|(I - A)^{-1}\right\|_\infty < \frac{1}{1 - \|A\|_\infty}$.

Theorem 1. Let $f \in C^4(I)$ and $k \in C^4(I \times I)$ such that. $\frac{3}{2}\|K\|_\infty \|L\|_\infty (b - a) < 1$, then equation (13) defines a unique approximate and the resulting error $\hat{e} = y - \hat{U}$ satisfies

$$\|\hat{e}\|_\infty < \gamma h^4, \quad \forall\ r \in I,$$

where γ is a constant.

Proof. It is essay to show that $\|A\|_\infty$, $\|A^*\|_\infty$, $\|D\|_\infty$, $\|D^*\|_\infty$, $\|E\|_\infty$, $\|E^*\|_\infty$ and $\|P\|_\infty$, $\|P^*\|_\infty \leq (\|K\|_\infty + \|L\|_\infty)(b - a)$ also $\|B\|_\infty$, $\|B^*\|_\infty$, $\|C\|_\infty$, $\|C^*\|_\infty \leq (\|K\|_\infty + \|L\|_\infty)\dfrac{(b-a)h}{2}$.
Hence

$$\|H_1\|_\infty \leq 2(\|K\|_\infty + \|L\|_\infty)(b - a)$$

and

$$\|H_2\|_\infty \leq \frac{48}{100}(\|K\|_\infty + \|L\|_\infty)(b - a),$$

then we have

$$(\|H_1 + H_2 Z\|_\infty) < \frac{3}{2}(\|K\|_\infty + \|L\|_\infty) < 1.$$

Now by Lemma 1, the system (12) has a unique solution $\hat{y}$. It follows that the equation (13) defines a unique solution $\hat{U}$.

Now, let $\hat{e} = y - \hat{y} = (y_0 - \hat{y}_0, y_1 - \hat{y}_1, \cdots, y_n - \hat{y}_n)^T$. Then from equation (11), we get

$$(I - (H_1 + H_2 Z))\hat{e} = O(h^4).$$

Therefore,

$$\hat{e} = (I - (H_1 + H_2 Z))^{-1} = O(h^4),$$

for which implies by Lemma 1, that there exists α_0 such that

$$\|\hat{e}\|_\infty \leq \underbrace{\frac{\alpha_1 h^4}{1 - \frac{3}{2}(\|K\|_\infty + \|L\|_\infty)(b-a)}}_{\alpha_2}.$$

On the other hand, from equation (8), we have $(M - \hat{M}) = (\frac{12}{h^2})Z\hat{e}$. Therefore,

$$\left\|Z - \hat{Z}\right\|_\infty \leq 12\alpha_2 h^4.$$

In consequence, for all $i = 0, 1, \cdots, n-1$ and $x \in [x_i, x_{i+1}]$, we have

$$\left|U_i(X) - \hat{U}_i(X)\right| \leq 12\alpha_2 h^4.$$

It follows that

$$\left\|Y - \hat{U}\right\|_\infty \leq \|Y - U\|_\infty + \left\|U - \hat{U}\right\|_\infty \leq \alpha_1 h^4 + 12\alpha_2 h^4.$$

Thus, the proof is completed by taking $\gamma = \alpha_1 + 12\alpha_2$.

4 Numerical results

In this section, we present three examples to illustrate the efficiency and accuracy of the proposed method. The computed errors e_i are defined by $e_i = |u_i - S_i|$, where u_i is the exact solution of equation (1) and S_i is an approximate solution of the same equation. Also we compute Least square error(LSE)$= \sum_{i=0}^{n}(u_i - S_i)^2$ and all computations are performed using the Python program.

Example 1. Consider the linear Volterra-Fredholm integral equation

$$u(x) = -\frac{x^2}{2} - \frac{7x}{2} + 2 + \int_0^x u(t)dt + \int_0^1 xu(t)dt.$$

The exact solution to this equation is given by $u(x) = x + 2$.

Example 2. Consider the linear Volterra-Fredholm integral equation

$$u(x) = 2\cos(x) - 1 + \int_0^x (x-t)u(t)dt + \int_0^\pi u(t)dt.$$

The exact solution to this equation is given by $u(x) = \cos(x)$.

Example 3. Consider the linear Volterra-Fredholm integral equation

$$u(x) = -\frac{9x^5}{10} + 2x^3 - \frac{3x^2}{2} - \frac{3x}{2} + \frac{19}{10} + \int_0^x (x+t)u(t)dt + \int_0^1 (x-t)u(t)dt.$$

The exact solution to this equation is given by $u(x) = 2x^3 + 1$.

Table 1

The Numerical Results for Example 1 with $n = 5$ and $\tau = 1$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	2	2	0	0
0.2	2.2	2.2	$8.8817842 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.4	2.4	2.4	0	0
0.6	2.6	2.6	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
0.8	2.8	2.8	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
1	3.	3.	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
LSE				$6.113672015 \times 10^{-30}$

Table 2

The Numerical Results for Example 1 with $n = 5$ and $\tau = 5$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	2	2	0	0
0.2	2.2	2.2	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.4	2.4	2.4	$4.44089210 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.6	2.6	2.6	$4.44089210 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.8	2.8	2.8	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
1	3.	3.	$2.22044605 \times 10^{-15}$	$4.93038066 \times 10^{-30}$
LSE				$6.902532920 \times 10^{-30}$

Table 3

The Numerical Results for Example 1 with $n = 5$ and $\tau = 179.7764$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	2	2	0	0
0.2	2.2	2.2	0	0
0.4	2.4	2.4	$4.4408921 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.6	2.6	2.6	$8.8817842 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.8	2.8	2.8	$8.8817842 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
1	3.	3.	0	0
LSE				$1.77493703674 \times 10^{-30}$

Table 4

The Numerical Results for Example 1 with $n = 10$ and $\tau = 1$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	2	2	0	0
0.1	2.2	2.2	$4.4408921 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.2	2.4	2.4	0	0
0.3	2.6	2.6	$4.4408921 \times 10^{-16}$	$1.97215226 \times 10^{-30}$
0.4	2.8	2.8	$4.4408921 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.5	3.	3.	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.6	2	2	$1.77635684 \times 10^{-15}$	$3.15544362 \times 10^{-30}$
0.7	2.2	2.2	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
0.8	2.4	2.4	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
0.9	2.6	2.6	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
1	2.8	2.8	$1.33226763 \times 10^{-15}$	$1.77493704 \times 10^{-30}$
LSE				$1.16356983520 \times 10^{-29}$

Table 5

The Numerical Results for Example 1 with $n = 10$ and $\tau = 5$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	2	2	0	0
0.1	2.2	2.2	0	$1.97215226 \times 10^{-31}$
0.2	2.4	2.4	0	0
0.3	2.6	2.6	0	0
0.4	2.8	2.8	$4.44089210 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.5	3.	3.	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.6	2	2	$4.44089210 \times 10^{-16}$	$1.97215226 \times 10^{-31}$
0.7	2.2	2.2	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.8	2.4	2.4	$1.77635684 \times 10^{-15}$	$3.15544362 \times 10^{-30}$
0.9	2.6	2.6	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
1	2.8	2.8	0	0
LSE				$6.113672015462 \times 10^{-30}$

Table 6

The Numerical Results for Example 1 with $n = 10$ and $\tau = 179.7764$

0	2	2	0	0
0.1	2.2	2.2	0	$1.97215226 \times 10^{-31}$
0.2	2.4	2.4	0	0
0.3	2.6	2.6	$1.77635684 \times 10^{-15}$	$3.15544362 \times 10^{-30}$
0.4	2.8	2.8	$4.44089210 \times 10^{-15}$	$1.97215226 \times 10^{-31}$
0.5	3.	3.	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.6	2	2	$1.33226763 \times 10^{-15}$	$1.97215226 \times 10^{-31}$
0.7	2.2	2.2	$8.88178420 \times 10^{-16}$	$7.88860905 \times 10^{-31}$
0.8	2.4	2.4	$1.77635684 \times 10^{-15}$	$3.15544362 \times 10^{-30}$
0.9	2.6	2.6	$2.22044605 \times 10^{-15}$	$7.88860905 \times 10^{-31}$
1	2.8	2.8	$1.77635684 \times 10^{-15}$	$3.15544362 \times 10^{-30}$
LSE				$6.113672015462 \times 10^{-29}$

Table 7

The Numerical Results for Example 2 with $n = 5$ and $\tau = 1$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	1	1.00692878	0.00692878	$4.80080263 \times 10^{-5}$
$\frac{\pi}{5}$	0.80901699	0.81390488	0.00488789	$2.38914209 \times 10^{-5}$
$\frac{2\pi}{5}$	0.30901699	0.31128372	0.00226672	$5.13803977 \times 10^{-5}$
$\frac{3\pi}{5}$	-0.3090169	-0.30814317	0.00087382	$7.63563793 \times 10^{-7}$
$\frac{4\pi}{5}$	-0.8090169	-0.80948866	0.00047166	$2.22464285 \times 10^{-7}$
π	$-1.$	-0.9488828	0.00057887	$3.35087778 \times 10^{-7}$
LSE				$7.835860281 \times 10^{-5}$

Table 8

The Numerical Results for Example 2 with $n = 5$ and $\tau = 5$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	-1.93012882	2.93012882	8.5856549
$\frac{\pi}{5}$	-2.08618179	0.81390488	2.89519878	8.3821759
$\frac{2\pi}{5}$	-2.24223476	0.31128372	2.55125175	6.5088855
$\frac{3\pi}{5}$	-2.39828773	-0.30814317	2.08927073	4.36505219
$\frac{4\pi}{5}$	-2.55434069	-0.80948866	1.7453237	3.04615481
π	$-1.$	-2.71039366	1.71039366	2.92544648
LSE				33.813369868

Table 9

The Numerical Results for Example 2 with $n = 5$ and $\tau = 179.7764$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	1.0178317	$1.78316984 \times 10^{-2}$	$3.17969468 \times 10^{-4}$
$\frac{\pi}{5}$	0.80901699	0.82402291	$1.50059187 \times 10^{-2}$	$2.25177596 \times 10^{-4}$
$\frac{2\pi}{5}$	0.30901699	0.31686699	$7.84999480 \times 10^{-3}$	$6.16224183 \times 10^{-5}$
$\frac{3\pi}{5}$	-0.30901699	-0.30908288	$6.58812345 \times 10^{-5}$	$4.34033706 \times 10^{-9}$
$\frac{4\pi}{5}$	-0.80901699	-0.8129492	$3.93220113 \times 10^{-3}$	$1.54622057 \times 10^{-5}$
π	$-1.$	-2.71039366	$1.21137812 \times 10^{-3}$	$1.46743694 \times 10^{-6}$
LSE				0.0006217034654

Table 10

The Numerical Results for Example 2 with $n = 10$ and $\tau = 1$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	1.00094592	$9.45917105 \times 10^{-4}$	8.9475917×10^{-7}
$\frac{\pi}{10}$	$9.51056516 \times 10^{-1}$	$9.51847558 \times 10^{-1}$	$7.91041543 \times 10^{-4}$	$6.25746722 \times 10^{-7}$
$\frac{2\pi}{10}$	$5.87785252 \times 10^{-1}$	$8.09581066 \times 10^{-1}$	$5.64072056 \times 10^{-4}$	$3.18177284 \times 10^{-7}$
$\frac{3\pi}{10}$	$3.09016994 \times 10^{-1}$	$3.09309388 \times 10^{-1}$	$4.08708268 \times 10^{-4}$	$1.67042448 \times 10^{-7}$
$\frac{4\pi}{10}$	$6.12323400 \times 10^{-17}$	$2.05322609 \times 10^{-4}$	$2.92393614 \times 10^{-4}$	$8.54940254 \times 10^{-8}$
$\frac{5\pi}{10}$	$-3.09016994 \times 10^{-1}$	$-3.08878314 \times 10^{-1}$	$2.05322609 \times 10^{-4}$	$4.21573737 \times 10^{-8}$
$\frac{6\pi}{5}$	$-5.87785252 \times 10^{-1}$	$-5.87699570 \times 10^{-1}$	$7.84999480 \times 10^{-4}$	$1.92321364 \times 10^{-8}$
$\frac{7\pi}{5}$	$-0.30901699 \times 10^{-1}$	$-8.08974167 \times 10^{-1}$	$1.38679978 \times 10^{-4}$	$7.34153214 \times 10^{-9}$
$\frac{8\pi}{5}$	$-8.09016994 \times 10^{-1}$	$-9.51067784 \times 10^{-1}$	$8.56827412 \times 10^{-5}$	$1.83414589 \times 10^{-10}$
9π	$9.51056516 \times 10^{-1}$	$-9.51915530 \times 10^{-1}$	$4.28269295 \times 10^{-5}$	$1.26963313 \times 10^{-9}$
π	$-1.$	$-9.99915530 \times 10^{-1}$	$8.44696327 \times 10^{-5}$	$7.13511885 \times 10^{-9}$
LSE				$2.169046919839 \times 10^{-6}$

Table 11

The Numerical Results for Example 2 with $n = 10$ and $\tau = 5$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	$9.99828516 \times 10^{-1}$	$1.71483912 \times 10^{-4}$	$2.940676226 \times 10^{-8}$
$\frac{\pi}{10}$	$9.87412008 \times 10^{-8}$	$9.51847558 \times 10^{-1}$	0.95184745	0.90601358
$\frac{2\pi}{10}$	$2.01267331 \times 10^{-7}$	$8.09581066 \times 10^{-1}$	0.80951045	0.65530718
$\frac{3\pi}{10}$	$1.60929543 \times 10^{-7}$	$3.09309388 \times 10^{-1}$	0.30930922	$9.56721979 \times 10^{-1}$
$\frac{4\pi}{10}$	$6.07030052 \times 10^{-8}$	$2.05322609 \times 10^{-4}$	$2.05261906 \times 10^{-4}$	$4.21324500 \times 10^{-8}$
$\frac{5\pi}{10}$	$1.39447547 \times 10^{-9}$	$-3.08849005 \times 10^{-1}$	0.3088490064	$9.53877087 \times 10^{-1}$
$\frac{6\pi}{5}$	$2.82203126 \times 10^{-8}$	$-5.87699570 \times 10^{-1}$	0.58769959	0.34539081
$\frac{7\pi}{5}$	$9.68987137 \times 10^{-8}$	$-8.08974167 \times 10^{-1}$	0.80897426	0.65443935
$\frac{8\pi}{5}$	$1.14481596 \times 10^{-7}$	$-9.51067784 \times 10^{-1}$	0.95106875	0.904531772
9π	$2.97953270 \times 10^{-8}$	$-9.51915530 \times 10^{-1}$	0.951915559	0.90614323
π	$-1.$	-1.00001557	1.557×10^{-5}	2.424249×10^{-10}
LSE				4.5628859

Table 12

The Numerical Results for Example 2 with $n = 10$ and $\tau = 179.7764$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	0	1.	1
$\frac{\pi}{10}$	$9.51056516 \times 10^{-1}$	0	9.5105651×10^{-1}	$9.04508497 \times 10^{-1}$
$\frac{2\pi}{10}$	$8.09016994 \times 10^{-1}$	0	$8.09016994 \times 10^{-1}$	$6.54508497 \times 10^{-1}$
$\frac{3\pi}{10}$	$5.87785252 \times 10^{-1}$	0	$5.87785252 \times 10^{-1}$	$3.45491503 \times 10^{-1}$
$\frac{4\pi}{10}$	$3.09016994 \times 10^{-1}$	0	$3.09016994 \times 10^{-1}$	$9.54915028 \times 10^{-2}$
$\frac{5\pi}{10}$	$6.12323400 \times 10^{-17}$	0	$6.12323400 \times 10^{-17}$	$3.74939946 \times 10^{-33}$
$\frac{6\pi}{5}$	$-3.09016994 \times 10^{-1}$	0	$3.09016994 \times 10^{-1}$	$9.54915028 \times 10^{-2}$
$\frac{7\pi}{5}$	$-5.87785252 \times 10^{-1}$	0	$5.87785252 \times 10^{-1}$	$3.45491503 \times 10^{-1}$
$\frac{8\pi}{5}$	$-8.09016994 \times 10^{-1}$	0	$8.09016994 \times 10^{-1}$	$6.54508497 \times 10^{-1}$
9π	$-9.51056516 \times 10^{-1}$	0	9.5105651×10^{-1}	$9.04508497 \times 10^{-1}$
π	$-1.$	0	1	1
LSE				5.9999999999

Table 13

The Numerical Results for Example 3 with $n = 5$ and $\tau = 1$

x_i	u_i	Q_i	$\|u_i - Q_i\|$	$\|u_i - Q_i\|^2$
0	1	0.99806294	0.00193706	$3.75221081 \times 10^{-6}$
0.2	1.016	1.01417071	0.00182929	$3.34628927 \times 10^{-6}$
0.4	1.128	1.12604257	0.00195743	$3.83151884 \times 10^{-6}$
0.6	1.432	1.42968468	0.00231532	$5.36072097 \times 10^{-6}$
0.8	2.024	2.02005699	0.00394301	$1.55473138 \times 10^{-5}$
1	3	3.00220523	0.00220523	$4.86302129 \times 10^{-6}$
LSE				$3.6701074960 \times 10^{-5}$

Table 14

The Numerical Results for Example 3 with $n = 5$ and $\tau = 5$

x_i	u_i	Q_i	$\|u_i - Q_i\|$	$\|u_i - Q_i\|^2$
0	1	0.99806294	0.00193706	$1.84295248 \times 10^{-7}$
0.2	1.016	1.01417071	0.00182929	$4.02037755 \times 10^{-7}$
0.4	1.128	1.12604257	0.00195743	$1.27690052 \times 10^{-6}$
0.6	1.432	1.42968468	0.00231532	$4.74011424 \times 10^{-6}$
0.8	2.024	2.02005699	0.00394301	$2.86127439 \times 10^{-5}$
1	3	3.00220523	0.00220523	$2.14332879 \times 10^{-6}$
LSE				$3.73594204345 \times 10^{-5}$

Table 15

The Numerical Results for Example 3 with $n = 5$ and $\tau = 179.7764$

x_i	u_i	Q_i	$\|u_i - Q_i\|$	$\|u_i - Q_i\|^2$
0	1	0.98349595	0.01650405	$2.72383633 \times 10^{-4}$
0.2	1.016	1.00261956	0.01338044	$31.79036304 \times 10^{-4}$
0.4	1.128	1.11802403	0.00997597	$9.95200562 \times 10^{-5}$
0.6	1.432	1.42833778	0.00366222	$1.34118860 \times 10^{-5}$
0.8	2.024	2.03365712	0.00965712	$9.32599357 \times 10^{-5}$
1	3	3.03761808	0.03761808	$1.41511981 \times 10^{-3}$
LSE				0.0020727316215

Table 16

The Numerical Results for Example 3 with $n = 10$ and $\tau = 1$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	0.9997679	0.0002321	5.3870937910^{-8}
0.1	1.002	1.0017762	0.0002238	$5.00871848 \times 10^{-8}$
0.2	1.016	1.0157778	0.0002222	$4.93746997 \times 10^{-8}$
0.3	1.054	1.05377251	0.00022749	$5.17498550 \times 10^{-8}$
0.4	1.128	1.12775968	0.00024032	$5.77557608 \times 10^{-8}$
0.5	1.25	1.249738	0.000262	$6.86443139 \times 10^{-8}$
0.6	1.432	1.43170555	0.00029445	$8.67031218 \times 10^{-8}$
0.7	1.686	1.68565824	0.00034176	$1.16799251 \times 10^{-7}$
0.8	2.024	2.02360244	0.00039756	$1.58055550 \times 10^{-7}$
0.9	2.458	2.45741921	0.00058079	$3.37315181 \times 10^{-7}$
1	3	3.00023429	0.00023429	$5.48933512 \times 10^{-8}$
LSE				$1.085249207159 \times 10^{-6}$

Table 17

The Numerical Results for Example 3 with $n = 10$ and $\tau = 5$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	0.99986069	$1.39312375 \times 10^{-4}$	$1.94079379 \times 10^{-8}$
0.1	1.002	1.0018592	$1.40802641 \times 10^{-4}$	$1.98253837 \times 10^{-8}$
0.2	1.016	1.01585209	$1.47913604 \times 10^{-4}$	$2.18784343 \times 10^{-8}$
0.3	1.054	1.0538375	$1.62496806 \times 10^{-4}$	$52.64052120 \times 10^{-8}$
0.4	1.128	1.12781298	$1.87016823 \times 10^{-4}$	$3.49752920 \times 10^{-8}$
0.5	1.25	1.24977517	$2.24826927 \times 10^{-4}$	$5.05471470 \times 10^{-8}$
0.6	1.432	1.43171956	$2.80436529 \times 10^{-4}$	$7.86446468 \times 10^{-8}$
0.7	1.686	1.68563868	$3.61320281 \times 10^{-4}$	$1.30552345 \times 10^{-7}$
0.8	2.024	2.02353461	$4.65394696 \times 10^{-4}$	$2.16592223 \times 10^{-7}$
0.9	2.458	2.45727901	$7.20992595 \times 10^{-4}$	$5.198303221 \times 10^{-7}$
1	3	3.00001603	$1.60253886 \times 10^{-5}$	$2.56813081 \times 10^{-10}$
LSE				$1.1189157570 \times 10^{-6}$

Table 18

The Numerical Results for Example 3 with $n = 10$ and $\tau = 179.7764$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	0.99986069	$0.13930000 \times 10^{-4}$	$1.940727610 \times 10^{-8}$
0.1	1.002	1.00185920	$1.40800000 \times 10^{-4}$	$1.98246400 \times 10^{-8}$
0.2	1.016	1.01585209	$1.47910000 \times 10^{-4}$	$2.18773681 \times 10^{-8}$
0.3	1.054	1.05383750	$1.62500000 \times 10^{-4}$	$2.64062500 \times 10^{-8}$
0.4	1.128	1.12781298	$1.87020000 \times 10^{-4}$	$3.49764804 \times 10^{-8}$
0.5	1.25	1.24977517	$2.24830000 \times 10^{-4}$	$5.05485289 \times 10^{-8}$
0.6	1.432	1.43171956	$2.80440000 \times 10^{-4}$	$7.86465936 \times 10^{-8}$
0.7	1.686	1.68563868	$3.61320000 \times 10^{-4}$	$13.05521424 \times 10^{-8}$
0.8	2.024	2.02353461	$4.65390000 \times 10^{-4}$	$21.65878521 \times 10^{-8}$
0.9	2.458	2.45727900	$7.21000000 \times 10^{-4}$	$51.9841000 \times 10^{-8}$
1	3	3.00001603	$1.60000000 \times 10^{-4}$	$2.56000000 \times 10^{-8}$
LSE				$10.40243756 \times 10^{-8}$

Conclusion

This paper presents numerical solutions for Volterra-Fredholm integral equations and investigates the convergence analysis. Three test examples from previous studies [25–27] are considered. The numerical results from Tables 1-18, indicate that accuracy decreases as τ increases and as n decreases. Additionally, we found that when the exact solution is a linear function, the accuracy is significantly high.

Author Contributions

S.H. Salim did the main part of this research. The results were audited and reviewed by R.K. Saeed and K.H.F. Jwamer. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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Double factorization of the Jonsson spectrum

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First of all, we have to note that in this article, we introduced the new concepts of relations between Jonsson theories in the class of cosemanticness for some considered Jonsson spectrum. All consideration of this new approach was done under sufficiently important class of Jonsson theories, which we called as normal Jonsson theories class. The main result, that we obtained, describes the model-theoretical properties of syntactical and semantical similarities inside the fixed cosemanticness class. For all new concepts in the article, we provided classical samples. The main result of this paper is considering normal Jonsson theories class by similarity to some fixed class of polygons (S-acts).

Keywords: Jonsson theory, perfect Jonsson theory, normal Jonsson theory, Jonsson set, almost Jonsson set, Jonsson fragment, syntactic similarity, semantic similarity, Jonsson specrtum, cosemanticness, S-act.

2020 Mathematics Subject Classification: 03C35, 03C48, 03C52, 03C65.

Introduction

The content of this article actually belongs to new approach and studying of generally speaking incomplete theories, and partically more exactly we focused our researches on studying Jonsson theories. Our new approach consists of applying syntactically and semantically similarities inside some considered class of cosemanticness from fixed Jonsson spectrum for some subclass of existentially closed models for considered fixed Jonsson theory. Such new method of studying Jonsson theories by our proposals allowed to penetrate in more details when we have operated with classical settlements of many tasks and problems which appears under considering and researching Jonsson spectra.

The notion of syntactical and semantical similarities was appeared in the works of T.G. Mustafin, for example in [1], when he introduced those notions for studying complete theories under stability consideration terms. By main result of this article it turned out that many concepts from stability theory saved their properties under semantical similarity, starting from basic notions of formulas and types and up to orthogonality, and independence, and forking, and spectral functions, which appears under studying of stability theories and their types. It turned out that, for any complete theories, it follows that there exists syntactically similar elementary theory of some given polygon (S-act) over a fixed monoid S. With the help of such consideration it became clear that all researches in the field of studying Model Theory in complete theories we can operate working with some fixed polygons. And in other side, in the case of incomplete theories after works [2–13] such implementation by using polygons is possible for Jonsson theories. This approach is sufficiently new, moreover not only Jonsson theories and some concepts which linked with Jonsson theories, for example hybrids of Jonsson theories, allowed us to wirk and describe Jonsson theories even for different signatures.

This paper consists of 3 sections. In Section 1, we give some basic information on Jonsson theories and related concepts, and introduce the new notion of normal Jonsson theory. In Section 2, the concepts

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of syntactic and semantic similarities for complete theories and for Jonsson theories are described. In Section 3, we present our results obtained for double factorization equivalence class of some fixed Jonsson spectrum and show its syntactic similarity to the class of theories of some polygon.

Note that here and after we will use the termin "S-act" instead of "polygon".

Now let us introduce the notation and determine the frame of our study.

We work in a first-order countable language L. By theory, we mean a consistent set of sentences in the given language.

If T is an L-theory, then E_T denotes a class of existentially closed models of the theory T.

Let A be an L-structure. By $T^0(A)$, we mean the theory $Th_{\forall\exists}(A)$ that is a set of all $\forall\exists$-sentences of L true for the structure A. The theory $T^0(A)$ is called a Kaiser hull of A.

1 Jonsson theories

We start with some basic information on Jonsson theories and related concepts. In this section the apparatus of the study of Jonsson theories is described.

Before presenting the concept of Jonsson theories let us remind the definitions of two properties that are essential for studying this class of incomplete theories.

Definition 1. [14; 80] A theory T has the joint embedding property, if, for any models A and B of T, there exists a model M of T and isomorphic embeddings $f : A \to M$, $g : B \to M$.

Definition 2. [14; 80] A theory T has the amalgamation property, if for any models A, B_1, B_2 of T and isomorphic embeddings $f_1 : A \to B_1$, $f_2 : A \to B_2$ there are $M \models T$ and isomorphic embeddings $g_1 : B_1 \to M$, $g_2 : B_2 \to M$, such that $g_1 \circ f_1 = g_2 \circ f_2$.

We write "JEP" and "AP" as shorter forms for the joint embedding and amalgamation properties, correspondingly.

Now let us recall the main definition of this section.

Definition 3. [14; 80] A theory T is called Jonsson, if:
1) the theory T has an infinite model;
2) the theory T is inductive;
3) the theory T has the joint embedding property (JEP);
4) the theory T has the amalgamation property (AP).

There are a lot of classical examples of Jonsson theories:
1) group theory;
2) the theory of abelian groups;
3) the theory of Boolean algebras;
4) the theory of linear orders;
5) field theory of characteristic p, where p is zero or a prime number;
6) the theory of ordered fields;
7) the theory of modules et cetera.

In [6], it is proved that the theory of differentially closed fields of the fixed characteristic is a Jonsson theory as well.

The special properties of Jonsson theories, namely AP and JEP, can be syntactically described by the following two theorems:

Theorem 1. [15] For the first order theory T of the language L (of arbitrary cardinality) the following conditions are equivalent:

1) T has JEP;

2) For all universal sentences α, β of L, if $T \vdash \alpha \vee \beta$ then $T \vdash \alpha$ or $T \vdash \beta$.

If φ and ψ are existential L-sentences such that $T \cup \{\varphi\}$ and $T \cup \{\psi\}$ are consistent then $T \cup \{\varphi, \psi\}$ is consistent.

Theorem 2. [16] The following are equivalent:

1) T has the Amalgamation property;

2) For all universal L-formulas $\alpha_1(\overline{x}), \alpha_2(\overline{x})$ with $T \vdash \forall x(\alpha_1(x) \vee \alpha_2(x))$ there are existential L-sentences $\beta_1(\overline{x}), \beta_2(\overline{x})$ such that

$$T \vdash \forall x(\beta_i(x) \to \alpha_i(x)),\ i = 1, 2,$$

and

$$T \vdash \forall x(\beta_1(x) \vee \beta_2(x)).$$

Another fundamental property of Jonsson theories follows from the theorem of W. Hodges and shows the connection between existentially closed models of such theories:

Theorem 3. [17; 363] Suppose T be an L-theory, and let T admit JEP. Let A and B be existentially closed model of T. Then each $\forall\exists$-sentence that is true in A is true in B as well.

In this paper, we study a special subclass of Jonsson theories, namely perfect Jonsson theories. To describe them we need the following definitions of Mustafin Ye.T.

Definition 4. [18] Let $\kappa \geq \omega$. Model $\mathcal{M}$ of theory T is called:

1) κ-universal for T, if each model of theory T with the power strictly less κ isomorphically imbedded in $\mathcal{M}$;

2) κ-homogeneous for T, if for any two models $\mathcal{A}$ and $\mathcal{A}_1$ of theory T, which are submodels of $\mathcal{M}$ with the power strictly less then κ and for isomorphism $f : \mathcal{A} \to \mathcal{A}_1$ for each extension $\mathcal{B}$ of model $\mathcal{A}$, which is a submodel of $\mathcal{M}$ and is model of T with the power strictly less then κ there exists the extension $\mathcal{B}_1$ of model $\mathcal{A}_1$, which is a submodel of $\mathcal{M}$ and an isomorphism $g : \mathcal{B} \to \mathcal{B}_1$ which extends f.

Definition 5. [18] A model $\mathcal{C}$ of the Jonsson theory T is called a semantic model, if it is ω^+-homogeneous-universal.

Definition 6. [18] The center of Jonsson theory T is an elementary theory of its semantic model $\mathcal{C}$ and denoted through T^*, i.e. $T^* = \mathrm{Th}(\mathcal{C})$.

Definition 7. [19] A Jonsson theory T is called perfect, if a semantic model of T is ω^+-saturated model of T.

The criterion for the perfectness of the Jonsson theory was obtained by Yeshkeyev A.R. and it is as follows:

Theorem 4. [19] For any Jonsson theory T following conditions are equivalent:

1) T is perfect;

2) T^* is the model companion of T.

Let us also demonstrate some properties of a perfect Jonsson theory and its center.

Theorem 5. [20; 1243] Let T be a Jonsson theory. Then for any model $A \in E_T$ theory $T^0(A)$ is Jonsson, where $T^0(A) = Th_{\forall\exists}(A)$.

We can see that in case of perfectness of T its center T^* is also a Jonsson theory.

Proposition 1. [21] Let T be a perfect Jonsson theory, then for every sentence $\varphi \in T^* \backslash T$ the theory $T' = T \cup \{\varphi\}$ is a Jonsson.

The following definition was introduced by Mustafin T.G.

Definition 8. We say that the Jonsson theory T_1 is cosemantic to the Jonsson theory T_2 ($T_1 \bowtie T_2$), if $\mathcal{C}_{T_1} = \mathcal{C}_{T_2}$, where $\mathcal{C}_{T_i}$ are semantic model of T_i, $i = 1, 2$.

The properties of cosemantic Jonsson theories were studied by Mustafin Ye.T. in [18]. This binary relation between two theories is an equivalence relation as it is easy to see. In the framework of study of Jonsson theories, it was introduced as a special tool for comparing Jonsson theories from the point of view of their semantic invariants, i.e. their semantic models. A lot of important model-theoretic properties coincide for Jonsson theories that are cosemantic. In this manner, when considering such properties, we may describe not just a Jonsson theory but the whole class of theories that are cosemantic to it.

In our research, we often apply so-called semantic method, whose essence is to study the properties of L-structures with the help of the theories of these structures. This is why the following notion was introduced by the first author of this paper.

Definition 9. [20] Let K be a class of L-structures. A Jonsson spectrum $JSp(K)$ of K is the followng set of theories

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } \forall A \in K\; A \models T\}.$$

The particular case of a Josson spectrum is a Robinson spectrum. Robinsonian theories are $\forall$-axiomatizable Jonsson theories.

Definition 10. [20] Let K be a class of L-structures. A Robinson spectrum $RSp(K)$ of K is the following set of theories

$$RSp(K) = \{T \mid T \text{ is a Robinsonian theory and } \forall A \in K\; A \models T\}.$$

The following proposition is important for studying Robinsonian theories and Robinson spectrum.

Proposition 2. [22] Let K be an arbitrary class of L-structures (possibly, it consists of one structure), $RSp(K)_{/\bowtie}$ be a factor set of the Robinson spectrum of K with respect to cosemanticness. Then every cosemanticness class $[\Delta]$ contains exactly one theory. In other words, for any two Robinsonian L-theories T and T', the relation of cosemanticness is equivalent to the equality (logical equivalence) of theories, i.e. $T \bowtie T' \Leftrightarrow T = T'$.

That is in Robinson spectrum factorized by cosemanticness, each cosemanticness class is single-element.

To study Jonsson theories through their semantic invariants, we often consider specific subsets of the semantic models of these theories. Let us describe them.

Definition 11. Let T be a Jonsson theory, C_T be its semantic model, $X \subseteq C$. X is said to be a Jonsson subset of C_T, if X is an $\exists$-definable set and $cl(X) = M$, where $M \in E_T$.

For each Jonsson subset $X \subseteq C_T$ for the theory T, we always can construct the fragment of X:

Definition 12. The fragment of a Jonsson subset X is a theory $Fr(X) = Th_{\forall\exists}(M)$, where $M = cl(X)$.

Let T be a Jonsson theory, $X \subseteq C_T$ and let $cl(X) = M \in E_T$. That is X is a Jonsson subset of C_T. Then $Fr(X) = Th_{\forall\exists}(M)$ and, moreover, the following lemma is true:

Lemma 1. [19; 299] For any Jonsson set $X \subset C_T$, the fragment $Fr(X)$ is a Jonsson theory.

Obviously, all axioms of T are true in a semantic model of $Fr(X)$, that is $C_{Fr(X)} \in Mod(T)$ and moreover $C_{Fr(X)}$ is existentially closed over T. It means that whenever X is a Jonsson set for T the semantic model of $Fr(X)$ is always embedded in C_T and an existentially closed submodel of C_T for any Jonsson theory T. To generalize this case and refine possible situations in the context of study of Jonsson theories, we introduce the following notions:

Definition 13. Let T be a Jonsson theory, C_T be its semantic model, $X \subseteq C$. X is called an almost Jonsson subset of C_T, if X is an $\exists$-definable set and $cl(X) = M$, where $M \in Mod(T)$, and $Th_{\forall\exists}(M)$ is a Jonsson theory.

By analogy with the concept of a Jonsson set, for an almost Jonsson subset $X \subseteq C_T$ of the theory T, we consider the fragment of X:

Definition 14. The fragment of an almost Jonsson subset X is a theory $Fr(X) = Th_{\forall\exists}(M)$, where $M = cl(X)$.

Thus the following definition refines the class of Jonsson theories whose properties we study in this paper:

Definition 15. A Jonsson theory T is called normal if for each almost Jonsson subset $X \subseteq C_T$, $C_{Fr(X)} \in Mod(T)$ and $C_{Fr(X)}$ is an existentially closed submodel of C_T.

There are natural examples of normal Jonsson theories, let us describe the following.

Example 1. Let T_{AG} be the theory of all abelian groups and let X be a set such as $cl(X) \in M \in Mod(T_{AG})$, i.e. X is an almost Jonsson set and M is an abelian group. It is well-known that $Fr(X) = Th_{\forall\exists}(M)$ is a Jonsson theory. Therefore, T_{AG} is a normal Jonsson theory.

Besides, there are non-normal Jonsson theories. The following example confirms this fact.

Example 2. Let T_V be the theory of all vector spaces. It is known that this theory is Jonsson. Let us consider a vector space that is the semantic model of T and its subspace $V \subseteq C_{T_V}$. The domain of V is a Jonsson set, and, consequently, is an almost Jonsson set. If X is the domain of V, then $cl(X) = V$. However, V is not an existentially closed submodel of C_{T_V}, since V may have another dimension that differs from dimension of C_{T_V}. Dimension of V can be formed by an $\forall\exists$-sentence, and this sentence fails in C_{T_V}. According to Theorem 3, V is not existentially closed in C_{T_V}). Thus, T_V is a Jonsson theory that is not normal.

The following theorem is necessary for our study.

Lemma 2. Let T be a perfect normal Jonsson theory. Then T^* is also a normal Jonsson theory.

Proof. Firstly we should note that it follows from Theorem 5 that T^* is also a perfect Jonsson theory. Moreover, it is easy to see that $T \bowtie T^*$, which means that

$$C_T = C_{T^*}. \tag{1}$$

Let X be an arbitrary almost Jonsson subset of C_{T^*}. Then $cl(X) = M \in Mod(T^*)$, and $C_{Fr(X)} \in Mod(T^*)$. According to Theorem 4, $Mod(T^*) = E_T$, therefore $C_{Fr(X)} \in E_T$, which means that $C_{Fr(X)}$ is an existentially closed submodel of C_T. By (1), $C_{Fr(X)}$ is also an existentially closed submodel of C_{T^*}. Thus T^* is a normal Jonsson theory.

2 *Syntactic and semantic similarities of Jonsson theories*

In this sections, we describe the notions of Mustafin T.G. that he introduced for complete theories, and Jonsson analogies of these notions proposed by the first author of this article.

To study and compare complete theories, especially theories of different languages, Mustafin T.G. [1]. used binary relations, which ha called syntactic similarity and semantic similarity. Let us describe them.

We start with the concept of syntactic similarity of complete theories. Let $F_n(T)$, $n < \omega$ be the Boolean algebra of formulas of T with exactly n free variables $v_1, \ldots, v_n$ and $F(T) = \bigcup_n F_n(T)$.

Definition 16. [1] Complete theories T_1 and T_2 are syntactically similar if and only if there exists a bijection $f : F(T_1) \to F(T_2)$ such that

1) $f \restriction F_n(T_1)$ is an isomorphism of the Boolean algebras $F_n(T_1)$ and $F_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$, $n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$.

The following example of syntactic similarity of complete theories was given in [1].

Example 1. The following theories T_1 and T_2 of the signature $\sigma = \langle \varphi, \psi \rangle$ are syntactically similar, where φ, ψ are binary functions:

$$T_1 = \mathrm{Th}(\langle Z; +, \cdot \rangle), \quad T_2 = \mathrm{Th}(\langle Z; \cdot, + \rangle).$$

Now we describe the concept of semantic similarity of complete theories. For this, we need the following definitions.

Definition 17. [1]

1) $\langle A, \Gamma, \mathcal{M} \rangle$ is called the pure triple, where A is not empty, Γ is the permutation group of A and $\mathcal{M}$ is the family of subsets of A such that from $M \in \mathcal{M}$ follows that $g(M) \in \mathcal{M}$ for every $g \in \Gamma$.

2) If $\langle A_1, \Gamma_1, \mathcal{M}_1 \rangle$ and $\langle A_2, \Gamma_2, \mathcal{M}_2 \rangle$ are pure triples and $\psi : A_1 \to A_2$ is a bijection then ψ is an isomorphism, if:

(i) $\Gamma_2 = \{\psi g \psi^{-1} : g \in \Gamma_1\}$;

(ii) $\mathcal{M}_2 = \{\psi(E) : E \in \mathcal{M}_1\}$.

Definition 18. [1] The pure triple $\langle C, Aut(\mathcal{C}), Sub(C) \rangle$ is called the semantic triple of complete theory T, where C is a domain of Monster model $\mathcal{C}$ of theory T, $Aut(\mathcal{C})$ is the automorphism group of $\mathcal{C}$, $Sub(C)$ is a class of all subsets of C each of which is a domain of the corresponding elementary submodel of $\mathcal{C}$.

Definition 19. [1] Complete theories T_1 and T_2 are semantically similar if and only if their semantic triples are isomorphic.

The following example of the semantic similarity of complete theories was given in [1].

Example 2. The following theories T_1 and T_2 are semantically similar, where

$$T_1 = Th(\langle \mathcal{M}_1; P_n, n < \omega; a_{nm}, n, m < \omega \rangle),$$
$$\mathcal{M}_1 = \{a_{nm} : n, m < \omega\},$$
$$P_n(\mathcal{M}_1) = \{a_{nm} : m < \omega\},$$

and

$$T_2 = Th(\langle \mathcal{M}_2; Q_n, n < \omega; Q_{nm}, n, m < \omega; b_{nmk}, n, m, k < \omega \rangle),$$
$$\mathcal{M}_2 = \{b_{nmk} : n, m, k < \omega\},$$
$$Q_n(\mathcal{M}_2) = \{b_{nmk} : m, k < \omega\},$$
$$Q_{nm}(\mathcal{M}_2) = \{b_{nmk} : k < \omega\}.$$

It turned out that the above types of similarity are not equivalent to each other.

Proposition 3. [1] If T_1 and T_2 are syntactically similar, then T_1 and T_2 semantically similar. The converse implication generally fails.

Let us recall the definition of semantic property.

Definition 20. [1] A property (or a notion) of theories (or models, or elements of models) is called semantic if and only if it is invariant relative to semantic similarity.

For example from [1] it is known that:

Proposition 4. The following properties and notions are semantic:

(1) type;

(2) forking;

(3) λ-stability;

(4) Lascar rank;

(5) Strong type;
(6) Morley sequence;
(7) Orthogonality, regularity of types;
(8) $I(\aleph_\alpha, T)$ is the spectrum function.

The following definition was introduced in the frame of Jonsson theories study by first author of this article in [19].

Let T be an arbitrary Jonsson theory, then $E(T) = \bigcup_{n<\omega} E_n(T)$, where $E_n(T)$ is a lattice of $\exists$-formulas with n free variables, T^* is a center of Jonsson theory T, i.e. $T^* = Th(\mathcal{C})$, where $\mathcal{C}$ is semantic model of Jonsson theory T in the sense of [18].

Definition 21. [19] Let T_1 and T_2 are arbitrary Jonsson theories. We say that T_1 and T_2 are Jonsson syntactically similar, if a bijection $f : E(T_1) \longrightarrow E(T_2)$ exists such that:
1) restriction f to $E_n(T_1)$ is isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$.

The examples of syntactic similarities of two Jonsson theories are given in [21].

As in the case of complete theories, the first author of this article defined in [19] a semantic similarity between two Jonsson theories.

Definition 22. [19] The pure triple $\langle C, Aut(\mathcal{C}), Sub(C)\rangle$ is called the Jonsson semantic triple, where C is a domain of semantic model $\mathcal{C}$ of theory T, $Aut(\mathcal{C})$ is the automorphism group of $\mathcal{C}$, $Sub(C)$ is a class of all subsets of C which are domains of the corresponding existentially closed submodels of $\mathcal{C}$.

Definition 23. [19] Two Jonsson theories T_1 and T_2 are called Jonsson semantically similar, if their Jonsson semantic triples are isomorphic as pure triples.

The correctness of this definition follows from the fact that the perfect Jonsson theory has a unique semantic model up to isomorphism. Otherwise, all semantic models are only elementary equivalent to each other.

For the convenience of further exposition we introduce the following notation. The syntactic and semantic similarities of the complete theories T_1 and T_2 will be denoted $T_1 \overset{S}{\bowtie} T_2$ and $T_1 \underset{S}{\bowtie} T_2$ respectively. In the case when we consider Jonsson theories T_1 and T_2, through $T_1 \overset{S}{\rtimes} T_2$ will denote the Jonsson syntactic similarity of theories T_1 and T_2, and through $T_1 \underset{S}{\rtimes} T_2$ Jonsson semantic similarity of theories T_1 and T_2.

Theorem 6. [19] Let T_1 and T_2 are $\exists$-complete perfect Jonsson theories, then following conditions are equivalent:
1) $T_1 \overset{S}{\rtimes} T_2$;
2) $T_1^* \overset{S}{\bowtie} T_2^*$.

An analogous result of Proposition 3 in the case of two Jonsson theories was obtained by Yeshkeyev A.R.

Theorem 7. Let T_1 and T_2 be two Jonsson theories and let T_1 and T_2 be Jonsson syntactically similar. Then T_1 and T_2 are Jonsson semantically similar.

Thus it is true that

$$T_1 \overset{S}{\rtimes} T_2 \Rightarrow T_1 \underset{S}{\rtimes} T_2$$

for any two Jonsson theories T_1 and T_2. That is Jonsson syntactic similarity is a sufficient condition of Jonsson semantic similarity of theories. There are also some cases when this condition is necessary.

In this paper, we consider such specific classes of Jonsson theories for which these two relations are equivalent. We denote this relation by the following:

$$T_1 \overset{SS}{\rtimes} T_2.$$

Lemma 3. [21] Any two cosemantic Jonsson theories are Jonsson semantically similar.

The proof follows from the definition of cosemantic Jonsson theories.

The converse result is also true:

Lemma 4. Let T_1 and T_2 be Jonsson theories and let $T_1 \underset{S}{\rtimes} T_2$. Then $T_1 \bowtie T_2$.

Proof. Let C_{T_1} and C_{T_2} be semantic models of T_1 and T_2, correspondingly. Let $T_1 \underset{S}{\rtimes} T_2$, then Jonsson semantic triples $(C_{T_1}, Aut(C_{T_1}), Sub(C_{T_1})$ and $(C_{T_2}, Aut(C_{T_2}), Sub(C_{T_2})$ are isomorphic as pure triples. Then it is clear that there exists an isomorphism between C_{T_1} and C_{T_1}, which means that $T_1 \bowtie T_2$.

Thus we obtain that, for any two Jonsson theories T_1 and T_2 of language L, it follows that

Corollary 1. $T_1 \overset{S}{\rtimes} T_2 \Rightarrow T_1 \underset{S}{\rtimes} T_2 \Leftrightarrow T_1 \bowtie T_2$.

The definitions of relations of Jonsson semantic and syntactic similarity were also generalized for classes of Jonsson theories in [21]:

Definition 24. [21] Let $\mathcal{A} \in \text{Mod}\sigma_1$, $\mathcal{B} \in \text{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is Jonsson syntactically similar to class $[T]_2$ and denote $[T]_1 \overset{S}{\rtimes} [T]_2$, if for any theory $\Delta \in [T]_1$ there is theory $\Delta' \in [T]_2$ such that $\Delta \overset{S}{\rtimes} \Delta'$.

Definition 25. [21] The pure triple $\langle C, Aut(\mathcal{C}), \overline{E}_{[T]} \rangle$ is called the Jonsson semantic triple for class $[T] \in JSp(A)/_{\bowtie}$, where C is the semantic model of $[T]$, $AutC$ is the group of all automorphisms of C, $\overline{E}_{[T]}$ is the class of isomorphically images of all existentially closed models of $[T]$.

Definition 26. [21] Let $\mathcal{A} \in \text{Mod}\sigma_1$, $\mathcal{B} \in \text{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is Jonsson semantically similar to class $[T]_2$ and denote $[T]_1 \underset{S}{\rtimes} [T]_2$, if their semantically triples are isomorphic as pure triples.

3 *Properties of classes of S-acts*

In this section, we show our main result using the concepts from the previous sections, for the special class of structures, namely for S-acts. Let us shortly describe this class.

Definition 27. [1] By an S-act over a monoid S (sometimes it is called S-acts) we mean a structure with only unary functions $\langle A; f_\alpha : \alpha \in S \rangle$ such that:

1) $f_e(a)\ \forall a \in A$, where e is the unit of S;

2) $f_{\alpha\beta}(a) = f_\alpha(f_\beta(a))\ \forall \alpha, \beta \in S$, $\forall a \in A$.

The following results show that any complete theory has some syntactic similar theory.

Theorem 8. [1] For every theory T_2 in a finite signature there is a theory T_1 of S-acts such that some inessential extension of T_1 is an almost envelope of T_2.

Theorem 9. [1] For every theory T_2 in an infinite signature there is a theory T_1 of S-acts such that some inessential extension of T_1 is an envelope of T_2.

Now we present the settlement of our research's problem and the main result of our paper.

Let T be a Jonsson L-theory, E_T be a class of the existentially closed models of T, $K \subseteq E_T$. Let us construct a Jonsson spectrum $JSp(K)$ of the class K and on this spectrum we introduce the following relations: cosemanticness, Jonsson syntactic similarity and Jonsson semantic similarity of Jonsson theories. It is obvious that all these relations are equivalence relations, so we obtain a factor-set of the Jonsson spectrum of K with respect to relations introduced that we denote by $JSp(K)/_{\underset{\bowtie}{SS}}$. $[T]$ is an equivalence class of a theory T from $JSp(K)/_{\underset{\bowtie}{SS}}$. The examples of such classes do exist and let us demonstrate some of them. Let us consider $RSp(K)$ which is a partial case of $JSp(K)$. We introduce the relation of Jonsson syntactic similarity of theories on $RSp(K)$. According to Corollary 1, all theories from the equivalence class $[T] \in RSp(K)/_{\underset{\rtimes}{S}}$ are also Jonsson semantically similar and cosemantic. So we get $RSp(K)/_{\underset{\bowtie}{SS}}$. According to Theorem 2, in any cosemanticness class in Robinsonian spectrum, there is only one theory with respect to logical equivalence.

Theorem 10. Let T be a Jonsson L-theory, $K \subseteq E_T$, $[T] \in JSp(K)/_{\underset{\bowtie}{SS}}$ be an $\exists$-complete perfect normal class (i.e. such that all $T_i \in [T]$ are perfect normal Jonsson theories). Then there exists $[T'_\Pi] \in JSp(K')/_{\underset{\bowtie}{SS}}$, where K' is a class of some S-acts in the corresponding language, such that $[T_\Pi]$ is an $\exists$-complete perfect class and $[T]$ is Jonsson syntactically similar to $[T'_\Pi]$.

Proof. Let $[T]$ be a perfect normal $\exists$-complete equivalence class in $JSp(K)/_{\underset{\bowtie}{SS}}$. Since the center T^* of this class is a complete theory, according to Theorem 8 in the case of a finite signature and Theorem 9 in the case of an infinite signature, there is a complete theory of the S-act T_Π such that $T^* \overset{S}{\bowtie} T_\Pi$. But then, according to Proposition 3, it follows that $T^* \underset{S}{\bowtie} T_\Pi$. Since the concept of type is a semantic notion (Proposition 4), the concept of a formula is also semantic. It follows from Theorem 1 and Theorem 2 that the properties of JEP and AP are formulated using some L-formulas, i.e. JEP and AP are semantic concepts. It is clear that $\forall\exists$-axiomatizability is also a semantic property, since all axioms are true in the semantic model. This means that the property "to be a Jonsson theory" is a semantic concept, and therefore T_Π is also a Jonsson theory.

Since $[T]$ is a normal class the center T^* is a normal theory as well according to Lemma 2. In this manner, the property of being normal Jonsson theory is also transferred to T_Π, as this notion is semantic. It means that the theory T_Π is a normal Jonsson theory.

Since T^* is a perfect Jonsson theory, then semantic model $\mathcal{C}_T$ of the class $[T]$ is ω^+-saturated. But $T^* \underset{S}{\bowtie} T_\Pi$ and, by definition, the semantic triples of these theories are isomorphic, then $\mathcal{C}_{[T]} \cong \mathcal{C}_{T_\Pi}$, therefore $\mathcal{C}_{T_\Pi}$ is also ω^+-saturated and therefore T_Π is a perfect Jonsson theory. Let K' be a class of S-acts such that $T_\Pi \in JSp(K)$. Then the equivalence class $[T_\Pi] \in JSp(K')/_{\underset{\bowtie}{SS}}$ is a perfect class, since all theories in this equivalence class has the same semantic model, which is ω^+-saturated.

Consider $JSp(C_{T_\Pi})$. Since the theory T_Π is perfect then $|JSp(C_{T_\Pi})/_{\bowtie}| = 1$ due to the fact that T_Π is normal. Let $\Delta \in JSp(C_{T_\Pi})$, i.e. Δ is Jonsson theory and $\Delta^* = T_\Pi$. We show that Δ is perfect $\exists$-complete Jonsson theory. By virtue of $T^* \underset{S}{\bowtie} \Delta^*$, then from the definition of semantic similarity for complete theories it follows that Δ is a perfect Jonsson theory. If Δ is $\exists$-complete, then we take Δ and then by Theorem 6 it follows that $T \overset{S}{\rtimes} \Delta = T'_\Pi$. If Δ is not $\exists$-complete, then we carry out the following replenishment procedure for this theory. As $\Delta \subset T_\Pi$, then for any existential sentence φ, of the signature language of Δ such that $\Delta \nvdash \varphi$ and $\Delta \nvdash \neg\varphi$, but $\varphi \in T_\Pi$, consider the theory $\Delta' = \Delta \cup \{\varphi\}$. Since $\Delta \subset \Delta' \subset T_\Pi$, and Δ, T_Π are Jonsson theories, it follows from Proposition 1 that Δ' is also a Jonsson theory. If Δ' is not $\exists$-complete, then we continue the procedure of adding existential sentences $\varphi \in T_\Pi$ until Δ' it becomes $\exists$-complete. We make this procedure for each $T' \in JSp(K')/_{\underset{\bowtie}{SS}}$ and obtain an $\exists$-complete equivalence class.

Let $\overline{\Delta} = \Delta \cup \{\varphi | \varphi \in \Sigma_1, \varphi \in T_{\Pi}\}$ is the result of replenishment procedure of the theory Δ, i.e. $\overline{\Delta}$ is $\exists$-complete and at the same time $\overline{\Delta}$ is a Jonsson theory. We show that $\overline{\Delta} \in JSp(C_{T_{\Pi}})$, hence the perfection of the theory of $\overline{\Delta}$ will follow from here. Suppose the contrary, let $\overline{\Delta} \notin JSp(C_{T_{\Pi}})$, then $C_{T_{\Pi}} \notin Mod(\overline{\Delta})$, but this is not true since $C_{T_{\Pi}} \models \Delta$ and for any sentence $\varphi \in \overline{\Delta} \backslash \Delta$, $\varphi \in T_{\Pi}$. Consequently, $C_{T_{\Pi}} \models \varphi$ and $C_{T_{\Pi}} \in \mathrm{Mod}(\overline{\Delta})$. We obtain a contradiction, i.e. $\overline{\Delta} \in JSp(C_{T_{\Pi}})$. But $C_{T_{\Pi}}$ is saturated, therefore, $\overline{\Delta}$ is a perfect Jonsson theory. Then by Theorem 6 we have $T^* \overset{S}{\bowtie} \overline{\Delta}^* \Leftrightarrow T \overset{S}{\rtimes} \overline{\Delta}$, where $\overline{\Delta} = T'_{\Pi}$. It follows that, for each theory $T \in [T] \in JSp(K)/_{\underset{\bowtie}{SS}}$, there exists such theory $T'_{\Pi} \in [T'_{\Pi}] \in JSp(K')/_{\underset{\bowtie}{SS}}$ such that $T \overset{S}{\rtimes} \overline{\Delta}$. Thus, according to Definition 24, the class $[T]$ is Jonsson syntactically similar to the class $[T'_{\Pi}]$.

It should be noted that, in this manner, the results of [21] are special cases of Theorem 10 considering the theories as single-element equivalence classes in some Jonsson spectrum.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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