ISSN 2518-7929 (Print)
ISSN 2663-5011 (Online)

BULLETIN
OF THE KARAGANDA UNIVERSITY

MATHEMATICS
SERIES

No. 2(114)/2024

ISSN 2518-7929 (Print)
ISSN 2663–5011(Online)
Индексі 74618
Индекс 74618

ҚАРАҒАНДЫ УНИВЕРСИТЕТІНІҢ **ХАБАРШЫСЫ**

ВЕСТНИК КАРАГАНДИНСКОГО УНИВЕРСИТЕТА

BULLETIN OF THE KARAGANDA UNIVERSITY

МАТЕМАТИКА сериясы

Серия МАТЕМАТИКА

MATHEMATICS Series

No. 2(114)/2024

1996 жылдан бастап шығады
Издается с 1996 года
Founded in 1996

Жылына 4 рет шығады
Выходит 4 раза в год
Published 4 times a year

Қарағанды / Караганда / Karaganda
2024

Postal address: 28, University Str., Karaganda, 100024, Kazakhstan.
E-mail: vestnikku@gmail.com. Web-site: mathematics-vestnik.ksu.kz

Bulletin of the Karaganda University. Mathematics series.
ISSN 2518-7929 (Print). ISSN 2663–5011 (Online).

Proprietary: NLC «Karagandy University of the name of academician E.A. Buketov».

Registered by the Ministry of Information and Social Development of the Republic of Kazakhstan.
Rediscount certificate No. KZ43VPY00027385 dated 30.09.2020.

Signed in print 28.06.2024. Format 60×84 1/8. Photocopier paper. Volume 30.38 p.sh. Circulation 200 copies.
Price upon request. Order № 45.

Printed in the Publishing house of NLC «Karagandy University of the name of academician E.A. Buketov».
28, University Str., Karaganda, 100024, Kazakhstan. Tel.: (7212) 35-63-16. E-mail: izd_kargu@mail.ru

CONTENT

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MATHEMATICS

https://doi.org/10.31489/2024M2/4-21 *Research article*

On estimates of M-term approximations of the Sobolev class in the Lorentz space

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In the paper spaces of periodic functions of several variables were considered, namely the Lorentz space $L_{2,\tau}(\mathrm{T}^m)$, the class of functions with bounded mixed fractional derivative $W^{\overline{r}}_{2,\tau}$, $1 \leq \tau < \infty$, and the order of the best M-term approximation of a function $f \in L_{p,\tau}(\mathrm{T}^m)$ by trigonometric polynomials was studied. The article consists of an introduction, a main part, and a conclusion. In the introduction, basic concepts, definitions and necessary statements for the proof of the main results were considered. One can be found information about previous results on the mentioned topic. In the main part, exact-order estimates are established for the best M-term approximations of functions of the Sobolev class $W^{\overline{r}}_{2,\tau_1}$ in the norm of the space $L_{p,\tau_2}(\mathrm{T}^m)$ for various relations between the parameters p, τ_1, τ_2.

Keywords: Lorentz space, Sobolev class, mixed derivative, trigonometric polynomial, M-term approximation.

2020 Mathematics Subject Classification: 41A10, 41A25, 42A05.

Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ be the sets of natural, integer, and real numbers, respectively, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{R}^m$ is m-dimensional Euclidean space of points $\bar{x} = (x_1, \ldots, x_m)$ with real coordinates; $\mathbb{T}^m = [0, 2\pi)^m$ and $\mathbb{I}^m = [0, 1)^m$ are m-dimensional cubes.

We denote by $L_{p,\tau}(\mathbb{T}^m)$ the Lorentz space of all real-valued Lebesgue measurable functions f that have 2π-period in each variable and for which the quantity

$$\|f\|_{p,\tau} = \left\{ \frac{\tau}{p} \int_0^1 \left(f^*(t)\right)^\tau t^{\frac{\tau}{p}-1} dt \right\}^{\frac{1}{\tau}}, \ 1 < p < \infty, \ 1 \leq \tau < \infty$$

is finite, where $f^*(t)$ is a non-increasing rearrangement of the function $|f(2\pi\overline{x})|$, $\overline{x} \in \mathbb{I}^m$ (see [1]).

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This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP22683029).

Received: 24 August 2023; *Accepted:* 05 February 2024.

In case when $\tau = p$, the Lorentz space $L_{p,\tau}(\mathbb{T}^m)$ coincides with the Lebesgue space $L_p(\mathbb{T}^m)$ with the norm (see, for example, [2])

$$\|f\|_p = \left[\int_0^{2\pi} ... \int_0^{2\pi} |f(x_1, ..., x_m)|^p dx_1...dx_m\right]^{\frac{1}{p}}, \ 1 \leq p < \infty.$$

Let us begin by introducing some notation: $a_{\overline{n}}(f)$ are Fourier coefficients of the function $f \in L_1(\mathbb{T}^m)$ by the system $\{e^{i\langle\overline{n},\overline{x}\rangle}\}_{\overline{n}\in\mathbb{Z}^m}$ and $\langle\overline{y},\overline{x}\rangle = \sum\limits_{j=1}^{m} y_j x_j$;

$$\delta_{\overline{s}}(f,\overline{x}) = \sum_{\overline{n}\in\rho(\overline{s})} a_{\overline{n}}(f)\, e^{i\langle\overline{n},\overline{x}\rangle},$$

where

$$\rho(\overline{s}) = \left\{\overline{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m \colon [2^{s_j-1}] \leq |k_j| < 2^{s_j}, j = 1, \dots, m\right\},$$

and $[a]$ is an integer part of a, $\overline{s} = (s_1, ..., s_m)$, $s_j = 0, 1, 2, \dots$

For a given vector $\overline{r} = (r_1, \dots, r_m) > \overline{0} = (0, \dots, 0)$ we set $\overline{\gamma} = \frac{\overline{r}}{r_1}$ and

$$Q_n^{(\overline{\gamma})} = \cup_{\langle\overline{s},\overline{\gamma}\rangle<n}\rho(\overline{s}),$$

$S_n^{(\overline{\gamma})}(f,\overline{x}) = \sum_{\overline{k}\in Q_n^{(\overline{\gamma})}} a_{\overline{k}}(f)e^{i\langle\overline{k},\overline{x}\rangle}$ is a partial sum of the Fourier series of the function f (see [2]).

Let us consider an one-dimensional Bernoulli kernel (see, for example, [2])

$$F_r(x) = 1 + 2\sum_{k=1}^{\infty} k^{-r}\cos(kx - r\pi/2), \ r > 0.$$

Next, for the vector $\overline{r} = (r_1, ..., r_m)$, $r_j > 0$, $j = 1, \dots, m$, we set

$$F_{\overline{r}}(\overline{x}) = \prod_{j=1}^{m} F_{r_j}(x_j).$$

Let us consider a Sobolev functional class

$$W_{p,\tau}^{\overline{r}} = \{f : \ f = \varphi \star F_{\overline{r}}, \ \|\varphi\|_{p,\tau} \leqslant 1\},$$

where $1 < p < \infty$, $1 \leq \tau < \infty$,

$$(\varphi \star F_{\overline{r}})(\overline{x}) = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} \varphi(\overline{x} - \overline{u})F_{\overline{r}})(\overline{u})d\overline{u}.$$

In case when $\tau = p$, the class $W_{p,\tau}^{\overline{r}}$ has been considered in [3] and [4], so in this case, instead of $W_{p,p}^{\overline{r}}$ we write $W_p^{\overline{r}}$.

The value

$$e_M(f)_{p,\tau} = \inf_{\bar{k}^{(j)},b_j}\left\|f - \sum_{j=1}^{M} b_j e^{i\langle\bar{k}^{(j)},\bar{x}\rangle}\right\|_{p,\tau}$$

is called the best M-term trigonometric approximation of the function $f \in L_{p,\tau}(\mathbb{T}^m)$, $n \in \mathbb{N}$.

If $F \subset L_{p,\tau}(\mathbb{T}^m)$ is some functional class, then we set $e_M(F)_{p,\tau} = \sup_{f\in F} e_M(f)_{p,\tau}$. In case when $\tau = p$, instead of $e_M(F)_{p,\tau}$ we wrire $e_M(F)_p$.

The best M-term approximation of a function $f \in L_2[0,1]$ by polynomials in an orthonormal system has been first determined by S.B. Stechkin [5] and he has established a criterion for the absolute convergence of the Fourier series in this system. The advantage of the M-term approximation with respect to the one-dimensional trigonometric system over the linear approximation by M-order trigonometric polynomials has been shown by R.S. Ismagilov [6].

Exact order estimates of the best M-term approximation of the Bernoulli kernel have been established by V.E. Maiorov [7] and Yu. Makovoz [8], E.S. Belinsky [9, 10]. In the one-dimensional case, the value $e_M(W_q^{\overline{r}})_p$ has been estimated by S. Belinsky [9]. At present, many important results on estimates of M-term approximations of functions from various Sobolev, Nikol'skii–Besov and Lizorkin–Triebel classes are known [11, 12]. In the multidimensional case, for $1 < q \le p < 2$ and $r_1 > \frac{1}{2}(\frac{1}{q} - \frac{1}{p})$, order-exact estimates of the best M-term approximation of functions of $W_q^{\overline{r}}$ in the norm of $L_p(\mathbb{T}^m)$ have been obtained by V.N. Temlyakov [3, 4], and for $1 < q \le p < 2$ and $r_1 \le \frac{1}{2}(\frac{1}{q} - \frac{1}{p})$, E.S. Belinsky [10] has proved the following theorem:

Theorem. Let $1 < q \le 2 < p < \infty$ and $r_1 = \ldots = r_\nu < r_{\nu+1} \le \ldots r_m$. Then

$$e_M(W_q^{\overline{r}})_p \asymp M^{-\frac{p}{2}(r_1 + \frac{1}{p} - \frac{1}{q})}(\log M)^{(\nu-1)(p-1)(r_1 - p'(\frac{1}{q} - \frac{1}{p}))_+}$$

in case $\frac{1}{q} - \frac{1}{p} < r_1 < \frac{1}{q}$, where $q' = \frac{q}{q-1}$.

Note that a generalization of this theorem on the Lorentz space $L_{p,\tau}(\mathbb{T}^m)$ has been proved in [13–15].

Throughout the paper, $A_n \asymp B_n$ means that there are positive numbers C_1, C_2 independent of $n \in \mathbb{N}$ such that $C_1 A_n \le B_n \le C_2 A_n$ for $n \in \mathbb{N}$ and $\log M$, where $\log M$ is the logarithm with base 2 of the number $M > 1$.

By the constructive method, V.N. Temlyakov [16,17] has established estimates for M–term approximations of functions of the class $W_q^{\overline{r}}$ in the space $L_p(\mathbb{T}^m)$ for $1 < q \le 2 < p < \infty$ and $(\frac{1}{q} - \frac{1}{p})p' < r_1 < \frac{1}{q}$, $p' = \frac{p}{p-1}$ and has raised the question of finding constructive evaluation method for $\frac{1}{q} - \frac{1}{p} < r_1 \le (\frac{1}{q} - \frac{1}{p})p'$. Further application of the constructive method is given in [18, 19].

In the first section, some auxiliary assertions are formulated that are necessary for proving main results. The main results of the article are formulated as a theorem and proved in the second section. In conclusion, we compare the proved Theorem 1 with previously known results.

1 Auxiliary statements

Theorem A. [20] Let $1 < q < \lambda < \infty$, $1 < \tau,\ \theta < \infty$. If a function $f \in L_{q,\tau}(\mathrm{T}^m)$, then

$$\|f\|_{q,\tau} \ge C\Big(\sum_{\overline{s} \in \mathrm{Z}_+^m} \prod_{l=1}^{m} 2^{s_l(1/\lambda - 1/q)\tau} \|\delta_{\overline{s}}(f)\|_{\lambda,\theta}^{\tau}\Big)^{1/\tau}.$$

Theorem B. [20] Let $1 < p < q < \infty$, $1 < \tau_1, \tau_2 < \infty$. If a function $f \in L_{p,\tau_1}(\mathrm{T}^m)$ satisfies the condition

$$\sum_{\overline{s} \in \mathrm{Z}_+^m} \prod_{j=1}^{m} 2^{s_j \tau_2(1/p - 1/q)} \|\delta_{\overline{s}}(f)\|_{p,\tau_1}^{\tau_2} < \infty,$$

then $f \in L_{q,\tau_2}(\mathrm{T}^m)$ and the inequality

$$\|f\|_{q,\tau_2} \le C\left(\sum_{\overline{s} \in \mathrm{Z}_+^m} \prod_{j=1}^{m} 2^{s_j \tau_2(1/p - 1/q)} \|\delta_{\overline{s}}(f)\|_{p,\tau_1}^{\tau_2}\right)^{1/\tau_2}$$

holds.

For a function $f \in L_1(\mathbb{T}^m)$ we set

$$f_{l,\bar{r}}(\bar{x}) = \sum_{l \leq \langle \bar{s}, \bar{\gamma} \rangle < l+1} \delta_{\bar{s}}(f, \bar{x}), l \in \mathbb{Z}_+,$$

where $\bar{\gamma} = (\gamma_1, \ldots, \gamma_m)$, $\gamma_1 = \ldots = \gamma_\nu < \gamma_{\nu+1} \leq \cdots \leq \gamma_m$, $\gamma_j = \frac{r_j}{r_1}$, $r_j > 0$, $j = 1, \ldots, m$.

Let us consider the following class defined in [5, 6]

$$W_A^{a,b,\bar{r}} = \left\{ f \in L_1(\mathbb{T}^m) \colon \|f_{l,\bar{r}}\|_A \leq 2^{-la} l^{(\nu-1)b} \right\},$$

where

$$\|f_{l,\bar{r}}\|_A = \sum_{l \leq \langle \bar{s}, \bar{\gamma} \rangle < l+1} \sum_{\overline{n} \in \rho(\overline{s})} |a_{\overline{n}}(f)|.$$

The following lemma is a consequence of Lemma 6.1 in [16] (see also Lemma 2.1 in [17]), which we often use in proofs of main results.

Lemma 1. [15] Let $2 \leq p < \infty$ and $1 < \tau < \infty$, $a > 0$. Then for $f \in W_A^{a,b,\bar{r}}$ there are constructive approximation methods of the greedy algorithm type of $G_M(f)$ with the property:

$$\|f - G_M(f)\|_{p,\tau} \leq C(m) M^{-a-\frac{1}{2}} (\log M)^{(\nu-1)(a+b)}.$$

2 Main results

Theorem 1. Let $0 < r_1 = \ldots = r_\nu < r_{\nu+1} \leq \ldots r_m$, $2 < p < \infty$, $1 < \max\{\tau_1, 2\} \leq \tau_2 < \infty$, $\tau_2^{'} = \frac{\tau_2}{\tau_2 - 1}$.

a) If $\frac{1}{2} - \frac{1}{p} < r_1 < (\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})\tau_2^{'}$, then

$$e_M(W_{2,\tau_1}^{\overline{r}})_{p,\tau_2} \leq C M^{-\frac{p}{2}(r_1 + \frac{1}{p} - \frac{1}{2})} (\log_2 M)^{\frac{1}{2} - \frac{1}{\tau_1}}, \ M > 1.$$

b) If $\tau_2' \left(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2} \right) < r_1 < \frac{1}{2}$, then

$$e_M(W_{2,\tau_1}^{\overline{r}})_{p,\tau_2} \leq C M^{-\frac{p}{2}(r_1 + \frac{1}{p} - \frac{1}{2})} (\log_2 M)^{\frac{1}{2} - \frac{1}{\tau_1}} (\log_2 M)^{(\nu-1)\frac{p}{\tau_2'} \left(r_1 - \tau_2' \left(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2} \right) \right)}.$$

Proof. Let us introduce some notation

$$Q_{n,\bar{\gamma}} = \cup_{\langle \bar{s}, \bar{\gamma} \rangle \leq n} \rho(\bar{s}), \ S_{Q_{n,\bar{\gamma}}}(f, \overline{x}) = \sum_{\langle \bar{s}, \bar{\gamma} \rangle \leq n} \delta_{\bar{s}}(f, \bar{x}).$$

For a natural number M, there exists a number $n \in \mathbb{N}$ such that $M \asymp 2^n n^{\nu-1}$.

Let $\nu \geq 2$. We set

$$n_1 = \frac{p}{2} n - p \Big(\frac{1}{2} - \frac{1}{\tau_2} \Big) (\nu - 1) \log n,$$

$$n_2 = \frac{p}{2} n + \frac{p}{2} (\nu - 1) \log n.$$

Also, let us introduce

$$S_l = \Big(2^{l r_1 \tau_1} \sum_{l \leq \langle \bar{s}, \bar{\gamma} \rangle < l+1} 2^{\langle \bar{s}, \bar{1} \rangle (\frac{1}{2} - \frac{1}{q}) \tau_1} \|\delta_{\bar{s}}(\varphi)\|_2^{\tau_1} \Big)^{1/\tau_1}$$

and

$$m_l = \left[2^{-l\frac{\tau_2'}{p}} S_l^{\tau_1} 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\right] + 1,$$

where $\langle \bar{s}, \bar{1}\rangle = \sum\limits_{j=1}^{m} s_j$, $p' = \frac{p}{p-1}$ and $[y]$ is an integer part of a number y.

By $G(l)$ is denoted the set of indices $\overline{s}$, $l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1$, with the largest $\|\delta_{\overline{s}}(\varphi)\|_2$, and $m_l = |G(l)|$ is the number of elements of $G(l)$.

Let us consider the functions

$$F_1(\overline{x}) = \sum_{n \leq l < n_1} f_l(\overline{x}),$$

$$F_2(\overline{x}) = \sum_{n_1 \leq l < n_2} \sum_{\overline{s} \notin G(l)} \delta_{\overline{s}}(f, \overline{x}),$$

$$F_3(\overline{x}) = \sum_{n_1 \leq l < n_2} \sum_{\overline{s} \in G(l)} \delta_{\overline{s}}(f, \overline{x}).$$

Let us estimate $\|F_1\|_A$. Applying Hölder's inequality for the sum and Parseval's equality, we have

$$\|F_1\|_A = \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} \sum_{\overline{k} \in \rho(\overline{s})} |a_{\overline{k}}(f)| \leq 2^{-\frac{m}{2}} \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} 2^{\langle \bar{s}, \bar{1}\rangle \frac{1}{2}} \|\delta_{\overline{s}}(f)\|_2 =$$

$$= 2^{-\frac{m}{2}} \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} 2^{\langle \bar{s}, \bar{1}\rangle (\frac{1}{2} - \frac{1}{q})} \|\delta_{\overline{s}}(f)\|_2 2^{\langle \bar{s}, \bar{1}\rangle \frac{1}{q}}. \quad (1)$$

It is known that the Fourier coefficients of the convolution $f = \varphi \star F_{\overline{r}}$ are equal to $a_{\overline{k}}(\varphi) a_{\overline{k}}(F_{\overline{r}})$, $\overline{k} \in \mathbb{Z}^m$. Therefore, using Parseval's equality, it is easy to verify that

$$\|\delta_{\overline{s}}(f)\|_2 << 2^{-\langle \bar{s}, \bar{r}\rangle} \|\delta_{\overline{s}}(\varphi)\|_2, \ \overline{s} \in \mathbb{Z}_+^m. \quad (2)$$

Hense, from (1) and (2) we get

$$\|F_1\|_A \leq 2^{-\frac{m}{2}} \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} 2^{\langle \bar{s}, \bar{1}\rangle \frac{1}{2}} \|\delta_{\overline{s}}(f)\|_2 \leq C \sum_{l=n}^{n_1-1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} 2^{\langle \bar{s}, \bar{1}\rangle \frac{1}{2}} 2^{-\langle \bar{s}, \bar{r}\rangle} \|\delta_{\overline{s}}(\varphi)\|_2. \quad (3)$$

If $2 < \tau_1 < \infty$, then according to the inequality of different metrics for trigonometric polynomials in the Lorentz space [20] we have

$$\|\delta_{\overline{s}}(\varphi)\|_2 \leq C \Big(\sum_{j=1}^{m} (s_j + 1)\Big)^{\frac{1}{2} - \frac{1}{\tau_1}} \|\delta_{\overline{s}}(\varphi)\|_{2,\tau_1}.$$

From Lemma 1.6 [21] for $p = 2$ and $2 < \tau_1 < \infty$ we get

$$\left(\sum_{\overline{s} \in \mathbb{Z}_+} \left(\sum_{j=1}^{m} (s_j + 1)\right)^{(\frac{1}{\tau_1} - \frac{1}{2})\tau_1} \|\delta_{\overline{s}}(\varphi)\|_2^{\tau_1}\right)^{\frac{1}{\tau_1}} \leq C \left(\sum_{\overline{s} \in \mathbb{Z}_+} \|\delta_{\overline{s}}(\varphi)\|_{2,\tau_1}^{\tau_1}\right)^{\frac{1}{\tau_1}} \leq C\|\varphi\|_{2,\tau_1}. \quad (4)$$

By virtue of inequality (4) and Hölder's inequality, we obtain

$$\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{\langle\bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\overline{s}}(f)\|_2 \leq \left(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\overline{s}}(\varphi)\|_2^{\tau_1}\right)^{\frac{1}{\tau_1}} \times$$
$$\times\left(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{-\langle\bar{s},\bar{\gamma}\rangle(r_1-\frac{1}{2})\tau_1'}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_1'}\right)^{\frac{1}{\tau_1'}} \leq \tag{5}$$
$$\leq C\|\varphi\|_{2,\tau_1}\left(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1} 2^{-\langle\bar{s},\bar{\gamma}\rangle(r_1-\frac{1}{2})\tau_1'}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_1'}\right)^{\frac{1}{\tau_1'}} \leq$$
$$\leq C2^{-l\left(r_1-\frac{1}{2}\right)}l^{(\nu-1)\frac{1}{\tau_1'}}l^{\frac{1}{2}-\frac{1}{\tau_1}}\|\varphi\|_{2,\tau_1},$$

where $\tau_1'=\frac{\tau_1}{\tau_1-1}$, $1<\tau_1<\infty$.

(3) and (5) imply that

$$\|F_1\|_A \leq C\sum_{l=n}^{n_1-1} 2^{\frac{l}{2}}l^{(\nu-1)\frac{1}{\tau_1'}}l^{\frac{1}{2}-\frac{1}{\tau_1}}2^{-lr_1} \leq C2^{-n_1(r_1-\frac{1}{2})}n_1^{(\nu-1)\frac{1}{\tau_1'}}n_1^{\frac{1}{2}-\frac{1}{\tau_1}} \tag{6}$$

for a function $f\in W^{\overline{r}}_{2,\tau_1}$ when $r_1<\frac{1}{2}$ and $2<\tau_1<\infty$.

By Lemma 1 for the function F_1 using a constructive method, one can find an M-term trigonometric polynomial $G_M(F_1)$ such that

$$\|F_1-G_M(F_1)\|_{p,\tau_2} \leq CM^{-\frac{1}{2}}2^{-n_1(r_1-\frac{1}{2})}n_1^{(\nu-1)\frac{1}{\tau_1'}}n_1^{\frac{1}{2}-\frac{1}{\tau_1}}. \tag{7}$$

Therefore, according to inequality (6) and (7) and taking into account the definition of the number n_1 and the relation $M\asymp 2^n n^{\nu-1}$, we obtain

$$\|F_1-G_M(F_1)\|_{p,\tau_2} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} \tag{8}$$

in case when $q=2<p<\infty$, $2<\tau_1<\infty$, $1<\tau_2<\infty$, $r_1<\frac{1}{2}$.

Let us estimate $\|F_3\|_A$. Applying Hölder's inequality for the sum and Parseval's equality, we obtain

$$\|F_3\|_A=\sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)}\sum_{\overline{k}\in\rho(\overline{s})}|a_{\overline{k}}(f)| \leq 2^{-\frac{m}{2}}\sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)} 2^{\langle\bar{s},\bar{1}\rangle\frac{1}{2}}\|\delta_{\overline{s}}(f)\|_2 \leq$$
$$\leq C\sum_{l=n_1}^{n_2-1}2^{\frac{l}{2}}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})}\|\delta_{\overline{s}}(f)\|_2. \tag{9}$$

Now, to the inner sum on the right side of inequality (9), applying Hölder's inequality for $\frac{1}{\tau_1}+\frac{1}{\tau_1'}=1$, $1<\tau_1<\infty$, we have

$$\|F_3\|_A \leq C\sum_{l=n_1}^{n_2-1}2^{\frac{l}{2}}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\Big(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\overline{s}\in G(l)}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\overline{s}}(f)\|_2^{\tau_1}\Big)^{\frac{1}{\tau_1}}|G(l)|^{\frac{1}{\tau_1'}}.$$

Then, using (2) we get

$$\|F_3\|_A \le C\sum_{l=n_1}^{n_2-1} 2^{\frac{l}{2}}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\times$$

$$\times\Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}2^{-\langle\bar{s},\bar{r}\rangle}\|\delta_{\bar{s}}(\varphi)\|_2^{\tau_1}\Big)^{\frac{1}{\tau_1}}|G(l)|^{\frac{1}{\tau_1'}}\le$$

$$\le C\sum_{l=n_1}^{n_2-1} 2^{l\left(\frac{1}{2}-r_1\right)}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\times \tag{10}$$

$$\times\Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\in G(l)}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\bar{s}}(\varphi)\|_2^{\tau_1}\Big)^{\frac{1}{\tau_1}}|G(l)|^{\frac{1}{\tau_1'}}.$$

We set

$$\tilde{S}_l=\Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_1}$$

and

$$m_l:=|G(l)|:=\Big[2^{-l\frac{\tau_2'}{p}}\tilde{S}_l^{\tau_1}2^{n\frac{\tau_2'}{2}}n^{(\nu-1)\frac{\tau_2'}{2}}\Big]+1.$$

Then (10) implies that

$$\|F_3\|_A\le C\sum_{l=n_1}^{n_2-1}2^{-l(r_1-\frac{1}{2})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l m_l^{\frac{1}{\tau_1'}}\le$$

$$\le C\sum_{l=n_1}^{n_2-1}2^{-l(r_1-\frac{1}{2})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l\Big\{2^{-l\frac{\tau_2'}{p}}\tilde{S}_l^{\tau_1}2^{n\frac{\tau_2'}{2}}n^{(\nu-1)\frac{\tau_2'}{2}}+1\Big\}^{\frac{1}{\tau_1'}}\le \tag{11}$$

$$\le C\Big\{\Big(2^n n^{\nu-1}\Big)^{\frac{\tau_2'}{2\tau_1'}}\sum_{l=n_1}^{n_2-1}2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}}+\sum_{l=n_1}^{n_2-1}2^{-l(r_1-\frac{1}{2})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l\Big\}.$$

Since $\tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}}=\tilde{S}_l^{\tau_1}$ and $-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'}=\tau_2'(-\frac{1}{2}+\frac{1}{p}-\frac{1}{p\tau_1}+\frac{1}{2\tau_2})$, then by (4) we have

$$\sum_{l=n_1}^{n_2-1}2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}}=\sum_{l=n_1}^{n_2-1}2^{-l(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l^{\tau_1}\le$$

$$\le C\sum_{l=n_1}^{n_2-1}2^{-l(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\|\varphi\|_{2,\tau_1}^{\tau_1}\le$$

$$\le C\sum_{l=n_1}^{n_2-1}2^{-l(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}l^{\frac{1}{2}-\frac{1}{\tau_1}}$$

for a function $f\in W_{2,\tau_1}^{\bar{r}}$ and $2<\tau_1<\infty$. Since $r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})<0$, then, taking into account the definition of the number n_2, from here we obtain

$$\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{p\tau_1'})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l^{1+\frac{\tau_1}{\tau_1'}} \leq C2^{-n_2(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}n_2^{\frac{1}{2}-\frac{1}{\tau_1}} \leq$$

$$\leq C2^{-n\frac{p}{2}(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}n^{-(\nu-1)\frac{p}{2}(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}n^{\frac{1}{2}-\frac{1}{\tau_1}} \quad (12)$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$, $2 < \tau_1 < \infty$.

Next, due to inequality (4), taking into account that a function $f \in W^{\overline{r}}_{2,\tau_1}$ and $r_1 - \frac{1}{2} < 0$, we have

$$\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l \leq C\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}}\|\varphi\|_{2,\tau_1} \leq$$

$$\leq C\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2})}(l+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \leq C2^{-n_2(r_1-\frac{1}{2})}(n_2+1)^{\frac{1}{2}-\frac{1}{\tau_1}} \leq \quad (13)$$

$$\leq C2^{-n\frac{p}{2}(r_1-\frac{1}{2})}n^{-(\nu-1)\frac{p}{2}(r_1-\frac{1}{2})}n^{\frac{1}{2}-\frac{1}{\tau_1}}.$$

Now it follows from inequalities (11)–(13) that

$$\|F_3\|_A \leq C\Big\{\Big(2^n n^{\nu-1}\Big)^{\frac{\tau_2'}{2\tau_1'}} 2^{-n\frac{p}{2}(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}n^{-(\nu-1)\frac{p}{2}(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}n^{\frac{1}{2}-\frac{1}{\tau_1}}+$$

$$+(2^n n^{\nu-1})^{-\frac{p}{2}(r_1-\frac{1}{2})}n^{\frac{1}{2}-\frac{1}{\tau_1}}\Big\}$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$, $2 < \tau_1 < \infty$, $1 < \tau_2 < \infty$, $r_1 - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < 0$.

Since $\frac{p}{2}(r_1 - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})) - \frac{\tau_2'}{2\tau_1'} = \frac{p}{2}(r_1 - \frac{1}{2})$, then it follows that

$$\|F_3\|_A \leq C(2^n n^{\nu-1})^{-\frac{p}{2}(r_1-\frac{1}{2})}n^{\frac{1}{2}-\frac{1}{\tau_1}}. \quad (14)$$

Since $2 < p < \infty$, then by Lemma 1 for the function F_3, by a constructive method, there is an M-term trigonometric polynomial $G_M(F_3)$ such that

$$\|F_3 - G_M(F_3)\|_{p,\tau_2} \leq CM^{-\frac{1}{2}}(2^n n^{\nu-1})^{-\frac{p}{2}(r_1-\frac{1}{2})}n^{\frac{1}{2}-\frac{1}{\tau_1}}.$$

Hence, in accordance with (14), we have

$$\|F_3 - G_M(F_3)\|_{p,\tau_2} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} \quad (15)$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ for $2 < p < \infty$, $2 < \tau_1 < \infty$, $1 < \tau_2 < \infty$ and $r_1 < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})$.

Let us estimate $\|F_2\|_{p,\tau_2}$. So,

$$\|F_2\|_{p,\tau_2} \leq C\Big(\sum_{l=n_1}^{n_2-1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(f)\|_2^{\tau_2-\tau_1}\|\delta_{\bar{s}}(f)\|_2^{\tau_1}\Big)^{1/\tau_2}.$$

Taking into account that

$$\|\delta_{\bar{s}}(f)\|_2 \leq m_l^{-\frac{1}{\tau_1}}2^{-lr_1}l^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l$$

for $\overline{s} \notin G(l)$ and substituting the values of the numbers m_l for $\tau_2 - \tau_1 \geq 0$, we have

$$\|F_2\|_{p,\tau_2} \leq C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\overline{s}}(f)\|_2^{\tau_1}\Big(m_l^{-\frac{1}{\tau_1}}2^{-lr_1}l^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l\Big)^{\tau_2-\tau_1}\Big)^{1/\tau_2} \leq$$

$$\leq C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)} 2^{l(\frac{1}{2}-\frac{1}{p})\tau_2}2^{-lr_1\tau_1}\|\delta_{\overline{s}}(\varphi)\|_2^{\tau_1}\Big(m_l^{-\frac{1}{\tau_1}}2^{-lr_1}l^{\frac{1}{2}-\frac{1}{\tau_1}}\tilde{S}_l\Big)^{\tau_2-\tau_1}\Big)^{1/\tau_2} \leq$$

$$\leq C\Big(\sum_{l=n_1}^{n_2-1}\Big(\Big(2^{-l\frac{\tau_2'}{p}}\tilde{S}_l^{\tau_1}2^{n\frac{\tau_2'}{2}}n^{(\nu-1)\frac{\tau_2'}{2}}\Big)^{-\frac{1}{\tau_1}}2^{-lr_1}\tilde{S}_l l^{\frac{1}{2}-\frac{1}{\tau_1}}\Big)^{\tau_2-\tau_1}\times \tag{16}$$

$$\times 2^{l(\frac{1}{2}-\frac{1}{p})\tau_2}2^{-lr_1\tau_1}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\bar{s}\notin G(l)}\|\delta_{\overline{s}}(\varphi)\|_2^{\tau_1}\Big)^{1/\tau_2} =$$

$$= C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}\Big(\sum_{l=n_1}^{n_2-1}2^{-l(r_1-\frac{\tau_2'}{p\tau_1})(\tau_2-\tau_1)}l^{(\frac{1}{2}-\frac{1}{\tau_1})(\tau_2-\tau_1)}l^{-(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\tilde{S}_l^{\tau_1}\Big)^{1/\tau_2}.$$

Using inequality (4), it is easy to verify that

$$\tilde{S}_l = \left(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\left(\sum_{j=1}^{m}(s_j+1)\right)^{(\frac{1}{\tau_1}-\frac{1}{2})\tau_1}\|\delta_{\overline{s}}(\varphi)\|_2^{\tau_1}\right)^{1/\tau_1} \leq$$

$$\leq C\Big\|\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\overline{s}}(\varphi)\Big\|_{2,\tau_1} \leq C\|\varphi\|_{2,\tau_1} \tag{17}$$

for a function $f \in W_{2,\tau_1}^{\overline{r}}$, $2 < \tau_1 \leq \tau_2 < \infty$.

Now it follows from inequalities (16) and (17) that

$$\|F_2\|_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}\times$$

$$\times\Big(\sum_{l=n_1}^{n_2-1}2^{-l(r_1-\frac{\tau_2'}{p\tau_1})(\tau_2-\tau_1)}l^{(\frac{1}{2}-\frac{1}{\tau_1})(\tau_2-\tau_1)}2^{l(\frac{1}{2}-\frac{1}{p})\tau_2}l^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_1}2^{-lr_1\tau_1}\Big)^{1/\tau_2} =$$

$$= C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}\Big(\sum_{l=n_1}^{n_2-1}2^{-l\tau_2(r_1-\frac{\tau_2'}{p\tau_1\tau_2}(\tau_2-\tau_1)-(\frac{1}{2}-\frac{1}{p}))}l^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_2}\Big)^{1/\tau_2}.$$

Since

$$r_1 - \frac{\tau_2'}{p\tau_1\tau_2}(\tau_2-\tau_1) - (\frac{1}{2}-\frac{1}{p}) = r_1 - \tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}),$$

then taking into account the definition of the number n_2, from here we get

$$\|F_2\|_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}}2^{-n_2(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}))}n_2^{\frac{1}{2}-\frac{1}{\tau_1}} \leq$$

$$\leq C2^{-n\frac{p}{2}(r_1-\frac{1}{p}-\frac{1}{2})}n^{\frac{1}{2}-\frac{1}{\tau_1}} \tag{18}$$

for function $f \in W_{2,\tau_1}^{\overline{r}}$ when $2 < p < \infty$, $2 < \tau_1 \leq \tau_2 < \infty$, $r_1 < \tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2})$.

Now it follows from inequalities (8), (15), and (18) that

$$\|f-(S_{Q_{n,\bar\gamma}}(f)+G_M(F_1)+G_M(F_3))\|_{p,\tau_2}\le$$
$$\le\|F_1-G_M(F_1)\|_{p,\tau_2}+\|F_3-G_M(F_3)\|_{p,\tau_2}+\|F_2\|_{p,\tau_2}+$$
$$+\Big\|\sum_{\langle\bar s,\bar\gamma\rangle\ge n_2}\delta_{\bar s}(f)\Big\|_{p,\tau_2}\le CM^{-\frac{p}{2}(r_1+\frac1p-\frac12)}(\log M)^{\frac12-\frac1{\tau_1}}+\Big\|\sum_{\langle\bar s,\bar\gamma\rangle\ge n_2}\delta_{\bar s}(f)\Big\|_{p,\tau_2}$$

for a function $f\in W^{\bar r}_{2,\tau_1}$ when $2<p<\infty$, $2<\tau_1\le\tau_2<\infty$, $\frac12-\frac1p<r_1<\tau_2'(\frac12-\frac1p+\frac1{p\tau_1}-\frac1{2\tau_2})$.

Further, taking into account that $2<\tau_1<\tau_2<\infty$ and $r_1+\frac1p-\frac12>0$, and successively applying Theorem B, Jensen's inequality, Theorem A, then Lemma 1.3 [21] and Theorem 1.1 [21], we obtain

$$\Big\|\sum_{\langle\bar s,\bar\gamma\rangle\ge n_2}\delta_{\bar s}(f)\Big\|_{p,\tau_2}=\Big\|\sum_{l=n_2}^{\infty}\sum_{l\leqslant\langle\bar s,\bar\gamma\rangle<l+1}\delta_{\bar s}(f)\Big\|_{p,\tau_2}\le$$
$$\le C\left(\sum_{l=n_2}^{\infty}\sum_{l\le\langle\bar s,\bar\gamma\rangle<l+1}\prod_{j=1}^{m}2^{s_j(\frac12-\frac1p)\tau_2}\|\delta_{\bar s}(f)\|^{\tau_2}_{2,\tau_1}\right)^{\frac1{\tau_2}}\le$$
$$\le C\left(\sum_{l=n_2}^{\infty}2^{l(\frac12-\frac1p)\tau_2}\left[\sum_{l\le\langle\bar s,\bar\gamma\rangle<l+1}\|\delta_{\bar s}(f)\|^{\tau_1}_{2,\tau_1}\right]^{\frac{\tau_2}{\tau_1}}\right)^{\frac1{\tau_2}}\le$$
$$\le C\Big(\sum_{l=n_2}^{\infty}2^{l(\frac12-\frac1p)\tau_2}\Big\|\sum_{l\le\langle\bar s,\bar\gamma\rangle<l+1}\delta_{\bar s}(f)\Big\|^{\tau_2}_{2,\tau_1}\Big)^{\frac1{\tau_2}}\le C\Big(\sum_{l=n_2}^{\infty}2^{-l(r_1+\frac1p-\frac12)p}\Big)^{\frac1p}\le$$
$$\le C2^{-n_2(r_1+\frac1p-\frac12)}\le CM^{-\frac{p}{2}(r_1+\frac1p-\frac12)},$$

that leads to

$$e_M(f)_{p,\tau_2}\le\|f-(S_{Q_{n,\bar\gamma}}(f)+G_M(F_1)+G_M(F_3))\|_{p,\tau_2}\le CM^{-\frac{p}{2}(r_1+\frac1p-\frac12)}(\log M)^{\frac12-\frac1{\tau_1}}$$

For a function $f\in W^{\bar r}_{2,\tau_1}$ when $2<p<\infty$, $2<\tau_1\le\tau_2<\infty$, $\frac12-\frac1p<r_1<\tau_2'(\frac12-\frac1p+\frac1{p\tau_1}-\frac1{2\tau_2})$.

Assume that $\tau_2'(\frac12-\frac1p+\frac1{p\tau_1}-\frac1{2\tau_2})<r_1<\frac12$. Then, taking into account the definition of the number n_1, we get

$$\sum_{l=n_1}^{n_2-1}2^{-l\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac1{p\tau_1}-\frac1{2\tau_2}\right)\right)}(l+1)^{\frac12-\frac1{\tau_1}}\le C2^{-n_1\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac1{p\tau_1}-\frac1{2\tau_2}\right)\right)}n_1^{\frac12-\frac1{\tau_1}}\le$$
$$\le C2^{-\frac{p}{2}n\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac1{p\tau_1}-\frac1{2\tau_2}\right)\right)}n^{p(\frac12-\frac1{\tau_2})(\nu-1)\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac1{p\tau_1}-\frac1{2\tau_2}\right)\right)}n^{\frac12-\frac1{\tau_1}}\qquad(19)$$

for a function $f\in W^{\bar r}_{2,\tau_1}$ when $\tau_2'(\frac12-\frac1p+\frac1{p\tau_1}-\frac1{2\tau_2})<r_1<\frac12$.

(11), (13) and (19) imply that

$$\|F_3\|_A\le(2^nn^{\nu-1})^{-\frac{p}{2}(r_1-\frac12)}n^{\frac{p}{\tau_2'}(\nu-1)\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac1{p\tau_1}-\frac1{2\tau_2}\right)\right)}n^{\frac12-\frac1{\tau_1}}$$

for a function $f\in W^{\bar r}_{2,\tau_1}$ when $\tau_2'(\frac12-\frac1p+\frac1{p\tau_1}-\frac1{2\tau_2})<r_1<\frac12$.

Hence, by Lemma 1 we obtain

$$\|F_3-G_M(F_3)\|_{p,\tau_2}\le CM^{-\frac{p}{2}(r_1+\frac1p-\frac12)}(\log M)^{\frac{p}{\tau_2'}(\nu-1)\left(r_1-\tau_2'\left(\frac12-\frac1p+\frac1{p\tau_1}-\frac1{2\tau_2}\right)\right)}(\log M)^{\frac12-\frac1{\tau_1}}\qquad(20)$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ when $\tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$.

Let us estimate $\|F_2\|_{p,\tau_2}$ in case when $\tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$. (16) and (17) imply that

$$\|F_2\|_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-\frac{\tau_2'}{2}\frac{\tau_2-\tau_1}{\tau_1\tau_2}} \Big(\sum_{l=n_1}^{n_2-1} 2^{-l\tau_2(r_1 - \frac{\tau_2'}{p\tau_1\tau_2}(\tau_2-\tau_1)-(\frac{1}{2}-\frac{1}{p}))} l^{(\frac{1}{2}-\frac{1}{\tau_1})\tau_2}\Big)^{1/\tau_2} \leq$$
$$\leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{p}{\tau_2'}(\nu-1)\left(r_1-\tau_2'\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} \tag{21}$$

in case when $\tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$.

Since $\tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$, (8) implies that

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} \leq$$
$$\leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{p}{\tau_2'}(\nu-1)\left(r_1-\tau_2'\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} \tag{22}$$

(20)–(22) (see (18)) imply that

$$\|f - (S_{Q_{n,\bar\gamma}}(f) + G_M(F_1) + G_M(F_3))\|_{p,\tau_2} \leq$$
$$\leq \|F_1 - G_M(F_1)\|_{p,\tau_2} + \|F_3 - G_M(F_3)\|_{p,\tau_2} + \|F_2\|_{p,\tau_2} + \Big\|\sum_{\langle\bar s,\bar\gamma\rangle \geq n_2} \delta_{\bar s}(f)\Big\|_{p,\tau_2} \leq$$
$$\leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{p}{\tau_2'}(\nu-1)\left(r_1-\tau_2'\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}} + \Big\|\sum_{\langle\bar s,\bar\gamma\rangle \geq n_2} \delta_{\bar s}(f)\Big\|_{p,\tau_2}.$$

Then, taking into account that $\tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$ and following the same steps as in [20], we have

$$\Big\|\sum_{\langle\bar s,\bar\gamma\rangle \geq n_2} \delta_{\bar s}(f)\Big\|_{p,\tau_2} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{p}{\tau_2'}(\nu-1)\left(r_1-\tau_2'\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}}.$$

Hence,

$$e_M(f)_{p,\tau_2} \leqslant \|f - (S_{Q_{n,\bar\gamma}}(f) + G_M(F_1) + G_M(F_3))\|_{p,\tau_2} \leq$$
$$\leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{p}{\tau_2'}(\nu-1)\left(r_1-\tau_2'\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{p\tau_1}-\frac{1}{2\tau_2}\right)\right)}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}}$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ when $2 < p < \infty$, $2 < \tau_1 \leq \tau_2 < \infty$, $\tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < r_1 < \frac{1}{2}$.

Let $1 < \tau_1 \leq 2$. Then by Lemma 1.5 [21] the inequality

$$\Big(\sum_{l\leq\langle\bar s,\bar\gamma\rangle<l+1} \|\delta_{\bar s}(f)\|^2_{2,\tau_1}\Big)^{1/2} \leq C\Big\|\sum_{l\leq\langle\bar s,\bar\gamma\rangle<l+1} \delta_{\bar s}(f)\Big\|_{2,\tau_1}. \tag{23}$$

Since $1 < \tau_1 \leq 2$, then (see [1; 217])

$$\|\delta_{\bar s}(f)\|_2 \leq C\|\delta_{\bar s}(f)\|_{2,\tau_1}. \tag{24}$$

It follows from inequalities (1), (23), and (24) that

$$\|F_1\|_A \leq C\sum_{l=n}^{n_1-1} 2^{l/2}\Big\|\sum_{l\leq\langle\bar s,\bar\gamma\rangle<l+1} \delta_{\bar s}(f)\Big\|_{2,\tau_1}.$$

Now, given that the function $f \in W^{\overline{r}}_{2,\tau_1}$ and the choice of the number n_1, we get

$$\|F_1\|_A \leq CM^{-\frac{p}{2}(r_1-\frac{1}{2})}(\log M)^{(\nu-1)\frac{p}{\tau_2'}(r_1-\frac{1}{2})}$$

for $r_1 < 1/2$. Further, arguing as in the proof of inequality (8), we obtain

$$\|F_1 - G_M(F_1)\|_{p,\tau_2} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})}(\log M)^{\frac{l}{2}-\frac{1}{\tau_1}} \leq CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})} \tag{25}$$

in case when $q = 2 < p < \infty$, $1 < \tau_1 \leq 2$, $1 < \tau_2 < \infty$, $r_1 < \frac{1}{2}$.

Let us estimate $\|F_3\|_A$. For this we set

$$\tilde{S}_l = \Big(2^{lr_1\tau_1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} \|\delta_{\bar{s}}(f)\|_2^2\Big)^{1/2}$$

and

$$\tilde{m}_l := |G(l)| := \Big[2^{-l\frac{\tau_2'}{p}}\tilde{S}_l^2 2^{n\frac{\tau_2'}{2}} n^{(\nu-1)\frac{\tau_2'}{2}}\Big] + 1.$$

In inequality (9) it is proved that

$$\|F_3\|_A \leq 2^{-\frac{m}{2}} \sum_{l=n_1}^{n_2-1} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1, \bar{s} \in G(l)} 2^{\langle \bar{s}, \bar{1}\rangle \frac{1}{2}} \|\delta_{\bar{s}}(f)\|_2 \leq$$

$$\leq 2^{-\frac{m}{2}} \sum_{l=n_1}^{n_2-1} 2^{(l+1)/2} \sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1, \bar{s} \in G(l)} \|\delta_{\bar{s}}(f)\|_2. \tag{26}$$

Applying Hölder's inequality to the inner sum and substituting the value of the number $\tilde{m}_l := |G(l)|$ from (26), we obtain

$$\|F_3\|_A \leq 2^{-\frac{m}{2}} \sum_{l=n_1}^{n_2-1} 2^{(l+1)/2} \Big(\sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1, \bar{s} \in G(l)} \|\delta_{\bar{s}}(f)\|_2^2\Big)^{1/2} |G(l)|^{1/2} \times$$

$$\times 2^{-\frac{m-1}{2}} \Big\{\sum_{l=n_1}^{n_2-1} 2^{l(\frac{1}{2}-r_1)} 2^{-l\frac{\tau_2'}{2p}} \tilde{S}_l^2 (2^n n^{(\nu-1)})^{\frac{\tau_2'}{4}} + \sum_{l=n_1}^{n_2-1} 2^{l(\frac{1}{2}-r_1)} \tilde{S}_l\Big\}. \tag{27}$$

Using inequalities (23) and (24) and taking into account the value of the numbers $\tilde{S}_l$, we obtain

$$\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{2p})} \tilde{S}_l^2 \leq \sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{2p})} \Big(2^{lr_1} \Big\|\sum_{l \leq \langle \bar{s}, \bar{\gamma}\rangle < l+1} \delta_{\bar{s}}(f)\Big\|_{2,\tau_1}\Big). \tag{28}$$

Since a function $f \in W^{\overline{r}}_{2,\tau_1}$ and

$$r_1 - \frac{1}{2} + \frac{\tau_2'}{2p} = r_1 - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2}) \leq r_1 - \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2}) < 0,$$

then from inequality (28) we have

$$\sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\frac{1}{2}+\frac{\tau_2'}{2p})} \tilde{S}_l^2 \leq C \sum_{l=n_1}^{n_2-1} 2^{-l(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}))} \leq C2^{-n_2(r_1-\tau_2'(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}))}. \tag{29}$$

Since the function $f \in W^{\overline{r}}_{2,\tau_1}$ and $r_1 - \frac{1}{2} < 0$, we can prove similarly that

$$\sum_{l=n_1}^{n_2-1} 2^{l(\frac{1}{2}-r_1)} \tilde{S}_l \le C 2^{n_2(\frac{1}{2}-r_1)}. \tag{30}$$

Now it follows from inequalities (27), (29), and (30) that

$$\|F_3\|_A \le C\Big\{(2^n n^{(\nu-1)})^{\frac{\tau_2'}{4}} 2^{-n_2\left(r_1-\tau_2'\left(\frac{1}{2}-\frac{1}{p}+\frac{1}{2p}-\frac{1}{2\tau_2}\right)\right)} + 2^{n_2(\frac{1}{2}-r_1)}\Big\} \le$$
$$\le C(2^n n^{\nu-1})^{-\frac{p}{2}(r_1-\frac{1}{2})}$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ when $2 < p < \infty$, $1 < \tau_1 \le 2$ and $1 < \tau_2 < \infty$, $r_1 < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$.

Therefore, according to Lemma 1, for the function F_3, by a constructive method, there is an M-term trigonometric polynomial $G_M(F_3)$ such that

$$\|F_3 - G_M(F_3)\|_{p,\tau_2} \le CM^{-\frac{1}{2}}(2^n n^{\nu-1})^{-\frac{p}{2}(r_1-\frac{1}{2})} \le CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})} \tag{31}$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ for $2 < p < \infty$, $1 < \tau_1 \le 2$, $1 < \tau_2 < \infty$, $r_1 < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{p\tau_1} - \frac{1}{2\tau_2})$.

Let us estimate $\|F_2\|_{p,\tau_2}$. To do this, note that if $\overline{s} \notin G(l)$, then

$$\|\delta_{\overline{s}}(f)\|_2 \le \tilde{m}_l^{-\frac{1}{2}} 2^{-lr_1} \tilde{S}_l \tag{32}$$

and

$$\|F_2\|_{p,\tau_2} \leqslant C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l \le \langle \bar{s},\bar{\gamma}\rangle < l+1, \bar{s}\notin G(l)} 2^{\langle \bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2} \|\delta_{\overline{s}}(f)\|_2^{\tau_2}\Big)^{1/\tau_2} =$$
$$= C\Big(\sum_{l=n_1}^{n_2-1} \sum_{l \le \langle \bar{s},\bar{\gamma}\rangle < l+1, \bar{s}\notin G(l)} 2^{\langle \bar{s},\bar{1}\rangle(\frac{1}{2}-\frac{1}{p})\tau_2} \|\delta_{\overline{s}}(f)\|_2^{\tau_2-2} \|\delta_{\overline{s}}(f)\|_2^{2}\Big)^{1/\tau_2}.$$

Further, if $\tau_2 - 2 \ge 0$, then using inequality (32) and repeating the arguments of the proof (18), we obtain

$$\|F_2\|_{p,\tau_2} \le C(2^n n^{\nu-1})^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})} \le CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{2})} \tag{33}$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ when $q = 2 < p < \infty$, $1 < \tau_1 \le 2 \le \tau_2 < \infty$, $r_1 < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$.

Now inequalities (25), (31), (33) imply that

$$e_M(f)_{p,\tau_2} \le \|f - (S_{Q_{n,\bar{\gamma}}}(f) + G_M^p(F_1) + G_M^p(F_3))\|_{p,\tau_2} \le CM^{-\frac{p}{2}(r_1+\frac{1}{p}-\frac{1}{q})}(\log M)^{\frac{1}{2}-\frac{1}{\tau_1}}$$

for a function $f \in W^{\overline{r}}_{2,\tau_1}$ when $2 < p < \infty$, $1 < \tau_1 \le 2 \le \tau_2 < \infty$, $r_1 < \tau_2'(\frac{1}{2} - \frac{1}{p} + \frac{1}{2p} - \frac{1}{2\tau_2})$. The proof is complete.

Remark 1. In case when $\tau_1 = 2$, Theorem 1 complements Theorem 4 in [14].

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP22683029).

Author Contributions

G. Akishev and A.Kh. Myrzagaliyeva collected and analyzed data. Both authors participated in the revision of the manuscript and approved the final submission.

All authors contributed equally to this work.

Conflict of Interest

The authors declare no relevant financial or non-financial competing interests.

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Лоренц кеңістігіндегі Соболев класының M-мүшелік жуықтауларын бағалау туралы

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Жұмыста бірнеше айнымалы периодты функциялар кеңістіктері зерделенген, атап айтқанда Лоренц кеңістігі $L_{2,\tau}(\mathbb{T}^m)$, шектеулі аралас бөлшек туындысы бар функциялар класы $W^{\overline{r}}_{2,\tau}$, $1 \leq \tau < \infty$ және $f \in L_{p,\tau}(\mathbb{T}^m)$ функциясының тригонометриялық көпмүшеліктермен ең жақсы M-мүшелік жуықтауларының реті зерттелген. Мақала кіріспеден, негізгі бөлімнен және қорытындыдан тұрады. Кіріспеде негізгі нәтижелерді дәлелдеу үшін ұғымдар, анықтамалар және қажетті тұжырымдар қарастырылған. Сонымен қатар, осы тақырып бойынша алдыңғы зерттеулер жайлы ақпаратты табуға болады. Негізгі бөлімде $W^{\overline{r}}_{2,\tau_1}$ Соболев класы функцияларының $L_{p,\tau_2}(\mathbb{T}^m)$ кеңістігінің нормасы бойынша p, τ_1, τ_2 параметрлері арасындағы қатынастар үшін ең жақсы M-мүшелік жуықтауларының нақты реттік бағалаулары анықталған.

Кілт сөздер: Лоренц кеңістігі, Соболев класы, аралас туынды, тригонометриялық көпмүшеліктер, M-мүшелік жуықтау.

Об оценках M-членных приближений класса Соболева в пространстве Лоренца

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В работе изучены пространства периодических функций нескольких переменных, а именно пространство Лоренца $L_{2,\tau}(\mathrm{T}^m)$, класс функций с ограниченной смешанной дробной производной $W^{\bar{r}}_{2,\tau}$, $1 \leq \tau < \infty$, и порядок наилучшего M-членного приближения функции $f \in L_{p,\tau}(\mathrm{T}^m)$ тригонометрическими полиномами. Статья состоит из введения, основной части и заключения. Во введении рассмотрены основные понятия, определения и необходимые утверждения для доказательства основных результатов. Также можно найти информацию о предыдущих результатах по этой теме. В основной части установлены точные по порядку оценки для наилучших M-членных приближений функций класса Соболева $W^{\bar{r}}_{2,\tau_1}$ по норме пространства $L_{p,\tau_2}(\mathrm{T}^m)$ для различных соотношений между параметрами p, τ_1, τ_2.

Ключевые слова: пространство Лоренца, класс Соболева, смешанная производная, тригонометрический полином, M-членное приближение.

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Bulletin of the Karaganda University. Mathematics Series, No. 2(114), 2024, pp. 22–39

https://doi.org/10.31489/2024M2/22-39 *Research article*

Solution of a boundary value problem for a third-order inhomogeneous equation with multiple characteristics with the construction of the Green's function

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In the paper the second boundary value problem in a rectangular domain for an inhomogeneous third-order partial differential equation with multiple characteristics with constant coefficients was considered. The uniqueness of the solution to the problem posed is proven by the method of energy integrals. A counterexample is constructed in case when the uniqueness theorem's conditions are violated. Using the method of separation of variables, the solution to the problem is sought in the form of a product of two functions $X(x)$ and $Y(y)$. To determine $Y(y)$, we obtain a second-order ordinary differential equation with two boundary conditions at the boundaries of the segment $[0,q]$. For this problem, the eigenvalues and the corresponding eigenfunctions are found for $n=0$ and $n \in N$. To determine $X(x)$, we obtain a third-order ordinary differential equation with three boundary conditions at the boundaries of the segment $[0,p]$. Using the Green's function method, we constructed solution of the specified problem. A separate Green's function for $n=0$ and a separate Green's function for the case when n is natural were constructed. The solution for $X(x)$ is written in terms of the constructed Green's function. After some transformations, an integral Fredholm equation of the second kind is obtained, the solution of which is written through the resolvent. Estimates for the resolvent and Green's function are obtained. The uniform convergence of the solution and the possibility of its term-by-term differentiation under certain conditions on given functions are proven. When justifying the uniform convergence of the solution, the absence of a "small denominator" is proven.

Keywords: differential equation, the third order, multiple characteristics, the second boundary value problem, regular solution, uniqueness, existence, Green's function.

2020 Mathematics Subject Classification: 35G15.

Introduction

Third-order partial differential equations are considered in solving problems in the theory of nonlinear acoustics and in the hydrodynamic theory of space plasma, fluid filtration in porous media [1].

In the aggregate, all third-order equations occupy a special place in terms of their specific character, equations with multiple characteristics.

The first results on a third-order equation with multiple characteristics were obtained by H. Block [2], E. Del Vecchio [3].

L. Cattabriga in [4] for equation $D_x^{2n+1}u - D_y^2 u = 0$ constructed a fundamental solution in the form of a double improper integral.

In [5], a fundamental solution of a third-order equation with multiple characteristics containing the second derivative with respect to time was constructed, their properties were studied, and estimates were found for $|t| \to \infty$.

In works [6–9], boundary value problems for third-order equations with multiple characteristics are considered using the construction of the Green's function. Also, we note the works [10–21], in which

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Received: 31 January 2024; *Accepted:* 04 March 2024.

the boundary value problems for third-order equations are considered. Boundary value problems close to the topic of this work were studied in [22, 23]. In [24, 25], a solution to the problem posed for a third-order equation was found with other boundary conditions.

1 *Formulation of the problem*

In the domain $D = \{(x, y):\ 0 < x < p,\ 0 < y < q\}$, we consider the following third-order equation of the form:

$$L(u) = U_{xxx} - U_{yy} + A_1 U_{xx} + A_2 U_x + A_3 U_y + A_4 U = g_1(x, y), \tag{1}$$

where $A_i,\ p,\, q \in R,\ i = \overline{1, 4}$, are given sufficiently smooth functions.

By the replacement

$$U\left(x, y\right) = u\left(x, y\right) e^{-\frac{A_1}{3}x + \frac{A_3}{2}y},$$

equation (1) can be reduced to the form

$$u_{xxx} - u_{yy} + a_1 u_x + a_2 u = g(x, y), \tag{2}$$

where $a_1 = -\frac{{A_1}^2}{3} + A_2,\ a_2 = \frac{2A_1^3}{27} + \frac{A_3^2}{2} - \frac{A_1 A_2}{3} + A_4,\ g(x, y) = g_1(x, y) \cdot e^{\frac{A_1}{3}x - \frac{A_3}{2}y}$.

Problem A_2. Find function $u\left(x, y\right)$ from class $C^{3,2}_{x,y}\left(D\right) \cap C^{2,1}_{x,y}\left(\overline{D}\right)$, that satisfies equation (2) and the following boundary conditions:

$$u_y\left(x, 0\right) = 0, \quad u_y\left(x, q\right) = 0, \quad 0 \le x \le p, \tag{3}$$

$$u\left(p, y\right) = \psi_2\left(y\right), \quad u_x\left(p, y\right) = \psi_3\left(y\right), \quad u_{xx}\left(0, y\right) = \psi_1\left(y\right), \quad 0 \le y \le q, \tag{4}$$

where $\psi_i\left(y\right),\ i = \overline{1, 3}$, $g\left(x, y\right)$ are given functions. Note that in works [9–12] the case $a_1 = a_2 = 0$ was considered.

2 *The uniqueness of solution*

Theorem 1. If problem A_2 has a solution, then if conditions $a_1 \le 0,\ a_2 \ge 0$ are met, it is unique.

Proof. Let's assume the opposite. Let problem A_2 have two solutions $u_1\left(x, y\right)$ and $u_2\left(x, y\right)$. Then function $u\left(x, y\right) = u_1\left(x, y\right) - u_2\left(x, y\right)$ satisfies the homogeneous equation (2) with homogeneous boundary conditions. Let's prove that $u\left(x, y\right) \equiv 0$ is in $\overline{D}$.

In the domain D the identity

$$uL\left[u\right] = uu_{xxx} - uu_{yy} + a_1 uu_x + a_2 u^2 = 0$$

or

$$\frac{\partial}{\partial x}\left(uu_{xx} - \frac{1}{2}u_x^2 + \frac{1}{2}a_1 u^2\right) - \frac{\partial}{\partial y}\left(uu_y\right) + u_y^2 + a_2 u^2 = 0 \tag{5}$$

holds. Integrating identity (5) over the domain D and taking into account homogeneous boundary conditions, we obtain

$$-\frac{1}{2}a_1 \int\limits_0^q u^2\left(0, y\right) dy + \frac{1}{2}\int\limits_0^q u_x^2\left(0, y\right) dy + \int\limits_0^p \int\limits_0^q u_y^2 dxdy + a_2 \int\limits_0^p \int\limits_0^q u^2 dxdy = 0.$$

If $a_1,\ a_2 \ne 0$, from the fourth term, we get $u\left(x, y\right) \equiv 0$, $(x, y) \in \overline{D}$. If $a_2 = 0$, then from the third term $u_y\left(x, y\right) = 0$. From the equation and taking into account the homogeneous boundary conditions (4) we obtain $u\left(x, y\right) \equiv 0$ is in $\overline{D}$. The theorem has been proven.

Remark 1. Note that if the conditions of Theorem 1 are violated, the homogeneous problem A_2 for the homogeneous equation (2) may have a nontrivial solution. For example, problem

$$\begin{cases} u_{xxx}(x,y) + \left(\frac{(2k+1)\pi}{2p}\right)^2 u_x(x,y) - \left(\frac{\pi n}{q}\right)^2 u(x,y) - u_{yy}(x,y) = 0, \\ u_y(x,0) = 0, \quad u_y(x,q) = 0, \quad 0 \le x \le p, \\ u(p,y) = 0, \quad u_x(p,y) = 0, \quad u_{xx}(0,y) = 0, \quad 0 \le y \le q \end{cases}$$

has a nontrivial solution in the form:

$$u(x,y) = \left(1 + (-1)^{k+1} \sin\left(\frac{(2k+1)\pi}{2p} x\right)\right) \cos\left(\frac{\pi n}{q} y\right), \quad n,k \in Z.$$

3 Existence of a solution

Theorem 2. If the following conditions are met:

1) $\psi_i(y) \in C^3[0,q], \quad {\psi_i}'(0) = {\psi_i}'(q) = 0, \quad i = \overline{1,3}$;

2) $\dfrac{\partial^3 g(x,y)}{\partial x \partial y^2} \in C[0,q], \quad g_y(x,0) = g_y(x,q) = 0, \quad 0 \le x \le p$;

3) $0 \le C < \min\left\{\dfrac{1}{p^2 + \frac{1}{2}p^3}, \dfrac{\lambda_1^2}{Kp(\lambda_1 + 1)}\right\}$,

then a solution to the problem exists.
Here $C = \max\{|a_1|, |a_2|\}$, $\lambda_1 = \sqrt[3]{\left(\frac{\pi}{q}\right)^2}$, $K = \frac{16}{3}\left(1 - \exp\left(-\frac{2\sqrt{3}\pi}{3}\right)\right)^{-1}$.

In works [9–12] $C = 0$. The 3rd condition is satisfied at $C = 0$.

Proof. Consider the following Sturm-Liouville problem taking into account the boundary conditions (3):

$$\begin{cases} Y''(y) + \lambda^3 Y(y) = 0, \\ Y_y(0) = Y_y(q) = 0, \end{cases} \tag{6}$$

eigenvalues and eigenfunctions of problem (6) have the form:

$$Y_n(y) = \begin{cases} \frac{1}{\sqrt{q}}, \quad {\lambda_0}^3 = 0, \quad n = 0, \\ \sqrt{\frac{2}{q}} \cos\left(\frac{\pi n}{q} y\right), \quad {\lambda_n}^3 = \left(\frac{\pi n}{q}\right)^2, \quad n \in N. \end{cases}$$

Let's expand $g(x,y)$ into a Fourier series of $\{Y_n(y)\}$:

$$g(x,y) = \sum_{n=1}^{\infty} g_n(x) Y_n(y),$$

here $g_n(x) = \sqrt{\frac{2}{q}} \int\limits_0^q g(x,\eta) \cos\left(\frac{\pi n}{q}\eta\right) d\eta$. We integrate function $g_n(x)$ by parts twice and taking into account condition 2, Theorem 2, we obtain the estimate $|g_n(x)| \le \frac{M}{n^2} |F_n(x)|$. Here $F_n(x) = \sqrt{\frac{2}{q}} \int\limits_0^q g_{\eta\eta}(x,\eta) \cos \frac{\pi n}{q} \eta d\eta$.

Further on we will denote all arbitrary positive constants by M.

We look for a solution to problem A_2 in the form

$$u\left(x,y\right)=\sum_{n=0}^{\infty}X_n\left(x\right)Y_n\left(y\right). \tag{7}$$

Substituting (7) into equation (2) and taking into account condition (4) we have the following problem:

$$\begin{cases} X'''+a_1X'+a_2X+\lambda_n^3X=g\left(x\right), \\ X''\left(0\right)=\psi_{1n},\ X\left(p\right)=\psi_{2n},\ X'\left(p\right)=\psi_{3n}, \end{cases} \tag{8}$$

where $\psi_{in}=\sqrt{\frac{2}{q}}\int\limits_0^q\psi_{in}\left(\eta\right)\cos\left(\frac{\pi n}{q}\eta\right)d\eta,\ i=\overline{1,3}.$

Using the function

$$V\left(x\right)=X\left(x\right)-\rho\left(x\right), \tag{9}$$

boundary conditions (8) are transformed into homogeneous ones. Function $\rho\left(x\right)$ looks like:

$$\rho_n\left(x\right)=\psi_{2n}-\psi_{3n}p+\frac{\psi_{1n}}{2}p^2+\left(\psi_{3n}-\psi_{1n}p\right)x+\frac{\psi_{1n}}{2}x^2.$$

Substituting (9) into (8) we obtain the problem

$$\begin{cases} V'''+\lambda_n^3V=\lambda_n^3f_n\left(x\right)-a_1V'-a_2V, \\ V''\left(0\right)=V\left(p\right)=V'\left(p\right)=0, \end{cases} \tag{10}$$

here

$$f_n\left(x\right)=\left(\frac{a_1p-a_1x+a_2px}{\lambda_n^3}-\frac{a_2p^2+a_2x^2}{2\lambda_n^3}-\frac{p^2+x^2}{2}+px\right)\psi_{1n}- \\ -\left(\frac{a_2}{\lambda_n^3}+1\right)\psi_{2n}+\left(\frac{a_2p-a_1-a_2x}{\lambda_n^3}+p-x\right)\psi_{3n}+\frac{g(x)}{\lambda_n^3}.$$

Then we have estimates

$$\begin{array}{c} \left|f_n\left(x\right)\right|\le\frac{M}{n^3}\left(\left|\Psi_{1n}\right|+\left|\Psi_{2n}\right|+\left|\Psi_{3n}\right|+\frac{1}{n}\left|F_n\left(x\right)\right|\right), \\ \left|f'_n\left(x\right)\right|\le\frac{M}{n^3}\left(\left|\Psi_{1n}\right|+\left|\Psi_{3n}\right|+\frac{1}{n}\left|F'_n\left(x\right)\right|\right). \end{array} \tag{11}$$

Let's consider cases $n=0$ and $n\in N$ separately. Problem (10) for $\lambda_0=0$ has the form:

$$\begin{cases} V_0'''=f_0\left(x\right)-a_1V_0'-a_2V_0, \\ V_0''\left(0\right)=V_0\left(p\right)=V_0'\left(p\right)=0, \end{cases} \tag{12}$$

here

$$f_0\left(x\right)=g_0\left(x\right)+\left(a_1\left(x-p\right)+a_2\left(px-\frac{p^2}{2}-\frac{x^2}{2}\right)\right)\psi_{1\,0}-a_2\psi_{2\,0}+\left(a_2\left(p-x\right)-a_1\right)\psi_{3\,0}.$$

Problem (12) is equivalent to the integro-differential equation

$$V_0\left(x\right)=\int\limits_0^pG_n\left(x,\xi\right)f_n\left(\xi\right)d\xi+a_1\int\limits_0^pG_0\left(x,\xi\right)V'_0\left(\xi\right)d\xi-a_2\int\limits_0^pG_0\left(x,\xi\right)V_0\left(\xi\right)d\xi, \tag{13}$$

here $G_0\left(x,\xi\right)$ is the Green's function of problem (12), it has the following properties:

$$\frac{\partial^3G_0\left(x,\xi\right)}{\partial x^3}=0,$$

$$G_{10xx}(0,\xi) = G_{20}(p,\xi) = G_{20x}(p,\xi) = 0,$$

$$\begin{aligned} &G_{20}(\xi,\xi) - G_{10}(\xi,\xi) = 0, \\ &G_{20x}(\xi,\xi) - G_{10x}(\xi,\xi) = 0, \\ &G_{20xx}(\xi,\xi) - G_{10xx}(\xi,\xi) = 1. \end{aligned}$$

Function $G_0(x,\xi)$ has the form

$$G_0(x,\xi) = \frac{1}{2}\begin{cases} (p-\xi)(p+\xi-2x), & 0 \le x < \xi \le p, \\ (x-p)^2, & 0 \le \xi < x \le p. \end{cases} \tag{14}$$

It is easy to verify that the function defined by formula (14) has all the properties formulated in the definition of the Green's function.

Integrating by parts the second integral in (13) and introducing the notation

$$\alpha_0(x) = \int_0^p G_0(x,\xi) f_0(\xi)\, d\xi,$$
$$\bar{G}_0(x,\xi) = a_1 G_{0\xi}(x,\xi) - a_2 G_0(x,\xi),$$

we get

$$V_0(x) = \alpha_0(x) + \int_0^p \bar{G}_0(x,\xi) V_0(\xi)\, d\xi. \tag{15}$$

Equation (15) is the Fredholm integral equation of the second kind. We solve (15) using the iteration method.

Taking the zero approximation $V_0(x) = \alpha_0(x)$, we write (15) as follows:

$$V_m(x) = \alpha_0(x) + \int_0^p \bar{G}_0(x,\xi) V_{m-1}(\xi)\, d\xi, \quad m = 1, 2, \ldots$$

The first approximation is

$$V_1(x) = \alpha_0(x) + \int_0^p \bar{G}_0(x,\xi)\, \alpha_0(\xi)\, d\xi,$$

the second approximation is

$$\begin{aligned} V_2(x) = \alpha_0(x) + \int_0^p \bar{G}_0(x,s) V_1(s)\, ds = \alpha_0(x) + \\ + \int_0^p \bar{G}_0(s,\xi)\left(\alpha_0(s) + \int_0^p \bar{G}_0(s,\xi)\, \alpha_0(\xi)\, d\xi\right) ds = \\ = \alpha_0(x) + \int_0^p \bar{G}_0(x,\xi)\, \alpha_0(\xi)\, d\xi + \int_0^p \bar{G}_0(x,s)\, ds \int_0^p \bar{G}_0(s,\xi)\, \alpha_0(\xi)\, d\xi, \end{aligned}$$

by changing the order of integration in the iterated integral and making the replacement

$$\bar{G}_1(x,\xi) = \int_0^p \bar{G}_0(x,s)\, \bar{G}_0(s,\xi)\, ds,$$

then we get

$$V_2(x) = \alpha_0(x) + \int_0^p \left(\bar{G}_0(x,\xi) + \bar{G}_1(x,\xi)\right) \alpha_0(\xi)\, d\xi.$$

If we continue the process indefinitely, we get

$$V_0(x) = \alpha_0(x) + \int_0^p \left(\bar{G}_0(x,\xi) + \sum_{m=1}^{\infty} \bar{G}_m(x,\xi)\right) \alpha_0(\xi)\, d\xi.$$

Here

$$\bar{G}_m(x,\xi) = \int_0^p \bar{G}_0(x,s)\, \bar{G}_{m-1}(s,\xi)\, ds, \quad m = 1, 2, 3, \ldots$$

If we denote

$$R_0(x,\xi) = \bar{G}_0(x,\xi) + \sum_{m=1}^{\infty} \bar{G}_m(x,\xi),$$

then we have a solution in the form

$$V_0(x) = \alpha_0(x) + \int_0^p R_0(x,\xi)\, \alpha_0(\xi)\, d\xi.$$

Then we get a solution for $\lambda_0 = 0$ in the form

$$u_0(x) = \frac{1}{\sqrt{q}} \left(V_0(x) + \rho_0(x)\right).$$

Let's evaluate this solution. First let's find the estimate $G_0(x,\xi)$:

$$|G_0(x,\xi)| \le \frac{1}{2}p^2, \quad |G_{0\xi}(x,\xi)| \le p.$$

For the resolvent $|R_0(x,\xi)| \le \left|\bar{G}_0(x,\xi)\right| + \left|\bar{G}_1(x,\xi)\right| + \ldots + \left|\bar{G}_m(x,\xi)\right| + \ldots$ we find an estimate using the majorant series:

$$\left|\bar{G}_0(x,\xi)\right| \le C\left(p + \frac{1}{2}p^2\right) \le \frac{1}{p}(J_0 p),$$

$$\left|\bar{G}_1(x,\xi)\right| \le \int_0^p \left|\bar{G}_0(x,s)\right| \left|\bar{G}_0(s,\xi)\right| ds \le \frac{1}{p}(J_0 p)^2,$$

$$\left|\bar{G}_2(x,\xi)\right| \le \int_0^p \left|\bar{G}_0(x,s)\right| \left|\bar{G}_1(s,\xi)\right| ds \le \frac{1}{p}(J_0 p)^3,$$

. . . .

$$\left|\bar{G}_m(x,\xi)\right| \le \int_0^p \left|\bar{G}_0(x,s)\right| \left|\bar{G}_{m-1}(s,\xi)\right| ds \le \frac{1}{p}(J_0 p)^{m+1},$$

. . . .

Here $C = \max\{|a_1|, |a_2|\}$, $J_0 = C\left(p + \frac{1}{2}p^2\right)$. Hence the majorant series looks

$$\frac{1}{p}\sum_{m=1}^{\infty}(J_0 p)^m.$$

Condition 3, Theorem 2 can be written as

$$C < \frac{2}{p^3 + 2p^2} \Rightarrow C\left|\frac{1}{2}p^2 + p\right| < \frac{1}{p},$$

hence

$$J_0 p < 1,$$

then the majorizing series is the sum of the terms of an infinite decreasing geometric progression. In this case, the resolvent converges uniformly, and its estimate has the form

$$|R_0(x,\xi)| \leq \frac{J_0}{1 - J_0 p} \leq M.$$

For $\alpha_0(x)$ the estimate is

$$|\alpha_0(x)| \leq \int_0^p |G_0(x,\xi)|\,|g_0(\xi)|\,d\xi \leq M.$$

Then

$$|u_0(x)| \leq M, \quad \left|u_0{'''}(x)\right| \leq M.$$

The solution to problem (10), at $n \in N$, is sought as follows:

$$V_n(x) = \lambda_n^3 \int_0^p G_n(x,\xi) f_n(\xi)\,d\xi - a_1 \int_0^p G_n(x,\xi)\,V_n{'}(\xi)\,d\xi - a_2 \int_0^p G_n(x,\xi)\,V_n(\xi)\,d\xi, \tag{16}$$

where $G_n(x,\xi)$ is the Green's function of problem (10), which has the following properties:

$$\frac{\partial^3 G_n(x,\xi)}{\partial x^3} + \lambda_n^3 G_n(x,\xi) = 0,$$

$$G_{1nxx}(0,\xi) = G_{2n}(p,\xi) = G_{2nx}(p,\xi) = 0, \tag{17}$$

$$\begin{gathered} G_{2n}(\xi,\xi) - G_{1n}(\xi,\xi) = 0, \\ G_{2nx}(\xi,\xi) - G_{1nx}(\xi,\xi) = 0, \\ G_{2nxx}(\xi,\xi) - G_{1nxx}(\xi,\xi) = 1. \end{gathered} \tag{18}$$

Let's construct the Green's function. Since linearly independent solutions to Equation $X'''_n + \lambda_n^3 X_n = 0$ have the form:

$$X_1(x) = e^{-\lambda_n x}, \quad X_2(x) = e^{\frac{\lambda_n}{2}x}\cos\beta_n x, \quad X_3(x) = e^{\frac{\lambda_n}{2}x}\sin\beta_n x, \quad \beta_n = \frac{\sqrt{3}}{2}\lambda_n,$$

let us represent the required Green's function in the form

$$G_n(x,\xi) = \begin{cases} a_1 e^{-\lambda_n x} + a_2 e^{\frac{\lambda_n}{2}x}\cos\beta_n x + a_3 e^{\frac{\lambda_n}{2}x}\sin\beta_n x\,, & 0 \leq x \leq \xi, \\ b_1 e^{-\lambda_n x} + b_2 e^{\frac{\lambda_n}{2}x}\cos\beta_n x + b_3 e^{\frac{\lambda_n}{2}x}\sin\beta_n x\,, & \xi \leq x \leq p, \end{cases} \tag{19}$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are currently unknown functions from ξ.

From properties (18) of the Green's function and setting $c_n(\xi) = b_n(\xi) - a_n(\xi)\,,\ n = 1,2,3$, we obtain a system of linear equations for finding the functions $c_n(\xi)$:

$$\begin{cases} c_1 e^{-\lambda_n \xi} + c_2 e^{\frac{\lambda_n}{2}\xi} cos\beta_n\xi + c_3 e^{\frac{\lambda_n}{2}\xi}\sin\beta_n\xi = 0, \\ -c_1 e^{-\lambda_n \xi} + c_2 e^{\frac{\lambda_n}{2}\xi} cos\left(\beta_n\xi + \frac{\pi}{3}\right) + c_3 e^{\frac{\lambda_n}{2}\xi}\sin\left(\beta_n\xi + \frac{\pi}{3}\right) = 0, \\ c_1 e^{-\lambda_n \xi} + c_2 e^{\frac{\lambda_n}{2}\xi} cos\left(\beta_n\xi + \frac{2\pi}{3}\right) + c_3 e^{\frac{\lambda_n}{2}\xi}\sin\left(\beta_n\xi + \frac{2\pi}{3}\right) = \frac{1}{\lambda_n^2}. \end{cases}$$

The determinant of this system is equal to the value of the Wronski determinant $W(X_1, X_2, X_3)$ at point $x = \xi$, and therefore is nonzero and equal to $W(X_1, X_2, X_3) = \frac{3\sqrt{3}}{2}$. Having calculated $\Delta c_i,\ \ i = 1,2,3$, we get:

$$c_1(\xi) = \frac{e^{\lambda_n \xi}}{3\lambda_n^2}, \quad c_2(\xi) = -\frac{2e^{-\frac{\lambda_n}{2}\xi}\sin\left(\beta_n\xi + \frac{\pi}{6}\right)}{3\lambda_n^2}, \quad c_3(\xi) = \frac{2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\beta_n\xi + \frac{\pi}{6}\right)}{3\lambda_n^2}.$$

Next, we will use property (17) of the Green's function; in our case, these relations take the form:

$$\begin{cases} 2b_1 - b_2 + \sqrt{3}b_3 = \frac{2}{3{\lambda_n}^2}\left(e^{\lambda_n\xi} + 2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)\right), \\ b_1 e^{-\lambda_n p} + b_2 e^{\frac{\lambda_n}{2}p}\cos\frac{\sqrt{3}}{2}\lambda_n p + b_3 e^{\frac{\lambda_n}{2}p}\sin\frac{\sqrt{3}}{2}\lambda_n p = 0, \\ -b_1 e^{-\lambda_n p} + b_2 e^{\frac{\lambda_n}{2}p}\cos\left(\frac{\sqrt{3}}{2}\lambda_n p + \frac{\pi}{3}\right) + b_3 e^{\frac{\lambda_n}{2}p}\sin\left(\frac{\sqrt{3}}{2}\lambda_n p + \frac{\pi}{3}\right) = 0. \end{cases}$$

Due to the linear independence of ${X_1}''(0)\,,\ {X_2}'(p)\,,\ {X_3}'(p)$, the determinant of this system is:

$$\Delta = \sqrt{3}e^{\lambda_n p}\left(1 + 2e^{\frac{-3\lambda_n}{2}p}\cos\left(\frac{\sqrt{3}}{2}\lambda_n p\right)\right) = \sqrt{3}e^{\lambda_n p}\overline{\Delta},$$

here $\overline{\Delta} = 1 + 2e^{\frac{-3\lambda_n}{2}p}\cos\left(\frac{\sqrt{3}}{2}\lambda_n p\right)$.

Consider the following function

$$\overline{\Delta} = 1 + 2e^{-\sqrt{3}t}\cos t, \quad t = \frac{\sqrt{3}}{2}\lambda_n t.$$

The critical points of this function are

$$t_k = \frac{2\pi}{3} + \pi k, \quad k = 0,1,2,3,\ldots$$

$\overline{\Delta}(t)$ takes minimum value only at $k = 0$. Then

$$\overline{\Delta} \geq 1 - \exp\left(-\frac{2\sqrt{3}\pi}{3}\right) > 0.$$

This proves the absence of a "small denominator", hence $\Delta \neq 0$.

Having calculated $\Delta b_i,\ \ i=1,2,3,$ we obtain:

$$b_1=\frac{e^{\lambda_n p}}{\sqrt{3}\lambda_n^2\Delta}\left(e^{\lambda_n\xi}+2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)\right),$$

$$b_2=-\frac{2e^{-\frac{\lambda_n}{2}p}}{\sqrt{3}\lambda_n^2\Delta}\left(e^{\lambda_n\xi}+2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\sin\left(\frac{\sqrt{3}}{2}\lambda_n p+\frac{\pi}{6}\right),$$

$$b_3=\frac{e^{-\frac{\lambda_n}{2}p}}{\sqrt{3}\lambda_n^2\Delta}\left(e^{\lambda_n\xi}+2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\cos\left(\frac{\sqrt{3}}{2}\lambda_n p+\frac{\pi}{6}\right).$$

Considering $a_k\left(\xi\right)=b_k\left(\xi\right)-c_k\left(\xi\right)\ ,\ \ k=1,2,3$ we have $a_k,\ k=1,2,3$:

$$a_1=\frac{2}{\sqrt{3}\lambda_n^2\Delta}\left(e^{\lambda_n\left(p-\frac{\xi}{2}\right)}\cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)-e^{\lambda_n\left(\xi-\frac{p}{2}\right)}\cos\left(\frac{\sqrt{3}}{2}\lambda_n p\right)\right),$$

$$a_2=\tfrac{2}{\sqrt{3}\lambda_n^2\Delta}\left(e^{-\lambda_n\left(\frac{\xi}{2}-p\right)}\left(1+2e^{\frac{-3\lambda_n}{2}p}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n p\right)\right)\sin\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi+\tfrac{\pi}{6}\right)-\right.$$
$$\left.-e^{-\lambda_n\left(\frac{p}{2}-\xi\right)}\left(1+2e^{-\frac{3\lambda_n}{2}\xi}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\sin\left(\tfrac{\sqrt{3}}{2}\lambda_n p+\tfrac{\pi}{6}\right)\right),$$

$$a_3=\tfrac{2}{\sqrt{3}\lambda_n^2\Delta}\left(e^{-\lambda_n\left(\frac{p}{2}-\xi\right)}\left(1+2e^{-\frac{3\lambda_n}{2}\xi}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n p+\tfrac{\pi}{6}\right)-\right.$$
$$\left.-e^{-\lambda_n\left(\frac{\xi}{2}-p\right)}\left(1+2e^{\frac{-3\lambda_n}{2}p}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n p\right)\right)\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi+\tfrac{\pi}{6}\right)\right).$$

Putting the found values into (19), we obtain function $G_n\left(x,\xi\right)$ in the form:

$$G_n\left(x,\xi\right)=\begin{cases}G_{1n}\left(x,\xi\right),\ 0\le x<\xi,\\ G_{2n}\left(x,\xi\right),\ \xi<x\le p,\end{cases}$$

here

$$G_{1n}\left(x,\xi\right)=\tfrac{1}{\sqrt{3}\lambda_n^2\Delta}\left(e^{-\lambda_n x}\left(2e^{\lambda_n\left(p-\frac{\xi}{2}\right)}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi\right)-2e^{\lambda_n\left(\xi-\frac{p}{2}\right)}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n p\right)\right)-\right.$$
$$-2e^{-\frac{\lambda_n}{2}(p-x)}\left(e^{\lambda_n\xi}+2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\sin\left(\tfrac{\sqrt{3}}{2}\lambda_n\left(p-x\right)+\tfrac{\pi}{6}\right)+$$
$$\left.+2e^{\lambda_n\left(\frac{x}{2}-\frac{\xi}{2}\right)}\left(e^{\lambda_n p}+2e^{\frac{-\lambda_n}{2}p}\cos\left(\tfrac{\sqrt{3}}{2}\lambda_n p\right)\right)\sin\left(\tfrac{\sqrt{3}}{2}\lambda_n\left(\xi-x\right)+\tfrac{\pi}{6}\right)\right),$$

$$G_{2n}(x,\xi)=\frac{1}{\sqrt{3}\lambda_n^2\Delta}\left(e^{\lambda_n\xi}+2e^{-\frac{\lambda_n}{2}\xi}\cos\left(\frac{\sqrt{3}}{2}\lambda_n\xi\right)\right)\left(e^{\lambda_n(p-x)}-2e^{-\frac{\lambda_n}{2}(p-x)}\sin\left(\frac{\sqrt{3}}{2}\lambda_n(p-x)+\frac{\pi}{6}\right)\right).$$

The estimate for $G_n\left(x,\xi\right)$ has the form

$$\left|G_n\left(x,\xi\right)\right|\le\frac{K}{\lambda_n^2},\quad\left|G_{n\xi}\left(x,\xi\right)\right|\le\frac{K}{\lambda_n}.\tag{20}$$

Integrating by parts the second integral in (16) and introducing the notation

$$V_{0n}\left(x\right)=\lambda_n^3\int\limits_0^p G_n\left(x,\xi\right)f_n\left(\xi\right)d\xi,$$

$$\bar{G}_n\left(x,\xi\right)=a_1G_{n\xi}\left(x,\xi\right)-a_2G_n\left(x,\xi\right),$$

then (16) has the form

$$V_n(x) = V_{0n}(x) + \int\limits_0^p \bar{G}_n(x,\xi)\, V_n(\xi)\, d\xi. \tag{21}$$

Equation (21) is the Fredholm integral equation of the second kind. Let us write the solution (21) using the resolvent in the form

$$V_n(x) = V_{0n}(x) + \int\limits_0^p R_n(x,\xi)\, V_{0n}(\xi)\, d\xi,$$

where

$$R_n(x,\xi) = \bar{G}_n(x,\xi) + \sum_{m=1}^{\infty} \bar{G}_{mn}(x,\xi), \tag{22}$$

here

$$\bar{G}_{mn}(x,\xi) = \int\limits_0^p \bar{G}_n(x,s)\, \bar{G}_{(m-1)n}(s,\xi)\, ds,\ \ m = 1, 2, \ldots, \quad \bar{G}_{0n}(x,\xi) = \bar{G}_n(x,s)\,.$$

The following relations are valid for functions $G_n(x,\xi)\,,\ \bar{G}_n(x,\xi)$

$$\begin{gathered} G_{nxx}(x, x-0) - G_{nxx}(x, x+0) = 1, \\ G_{n\xi\xi}(x, x-0) - G_{n\xi\xi}(x, x+0) = 1, \\ G_{nx\xi}(x, x-0) - G_{nx\xi}(x, x+0) = -1, \end{gathered} \tag{23}$$

$$\begin{gathered} \bar{G}_n(x, x-0) - \bar{G}_n(x, x+0) = 0, \\ \bar{G}_{nx}(x, x-0) - \bar{G}_{nx}(x, x+0) = -a_1, \\ \bar{G}_{nxx}(x, x-0) - \bar{G}_{nxx}(x, x+0) = -a_2, \\ \bar{G}_{nxxx}(x, x-0) - \bar{G}_{nxxx}(x, x+0) = 0. \end{gathered} \tag{24}$$

Let us evaluate solution (22). From

$$R_n(x,\xi) = \bar{G}_{1n}(x,\xi) + \bar{G}_{2n}(x,\xi) + \ldots + \bar{G}_{mn}(x,\xi) + \ldots,$$

let's find the estimate

$$|R_n(x,\xi)| \le \left|\bar{G}_{1n}(x,\xi)\right| + \left|\bar{G}_{2n}(x,\xi)\right| + \ldots + \left|\bar{G}_{mn}(x,\xi)\right| + \ldots, \tag{25}$$

using equality $\bar{G}_n(x,\xi) = a_1 G_{n\xi}(x,\xi) - a_2 G_n(x,\xi)$ taking into account (20), we have an estimate for $\bar{G}_n(x,\xi)$ in the form

$$\left|\bar{G}_n\right| \le |a_1|\,|G_{n\xi}| + |a_2|\,|G_n| \le \left(\frac{1}{\lambda_n} + \frac{1}{\lambda_n^2}\right) M.$$

For the right side of inequality (25), we construct a majorizing series. By entering the designation

$$J = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_1^2}\right) M,$$

we have

$$\left|\bar{G}_{1n}(x,\xi)\right| \leq \left|\bar{G}_n(x,\xi)\right| \leq MN\left(\frac{1}{\lambda_n}+\frac{1}{\lambda_n^2}\right) \leq \frac{1}{p}Jp,$$

$$\left|\bar{G}_{2n}(x,\xi)\right| \leq \int_0^p \left|\bar{G}_{1n}(x,s)\right| \left|\bar{G}_{1n}(s,\xi)\right| ds \leq \frac{1}{p}J^2p^2,$$

$$\left|\bar{G}_{3n}(x,\xi)\right| \leq \int_0^p \left|\bar{G}_{1n}(x,s)\right| \left|\bar{G}_{2n}(s,\xi)\right| ds \leq \frac{1}{p}J^3p^3,$$

. . . .

$$\left|\bar{G}_{mn}(x,\xi)\right| \leq \int_0^p \left|\bar{G}_{1n}(x,s)\right| \left|\bar{G}_{(m-1)n}(s,\xi)\right| ds \leq \frac{1}{p}J^mp^m,$$

. . . .

Then the majorizing series has the form

$$\frac{1}{p}\sum_{m=1}^{\infty}(Jp)^m.$$

Condition 3, Theorem 2 can be written as

$$C < \frac{\lambda_1^2}{Kp(\lambda_1+1)} \quad \Rightarrow \quad \left(\frac{1}{\lambda_1}+\frac{1}{\lambda_1^2}\right)KC < \frac{1}{p},$$

from here

$$Jp < 1,$$

then the majorizing series is the sum of the terms of an infinite decreasing geometric progression. In this case, the resolvent converges uniformly, and its estimate has the form

$$|R(x,\xi)| \leq \frac{J}{1-Jp} \leq M. \tag{26}$$

Substituting $G_n(x,\xi) = -\frac{1}{\lambda_n^3}G_{n\xi\xi\xi}(x,\xi)$ into $V_{0n}(x)$ and integrating, we have

$$V_{0n}(x) = -f_n(x) + f_n(0)\,G_{2n\xi\xi}(x,0) - f_n(p)\,G_{1n\xi\xi}(x,p) + \int_0^p G_{n\xi\xi}(x,\xi)\,{f_n}'(\xi)\,d\xi.$$

Taking into account estimates (11) and

$$|G_{2n\xi\xi}(x,0)| \leq K, \quad |G_{1n\xi\xi}(x,p)| \leq K,$$

we get

$$|V_{0n}(x)| \leq \frac{M}{n^4}\left(1+|F_n(x)|+|F_n(0)|+|F_n(p)|\right) + \frac{M}{n^3}\left(|\Psi_{1n}|+|\Psi_{2n}|+|\Psi_{3n}|\right). \tag{27}$$

From (26) and (27) we obtain the estimate

$$\begin{gathered}|V_n(x)| \leq |V_{0n}(x)| + \int_0^p |R(x,\xi)|\,|V_{0n}(\xi)|\,d\xi \leq \\ \leq \frac{M}{n^4}\left(1+|F_n(x)|+|F_n(0)|+|F_n(p)|\right) + \frac{M}{n^3}\left(|\Psi_{1n}|+|\Psi_{2n}|+|\Psi_{3n}|\right).\end{gathered}$$

Due to (7) and (9), the solution to the problem has the form

$$u(x,y)=\sum_{n=1}^{\infty}\left(V_n(x)+\rho_n(x)\right)\cos\left(\frac{\pi n}{q}y\right).$$

Let's check this solution for convergence. Considering the assessment

$$|\rho_n(x)|\leq\frac{M}{n^3}\left(|\Psi_{1n}|+|\Psi_{2n}|+|\Psi_{3n}|\right),$$

we have

$$|u(x,y)|\leq M\sum_{n=1}^{\infty}\frac{1}{n^4}\left(1+|F_n(x)|+|F_n(0)|+|F_n(p)|\right)+M\sum_{n=1}^{\infty}\frac{1}{n^3}\left(|\Psi_{1n}|+|\Psi_{2n}|+|\Psi_{3n}|\right).$$

Let's show the convergence $u_{xxx}(x,y)$. Taking into account (23) and (24), we find the derivatives of $V_n(x)$ with respect to x of the third order.

$$V'''_n(x)=\lambda_n^3 f_n(x)-a_1\left(V'_{0n}(x)+\int_0^p R_{nx}(x,\xi)V_{0n}(\xi)\,d\xi\right)-$$
$$-a_2\left(V_{0n}(x)+\int_0^p R_n(x,\xi)V_{0n}(\xi)\,d\xi\right)-\lambda_n^3\left(V_{0n}(x)+\int_0^p R_n(x,\xi)V_{0n}(\xi)\,d\xi\right).$$

Using estimate (23) and the properties of the Green's function, we get

$$\left|V'_{0n}(x)\right|\leq\frac{M}{n^{\frac{7}{3}}}\left(|\Psi_{1n}|+|\Psi_{2n}|+|\Psi_{3n}|+\frac{|F_n(0)|}{n}+1\right),$$

$$|R_{nx}(x,\xi)|\leq n^{\frac{2}{3}}M,$$

next we have

$$\left|V'''_n(x)\right|\leq\frac{M^2}{n}\sum_{i=1}^{3}|\Psi_{in}|+\frac{M}{n^2}\left(|F_n(x)|+|F_n(0)|+|F_n(p)|+1\right).$$

From here

$$|u_{xxx}(x,y)|\leq\sum_{n=1}^{\infty}\frac{M}{n}\left(|\Psi_{1n}|+|\Psi_{2n}|+|\Psi_{3n}|\right)+\sum_{n=1}^{\infty}O\left(n^{-2}\right).$$

Using the Cauchy-Bunyakovsky and Bessel inequalities, we obtain:

$$|u_{xxx}(x,y)|\leq M\left(\sqrt{\sum_{n=1}^{\infty}|\Psi_{1n}|^2}+\sqrt{\sum_{n=1}^{\infty}|\Psi_{2n}|^2}+\sqrt{\sum_{n=1}^{\infty}|\Psi_{3n}|^2}\right)\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}+\sum_{n=1}^{\infty}O\left(n^{-2}\right)\leq$$
$$\leq M\sqrt{\frac{\pi^2}{6}}\left(\|\psi'''_1(y)\|+\|\psi'''_2(y)\|+\|\psi'''_3(y)\|\right)+\sum_{n=1}^{\infty}O\left(n^{-2}\right)<\infty.$$

Here

$$\sum_{n=1}^{\infty}|\Psi_{in}|^2\leq\left\|\psi_i'''\right\|^2_{L_2[0,q]},\ i=\overline{1,3},\quad\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Given the inequality

$$|u_{yy}(x,y)|\leq|u_{xxx}(x,y)|+|a_1|\,|u_x(x,y)|+|a_2|\,|u(x,y)|,$$

we can conclude that u_{yy} also converge.

From the solution of problems (11) and (13) we obtain a solution to problem A_2 in explicit form:

$$u(x,y)=\frac{1}{\sqrt{q}}\left(\alpha_0(x)+\int\limits_0^p R_0(x,\xi)\,\alpha_0(\xi)\,d\xi.+\rho_0(x)\right)+$$
$$+\sqrt{\frac{2}{q}}\sum_{n=1}^{\infty}\cos\left(\frac{\pi n}{q}y\right)\int\limits_0^p G_n(x,\xi)\,\lambda_n^3 f_n(\xi)\,d\xi+$$
$$+\sqrt{\frac{2}{q}}\sum_{n=1}^{\infty}\cos\left(\frac{\pi n}{q}y\right)\left(\int\limits_0^p R_n(x,\xi)\int\limits_0^p G_n(x,s)\,\lambda_n^3 f_n(s)\,ds d\xi\right)+\sqrt{\frac{2}{q}}\sum_{n=1}^{\infty}\rho_n(x)\cos\left(\frac{\pi n}{q}y\right).$$

Thus, Theorem 2 is proved.

Conclusion

In this paper, we consider a boundary value problem for a third-order inhomogeneous equation with multiple characteristics, containing low-order terms with constant coefficients. The uniqueness and existence of a solution to the problem posed are investigated. Sufficient conditions are found for the coefficients under which the problem posed is uniquely solvable, and in the case of violating these conditions, an example of a nontrivial solution to a homogeneous problem is constructed. The solution to the problem is constructed in the form of a eigenfunctions' series for a one-dimensional spectral problem.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Грин функциясын құра отырып, еселі сипаттамалары бар үшінші ретті біртекті емес теңдеу үшін шеттік есептің шешімі

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Жұмыста тұрақты коэффициенттерімен еселі сипаттамалары бар дербес туындылы үшінші ретті біртекті емес дифференциалдық теңдеу үшін тікбұрышты облыста екінші шеттік есеп қарастырылған. Қойылған есептің шешімінің жалғыздығы энергия интегралдары әдісімен дәлелденді. Жалғыздық теоремасының шарттары бұзылған жағдайға қарсы мысал құрастырылды. Айнымалыларды бөліктеу әдісін қолданып, есептің шешімі $X(x)$ және $Y(y)$ екі функцияның көбейтіндісі ретінде ізделеді. $Y(y)$ анықтау үшін $[0, q]$ сегментінің шекараларында екі шекаралық шарттары бар екінші ретті қарапайым дифференциалдық теңдеуді аламыз. Бұл есеп үшін меншікті мәндері және оған сәйкес $n = 0$ және $n \in N$ үшін меншікті функциялары табылды. $X(x)$ анықтау үшін $[0, p]$ сегментінің шекараларында үш шекаралық шарты бар үшінші ретті қарапайым дифференциалдық теңдеуді аламыз. Көрсетілген есептің шешімі Грин функциясы әдісі көмегімен шығарылған. $n = 0$ үшін бөлек Грин функциясы және n натурал сан болған жағдай үшін бөлек Грин функциясы құрылды. $X(x)$ үшін шешім құрылған Грин функциясы арқылы жазылған. Кейбір түрлендірулерден кейін шешімі резольвента арқылы жазылған екінші текті интегралды Фредгольм теңдеуі алынды. Резольвента мен Грин функциясы үшін бағалаулар табылды. Шешімнің бірқалыпты жинақтылығы және берілген функцияларда кейбір шарттар үшін мүшелеп дифференциалдану мүмкіндігі дәлелденді. Шешімнің бірқалыпты жинақтылығын негіздеу кезінде «кіші бөлімнің» жоқтығы дәлелденген.

Кілт сөздер: дифференциалдық теңдеу, үшінші рет, еселі сипаттамалар, екінші шеттік есеп, тұрақты шешім, жалғыздық, бар болу, Грин функциясы.

Решение краевой задачи для неоднородного уравнения третьего порядка с кратными характеристиками с построением функции Грина

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В работе рассмотрена вторая краевая задача в прямоугольной области для неоднородного дифференциального уравнения в частных производных третьего порядка с постоянными коэффициентами с кратными характеристиками. Единственность решения поставленной задачи доказана методом интегралов энергии. Построен контрпример в случае нарушения условий теоремы единственности. Используя метод разделения переменных, решение задачи ищется в виде произведения двух функций $X(x)$ и $Y(y)$. Для определения $Y(y)$ получаем обыкновенное дифференциальное уравнение второго порядка с двумя граничными условиями на границах сегмента $[0, q]$. Для этой задачи найдены собственные значения и соответствующие им собственные функции при $n = 0$ и $n \in N$. Для определения $X(x)$ получаем обыкновенное дифференциальное уравнение третьего порядка с тремя граничными условиями на границах сегмента $[0, p]$. Методом функции Грина получено решение указанной задачи. Были построены отдельная функция Грина для $n = 0$ и отдельная функция Грина для случая, когда n – натуральное. Решение для $X(x)$ выписано через построенную функцию Грина. После некоторых преобразований получено интегральное уравнение Фредгольма второго рода, решение которой выписано через резольвенту. Получены оценки резольвенты и функции Грина. Доказаны равномерная сходимость решения и возможность его почленного дифференцирования при некоторых условиях на заданные функции. При обосновании равномерной сходимости решения доказано отсутствие «малого знаменателя».

Ключевые слова: дифференциальное уравнение, третий порядок, кратные характеристики, вторая краевая задача, регулярное решение, единственность, существование, функция Грина.

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https://doi.org/10.31489/2024M2/40-56 *Research article*

Homogenization of Attractors to Ginzburg-Landau Equations in Media with Locally Periodic Obstacles: Sub- and Supercritical Cases

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The Ginzburg-Landau equation with rapidly oscillating terms in the equation and boundary conditions in a perforated domain was considered. Proof was given that the trajectory attractors of this equation converge weakly to the trajectory attractors of the homogenized Ginzburg-Landau equation. To do this, we use the approach from the articles and monographs of V.V. Chepyzhov and M.I. Vishik about trajectory attractors of evolutionary equations, and we also use homogenization methods that appeared at the end of the 20th century. First, we use asymptotic methods to construct asymptotics formally, and then we justify the form of the main terms of the asymptotic series using functional analysis and integral estimates. By defining the corresponding auxiliary function spaces with weak topology, we derive a limit (homogenized) equation and prove the existence of a trajectory attractor for this equation. Then, we formulate the main theorems and prove them by using auxiliary lemmas. We prove that the trajectory attractors of this equation tend in a weak sense to the trajectory attractors of the homogenized Ginzburg-Landau equation in the subcritical case, and they disappear in the supercritical case.

Keywords: attractors, homogenization, Ginzburg-Landau equations, nonlinear equations, weak convergence, perforated domain, porous medium.

2020 Mathematics Subject Classification: 35B40; 35B41; 35Q80.

Introduction

This work is devoted to investigating boundary value initial problems in the perforated domain. Assuming Robin (Fourier) type of boundary conditions to be set on the boundary of holes, we write down the homogenized (limit) problem and prove the Hausdorff convergence of attractors (Fig.) as the small parameter tends to zero. Thus, we define the homogenized attractor and prove the convergence of the initial attractors to the attractor of the homogenized problem. The asymptotic behaviour of attractors to an initial boundary value problem for complex Ginzburg-Landau equations in perforated domains for the critical case (appearance of additional potential in the homogenized equation) is studied in [1]. In this paper, we investigate subcritical and supercritical cases. For the asymptotic analysis of problems in perforated domains, see, for instance, [2, 3] and [4–7].

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The results of K.A. Bekmaganbetov and A.A. Tolemis in Sections 1 and 2 are supported in part by the Committee of Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan (project AP14869553). The results of G.A. Chechkin in Section 3 is partially supported by Russian Science Foundation (project 20-11-20272) and in Section 4, the work was financially supported by the Ministry of Science and Higher Education of the Russian Federation as part of the program of the Moscow Center for Fundamental and Applied Mathematics under the agreement № 075-15-2022-284. The results of V.V. Chepyzhov in Sections 3 and 4 are partially supported by the Russian Science Foundation (project 23-71-30008).

Received: 05 October 2023; *Accepted:* 12 February 2024.

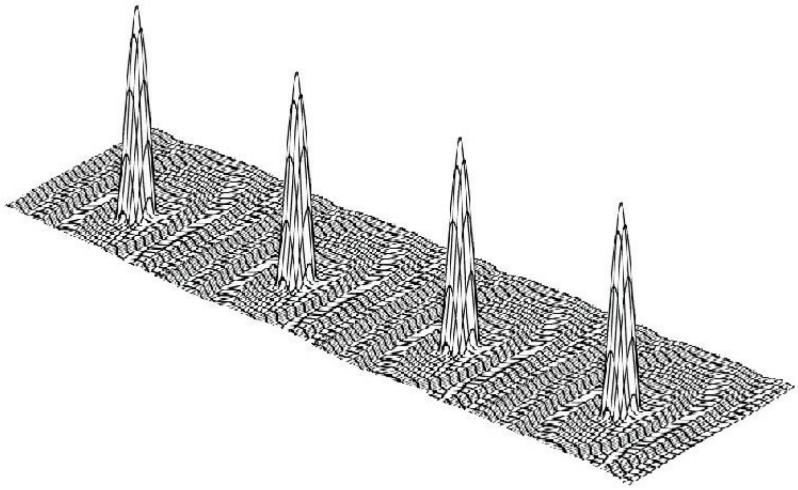

Figure. Attractor of the Ginzburg-Landau equation

About attractors, see, for example, monographs [8–10] and the references therein. Homogenization of attractors were studied in [9, 11–16] (see also [17, 18]).

In the paper, we prove that the trajectory attractor $\mathfrak{A}_\mu$ of the Ginzburg-Landau equation in the perforated domain converges in a weak sense as $\mu \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized equation in an appropriate functional space. Here, μ characterizes the diameter of cavities and the distance between them in the perforated medium.

The results are announced in [19].

1 *Statement of the problem*

First, we define a perforated domain. Let $\Omega \subset \mathbb{R}^d, d \geq 2$ be a smooth bounded domain. Denote

$$\Upsilon_\mu = \left\{ j \in \mathbb{Z}^d \,:\, \mathrm{dist}\,(\mu j, \partial\Omega) \geq \mu\sqrt{d} \right\},\ \square \equiv \left\{ \xi : -\frac{1}{2} < \xi_j < \frac{1}{2},\ j = 1, \ldots, d \right\}.$$

Given a 1-periodic in ξ smooth function $F(x, \xi)$ such that $F(x,\xi)\Big|_{\xi \in \partial\square} \geq \mathrm{const} > 0$, $F(x, 0) = -1$, $\nabla_\xi F \neq 0$ as $\xi \in \square \backslash \{0\}$, we set

$$D_j^\mu = \left\{ x \in \mu\,(\square + j) \mid F(x, \frac{x}{\mu}) \leq 0 \right\}$$

and introduce the perforated domain as follows:

$$\Omega_\mu = \Omega \backslash \bigcup_{j \in \Upsilon_\mu} D_j^\mu.$$

Denote by ω the set $\left\{ \xi \in \mathbb{R}^d \mid F(x, \xi) < 0 \right\}$, and by S the set $\left\{ \xi \in \mathbb{R}^d \mid F(x, \xi) = 0 \right\}$.

Afterwards, we will often interprete 1-periodic in ξ functions as functions defined on d-dimensional torus $\mathbb{T}^d \equiv \left\{ \xi : \xi \in \mathbb{R}^d / \mathbb{Z}^d \right\}$.

According to the above construction, the boundary $\partial\Omega_\mu$ consists of $\partial\Omega$ and the boundary of the cavities $S_\mu \subset \Omega$, $S_\mu = (\partial\Omega_\mu) \cap \Omega$.

We study the asymptotic behaviour of attractors to the problem

$$\begin{cases} \dfrac{\partial u_\mu}{\partial t} = (1+\alpha \mathrm{i})\Delta u_\mu + R(x,\dfrac{x}{\mu})\, u_\mu - \left(1+\beta(x,\dfrac{x}{\mu})\mathrm{i}\right)|u_\mu|^2 u_\mu + g(x), & x \in \Omega_\mu, \\ (1+\alpha \mathrm{i})\dfrac{\partial u_\mu}{\partial \nu} + \mu^\theta q(x,\dfrac{x}{\mu}) u_\mu = 0, & x \in S_\mu, t > 0, \\ u_\mu = 0, & x \in \partial\Omega, \\ u_\mu = U(x), & x \in \Omega_\mu, t = 0, \end{cases} \qquad (1)$$

where $\theta > 1$ (subcritical case) and $0 < \theta < 1$ (supercritical case). Here α is a real constant, ν is the outward unit vector to the boundary, $u = u_1 + \mathrm{i}u_2 \in \mathbb{C}$, $g(x) \in C^1(\Omega;\mathbb{C})$, $q(x,\xi) \in C^1(\Omega;\mathbb{R}^d)$ and $q(x,\xi)$ is a nonnegative 1-periodic in ξ function. We assume that

$$-R_1 \le R(x,\xi) \le R_2,\ -\beta_1 \le \beta(x,\xi) \le \beta_2 \ \ (R_0, R_1, \beta_1, \beta_2 > 0), \qquad (2)$$

for $x \in \Omega$, $\xi \in \mathbb{R}^d$ and the functions $R\,(x,\xi)$ and $\beta\,(x,\xi)$ have the averages $\bar{R}(x)$ and $\bar{\beta}(x)$ in $L_{\infty,*w}(\Omega)$ respectively, i.e.,

$$\int_\Omega R\,(x,\xi)\,\varphi_1(x)dx \ \to \ \int_\Omega \bar{R}(x)\varphi_1(x)dx, \int_\Omega \beta\,(x,\xi)\,\varphi_1(x)dx \ \to \ \int_\Omega \bar{\beta}(x)\varphi_1(x)dx$$

as $\mu \to 0+$ for any function $\varphi_1(x) \in L_1(\Omega)$, where $\xi = \dfrac{x}{\mu}$.

We denote the spaces $\mathbf{H} := L_2(\Omega;\mathbb{C})$, $\mathbf{H}_\mu := L_2(\Omega_\mu;\mathbb{C})$, $\mathbf{V} := H_0^1(\Omega;\mathbb{C})$, $\mathbf{V}_\mu := H^1(\Omega_\mu;\mathbb{C};\partial\Omega)$ – set of functions from $H^1(\Omega_\mu;\mathbb{C})$ with zero trace on $\partial\Omega$, and $\mathbf{L}_p := L_p(\Omega;\mathbb{C})$, $\mathbf{L}_{p,\mu} := L_p(\Omega_\mu;\mathbb{C})$. The norms in these spaces are denoted, respectively, by

$$\|v\|^2 := \int_\Omega |v(x)|^2 dx,\ \|v\|_\mu^2 := \int_{\Omega_\mu} |v(x)|^2 dx,\ \|v\|_1^2 := \int_\Omega |\nabla v(x)|^2 dx,$$

$$\|v\|_{1\mu}^2 := \int_{\Omega_\mu} |\nabla v(x)|^2 dx,\ \|v\|_{\mathbf{L}_p}^p := \int_\Omega |v(x)|^p dx,\ \|v\|_{\mathbf{L}_p\,\mu}^p := \int_{\Omega_\mu} |v(x)|^p dx.$$

Recall that $\mathbf{V}' := H^{-1}(\Omega;\mathbb{C})$ and $\mathbf{L}_q$ are the dual spaces of $\mathbf{V}$ and $\mathbf{L}_p$ respectively, where $q = p/(p-1)$, moreover, $\mathbf{V}'_\mu$ and $\mathbf{L}_{q,\mu}$ are the dual spaces for $\mathbf{V}_\mu$ and $\mathbf{L}_{p,\mu}$.

As in [9], we study weak solutions of the initial boundary value problem (1), that is, the functions

$$u_\mu(x,s) \in L_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\mu) \cap L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\mu) \cap L_4^{loc}(\mathbb{R}_+;\mathbf{L}_{4,\mu})$$

which satisfy the problem (1) in the distributional sense, i.e.

$$-\int_0^\infty \int_{\Omega_\mu} u_\mu \frac{\partial \psi}{\partial t}\, dxdt + (1+\alpha\mathrm{i})\int_0^\infty \int_{\Omega_\mu} \nabla u_\mu \nabla \psi\, dxdt -$$
$$-\int_0^\infty \int_{\Omega_\mu} \left(\left(R\left(x,\frac{x}{\mu}\right)u_\mu - \left(1+\beta\left(x,\frac{x}{\mu}\right)\mathrm{i}\right)|u_\mu|^2 u_\mu\right)\right)\psi\, dxdt +$$
$$+\mu^\theta \int_0^{+\infty} \int_{S_\mu} q\left(x,\frac{x}{\mu}\right) u_\mu \psi\, d\sigma dt = \int_0^\infty \int_{\Omega_\mu} g(x)\psi\, dxdt \qquad (3)$$

for any function $\psi \in C_0^\infty(\mathbb{R}_+;\mathbf{V}_\mu \cap \mathbf{L}_{4,\mu})$.

If $u_\mu(x,t) \in L_4(0,M;\mathbf{L}_{4,\mu})$, then it follows that

$$R\left(x,\frac{x}{\mu}\right)u_\mu(x,t) - \left(1+\beta\left(x,\frac{x}{\mu}\right)\mathrm{i}\right)|u_\mu(x,t)|^2 u_\mu(x,t) \in L_{4/3}(0,M;\mathbf{L}_{4/3,\mu}).$$

At the same time, if $u_\mu(x,t) \in L_2(0,M;\mathbf{V}_\mu)$, then $(1+\alpha\mathrm{i})\Delta u_\mu(x,t)+g(x) \in L_2(0,M;\mathbf{V}'_\mu)$. Therefore, for an arbitrary weak solution $u_\mu(x,s)$ of the problem (1) we have

$$\frac{\partial u_\mu(x,t)}{\partial t} \in L_{4/3}(0,M;\mathbf{L}_{4/3,\mu}) + L_2(0,M;\mathbf{V}'_\mu).$$

The Sobolev embedding theorem implies that

$$L_{4/3}(0,M;\mathbf{L}_{4/3,\mu}) + L_2(0,M;\mathbf{V}'_\mu) \subset L_{4/3}\left(0,M;\mathbf{H}_\mu^{-r}\right),$$

where the space $\mathbf{H}_\mu^{-r} := H^{-r}(\Omega_\mu;\mathbb{C})$ and $r = \max\{1, d/4\}$. Hence, for any weak solution $u_\mu(x,t)$ of (1) we have $\frac{\partial u_\mu(x,t)}{\partial t} \in L_{4/3}\left(0,M;\mathbf{H}_\mu^{-r}\right)$.

Remark 1.1. The existence of weak solution $u(x,s)$ to the problem (1) for every $U \in \mathbf{H}_\mu$ and fixed μ, such that $u(x,0) = U(x)$ can be proved by standard approach (see for instance [8]).

The following key Lemma can be proved similar to Proposition 3 from [17].

Lemma 1.1. Let $u_\mu(x,t) \in L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\mu) \cap L_4^{loc}(\mathbb{R}_+;\mathbf{L}_{4,\mu})$ be a weak solution to the problem (1). Then

(i) $u \in C(\mathbb{R}_+;\mathbf{H}_\mu)$;

(ii) the function $\|u_\mu(\cdot,t)\|_\mu^2$ is absolutely continuous on $\mathbb{R}_+$ and, moreover,

$$\frac{1}{2}\frac{d}{dt}\|u_\mu(\cdot,t)\|_\mu^2 + \|\nabla u_\mu(\cdot,t)\|_\mu^2 + \|u_\mu(\cdot,t)\|_{\mathbf{L}_{4,\mu}}^4 - \int_{\Omega_\mu} R\left(x,\frac{x}{\mu}\right)|u_\mu(x,t)|^2 dx +$$
$$+\mu^\theta \int_{S_\mu} q\left(x,\frac{x}{\mu}\right)|u_\mu(x,t)|^2 d\sigma = \int_{\Omega_\mu} Re\left(g(x)\bar{u}_\mu(x,t)\right)dx,$$

for almost every $t \in \mathbb{R}_+$.

Let us fix μ. In further analysis, we shall omit the index μ in the notation of the spaces, where it is natural. We now apply the scheme described in [1; Section 2] to construct the trajectory attractor for the problem (1), which has the form from the scheme, if we set $E_1 = \mathbf{L}_p \cap \mathbf{V}$, $E_0 = \mathbf{H}^{-r}$, $E = \mathbf{H}$ and $A(u) = (1+\alpha\mathrm{i})\Delta u + R(\cdot)\,u - (1+\beta(\cdot)\mathrm{i})\,|u|^2 u + g(\cdot)$.

To describe the trajectory space $\mathcal{K}_\mu^+$ for the problem (1), we follow the general framework of [1; Section 2] and define the Banach spaces for every $[t_1,t_2] \in \mathbb{R}$

$$\mathcal{F}_{t_1,t_2} := L_4(t_1,t_2;\mathbf{L}_4) \cap L_2(t_1,t_2;\mathbf{V}) \cap L_\infty(t_1,t_2;\mathbf{H}) \cap \left\{v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}\left(t_1,t_2;\mathbf{H}^{-r}\right)\right\}$$

with norm

$$\|v\|_{\mathcal{F}_{t_1,t_2}} := \|v\|_{L_4(t_1,t_2;\mathbf{L}_4)} + \|v\|_{L_2(t_1,t_2;\mathbf{V})} + \|v\|_{L_\infty(0,M;\mathbf{H})} + \left\|\frac{\partial v}{\partial t}\right\|_{L_{4/3}(t_1,t_2;\mathbf{H}^{-r})}. \tag{4}$$

According to the scheme, we use the norm (4); in this case, the translation semigroup $\{S(h)\}$ satisfies the conditions from the scheme.

Setting $\mathcal{D}_{t_1,t_2} = L_2\left(t_1,t_2;\mathbf{V}\right)$ we have that $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$ and if $u(s) \in \mathcal{F}_{t_1,t_2}$, then $A(u(s)) \in \mathcal{D}_{t_1,t_2}$. We can consider a weak solutions of the problem (1) as a solution of an equation in the general scheme from [1; Section 2].

Define the spaces

$$\mathcal{F}_+^{loc} = L_4^{loc}(\mathbb{R}_+;\mathbf{L}_4) \cap L_2^{loc}(\mathbb{R}_+;\mathbf{V}) \cap L_\infty^{loc}(\mathbb{R}_+;\mathbf{H}) \cap \left\{v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}^{loc}(\mathbb{R}_+;\mathbf{H}^{-r})\right\},$$

$$\mathcal{F}_{\mu,+}^{loc} = L_4^{loc}(\mathbb{R}_+;\mathbf{L}_{4,\mu}) \cap L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\mu) \cap L_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\mu) \cap \left\{v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}^{loc}(\mathbb{R}_+;\mathbf{H}_\mu{}^{-r})\right\}.$$

We denote by $\mathcal{K}_\mu^+$ the set of all weak solutions of the problem (1). Recall that for any $U \in \mathbf{H}$ there exist at least one trajectory $u(\cdot) \in \mathcal{K}_\mu^+$ such that $u(0) = U(x)$. Therefore, the trajectory space $\mathcal{K}_\mu^+$ of the problem (1) is not empty and is sufficiently large.

It is clear that $\mathcal{K}_\mu^+ \subset \mathcal{F}_+^{loc}$ and the trajectory space $\mathcal{K}_\mu^+$ is translation invariant, that is, if $u(s) \in \mathcal{K}_\mu^+$, then $u(h+s) \in \mathcal{K}_\mu^+$ for all $h \geq 0$. Therefore,

$$S(h)\mathcal{K}_\mu^+ \subseteq \mathcal{K}_\mu^+, \quad \forall h \geq 0.$$

We now define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ on the spaces $\mathcal{F}_{t_1,t_2}$ using the norms of the spaces $L_2(t_1,t_2;\mathbf{H})$:

$$\rho_{0,M}(u,v) = \left(\int_0^M \|u(s) - v(s)\|_{\mathbf{H}}^2 ds\right)^{1/2}, \quad \forall u(\cdot), v(\cdot) \in \mathcal{F}_{0,M}.$$

These metrics generate the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$ (respectively $\Theta_{\mu,+}^{loc}$ in $\mathcal{F}_{\mu,+}^{loc}$). Recall that a sequence $\{v_k\} \subset \mathcal{F}_+^{loc}$ converges to $v \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\|v_k(\cdot) - v(\cdot)\|_{L_2(0,M;\mathbf{H})} \to 0$ $(k \to \infty)$ for each $M > 0$. The topology Θ_+^{loc} is metrizable. We consider this topology in the trajectory space $\mathcal{K}_\mu^+$ of (1). The translation semigroup $\{S(t)\}$ acting on $\mathcal{K}_\mu^+$ is continuous in the topology Θ_+^{loc}.

Following the general scheme of [1; Section 2], we define bounded sets in $\mathcal{K}_\mu^+$ using the Banach space $\mathcal{F}_{+,\mu}^b$. We clearly have

$$\mathcal{F}_{+,\mu}^b = L_4^b(\mathbb{R}_+;\mathbf{L}_{4,\mu}) \cap L_2^b(\mathbb{R}_+;\mathbf{V}_\mu) \cap L_\infty(\mathbb{R}_+;\mathbf{H}_\mu) \cap \left\{v \;\Big|\; \frac{\partial v}{\partial t} \in L_{4/3}^b(\mathbb{R}_+;\mathbf{H}_\mu^{-r})\right\}.$$

In an analogous way, we have

$$\mathcal{F}_+^b = L_4^b(\mathbb{R}_+;\mathbf{L}_4) \cap L_2^b(\mathbb{R}_+;\mathbf{V}) \cap L_\infty(\mathbb{R}_+;\mathbf{H}) \cap \left\{v \;\Big|\; \frac{\partial v}{\partial t} \in L_{4/3}^b(\mathbb{R}_+;\mathbf{H}^{-r})\right\};$$

$\mathcal{F}_+^b$ and $\mathcal{F}_{+,\mu}^b$ are subspaces of $\mathcal{F}_+^{loc}$ and $\mathcal{F}_{+,\mu}^{loc}$, respectively.

Consider the translation semigroup $\{S(t)\}$ on $\mathcal{K}_\mu^+$, $S(t): \mathcal{K}_\mu^+ \to \mathcal{K}_\mu^+,\ t \geq 0$.

Let $\mathcal{K}_\mu$ be the kernel of the problem (1) that consists of all weak complete solutions $u(s), \in \mathbb{R}$, of the system bounded in the space

$$\mathcal{F}_\mu^b = L_4^b(\mathbb{R};\mathbf{L}_{4,\mu}) \cap L_2^b(\mathbb{R};\mathbf{V}_\mu) \cap L_\infty(\mathbb{R};\mathbf{H}_\mu) \cap \left\{v \;\Big|\; \frac{\partial v}{\partial t} \in L_{4/3}^b(\mathbb{R};\mathbf{H}_\mu^{-r})\right\}.$$

In analogous way we define $\mathcal{F}^b$.

The definition of trajectory attractor was given in [1] (see also [9]).

Proposition 1.1. The problem (1) has the trajectory attractors $\mathfrak{A}_\mu$ in the topological space Θ_+^{loc}. The set $\mathfrak{A}_\mu$ is uniformly (w.r.t. $\mu \in (0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover,

$$\mathfrak{A}_\mu = \Pi_+\mathcal{K}_\mu,$$

the kernel $\mathcal{K}_\mu$ is non-empty and uniformly (w.r.t. $\mu \in (0,1)$) bounded in $\mathcal{F}^b$. Recall that the spaces $\mathcal{F}_+^b$ and Θ_+^{loc} depend on μ.

The proof of this proposition almost coincides with the proof given in [9] for a particular case. The existence of an absorbing set that is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc} is proved using Lemma 1.1 similar to [9].

We note that

$$\mathfrak{A}_\mu \subset \mathcal{B}_0(R), \quad \forall \mu \in (0,1),$$

where $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_+^b$ with a sufficiently large radius R. The Aubin-Lions-Simon Lemma from [1; Section 2] implies that

$$\mathcal{B}_0(R) \Subset L_2^{loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta}), \tag{5}$$

$$\mathcal{B}_0(R) \Subset C^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta}), \quad 0 < \delta \le 1. \tag{6}$$

Using compact inclusions (5) and (6), we strengthen the attraction to the constructed trajectory attractor.

Corollary 1.1. For any set $\mathcal{B} \subset \mathcal{K}_\mu^+$ bounded in $\mathcal{F}_+^b$ we have

$$\operatorname{dist}_{L_2(0,M;H^{1-\delta})} (\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\mu) \to 0 \ (t \to \infty),$$
$$\operatorname{dist}_{C([0,M];H^{-\delta})} (\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\mu) \to 0 \ (t \to \infty),$$

where M is an arbitrary positive number.

2 Homogenized (limit) problem

Let M_i be 1-periodic solution to a problem

$$\Delta_\xi (M_i + \xi_i) = 0 \text{ in } \square \setminus \omega, \qquad \frac{\partial M_i}{\partial \nu_\xi} = \nu_i \text{ on } S(x), \tag{7}$$

having zero mean values over the cell of periodicity. Denote by $\langle \cdot \rangle$ the integral over the set $\square \cap \omega$.

The case $\theta > 1$. The homogenized (limit) problem has the form

$$\begin{cases} \dfrac{\partial u_0}{\partial t} - (1+\alpha \mathrm{i}) \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle \frac{\partial u_0}{\partial x_j} \right) - \\ \qquad\qquad -R(x)u_0 + (1+\beta(x)\mathrm{i}) |u_0|^2 u_0 = |\square \cap \omega| \, g(x), & x \in \Omega, \\ u_0 = 0, & x \in \partial\Omega, t > 0, \\ u_0 = U(x), & x \in \Omega, t = 0. \end{cases} \tag{8}$$

We consider weak solution to the problem (8), i.e. the function $u_0 = u_0(x,t)$, $x \in \Omega$, $t \ge 0$,

$$u_0 \in L_4^{loc}(\mathbb{R}_+; \mathbf{L}_4) \cap L_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap L_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}^{loc}(\mathbb{R}_+; \mathbf{H}^{-r}) \right\},$$

satisfying the integral identity

$$- \int_{\mathbb{R}_+} \int_\Omega u_0 \frac{\partial v}{\partial t} dt dx + (1+\alpha \mathrm{i}) \int_{\mathbb{R}_+} \int_\Omega \sum_{i,j=1}^{d} \left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_j} dt dx -$$
$$- \int_{\mathbb{R}_+} \int_\Omega \Big(R(x)u_0 - (1+\beta(x)\mathrm{i}) |u_0|^2 u_0 \Big) v dt dx = \int_{\mathbb{R}_+} \int_\Omega |\square \cap \omega| \, g(x) \, v \, dt dx$$

for any function $v \in C_0^\infty(\mathbb{R}_+; \mathbf{V} \cap \mathbf{L}_4)$.

Remark 2.1. It should be noted that $M_i(x,\xi)$ are not defined in the whole Ω. Applying the technique of the symmetric extension allows to extend $M_i(x,\xi)$ into the interior of the "holes" retaining the regularity of these functions. We keep the same notation for the extended functions.

3 Auxiliaries

3.1 General reasoning

We investigate the asymptotic behavior of the solution $u_\mu(x)$ as $\mu \to 0$ of the following boundary-value problem in the domain Ω_μ :

$$\begin{cases} -(1+\alpha \mathrm{i})\, \Delta u_\mu = g(x) & \text{in} \quad \Omega_\mu, \\ (1+\alpha \mathrm{i})\, \dfrac{\partial u_\mu}{\partial \nu_\mu} + \mu^\theta q\left(x, \dfrac{x}{\mu}\right) u_\mu = 0 & \text{on} \quad S_\mu, \\ u_\mu = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{9}$$

where n_μ is the internal normal to the boundary of "holes" $q(x,\xi)$ is a sufficiently smooth 1-periodic in ξ function.

Definition 3.1. Function $u_\mu \in H^1(\Omega_\mu, \partial\Omega)$ is a solution of problem (9), if the following integral identity

$$(1+\alpha \mathrm{i}) \int_{\Omega^\mu} \nabla u_\mu(x)\, \nabla v(x)\, dx + \mu^\theta \int_{S_\mu} q\left(x, \frac{x}{\mu}\right) u_\mu(x) v(x)\, ds = \int_{\Omega^\mu} g(x)\, v(x)\, dx$$

holds true for any function $v \in H^1(\Omega_\mu, \partial\Omega)$.

Here, we use the standard notation $H^1(\Omega^\mu, \partial\Omega)$ for the closure of the set of $C^\infty(\overline{\Omega}^\mu)$-functions vanishing in a neighborhood of $\partial\Omega$, by the $H^1(\Omega^\mu)$ norm.

In [1] we showed that $\theta = 1$ is a critical value for problem (9); in what follows we prove that the dissipation dominates if $\theta < 1$ and is neglectable if $\theta > 1$.

3.2 Subcritical case $\theta > 1$

This section deals with problem (9) in the case $\theta > 1$. Substituting the expression

$$u_\mu(x) = u_0(x) + \mu^{\theta-1} u_{1,-1}\left(x, \frac{x}{\mu}\right) + \dots + \mu u_{0,1}\left(x, \frac{x}{\mu}\right) + \mu^\theta u_{1,0}\left(x, \frac{x}{\mu}\right) + \\ + \mu^2 u_{0,2}(x, \frac{x}{\mu}) + \mu^{\theta+1} u_{1,1}(x, \frac{x}{\mu}) + \dots + \mu^{k\theta+l} u_{k,l}(x, \frac{x}{\mu}) + \dots \tag{10}$$

in equation (9) and taking into account an evident relation

$$\frac{\partial}{\partial x} \zeta\left(x, \frac{x}{\mu}\right) = \left(\frac{\partial}{\partial x} \zeta(x,\xi) + \frac{1}{\mu} \frac{\partial}{\partial \xi} \zeta(x,\xi)\right)\Bigg|_{\xi = \frac{x}{\mu}},$$

we obtain, after simple transformations, the following formal equality

$$
\begin{aligned}
-\frac{g(x)}{1+\alpha\mathrm{i}} = \Delta_x u_\mu(x) \cong \Delta_x u_0(x) + \mu^{\theta-1}\left(\Delta_x u_{1,-1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + 2\mu^{\theta-2}\left(\nabla_x, \nabla_\xi u_{1,-1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+ \mu^{\theta-3}\left(\Delta_\xi u_{1,-1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \mu\left(\Delta_x u_{0,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + 2\left(\nabla_x, \nabla_\xi u_{0,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+ \frac{1}{\mu}\left(\Delta_\xi u_{0,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \mu^{\theta}\left(\Delta_x u_{1,0}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + 2\mu^{\theta-1}\left(\nabla_x, \nabla_\xi u_{1,0}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+ \mu^{\theta-2}\left(\Delta_\xi u_{1,0}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \mu^{2}\left(\Delta_x u_{0,2}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \mu 2\left(\nabla_x, \nabla_\xi u_{0,2}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+ \left(\Delta_\xi u_{0,2}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \mu^{\theta+1}\left(\Delta_x u_{1,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + 2\mu^{\theta}\left(\nabla_x, \nabla_\xi u_{1,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+ \mu^{\theta-1}\left(\Delta_\xi u_{1,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \dots + \mu^{k\theta+l}\left(\Delta_x u_{k,l}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + 2\mu^{k\theta+l-1}\left(\nabla_x, \nabla_\xi u_{k,l}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+ \mu^{k\theta+l-2}\left(\Delta_\xi u_{k,l}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \dots \quad (11)
\end{aligned}
$$

Similarly, on S_μ we get

$$
\begin{aligned}
0 = \frac{\partial u_\mu}{\partial \nu_\mu} + \mu^{\theta}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha\mathrm{i}}u_\mu \cong \left(\nabla_x u_0, \nu_\mu\right) + \mu^{\theta-1}\left(\nabla_x u_{1,-1}, \nu_\mu\right) + \mu^{\theta}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha\mathrm{i}}u_0 + \dots + \\
+ \mu^{\theta-2}\left(\nabla_\xi u_{1,-1}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \mu^{2\theta-1}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha\mathrm{i}}u_{1,-1} + \mu\left(\nabla_x u_{0,1}, \nu_\mu\right) + \\
+ \left(\nabla_\xi u_{0,1}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \mu^{\theta+1}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha\mathrm{i}}u_{0,1} + \mu^{\theta}\left(\nabla_x u_{1,0}, \nu_\mu\right) + \mu^{\theta-1}\left(\nabla_\xi u_{1,0}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \\
+ \mu^{2\theta}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha\mathrm{i}}u_{1,0} + \mu^{2}\left(\nabla_x u_{0,2}, \nu_\mu\right) + \mu\left(\nabla_\xi u_{0,2}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \mu^{\theta+2}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha\mathrm{i}}u_{0,2} + \\
+ \mu^{\theta+1}\left(\nabla_x u_{1,1}, \nu_\mu\right) + \mu^{\theta}\left(\nabla_\xi u_{1,1}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \mu^{2\theta+1}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha\mathrm{i}}u_{1,1} + \dots + \\
+ \mu^{k\theta+l}\left(\nabla_x u_{k,l}, \nu_\mu\right) + \mu^{k\theta+l-1}\left(\nabla_\xi u_{k,l}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \mu^{(k+1)\theta+l}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha\mathrm{i}}u_{k,l} + \dots \quad (12)
\end{aligned}
$$

Note that the normal vector ν_μ depends on x and $\frac{x}{\mu}$ in Ω_μ. Considering, as usually, x and $\xi = \frac{x}{\mu}$ as independent variables, we represent ν_μ in Ω_μ in the following form:

$$\nu_\mu(x, \frac{x}{\mu}) = \widetilde{\nu}(x,\xi)\Big|_{\xi=\frac{x}{\mu}} + \mu\nu'_\mu(x,\xi)\Big|_{\xi=\frac{x}{\mu}},$$

where $\widetilde{\nu}$ is a normal to $S(x) = \{\xi \mid F(x,\xi) = 0\}$,

$$\nu'_\mu = \nu' + O(\mu).$$

Collecting all the terms with like powers of μ in (11) and (12), we arrive at the following auxiliary problems:

$$
\left\{
\begin{array}{rcll}
\Delta_\xi u_{1,-1}(x,\xi) & = & 0 & \text{in} \quad \omega, \\
\dfrac{\partial u_{1,-1}(x,\xi)}{\partial \nu} & = & 0 & \text{on} \quad S(x),
\end{array}
\right. \quad (13)
$$

$$\left\{\begin{array}{lcll} \Delta_\xi u_{1,0}\,(x,\xi) & = & -2\left(\nabla_\xi, \nabla_x u_{1,-1}(x,\xi)\right) & \text{in} \quad \omega, \\ \dfrac{\partial u_{1,0}(x,\xi)}{\partial \nu} & = & -\left(\nabla_x u_{1,-1}(x,\xi), \tilde{\nu}\right) & \text{on} \quad S(x), \end{array}\right. \tag{14}$$

and problem

$$\left\{\begin{array}{lll} \Delta_\xi u_{0,1}\,(x,\xi) = 0 & \text{in} & \omega, \\ \dfrac{\partial u_{0,1}(x,\xi)}{\partial \nu} = -\left(\nabla_x (u_0(x)), \tilde{n}\right) & \text{on} & S, \end{array}\right. \tag{15}$$

to be solved in the space of 1-periodic in ξ functions; here x is a parameter, $\omega := \Big\{\xi \in \mathbb{T}^d \,|\, F(x,\xi) > 0\Big\}$. The problem (15) is the standard "cell" problem appearing in the case of Neumann conditions on the boundary of holes. The solvability condition

$$\int_{S(x)} \left(\nabla_x u_0(x), \tilde{\nu}(\xi)\right)\, d\sigma = 0$$

for problem (15) is clearly satisfied, and its solution forms the first "internal" corrector in (10).

It follows from (13) that $u_{1,-1}$ does not depend on ξ. In fact, for our purposes, it suffices to put $u_{1,-1} \equiv 0$. Then $u_{1,0} \equiv 0$ solves (14).

In the next step, we collect all the terms of order μ^0 in (11) and of order μ^1 in (12). This yields

$$\left\{\begin{array}{lcll} \Delta_\xi u_{0,2}\,(x,\xi) & = & -\dfrac{g(x)}{1+\alpha \mathrm{i}} - \Delta_x u_0(x) - 2\left(\nabla_\xi, \nabla_x u_{0,1}(x,\xi)\right) & \text{in} \quad \omega, \\ \dfrac{\partial u_{0,2}(x,\xi)}{\partial \nu} & = & -\left(\nabla_x u_{0,1}(x,\xi), \tilde{\nu}\right) - \left(\nabla_\xi u_{0,1}(x,\xi), \nu'\right) - \left(\nabla_x u_0(x), \nu'\right) & \text{on} \quad S(x). \end{array}\right. \tag{16}$$

If we represent $u_{0,1}(x,\xi) = \left(\nabla_x u_0(x), M(x,\xi)\right)$, where $M(x,\xi) = (M_1(x,\xi), \dots, M_d(x,\xi))$ solves problem (7), then (16) takes the form

$$\left\{\begin{array}{lcll} \Delta_\xi u_{0,2}\,(x,\xi) & = & -\dfrac{g(x)}{1+\alpha \mathrm{i}} - \Delta_x u_0(x) - 2\displaystyle\sum_{i,j=1}^{d} \frac{\partial^2 u_0(x)}{\partial x_i\, \partial x_j}\, \frac{\partial M_i(x,\xi)}{\partial \xi_j} - & \\ & & -2\displaystyle\sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\, \frac{\partial^2 M_i(x,\xi)}{\partial \xi_j\, \partial x_j} & \text{in} \quad \omega, \\ \dfrac{\partial u_{0,2}(x,\xi)}{\partial \nu} & = & -\displaystyle\sum_{i,j=1}^{d} \frac{\partial^2 u_0(x)}{\partial x_i\, \partial x_j} M_i(x,\xi)\tilde{\nu}_j - \sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\, \frac{\partial M_i(x,\xi)}{\partial x_j}\, \tilde{\nu}_j - & \\ & & -\displaystyle\sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\, \frac{\partial M_i(x,\xi)}{\partial \xi_j}\, \nu'_j - \sum_{i=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\, \nu'_i & \text{on} \quad S(x). \end{array}\right.$$

Writing down the compatibility condition in the last problem, we get the following equation:

$$\int_{\square\cap\omega} \left(\frac{g(x)}{1+\alpha \mathrm{i}} + \Delta_x u_0(x) + 2\sum_{i,j=1}^{d} \frac{\partial^2 u_0(x)}{\partial x_i\, \partial x_j}\, \frac{\partial M_i(x,\xi)}{\partial \xi_j} + 2\sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\, \frac{\partial^2 M_i(x,\xi)}{\partial \xi_j\, \partial x_j} \right) d\xi =$$

$$= \int_Q \left(\sum_{i,j=1}^{d} \frac{\partial^2 u_0(x)}{\partial x_i\, \partial x_j} M_i(x,\xi)\tilde{\nu}_j + \sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\, \frac{\partial M_i(x,\xi)}{\partial x_j}\, \tilde{\nu}_j + \right.$$

$$\left. + \sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\, \frac{\partial M_i(x,\xi)}{\partial \xi_j}\, \nu'_j + \sum_{i=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\, \nu'_i \right) d\sigma.$$

In the same way, as in [1] we find the homogenized problem:

$$\begin{cases} (1+\alpha \mathrm{i}) \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle \frac{\partial u_0(x)}{\partial x_i} \right) + |\square \cap \omega| g(x) = 0 \quad \text{in} \quad \Omega, \\ \qquad\qquad u_0(x) = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{17}$$

The integral identity for problem (17) reads

$$(1+\alpha \mathrm{i}) \int_{\Omega} \sum_{i,j=1}^{d} \left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle \frac{\partial u_0(x)}{\partial x_i} \frac{v(x)}{x_j} = \int_{\Omega} |\square \cap \omega|\, g(x)\, v(x)\, dx$$

for any function $v \in \overset{\circ}{H^1} (\Omega)$.

Theorem 3.1. Suppose that $g(x) \in C^1(\mathbb{R}^d)$ and that $q(x,\xi)$ is smooth enough nonnegative function. Then, for any sufficiently small μ problem (9) has the unique solution and the following convergence

$$\|u_0 - u_\mu\|_{H^1(\Omega_\mu)} \longrightarrow 0$$

takes place, where u_0 is a solution of the problem (17).

3.2.1 Auxiliary propositions

Lemma 3.1. Under the conditions of Theorem 3.1 the inequality

$$\int_{\Omega^\mu} |\nabla v|^2 dx + \mu^\theta \int_{S_\mu} q\left(x, \frac{x}{\mu}\right) v^2 ds \geq C_{13} \|v\|^2_{H^1(\Omega_\mu)}$$

holds for any $v \in H^1(\Omega_\mu, \partial\Omega)$.

Lemma 3.2. For any $v \in H^1(\Omega_\mu)$

$$\left| \int_{S_\mu} q\left(x, \frac{x}{\mu}\right) u_0(x)\, v(x)\ ds \right| \leq C_{14}\, \mu^{-1}\, \|u_0\|_{H^1(\Omega_\mu)} \|v\|_{H^1(\Omega_\mu)}.$$

Proof of the Theorem 3.1. The proof of this assertion is based on this lemma, and it can be found in [20].

We omit their proof.

3.3 Supercritical case $\theta < 1$

This section deals with problem (9) in the case $\theta < 1$. The following assertion is valid.

Theorem 3.2. Suppose that $g(x) \in C^1(\mathbb{R}^d)$ and that $q(x,\xi)$ is smooth enough nonnegative function. Then, for any sufficiently small μ problem (9) has the unique solution and the following convergence

$$\|u_\mu\|_{H^1(\Omega_\mu)} \longrightarrow 0$$

takes place as $\mu \to 0$.

Proof. Keeping in mind Lemma 5 from [21], we get from the integral identity the estimate

$$\|u_\mu\|_{H^1(\Omega_\mu)} \leq C.$$

Acting in the same way as in [21], we deduce

$$\int\limits_{\Omega_\mu} u_\mu^2\,dx \le C\Big(\mu\int\limits_{S_\mu} q\Big(x,\frac{x}{\mu}\Big)u_\mu^2 ds + \mu\|u_\mu\|^2_{H^1(\Omega_\mu)}\Big).$$

On the other hand,

$$\Big|\mu\int\limits_{S_\mu} q\Big(x,\frac{x}{\mu}\Big)u_\mu^2 ds\Big| \le \mu^{1-\theta}\|g(x)\|_{L_2(\Omega)}\|u_\mu\|_{L_2(\Omega_\mu)} + O(\mu^{1-\theta}).$$

Combining these estimates, bearing in mind the uniform boundedness of u_μ in $H^1(\Omega_\mu)$, we complete the proof.

4 The main assertion

4.1 The case $\theta > 1$

Theorem 4.1. The following limit holds in the topological space Θ_+^{loc}

$$\mathfrak{A}_\mu \to \overline{\mathfrak{A}} \ \text{ as } \mu \to 0+\,. \tag{18}$$

Moreover,

$$\mathcal{K}_\mu \to \overline{\mathcal{K}} \ \text{ as } \mu \to 0+ \ \text{ in } \ \Theta^{loc}. \tag{19}$$

Remark 4.1. Recall that the functions from the sets $\mathfrak{A}_\mu$ and $\mathcal{K}_\mu$ are defined in the perforated domains Ω_μ. However, all these functions can be prolonged insides the holes in such a way that their norms in the spaces $\mathbf{H}, \mathbf{V}$, and $\mathbf{L}_p$ (without perforation) remain almost the same (are equivalent with the constants independent of the small parameter) as in the perforated spaces $\mathbf{H}_\mu, \mathbf{V}_\mu$, and $\mathbf{L}_{p,\mu}$ (the prolongation of functions defined in perforated domains, see, for instance, in [5; Ch.VIII]). So, in Theorem 4.1, we measure all the distances in the spaces without perforation.

Proof. It is clear that (19) implies (18). Therefore it is sufficient to prove (19), that is, for every neighbourhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc} there exists $\mu_1 = \mu_1(\mathcal{O}) > 0$ such that

$$\mathcal{K}_\mu \subset \mathcal{O}(\overline{\mathcal{K}}) \text{ for } \mu < \mu_1. \tag{20}$$

Suppose that (20) is not true. Then, there exists a neighbourhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\mu_k \to 0+ \ (k \to \infty)$, and a sequence $u_{\mu_k}(\cdot) = u_{\mu_k}(s) \in \mathcal{K}_{\mu_k}$ such that

$$u_{\mu_k} \notin \mathcal{O}'(\overline{\mathcal{K}}) \ \text{ for all } k \in \mathbb{N}. \tag{21}$$

The function $u_{\mu_k}(s), s \in \mathbb{R}$ is the solutions to the problem

$$\begin{cases} \dfrac{\partial u_{\mu_k}}{\partial t} = (1+\alpha \mathrm{i})\Delta u_{\mu_k} + R\left(x, \dfrac{x}{\mu_k}\right) u_{\mu_k} - \left(1 + \beta\left(x, \dfrac{x}{\mu_k}\right)\mathrm{i}\right)|u_{\mu_k}|^2 u_{\mu_k} + g\,(x)\,, & x \in \Omega_{\mu_k}, \\ (1+\alpha \mathrm{i})\dfrac{\partial u_{\mu_k}}{\partial \nu} + \mu_k^\theta q\Big(x, \dfrac{x}{\mu_k}\Big)u_{\mu_k} = 0, & x \in S_{\mu_k}, t > 0, \\ u_{\mu_k} = 0, & x \in \partial\Omega, \\ u_{\mu_k} = U(x), & x \in \Omega_{\mu_k}, t = 0. \end{cases} \tag{22}$$

on the entire time axis $t \in \mathbb{R}$. To obtain the uniform in μ estimate of the solution, we use the following Lemmata (see [22; Ch. III, §5] and [23] respectively).

We obtain the estimate using the integral identity (3), by means of Lemma 1.1. More precise the sequence $\{u_{\mu_k}(x,s)\}$ is bounded in $\mathcal{F}^b$, that is,

$$\|u_{\mu_k}\|_{\mathcal{F}^b} = \sup_{t\in\mathbb{R}} \|u_{\mu_k}(t)\| + \\ + \sup_{t\in\mathbb{R}}\left(\int_t^{t+1} \|u_{\mu_k}(s)\|_1^2 ds\right)^{1/2} + \sup_{t\in\mathbb{R}}\left(\int_t^{t+1} \|u_{\mu_k}(s)\|_{\mathbf{L}_4}^4 ds\right)^{1/4} + \\ + \sup_{t\in\mathbb{R}}\left(\int_t^{t+1} \left\|\frac{\partial u_{\mu_k}}{\partial t}(s)\right\|_{\mathbf{H}^{-r}}^{4/3} ds\right)^{3/4} \leq C \text{ for all } k\in\mathbb{N}. \quad (23)$$

The constant C must not depend on μ.

Hence there exists a subsequence $\{u_{\mu'_k}(x,s)\} \subset \{u_{\mu_k}(x,s)\}$ which we label the same such that

$$u_{\mu_k}(x,s) \to u(s) \text{ as } k\to\infty \text{ in } \Theta^{loc},$$

where $u(x,s)\in\mathcal{F}^b$ and $u(s)$ satisfies (23) with the same constant C. Due to (23) we have $u_{\mu_k}(x,s) \rightharpoonup u(x,s)$ $(k\to\infty)$ weakly in $L_2^{loc}(\mathbb{R};\mathbf{V})$, weakly in $L_4^{loc}(\mathbb{R};\mathbf{L}_4)$, $*$-weakly in $L_\infty^{loc}(\mathbb{R}_+;\mathbf{H})$ and $\frac{\partial u_{\mu_k}(x,s)}{\partial t} \rightharpoonup \frac{\partial u(x,s)}{\partial t}$ $(k\to\infty)$ weakly in $L_{4/3,w}^{loc}(\mathbb{R};\mathbf{H}^{-r})$. We claim that $u(x,s)\in\overline{\mathcal{K}}$. We have already proved that $\|u\|_{\mathcal{F}^b}\leq C$. So we have to establish that $u(x,s)$ is a weak solution of (8).

According to the auxiliary problem in the case $\theta>1$, we have

$$(1+\alpha\mathrm{i})\int_{-M}^{M}\int_{\Omega_{\mu_k}} \nabla u_{\mu_k}\nabla\psi dxdt + \mu_k^\theta \int_{-M}^{M}\int_{S_{\mu_k}} q\Big(x,\frac{x}{\mu_k}\Big) u_{\mu_k}\psi d\sigma dt + \int_{-M}^{M}\int_{\Omega_{\mu_k}} g(x)\psi dxdt \to$$

$$(1+\alpha\mathrm{i})\int_{-M}^{M}\int_{\Omega}\sum_{i,j=1}^{d}\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle \frac{\partial u_0(x,t)}{\partial x_i}\frac{\partial\psi}{\partial x_j}dxdt + \int_{-M}^{M}\int_{\Omega} |\Box\cap\omega|\, g(x)\psi dxdt$$

as $k\to\infty$.

Let us prove that

$$R\left(x,\frac{x}{\mu_k}\right) u_{\mu_k}(x,s) \rightharpoonup \bar{R}(x)u(x,s) \quad (24)$$

and

$$\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)|u_{\mu_k}(x,s)|^2 u_{\mu_k}(x,s) \rightharpoonup \left(1+\bar{\beta}(x)\mathrm{i}\right)|u(x,s)|^2 u(x,s) \quad (25)$$

as $k\to\infty$ weakly in $L_{4/3,w}^{loc}\left(\mathbb{R};\mathbf{L}_{4/3}\right)$.

We fix an arbitrary number $M>0$. The sequence $\{u_{\mu_k}(x,s)\}$ is bounded in $L_4(-M,M;\mathbf{L}_4)$ (see (23)). Then the sequence $\{|u_{\mu_k}(x,s)|^2 u_{\mu_k}(x,s)\}$ is bounded in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$. Since $\{u_{\mu_k}(x,s)\}$ is bounded in $L_2(-M,M;\mathbf{V})$ and $\left\{\dfrac{\partial u_{\mu_k}(x,s)}{\partial t}\right\}$ is bounded in $L_{4/3}(-M,M;\mathbf{H}^{-r})$ we can assume that $u_{\mu_k}(x,s)\to u(x,s)$ as $k\to\infty$ strongly in $L_2(-M,M;\mathbf{L}_2)$ and therefore

$$u_{\mu_k}(x,s)\to u(x,s) \text{ a.e. in } (x,s)\in\Omega\times(-M,M).$$

It follows that

$$|u_{\mu_k}(x,s)|^2 u_{\mu_k}(x,s) \to |u(x,s)|^2 u(x,s) \text{ a.e. in } (x,s)\in\Omega\times(-M,M). \quad (26)$$

We have

$$\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)|u_{\mu_k}(x,s)|^2u_{\mu_k}(x,s)-\left(1+\bar{\beta}(x)\mathrm{i}\right)|u(x,s)|^2u(x,s)=$$
$$=\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)\left(|u_{\mu_k}(x,s)|^2u_{\mu_k}(x,s)-|u(x,s)|^2u(x,s)\right)+$$
$$+\left(\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)-\left(1+\bar{\beta}(x)\mathrm{i}\right)\right)|u(x,s)|^2u(x,s). \quad (27)$$

Let us show that both summand in the right-hand side of (27) converges to zero as $k\to\infty$ weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$.

The sequence

$$\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)\left(|u_{\mu_k}(x,s)|^2u_{\mu_k}(x,s)-|u(x,s)|^2u(x,s)\right)$$

tends to zero as $k\to\infty$ almost everywhere in $(x,s)\in\Omega\times(-M,M)$ (see (26)) and is bounded in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$ (see (2)). Therefore Lemma 1.3 from [24] implies that

$$\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)\left(|u_{\mu_k}(x,s)|^2u_{\mu_k}(x,s)-|u(x,s)|^2u(x,s)\right)\rightharpoonup 0 \ \text{ as } \ k\to\infty$$

weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$.

The sequence

$$\left(\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)-\left(1+\bar{\beta}(x)\mathrm{i}\right)\right)|u(x,s)|^2u(x,s)$$

also approaches zero as $k\to\infty$ weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$ because, by the assumption $\beta\left(x,\frac{x}{\mu}\right)\rightharpoonup\bar{\beta}(x)$ as $k\to\infty$ *-weakly in $L_{\infty,w}\left(-M,M;\mathbf{L}_2\right)$ and $|u(x,s)|^2u(x,s)\in L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$.

We have proved (25). The convergence (24) is proved similarly. Using (24) and (25), we pass to the limit in the equation (22) as $k\to\infty$ in the space $D'\left(\mathbb{R}_+;\mathbf{H}^{-r}\right)$ and obtain that the function $u(x,s)$ satisfies the equation (8).

Consequently, $u\in\overline{\mathcal{K}}$. We have proved above that $u_{\mu_k}\longrightarrow u$ as $k\to\infty$ в Θ^{loc}. Assumption $u_{\mu_k}\notin\mathcal{O}'(\overline{\mathcal{K}})$ (see (21)) implies $u\notin\mathcal{O}'(\overline{\mathcal{K}})$, and, hence, $u\notin\overline{\mathcal{K}}$. We arrive at the contradiction that completes the proof of the theorem.

4.2 The case $\theta<1$

Considering the convergence in Theorem 3.2, we get the following assertion.

Theorem 4.2. The following limit holds in the topological space Θ_+^{loc}

$$\mathfrak{A}_\mu\to 0 \ \text{ as } \mu\to 0+\,.$$

Moreover,

$$\mathcal{K}_\mu\to 0 \ \text{ as } \mu\to 0+ \ \text{ in } \ \Theta^{loc}.$$

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Теңдеуде және шекаралық шарттарында тез тербелмелі мүшелері бар Гинсбург-Ландау теңдеуі тесік облыста қарастырылған. Бұл теңдеудің траекториялық аттракторлары әлсіз мағынада «оғаш мүшесі» (әлеуеті) бар орташаланған Гинсбург-Ландау теңдеуінің траекториялық аттракторларына жуықтайтыны дәлелденді. Ол үшін В.В. Чепыжовтың және М.И. Вишиктің эволюциялық теңдеулердің траекториялық аттракторлары туралы мақалалары мен монографияларының әдістемесі қолданылған. Сондай–ақ, XX ғасырдың соңында пайда болған орташалау әдістері пайдаланылған. Алдымен асимптотикалық әдістерді асимптотиканы формальды құру үшін қолданамыз, содан кейін асимптотикалық қатарлардың негізгі мүшелерін функционалды талдау және интегралды бағалау әдістерін қолдана отырып таңдаймыз. Сәйкесінше, көмекші әлсіз топологиялы функционалды кеңістікті анықтай отырып, шекті (орташаланған) теңдеуін аламыз және осы теңдеу үшін траекториялық аттракторы бар екенін дәлелдейміз. Содан кейін негізгі теоремаларды тұжырымдап, оны көмекші леммалардың көмегімен дәлелдейміз. Бұл теңдеудің траекториялық аттракторлары субкритикалық жағдайда орташаланған Гинсбург-Ландау теңдеуінің траекториялық аттракторына әлсіз түрде жинақталатынын және суперкритикалық жағдайда жоғалып кететінін дәлелдейміз.

Кілт сөздер: аттракторлар, орташалау, Гинсбург-Ландау теңдеулері, сызықтық емес теңдеулер, әлсіз жинақтылық, тесік облыс, кеуекті орта.

Усреднение аттракторов уравнений Гинзбурга-Ландау в средах с локально периодическими препятствиями: суб- и суперкритические случаи

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Рассмотрено уравнение Гинзбурга-Ландау с быстро осциллирующими членами в уравнении и граничных условиях в перфорированной области. Приведено доказательство того, что траекторные аттракторы этого уравнения в слабом смысле сходятся к траекторным аттракторам усредненного уравнения Гинзбурга-Ландау. Для этого мы используем подход из статей и монографий В.В. Чепыжова и М.И. Вишика о траекторных аттракторах эволюционных уравнений, а также применяем методы усреднения, появившиеся в конце XX века. Сначала используем асимптотические методы для формального построения асимптотик, далее обосновываем вид главных членов асимптотических рядов с помощью методов функционального анализа и интегральных оценок. Определяя соответствующие вспомогательные функциональные пространства со слабой топологией, мы выводим предельное (усредненное) уравнение и доказываем существование траекторного аттрактора для этого уравнения. Затем формулируем основные теоремы и доказываем их с помощью вспомогательных лемм. Кроме того, доказываем, что траекторные аттракторы этого уравнения сходятся в слабом смысле к траекторным аттракторам усреднённого уравнения Гинзбурга-Ландау в субкритическом случае и исчезают — в суперкритическом.

Ключевые слова: аттракторы, усреднение, уравнения Гинзбурга-Ландау, нелинейные уравнения, слабая сходимость, перфорированная область, пористая среда.

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https://doi.org/10.31489/2024M2/57-70 *Research article*

On the time-optimal control problem for a fourth order parabolic equation in a two-dimensional domain

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Previously, boundary control problems for the second order parabolic type equation in the bounded domain were studied. In this paper, a boundary control problem associated with a fourth-order parabolic equation in a bounded two-dimensional domain was considered. On the part of the considered domain's boundary, the value of the solution with control function is given. Restrictions on the control are given in such a way that the average value of the solution in the considered domain gets a given value. By the method of separation of variables the given problem is reduced to a Volterra integral equation of the first kind. The existence of the control function was proved by the Laplace transform method and an estimate was found for the minimal time at which the given average temperature in the domain is reached.

Keywords: initial-boundary problem, fourth-order parabolic equation, minimal time, admissible control, Volterra integral equation, Laplace transform method.

2020 Mathematics Subject Classification: 35K25, 35K35.

Introduction

In this paper, we consider the fourth order parabolic equation in the domain $\Omega = \{(x, y) : 0 < x < \pi, \quad 0 < y < \pi\}$

$$u_t(x, y, t) + \Delta^2 u(x, y, t) = 0, \quad (x, y, t) \in \Omega_T := \Omega \times (0, \infty), \tag{1}$$

with boundary value conditions

$$u(0, y, t) \ = \ \psi(y)\, \nu(t), \quad u_x(\pi, y, t) \ = \ 0, \quad u_{xx}(0, y, t) \ = \ 0, \quad u_{xxx}(\pi, y, t) \ = \ 0, \tag{2}$$

$$u(x, 0, t) \ = \ 0, \quad u_y(x, \pi, t) \ = \ 0, \quad u_{yy}(x, 0, t) \ = \ 0, \quad u_{yyy}(x, \pi, t) \ = \ 0, \tag{3}$$

and initial value condition

$$u(x, y, 0) \ = \ 0, \quad 0 \le x, y \le \pi, \tag{4}$$

where $\Delta^2 u(x, y, t) = u_{xxxx}(x, y, t) + u_{yyyy}(x, y, t)$, $\psi(y)$ is a given function and $\nu(t)$ is the control function.

Suppose $M > 0$ is a given constant. If the control function $\nu(t) \in W_2^1(\mathbb{R}_+)$ satisfies the conditions $\nu(0) = 0$ and $|\nu(t)| \le M$ on the half-line $t \ge 0$, we call it *admissible control*. We will prove later in Section 2 that the function ν belongs to the class $W_2^1(\mathbb{R}_+)$.

Now we present the following minimum time problem.

The work supported by the fundamental project (number: FZ-20200929243) of The Ministry of Innovative Development of the Republic of Uzbekistan.

Received: 21 December 2023; *Accepted:* 27 February 2024.

Time-Optimal Problem. Assume that $\theta > 0$ is given constant. Then, find the minimal value of $T > 0$ such that for $t > 0$ the solution $u(x, y, t)$ of the problem (1)–(4) with a control function $\nu(t)$ exists and for some $T_1 > T$ satisfies the equation

$$\int_0^\pi \int_0^\pi u(x, y, t)\, dy\, dx = \theta, \quad T \le t \le T_1. \tag{5}$$

It is known that fourth-order parabolic equations were introduced to describe the epitaxial growth of nanoscale thin films [1]. Therefore, interest in materials science has been increasing in recent years.

Control problems related to second-order parabolic type equations were first studied by Fattorini and Friedman [2,3]. Control problems for the infinite-dimensional case were studied by Egorov [4], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown in the particular conditions.

The optimal time problem related to the second-order parabolic type equation in the bounded $n-$dimensional domain was studied in a new method by Albeverio and Alimov [5] and the optimal time's estimate for achieving a given average temperature was found. In [6,7], mathematical models of thermocontrol processes for the second order parabolic equation are considered. The control problem for the second-order parabolic equation associated with the Neumann boundary condition in a bounded three-dimensional domain is studied in [8]. In this work, an estimate of the optimal time was found when the average temperature is close to the critical value.

In [9, 10], the control problems of the second-order parabolic type equation associated with the Dirichlet boundary condition in the two-dimensional domain are studied. In these articles, an estimate of the minimum time for achieving a given average temperature was found, and the existence of a control function is proved by the Laplace transform method. The boundary control problem related to the fast heating of the thin rod for the inhomogeneous heat conduction equation was studied in works [11, 12] and the existence of the admissible control function was proved.

The optimal time problem for the heat equation with the Neumann boundary condition in a one-dimensional domain is studied in [13]. The difference of this work from the previous works is that the required estimate for the minimum time is found with a non-negative definite weight function under the integral condition. In [14], the control problem for a second-order parabolic type equation with two control functions was studied and the existence of admissible control functions was proved by the Laplace transform method.

A lot of information on the optimal control problems was given in detail in the monographs of Lions and Fursikov [15, 16]. Practical approaches to general numerical optimization and optimal control for equations of the second order parabolic type are studied in works such as [17, 18].

Boundary control problems related to the second-order pseudo-parabolic equation in a bounded domain are studied in detail in works [19–21]. In these works, the existence of the control function is proved using the method of Laplace transform.

In [22], Guo considered the null boundary control problem for a fourth order parabolic equation in one-dimensional bounded domain by the method reducing the control problem to the well-posed problems, proposed by Guo and Littman [23]. In [24], the null interior controllability for a fourth order parabolic equation was studied. The method that they used is based on Lebeau-Rabbiano inequality. The initial boundary value problem for equations from a class of fourth order semilinear parabolic equations was studied by Xu, et al. [25], and the global existence and nonexistence of solutions with initial data in the potential well are derived. Further research results on the global dynamic behavior of solutions associated with fourth-order parabolic equations for the epitaxial thin film model were studied by Chen [26].

In this work, the boundary control problem for the fourth-order parabolic equation is considered. The difference between this work and the previous works is that in this problem, the control problem associated with the fourth order parabolic type equation is studied. In Section 1, the boundary control problem studied is reduced to the Volterra integral equation of the first kind by the Fourier method. In Section 2, the existence of a solution to the Volterra integral equation is proved using the Laplace transform method. Section 3 gives an estimate of the minimum time required to reach a given average temperature of the plate.

We now consider the eigenvalue problem

$$\Delta^2 X(x,y) = \lambda\, X(x,y), \quad (x,y) \in \Omega,$$

with the boundary value conditions

$$X(0,y) = X_{xx}(0,y) = 0, \quad X_x(\pi,y) = X_{xxx}(\pi,y) = 0,$$

and

$$X(x,0) = X_{yy}(x,0) = 0, \quad X_y(x,\pi) = X_{yyy}(x,\pi) = 0, \quad (x,y) \in \partial\Omega.$$

Then we have the eigenvalue and eigenfunctions defined as follows

$$\lambda_{mn} = \left(\frac{2\,m+1}{2}\right)^4 + \left(\frac{2\,n+1}{2}\right)^4, \quad X_{mn}(x,y) = \sin\frac{2\,m+1}{2}x\,\sin\frac{2\,n+1}{2}y, \quad m,n = 0,1,\dots$$

Suppose that the function $\psi \in H^4(\Omega)$ satisfies the following conditions

$$\psi(0) = \psi^{(1)}(\pi) = \psi^{(2)}(0) = \psi^{(3)}(\pi) = 0, \quad \psi_n \geq 0,$$

where ψ_n is the Fourier coefficient of the function $\psi(y)$ and as follows

$$\psi_n = \frac{2}{\pi}\int\limits_0^\pi \psi(y)\,\sin\frac{2n+1}{2}y\,dy, \quad n = 0,1,\dots \tag{6}$$

We set

$$\beta_{mn} \;=\; \frac{1}{\pi}\frac{(2m+1)^2\,\psi_n}{2n+1}, \quad m,n = 0,1,\dots, \tag{7}$$

where ψ_n is defined by (6).

Theorem 1. Let be

$$0 < \theta < \frac{\beta_0\,M}{\lambda_0}.$$

Set

$$T_0 \;=\; -\frac{1}{\lambda_0}\ln\left(1 - \frac{\theta\,\lambda_0}{\beta_0\,M}\right).$$

Then a solution T_{min} of the time-optimal problem exists and the estimate $T_{min} \leq T_0$ is valid.

1 *Main integral equation*

In this section, we consider how the given control problem can be reduced to a Volterra integral equation of the first kind.

By the solution of the initial-boundary problem (1)–(4), we mean the function $u(x,y,t)$, which is expressed in the following form

$$u(x,y,t) \;=\; \psi(y)\,\nu(t) - w(x,y,t), \tag{8}$$

where the function $w(x,y,t)$ with the regularity $w(x,y,t)\in C^{4,4,1}_{x,y,t}(\Omega_T)\cap C(\bar\Omega_T)$ and $w_{xx}, w_{yy}\in C(\bar\Omega)$ is the solution to the initial-boundary problem

$$w_t(x,y,t)+\Delta^2 w(x,y,t) \;=\; \psi(y)\,\nu'(t)+\psi^{(4)}(y)\,\nu(t),$$

with the boundary value conditions

$$w(0,y,t) = w_{xx}(0,y,t) = 0, \quad w_x(\pi,y,t) = w_{xxx}(\pi,y,t) = 0,$$

$$w(x,0,t) = w_{yy}(x,0,t) = 0, \quad w_y(x,\pi,t) = w_{yyy}(x,\pi,t) = 0,$$

and the initial condition

$$w(x,y,0)=0.$$

As a result, we get the following solution

$$w(x,y,t) \;=\; \frac{4}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_n}{2m+1}\Big(\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu'(s)\,ds\Big)\sin\frac{2m+1}{2}x\,\sin\frac{2n+1}{2}y+$$

$$+\frac{1}{4\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)^4\,\psi_n}{2m+1}\Big(\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu(s)\,ds\Big)\sin\frac{2m+1}{2}x\,\sin\frac{2n+1}{2}y. \tag{9}$$

By (8) and (9), we have the solution of the initial-boundary problem (1)–(4) (see, [27]):

$$u(x,y,t)=\psi(y)\,\nu(t)-$$

$$-\frac{4}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_n}{2m+1}\Big(\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu'(s)\,ds\Big)\sin\frac{2m+1}{2}x\,\sin\frac{2n+1}{2}y-$$

$$-\frac{1}{4\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)^4\,\psi_n}{2m+1}\Big(\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu(s)\,ds\Big)\sin\frac{2m+1}{2}x\,\sin\frac{2n+1}{2}y.$$

Using condition (5) and the solution to problem (1)–(4), we can write

$$h(t) \;=\; \int_0^\pi\int_0^\pi u(x,y,t)\,dx\,dy=\nu(t)\int_0^\pi\int_0^\pi\psi(y)\,dx\,dy-$$

$$-\frac{16}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_n}{(2m+1)^2\,(2n+1)}\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu'(s)\,ds-$$

$$-\frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n+1)^3\,\psi_n}{(2m+1)^2}\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu(s)\,ds. \tag{10}$$

From the definition of the function $\nu(t)$ and from (10), we may write

$$h(t) \;=\; \nu(t)\int_0^\pi\int_0^\pi\psi(y)\,dx\,dy-\nu(t)\,\frac{16}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_n}{(2m+1)^2\,(2n+1)}+$$

$$+\,\frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2m+1)^2\,\psi_n}{2n+1}\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu(s)\,ds. \tag{11}$$

Note that

$$\int_0^{\pi}\int_0^{\pi}\psi(y)\,dx\,dy \;=\; \frac{16}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\psi_n}{(2m+1)^2\,(2n+1)}. \tag{12}$$

Then, from (11) and (12), we obtain

$$h(t) \;=\; \frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2m+1)^2\,\psi_n}{2n+1}\int_0^t e^{-\lambda_{mn}(t-s)}\,\nu(s)\,ds.$$

We set

$$B(t) \;=\; \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{mn}\,e^{-\lambda_{mn}t}, \quad t>0, \tag{13}$$

where β_{mn} is defined by (7).

Let there exist $M_0>0$ constant. Denote by $W(M_0)$ the set of function $h\in W_2^2(-\infty,+\infty)$, which satisfies the condition

$$\|h\|_{W_2^2(R_+)}\le M_0, \quad h(t)=0 \quad \text{for all} \quad t\le 0.$$

Thus, we have the following Volterra integral equation

$$\int_0^t B(t-s)\,\nu(s)\,ds \;=\; h(t), \quad t>0, \tag{14}$$

where $h(t)=\theta$ for $T\le t\le T_1$.

Theorem 2. Assume that $M_0>0$ exists. Then, for any function $h\in W(M_0)$ the solution $\nu(t)$ of integral equation (14) exists and satisfies the condition

$$|\nu(t)| \;\le\; M.$$

2 *Proof of Theorem 2*

Proposition 1. Suppose that $\alpha\in(\frac{3}{4},1)$. Then for the function $B(t)$ defined by (13) the following estimate

$$0<B(t)\le\; C_\alpha\,t^{-\alpha}, \quad 0<t\le 1, \tag{15}$$

is valid.

Proof. Using the definition (13) and $\lambda_{mn}=\big(\frac{2\,m+1}{2}\big)^4+\big(\frac{2\,n+1}{2}\big)^4$, we may write

$$B(t)=\frac{1}{\pi}\sum_{m=0}^{\infty}(2m+1)^2\,e^{-(\frac{2m+1}{2})^4\,t}\sum_{n=0}^{\infty}\frac{\psi_n}{2n+1}\,e^{-(\frac{2n+1}{2})^4\,t}.$$

We set

$$A(t) \;=\; \sum_{n=0}^{\infty}\frac{\psi_n}{2n+1}\,e^{-(\frac{2n+1}{2})^4\,t}, \quad t>0.$$

Clearly, for any $0 < t \leq T$, this function satisfies the following inequality

$$0 \; < \; A(T) \; \leq \; A(t) \; < \; A(0). \tag{16}$$

Let $\delta > 0$ be constant. We know the maximum value of the function $g(t, \delta) = t^{\alpha} e^{-\delta t}$ is reached at the point $t = \frac{\alpha}{\delta}$ and this value is equal to $\frac{\alpha^{\alpha}}{\delta^{\alpha}} e^{-\alpha}$.

As a result, for any $\alpha \in (\frac{3}{4}, 1)$, we have the following estimate

$$\sum_{m=0}^{\infty} (2m+1)^2 \, e^{-(\frac{2m+1}{2})^4 \, t} \; = \; t^{-\alpha} \sum_{m=0}^{\infty} (2m+1)^2 \, t^{\alpha} \, e^{-(\frac{2m+1}{2})^4 \, t} \leq$$

$$\leq \frac{16^{\alpha} \, \alpha^{\alpha} \, e^{-\alpha}}{t^{\alpha}} \sum_{m=0}^{\infty} \frac{(2m+1)^2}{(2m+1)^{4\,\alpha}} \leq C_{\alpha} \, t^{-\alpha}, \tag{17}$$

where

$$\sum_{m=0}^{\infty} \frac{(2m+1)^2}{(2m+1)^{4\,\alpha}} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{4\alpha-2}} < +\infty.$$

Then the required estimate (15) follows from (16) and (17).

Proposition 1 is proved.

As we know, the Laplace transform of the function $\nu(t)$ is defined as follows

$$\widetilde{\nu}(p) = \int_0^{\infty} e^{-pt} \, \nu(t) \, dt, \quad \text{where} \;\; p = \sigma + i\,\tau, \quad \sigma > 0, \quad \tau \in \mathbb{R}.$$

We rewrite integral equation (14) as follows

$$\int_0^t B(t-s) \, \nu(s) ds \; = \; h(t), \quad t > 0.$$

Then we use Laplace transform and obtain the following equation

$$\widetilde{h}(p) = \int_0^{\infty} e^{-pt} \, dt \int_0^t B(t-s) \, \nu(s) ds = \widetilde{B}(p) \, \widetilde{\nu}(p).$$

Thus, we have

$$\widetilde{\nu}(p) = \frac{\widetilde{h}(p)}{\widetilde{B}(p)},$$

and

$$\nu(t) = \frac{1}{2\pi i} \int_{\sigma - i\,\infty}^{\sigma + i\,\infty} \frac{\widetilde{h}(p)}{\widetilde{B}(p)} \, e^{pt} dp = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\widetilde{h}(\sigma + i\,\tau)}{\widetilde{B}(\sigma + i\,\tau)} \, e^{(\sigma + i\,\tau)t} \, d\tau. \tag{18}$$

Then we can write

$$\widetilde{B}(p) = \int_0^{\infty} B(t) \, e^{-pt} \, dt = \sum_{m,n=0}^{\infty} \beta_{mn} \int_0^{\infty} e^{-(p+\lambda_{mn})t} \, dt = \sum_{m,n=0}^{\infty} \frac{\beta_{mn}}{p + \lambda_{mn}},$$

where $B(t)$ is defined by (13) and

$$\widetilde{B}(\sigma+i\,\tau)=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{\sigma+\lambda_{mn}+i\,\tau}=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}\,(\sigma+\lambda_{mn})}{(\sigma+\lambda_{mn})^2+\tau^2}-i\,\tau\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{(\sigma+\lambda_{mn})^2+\tau^2}$$
$$=\mathrm{Re}\widetilde{B}(\sigma+i\,\tau)+i\,\mathrm{Im}\widetilde{B}(\sigma+i\,\tau),$$

where

$$\mathrm{Re}\widetilde{B}(\sigma+i\,\tau)=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}\,(\sigma+\lambda_{mn})}{(\sigma+\lambda_{mn})^2+\tau^2},\quad \mathrm{Im}\widetilde{B}(\sigma+i\,\tau)=-\tau\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{(\sigma+\lambda_{mn})^2+\tau^2}.$$

Obviously, the following inequality holds

$$(\sigma+\lambda_{mn})^2+\tau^2\le[(\sigma+\lambda_{mn})^2+1](1+\tau^2),$$

and we further have

$$\frac{1}{(\sigma+\lambda_{mn})^2+\tau^2}\ge\frac{1}{1+\tau^2}\,\frac{1}{(\sigma+\lambda_{mn})^2+1}. \tag{19}$$

Thus, due to (19), we can obtain the following estimates

$$|\mathrm{Re}\widetilde{B}(\sigma+i\,\tau)|=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}\,(\sigma+\lambda_{mn})}{(\sigma+\lambda_{mn})^2+\tau^2}\ge$$
$$\ge\frac{1}{1+\tau^2}\sum_{m,n=0}^{\infty}\frac{\beta_{mn}\,(\sigma+\lambda_{mn})}{(\sigma+\lambda_{mn})^2+1}=\frac{C_{1,\sigma}}{1+\tau^2}, \tag{20}$$

and

$$|\mathrm{Im}\widetilde{B}(\sigma+i\,\tau)|=|\tau|\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{(\sigma+\lambda_{mn})^2+\tau^2}\ge$$
$$\ge\frac{|\tau|}{1+\tau^2}\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{(\sigma+\lambda_{mn})^2+1}=\frac{C_{2,\sigma}\,|\tau|}{1+\tau^2}, \tag{21}$$

where $C_{1,\sigma}$, $C_{2,\sigma}$ are defined as follows

$$C_{1,\sigma}=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}\,(\sigma+\lambda_{mn})}{(\sigma+\lambda_{mn})^2+1},\quad C_{2,\sigma}=\sum_{m,n=0}^{\infty}\frac{\beta_{mn}}{(\sigma+\lambda_{mn})^2+1}.$$

From (20) and (21), we have the following estimate

$$|\widetilde{B}(\sigma+i\,\tau)|^2=|\mathrm{Re}\widetilde{B}(\sigma+i\,\tau)|^2+|\mathrm{Im}\widetilde{B}(\sigma+i\,\tau)|^2\ge\frac{\min(C_{1,\sigma}^2,C_{2,\sigma}^2)}{1+\tau^2},$$

and

$$|\widetilde{B}(\sigma+i\,\tau)|\ge\frac{C_\sigma}{\sqrt{1+\tau^2}},\quad \text{where}\quad C_\sigma=\min(C_{1,\sigma},C_{2,\sigma}). \tag{22}$$

Proceeding to the limit as $\sigma\to 0$ from (18), we have

$$\nu(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{h}(i\,\tau)}{\widetilde{B}(i\,\tau)}\,e^{i\,\tau t}\,d\tau. \tag{23}$$

Proposition 2. [20] Assume that $h(t) \in W(M_0)$. Then for the imaginary part of the Laplace transform of function $h(t)$ the inequality

$$\int_{-\infty}^{+\infty} |\widetilde{h}(i\tau)| \sqrt{1+\tau^2}\, d\tau \le C_1 \, \|h\|_{W_2^2(R_+)}$$

is valid, where $C_1 > 0$ is a constant.

Proof of the Theorem 2. Now we prove that $\nu \in W_2^1(\mathbb{R}_+)$. By (22) and (23), we obtain

$$\int_{-\infty}^{+\infty} |\widetilde{\nu}(\tau)|^2 (1+|\tau|^2)\, d\tau \ = \ \int_{-\infty}^{+\infty} \left| \frac{\widetilde{h}(i\tau)}{\widetilde{B}(i\tau)} \right|^2 (1+|\tau|^2)\, d\tau \ \le C_0 \int_{-\infty}^{+\infty} |\widetilde{h}(i\tau)|^2 (1+|\tau|^2)^2\, d\tau \ = \ C_0 \|h\|^2_{W_2^2(\mathbb{R})},$$

$C_0 = \min(C_{1,0}, C_{2,0})$ which is defined by (22). Further,

$$|\nu(t) - \nu(s)| \ = \ \left| \int_s^t \nu'(\xi)\, d\xi \right| \ \le \ \|\nu'\|_{L_2} (t-s)^{1/2}.$$

From (22), (23) and Proposition 2, we have the estimate

$$|\nu(t)| \le \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|\widetilde{h}(i\tau)|}{|\widetilde{B}(i\tau)|} d\tau \le \frac{1}{2\pi C_0} \int_{-\infty}^{+\infty} |\widetilde{h}(i\tau)| \sqrt{1+\tau^2} d\tau \le$$

$$\le \frac{C_1}{2\pi C_0} \|h\|_{W_2^2(R_+)} \le \frac{C_1 \, M_0}{2\pi C_0} = M,$$

where

$$M_0 = \frac{2\pi C_0}{C_1} \, M.$$

Theorem 2 is proved.

3 Estimate for the Minimal Time

Now we introduce the following integral equation

$$\int_0^t B(t-s)\, \nu(s)\, ds \ = \ \theta, \quad T \le t \le T_1,$$

where $B(t)$ is defined by (13).

We set

$$\beta_0 = \beta_{00}, \quad \lambda_0 = \lambda_{00},$$

where β_{mn} defined by (7).

Proposition 3. The following estimate is valid:

$$B(t) \ge \beta_0 \, e^{-\lambda_0 t},$$

where the function $B(t)$ is defined by Eq. (13).

The proof of this proposition follows from the fact that the functional series defined by (13) is positive for all $t \geq 0$.

We introduce the following function

$$H(t) = \int_0^t B(t-s)\,ds = \int_0^t B(s)\,ds.$$

It is known that the physical meaning of this function is the average temperature in a bounded domain Ω (see, [5]). It is known $H(0) = 0$ and $H'(t) = B(t) > 0$.

We set

$$H^* = \lim_{t\to\infty} H(t) = \int_0^\infty B(s)ds.$$

The average temperature in the bounded domain does not exceed H^*. Clearly, H^* is finite. Indeed,

$$H^* = \int_0^\infty B(s)\,ds = \sum_{m,n=0}^{\infty} \frac{\beta_{mn}}{\lambda_{mn}} < +\infty,$$

where β_{mn} is defined by (7) and $\lambda_{mn} = \left(\frac{2m+1}{2}\right)^4 + \left(\frac{2n+1}{2}\right)^4$.

Proposition 4. Assume that $0 < \theta < MH^*$. Then there exist $T > 0$ and a control function $\nu(t)$ and the following equality

$$\int_0^T B(T-s)\,\nu(s)ds = \theta \tag{24}$$

is valid.

Proof. The proof of this follows directly from the properties of the function H. Indeed, if we set $\nu(t) = M$ then

$$\int_0^t B(t-s)\,\nu(s)ds = M\int_0^t B(t-s)ds = M\,H(t),$$

and because of (24) there exists $T > 0$ so that $M\,H(T) = \theta$.

Proposition 4 is proved.

Remark 1. It is clear that the value T, which was found in Proposition 4, gives a solution to the problem. That is T is the root of the following equation

$$H(T) = \frac{\theta}{M}. \tag{25}$$

Lemma 1. Let

$$0 < \theta < \frac{\beta_0\,M}{\lambda_0}.$$

Then there exists $T > 0$ so that

$$T < -\frac{1}{\lambda_0}\ln\left(1 - \frac{\theta\,\lambda_0}{\beta_0\,M}\right),$$

and Eq. (25) is fulfilled.

Proof. Using Proposition 3, we can write the following inequality

$$H(t) = \int_0^t B(s)\,ds \geq \beta_0 \int_0^t e^{-\lambda_0 s} ds =$$

$$= \frac{\beta_0}{\lambda_0}\left(1 - e^{-\lambda_0 t}\right). \tag{26}$$

To determine T_0, we consider the following equation:

$$\frac{\beta_0}{\lambda_0}\left(1 - e^{-\lambda_0\, T_0}\right) = \frac{\theta}{M}. \tag{27}$$

Then we get

$$T_0 = -\frac{1}{\lambda_0}\ln\left(1 - \frac{\theta\,\lambda_0}{\beta_0\, M}\right).$$

In accordance with (26) and (27) we have

$$0 < \frac{\theta}{M} \leq H(T_0).$$

Then obviously there exists T, $0 < T < T_0$, which is a solution to equation (25).
Lemma 1 is proved.
The proof of Theorem 1 follows from Lemma 1.

Acknowledgments

The author is grateful to Academician Sh.A. Alimov for his valuable comments.
The work supported by the fundamental project (number: FZ-20200929243) of The Ministry of Innovative Development of the Republic of Uzbekistan.

Conflict of Interest

The author declare no conflict of interest.

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Екіөлшемді облыстағы төртінші ретті параболалық теңдеу үшін оңтайлы уақытты басқару мәселесі туралы

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Бұрын шектелген облыстағы екінші ретті параболалық типті теңдеу үшін шекаралық бақылау есептері зерттелді. Бұл жұмыста шектелген екіөлшемді облыстағы төртінші ретті параболалық теңдеумен байланысты шекаралық бақылау есебі қарастырылған. Қарастырылатын облыс шекарасының бөлігінде басқару функциясы бар шешімнің мәні берілген. Бақылаудағы шектеулер қарастырылатын облыстағы шешімнің орташа мәні нақты мәнді алатындай етіп берілді. Айнымалыларды бөлу әдісімен берілген есеп бірінші текті Вольтерра интегралдық теңдеуіне келтіріледі. Басқару функциясының бар болуы Лаплас түрлендіру әдісімен дәлелденді және облыста берілген орташа температураға жетудің ең аз уақытының бағасы табылды.

Кілт сөздер: бастапқы-шекаралық есеп, төртінші ретті параболалық теңдеу, ең аз уақыт, рұқсат етілген бақылау, Вольтерра интегралдық теңдеуі, Лапластың түрлендіру әдісі.

О задаче быстродействия параболического уравнения четвертого порядка в двумерной области

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Ранее были исследованы задачи граничного управления для уравнения параболического типа второго порядка в ограниченной области. В данной работе рассмотрена задача граничного управления, связанная с параболическим уравнением четвертого порядка в ограниченной двумерной области. На части границы рассматриваемой области дано значение решения с функцией управления. Ограничения на управление задаются таким образом, чтобы среднее значение решения в рассматриваемой области получало заданное значение. Задача, заданная методом разделения переменных, сводится к интегральному уравнению Вольтерра первого рода. Методом преобразования Лапласа доказано существование функции управления и найдена оценка минимального времени достижения заданной средней температуры в области.

Ключевые слова: начально-краевая задача, параболическое уравнение четвертого порядка, минимальное время, допустимое управление, интегральное уравнение Вольтерра, метод преобразования Лапласа.

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Bulletin of the Karaganda University. Mathematics Series, No. 2(114), 2024, pp. 71–85

https://doi.org/10.31489/2024M2/71-85 *Research article*

On some linear two-point inverse problem for a multidimensional heat conduction equation with semi-nonlocal boundary conditions

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It is known that V.A. Ilyin and E.I. Moiseev studied generalized nonlocal boundary value problems for the Sturm-Liouville equation, the nonlocal boundary conditions specified at the interior points of the interval under consideration. For such problems, uniqueness and existence theorems for a solution to the problem were proven. There are many difficulties in studying these generalized nonlocal boundary value problems for partial differential equations, especially in obtaining a priori estimates. Therefore, it is necessary to use new methods for solving generalized nonlocal problems (forward problems). As we know, it is not difficult to establish a connection between forward and inverse problems. Therefore, when solving generalized nonlocal boundary value problems for partial differential equations, reducing them to multipoint inverse problems is necessary. The first results in the direction belong to S.Z. Dzhamalov. In his works, he proposed and investigated multipoint inverse problems for some equations of mathematical physics. In this article, the authors studied the correctness of one linear two-point inverse problem for the multidimensional heat conduction equation. Using the methods of a priori estimates, Galerkin's method, a sequence of approximations and contracting mappings, the unique solvability of the generalized solution of the linear two-point inverse problem for the multidimensional heat equation was proved.

Keywords: multidimensional heat conduction equation, linear two-point inverse problem, unique solvability of a generalized solution, methods of a priori estimates, Galerkin's method, sequences of approximations and contracting mappings.

2020 Mathematics Subject Classification: 35K05, 35R30.

Introduction

Due to the significant increase in the capabilities of computer technology over the past decades, complex mathematical models that take into account a more significant number of physical factors are beginning to be used in applied mathematics. In [1–4], mathematical models that arise in the study of several applied problems and lead to the consideration of nonlocal boundary value problems were first proposed. As is known, it is not difficult to establish a connection between nonlocal boundary value problems and multipoint inverse problems [3–6]. In this regard, it should be especially noted that heat propagation processes are closely related precisely to multipoint inverse problems for parabolic equations [4]. For parabolic equations, particularly heat equations, the difference between inverse problems was studied in [7–19].

To this end, in this work, using the results of [5, 6], we study the unique solvability of a particular linear two-point inverse problem (LTIP) for a multidimensional heat equation.

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This research was funded by the Ministry of Innovative Development of the Republic of Uzbekistan, Grant no. F-FA-2021-424.

Received: 23 October 2023; *Accepted:* 24 January 2024.

Let Ω be a simply connected domain in space R^n with sufficiently smooth boundary $\partial\Omega$. Consider the multidimensional heat conduction equation in domain $G = \Omega \times (0,T) \times (0,l) = Q \times (0,l) \subset R^{n+2}$:

$$Lu = u_t - \Delta_x u - u_{yy} + c(x,t)u = g(x,t,y) + \sum_{i=1}^{2} h_i(x,t) f_i(x,t,y), \tag{1}$$

where $\Delta_x u = \sum\limits_{m=1}^{n} u_{x_m x_m}$ is the Laplace operator with regard to variables x, here $c(x,t)$, $g(x,t,y)$ and $f_i(x,t,y)$ $i = 1,2$ are given functions, and $h_1(x,t)$, $h_2(x,t)$ are the unknown functions.

1 *Linear two-point inverse problem*

It is required to find functions $\{u(x,t,y),\, h_1(x,t),\, h_2(x,t)\}$, that satisfy equation (1) in domain G, such that function $u(x,t,y)$ satisfies the following semi-nonlocal boundary conditions:

$$\gamma u\,|_{t=0} = u\,|_{t=T}, \tag{2}$$

$$u\,|_{\partial\Omega} = 0, \tag{3}$$

$$u\,|_{y=0} = u\,|_{y=l} = 0, \tag{4}$$

where γ is some constant nonzero number, the value of which will be specified below.

In addition, the solution to problem (1)–(4) satisfies the following auxiliary conditions:

$$u(x,t,\ell_j) = \varphi_j\,(x,t), \tag{5}$$

where $\ell_j \in (0,\ell)$, $j = 1,2$ are such that $0 < \ell_1 < \ell_2 < \ell < +\infty$, and functions $u(x,t,y)$ and $h_i(x,t)$, $i = 1,2$ belong to the following class:

$$U = \left\{(u, h_i,\, i = 1,2);\, u \in W_2^{2,1}(G),\, D_y^3(u_t, u_x, u_{xx}) \in L_2(G),\, h_i \in W_2^{2,1}(Q)\right\},$$

here $W_2^{2,1}(G)$ is the Sobolev space with norm

$$\|u\|^2_{W_2^{2,1}(G)} = \int\limits_G \left(u_{xx}^2 + u_{yy}^2 + u_{xy}^2\right) dxdtdy + \int\limits_G \left(u_x^2 + u_t^2 + u_y^2 + u^2\right) dxdtdy.$$

Let us introduce the following notation.
Let $g_j(x,t) = g(x,t,\ell_j), f_{i\,j}(x,t) = f_i(x,t,\ell_j)$, $\forall i,j = 1,2$.
$\mathfrak{F}^2 = \max\{\|f_{11}\|^2_{C(Q)}, \|f_{12}\|^2_{C(Q)}, \|f_{21}\|^2_{C(Q)}, \|f_{22}\|^2_{C(Q)}\}$.
Then we define a square matrix of the second order by $\mathbb{F} = \{f_{i\,j}\}_{i,j=1}^2$, i.e., $\mathbb{F} = \begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{pmatrix}$, and we denote its determinant by $H = \det \mathbb{F} = \begin{vmatrix} f_{11} \, f_{21} \\ f_{12} \, f_{22} \end{vmatrix}$.

Definition 1. Function $u(x,t,y) \in U$ that satisfies equation (1) almost everywhere in domain G with conditions (2)–(5), is called a generalized solution to problem (1)–(5).

Let all the coefficients of equation (1) be sufficiently smooth functions in domain Q and let the following conditions be satisfied regarding the coefficients on the right-hand sides of equation (1) and the given function $\varphi_j(x,t)$, $j = 1,2$.

Condition 1:
Periodicity: $c(x,0) = c(x,T)$, for all $x \in \bar{\Omega}$.

Nonlocal conditions: $\gamma g(x,0,y) = g(x,T,y)$, $\gamma f_j(x,0,y) = f_j(x,T,y)$, $j = 1,2$.
Smoothness: $g_j(x,t) = g(x,t,\ell_j) \in C^{0,1}_{x,t}(Q)$, $f_{ij}(x,t) = f_i(x,t,\ell_j) \in C^{0,1}_{x,t}(Q)$, $i,j = 1,2$;
$H = |\det \mathbb{F}| \geq \eta > 0$, $(1 + D_y^3)g \in W_2^1(G)$, $(1 + D_y^3)f_i \in W_2^2(G)$, $i = 1,2$.

Condition 2:
$\varphi_j(x,t) \in W_2^{2,1}(Q)$; $\gamma\varphi_j|_{t=0} = \varphi_j|_{t=T}$; $\varphi_j|_{\partial\Omega} = 0$, $j = 1,2$;
here $W_2^{2,1}(Q)$ is the Sobolev space with norm $\|u\|^2_{W_2^{2,1}(Q)} = \int\limits_Q \left(u_{xx}^2 + u_x^2 + u_t^2 + u^2\right)dxdt$.

2 Unique solvability to problem (1)–(5)

Theorem 1. Let the above conditions 1 and 2 be satisfied for the coefficients of equation (1), in addition, let $\lambda c(x,t) - c_t(x,t) \geq \delta_1 > 0$ for all $(x,t) \in \overline{Q}$, where $\lambda = \frac{2}{T}\ln|\gamma| > 0$, $|\gamma| > 1$ and let there exist a small positive number σ such that the following estimates hold: $\delta_0 - 10\sigma^{-1} \geq \delta > 0$, $q \equiv M \cdot \sum\limits_{i=1}^{2} \left\|(1 + D_y^3)f_i\right\|_{W_2^2(G)} < 1$, (where $\delta_0 = \min\left\{2, \lambda, \delta_1 + \left(\frac{\pi}{\ell}\right)^2\right\}$, $M = 4\sigma\,\eta^{-2}\mathfrak{F}^2\ c_1c_2$; where $c_1 = \sum\limits_{k=1}^{\infty} \frac{\mu_k^4}{\left(1+\mu_k^2\right)^3}$, $\mu_k = \frac{k\pi}{\ell}$, c_2 is a constant number determined from the Sobolev embedding theorem). Then, there is a unique solution to problem (1)–(5) from the specified class U.

We first use the Fourier method to prove the solvability of problem (1)–(5). Namely, the solution to problem (1)–(5) is sought in the following form:

$$u(x,t,y) = \sum_{k=1}^{\infty} u_k(x,t)Y_k(y),$$

where functions $Y_k(y) = \left\{\sqrt{\frac{2}{\ell}}\sin\mu_k y\right\}$, $\mu_k = \frac{\pi k}{\ell}$, $k = 1,2,3,...$ are solutions of the Sturm- Liouville spectral problem with Dirichlet conditions. It is known that the system of eigenfunctions $\{Y_k(y)\}$ is complete in space $L_2(0,\ell)$ and forms an orthonormal basis in it [7–10].

In order to determine unknown functions, some construction formalities must be performed.

Let us consider the traces of equation (1) for $y = \ell_j$, $j = 1,2$.

$$\begin{gathered} Lu(x,t,\ell_j) = u_t(x,t,\ell_j) - \Delta_x u(x,t,\ell_j) - u_{yy}(x,t,\ell_j) + \\ +c(x,t)u(x,t,\ell_j) = g(x,t,\ell_j) + h_1(x,t)f_{1j}(x,t) + h_2(x,t)f_{1j}(x,t). \end{gathered} \tag{6}$$

Now, considering condition (5), $H = |\det \mathbb{F}| \geq \eta > 0$, and the corresponding notation, we define the formally unknown functions $h_j(x,t)$, $j = 1,2$ from the equation (6) in the following form:

$$h_1(x,t) = \frac{1}{H}[\Phi_1(x,t)f_{22}(x,t) - \Phi_2(x,t)f_{21}(x,t)],$$

$$h_2(x,t) = \frac{1}{H}[\Phi_2(x,t)f_{11}(x,t) - \Phi_1(x,t)f_{12}(x,t)],$$

here

$$\begin{gathered} \Phi_j(x,t) = \varphi_{jt}(x,t) - \Delta_x\varphi_j(x,t) + c(x,t)\varphi_j(x,t) - g_j(x,t) + \sum\limits_{k=1}^{\infty} \mu_k^2 u_k(x,t)\sin\mu_k\ell_j = \\ = L_0\varphi_j(x,t) - g_j(x,t) + \sum\limits_{k=1}^{\infty} \mu_k^2 u_k(x,t)\sin\mu_k\ell_j, \end{gathered}$$

$$L_0\varphi_j \equiv \varphi_{jt}(x,t) - \Delta_x\varphi_j(x,t) + c(x,t)\varphi_j(x,t),\ j = 1,2,$$

where functions $u_k(x,t)$ are defined in domain $Q = \Omega \times (0,T)$ as a solution to the following infinite system of loaded heat equations [3], [11]:

$$\begin{gathered}
Lu_k = u_{kt} - \Delta_x u_k + (c(x,t) + \mu_k^2)u_k = g_k + \\
+\tfrac{f_{1k}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_2)] + \\
\tfrac{f_{2k}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_1)]
\end{gathered} \tag{7}$$

with semi-nonlocal boundary conditions

$$\gamma u_k \left|_{t=0} \right. = u_k \left|_{t=T} \right., \tag{8}$$

$$u_k \left|_{\partial\Omega} \right. = 0, \tag{9}$$

where $f_1(x,t,y) = \sum\limits_{k=1}^{\infty} f_{1k}(x,t)\sin\mu_k y$, $\quad f_1(x,t,\ell_1) = f_{11}(x,t) = \sum\limits_{k=1}^{\infty} f_{1k}(x,t)\sin\mu_k\ell_1$,
$f_2(x,t,y) = \sum\limits_{k=1}^{\infty} f_{2k}(x,t)\sin\mu_k y$, $\quad f_2(x,t,\ell_1) = f_{21}(x,t) = \sum\limits_{k=1}^{\infty} f_{2k}(x,t)\sin\mu_k\ell_1$,
$f_{ik} = \sqrt{\frac{2}{\ell}}\int\limits_0^{\ell} f_i \sin\mu_k y dy$, for any $i = 1,2$; $g_k = \sqrt{\frac{2}{\ell}}\int\limits_0^{\ell} g\sin\mu_k y dy$, $k = 1,2,3,\ldots$

Proof. Let us prove the theorem 1 step by step. First, we show that function $u(x,t,y)$ for any $j = 1,2$ satisfies condition (5) i.e. $u|_{y=\ell_j} = u(x,t,\ell_j) = \varphi_j(x,t)$.

Let us prove the fulfilment of these conditions using inverse assumptions. Let there be function $\vartheta_j(x,t)$ satisfying condition (5): $\vartheta_j(x,t)$, such that $u|_{y=\ell_j} = \vartheta_j(x,t) \neq \varphi_j(x,t)$, i.e.,
$u|_{y=\ell_j} = \sum\limits_{k=0}^{\infty} u_k(x,t)\sin\mu_k\ell_j = \vartheta_j(x,t) \neq \varphi_j(x,t)$.

Then for functions $z_j(x,t) = \vartheta_j(x,t) - \varphi_j(x,t)$ in domain Q, considering conditions (8)-(9), multiplying equation (7) by $\sin\mu_k\ell_j$ and summing over k from 1 to ∞, we obtain the following loaded equations:

$$\begin{gathered}
\sum_{k=1}^{\infty} u_{kt}\sin\mu_k\ell_j - \sum_{k=1}^{\infty} \Delta_x u_k \sin\mu_k\ell_j + \sum_{k=1}^{\infty} (c + \mu_k^2)u_k \sin\mu_k\ell_j = \sum_{k=1}^{\infty} g_k \sin\mu_k\ell_j + \\
+\frac{\sum\limits_{k=1}^{\infty} f_{1k}\sin\mu_k\ell_j}{H}[\Phi_1 f_{22} - \Phi_2 f_{21}] + \frac{\sum\limits_{k=1}^{\infty} f_{2k}\sin\mu_k\ell_j}{H}[\Phi_2 f_{11} - \Phi_1 f_{12}] = \sum_{k=1}^{\infty} g_k \sin\mu_k\ell_j + \\
+\frac{\sum\limits_{k=1}^{\infty} f_{1k}\sin\mu_k\ell_j}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_2)] + \\
+\frac{\sum\limits_{k=1}^{\infty} f_{2k}\sin\mu_k\ell_j}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_1)] = \\
= g_j + \tfrac{f_{1j}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_2)] + \\
+\tfrac{f_{2j}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m \sin\mu_m\ell_1)].
\end{gathered} \tag{10}$$

We consider each case separately to make it easier to understand the formula (10). First, we consider the case for $j = 1$. Then, from formula (10), we obtain:

$$\begin{gathered}\vartheta_{1t} - \Delta_x\vartheta_1 + c(x,t)\vartheta_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_1 = g_1 + \\ +\tfrac{f_{11}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_2)] + \\ +\tfrac{f_{21}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_1)] = \\ = g_1 + \tfrac{L_0\varphi_1 - g_1}{H}[f_{11}f_{22} - f_{12}f_{21}] + \frac{\sum_{k=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_1}{H}[f_{11}f_{22} - f_{12}f_{21}] + \\ \tfrac{L_0\varphi_2 - g_2}{H}[f_{21}f_{11} - f_{21}f_{11}] + \frac{\sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_2}{H}[f_{21}f_{11} - f_{21}f_{11}] = \\ = g_1 + L_0\varphi_1 - g_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_1 = L_0\varphi_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_1.\end{gathered} \tag{11}$$

Then from formulas (7)–(11) for function $z_1(x,t) = \vartheta_1(x,t) - \varphi_1(x,t) \Rightarrow \vartheta_1 = z_1 + \varphi_1$ in domain Q, we obtain the following identity

$$L_0(z_1 + \varphi_1) + \sum_{k=1}^{\infty}\mu_k^2 u_k\sin\mu_k\ell_1 = L_0\varphi_1 + \sum_{k=1}^{\infty}\mu_k^2 u_k \sin\mu_k\ell_1.$$

Hence, we obtain the following problem:

$$L_0 z_1 = z_{1t} - z_{1xx} + c(x,t)z_1 = 0, \tag{12}$$

$$\gamma\, z_1|_{t=0} = z_1|_{t=T}, \tag{13}$$

$$z_1\,|_{\partial\Omega} = 0. \tag{14}$$

Now we will prove the uniqueness of the solution to problem (12)–(14) using the method of energy integrals [3], [4], [8]. To do this, consider identity $2(L_0 z_1, e^{-\lambda t} z_{1\,t}) = 0$ and, integrating identity (12) by parts, considering conditions of Theorem 1 and boundary conditions (13), (14) for $|\gamma| > 1$, we obtain the inequality $\|z_j\|_{W_2^1(Q)} \le 0$, which implies that $z_1(x,t) = 0$.

So, problem (12)–(14) has a unique solution, i.e. $\vartheta_1(x,t) \equiv \varphi_1(x,t)$. From this, we obtain that problem (1)–(4) satisfies condition (5) for $j = 1$, i.e. $u(x,t,\ell_1) = \varphi_1(x,t)$. $u(x,t,\ell_2) = \varphi_2(x,t)$ is proved similarly for $j = 2$.

Now we will prove the solvability of problem (7)–(9) using the methods of a priori estimates, Galerkin's, and successive approximations [3], [8], namely, in domain Q, we consider a family of infinite loaded heat conduction equations:

$$\begin{gathered} Lu_k^{(l)} = u_{kt}^{(l)} - \Delta_x u_k^{(l)} + (c(x,t) + \mu_k^2)u_k^{(l)} = g_k + \\ + \frac{f_{1k}}{H}[f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)}\sin\mu_m\ell_1) - f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)}\sin\mu_m\ell_2)] + \\ + \frac{f_{2k}}{H}[f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)}\sin\mu_m\ell_2) - f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty}\mu_m^2 u_m^{(l-1)} u_m \sin\mu_m\ell_1)] = F(u_k^{(l-1)}) \end{gathered} \tag{15}$$

with semi-nonlocal boundary conditions

$$\gamma u_k^{(l)}\,|_{t=0} = u_k^{(l)}\,|_{t=T}, \tag{16}$$

$$u_k^{(l)}|_{\partial\Omega} = 0, \tag{17}$$

where $l \in N \cup \{0\}$, N is the set of natural numbers. In the future, to prove the unique solvability of problem (15)–(17), we need the following notation and lemmas.

Let us define the space of vector functions

$$W_{p,q}(Q) = \{\vartheta_k | \vartheta_k \in W_{2,x,t}^{p,q}(Q),\ k \in N;\ p,q = 0,1,2\}$$

with norm

$$\langle \vartheta_k \rangle_{p,q}^2 = \sqrt{\frac{2}{\ell}} \sum_{k=1}^{\infty} (1+\mu_k^2)^3 \|\vartheta_k\|_{W_{2,x,t}^{p,q}(Q)}^2, \tag{18}$$

where $W_{2,x,t}^{p,q}(Q)$ may be one of the following Sobolev spaces

$$W_{2,x,t}^{2,2}(Q) \equiv W_2^{2,2}(Q) \equiv W_2^2(Q);\ W_{2,x,t}^{2,1}(Q) \equiv W_2^{2,1}(Q);\ W_{2,x,t}^{1,1}(Q) \equiv W_2^1(Q);\ W_{2,x,t}^{0,0}(Q) = W_2^0 = L_2(Q).$$

The norm in space $W_{2,1}(Q)$ is defined as follows

$$\langle \vartheta_k \rangle_{2,1}^2 = \sqrt{\frac{2}{\ell}} \sum_{k=1}^{\infty} (1+\mu_k^2)^3 \|\vartheta_k\|_{W_2^{2,1}(Q)}^2,$$

and the norm in space $W_{0,0}(Q)$ is defined as follows

$$\langle \vartheta_k \rangle_{0,0}^2 = \sqrt{\frac{2}{\ell}} \sum_{k=1}^{\infty} (1+\mu_k^2)^3 \|\vartheta_k\|_{L_2(Q)}^2.$$

It is obvious that the space $W_{p,q}(Q)$ with a certain norm (18) is a Banach space [3], [8]. From the definition of spaces $W_{p,q}(Q)$ it follows that $W_{2,2}(Q) \subset W_{2,1}(Q) \subset W_{1,1}(Q) \subset W_{0,0}(Q)$.

Now let us denote the class of vector functions $\{\vartheta_k(x,t)\}_{k=1}^{\infty}$ such that $\{\vartheta_k(x,t)\}_{k=1}^{\infty} \in W_{2,1}(Q)$, satisfying the corresponding conditions (16), (17) by $W(Q)$.

Definition 2. The solution to problem (15)–(17) is called vector function $\{\vartheta_k(x,t)\}_{k=1}^{\infty} \in W(Q)$ that satisfies equation (15) almost everywhere in domain Q.

Lemma 1. Let all the conditions of the theorem be satisfied. Then, to solve problem (15)–(17), the following estimates are valid:

I) $\left\langle u_k^{(l)} \right\rangle_{1,1}^2 \le const(\hat{k},\ \hat{l}) < +\infty$;

II) $\left\langle u_k^{(l)} \right\rangle_{2,1}^2 \le const(\hat{k},\ \hat{l}) < +\infty$.

Here and below, we will use the symbol $const(\hat{k},\ \hat{l})$ to denote the constant independent on parameters k, l.

Proof. Consider the following identity

$$2(Lu_k^{(l)}, e^{-\lambda t}u_{kt}^{(l)})_0 = 2(F(u_k^{(l-1)}), e^{-\lambda t}u_{kt}^{(l)})_0, \tag{19}$$

where constant $\lambda > 0$ will be chosen later.

Considering the conditions of the theorem, integrating identity (19) by parts and applying Cauchy's inequality with σ [8], it is easy to obtain the lower bound of the following inequality

$$\begin{aligned} 2\int_Q Lu_k^{(l)} \cdot e^{-\lambda t} \cdot u_{kt}^{(l)} dxdt \ge \int_Q e^{-\lambda t}\{2 \cdot u_{kt}^{2(l)} + \lambda \cdot u_{kx}^{2(l)} + (\lambda c - c_t + \lambda\mu_k^2) \cdot u_k^{2(l)}\}dxdt - \\ - \int_{\partial Q} e^{-\lambda t}\{2u_{kt}^{(l)}u_{kx}^{(l)}e_x - 2u_{kx}^{2(l)}e_t - (c+\mu_k^2)u_k^{2(l)}e_t\}ds, \end{aligned} \tag{20}$$

where $\overrightarrow{e} = ((e_x, e_t); (e_x = (\overrightarrow{e}, x); e_t = (\overrightarrow{e}, t))$ is the unit vector of the internal normal to boundary ∂Q. The conditions of Theorem 1 ensure that the integral over domain Q is not negative. Considering the semi-nonlocal boundary conditions (16), (17) and conditions of Theorem 1, with the choice of $\gamma^2 = e^{\lambda T}$, we obtain the conversion of the boundary integrals to zero. Thus, from inequality (20), we obtain the lower bound of the following inequality

$$\begin{gathered} 2\int\limits_Q Lu_k^{(l)} \cdot e^{-\lambda t} \cdot u_{kt}^{(l)} dxdt \geq \\ \geq \int\limits_Q e^{-\lambda t} \left\{ 2 \cdot u_{kt}^{2(l)} + \lambda \cdot u_{kx}^{2(l)} + \left(\delta_1 + \lambda \left(\tfrac{\pi}{\ell}\right)^2\right) \cdot u_k^{2(l)} \right\} dxdt \geq \delta_0 \left\| u_k^{(l)} \right\|^2_{W_2^{1,1}(Q)}, \end{gathered} \tag{21}$$

where $\delta_0 = \min\left\{2,\ \lambda, \delta_1 + \left(\frac{\pi}{\ell}\right)^2\right\}$, $\lambda c - c_t \geq \delta_1 > 0$.

Applying Cauchy's inequality with σ to identity (19), we obtain the upper bound

$$\begin{gathered} \left| 2(F(u_k^{(l-1)}), e^{-\lambda t} u_{kt}^{(l)})_0 \right| \leq \left| 2 \left(g_k + \tfrac{f_{1k}}{H} [f_{22}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} \sin \mu_m \ell_1) - \right.\right. \\ -f_{21}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} \sin \mu_m \ell_2)] + \tfrac{f_{2k}}{H} [f_{11}(L_0\varphi_2 - g_2 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} \sin \mu_m \ell_2) - \\ \left.\left. -f_{12}(L_0\varphi_1 - g_1 + \sum_{m=1}^{\infty} \mu_m^2 u_m^{(l-1)} u_m \sin \mu_m \ell_1)], e^{-\lambda t} u_{kt}^{(l)} \right)_0 \right| \leq 9\sigma^{-1} \left\| u_k^{(l)} \right\|^2_{W_2^1(Q)} + \\ +\sigma \left[\|g_k\|_0^2 + \eta^{-2} \mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2 \right) \|f_{ik}\|^2_{C(Q)} \right] + \\ +2c_1 \eta^{-2} \sigma\, \mathfrak{F}^2 \sum_{i=1}^{2} \|f_{ik}\|^2_{C(Q)} \sum_{m=1}^{\infty} \left(1 + \mu_m^2\right)^3 \left\| u_m^{(l-1)} \right\|^2_{W_2^{1,1}(Q)}, \end{gathered} \tag{22}$$

where $T_0 = \max\{1, \|c\|_{C(Q)}\}$, $\mathfrak{F}^2 = \max\{\|f_{11}\|^2_{C(Q)}, \|f_{12}\|^2_{C(Q)}, \|f_{21}\|^2_{C(Q)}, \|f_{22}\|^2_{C(Q)}\}$.

Combining inequalities (21) and (22), we obtain

$$\begin{gathered} (\delta_0 - 9\sigma^{-1}) \left\| u_k^{(l)} \right\|^2_{W_2^{1,1}(Q)} \leq \sigma \left[\|g_k\|_0^2 + \eta^{-2} \mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2 \right) \|f_{ik}\|^2_{C(Q)} \right] + \\ +2c_1 \eta^{-2} \sigma\, \mathfrak{F}^2 \sum_{i=1}^{2} \|f_{ik}\|^2_{C(Q)} \sum_{m=1}^{\infty} \left(1 + \mu_m^2\right)^3 \left\| u_m^{(l-1)} \right\|^2_{W_2^{1,1}(Q)}. \end{gathered} \tag{23}$$

Applying the Sobolev embedding theorem $\|f_{ik}\|^2_{C(Q)} \leq c_2 \|f_{ik}\|^2_{W_2^2(Q)}$ [8,9] to inequality (23), we obtain

$$\begin{gathered} (\delta_0 - 9\sigma^{-1}) \left\| u_k^{(l)} \right\|^2_{W_2^{1,1}(Q)} \leq \sigma \left[\|g_k\|_0^2 + 2c_2 \eta^{-2} \mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2 \right) \|f_{ik}\|^2_{W_2^2(Q)} \right] + \\ +2c_1 c_2 \eta^{-2} \sigma\, \mathfrak{F}^2 \sum_{i=1}^{2} \|f_{ik}\|^2_{W_2^2(Q)} \sum_{m=1}^{\infty} \left(1 + \mu_m^2\right)^3 \left\| u_m^{(l-1)} \right\|^2_{W_2^{1,1}(Q)}. \end{gathered} \tag{24}$$

Taking into account the condition of Theorem 1 $\delta_0 - 9\sigma^{-1} > \delta_0 - 10\sigma^{-1} \geq \delta > 0$, dividing inequalities (24) by δ, multiplying inequalities (24) by $\left(1 + \mu_m^2\right)^3$ and summing over k from 1 to ∞, we obtain the first recurrent formula

$$\begin{gathered} \left\langle u_k^{(l)} \right\rangle^2_{1,1} \leq \sigma \delta^{-1} \left[\langle g_k \rangle_0^2 + c_2 \eta^{-2} \mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \|\varphi_j\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2 \right) \langle f_{ik} \rangle_2^2 \right] + \\ 2c_1 c_2 \eta^{-2} \sigma \delta^{-1} \mathfrak{F}^2 \sum_{i=1}^{2} \langle f_{ik} \rangle_2^2 \left\langle u_m^{(l-1)} \right\rangle^2_{1,1}, \end{gathered} \tag{25}$$

where $c_1 = \sum\limits_{k=1}^{\infty} \frac{\mu_k^4}{\left(1+\mu_k^2\right)^3}$, c_2 is the Sobolev embedding coefficient.

Introduce notation $\sigma\delta^{-1}\left[\langle g_k\rangle_0^2 + c_2\eta^{-2}\mathfrak{F}^2 \sum\limits_{i,j=1}^{2}\left(T_0\left\|\varphi_j\right\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2\right)\langle f_{ik}\rangle_2^2\right] \equiv A$ and, considering the conditions of Theorem 1 $2c_1c_2\eta^{-2}\sigma\delta^{-1}\mathfrak{F}^2\sum\limits_{i=1}^{2}\langle f_{ik}\rangle_2^2 \le q = M\sum\limits_{i=1}^{2}\langle f_{ik}\rangle_2^2 < 1$, from recurrent formula (25), we obtain the validity of estimate I), i.e. we get the first estimate. Indeed, for this purpose we take function $\{u_k^{(-1)}\} \equiv \{0\}$ as an initial approximation.

Then, for the zero approximation, we obtain

$$\left\langle u_k^{(0)}\right\rangle_{1,1}^2 \le \sigma\delta^{-1}\left[\langle g_k\rangle_0^2 + 2c_2\eta^{-2}\mathfrak{F}^2\sum_{i,j=1}^{2}\left(T_0\left\|\varphi_j\right\|^2_{W_2^{2,1}(Q)} + \|g_j\|_0^2\right)\langle f_{ik}\rangle_2^2\right] \equiv A.$$

Continuing this process, by induction, we obtain the first a priori estimate for any function $u_k^{(l)}$, $\forall l \ge 1$

$$\left\langle u_k^{(l)}\right\rangle_{1,1}^2 \le A\cdot\sum_{n=0}^{l} q^n \le \frac{A}{1-q}.$$

Now, let us prove the validity of the second estimate II). To do this, consider the following identity

$$-2\int\limits_Q e^{-\lambda t} L u_k^{(l)}\cdot\Delta_x u_k^{(l)}dxdt = -2\int\limits_Q e^{-\lambda t} F(u_k^{(l-1)})\cdot\Delta_x u_k^{(l)}dxdt. \tag{26}$$

Reasoning similarly to the proof of estimate I), based on integration by parts (26), considering the conditions of the theorem and semi-nonlocal boundary conditions (16), (17), we arrive at the following lower bound

$$\begin{aligned}\left|-2\int\limits_Q e^{-\lambda t}Lu_k^{(l)}\cdot\Delta_x u_k^{(l)}dxdt\right| &\ge \int\limits_Q\left(2\Delta_x u_k^{2(l)} + (\lambda+\mu_k^2)u_{kx}^{2(l)}\right)dxdt - \sigma^{-1}\int\limits_Q\Delta_x u_k^{2(l)}dxdt - \\ &-\sigma\|c\|^2_{C(Q)}\left\|u_k^{(l)}\right\|_0^2 + 2\int\limits_{\partial Q} e^{-\lambda t}[u_{kt}^{(l)}u_{kx}^{(l)}e_x + (u_{kx}^{2(l)} - u_{kt}^{2(l)})e_t + u_k^{(l)}u_{kx}^{(l)}e_x]ds \ge \\ &\ge \int\limits_Q\left(2\Delta_x u_k^{2(l)} + \left(\lambda+\left(\tfrac{\pi}{\ell}\right)^2\right)u_{kx}^{2(l)}\right)dxdt - \sigma^{-1}\int\limits_Q\Delta_x u_k^{2(l)}dxdt - \sigma\|c\|^2_{C(Q)}\left\|u_k^{(l)}\right\|_0^2 \ge \\ &\ge \delta_0\left\|u_k^{(l)}\right\|^2_{W_2^{2,1}(Q)} - \sigma^{-1}\left\|\Delta_x u_k^{(l)}\right\|_0^2 - \sigma\|c\|^2_{C(Q)}\left\|u_k^{(l)}\right\|_0^2,\end{aligned} \tag{27}$$

where $\delta_0 = \min\left\{2,\, \delta_1,\, \lambda + \left(\frac{\pi}{\ell}\right)^2\right\}$. The conditions of Theorem 1 ensure that the integral over domain Q is not negative. Considering the semi-nonlocal boundary conditions (16), (17) and the conditions of Theorem 1, with the choice of $\gamma^2 = e^{\lambda T}$, we obtain the conversion of the boundary integrals to zero. Thus, from inequalities (21) and (27), we obtain the lower bound of the following inequality

$$\left|-2\int\limits_Q e^{-\lambda t}Lu_k^{(l)}\cdot\Delta_x u_k^{(l)}dxdt\right| \ge \delta_0\left\|u_k^{(l)}\right\|^2_{W_2^{2,1}(Q)} - \sigma^{-1}\left\|\Delta_x u_k^{(l)}\right\|_0^2 - \sigma\|c\|^2_{C(\bar{Q})}\left\|u_k^{(l)}\right\|_0^2. \tag{28}$$

Now, applying the Cauchy inequality with σ to identity (27), we obtain the upper bound of the following inequality

$$\left| -2 \int_Q e^{-\lambda t} F(u_k^{(l-1)}) \cdot \Delta_x u_k^{(l)} dxdt \right| \leq 9\sigma^{-1} \left\| \Delta_x u_k^{(l)} \right\|_0^2 + \sigma \left\| F(u_k^{(l-1)}) \right\|_0^2 \leq$$
$$\leq 9\sigma^{-1} \left\| \Delta_x u_k^{(l)} \right\|_0^2 + \sigma \left\| g_k \right\|_0^2 + \sigma\eta^{-2}\mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \left\| \varphi_j \right\|_{W_2^{2,1}(Q)}^2 + \left\| g_j \right\|_0^2 \right) \left\| f_{ik} \right\|_{C(Q)}^2 + \tag{29}$$
$$+ 2\sigma\eta^{-2} c_1 \mathfrak{F}^2 \sum_{i=1}^{2} \left\| f_{ik} \right\|_{C(Q)}^2 \sum_{m=1}^{\infty} \left(1+\mu_m^2\right)^3 \left\| u_m^{(l-1)} \right\|_{W_2^{2,1}(Q)}^2 .$$

Combining inequalities (28) and (29), we obtain

$$\left(\delta_0 - 10\sigma^{-1}\right) \left\| u_k^{(l)} \right\|_{W_2^{2,1}(Q)}^2 \leq \sigma \left\| c \right\|_{C(\overline{Q})}^2 \left\| u_k^{(l)} \right\|_0^2 + \sigma \left\| g_k \right\|_0^2 +$$
$$+ \sigma\eta^{-2}\mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \left\| \varphi_j \right\|_{W_2^{2,1}(Q)}^2 + \left\| g_j \right\|_0^2 \right) \left\| f_{ik} \right\|_{C(Q)}^2 + \tag{30}$$
$$+ 2\sigma\eta^{-2} c_1 \mathfrak{F}^2 \sum_{i=1}^{2} \left\| f_{ik} \right\|_{C(Q)}^2 \sum_{m=1}^{\infty} \left(1+\mu_m^2\right)^3 \left\| u_m^{(l-1)} \right\|_{W_2^{2,1}(Q)}^2 .$$

Applying the Sobolev embedding theorem $\left\| f_{ik} \right\|_{C(Q)}^2 \leq c_2 \left\| f_{ik} \right\|_{W_2^2(Q)}^2$ to inequality (30), we obtain

$$\left(\delta_0 - 10\sigma^{-1}\right) \left\| u_k^{(l)} \right\|_{W_2^{2,1}(Q)}^2 \leq \sigma\delta^{-1} \left\| c \right\|_{C(\overline{Q})}^2 \left\| u_k^{(l)} \right\|_{W_2^{2,1}(Q)}^2 +$$
$$+ \sigma \left\| g_k \right\|_0^2 + \sigma\eta^{-2} c_2 \mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \left\| \varphi_j \right\|_{W_2^{2,1}(Q)}^2 + \left\| g_j \right\|_0^2 \right) \left\| f_{ik} \right\|_{W_2^2(Q)}^2 + \tag{31}$$
$$+ 2\sigma\eta^{-2} c_1 c_2 \mathfrak{F}^2 \sum_{i=1}^{2} \left\| f_{ik} \right\|_{W_2^2(Q)}^2 \sum_{m=1}^{\infty} \left(1+\mu_m^2\right)^3 \left\| u_m^{(l-1)} \right\|_{W_2^{2,1}(Q)}^2 .$$

Considering the conditions of the theorem and $\delta_0 - 10\sigma^{-1} \geq \delta > 0$, dividing inequalities (31) by δ, multiplying by $\left(1+\mu_m^2\right)^3$ and summing over k from 1 to ∞, we obtain the second recurrent formula

$$\left\langle u_k^{(l)} \right\rangle_{2,1}^2 \leq 2\sigma\delta^{-1} \left\| c \right\|_{C(\overline{Q})}^2 \left[\left\langle g_k \right\rangle_0^2 + c_2\eta^{-2}\mathfrak{F}^2 \sum_{i,j=1}^{2} \left(\left(T_0 \left\| \varphi_j \right\|_{W_2^{2,1}(Q)}^2 + \left\| g_j \right\|_0^2 \right) \left\langle f_{ik} \right\rangle_2^2 \right) \right] +$$
$$+ \sigma\delta^{-1} \left[\left\langle g_k \right\rangle_0^2 + \eta^{-2} c_2 \mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \left\| \varphi_j \right\|_{W_2^{2,1}(Q)}^2 + \left\| g_j \right\|_0^2 \right) \left\langle f_{ik} \right\rangle_2^2 \right] + \tag{32}$$
$$+ 2\sigma\delta^{-1}\eta^{-2} c_1 c_2 \mathfrak{F}^2 \sum_{i=1}^{2} \left\langle f_{ik} \right\rangle_2^2 \left\langle u_k^{(l-1)} \right\rangle_{2,1}^2 .$$

From estimate (32), considering (24), we obtain the following recurrent formulas

$$\left\langle u_k^{(l)} \right\rangle_{2,1}^2 \leq 3\sigma\delta^{-1} \left\| c \right\|_{C(\overline{Q})}^2 \left[\left\langle g_k \right\rangle_0^2 + c_2\eta^{-2}\mathfrak{F}^2 \sum_{i,j=1}^{2} \left(\left(T_0 \left\| \varphi_j \right\|_{W_2^{2,1}(Q)}^2 + \left\| g_j \right\|_0^2 \right) \left\langle f_{ik} \right\rangle_2^2 \right) \right] + \tag{33}$$
$$+ 2\sigma\delta^{-1}\eta^{-2} c_1 c_2 \mathfrak{F}^2 \sum_{i=1}^{2} \left\langle f_{ik} \right\rangle_2^2 \left\langle u_k^{(l-1)} \right\rangle_{2,1}^2 .$$

Introducing the following notation

$$3\sigma\delta^{-1} \left\| c \right\|_{C(\overline{Q})}^2 \left[\left\langle g_k \right\rangle_0^2 + c_2\eta^{-2}\mathfrak{F}^2 \sum_{i,j=1}^{2} \left(T_0 \left\| \varphi_j \right\|_{W_2^{2,1}(Q)}^2 + \left\| g_j \right\|_0^2 \right) \left\langle f_{ik} \right\rangle_2^2 \right] \equiv A_1$$

and, considering the conditions of Theorem 1 and

$$2\sigma\delta^{-1}\eta^{-2}c_1c_2\mathfrak{F}^2\sum_{i,j=1}^{2}\langle f_{ik}\rangle_2^2 \le q \equiv M\sum_{i,j=1}^{2}\langle f_{ik}\rangle_2^2<1$$

from recurrent formula (33), we obtain the validity of estimate II), taking $\{u_k^{(-1)}\}\equiv\{0\}$ as an initial approximation. As a result, for the zero approximation, we obtain

$$\left\langle u_k^{(0)}\right\rangle_{2,1}^2 \le 3\sigma\delta^{-1}\|c\|^2_{C(\overline{Q})}\left[\langle g_k\rangle_0^2+c_2\eta^{-2}\mathfrak{F}^2\sum_{i,j=1}^{2}\left(T_0\|\varphi_j\|^2_{W_2^{2,1}(Q)}+\|g_j\|_0^2\right)\langle f_{ik}\rangle_2^2\right]\equiv A_1.$$

Continuing this process, by induction, we obtain the second a priori estimate for any function $u_k^{(l)}$, $\forall l\ge 1$

$$\left\langle u_k^{(l)}\right\rangle_{2,1}^2 \le A_1\cdot\sum_{n=0}^{l}q^n\le\frac{A_1}{1-q}.$$

Similar to the proof of estimate I), estimate II) is easily obtained. Lemma 1 is proven.

Let us now introduce a new function from $W(Q)$ according to formula $\vartheta_k^{(l)} = u_k^{(l)} - u_k^{(l-1)}$, $\forall l = N\cup\{0\}$, $k=1,2,\ldots$ $\{u_k^{(-1)}\}\equiv\{0\}$. Then the following Lemma holds for it.

Lemma 2. Let all the conditions of Theorem 1 and Lemma 1 be satisfied. Then the following a priori estimates are valid for functions $\{\vartheta_k^{(l)}\}\in W(Q)$:

III) $\left\langle \vartheta_k^{(l)}\right\rangle_{1,1}^2 \le A\cdot q^{(l)}$;

IV) $\left\langle \vartheta_k^{(l)}\right\rangle_{2,1}^2 \le A_1\cdot q^{(l)}$.

Here and below we will use symbol $const(\hat{k},\ \hat{l})$ to denote the constant independent on parameters k,l.

Proof. From (15)–(17) for function $\{\vartheta_k^{(l)}\}\in W(Q)$, we obtain the following problem

$$\begin{gathered}L\vartheta_k^{(l)}=\vartheta_{kt}^{(l)}-\Delta_x\vartheta_k^{(l)}+(c(x,t)+\mu_k^2)\vartheta_k^{(l)}=\\ =\tfrac{f_{1k}}{H}[f_{22}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_1-f_{21}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_2]+\\ +\tfrac{f_{2k}}{H}[f_{11}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_2-f_{12}\sum_{m=1}^{\infty}\mu_m^2\vartheta_m^{(l-1)}\sin\mu_m\ell_1]=T(\vartheta_k^{(l-1)})\end{gathered}\tag{34}$$

with semi-nonlocal boundary conditions

$$\gamma\vartheta_k^{(l)}\,|_{t=0}=\vartheta_k^{(l)}\,|_{t=T},\tag{35}$$

$$\vartheta_k\,|_{\partial\Omega}=0,\tag{36}$$

where $l=0,1,2,\ldots$

Therefore, as in the proof of Lemma 1, for the function $\{\vartheta_k^{(l)}\}=\{u_k^{(l)}\}-\{u_k^{(l-1)}\}\in W(Q)$ from (34)–(36), as a proof of Lemma 1, consider the following identity

$$2\Big(L\vartheta_k^{(l)},e^{-\lambda t}\vartheta_{kt}^{(l)}\Big)_0=2\Big(T(\vartheta_k^{(l-1)}),e^{-\lambda t}\vartheta_{kt}^{(l)}\Big)_0.\tag{37}$$

Integrating by parts (37), taking into account the conditions of Theorem 1, we obtain the third recurrent formula

$$\left\langle \vartheta_k^{(l)}\right\rangle_{1,1}^2\le q\left\langle \vartheta_k^{(l-1)}\right\rangle_{1,1}^2.\tag{38}$$

Repeating the reasoning, similar to the proof of Lemma 1, from (38), we obtain a priori estimate III) for the function $\{\vartheta_k^{(l)}\}$, $k = 1, 2, 3, \ldots$ Estimate IV) is proven similarly. Lemma 2 is proven.

Theorem 2. Let all the conditions of Theorem 1 be satisfied. Then problem (15)–(17) is uniquely solvable in $W(Q)$.

Proof. Let us define the following mapping in space $W(Q)$

$$u_k^{(l)} = L^{-1}F(u_k^{(l-1)}) = \mathcal{F}u_k^{(l-1)}.$$

1. Let us show that operator $\mathcal{F}$ maps space $W(Q)$ into itself. Let $\left\{ u_k^{(l-1)} \right\} \in W(Q)$, then to solve problem (15)–(17) the statement of Lemma 1 is true, i.e. estimate II) is valid for the function $\{u_k^{(l)}\}$, $k = 1, 2, 3, \ldots$ It follows that for any $l = 1, 2, 3 \ldots$ we obtain $\left\{ u_k^{(l)} \right\} \in W(Q)$. Thus, $\mathcal{F} : W(Q\,) \to W(Q)$.

2. Let us show that $\mathcal{F}$ is a contraction operator. Let $\left\{ u_k^{(l)} \right\}, \left\{ u_k^{(l-1)} \right\} \in W(Q)$. Consider new function $\left\{ \vartheta_k^{(l)} \right\} = \left\{ u_k^{(l)} \right\} - \left\{ u_k^{(l-1)} \right\}$, the statement of Lemma 2 is valid for it, i.e. estimate IV) is true for the function $\{\vartheta_k^{(l)}\}$, $k = 1, 2, 3, \ldots$, and

$$\left|\left|\left|\vartheta_k^{(l)}\right|\right|\right|_{2,1}^2 = \left\langle \vartheta_k^{(l)} \right\rangle_{2,1}^2 \leq A_1 \cdot q^{(l)} \tag{39}$$

is true.

Now let us establish the fundamentality of sequence $\left\{ u_k^{(l)} \right\} \in W(Q)$. From (34)–(36), the triangle inequality and a priori estimates (39), we obtain

$$\left|\left|\left|u_k^{(l+p+1)} - u_k^{(l)}\right|\right|\right|_{2,1}^2 \leq \left|\left|\left|u_k^{(l+p+1)} - u_k^{(l+p)}\right|\right|\right|_{2,1}^2 + \left|\left|\left|u_k^{(l+p)} - u_k^{(l+p-1)}\right|\right|\right|_{2,1}^2 + \ldots + \left|\left|\left|u_k^{(l+1)} - u_k^{(l)}\right|\right|\right|_{2,1}^2 \leq$$
$$\leq A_1(q^{(l+p+1)} + q^{(l+p)} + \ldots + q^{(l)}) = A_1 q^{(l)}(1 + q + \ldots + q^{(p+1)}) \leq \tfrac{A_1 q^{(l)}}{1-q}.$$

This implies the fundamental nature of sequence $\left\{ u_k^{(l)} \right\}$. Thus, $\mathcal{F}$ is a contraction operator according to the well-known principle of contracting mappings [3], [9], problem (15)–(17) has a unique solution belonging to space $W(Q)$. Here $u_k^{(l)} \to u_k$ as $l \to \infty$, and $u_k(x,t)$ is a unique solution to problem (7)–(9) for fixed k.

From the principle of contraction mappings, we conclude that problem (7)–(9) has a unique solution from $W(Q)$. Theorem 2 is proven.

Now we prove Theorem 1. Applying the Parseval–Steklov equality to functions $\{u_k\} \in W(Q)$, we obtain the assertion of the theorem, that is, $u(x,t,y) \in U$ [8,9]. Theorem 1 is proven.

Remark 1. If we take function $\varphi_j(x,t)$ as a solution to the following problem $\varphi_j(x,t) \in W_2^{2,1}(Q)$, $g_j \in W_2^1(Q)$

$$L_0\varphi = \varphi_{jt} - \Delta_x\varphi_j + c(x,t)\varphi = g_j,$$
$$\gamma\varphi_j|_{t=0} = \varphi_j|_{t=T},$$
$$\varphi_j\,|_{\partial\Omega} = 0,$$

then function $\Phi_j(x,t)$ is defined as follows: $\Phi_j(x,t) = L_0\varphi_j - g_j + \sum\limits_{k=1}^{\infty} \mu_k^2 u_k \sin\mu_k\ell_j = \sum\limits_{k=1}^{\infty} \mu_k^2 u_k \sin\mu_k\ell_j$, $j = 1, 2$, and the proof of the theorem is greatly simplified.

Remark 2. For equation (1), LTPIPs with the Cauchy condition are studied similarly; in this case, instead of condition (2), the Cauchy condition $u\,|_{t=0} = u_0(x)$ is proposed.

Conclusion

In this article, the authors studied the correctness of one linear two-point inverse problem for the multidimensional heat conduction equation. Using the methods of a priori estimates, Galerkin's method, and successive approximations and contraction mappings, the theorem of unique solvability of the generalized solution in the specified class of integrable functions is proved.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Жартылай локольдыемес шектік шарттары бар көпөлшемді жылуөткізгіштік теңдеуіне қойылған сызықты екінүктелі кері есептер туралы

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В.А. Ильин және Е.И. Моисеевтер Штурм-Лиувилл теңдеулері үшін жалпылама локальдыемес шектік есептердің шешімінің бар болуын және жалғыздығын дәлелдеген. Дербес туындылы дифференциальдық теңдеулер үшін жалпылама локальдыемес шектік есептерді қарастырғанда априорлық бағаларды алуда көп қиындықтарға тап боламыз. Сондықтан, дербес туындылы дифференциальдық теңдеулерге қойылған локальдыемес шектік есептерді шешу үшін көп нүктелі кері есептерге келтіру қажет. Бұл бағыттағы алғашқы нәтижелер С.З. Джамаловқа тиесілі. Ол өз жұмысында математикалық физиканың көп нүктелі қисықтар сияқты көптеген параметрлерін де зерттеді. Мақалада көп өлшемді жылуөткізгіштік теңдеуіне қойылған сызықты екінүктелі кері есептің қисындылығы қарастырылған. Априорлық бағалау, Галеркин, біртіндеп жуықтау және қысушы бейнелеу әдістерін қолданып, көпөлшемді жылуөткізгіштік теңдеуіне қойылған сызықты екінүктелі кері есептің жалғыз шешімінің бар болуы дәлелденген.

Кілт сөздер: көп өлшемді жылуөткізгіштік теңдеуі, сызықты екінүктелі кері есеп, жалпылама шешімнің жалғыз болуы, априорлық бағалау, Галеркин әдісі, біртіндеп жуықтау және қысушы бейнелеу әдістері.

О некоторой линейной двухточечной обратной задаче для многомерного уравнения теплопроводности с полунелокальными краевыми условиями

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Известно, что В.А. Ильин и Е.И. Моисеев изучали обобщенные нелокальные краевые задачи для уравнения Штурма-Лиувилля, нелокальные краевые условия которого задаются во внутренних точках рассматриваемого интервала. Для таких задач доказаны теоремы единственности и существования решения задачи. Существует много проблем при исследовании этих обобщенных нелокальных краевых задач для дифференциальных уравнений с частными производными, особенно при получении априорных оценок. Поэтому необходимо использовать новые методы для решения обобщенных нелокальных задач (прямых задач). Как нам известно, нетрудно установить связь между прямыми и обратными задачами. Поэтому при решении обобщенных нелокальных краевых задач для дифференциальных уравнений в частных производных необходимо свести их к многоточечным обратным задачам. В этом направлении первые результаты принадлежат С.З. Джамалову. Он в своих работах предложил и исследовал многоточечные обратные задачи для некоторых уравнений математической физики. В настоящей работе исследована корректность одной линейной двухточечной обратной задачи для многомерного уравнения теплопроводности. Методами априорных оценок, Галеркина, последовательности приближений и сжимающихся отображений доказана однозначная разрешимость обобщённого решения одной линейной двухточечной обратной задачи для многомерного уравнения теплопроводности.

Ключевые слова: многомерное уравнение теплопроводности, линейная двухточечная обратная задача, однозначная разрешимость обобщённого решения, методы априорных оценок, Галеркина, последовательности приближений и сжимающихся отображений.

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Bulletin of the Karaganda University. Mathematics Series, No. 2(114), 2024, pp. 86–104

https://doi.org/10.31489/2024M2/86-104 *Research article*

On the spectral problem for three-dimesional bi-Laplacian in the unit sphere

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In this work, we introduce a new concept of the stream function and derive the equation for the stream function in the three-dimensional case. To construct a basis in the space of solutions of the Navier-Stokes system, we solve an auxiliary spectral problem for the bi-Laplacian with Dirichlet conditions on the boundary. Then, using the formulas employed for introducing the stream function, we find a system of functions forming a basis in the space of solutions of the Navier-Stokes system. It is worth noting that this basis can be utilized for the approximate solution of direct and inverse problems for the Navier-Stokes system, both in its linearized and nonlinear forms. The main idea of this work can be summarized as follows: instead of changing the boundary conditions (which remain unchanged), we change the differential equations for the stream function with a spectral parameter. As a result, we obtain a spectral problem for the bi-Laplacian in the domain represented by a three-dimensional unit sphere, with Dirichlet conditions on the boundary of the domain. By solving this problem, we find a system of eigenfunctions forming a basis in the space of solutions to the Navier-Stokes equations. Importantly, the boundary conditions are preserved, and the continuity equation for the fluid is satisfied. It is also noteworthy that, for the three-dimensional case of the Navier-Stokes system, an analogue of the stream function was previously unknown.

Keywords: Navier-Stokes system, bi-Laplacian, spectral problem, stream function.

2020 Mathematics Subject Classification: 35K40, 35K51, 58J50.

Introduction

Previously, we solved the spectral problem for the bi-Laplacian in the unit circle with Dirichlet conditions on the boundary. As is known, in the two-dimensional case the linear Navier-Stokes system can be transformed into a single equation for the stream function [1–3]. Note that the spectral problem for the two-dimensional bi-Laplacian in the unit circle was solved in [4–6], and its results were applied to an approximate solution of the inverse problem with final redefinition conditions for the two-dimensional system of Navier-Stokes equations. For the bi-Laplacian, the solvability of two-dimensional spectral problems for square domains was considered in [7–12], and for the $2m-$Laplacian, spectral problems for multidimensional domains with smooth and non-smooth boundaries – in [13–16]. In [8, 10, 11], lower bounds for eigenvalues were obtained by introducing intermediate spectral problems (the main thing was the fact that one of the boundary conditions was replaced by a family of approximate conditions on the boundary, which in the limit tended to original). In [13–16], estimates were given for the number of eigenvalues not exceeding a given number. However, the calculation of eigenvalues and eigenfunctions in the above spectral problems has remained open. This issue is dedicated to submitted work.

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This research has funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19674862).

Received: 18 December 2023; *Accepted:* 22 February 2024.

The question of constructing a basis applicable to domains with time-varying boundaries also remains open. For example, problems of this kind in degenerate domains or in domains with time-varying boundaries were considered in papers [17–29]. Note that the results of this work can be used in the construction of this basis.

1 Stream function for a three-dimensional linearized Navier-Stokes system. Statement of the spectral problem

Let $y = (y_1, y_2, y_3)$, $Q_{yt} = \{y, t : |y| < 1,\ 0 < t < T\}$ be a cylindrical domain, and Ω be a section (sphere with unit radius) of the cylinder Q_{yt} for any fixed time $t \in [0, T]$ with boundary $\partial\Omega$, $\Sigma_{yt} = \partial\Omega \times (0, T)$. In the cylindrical domain Q_{yt} we consider the following initial boundary value problem for the linear three-dimensional Navier-Stokes equation of determining the vector function $w(y,t) = \{w_1(y,t), w_2(y,t), w_3(y,t)\}$ and scalar function $P(y,t)$:

$$\partial_t w - \Delta w = f - \nabla P, \quad (y,t) \in Q_{yt}, \tag{1.1}$$

$$\operatorname{div} w = 0, \ \ (y,t) \in Q_{yt}, \tag{1.2}$$

$$w = 0, \ \ (y,t) \in \Sigma_{yt} \text{ is a lateral surface of the cylinder}, \tag{1.3}$$

$$w = 0, \ \ y \in \Omega \text{ is a unit sphere, base of cylinder}. \tag{1.4}$$

Let's introduce the notations of spaces $\mathbf{V}$, $\mathbf{H}$, $\mathbf{L}^2(\Omega)$, $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}^2(\Omega)$, used in studying the solvability of the initial boundary value problem (1.1)–(1.4), and which we will use in the future:

$$\mathbf{V} = \{v : \ v \in \mathbf{H}_0^1(\Omega) = \left(H_0^1(\Omega)\right)^3, \operatorname{div}\, v = 0\},$$

$$\mathbf{H} = \left\{v : \ v \in \mathbf{L}^2(\Omega), \operatorname{div}\, v = 0\right\},$$

$$\mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^3, \ \ \mathbf{H}^2(\Omega) = \left(H^2(\Omega)\right)^3.$$

The following dense embeddings take place

$$\mathbf{V} \subset \mathbf{H} \equiv \mathbf{H}' \subset \mathbf{V}', \ \ \mathbf{H}_0^1(\Omega) \subset \mathbf{L}^2(\Omega) \equiv \left(\mathbf{L}^2(\Omega)\right)' \subset \mathbf{H}^{-1}(\Omega),$$

and $(\cdot,\cdot)$, $((\cdot,\cdot))$ are scalar products in spaces $\mathbf{H}$, $\mathbf{L}^2(\Omega)$ and $\mathbf{V}$, $\mathbf{H}_0^1(\Omega)$, respectively. The Helmholtz decomposition of space $\mathbf{L}^2(\Omega)$: $\mathbf{L}^2(\Omega) = \mathbf{H} \oplus \mathbf{H}^\perp$, where

$$\mathbf{H}^\perp \text{ is an orthogonal complement to } \mathbf{H} \text{ in the space } \mathbf{L}^2(\Omega),$$

$$\mathbf{H}^\perp = \{v : \ v \in \mathbf{L}^2(\Omega), \ v = \nabla u, \ u \in \mathbf{H}^1(\Omega)\},$$

$$\left(\mathbf{H} \oplus \mathbf{H}^\perp\right)' \equiv \left(\mathbf{L}^2(\Omega)\right)' \equiv \mathbf{L}^2(\Omega) \equiv \mathbf{H} \oplus \mathbf{H}^\perp,$$

and the "prime" symbol denotes a topologically dual space.

So, we will look for a solution of the initial boundary value problem (1.1)–(1.4) in the spaces of the vector functions of liquid velocities $w(y,t) = \{w_1(y,t), w_2(y,t), w_3(y,t)\} \in L^2(0,T; \mathbf{V} \cap \mathbf{H}^2(\Omega)) \cap H^1(0,T; \mathbf{H}(\Omega))$, and scalar liquid pressure function $P(y,t) \in L^2(0,T; \mathbf{H}^1(\Omega))$ for a given vector functions of the acting forces $f(y,t) = \{f_1(y,t), f_2(y,t), f_3(y,t)\} \in L^2(0,T; \mathbf{H}(\Omega))$.

Let us transform boundary value problem (1.1)–(1.4). For this purpose, in the domain Q_{yt} we introduce the scalar stream function $U(y,t)$, defined up to an additive constant, by the equations:

$$w_1 = \partial_{y_2} U - \partial_{y_3} U, \quad w_2 = \partial_{y_3} U - \partial_{y_1} U, \quad w_3 = \partial_{y_1} U - \partial_{y_2} U. \tag{1.5}$$

We will act with the operators $\partial_{y_2} - \partial_{y_3}$, $\partial_{y_3} - \partial_{y_1}$, $\partial_{y_1} - \partial_{y_2}$ to equations (1.1) respectively and add the obtained results. Then for $U(y,t)$ we obtain the equation

$$(\partial_t - \Delta)\left(\Delta - \partial^2_{y_1y_2} - \partial^2_{y_2y_3} - \partial^2_{y_3y_1}\right)U = G(y,t), \quad \{y,t\} \in Q_{yt}, \tag{1.6}$$

where

$$2G(y,t) \equiv (\partial_{y_2} - \partial_{y_3})\, f_1 + (\partial_{y_3} - \partial_{y_1})\, f_2 + (\partial_{y_1} - \partial_{y_2})\, f_3.$$

From relations (1.3) and (1.5) we have the identities:

$$(\partial_{y_1} - \partial_{y_2})\, U \equiv (\partial_{y_2} - \partial_{y_3})\, U \equiv (\partial_{y_3} - \partial_{y_1})\, U \equiv 0, \quad (y,t) \in \Sigma_{yt} \tag{1.7}$$

or

$$\partial_{y_1} U \equiv \partial_{y_2} U \equiv \partial_{y_3} U, \ \ (y,t) \in \Sigma_{yt}. \tag{1.8}$$

Note that relations (1.7)–(1.8) do not completely determine the boundary conditions on the lateral surface of the cylinder Q_{yt}. In addition to (1.7)–(1.8) we will require that $\partial_{y_1} U \equiv 0$ on Σ_{yt}, which do not contradict relations (1.7)–(1.8). So, instead of (1.8) we will have:

$$\partial_{y_1} U \equiv \partial_{y_2} U \equiv \partial_{y_3} U \equiv 0, \ \ (y,t) \in \Sigma_{yt}. \tag{1.9}$$

Thus, equalities (1.9) allow us to set the following boundary conditions for equation (1.6)

$$\partial_{\vec{n}} U = 0, \ \ (y,t) \in \Sigma_{yt}, \tag{1.10}$$

$$U = 0, \ \ (y,t) \in \Sigma_{yt}, \tag{1.11}$$

where $\vec{n}$ is the outer unit normal to the sphere $|y| = 1$, and from (1.4) (doing the same thing as when establishing conditions (1.10)–(1.11)) we obtain the initial condition

$$U = 0, \ \ y \in \Omega \equiv \{|y| < 1\}, \ \ t = 0. \tag{1.12}$$

To numerically solve the initial boundary value problem (1.1)–(1.4) we will need to be able to solve approximately the initial boundary value problem (1.6), (1.10)–(1.12). We will look for a solution to this problem using the method of separation of variables. We have

$$U(y,t) = \sum_{k=1}^{\infty} c_k(t) u_k(y).$$

Then from equation (1.6) we obtain

$$c_k'(t)\left[\triangle u_k(y) - \partial^2_{y_1y_2} u_k(y) - \partial^2_{y_2y_3} u_k(y) - \partial^2_{y_3y_1} u_k(y)\right] =$$

$$= c_k(t)\triangle\left[\triangle u_k(y) - \partial^2_{y_1y_2} u_k(y) - \partial^2_{y_2y_3} u_k(y) - \partial^2_{y_3y_1} u_k(y)\right].$$

Further, we have

$$\frac{c_k'(t)}{c_k(t)} = \frac{\triangle\left(\triangle - \partial^2_{y_1y_2} - \partial^2_{y_2y_3} - \partial^2_{y_3y_1}\right) u_k(y)}{\left(\triangle - \partial^2_{y_1y_2} - \partial^2_{y_2y_3} - \partial^2_{y_3y_1}\right) u_k(y)} = -\lambda_k, \ \ \lambda_k > 0 \quad \text{for each} \quad k \in \mathbb{N},$$

i.e., we finally come to the need to solve the following spectral problem:

$$\triangle\left(\triangle - \partial^2_{y_1y_2} - \partial^2_{y_2y_3} - \partial^2_{y_3y_1}\right) u(y) = -\lambda\left(\triangle - \partial^2_{y_1y_2} - \partial^2_{y_2y_3} - \partial^2_{y_3y_1}\right) u(y), \tag{1.13}$$

$$u(y)_{|\partial\Omega} = \partial_{\vec{n}} u(y)_{|\partial\Omega} = 0. \tag{1.14}$$

Solving the spectral problem (1.13)–(1.14) poses certain difficulties (details in Appendix A). We actually need to construct a basis in the space solutions of the Navier-Stokes system $\mathbf{V} \cap \mathbf{H}^2(\Omega)$, the elements of which would ensure the fulfillment of equation (1.2) and boundary conditions (1.3). Therefore, it will be enough for us to use the solution to the following spectral problem, also posed on a unit sphere (but with a simplification of the equation in which there are no terms with mixed derivatives of the desired function):

$$(-\Delta)^2 Z(y) = \mu^2(-\Delta\, Z(y)),\ y \in \Omega = \{|y| < 1\},$$

$$\partial_{\vec{n}} Z(y) = 0,\ \text{at } |y| = 1, \tag{1.15}$$

$$Z(y) = 0,\ \text{at } |y| = 1. \tag{1.16}$$

Let us rewrite the equation in the form of a system for unknown functions $\{Z(y), Y(y)\}$:

$$-\Delta\, Z(y) = Y(y),\ \ -\Delta\, Y(y) = \mu^2 Y(y) \quad y \in \Omega. \tag{1.17}$$

So, we got spectral problem (1.17), (1.15) and (1.16).

2 Transition to spherical coordinates in the spectral problem

Let us write spectral problem (1.17), (1.15) and (1.16) in a spherical coordinate system $\{r, \theta, \zeta\} \in \Omega \equiv \{0 \le r < 1,\ \theta \in (0, \pi],\ \zeta \in (0, 2\pi]\}$ using transformation formulas

$$y_1 = r\sin\theta\cos\zeta,\ \ y_2 = r\sin\theta\sin\zeta,\ \ y_3 = r\cos\theta,$$

regarding the functions $Z(r, \theta, \zeta),\ \ Y(r, \theta, \zeta)$ (in this case, for the sake of simplicity, we leave the function designations unchanged):

$$-\frac{1}{r^2}\partial_r\left(r^2\partial_r Z\right) - \frac{1}{r^2}\Delta_{\theta,\zeta} Z = Y, \quad \{r, \theta, \zeta\} \in \Omega, \tag{2.1}$$

$$\Delta_{\theta,\zeta} Z \equiv \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta Z\right) + \frac{1}{\sin^2\theta}\partial_\zeta^2 Z, \quad \{r, \theta, \zeta\} \in \Omega, \tag{2.2}$$

$$-\frac{1}{r^2}\partial_r\left(r^2\partial_r Y\right) - \frac{1}{r^2}\Delta_{\theta,\zeta} Y = \mu^2 Y, \quad \{r, \theta, \zeta\} \in \Omega, \tag{2.3}$$

$$\Delta_{\theta,\zeta} Y \equiv \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta Y\right) + \frac{1}{\sin^2\theta}\partial_\zeta^2 Y, \quad \{r, \theta, \zeta\} \in \Omega, \tag{2.4}$$

$$Z \ \text{ is bounded in the neighborhood of the point } \ r = 0, \tag{2.5}$$

$$\partial_r Z = 0 \ \text{ at } \ r = 1, \tag{2.6}$$

$$Z = 0 \ \text{ at } \ r = 1. \tag{2.7}$$

3 Solution of the spectral problem in spherical coordinates

We will solve problem (2.1)–(2.7) using the method of separation of variables:

$$Z(r,\theta,\zeta)=\sum_j R_{Z_j}(r)\Theta_{Z_j}(\theta,\zeta),\;\; Y(r,\theta,\zeta)=\sum_j R_{Y_j}(r)\Theta_{Y_j}(\theta,\zeta), \tag{3.1}$$

$$\frac{\left(r^2R'_{Y_j}\right)'+\mu_j^2r^2R_{Y_j}}{R_{Yj}}=-\frac{\Delta_{\theta,\zeta}\Theta_{Y_j}}{\theta_{Y_j}}=\mu^2_{Y_j},\;\; \frac{\left(r^2R'_{Z_j}\right)'+r^2R_{Y_j}}{R_{Z_j}}=-\frac{\Delta_{\theta,\zeta}\Theta_{Z_j}}{\theta_{Z_j}}=\mu^2_{Z_j}, \tag{3.2}$$

where the "prime" symbol here and below denotes the derivative with respect to the variable r.

The second relation from (3.2) follows from the fact that the boundary value problems (3.3)–(3.4) and (3.5)–(3.6) for the functions $\Theta_{Z_j}(\theta,\zeta)$ and $\Theta_{Y_j}(\theta,\zeta)$ coincide, then their solutions can be taken equal to each other, i.e. $\Theta_{Z_j}(\theta,\zeta)=\Theta_{Y_j}(\theta,\zeta)$ and $\mu^2_{Z_j}=\mu^2_{Y_j}$.

Substituting (3.1) into (2.1)–(2.7) and taking (3.2) into account, we obtain

$$-\Delta_{\theta,\zeta}\Theta_{Z_j}=\mu^2_{Z_j}\Theta_{Z_j},\;\; \theta\in(0,\pi),\;\; \zeta\in(0,2\pi),\;\; \Theta_{Z_j}(\theta,\zeta)=\Theta_{Z_j}(\theta,\zeta+2\pi), \tag{3.3}$$

$$\text{conditions of boundedness}\;\; \Theta_{Z_j}(\theta,\zeta)\;\; \text{at}\;\; \theta=0,\;\; \theta=\pi, \tag{3.4}$$

$$-\Delta_{\theta,\zeta}\Theta_{Y_j}=\mu^2_{Y_j}\Theta_{Y_j},\;\; \theta\in(0,\pi),\;\; \zeta\in(0,2\pi),\Theta_{Y_j}(\theta,\Theta_{Y_j}(\theta,\zeta+2\pi), \tag{3.5}$$

$$\text{conditions of boundedness}\;\; \Theta_{Z_j}(\theta,\zeta)\;\; \text{at}\; \theta=0,\;\; \theta=\pi, \tag{3.6}$$

$$r^2R''_{Z_j}(r)+2rR'_{Z_j}(r)-\mu^2_{Z_j}R_{Z_j}(r)=-r^2R_{Y_j}(r), \tag{3.7}$$

$$r^2R''_{Y_j}(r)+2rR'_{Y_j}(r)+\left(\mu_j^2r^2-\mu^2_{Y_j}\right)R_{Y_j}(r)=0, \tag{3.8}$$

$$R_{Z_j}(r)\;\; \text{are bounded in the neighborhood of zero,}\;\; R_{Z_j}(1)=0,\;\; R'_{Z_j}(1)=0. \tag{3.9}$$

Let us deal with the solution of boundary value problems (3.3)–(3.4) and (3.5)–(3.6). Let us use the variable separation method:

$$\Theta_{Z_j}(\theta,\zeta)=\sum_m P_{Z_{jm}}(\theta)Q_{Z_{jm}}(\zeta),\;\; \Theta_{Y_{jm}}(\theta,\zeta)=\sum_m P_{Y_{jm}}(\theta)Q_{Y_{jm}}(\zeta). \tag{3.10}$$

Then (3.3)–(3.4) and (3.5)–(3.6) are reduced to the following systems:

$$Q''_{Z_{jm}}(\zeta)+m^2Q_{Z_{jm}}(\zeta)=0,\;\; \zeta\in[0,2\pi),\;\; m^2\in\{0,1,2,\ldots\},\;\; Q_{Z_{jm}}(\zeta)=Q_{Z_{jm}}(\zeta+2\pi), \tag{3.11}$$

$$\frac{1}{\sin\theta}\left(\sin\theta\, P'_{Z_{jm}}(\theta)\right)'+\left[\mu^2_{Z_j}-\frac{m^2}{\sin^2\theta}\right]P_{Z_{jm}}(\theta)=0, \tag{3.12}$$

$$\text{conditions of boundedness}\;\; P_{Z_{jm}}(\theta)\;\; \text{at points}\;\; \theta=0,\;\; \theta=\pi, \tag{3.13}$$

$$Q''_{Y_{jm}}(\zeta)+m^2Q_{Y_{jm}}(\zeta)=0,\;\; \zeta\in[0,2\pi),\;\; m^2\in\{0,1,2,\ldots\},\;\; Q_{Y_{jm}}(\zeta)=Q_{Y_{jm}}(\zeta+2\pi), \tag{3.14}$$

$$\frac{1}{\sin\theta}\left(\sin\theta\, P'_{Y_{jm}}(\theta)\right)'+\left[\mu^2_{Y_j}-\frac{m^2}{\sin^2\theta}\right]P_{Y_{jm}}(\theta)=0, \tag{3.15}$$

$$\text{conditions of boundedness}\;\; P_{Y_{jm}}(\theta)\;\; \text{at points}\;\; \theta=0,\;\; \theta=\pi, \tag{3.16}$$

where the "prime" symbol denotes the derivative with respect to the variables ζ and θ.

The solutions of boundary value problems (3.11) and (3.14) coincide and are equal:

$$Q_{Z_{jm}}(\zeta)=Q_{Y_{jm}}(\zeta)=\{\cos m\zeta,\;\sin m\zeta\},\;\; \zeta\in[0,2\pi),\;\; m\in\{0,1,2,\ldots\}. \tag{3.17}$$

In addition, it is easy to see that relations (3.12)–(3.13) and (3.15)–(3.16) also coincide, and their solutions were found, for example, in ([30], p. 374–376) with using Legendre polynomials $P_{Z_j}(\theta)$ and $P_{Y_j}(\theta)$.

If in the equation (3.12) we make the substitution $t=\cos\theta$ and denote $X(t)|_{t=\cos\theta}=X(\cos\theta)=P_{Z_j}(\theta)$, so we get the equation

$$\left(\left(1-t^2\right)X'(t)\right)'+\left(\mu_{Zj}^2-\frac{m^2}{1-t^2}\right)X(t)=0,\ \ |t|<1. \tag{3.18}$$

Relation (3.12)–(3.13) admits bounded solutions only if and only if $\mu_{Z_j}^2=j(j+1)$ (3.20):

$$X(t)_{|t=\cos\theta}=P_j^{(m)}(t)_{|t=\cos}=P_j^{(m)}(\cos\theta)=P_{Z_j}(\theta),\quad \text{where}\quad m=0,1,2,\ldots,j. \tag{3.19}$$

Thus, according to (3.10) and (3.17)–(3.19) we obtain the eigenvalues

$$\mu_{Zj}^2=\mu_{Yj}^2=j(j+1), \tag{3.20}$$

each of which corresponds to $2j+1$ spherical functions

$$\begin{gathered}\Theta_{Z_j}^{(0)}(\theta,\zeta)=P_j(\theta),\\ \Theta_{Z_j}^{(-1)}(\theta,\zeta)=P_j^{(1)}(\cos\theta)\cos\zeta,\ \ \Theta_{Z_j}^{(1)}(\theta,\zeta)=P_j^{(1)}(\cos\theta)\sin\zeta,\\ \Theta_{Z_j}^{(-2)}(\theta,\zeta)=P_j^{(2)}(\cos\theta)\cos 2\zeta,\ \ \Theta_{Z_j}^{(2)}(\theta,\zeta)=P_j^{(2)}(\cos\theta)\sin 2\zeta,\\ \cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\ \Theta_{Z_j}^{(-l)}(\theta,\zeta)=P_j^{(l)}(\cos\theta)\cos l\zeta,\ \ \Theta_{Z_j}^{(l)}(\theta,\zeta)=P_j^{(l)}(\cos\theta)\sin l\zeta,\\ l=1,2,\ldots,j,\end{gathered} \tag{3.21}$$

where $P_j^{(\pm l)}(\cos\theta)$ are Legendre polynomials.

It should be noted that the system of spherical functions $\left\{\Theta_{Z_j}(\theta,\zeta),\ \ j=0,1,2,\ldots\right\}$ is orthogonal with weight $\sin\theta$ and forms an orthogonal basis in $L_2(\Sigma)$, where $\{1,\theta,\zeta\}\in\Sigma$ is the surface of the unit sphere. We can normalize this system of functions using the condition

$$\int\limits_0^{\pi}\int\limits_0^{2\pi}\left|\Theta_{Z_j}^{(\mp l)}(\theta,\zeta)\right|^2\sin\theta\,d\theta\,d\zeta=1.$$

Functions $\Theta_{Z_j}^{(0)}(\theta,\zeta)=P_j(\cos\theta)$ do not depend on ζ and called zonal. Since $P_j(t)$ has exactly j zeros inside the interval $(-1,1)$, the unit sphere is divided into $(j+1)$ latitude zones, inside which the zonal function retains its sign.

Let us consider the behavior of the function on the sphere

$$\Theta_{Z_j}^{(-l)}(\theta,\zeta)=\sin^l\theta\left[\frac{d^l}{dt^l}P_j(t)\right]\Bigg|_{t=\cos\theta}\cos l\zeta,\ \ \Theta_{Z_j}^{(+l)}(\theta,\zeta)=\sin^l\theta\left[\frac{d^l}{dt^l}P_j(t)\right]\Bigg|_{t=\cos\theta}\sin l\zeta.$$

Since $\sin\theta$ becomes zero at the poles and $\sin l\zeta$ or $\cos l\zeta$ becomes zero at $2l$ meridians, and $\frac{d^l}{dt^l}P_j(t)$ at $(j-l)$ latitudes, the entire sphere is divided into cells in which $\Theta_{Z_j}^{(\mp l)}(\theta,\zeta)$ maintains a constant sign. Functions $\Theta_{Z_j}^{(\pm l)}(\theta,\zeta)$ at $l>0$ are called tesseral.

Similar constructions are valid for boundary value problem (3.15)–(3.16).

Now we transform equations (3.7)–(3.8), by making the following substitutions

$$R_{Y_j}(r) = \frac{\Phi_{Y_j}(r)}{\sqrt{r}}, \;\; R_{Z_j}(r) = \frac{\Phi_{Z_j}(r)}{\sqrt{r}}. \tag{3.22}$$

Then, taking into account (3.20), instead of (3.7)–(3.9), we obtain the following equations with boundary conditions:

$$r^2\Phi''_{Z_j}(r) + r\Phi'_{Z_j}(r) - \nu^2_{Z_j}\Phi_{Z_j}(r) = -r^2\Phi_{Y_j}(r), \;\; \nu^2_{Z_j} = (j+1/2)^2\,, \tag{3.23}$$

$$r^2\Phi''_{Y_j}(r) + r\Phi'_{Y_j}(r) + \left(\mu_j^2 r^2 - \nu^2_{Y_j}\right)\Phi_{Y_j}(r) = 0, \;\; \nu^2_{Y_j} = (j+1/2)^2\,, \tag{3.24}$$

$$r^{-\frac{1}{2}}\Phi_{Z_j}(r) \quad \text{are bounded in the neighborhood of zero,}$$

$$\Phi_{Z_j}(1) = 0, \;\; \Phi'_{Z_j}(1) = 0.$$

If in (3.24) we make the replacement $\rho = \mu_j r$, then by definition the cylindrical function $\Phi_{Y_j}(r) = J_{\nu_{Y_j}}(\mu_j r)$ will satisfy the equation (3.24), here $\nu_{Y_j} = \nu_{Z_j} = j + \frac{1}{2}$, $j = 0, 1, 2, \ldots$

So, according to the definition of cylindrical functions ([31], chapter VII, § 3) for the equation (3.24) the following statement is true.

Lemma 1. Equation (3.24) has a general solution in the form of a cylindrical function $\Phi_{Y_j}(r) = J_{j+\frac{1}{2}}(\mu_j r)$, $j = 0, 1, 2, \ldots$

Substituting this solution into equation (3.23), we will have a boundary value problem for a second-order nonhomogeneous ordinary differential equation:

$$\begin{gathered} r^2\Phi''_{Z_j}(r) + r\Phi'_{Z_j}(r) - \nu^2_{Z_j}\Phi_{Z_j}(r) = -r^2 J_{j+\frac{1}{2}}\left(\mu_j r\right), r \in (0,1), \\ r^{-\frac{1}{2}}\Phi_{Z_j}(r) \quad \text{are bounded in the neighborhood of zero,} \\ \Phi_{Z_j}(1) = 0, \;\; \Phi'_{Z_j}(1) = 0, \end{gathered} \tag{3.25}$$

where $j = 0, 1, 2, \ldots$

For boundary value problem (3.25) we establish the following lemma.

Lemma 2. For each $j \in \{0, 1, 2, \ldots\}$ the boundary value problem (3.25) has a countable family of solutions

$$\left\{\Phi_{Z_j}(r) = \int_0^1 G_j(r,\rho) J_{j+\frac{1}{2}}(\mu_{j+1,k}\,\rho)\,d\rho, \;\; \mu^2_{j+1,k}\right\}, \;\; k = 1, 2, \ldots,$$

where $\mu_{j+1,k}$ are the roots of the equations $J_{j+\frac{3}{2}}(\mu) = 0$, and G_j, $j = 0, 1, 2, \ldots$ is the corresponding Green's function.

Proof. We look for fundamental solutions for (3.25) in the form $\Phi_{j\,\mathrm{f.s.}}(r) = r^\sigma$, where σ is whole unknown number. Substituting r^σ into the homogeneous case of equation (3.25), we find: for $j \neq 0$ $\sigma = j + \frac{1}{2}$, $\sigma = -j - \frac{1}{2}$; for $j = 0$ $\sigma = \frac{1}{2}$, $\sigma = -\frac{1}{2}$, i.e. fundamental solutions are equal

$$z_{1j}(r) = r^{j+\frac{1}{2}}, \;\; z_{2j}(r) = r^{-j-\frac{1}{2}} \quad \text{for each} \quad j \neq 0, \;\; z_{10}(r) = r^{\frac{1}{2}}, \;\; z_{20}(r) = r^{-\frac{1}{2}}. \tag{3.26}$$

Thus, the general solution of homogeneous equation (3.25) according to (3.26) is written in the form

$$\Phi_{Z_j\,f.s.}(r) = C_{1j}r^{j+\frac{1}{2}} + C_{2j}r^{-j-\frac{1}{2}}, \;\; j \in \{1, 2, \ldots\}, \;\; \Phi_{Z_0\,f.s.}(r) = C_{10}r^{\frac{1}{2}} + C_{20}r^{-\frac{1}{2}}. \tag{3.27}$$

Thus, general solutions for the equation from (3.25), obtained on the basis of fundamental solutions (3.26)–(3.27) ([30], chapter 1, § 5, Cauchy method), have the form:

$$\Phi_{\mathrm{j,\,gen.s.}}(r) = \begin{cases} C_{1j} r^{j+\frac{1}{2}} + \Phi_{j\ \mathrm{part.s.}}(r), & j \neq 0, \\ C_{10} r^{\frac{1}{2}} + \Phi_{0\ \mathrm{part.s.}}(r), & j = 0, \end{cases} = \int_0^1 G_j(r,\rho) J_{j+\frac{1}{2}}(\mu_j \rho)\, d\,\rho, \;\; j = 0,1,2,\ldots, \tag{3.28}$$

where

$$G_j(r,\rho) = \begin{cases} -\frac{1}{2j+1} r^{j+\frac{1}{2}} \left[\rho^{-j+\frac{1}{2}} - \rho^{j+\frac{3}{2}}\right], & 0 < r < \rho < 1, \\ -\frac{1}{2j+1} \rho^{j+\frac{3}{2}} \left[r^{-j-\frac{1}{2}} - r^{j+\frac{1}{2}}\right], & 0 < \rho < r < 1, \end{cases} \quad j = 1,2,3,\ldots, \tag{3.29}$$

$$G_0(r,\rho) = \begin{cases} -r^{\frac{1}{2}} \left[\rho^{\frac{1}{2}} - \rho^{\frac{3}{2}}\right], & 0 < r < \rho < 1, \\ -\rho^{-\frac{1}{2}} \left[r^{-\frac{1}{2}} - r^{\frac{1}{2}}\right], & 0 < \rho < r < 1, \end{cases} \quad j = 0 \tag{3.30}$$

$$C_{1j} = \begin{cases} -\frac{1}{2j+1} \int\limits_0^1 \left[\rho^{-j+\frac{1}{2}} - \rho^{-j+\frac{3}{2}}\right] J_{j+1/2}(\mu\rho)\, d\,\rho, & j = 1,2,3,..., \\ -\int\limits_0^1 \left[\rho^{\frac{1}{2}} - \rho^{\frac{3}{2}}\right] J_{1/2}(\mu\rho)\, d\,\rho, & j = 0, \end{cases} \tag{3.31}$$

$$C_{2j} = 0, \;\; j = 0,1,2,3,\ldots, \tag{3.32}$$

the equality of the coefficients C_{2j} to zero follow from the conditions of boundedness in the neighborhood of the point $r = 0$ from (3.25).

We have included the details of the calculations contained in (3.27)–(3.32) in Appendix B.

Next, taking into account the solution formulas (3.28)–(3.31) and satisfying their second boundary conditions at $r = 1$ from (3.25), we obtain

$$J_{j+\frac{3}{2}}(\mu_{j+1}) = 0, \text{ for each } j \in \{0,1,2,\ldots\}. \tag{3.33}$$

Really, we have

$$\Phi'_{Z_j}(1) = 0 = -\int_0^1 \rho^{j+\frac{3}{2}} J_{j+\frac{1}{2}}(\mu\rho)\; d\rho, \;\; j = 0,1,2,\ldots$$

According to formula (20) from ([31], chapter VII, § 3) the last relations are equivalent to the equalities (3.33).

Finally, as a solution of spectral problem (3.7)–(3.9) and taking into account formula (3.22) as eigenfunctions $R_{Zjk}(r)$ from (3.28)–(3.30), we obtain:

$$R_{Zjk}(r) = r^{-\frac{1}{2}} \int_0^1 G_j(r,\rho) J_{j+\frac{1}{2}}(\mu_{j+1,k}\rho)\, d\rho, \;\; J_{j+\frac{1}{2}}(\mu_{j+1,k}) = 0, \;\; j,k = 1,2,3,\ldots, \tag{3.34}$$

$$R_{Z0k}(r) = r^{-\frac{1}{2}} \int_0^1 G_0(r,\rho) J_{\frac{1}{2}}(\mu_{1,k}\rho)\, d\rho, \;\; J_{\frac{1}{2}}(\mu_{1,k}) = 0, \;\; k = 1,2,3,\ldots \tag{3.35}$$

As the roots of the equations $J_{j+\frac{1}{2}}(\mu_{j+1}) = 0, \; j = 0,1,2,\ldots,$ (into (3.34)–(3.35)) we find the eigenvalues

$$\mu^2_{j+1,k}, \;\; j = 0,1,2,\ldots, \;\; k = 1,2,3,\ldots \tag{3.36}$$

Thus, from solutions (3.7)–(3.9), problems (3.11)–(3.21), (3.25) and (3.34)–(3.36) we obtain the following system of eigenfunctions and the corresponding its eigenvalues:

$$\begin{gathered}\left\{Z^{(\pm)}_{jkm}(r,\theta,\zeta) = R_{Zjk}(r)\Theta^{(\pm m)}_{Zj}(\theta,\zeta),\ \mu^2_{j+1,k}\right\}\\ j\in\{0,1,2,\ldots\},\ \ m\in\{0,1,2,\ldots,j\},\ \ k\in\{1,2,3,\ldots\}.\end{gathered} \tag{3.37}$$

Note that the system of eigenfunctions (3.37) satisfies the orthogonality conditions with weight $r^2\sin\theta$.

4 Construction of eigenfunctions in Cartesian coordinates. Main result

Now in (3.37) let us move on to the Cartesian coordinate system.

The system of eigenfunctions and eigenvalues has the form

$$\begin{gathered}\left\{u^{(\pm)}_{jkm}(y) \equiv R_{Zjk}(|y|)\Theta^{(\pm m)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right),\ \mu^2_{jk}\right\},\\ j\in\{1,2,\ldots\},\ \ m\in\{0,1,2,3,\ldots,j\},\ \ k\in\{1,2,3,\ldots\},\ \ |y|<1,\end{gathered} \tag{4.1}$$

$$\begin{gathered}\left\{u_{0k0}(y) \equiv R_{Z0k}(|y|)\Theta^{(0)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right),\ \mu^2_{0k}\right\},\\ j=0,\ \ m=0,\ \ k\in\{1,2,3,\ldots\},\ \ |y|<1,\end{gathered} \tag{4.2}$$

$$\operatorname{arctg}\frac{y_2}{y_1} = \operatorname{arctg}\zeta, \quad \text{where} \quad \zeta\in\begin{cases}[0,\frac{\pi}{2}), & y_1>0,\ y_2\ge 0;\\ (\frac{3\pi}{2},2\pi), & y_1>0,\ y_2<0;\\ (\frac{\pi}{2},\frac{3\pi}{2}), & y_1<0;\\ \frac{\pi}{2}, & y_1=0,\ y_2>0;\\ \frac{3\pi}{2}, & y_1=0,\ y_2<0.\end{cases} \tag{4.3}$$

Note that under the conditions of orthogonality of the system of eigenfunctions (4.1)–(4.2) there will be missing weight $|y|^2\sin\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}\right)$, since the Jacobian of the transformation when passing from the Cartesian system to the spherical coordinate system is equal to $r^2\sin\theta$.

Thus, we have established the validity of the following theorem.

Theorem 1. From the solution formulas (3.17), (3.20), (3.21), (3.34)–(3.37) for boundary value problems (3.3)–(3.4), (3.5)–(3.6) and (3.7)–(3.9) respectively, we obtain the following system of eigenfunctions and the corresponding eigenvalues:

$$\begin{gathered}\left\{u^{(\pm)}_{jkm}(y) \equiv Z^{(\pm)}_{jkm}(r,\theta,\zeta) = R_{Zjk}(r)\Theta^{(\pm m)}_{Zj}(\theta,\zeta),\ \mu^2_{j+1,k}\right\}\\ j\in\{0,1,2,\ldots\},\ \ m\in\{0,1,2,\ldots,j\},\ \ k\in\{1,2,3,\ldots\}.\end{gathered}$$

In the Cartesian system, accordingly, we obtain the relations (4.1)–(4.3).

Now, according to the formulas (1.5), (4.1)–(4.2) we define the system of eigenfunctions $w(y) = \{w_1(y), w_2(y), w_3(y)\}$ for the spectral problem (1)–(1.16).

Using the statement of Theorem 1, we establish the following result.

Theorem 2 (Main result). For all $j \in \{0,1,2,\ldots\}$, $m \in \{0,1,2,\ldots,j\}$, $k \in \{1,2,3,\ldots\}$, $|y| < 1$, we have that each triple of eigenfunctions $\left\{w^{(\pm)}_{1jkm}(y), w^{(\pm)}_{2jkm}(y), w^{(\pm)}_{3jkm}(y)\right\}$:

$$w^{(-)}_{1jkm} = (\partial_{y_2} - \partial_{y_3})\, u^{(-)}_{jkm}(y),\;\; w^{(-)}_{2jkm} = (\partial_{y_3} - \partial_{y_1})\, u^{(-)}_{jkm}(y),\;\; w^{(-)}_{3jkm} = (\partial_{y_1} - \partial_{y_2})\, u^{(-)}_{jkm}(y), \tag{4.4}$$

$$w^{(+)}_{1jkm} = (\partial_{y_2} - \partial_{y_3})\, u^{(+)}_{jkm}(y),\;\; w^{(+)}_{2jkm} = (\partial_{y_3} - \partial_{y_1})\, u^{(+)}_{jkm}(y),\;\; w^{(+)}_{3jkm} = (\partial_{y_1} - \partial_{y_2})\, u^{(+)}_{jkm}(y), \tag{4.5}$$

where for $j \neq 0$:

$$(\partial_{y_2} - \partial_{y_3})\, u^{(\pm)}_{jkm}(y) \equiv (\partial_{y_2} - \partial_{y_3})\, R_{Zjk}(|y|)\Theta^{(\pm m)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right), \tag{4.6}$$

$$(\partial_{y_3} - \partial_{y_1})\, u^{(\pm)}_{jkm}(y) \equiv (\partial_{y_3} - \partial_{y_1})\, R_{Zjk}(|y|)\Theta^{(\pm m)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right), \tag{4.7}$$

$$(\partial_{y_1} - \partial_{y_2})\, u^{(\pm)}_{jkm}(y) \equiv (\partial_{y_1} - \partial_{y_2})\, R_{Zjk}(|y|)\Theta^{(\pm m)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right), \tag{4.8}$$

and for $j = 0$:

$$(\partial_{y_2} - \partial_{y_3})\, u_{0k0}(y) \equiv (\partial_{y_2} - \partial_{y_3})\, R_{Z0k}(|y|)\Theta^{(0)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right), \tag{4.9}$$

$$(\partial_{y_3} - \partial_{y_1})\, u_{0k0}(y) \equiv (\partial_{y_3} - \partial_{y_1})\, R_{Z0k}(|y|)\Theta^{(0)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right), \tag{4.10}$$

$$(\partial_{y_2} - \partial_{y_3})\, u_{0k0}(y) \equiv (\partial_{y_1} - \partial_{y_2})\, R_{Z0k}(|y|)\Theta^{(0)}_{Zj}\left(\operatorname{arctg}\frac{\sqrt{y_1^2+y_2^2}}{y_3}, \operatorname{arctg}\frac{y_2}{y_1}\right) \tag{4.11}$$

form an orthogonal basis in the space $\mathbf{V} \cap \mathbf{H}^2(\Omega)$.

Remark 1. From (3.34)–(3.35), (4.1)–(4.2) and (3.25) it follows that the boundary conditions from (3.9) are valid for $r = |y| = 1$, and from (1.5), (4.4)–(4.11) we obtain the satisfiability of the equation (1.2), i.e. $\operatorname{div} w = 0$.

It is obvious that each triple of functions from (4.4)–(4.11) satisfies the homogeneous Dirichlet condition on the boundary of the unit sphere, with the possible exception of the following six points on the sphere $\{y_1, y_2, y_3\}$: $\{1,0,0\}$, $\{-1,0,0\}$, $\{0,1,0\}$, $\{0,-1,0\}$, $\{0,0,1\}$ and $\{0,0,-1\}$.

5 *Towards an approximate solution of the initial boundary value problem* (1.1)–(1.4)

We have constructed the orthogonal basis $w^{(\pm)}_{jkm}(y)$, $j = 0,1,2,\ldots, m = 0,1,2,\ldots,j, k = 1,2,3,\ldots$ in the space $\mathbf{V} \cap \mathbf{H}^2(\Omega)$. And based on this basis, we will introduce an approximate solution and given functions for the initial boundary value problem (1.1)–(1.4), formulated in weak form (in terms of the integral identity):

$$w^{(\pm)}_N(y,t) = \sum_{j=-N,k=1}^{N} \sum_{m=0}^{j} c^{(\pm)}_{jkmN}(t) w^{(\pm)}_{jkm}(y), \tag{5.1}$$

$$f^{(\pm)}_N(y,t) = \sum_{j=-N,k=1}^{N} \sum_{m=0}^{j} d^{(\pm)}_{jkmN}(t) w^{(\pm)}_{jkm}(y), \tag{5.2}$$

$$P_N^{(\pm)}(y,t) = \sum_{j=-N,k=1}^{N} \sum_{m=0}^{j} e_{jkmN}^{(\pm)}(t) w_{jkm}^{(\pm)}(y), \tag{5.3}$$

$$\left(\partial_t w_N^{(\pm)}, w_{lnp}^{(\pm)}\right) + \left(\left(w_N^{(\pm)}, w_{lnp}^{(\pm)}\right)\right) = \left(f_N^{(\pm)}, w_{lnp}^{(\pm)}\right), \ \ 0 \le l \le N, \ \ n = 1,\dots,N, \ \ p = 0,\dots,l, \tag{5.4}$$

$$w_N^{(\pm)}(y,0) = 0, \tag{5.5}$$

where the expansion coefficients $c_{jkmN}^{(\pm)}(t)$ (5.1) are to be determined at given coefficients $d_{jkmN}^{(\pm)}(t)$ (5.2) from the Cauchy problem for ordinary differential equations (5.4)–(5.5). And the expansion coefficients $e_{jkmN}^{(\pm)}(t)$ (5.3) are determined from equations (1.1). Thus, it is possible to find an approximate solution to the initial boundary value problem for the linearized system of Navier-Stokes equations (1.1)–(1.4).

Conclusion

In this work, a basis is constructed in the space solutions of the system of Navier-Stokes equations $\mathbf{V} \cap \mathbf{H}^2(\Omega)$, composed of eigenfunctions of the generalized spectral problem for a three-dimensional bi-Laplacian with Dirichlet boundary conditions in the unit sphere $\Omega = \{y = (y_1, y_2, y_3) : \ |y| < 1\}$. It is shown that these eigenfunctions satisfy the boundary conditions for the liquid velocity vector $w(y) = \{w_1(y), w_2(y), w_3(y)\}$ and the continuity equation $\operatorname{div} w(y) = 0, \ y \in \Omega$.

Appendix A. Spectral problem (1.13)–(1.14) *in spherical coordinates*

Let us recall the well-known formulas for gradient and divergence in spherical coordinates (r,θ,ζ):

$$\nabla u(y) = \partial_r u(r,\theta,\zeta)\cdot i_1 + \frac{1}{r}\partial_\theta u(r,\theta,\zeta)\cdot i_2 + \frac{1}{r\sin\theta}\partial_\zeta u(r,\theta,\zeta)\cdot i_3, \tag{A.1}$$

$$\operatorname{div}\vec{D}(y) = \frac{1}{r^2}\partial_r\left(r^2 D_1(r,\theta,\zeta)\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta D_2(r,\theta,\zeta)\right) + \frac{1}{r\sin\theta}\partial_\zeta D_3(r,\theta,\zeta), \tag{A.2}$$

where the vector $\vec{D} = \left\{\partial_r u(r,\theta,\zeta), \ \frac{1}{r}\partial_\theta u(r,\theta,\zeta), \ \frac{1}{r\sin\theta}\partial_\zeta u(r,\theta,\zeta)\right\}$ defined by the gradient vector. In addition, it is known that if $u(r,\theta,\zeta) = R(r)\Theta(\theta,\zeta)$, then

$$\Delta u(y) = \operatorname{div}\nabla u(y) = \frac{1}{r^2}\left(r^2 R'(r)\right)'\Theta(\theta,\zeta) + \frac{1}{r^2}R(r)\triangle_{\theta,\zeta}\Theta(\theta,\zeta),$$

where

$$\Delta_{\theta,\zeta} Z \equiv \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta Z\right) + \frac{1}{\sin^2\theta}\partial_\zeta^2 Z.$$

Now, instead of gradient (A.1), we introduce a new vector (modified gradient vector):

$$\widetilde{\nabla} u(y) = \frac{1}{r}\partial_\theta u(r,\theta,\zeta)\cdot i_1 + \frac{1}{r\sin\theta}\partial_\zeta u(r,\theta,\zeta)\cdot i_2 + \partial_r u(r,\theta,\zeta)\cdot i_3, \tag{A.3}$$

where $\frac{1}{r}\partial_\theta u = \tilde{D}_1, \quad \frac{1}{r\sin\theta}\partial_\zeta = \tilde{D}_2, \quad \partial_r u = \tilde{D}_3$.

Then, using (A.2) and (A.3), we have:

$$\operatorname{div}\widetilde{\nabla} u(y) \equiv \left(\partial_{y_1y_2}^2 + \partial_{y_2y_3}^2 + \partial_{y_3y_1}^2\right) u(y) =$$

$$= \frac{1}{r^2}\partial_r\left(r^2\frac{1}{r}\partial_\theta u(r,\theta,\zeta)\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta\frac{1}{r\sin\theta}\partial_\zeta u(r,\theta,\zeta)\right) + \frac{1}{r\sin\theta}\partial_\zeta\left(\partial_r u(r,\theta,\zeta)\right).$$

And finally, we have for the required operator (1.13):

$$\left(\triangle-\partial^2_{y_1y_2}-\partial^2_{y_2y_3}-\partial^2_{y_3y_1}\right)u=\frac{1}{r^2}\partial_r\left(r^2\partial_r u(r,\theta,\zeta)\right)+\frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta u(r,\theta,\zeta)\right)+$$

$$+\frac{1}{r^2\sin^2\theta}\partial^2_\zeta u(r,\theta,\zeta)-\frac{1}{r^2}\partial_r\left(r\partial_\theta u(r,\theta,\zeta)\right)-\frac{1}{r^2\sin\theta}\partial^2_{\theta\zeta}u-\frac{1}{r\sin\theta}\partial^2_{\zeta r}u(r,\theta,\zeta).$$

Having separated the variables $u(r,\theta,\zeta)=R(r)\Theta(\theta,\zeta)$, we obtain

$$\left(\triangle-\partial^2_{y_1y_2}-\partial^2_{y_2y_3}-\partial^2_{y_3y_1}\right)u(y)=\frac{1}{r^2}\left(r^2R'(r)\right)'\Theta\left(\theta,\zeta\right)+\frac{1}{r^2}R(r)\triangle_{\theta,\zeta}\Theta\left(\theta,\zeta\right)-$$

$$-\frac{1}{r}R'(r)\left(\partial_\theta\Theta\left(\theta,\zeta\right)+\frac{1}{\sin\theta}\partial_\zeta\Theta\left(\theta,\zeta\right)\right)-\frac{1}{r^2}R(r)\left(\partial_\theta\Theta\left(\theta,\zeta\right)+\frac{1}{\sin\theta}\partial^2_{\theta\zeta}\Theta\left(\theta,\zeta\right)\right);$$

$$\triangle_{\theta,\zeta}\Theta\left(\theta,\zeta\right)=\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Theta\left(\theta,\zeta\right)\right)+\frac{1}{\sin^2\theta}\partial^2_\zeta\Theta\left(\theta,\zeta\right). \qquad (A.4)$$

Thus, we have obtained the spectral problem (A.4) and (1.14), which (in our opinion) is an unsolvable problem to solve. Naturally, the boundary conditions (1.14) must be written on the surface of the unit sphere and at the center of the sphere (in spherical coordinates):

$$u(r,\theta,\zeta)_{|r=1}=0,\ \ \partial_r u(r,\theta,\zeta)_{|r=1}=0,$$

$u(r,\theta,\zeta)$ is bounded in the neighborhood of the center of sphere.

Appendix B. Cauchy Method

According to [23, chapter 1, § 5] a particular solution to the equation (3.25) has the form

$$\Phi_{j\,ch.s.}(r)=-\int\limits_0^r\eta_j(r,\rho)J_{j+\frac{1}{2}}(\mu_j\rho)d\rho, \qquad (B.1)$$

where for the Cauchy function $\eta_j(r,\rho)$ we have

$$\eta_j(r,\rho)=C_{1j}(\rho)r^{j+\frac{1}{2}}+C_{2j}(\rho)r^{-j-\frac{1}{2}}. \qquad (B.2)$$

Using (B.2), we obtain a system of equations for determining the unknown coefficients $C_{1j}(\rho)$ and $C_{2j}(\rho)$:

$$\begin{cases}\eta_j(\rho,\rho) &= C_{1j}(\rho)\rho^{j+\frac{1}{2}}+C_{2j}(\rho)\rho^{-j-\frac{1}{2}} &= 0,\\ \partial_r\eta_j(\rho,\rho) &= \left(j+\frac{1}{2}\right)\left[C_{1j}(\rho)\rho^{j-\frac{1}{2}}-C_{2j}(\rho)\rho^{-j-\frac{3}{2}}\right] &= 1.\end{cases} \qquad (B.3)$$

From (B.3) we have:

$$C_{1j}(\rho)=\frac{1}{2j+1}\rho^{-j+\frac{1}{2}},\quad C_{2j}(\rho)=-\frac{1}{2j+1}\rho^{j+\frac{3}{2}}. \qquad (B.4)$$

Thus, from (B.2)–(B.4) for the Cauchy function we obtain

$$\eta_j(r,\rho)=\frac{1}{2j+1}\left[\rho^{-j+\frac{1}{2}}r^{j+\frac{1}{2}}-\rho^{j+\frac{3}{2}}r^{-j-\frac{1}{2}}\right],$$

respectively for the particular solution $\Phi_{j\,ch.s.}(r)$ (B.1):

$$\Phi_{j\,ch.s.}(r) = -\frac{1}{2j+1}\int\limits_0^r \left[\rho^{-j+\frac{1}{2}}r^{j+\frac{1}{2}} - \rho^{j+\frac{3}{2}}r^{-j-\frac{1}{2}}\right] J_{j+\frac{1}{2}}(\mu_j\rho)d\rho. \qquad (B.5)$$

Now, using (B.5) and (3.22), we write the formulas for general solutions of the nonhomogeneous equations (3.25) and (3.7), respectively. We have

$$\Phi_{j\,gen.s.}(r) = C_{1j}r^{j+\frac{1}{2}} + C_{2j}r^{-j-\frac{1}{2}} - \frac{1}{2j+1}\int\limits_0^r \left[\rho^{-j+\frac{1}{2}}r^{j+\frac{1}{2}} - \rho^{j+\frac{3}{2}}r^{-j-\frac{1}{2}}\right] J_{j+\frac{1}{2}}(\mu_j\rho)d\rho, \qquad (B.6)$$

$$R_{j\,gen.s.}(r) = C_{1j}r^{j} + C_{2j}r^{-j-1} - \frac{1}{2j+1}\int\limits_0^r \left[\rho^{-j+\frac{1}{2}}r^{j} - \rho^{j+\frac{3}{2}}r^{-j-1}\right] J_{j+\frac{1}{2}}(\mu_j\rho)d\rho, \qquad (B.7)$$

where in (B.6)-(B.7) C_{1j} and C_{2j} are the unknown constants that need to be found. To do this, we will use the boundary conditions from (3.25). Due to the boundedness of the solution (B.7) in the neighborhood of zero, it is necessary that the coefficients C_{2j} be equal to zero, i.e., $C_{2j} = 0$. According to the boundary condition $R_j(1) = 0$ from (3.25) from (B.7) we get

$$C_{1j} = \frac{1}{2j+1}\int\limits_0^1 \left[\rho^{-j+\frac{1}{2}} - \rho^{j+\frac{3}{2}}\right] J_{j+\frac{1}{2}}(\mu\rho)d\rho,$$

$$\begin{aligned} R_{j\,gen.s.}(r) &= \tfrac{1}{2j+1}\int\limits_0^r \left[r^{-j-\frac{1}{2}} - r^{j+\frac{1}{2}}\right] \rho^{j+\frac{3}{2}} J_{j+\frac{1}{2}}(\mu_j\rho)d\rho + \\ &+ \tfrac{1}{2j+1}\int\limits_r^1 \left[\rho^{-j+\frac{1}{2}} - \rho^{j+\frac{3}{2}}\right] r^{j+\frac{1}{2}} J_{j+\frac{1}{2}}(\mu_j\rho)d\rho. \end{aligned}$$

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Бірлік шардағы үшөлшемді би-Лапласиан үшін қойылған спектрлік есеп туралы

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Мақалада ток функциясының жаңа түсінігін енгіземіз және үшөлшемді жағдайда ток функциясының теңдеуін шығарамыз. Навье-Стокс жүйесінің шешімдерінің кеңістігінде базис құру үшін шекарада Дирихле шарттары бар би-Лапласиан үшін көмекші спектрлік есепті шешеміз. Әрі қарай, ток функциясын енгізу үшін қолданылған формулаларды пайдалана отырып, Навье-Стокс жүйесінің шешімдерінің кеңістігінде базис болатын функциялар жүйесін табамыз. Бұл базисті Навье-Стокс жүйесі үшін сызықты және сызықты емес тура және кері есептерді жуықтап шешу үшін қолдануға болатынын атап өткен жөн. Ұсынылған жұмыстың негізгі идеясы келесідей: шекаралық шарттарды емес (оларды өзгеріссіз қалдырамыз) спектрлік параметрі бар ток функциясының дифференциалдық теңдеулерін өзгерту. Нәтижесінде біз облыс шекарасында Дирихле шарттарымен үшөлшемді бірлік шармен бейнеленген облыстағы би-Лапласианда спектрлік есеп аламыз, оны шешу кезінде Навье-Стокс теңдеулер жүйесінің шешімдерінің кеңістігінде базис құрайтын меншікті функциялар жүйесін табамыз. Бұл жағдайда шекаралық шарттар сақталып, сұйықтың үзіліссіздігі шартымен берілген теңдеу дің орындалғаны маңызды. Навье-Стокс жүйесінің үшөлшемді жағдайы үшін ток функциясының аналогы белгісіз болғанын да ескереміз.

Кілт сөздер: Навье-Стокс жүйесі, би-Лапласиан, спектрлік есеп, ток функциясы.

О спектральной задаче для трехмерного би-Лапласиана в единичном шаре

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В статье мы вводим новое понятие функции тока и выводим уравнение для функции тока в трехмерном случае. Для построения базиса в пространстве решений системы Навье-Стокса мы решаем вспомогательную спектральную задачу для би-Лапласиана с условиями Дирихле на границе. Далее, с помощью формул, которые использовались для введения функции тока, мы находим систему функций, образующую базис в пространстве решений системы Навье-Стокса. Следует отметить, что этот базис может быть использован для приближенного решения прямых и обратных задач для системы Навье-Стокса, как линеаризованной, так и нелинейной. Основная идея представленной работы заключается в следующем: изменять не граничные условия (их оставляем без изменений), а менять дифференциальные уравнения для функции тока со спектральным параметром. В результате мы получаем спектральную задачу для би-Лапласиана в области, представленной трехмерным единичным шаром, с условиями Дирихле на границе области, решая которую, мы находим систему собственных функций, образующих базис в пространстве решений системы уравнений Навье-Стокса. При этом является важным, что сохраняются граничные условия, и выполняется уравнение, представленное условием неразрывности жидкости. Заметим также, что для трехмерного случая системы Навье-Стокса аналог функции тока был неизвестен.

Ключевые слова: система Навье-Стокса, би-Лапласиан, спектральная задача, функция тока.

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https://doi.org/10.31489/2024M2/105-113 *Research article*

On a method for constructing the Green function of the Dirichlet problem for the Laplace equation

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The study of boundary value problems for elliptic equations is of both theoretical and applied interest. A thorough study of model physical and spectral problems requires an explicit and effective representation of the problem solution. Integral representations of solutions of problems of differential equations are one of the main tools of mathematical physics. Currently, the integral representation of the Green function of classical problems for the Laplace equation for an arbitrary domain is obtained only in a two-dimensional domain by the Riemann conformal mapping method. Starting from the three-dimensional case, these classical problems are solved only for spherical sectors and for the regions lying between the faces of the hyperplane. The problem of constructing integral representations of general boundary value problems and studying their spectral problems remains relevant. In this work, using the boundary condition of the Newtonian (volume) potential and the spectral property of the potential of a simple layer, the Green function of the Dirichlet problem for the Laplace equation was constructed.

Keywords: Laplace equation, Green function, Dirichlet problem, simple layer potential.

2020 Mathematics Subject Classification: 35C15, 35J05, 35J08.

Introduction

Let $\Omega \subset R^n$ is a bounded domain with a smooth boundary $\partial\Omega$.

The Dirichlet problem. Find in Ω the solution $u(x)$ of the Laplace equation

$$-\Delta_x u = f(x), \quad x \in \Omega,$$

satisfying the boundary condition

$$u|_{x\in\partial\Omega} = 0.$$

The function $G(x, y), x, y \in \Omega$ is called the Green function of the Dirichlet problem if

$$-\Delta_x G(x, y) = 0, \quad x \in \Omega, \quad G(x, y)|_{x\in\partial\Omega, y\in\Omega} = 0.$$

The solution of the Dirichlet problem using the Green's function $G(x, y)$ is representable in the following integral form

$$u(x) = \int_\Omega G(x, y) f(y) dy.$$

In the two-dimensional case, the method of conformal mapping of the analytical function is used to construct the Green's function. Starting from the three-dimensional case, the construction of the Green function is carried out by the method of Fredholm integral equations of the second kind, or by

This research was funded by the grant no. AP14871460 of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

Received: 08 January 2024; *Accepted:* 04 March 2024.

the method of maps, which are ineffective. Therefore, in multidimensional cases, $G(x, y)$ is constructed only for spherical sectors and for half-spaces.

In this paper, we present a method for constructing the Green function, which essentially uses the boundary properties of the Newtonian potential (volume potential) and the spectral properties of the trace of the potential of a simple layer.

By $u = L_N^{-1} f$ we shall call the Newtonian potential (volumetric potential)

$$u(x) = L_N^{-1} f = \int\limits_{\Omega} \varepsilon(x, y) f(y) dy, \tag{1}$$

where $\varepsilon(x, y)$ is the fundamental solution of the Laplace equation

$$-\Delta_x \varepsilon(x, y) = \delta(x, y), \tag{2}$$

the function $\varepsilon(x, y)$ in (2) has the following form

$$\varepsilon(x, y) = \begin{cases} -\frac{1}{2\pi} \ln |x - y|, & n = 2, \\ \frac{1}{\omega_n (n-2)|x-y|^{n-2}}, & n > 2. \end{cases} \tag{3}$$

Next, we will use the following statement from the work of T.Sh. Kal'menov, D. Suragan [1].

Theorem 1. The Newtonian potential $u(x) \in W_2^2(\Omega)$ at $x \in \Omega$ satisfies the Laplace equation

$$-\Delta_x u = f(x) \tag{4}$$

and the boundary condition

$$-\frac{u(x)}{2} + \int\limits_{\partial\Omega} \left(\frac{\partial \varepsilon}{\partial n_y}(x - y) u(y) - \varepsilon(x - y) \frac{\partial u(y)}{\partial n_y} \right) dS_y = 0, \quad x \in \partial\Omega. \tag{5}$$

Inversely, if $u \in W_2^2(\Omega)$ satisfies equation (3) and boundary condition (4), then $u(x)$ coincides with the Newtonian potential (1).

Note that in the work of the Saito [2] it is also established that $u(x) = L_N^{-1} f(x)$ satisfies the boundary condition (4). In contrast to the work of the Saito, in our work it was found that if the solution satisfies equation (3) and boundary condition (4), it coincides with the Newton potential $u(x) = L_N^{-1} f(x)$.

It follows from Theorem 1 that the Green function of problem (3)-(4) in an arbitrary domain is the fundamental solution $\varepsilon(x, y)$.

Similarly, the lateral boundary conditions of the wave and heat potentials are found in [3–6].

Let $-\Delta_0$ be the closure in $L_2(\Omega)$ of the differential operator $-\Delta$ on subset of functions $u \in C^{2+\alpha}(\overline{\Omega})$, $u|_{\partial\Omega} = \frac{\partial u}{\partial n}|_{\partial\Omega} = 0$, and $-\Delta_0^*$ is its adjoint operator in $L_2(\Omega)$.

The operator L is called a correct restriction if

$$L \subset -(\Delta_0)^*, \; L^{-1} \text{ is invertible on all } L_2(\Omega).$$

Correct restriction L of the operator $-(\Delta_0)^*$ we call a regular boundary extension if

$$-\Delta_0 \subset L, \; \|L^{-1}\|_{L_2(\Omega) \to L_2(\Omega)} < \infty.$$

The description of correct boundary value problems for general elliptic operators by the method of regular extensions of operators in Hilbert space is given by M.M. Vishik [7], and the description of correct restrictions for maximal operators is given by M.O. Otelbaev [8].

Next, we look for regular solutions of equation (3) in the form

$$u(x) = \int\limits_{\Omega} \varepsilon(x,y)f(y)dy + \int\limits_{\partial\Omega} \nu(\xi)\varepsilon(x,\xi)dS_\xi, \qquad (6)$$

where

$$u_\nu(x) = \int\limits_{\partial\Omega} \nu(\xi)\varepsilon(x,\xi)dS_\xi \qquad (7)$$

is the potential of a simple layer, and $\nu(\xi)$ is the density of the potential of the simple layer (6).

Suppose first that $\nu(x) \in C(\partial\Omega)$ and for each $x \in \partial\Omega$, and $\nu(x)$ is a linear continuous functional of $f(x) \in L_2(\Omega)$, i.e. $\nu(x) = \nu(x, f)$.

According to Riesz's theorem, $\nu(\xi, f)$ is representable as

$$\nu(\xi) = \nu(\xi, f) = \int\limits_{\Omega} \tilde{q}(\xi, y)f(y)dy, \qquad (8)$$

where $\tilde{q}$ is continuous over $\xi \in \partial\Omega$ and $\tilde{q}(\xi, y) \in L_2(\Omega)$ over variable $y \in \Omega$, i.e.,

$$||\tilde{q}(\xi, y)||_{L_2(\Omega)\cap C(\partial\Omega)} = \|\nu(\xi)\|_{C(\partial\Omega)}.$$

Substituting the right part (7) into the formula (6), we get

$$u_\nu(x) = \int\limits_{\partial\Omega} \varepsilon(x,\xi) \int\limits_{\Omega} \tilde{q}(\xi,y)f(y)dydS_\xi =$$

$$= \int\limits_{\Omega} f(y) \int\limits_{\partial\Omega} \varepsilon(x,\xi)\tilde{q}(\xi,y)dS_\xi dy = \int\limits_{\Omega} \tilde{q}(x,y)f(y)dy,$$

$$q(x,y) = \int\limits_{\partial\Omega} \varepsilon(x,\xi)\tilde{q}(\xi,y)dS_\xi. \qquad (9)$$

Thus, the operator

$$u_\nu = \mathcal{L}^{-1}f = \int\limits_{\Omega} q(x,y)f(y)dy, \quad x \in \Omega$$

converts an arbitrary function $f \in L_2(\Omega)$ to $\ker\triangle_0^*$, i.e. $-\triangle_y\mathcal{L}^{-1}f \equiv 0$.

Now we will rewrite the integral operator (5) in the form

$$u(x) = L_R^{-1}f = \int\limits_{\Omega} (\varepsilon(x,y) + q(x,y))f(y)dy.$$

By construction $-\Delta u = f(x)$. Therefore, the operator $u = L_R^{-1}f$ is a correct restriction of the maximal operator $-\Delta_0^*$, i.e. a invertible generalized solution of equation (3).

Remark. It is not difficult to establish that in the representation (8) we can consider $\widetilde{q} \in L_2(\partial\Omega) \cap L_2(\Omega)$.

According to the theory of correct restrictions generated by integral operators (T.Sh. Kal'menov, M. Otelbaev [9]), a correct restriction of L_R^{-1} generates regular boundary operators if and only if adjoint to $\left(L_R^{-1}\right)$ operator $\left(L_R^{-1}\right)^*$ is a correct restriction, i.e. the operator

$$\left(L_R^{-1}\right)^* g = \int\limits_{\Omega} \varepsilon(y,x)g(x)dx + \int\limits_{\Omega} q(x,y)g(x)dx$$

is a correct restriction.

According to [8], this can only be the case when

$$-\Delta_y q(x,y) = 0,$$

i.e.

$$-\Delta_y \left[\int\limits_{\partial\Omega} \varepsilon(x,\xi) q_0(\xi,y) dS_\xi \right] = -\int\limits_{\partial\Omega} \varepsilon(x,\xi) \Delta_y \widetilde{q}(\xi,y) dS_\xi = 0. \tag{10}$$

The following statement takes place

Lemma 1. The trace of the potential operator of a simple layer on $\partial\Omega$, given by the integral

$$(D_S^{-1}\nu)(x) = \int\limits_{\partial\Omega} \varepsilon(x,\xi)\nu(\xi) dS_\xi, \quad x \in \partial\Omega$$

is a completely continuous self-adjoint operator in $L_2(\Omega)$ and its kernel $\varepsilon(x,\xi), x, \xi \in \partial\Omega$ is represented as

$$\varepsilon(x,\xi) = \sum_{|m|=1}^{\infty} \frac{e_m(x)e_m(\xi)}{\lambda_m},$$

where $e_m(x)$ is a complete orthonormal system of eigenfunctions of the operator D_S^{-1} corresponding to the eigenvalues of $\frac{1}{\lambda_m}$.

Indeed, from $\varepsilon(x,\xi) = \varepsilon(\xi,x)$ and its weak divergence on $\partial\Omega$ follows the validity of Lemma 1.

It is easy to check that

$$D_S^{-1} e_m(x) = \frac{e_m(x)}{\lambda_m}, \quad D_S e_m(x) = \lambda_m e_m(x), \tag{11}$$

where D_S is inverse operator to D_S^{-1}.

Using Fourier series expansions

$$\varepsilon(x,\xi) = \sum_{|m|=1}^{\infty} \frac{e_m(x)e_m(\xi)}{\lambda_m}, \quad x \in \partial\Omega, \quad \xi \in \partial\Omega$$

and

$$-\Delta_y \tilde{g}(\xi,y) = -\sum_{|m|=1}^{\infty} \frac{(-\Delta_y \tilde{g})_m(y) e_m(\xi)}{\lambda_m}, \quad y \in \partial\Omega, \quad \xi \in \partial\Omega,$$

$$(-\Delta_y g(y))_m = \int\limits_{\partial\Omega} (-\Delta_y g(\xi,y)) e_m(\xi) dS_\xi.$$

From the equality (9) at $x \in \partial\Omega$ it follows that $(-\Delta_y \tilde{g}(y))_m = 0$, which is equivalent to $-\Delta_y \tilde{g}(\xi,y) \equiv 0$.

In particular,

$$\int\limits_{\partial\Omega} \varepsilon(x,\xi) \Delta_y \tilde{g}(\xi,y) dS_\xi = 0, \quad x \in \Omega.$$

Now we are looking for the Green function $G(x,y)$ in the form

$$G(x,y) = \varepsilon(x,y) - \int\limits_{\partial\Omega} \varepsilon(x,\xi) D_S \tilde{q}(\xi,y) dS_\xi. \tag{12}$$

Since $G(x,y) \equiv G(y,x)$, it follows from (11) that

$$\begin{aligned} G(x,y) =& \varepsilon(x,y) - \int\limits_{\partial\Omega} \varepsilon(x,\xi) D_S \tilde{q}(\xi,y) dS_\xi = \\ =& \varepsilon(y,x) - \int\limits_{\partial\Omega} \tilde{q}(\xi,x) D_S \varepsilon(y,\xi) dS_\xi = \\ =& \varepsilon(x,y) - \int\limits_{\partial\Omega} \tilde{q}(\xi,x) D_S \varepsilon(y,\xi) dS_\xi . \end{aligned}$$

It follows that

$$\tilde{q}(\xi,x) = \varepsilon(x,\xi).$$

Therefore,

$$G(x,y) = \varepsilon(x,y) - \int\limits_{\partial\Omega} \varepsilon(x,\xi) D_S \varepsilon(y,\xi) dS_\xi . \tag{13}$$

From (12) it is easy to verify that

$$\begin{aligned} &-\Delta_x G = \delta(x-y), \\ &-\Delta_x \int\limits_{\partial\Omega} \varepsilon(x,\xi) D_S \varepsilon(y,\xi) dS_\xi = -\Delta_y \int\limits_{\partial\Omega} \varepsilon(x,\xi) D_S \varepsilon(y,\xi) dS_\xi = 0, \ x \in \Omega, \ y \in \Omega. \end{aligned} \tag{14}$$

It takes place

Lemma 2. The following equality is true

$$-\int\limits_{x\in\partial\Omega, y\in\Omega} \varepsilon(x,\xi) D_S \varepsilon(y,\xi) dS_\xi = -\varepsilon(y,x) = -\varepsilon(x,y). \tag{15}$$

Proof. Let us set

$$\widetilde{e_m}(y) = \int\limits_{\partial\Omega} \varepsilon(y,\xi) e_m(\xi) dS_\xi, \quad y \in \Omega,$$

it is obvious that

$$-\Delta_y \widetilde{e_m}(y) = 0, \quad y \in \Omega,$$

$$\varepsilon(y,\xi) = \sum_{|m|=1}^{\infty} \widetilde{e_m}(y) e_m(\xi). \tag{16}$$

By construction

$$\varepsilon(y,\xi)|_{y\in\partial\Omega} = \sum_{|m|=1}^{\infty} \frac{e_m(y) e_m(\xi)}{\lambda_m}.$$

Taking into account the formula (10), from (15) we obtain

$$D_S \varepsilon(y,\xi) = \sum_{|m|=1}^{\infty} \lambda_m \widetilde{e_m}(y) e_m(\xi), \quad y \in \Omega.$$

Based on (16) and the ratio

$$\varepsilon(x,\xi) = \sum_{|m|=1}^{\infty} \frac{e_m(x)e_m(\xi)}{\lambda_m}, \quad x \in \partial\Omega,$$

from (14) at $y \in \Omega$ we have

$$-\int_{\partial\Omega} \varepsilon(x,\xi) D_S \varepsilon(y,\xi) dS_\xi \Big|_{x\in\partial\Omega,\ y\in\Omega} =$$
$$= -\left(\sum_{|m|=1}^{\infty} \frac{e_m(x)e_m(\xi)}{\lambda_m}, \sum_{|\bar{m}|=1}^{\infty} \lambda_m \widetilde{e}_{\bar{m}}(y) e_{\bar{m}}(\xi) \right)_{L_2(\partial\Omega)} =$$
$$= -\sum_{|m|=1}^{\infty} e_m(x)\widetilde{e_m}(y) = -\varepsilon(y,x) = -\varepsilon(x,y), \quad y \in \Omega, \quad x \in \partial\Omega.$$

Using this, from (11) we will make sure that

$$G(x,y)|_{x\in\partial\Omega} = \varepsilon(x,y) - \int_{\partial\Omega} \varepsilon(x,y) D_S \varepsilon(y,\xi) d\xi_y =$$
$$= \varepsilon(x,y) - \varepsilon(x,y)|_{y\in\Omega, x\in\partial\Omega} = 0.$$

Lemma 2 is proved.

Equality (13) and Lemma 2 follow

Theorem 2. The Green function $G(x,y)$ of the Dirichlet problem is given by the formula

$$G(x,y) = \varepsilon(x,y) - \int_{\partial\Omega} \varepsilon(x,\xi) D_S \varepsilon(y,\xi) dS_\xi,$$

where $\varepsilon(x,y)$ is the fundamental solution of the Laplace equation, and D_S is the operator defined by the formula (10).

Acknowledgments

This work is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP14871460).

Conflict of Interest

The author declares no conflict of interest.

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Лаплас теңдеуі үшін Дирихле есебінің Грин функциясының интегралдық көрсетілімі туралы

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Эллиптикалық теңдеулерге арналған шеткі есептерді зерттеу теориялық және қолданбалы қызығушылық тудырады. Модельдік физикалық және спектрлік есептерді мұқият зерттеу үшін есептің шешімін нақты және тиімді ұсыну қажет. Дифференциалдық теңдеулер есептерінің шешімдерінің интегралдық көрсетілімі математикалық-физиканың негізгі құралдарының бірі. Қазіргі уақытта еркін аймақ үшін Лаплас теңдеуі үшін классикалық есептердің Грин функциясының интегралды көрсетілімі Риманның конформды бейнелеу әдісімен тек екі өлшемді аймақта алынды. Үш өлшемді жағдайдан бастап, бұл классикалық есептер тек шар секторлары үшін және гипержазықтықтың беттері арасында орналасқан аймақтар үшін шешіледі. Жалпы шеткі есептердің интегралды көрсетілімін құру және олардың спектрлік мәселелерін зерттеу мәселесі өзекті болып қала береді. Жұмыста ньютондық (көлемдік) потенциалдың шекаралық шартын және қарапайым қабат потенциалының спектрлік қасиеттерін пайдалана отырып, Лаплас теңдеуі үшін Дирихле есебінің Грин функциясы құрастырылған.

Кілт сөздер: Лаплас теңдеуі, Грин функциясы, Дирихле есебі, қарапайым қабат потенциалы.

Об интегральном представлении функции Грина задачи Дирихле для уравнения Лапласа

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Изучение краевых задач для эллиптических уравнений представляет и теоретический, и прикладной интерес. Для тщательного изучения модельных физических и спектральных задач требуется явное и эффективное представление решения задачи. Интегральные представления решений задач дифференциальных уравнений являются одними из основных инструментов математической физики. В настоящее время интегральное представление функции Грина классических задач для уравнения Лапласа для произвольной области получено только в двумерной области методом конформного отображения Римана. Начиная с трехмерного случая, эти классические задачи решены только для шаровых секторов и для областей, лежащих между гранями гиперплоскости. Вопрос построения интегральных представлений общих краевых задач и изучения их спектральных проблем остается актуальным. В работе, пользуясь граничным условием ньютонового (объемного) потенциала и спектральным свойством потенциала простого слоя, построена функция Грина задачи Дирихле для уравнения Лапласа.

Ключевые слова: уравнение Лапласа, функция Грина, задача Дирихле, потенциал простого слоя.

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https://doi.org/10.31489/2024M2/114-123 *Research article*

Model companion properties of some theories

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The class $\mathcal{K}$ of algebraic systems of signature σ is called a formula-definable class if there exists an algebraic system A of signature σ such that for any algebraic system B of signature σ it is $B \in \mathcal{K}$ if and only if $Th(B) \cdot Th(A) = Th(A)$. The paper shows that the formula-definable class of algebraic systems is idempotently formula-definable and is an axiomatizable class of algebraic systems. Any variety of algebraic systems is an idempotently formula-definite class. If the class $\mathcal{K}$ of all existentially closed algebraic systems of a theory T is formula-definable, then a theory of the class $\mathcal{K}$ is a model companion of the theory T. Also, in the paper the examples of some theories on the properties of formula-definability, pseudofiniteness and smoothly approximability of their model companion were discussed.

Keywords: model companion, pseudofinite theory, formula-definable class, smoothly approximated structure.

2020 Mathematics Subject Classification: 03C30, 03C15, 03C50, 54A05.

Introduction

In the literature on model theory and universal algebra, after the theorem of Feferman S., Vaught R.L. [1], the product of complete theories is considered in various articles. In particular, in [2], it is shown that the product of two stable (superstable, ω-stable) theories will be a stable (superstable, ω-stable) theory, that is, the set of all stable (superstable, ω-stable) theories with the operation of the product of theories is a commutative semigroup.

A. Robinson introduced the definition of a model companion for a theory [3]. In articles by various authors, results are obtained regarding the existence of a model companion for a theory. In particular, in [4], there is the following criterion for the existence of a model companion for inductive theories.

Theorem 1. (P. Eklof, G. Sabbagh [4]) Let T be an inductive theory. Then T has a model companion T' if and only if the class of existentially closed models of a theory T is elementary.

Various properties of model companions from different points of view have been studied in the works of [5–7]. Pseudofinite models and ω-categorical smoothly approximated models were considered in [8–12].

1 *Background information*

Let us give the necessary definitions and known results on the theory of models and universal algebra. For brevity, by the word model, we mean an algebraic system.

Let L be a countable language of first-order signature σ. For any model A of language L, we denote by $Th(A)$ the set of all sentences (bounded formulas) of language L that are true in model A, that is,

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The work was partially supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan under Grants AP19677451.

Received: 25 January 2024; *Accepted:* 04 March 2024.

$Th(A)$ the complete theory of model A. For models B, A of language L, the notation $B \equiv A$ means $Th(B) = Th(A)$.

For a class $\mathcal{K}$ (we assume that all classes are abstract, that is, closed with respect to isomorphism), $Th(\mathcal{K})$ is the set of complete theories of all models of class $\mathcal{K}$. $Th(L)$ is the set of all complete theories of the language L. Since the language L is countable, the power is $|Th(L)| \leq 2^{\omega}$. If $\mathcal{K}$ is an axiomatizable class of models of a language L, then $Th(\mathcal{K})$ is the theory of class $\mathcal{K}$.

Definition 1. [13] A class $\mathcal{K}$ of models of signature σ is called a *formula-definable class* if there exists a model A of signature σ such that for any model B of signature σ, $B \in \mathcal{K}$ if and only if $Th(B) \cdot Th(A) = Th(A)$. The model A is then called the determinant of the class $\mathcal{K}$, and if $Th(A) \cdot Th(A) = Th(A)$, then the class $\mathcal{K}$ is called *idempotently formula-definable*.

Preliminary results in this direction were obtained in works [14–16].

Definition 2. If $S \subset Th(L)$, then $M(S)$ is the class of all models of all theories from S. We call the set S of theories axiomatizable if $M(S)$ is an axiomatizable class. A class $\mathcal{K}$ of models is called *inductive* if $Th(\mathcal{K})$ is an inductive theory, that is, $Th(\mathcal{K})$ is a $\forall\exists$-theory. Not every set of theories is axiomatizable.

Theorem 2. (S. Feferman, R. Vaught [1]) Filtered products and direct products of models of a language L preserve elementary equivalence.

This theorem allows us to introduce the product operation $Th(A) \cdot Th(B) \Leftrightarrow Th(A \times B)$, (the symbol $\Leftrightarrow$ means by definition), the direct product $\prod_{i \in I} T_i$ of complete theories T_i, $i \in I \Leftrightarrow Th(\prod_{i \in I} T_i)$, the ultraproduct $\prod_{i \in I} T_i/D$ of complete theories T_i, $i \in I$ by ultrafilter D over set $I \Leftrightarrow Th(\prod_{i \in I} T_i/D)$, the ultradegree T^I/D of complete theory T by ultrafilter D over set $I \Leftrightarrow Th(\prod_{i \in I} T_i/D)$, where $T_i = T$ for all $i \in I$.

We assume that the direct product of models is the direct product of a non-empty set of models. The direct product of an empty set of models is a trivial model.

It is clear that $S \subset Th(L)$ is axiomatizable if and only if S is closed with respect to ultraproducts of theories.

A theory T is called an *idempotent* theory if $T \cdot T = T$. A model A is called an *idempotent model* if $Th(A \times A) = Th(A)$.

The set $Th(L)$ with the operation $\cdot$ product of theories is a commutative semigroup with identity (we will not take much into account the theory of the trivial model, although, of course, it is a neutral element for the operation $\cdot$).

Subsemigroups of semigroups $< Th(L); \cdot >$ we call them semigroups of complete theories.

Definition 3. [17] A set $S \subset Th(L)$ is called a *formula-definable* set of theories if there is a theory $T \in Th(L)$ such that for any theory $T_1 \in Th(L)$ it holds, $T_1 \in S$ if and only if $T_1 \cdot T = T$. The theory T, in this case, is called the determinant of the set S. If the determinant of the set S is an idempotent theory T, then S is called an idempotent formula-definable set of theories, and T in this case is called the idempotent determinant of the set S.

It is clear that the class of models $\mathcal{K}$ is formula-definable if and only if $Th(\mathcal{K})$ is formula-definable. Furthermore, the class of models $\mathcal{K}$ is idempotent formula-definable if and only if $Th(\mathcal{K})$ is idempotent formula-definable.

In proving the results of the article, we will use the following theorems:

Theorem 3. (J. Keisler [18]) For any model A and any ultrafilter D over I, $A \equiv A^I/D$.

Theorem 4. (J. Keisler [18]) By any sentence ϕ there is a number n such that for any index set I and any models $A_i, i \in I$, there is a subset J in I that contains at most n elements, and for any $V, J \subset V \subset I$, $\prod_{i \in V} A_i \models \phi$ if and only if $\prod_{i \in I} A_i \models \phi$.

Theorem 5. (S. Feferman – R. Vaught [1]) For any two sets of models $\{A_i|i \in I\}$, $\{B_i|i \in I\}$ and for any ultrafilter D on I, $\prod_{i\in I}(A_i \times B_i)/D \cong \prod_{i\in I} A_i/D \times \prod_{i\in I} B_i/D$.

Theorem 6. (F. Galvin, J. Weinstein [19]) Let A, B, C be models of the language L. If $A\times B\times C \equiv A$, then $A \times B \equiv A$.

2 *Formula-definable semigroups of complete theories*

This section presents the results obtained on formula-definable semigroups of complete theories [14] and formula-definable classes of models.

Let T^n mean $\prod_{i\in I} T_i$, where $|I| = n$, $T_i = T$, for all $i \in I$, and T^I mean $\prod_{i\in I} T_i$, where $T_i = T$ for all $i \in I$.

Lemma 2.1. For any theory $\in Th(L)$ it holds

1) $T^I/D = T$ for any ultrafilter D over the set I.

2) If T is an idempotent theory, then $T^I = T$ for any set I.

Proof. 1) $T^I/D = T$. To prove it, you should use the fact that $T^I/D \Leftrightarrow Th(\prod_{i\in I} T_i/D)$, where $T_i = T$ for all $i \in I$ and apply Theorem 4, relying on Theorem 3.

2) Let T be an idempotent theory. It is clear that for any finite $n, T^n = T$.

Let I be an infinite set. And for some sentence $\phi \in T$, sentence $\phi \notin T^I$, then by Theorem 5, this contradicts the fact that for all finite m greater than a sufficiently large n, $\phi \in T^m = T$ holds. This means $T^I = T$.

Lemma 2.2. For any two sets of complete theories $\{T_i|i \in I\}$ and $\{T'_i|i \in I\}$ and for any ultrafilter D on I, $\prod_{i\in I}(T_i \cdot T'_i)/D = \prod_{i\in I} T_i/D \cdot \prod_{i\in I} T'_i/D$.

Proof. Follows directly from Theorem 6, relying on Theorem 3.

Lemma 2.3. Let T_1, T_2, T_3 be complete theories. If $T_1 \cdot T_2 \cdot T_3 = T_3$, then $T_1 \cdot T_3 = T_3$.

Proof. Follows from Theorem 7, based on Theorem 3.

Theorem 7. The formula-definable set of complete theories S is closed under finite, arbitrary direct products of theories.

Proof. Let the theory T be the determinant of the set S. The finite closedness of S with respect to the product is beyond doubt due to the associativity and commutativity of the direct product of theories.

Let $\{T_i|i \in I\} \subset S$ be an infinite set. If T is an idempotent theory, which means $T \in S$, then to prove the infinite closedness of S with respect to the product, one should use the same reasoning as in the proof of Lemma 2.1.

If the determinant of $T \notin \{T_i|i \in I\}$, then consider the set $\{T_i|i \in I\} \cup \{T\}$. Let for some sentence $\phi \in T$, sentence $\phi \notin \prod_{i\in I} T_i \cdot T$, then by Theorem 5, there exists a finite $J \subset I$ such that for any $V, J \subseteq V \subseteq I$, $\phi \notin \prod_{i\in I} T_i \cdot T$. However, this contradicts the fact that for all finite $V, J \subseteq V \subseteq I$ and the power V is greater than a sufficiently large n, $\phi \in \prod_{i\in I} T_i \cdot T$ holds.

Corollary 2.1. The formula-definable class of models $\mathcal{K}$ is closed under finite, arbitrary direct products of models. Its set of complete theories $Th(\mathcal{K})$ is also closed with respect to finite, arbitrary direct products of theories.

Lemma 2.4. The set of complete theories, closed under arbitrary direct products of theories, contains an idempotent theory $T' \in S$ such that for each theory $T \in S$, the following holds: $T \cdot T' = T'$.

Proof. Let us take the direct product of all theories from S, that is $\prod_{T\in S} T$. Since S is closed with respect to arbitrary direct products of theories, then $\prod_{T\in S} T \in S$. (In general, $|S| \leq 2^\omega$). Due to the closedness of S, the product $\prod_{T\in S} T \cdot \prod_{T\in S} T \in S$. This means there is a theory $T' \in S$ and $\prod_{T\in S} T \cdot \prod_{T\in S} T = T'$, which is present in both products. Now applying Lemma 2.3, we obtain that for any theory $T \in S$, the following holds: $T \cdot T' = T'$, including $T' \cdot T' = T'$.

Corollary 2.2. The class of models $\mathcal{K}$, which is closed with respect to arbitrary direct products of models, contains an idempotent model $A \in \mathcal{K}$ such that for each model $B \in \mathcal{K}, Th(B \times A) = Th(A)$ holds.

Theorem 8. A formula-definable set of complete theories S is an idempotent formula-definable set of theories. And the idempotent determinant of the set S is unique.

Proof. Let T^* be the determinant of the set S. By Theorem 7, S is closed under arbitrary direct products of theories. By Lemma 2.4, there is an idempotent theory $T' \in S$ such that for any theory $T \in S$, the following holds: $T \cdot T' = T'$. Now, if for some complete theory $T_1 \notin S$, $T_1 \cdot T' = T'$, then since $T_1 \cdot T' \cdot T^* = T^*$, then by Lemma 2.3, $T_1 \cdot T^* = T^*$ holds. That is, $T_1 \in S$. We have a contradiction. This means that the theory T' is an idempotent determinant of the set S.

There is only one idempotent determinant for S. Indeed, if there are two idempotent determinants T_1 and T_2 for S, then since $T_1 \in S$ and $T_2 \in S$ we have $T_1 = T_1 \cdot T_2 = T_2$.

Corollary 2.3. A formula-definable class of models of complete theories S is an idempotent formula-definable class of models.

Theorem 9. A formula-definable set of complete theories S is an axiomatizable set of complete theories.

Proof. Let $\{T_i | i \in I\} \subseteq S$ and $\prod_{i\in I} T_i/D$ be the ultraproduct of theories over the ultrafilter D over I. Using Lemmas 2.1 and 2.2, we obtain $\prod_{i\in I} T_i/D \cdot T = \prod_{i\in I} T_i/D \cdot T^I/D = \prod_{i\in I}(T_i \cdot T)/D = T$. This means that S is closed under the ultraproduct of theories, that is, S is an axiomatizable set of theories.

Corollary 2.4. A formula-definable class of models is an axiomatizable class.

Theorem 10. Each variety V is an idempotent formula-definable class of models.

Proof. The variety V is closed under arbitrary direct products. This means that $Th(V)$ is closed under the product of complete theories. Then, by Lemma 2.4, there is an idempotent theory $T \in Th(V)$ such that for any model $B \in V, Th(B) \cdot T = T$. Let A be a model of a theory T, then A is an idempotent model, and for any model $B \in V$, it is true $Th(B \times A) = Th(A)$. Since $T \in Th(V)$, then in model A, the truths are all the identities that define the variety V. Therefore, if $B \notin V$, then $Th(B \times A) \neq Th(A)$. This means that the variety V is an idempotent formula-definable class of models.

3 Some examples of theories with a model companion

Here, we study examples of some theories and their model companions for fulfilling formula-definable, pseudofinite and smoothly approximable properties. In what follows, T is not necessarily a complete theory.

Definition 4. (model companion of theory [3]) Theory T_1 is called a *model companion* of theory T if T_1 and T are mutually model consistent (i.e. models of theory T_1 are embedded in models of theory T, and models of theory T_1 are embedded in models of theory). The theory T_1 is model complete.

A model companion to a theory does not always exist, but if it does, it is unique.

Theorem 11. If the class $\mathcal{K}$ of existentially closed models of a theory T is a formula-definable class, then $\mathcal{K}$ is a model companion of the theory T.

Proof. Follows from Corollary 2.4 and Theorem 2. (P. Eklof, G. Sabbagh [4]).

Some important types of companions of incomplete theories and their model-theoretic properties have been studied in the works [5–7].

In the work of J. Ax [8], the concept of pseudofiniteness was first defined. The groundworks obtained to date for pseudofinite structures directly depend on the results of J. Ax. The basic definitions of pseudofiniteness are as follows.

Definition 5. [8] An infinite structure $\mathcal{M}$ of a fixed language L is *pseudofinite* if for all L-sentences φ, $\mathcal{M} \models \varphi$ implies that there is a finite L-structure $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi$. The theory $T = Th(\mathcal{M})$ of the pseudofinite structure $\mathcal{M}$ is called *pseudofinite*.

Many beautiful theorems in model theory of the 1950s-60s were proved using ultraproducts. Set theorists love ultraproducts because they give rise to elementary embeddings, a staple of large cardinal theory. J. Ax in [8] connect the notion of pseudofiniteness and the construction of ultraproducts.

Proposition 3.1. [8] Fix a language L and an L-structure $\mathcal{M}$. Then the following are equivalent:

1) an L-structure $\mathcal{M}$ is pseudofinite;

2) $\mathcal{M} \models T_f$, where T_f is the common theory of all finite L-structures;

3) $\mathcal{M}$ is elementarily equivalent to an ultraproduct of finite L-structures.

In classical logic, the following property is a straightforward consequence of pseudofiniteness.

Proposition 3.2. Let $\mathcal{M}$ be a pseudofinite structure and $f : M^k \to M^k$ be a definable function. Then f is injective if and only if f is surjective.

The study of countably infinite and countably categorical smoothly approximable structures is relevant in many areas of mathematics, including topology, analysis, and algebra.

Definition 6. [10] Let Σ be a countable signature and let $\mathcal{M}$ be a countable and ω-categorical Σ-structure. Σ-structure $\mathcal{M}$ (or $Th(\mathcal{M})$) is said to be *smoothly approximable* if there is an ascending chain of finite substructures $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \ldots \subseteq \mathcal{M}$ such that $\bigcup_{i\in\omega} \mathcal{M}_i = \mathcal{M}$ and for every i, and for every $\bar{a}, \bar{b} \in \mathcal{M}_i$ if $tp_{\mathcal{M}}(\bar{a}) = tp_{\mathcal{M}}(\bar{b})$, then there is an automorphism σ of M such that $\sigma(\bar{a}) = \bar{b}$ and $\sigma(\mathcal{M}_i) = \mathcal{M}_i$, or equivalently, if it is the union of an ω-chain of finite homogeneous substructures; or equivalently, if any sentence in $Th(\mathcal{M})$ is true of some finite homogeneous substructure of $\mathcal{M}$.

It is noted that the concept of a "finitely homogeneous substructure" does not mean that the substructure is homogeneous.

Smoothly approximated structures were first examined in generality in [10], subsequently in [11]. The model theory of smoothly approximable structures has been developed much further by G. Cherlin and E. Hrushovski [12].

A. Lachlan introduced the concept of *smoothly approximable* structures to change the direction of analysis from finite to infinite, that is, to classify large finite structures that appear to be *smooth approximations* to an infinite limit.

When proving the above properties for examples, in order to avoid textual routine, the following known results are used.

Corollary 3.1. [10] Every ω-categorical, ω-stable structure over a language with just finitely many function symbols is smoothly approximated.

Corollary 3.2. [10] If $\mathcal{M}$ is smoothly approximated, then $Th(\mathcal{M})$ is not finitely axiomatisable.

Remark. Any smoothly approximable structures are pseudofinite, but the converse is not always true.

Example 1. Theory T of the class of all Boolean algebras, T_1 theory of atomless Boolean algebras. It is known that T_1 is a model companion for T. It is clear that $T_1 \cdot T_1$ will be the theory of atomless

Boolean algebra, and all countable atomless Boolean algebras are isomorphic. If some Boolean algebra A has an atom, then its theory $Th(A)$ will satisfy $Th(A) \cdot T_1 \neq T_1$. This means that the class of models T_1 is a formula-definable class. Since the class of models of a theory T is a variety, then by Theorem 11, this class is a formula-definable class. Thus, we have obtained an example of a formula-definable class of models of theory T in which theory T_1 is a model companion and the class of all models of theory T_1 is a formula-definable class. A Boolean algebra is known to be pseudofinite if and only if each element has an atom [20]. It is clear that the theory of this model companion is not pseudofinite. Since the T_1 theory is finitely axiomatizable, the countable model of the model companion is not smoothly approximable by Corollary 3.2.

Example 2. Theory of T abelian groups of exponent of a prime number p. The complete theory T_1 of the infinite model of a theory T is a model companion of a theory T since the infinite model of a theory T is an existentially closed model and categorical. It is clear that the class of models of the theory T is formula-definable, the determinant of this class is the infinite model of the theory T_1. However, the model companion of T_1 is not a formula-definable class. The theory of this model companion is, of course, pseudofinite. The infinite countable model of the model companion is ω-categorical, ω-stable, and by Corollary 3.1. is smoothly approximable.

Example 3. Theory T of one equivalence relation. The class of models of theory T is a formula-definable class; its determinant is a model with an infinite number of classes, and each class contains an infinite number of elements. The theory of the T_1 model, in which the infinite countable model contains for each $1 \leq n \leq \omega$ an infinite number of n - element classes, is a model companion of the theory of T. The class of models of the theory of T_1 is not formula-definable since for some non-existentially closed models B in the theory of T, $Th(B) \cdot T_1 = T_1$ holds. In the work [21], it is proved that any theory with one equivalence relation is pseudofinite. It is clear that theory T_1 is pseudofinite. Also, this work proves that any countably categorical model of this theory is smoothly approximable. Therefore, an infinite countable model of T_1 theory is smoothly approximable by [21].

Example 4. Theory T of linear order. The model companion of theory T is the theory T_1 of dense linear order without endpoints. The classes of models of theory T and the class of models of theory T_1 are not formula-definable classes of models. If it is a formula-definable class of models, it must be closed under the product of models, but this is not the case. Theory T_1 is not pseudofinite (see [22]). The infinite countable model of theory T is not smoothly approximable since no automorphism permutes elements.

Conclusion

The paper shows that the formula-definable class of algebraic systems is idempotently formula-definable and is an axiomatizable class of algebraic systems. Any variety of algebraic systems is an idempotently formula-definite class. If the class $\mathcal{K}$ of all existentially closed algebraic systems of a theory T is formula-definable, then a theory of the class $\mathcal{K}$ is a model companion of the theory T. Also, the paper discusses examples of some theories on the properties of formula-definability, pseudofiniteness and smoothly approximability of their model companion.

Acknowledgments

The work was partially supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan under Grants AP19677451. The authors express sincere gratitude to the seminar participants on model theory named after academician A.D. Taimanov for their useful discussions and to the reviewers for their useful comments.

Author Contributions

All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Кейбір теориялардың модельдік компаньондарының қасиеттері

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σ сигнатурасының алгебралық жүйелерінің $\mathcal{K}$ класы формуламен анықталатын класс деп аталады, егер σ сигнатурасының кез келген B алгебралық жүйесі бар болса, онда тек $B \in \mathcal{K}$ үшін, яғни $Th(B) \cdot Th(A) = Th(A)$ орындалатындай σ сигнатурасының A алгебралық жүйесі табылса. Мақалада алгебралық жүйелердің формуламен анықталатын класы идемпотентті түрде формуламен айқындалатын класс және алгебралық жүйелердің аксиоматизацияланатын класы екендігі көрсетілген. Алгебралық жүйелердің кез келген түрі идемпотентті түрде формуламен анықталатын класс болып саналады. T теориясының барлық экзистенциалды тұйық алгебралық жүйелерінің $\mathcal{K}$ класы формуламен анықталатын болса, онда $\mathcal{K}$ класының теориясы T теориясының модельдік компаньоны болып табылады. Сондай-ақ, мақалада формуламен анықталатын, псевдоақырлы және олардың модельдік компаньонының тегіс аппроксимациялану қасиеттері туралы кейбір теориялардың мысалдары талқыланған.

Кілт сөздер: модельдік компаньон, псевдоақырлы теория, формула бойынша анықталатын класс, тегіс аппроксимацияланатын құрылым.

Свойства модельного компаньона некоторых теорий

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Класс $\mathcal{K}$ алгебраических систем сигнатуры σ называется формульно-определимым, если существует алгебраическая система сигнатуры σ, такая что для любой алгебраической системы сигнатуры σ выполняется $B \in \mathcal{K}$ тогда и только тогда, когда $Th(B) \cdot Th(A) = Th(A)$. В статье показано, что формульно-определимый класс алгебраических систем является идемпотентно формульно-определимым и аксиоматизируемым классом алгебраических систем. Любое многообразие алгебраических систем является идемпотентно формульно-определимым классом. Если класс $\mathcal{K}$ всех экзистенциально замкнутых алгебраических систем теории формульно-определим, то теория класса $\mathcal{K}$ является модельным компаньоном теории T. Также в статье рассмотрены примеры некоторых теорий на свойства формульно-определимости, псевдоконечности и гладкой аппроксимируемости моделей их модельного компаньона.

Ключевые слова: модельный компаньон, псевдоконечная теория, формульно-определимый класс, гладко аппроксимируемая структура.

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Bulletin of the Karaganda University. Mathematics Series, No. 2(114), 2024, pp. 124–134

https://doi.org/10.31489/2024M2/124-134 *Research article*

Boundary value problem for the time-fractional wave equation

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In the article, the boundary value problem for the wave equation with a fractional time derivative and with initial conditions specified in the form of a fractional derivative in the Riemann-Liouville sense is solved. The definition domain of the desired function is the upper half-plane (x,t). To solve the problem, the Fourier transform with respect to the spatial variable was applied, then the Laplace transform with respect to the time variable was used. After applying the inverse Laplace transform, the solution to the transformed problem contains a two-parameter Mittag-Leffler function. Using the inverse Fourier transform, a solution to the problem was obtained in explicit form, which contains the Wright function. Next, we consider limiting cases of the fractional derivative's order which is included in the equation of the problem.

Keywords: fractional derivative, Laplace transform, Fourier transform, Mittag-Leffler function, Wright function.

2020 Mathematics Subject Classification: 45D05, 35K20, 26A33.

Introduction

The mathematical apparatus of fractional order integrodifferentiation plays a significant role in various fields of science and engineering, including physics, biology, economics, etc [1]. Its application makes it possible to more accurately model and analyze phenomena that cannot be described by classical differential equations or integrals. Applications include: modeling the dynamics of complex systems with long-term dependence and memory, such as financial markets, environmental systems, communication networks, etc., analysis of nonlinear processes and phenomena, including diffusion, thermal conductivity, wave propagation, etc., solving optimization and control problems under conditions of uncertainty and changing conditions.

Fractional derivatives can be interpreted as a way to account for memory effects and temporal nonlocality in systems. In the classical differential model, all changes in the system instantly affect its state. However, in reality, many systems have memories and histories that influence their future behavior. Fractional order derivatives take this memory into account, allowing the modeling of systems with long-term dependencies and time delays in response to external influences. In addition, they can also take into account spatial correlations and coordinate nonlocality in systems where the influence on the state at a given point in space depends not only on neighboring points, but also on more distant ones [2].

Fractional derivative equations are a way to describe the evolution of physical systems with losses. They can model systems in which energy, mass, or other physical quantities are lost over time or space. The fractional derivative usually characterizes the degree of loss or dissipation in the system. For example, in diffusion processes, fractional derivatives can describe an anomalous distribution of particles due to long-term correlations or heterogeneity of the medium. Wave processes with losses can also be described using fractional derivatives, which makes it possible to take into account energy

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Received: 21 December 2023; *Accepted:* 04 March 2024.

dissipation in the system [3]. In mathematical modeling of continuous media with memory, equations arise describing a new type of wave motion that occupies an intermediate position between ordinary diffusion and classical waves [4, 5].

A loaded differential equation is an equation with a loaded term, which can contain differential or integrodifferential operators. This loaded term can be expressed as a function containing both the variables themselves and their derivatives.

Loaded equations allow you to model more complex physical or mathematical systems that cannot always be described by simple equations. For example, in problems of mathematical physics or control theory, loaded differential equations can be used to take into account the influence of external factors or additional conditions on the dynamics of the system.

Such equations play an important role in research related to the theory of boundary value problems, stability and control of dynamic systems, as well as in other areas of science and engineering where adequate consideration of the load on the system under study is required. In [6], the class of flat problems on the effect of moving loads on the surface of an aminated plate is studied. However, the presence of a loaded operator is accompanied by some difficulties during research, since it is not always possible to use direct research methods. For problems with loads, adaptation and development of specialized numerical methods are required [7]. All this emphasizes both the theoretical and practical significance of studying various boundary value problems for loaded differential equations. It is obvious that the presence of a loaded term gives rise to new, still unexplored problems in the theory of boundary value problems, therefore there is a need to develop new methods for solving the evolving theory of loaded differential equations [8].

Loaded differential equations can be considered as weak or strong perturbations of differential equations. In some cases, boundary value problems remain correct in natural classes of functions, where the loaded term is interpreted as a weak perturbation [9]. If the uniqueness of the solution to the boundary value problem is violated, then the load can be considered as a strong perturbation [10]. It turns out that the nature of the load (weak or strong perturbation) depends both on the order of the derivatives included in the loaded (perturbed) part of the operator, and on the manifold on which the trace of the desired function is specified.

The study of boundary value problems with loaded terms, presented in the form of integrals or fractional derivatives, can lead to different results depending on the specifics of the equation and the conditions of the problem. There may also be difficulties associated with the analysis and evaluation of integral operators in the resulting integral equations, since their kernels contain special functions. In [11, 12], the intervals for changing the order of the fractional derivative, that is contained in the loaded term, are determined, for which the theorems of existence and uniqueness of solutions to boundary value problems and arising integral equations are valid. We also note that the boundary value problems of heat conduction and the Volterra integral equations arising in their study with singularities in the kernel, similar to the singularities in this paper, were considered in [13, 14].

Also, integral equations with singularities in the kernel arise when studying boundary value problems in non-cylindrical domains that degenerate into a point at the initial moment of time [15–20].

Fractional derivatives in equations add new aspects and difficulties in the study of boundary value problems, since they take into account not only the previous state of the system, but also its history. The fractional order differentiation operation is a combination of differentiation and integration operations. Recently, work has appeared on the study of inverse boundary value problems with a load of fractional order. In [21], the inverse problem with a nonlinear gluing condition for a loaded equation of parabolic-hyperbolic type is studied for solvability. The problem is reduced to the study of the nonlinear Fredholm integral equation of the second kind. In [22], as an application of the analyticity of the solution, the uniqueness of an inverse problem in determining the fractional orders in the multi-term time-fractional diffusion equations from one interior point observation is established.

In this article, the boundary value problem for the fractional wave equation was solved, and two limiting cases were considered. The article is structured as follows. In Section 1, we introduce some necessary definitions and mathematical preliminaries of fractional calculus, special functions and boundary value problems which will be needed in the forthcoming Sections. The problem statement for the Riemann-Liouville fractional derivative wave equation in the upper half-plane (x,t) is given in Section 2. The initial conditions are given as a fractional derivative. Solving the problem is the content of Section 3: the Fourier transform for a spatial variable was consistently applied, followed by the Laplace transform for a temporal variable, the inverse Laplace transform and the inverse Fourier transform. Next, the limiting cases of the order of the fractional derivative are considered in Section 4. In the last Section the main result is formulated.

1 *Preliminaries*

Definition 1. [23] Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville integral of the order β is defined as follows

$$ {}_rD_{a,t}^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau, \quad \beta, a \in R, \quad \beta > 0. \tag{1}$$

Definition 2. Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville derivative of the order β is defined as follows

$$ {}_rD_{a,t}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}} d\tau, \quad \beta, a \in R, \quad n-1 < \beta < n. \tag{2}$$

From formula (2) it follows that

$$ {}_rD_{a,t}^{0} f(t) = f(t), \quad {}_rD_{a,t}^{n} f(t) = f^{(n)}(t), \quad n \in N.$$

Taking into account formula (1), formula (2) can be rewritten as

$$ {}_rD_{a,t}^{\beta} f(t) = \frac{d^n}{dt^n} {}_rD_{a,t}^{\beta-n} f(t), \quad \beta, a \in R, \quad n-1 < \beta < n.$$

The entire function of the form

$$E_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \mu)}, \quad \lambda > 0, \quad \mu \in C, \tag{3}$$

is called the Mittag-Leffler function.

The entire function of the form

$$\phi(\lambda, \mu; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in C, \tag{4}$$

is called the Wright function.

The formula for the integral Laplace transform of the Mittag-Leffler function is valid [24]

$$L\left[t^{\gamma-1} E_{a,\gamma}(\lambda t^a)\right] = \frac{s^{a-\gamma}}{s^a - \lambda}, \quad |\lambda| < |s|^a, \quad a > 0, \quad \gamma > -1. \tag{5}$$

Also the formula for the integral Laplace transform of the Wright function is valid [25]

$$L\left[t^{\beta-1} \phi(\rho, \beta, -\lambda t^\rho)\right] = s^{-\beta} \exp(-\lambda s^{-\rho}), \quad -1 < \rho < 0, \quad \lambda > 0. \tag{6}$$

2 Statement of the problem

In the domain $\Omega = \{(x,t) \mid -\infty < x < +\infty;\ t > 0\}$ find a solution to the problem:

$$D_{0t}^{\alpha} u(x,t) - u_{xx}(x,y) = f(x,t), \tag{7}$$

$$D_{0t}^{\alpha-1} u|_{t=0} = \varphi(x); \quad D_{0t}^{\alpha-2} u|_{t=0} = \psi(x), \quad \lim_{x\to\infty} u(x,t) = 0, \tag{8}$$

where $D_{0t}^{\alpha} f(t)$ is the Riemann-Liouville derivative of an order $\alpha \in (1;2)$.

We call a function $u(x,t)$ a *regular solution to equation (7) in the domain* G if $t^{1-\mu}u(x,t) \in C\left(\overline{G}\right)$ for some $\mu > 0$; in G, $u(x,t)$ has continuous derivatives with respect to x of the first and second order; the functions $D_{0t}^{\alpha-1}u(x,t)$ and $D_{0t}^{\alpha-2}u(x,t)$ are continuously differentiable as functions of t for a fixed x at interior points of G; and $u(x,t)$ satisfies equation (7) at all points of G.

3 Solving the problem

We apply Fourier transform to problem (7)-(8) with respect to the variable x:

$$D_{0t}^{\alpha} U(p,t) + p^2 U(p,t) = F(p,t), \tag{9}$$

$$D_{0t}^{\alpha-1} U|_{t=0} = \bar{\varphi}(p), \quad D_{0t}^{\alpha-2} U|_{t=0} = \bar{\psi}(p), \tag{10}$$

where $F(p,t); \bar{\varphi}(p); \bar{\psi}(p)$ are the Fourier images of input data in problem (7)-(8).

Let's apply Laplace transform to equation (9) with respect to the variable t taking into account conditions (10). Then we obtain

$$s^{\alpha}\bar{u}(p,s) - \bar{\varphi}(p) - s\bar{\psi}(p) + p^2\bar{u}(p,s) = \bar{f}(p,s),$$

where $\bar{f}(p,s)$ is the image of the function $F(p,t)$, or

$$\bar{u}(p,s) = \frac{\bar{f}(p,s)}{s^{\alpha}+p^2} + \frac{\bar{\varphi}(p)}{s^{\alpha}+p^2} + \frac{s}{s^{\alpha}+p^2}\bar{\psi}(p). \tag{11}$$

Applying the inverse Laplace transform to (11) with respect to the variable s and taking into account formula (5), we get

$$U(p,t) = \left((t^{\alpha-1}E_{\alpha,\alpha}(-p^2t^{\alpha})) * F(p,t)\right)(t) +$$
$$+ t^{\alpha-1}E_{\alpha,\alpha}(-p^2t^{\alpha})\bar{\varphi}(p) + t^{\alpha-2}E_{\alpha,\alpha-1}(-p^2t^{\alpha})\bar{\psi}(p), \tag{12}$$

where $E_{\lambda,\mu}(z)$ is the Mittag-Leffler function (3) and $*$ is the convolution operation.

Applying the inverse Fourier transform to (12) with respect to the variable p, we obtain

$$u(x,t) = \int_0^t \int_{-\infty}^{+\infty} G_1(x-\xi,\tau) f(\xi, t-\tau) d\xi d\tau + \int_{-\infty}^{+\infty} G_1(x-\xi,\tau)\varphi(\xi)d\xi +$$
$$+ \int_{-\infty}^{+\infty} G_2(x-\xi,\tau)\psi(\xi)d\xi, \tag{13}$$

where

$$G_1(x,t) = \frac{1}{\pi}\int_0^{+\infty} t^{\alpha-1}E_{\alpha,\alpha}(-p^2t^{\alpha})\cos(px)dp;$$

$$G_2(x,t) = \frac{1}{\pi}\int_0^{+\infty} t^{\alpha-2}E_{\alpha,\alpha-1}(-p^2t^\alpha)\cos(px)dp.$$

While derivation of formula (13) the well-known formula for the inverse Fourier transform with respect to the function $f(p)$ was used

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-ipx}f(p)dp = \frac{1}{\pi}\int_0^{+\infty} f(p)\cos(px)dp.$$

The function $G_1(x,t)$ was found in [24; 141]

$$G_1(x,t) = \frac{1}{2}t^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x|}{t^{\frac{\alpha}{2}}}\right),$$

where $\phi(\lambda,\mu;z)$ is the Wright function (4), according to the following scheme.

Let's apply Laplace transform to the second term in (13) with respect to the variable t and use formula:

$$L[t^{\gamma-1}E_{a,\gamma}(\lambda t^a)] = \frac{s^{a-\gamma}}{s^a-\lambda}$$

with $a=\alpha,\ \gamma=\alpha-1,\ \lambda=-p^2$.

Subtracting the last integral and taking into account formula 3.723 from [26], we get

$$g_1(x,s) = \frac{1}{2}s^{-\frac{\alpha}{2}}\exp(-|x|s^{\frac{\alpha}{2}}). \tag{14}$$

Applying the inverse Laplace transform to (14) with respect to the variable s taking into account formula (6) with $\lambda=x,\ \beta=\frac{\alpha}{2},\ \rho=-\frac{\alpha}{2},\ \lambda\in(1;2)$, we get

$$G_1(x,t) = \frac{1}{2}t^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2};\frac{\alpha}{2};-|x|t^{-\frac{\alpha}{2}}\right). \tag{15}$$

Similarly, applying Laplace transform to the third term in (13) with respect to the variable t taking into account formula (6) with $k=0,\ a=\alpha,\ b=\alpha-1,\ \lambda=p^2$, we obtain

$$g_2(x,s) = \frac{1}{\pi}\int_0^\infty \frac{s\cos px}{s^\alpha+p^2}dp = \frac{1}{2}s^{1-\frac{\alpha}{2}}\exp(-|x|s^{\frac{\alpha}{2}}).$$

Applying the inverse Laplace transform and taking into account the formula (6) with $\lambda=x$, $\rho=-\frac{\alpha}{2}$, $\beta=-\frac{\alpha}{2}$, we get

$$G_2(x,t) = \frac{1}{2}t^{\frac{\alpha}{2}-2}\phi\left(-\frac{\alpha}{2};\frac{\alpha}{2}-1;-|x|t^{-\frac{\alpha}{2}}\right). \tag{16}$$

Substituting (15) and (16) into (13), we obtain a solution to the original problem (7)-(8):

$$u(x,t) = \frac{1}{2}\int_0^t\int_{-\infty}^{+\infty}\tau^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x-\xi|}{\tau^{\frac{\alpha}{2}}}\right)f(\xi,t-\tau)d\xi d\tau+$$

$$+\frac{1}{2}\int_{-\infty}^{+\infty}t^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)\varphi(\xi)d\xi+\frac{1}{2}\int_{-\infty}^{+\infty}t^{\frac{\alpha}{2}-2}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2}-1;-\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)\psi(\xi)d\xi.$$

4 Limiting cases

Let's consider the limiting cases of the fractional derivative's order α.

I. $\alpha = 1$. Then problem (7)-(8) will take the form:

$$u_t - u_{xx} = f(x,t), \tag{17}$$

$$u|_{t=0} = \varphi(x), \tag{18}$$

$$\int_0^t u(x,\tau)d\tau|_{t=0} = \psi(x). \tag{19}$$

In the domain Ω the solution of problem (17)-(18) has the form [27]:

$$u(x,t) = \int_{-\infty}^{\infty} \varphi(\xi)G(x,\xi,t)d\xi + \int_0^t \int_{-\infty}^{\infty} f(\xi,\tau)G(x,\xi,t-\tau)d\xi d\tau, \tag{20}$$

where

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{(x-\xi)^2}{4t}\right). \tag{21}$$

We show that condition (19) is the overdetermination condition for $\alpha = 1$ in problem (17)-(18).

The solution of problem (17)-(18) has the form (20).

By virtue of Fubini's theorem, we have:

$$\int_0^t u(x,\tau)d\tau = \int_{-\infty}^{\infty} \varphi(\xi) \int_0^t G(x,\xi,t)\, d\tau d\xi + \int_{-\infty}^{\infty} \int_0^t f(\xi,\theta) \int_0^t G(x,\xi,\tau-\theta)\, d\tau d\theta d\xi,$$

where function $G(x,\xi,t)$ is defined by formula (21).

We calculate it using formula 3.471(2) [26; 354]

$$\int_0^t G(x,\xi,\tau)d\tau = \int_0^t \frac{1}{2\sqrt{\pi\tau}} \exp\left(-\frac{(x-\xi)^2}{4\tau}\right) d\tau = \sqrt{\frac{2}{x-\xi}}\, t^{\frac{3}{4}}\, exp\left(-\frac{(x-\xi)^2}{8\tau}\right) W_{-\frac{3}{4},\frac{1}{4}}\left(\frac{(x-\xi)^2}{4(t-\tau)}\right),$$

and

$$\int_\theta^t G(x,\xi,\tau-\theta)d\tau = \int_0^{t-\theta} G(x,\xi,\lambda)d\lambda = \sqrt{\frac{2}{x-\xi}}(t-\theta)^{\frac{3}{4}} exp\left(-\frac{(x-\xi)^2}{8(t-\tau)}\right) W_{-\frac{3}{4},\frac{1}{4}}\left(\frac{(x-\xi)^2}{4(t-\tau)}\right),$$

where $W_{\alpha,\beta}(z)$ is the Whittaker function [26; 1073].

Since for large values of z [26; 1075]

$$W_{\alpha,\beta} \sim e^{-\frac{z}{2}} z^{\alpha} \left(1 + \sum_{k=1}^{\infty} \frac{(\beta^2 - (\alpha - \frac{1}{2})^2)(\beta^2 - (\alpha - \frac{3}{2})^2)...(\beta^2 - (\alpha - k + \frac{1}{2})^2)}{k! z^{\alpha}}\right)$$

and with given $\lim_{t\to 0} \frac{(x-\xi)^2}{4t}$ and $0 < \theta < t$, then

$$\lim_{t\to 0} \int_0^t G(x,\xi,\tau)d\tau = \left\| z = \frac{(x-\xi)^2}{4t} \Rightarrow t = \frac{(x-\xi)^2}{4t} \right\| =$$

$$= \lim_{z\to\infty} \frac{\sqrt{2}}{\sqrt{|x-\xi|}} \frac{\sqrt{|x-\xi|}}{\sqrt{2} z^{\frac{3}{4}}} \exp\left(-\frac{z}{4}\right) z^{\frac{3}{4}} = \lim_{z\to\infty} e^{-\frac{z}{4}} = 0.$$

Similarly

$$\lim_{t\to 0}\int_0^t G(x,\xi,\tau-\theta)d\tau = 0.$$

Then the condition $D_{0t}^{\alpha-2}u|_{t=0}=\psi(x)$ in problem (17)-(18) is excess when $\alpha=1$.

On the other hand, for $\alpha=1$ out of (15)-(16) we have [28; 9]

$$G_1(x,t)=\frac{1}{2\sqrt{t}}\phi\left(-\frac{1}{2},\frac{1}{2};-\frac{|x|}{\sqrt{t}}\right)=\frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right).$$

Then, for $\alpha=1$, the solution of (20) coincides with (21).

II. $\alpha=2$. The problems (7)-(8) become as follows:

$$u_t-u_{xx}=f(x,t),\ u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\varphi(x).$$

The solution has the form [27; 258]

$$u(x,t)=\frac{1}{2}[\psi(x-t)+\psi(x+t)]+\frac{1}{2}\int_{x-t}^{x+t}\varphi(\xi)d\xi+\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(\xi,\tau)d\xi d\tau. \tag{22}$$

On the other hand, for $\alpha=2$ out of (15) we consider, that the function

$$G_1(x,t)=\frac{1}{2}\phi\left(-1,1;\frac{x}{t}\right)$$

doesn't exist.

Let's apply $\alpha=2$ to (8).

$$U(p,t)=((tE_{2,2}(-p^2t^2))*F(p,t))(t)+tE_{2,2}(-p^2t^2)\overline{\varphi}(p)+tE_{2,1}(-p^2t^2)\overline{\psi}(p).$$

Known that $\sin z=zE_{2,2}(-z^2),\ \cos z=E_{2,1}(-z^2)$. And we consider $z=pt$. Then

$$U(p,t)=\left(\frac{1}{p}\sin(pt)*F(p,t)\right)(t)+\frac{1}{p}\sin(pt)\overline{\varphi}(p)+\cos(pt)\overline{\psi}(p).$$

Apply the inverse Laplace transform.

Since

$$\sin(pt)=\frac{1}{2i}(e^{ipt}-e^{-ipt}),\cos(pt)=\frac{1}{2}(e^{ipt}+e^{-ipt}),$$

then

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2ip}((e^{ipt}-e^{-ipt})*F(p,t))(t)e^{ipx}dp+$$

$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2ip}((e^{ipt}-e^{-ipt})\overline{\varphi}(p)e^{ipx}dp+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{2}((e^{ipt}+e^{-ipt})\overline{\psi}(p)e^{ipx}dp.$$

Note that

$$\frac{1}{ip}((e^{ip(x+t)}-e^{-ip(x-t)})=\int_{x-t}^{x+t}e^{ip\eta}d\eta.$$

Then, given the convolution formula with respect to the variable t, we get

$$u(x,t)=\frac{1}{4\pi}\int_0^t\left\{\int_{-\infty}^{+\infty}\int_{x-(t-\tau)}^{x+(t-\tau)}e^{ip\eta}d\eta F(p,\tau)dp\right\}d\tau+$$

$$+\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{x-t}^{x+t)} e^{ip\eta}\overline{\varphi}(p)d\eta dp+\frac{1}{4\pi}\int_{-\infty}^{+\infty}(e^{ip(x+t)}+e^{ip(x-t)})\overline{\psi}(p)dp.$$

Changing the order of integration in the first and second integrals and considering that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ip\eta}F(p,t)dp=f(\eta,t),$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ip\eta}\overline{\varphi}(p)dp=\varphi(\eta),$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ip(x\pm t)}\overline{\psi}(p)dp=\psi(x\pm t)$$

are the originals of the function, we finally get

$$u(x,t)=\frac{1}{2}\int_{0}^{t}\int_{x-(t-\tau)}^{x+(t-\tau)} f(\eta,\tau)d\eta d\tau+\frac{1}{2}\int_{x-t}^{x+t}\varphi(\eta)d\eta+\frac{1}{2}[\psi(x+t)+\psi(x-t)].$$

Same as formula (22).

5 The main result

So, the following theorem has been proven.

Theorem 1. Let the function $u(x,t)$ be a regular solution to equation (7), and satisfies the conditions (8). Then for any point $(x,t)\in\Omega$ and $\alpha\in[1;2]$ the relation holds

$$u(x,t)=\frac{1}{2}\int_{0}^{t}\int_{-\infty}^{+\infty}\tau^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x-\xi|}{\tau^{\frac{\alpha}{2}}}\right)f(\xi,t-\tau)d\xi d\tau+$$

$$+\frac{1}{2}\int_{-\infty}^{+\infty}t^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)\varphi(\xi)d\xi+\frac{1}{2}\int_{-\infty}^{+\infty}t^{\frac{\alpha}{2}-2}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2}-1;-\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)\psi(\xi)d\xi, \quad (23)$$

where $\phi(\lambda,\mu;z)$ is the Wright function (4).

Conclusion

It can be shown that the function

$$G(x,t,\xi)=\frac{1}{2}t^{\frac{\alpha}{2}-1}\phi\left(-\frac{\alpha}{2},\frac{\alpha}{2};-\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)$$

is a fundamental solution to the equation

$$D_{0t}^{\alpha}u(x,t)-u_{xx}(x,y)=0,\quad \alpha\in(1;2).$$

In the future, we plan to solve a BVP in which the equation contains a loaded term in the form of a fractional derivative. When solving the problem, we will use the representation of the solution in the form (23). We assume that for certain values of the fractional derivative's order and of the type of manifold on that the load is specified, the uniqueness of the BVP's solution will be violated.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Уақыт бойынша бөлшек туындысы бар толқындық теңдеудің шеткі есебі

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Мақалада уақыт бойынша бөлшек туындысы бар және Риман-Лиувилл мағынасында бөлшек туынды ретінде берілген бастапқы шарттары бар толқындық теңдеудің шеткі есебі шешілді. Қажетті функцияны анықтау аймағы жоғарғы жартылай жазықтық (x, t) болып табылады. Есепті шешу үшін кеңістіктік айнымалы бойынша Фурье түрлендіруі дәйекті түрде қолданылады, содан кейін уақыт айнымалысы бойынша Лаплас түрлендіріледі. Кері Лаплас түрлендіруін қолданғаннан кейін түрлендірілген есепті шешу Миттаг-Леффлердің екіпараметрлі функциясын қамтиды. Кері Фурье түрлендіруін пайдаланғаннан кейін, Райт функциясын қамтитын тапсырманың шешімі айқын түрде алынады. Әрі қарай, есептің теңдеуіне кіретін бөлшек туынды ретінің шеткі жағдайлары қарастырылған.

Кілт сөздер: бөлшек туынды, Лаплас түрлендіруі, Фурье түрлендіруі, Миттаг-Леффлер функциясы, Райт функциясы.

Краевая задача для волнового уравнения с дробной производной по времени

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В статье решена краевая задача для волнового уравнения с дробной производной по времени и с начальными условиями, заданными в виде дробной производной в смысле Римана-Лиувилля. Областью определения искомой функции является верхняя полуплоскость (x, t). Для решения задачи последовательно применено преобразование Фурье по пространственной переменной, затем — преобразование Лапласа по временной переменной. Решение преобразованной задачи после применения обратного преобразования Лапласа содержит двухпараметрическую функцию Миттаг-Леффлера. После применения обратного преобразования Фурье получено решение поставленной задачи в явном виде, которое содержит функцию Райта. Далее рассмотрены предельные случаи порядка дробной производной, входящей в уравнение задачи.

Ключевые слова: дробная производная, преобразование Лапласа, преобразование Фурье, функция Миттаг-Леффлера, функция Райта.

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Bulletin of the Karaganda University. Mathematics Series, No. 2(114), 2024, pp. 135–146

https://doi.org/10.31489/2024M2/135-146 *Research article*

On the formulation and investigation of a boundary value problem for a third-order equation of a parabolic-hyperbolic type

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In the paper a novel boundary value problem for a third-order partial differential equation (PDE) of a parabolic-hyperbolic type, within a pentagonal domain consisting of both parabolic and hyperbolic regions was investigated. Such equations are pivotal in modeling complex physical phenomena across diverse fields such as physics, engineering, and finance due to their ability to encapsulate a wide range of dynamics through their mixed-type nature. By employing a constructive solution approach, we demonstrate the unique solvability of the posed problem. The significance of this study lies in its extension of the mathematical framework for understanding and solving higher-order mixed PDEs in complex geometrical domains, thus offering new avenues for theoretical and applied research in mathematical physics and related disciplines.

Keywords: differential equations, parabolic-hyperbolic type, a third-order parabolic-hyperbolic type.

2020 Mathematics Subject Classification: 35K30, 35L20, 35R05, 35J25.

Introduction

The study of non-classical equations of mathematical physics refers to the investigation of partial differential equations (PDEs) that exhibit behaviors beyond the standard classifications of parabolic, hyperbolic, and elliptic equations. These equations are often referred to as non-classical or degenerate equations. At present, the study of non-classical equations of mathematical physics is being intensively developed — equations of mixed, composite and mixed-composite types. One of the main reasons is the emergence of applied applications of boundary value problems posed for equations of these types. Many problems in physics, technology, mechanics and other areas require the study of such equations.

First, they began to study second-order mixed equations of the elliptic-hyperbolic type. The Italian mathematician Tricomi began to study fundamental studies of equations of such types in the 1920s.

After that, we began to study many different problems for equations of these types. A review of theoretical and applied research is given in the works and books of A.V. Bitsadze, L. Bers, M.M. Smirnov, as well as, in Uzbekistan, in the books of M.S. Salokhitdinov, T.D. Juraev.

Research into equations of elliptic-parabolic, parabolic-hyperbolic types began in the 1950s and 1960s. In 1959, I.M. Gelfand [1] pointed out the need for joint consideration of equations in one part of the domain of parabolic, and the other part of hyperbolic types. He gives an example related to the movement of gas in a channel surrounded by a porous medium: in the channel, the movement of gas is described by the wave equation, and outside it — by the diffusion equation.

Then, in the 1970s and 1980s, they began to study various problems for equations of the third and higher orders of the parabolic-hyperbolic type. Such problems were studied mainly by T.D. Dzhuraev and his students (for example, see [2, 3]).

At present, the study of various boundary value problems for equations of the third and higher orders of the parabolic-hyperbolic type has been developed on a broad scale (for example, see [4–15]).

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Received: 05 January 2024; *Accepted:* 04 March 2024.

1 Formulation of the problem

In this article, we consider one boundary value problem for a third-order parabolic-hyperbolic type equation of the form

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + c\right)(Lu) = 0 \tag{1}$$

in the pentagonal region G of the plane xOy, where $G = G_1 \cup G_2 \cup G_3 \cup J_1 \cup J_2$,

$$Lu = \begin{cases} u_{xx} - u_y, & (x,y) \in D_1, \\ u_{xx} - u_{yy}, & (x,y) \in D_i, \quad i = 2,3, \end{cases}$$

$c \in R$, and G_1 is a rectangle with vertices at points $A(0;0)$, $B(1;0)$, $B_0(1,1)$, $A_0(0,1)$; G_2 — triangle with vertices at points B, $C(0,-1)$, $D(-1,0)$; G_3 — rectangle with vertices at points A, D, $D_0(-1,1)$, A_0; J_1 — open segment with vertices at points B, D; J_2 — an open segment with vertices at points A, A_0.

The equation (1) is a special case of the equation $\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\right)(Lu) = 0$ when $\gamma = \frac{b}{a} = -1$, that is, the angular coefficient of the characteristic of the operator $a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ is equal to $\gamma = -1$.

For the equation (1), the following problem is posed:

Problem 1. It is required to find the function $u(x,y)$ which is 1) continuous in $\overline{G}$ and in the domain of $G \backslash J_1 \backslash J_2$ has continuous derivatives involved in the equation (1), and u_x and u_y are continuous in G up to part of the boundary of the domain G specified in the boundary conditions; 2) satisfies the equation (1) in the domain $G \backslash J_1 \backslash J_2$; 3) satisfies the following boundary conditions:

$$u(1,y) = \varphi_1(y), \quad 0 \le y \le 1, \tag{2}$$

$$u(-1,y) = \varphi_2(y), \quad 0 \le y \le 1,$$

$$u_x(1,y) = \varphi_3(y), \quad 0 \le y \le 1,$$

$$u|_{BC} = \psi_1(x), \quad 0 \le x \le 1, \tag{3}$$

$$u|_{DF} = \psi_2(x), \quad -1 \le x \le -\frac{1}{2}, \tag{4}$$

$$\left.\frac{\partial u}{\partial n}\right|_{BC} = \psi_3(x), \quad -1 \le x \le 0; \tag{5}$$

4) satisfies the following gluing conditions on the lines of type changing:

$$u(x,+0) = u(x,-0) = T(x), \quad -1 \le x \le 1, \tag{6}$$

$$u_y(x,+0) = u_y(x,-0) = N(x), \quad -1 \le x \le 1, \tag{7}$$

$$u_{yy}(x,+0) = u_{yy}(x,-0) = M(x), \quad -1 \le x \le 1, \tag{8}$$

$$u(+0,y) = u(-0,y) = \tau_3(y), \quad 0 \le y \le 1, \tag{9}$$

$$u_x(+0,y) = u_x(-0,y) = \nu_3(y), \quad 0 \le y \le 1, \tag{10}$$

$$u_{xx}(+0,y) = u_{xx}(-0,y) = \mu_3(y), \quad 0 \le y \le 1, \tag{11}$$

where

$$T(x) = \begin{cases} \tau_1(x), & if \quad 0 \le x \le 1, \\ \tau_2(x), & if \quad -1 \le x \le 0; \end{cases}$$

$N(x) = \begin{cases} \nu_1(x), & if \quad 0 \le x \le 1, \\ \nu_2(x), & if \quad -1 \le x \le 0; \end{cases}$ $M(x) = \begin{cases} \mu_1(x), & if \quad 0 < x < 1, \\ \mu_2(x), & if \quad -1 < x < 0, \end{cases}$ φ_i, ψ_i $(1,2,3)$ are given sufficiently smooth functions, τ_i, ν_i, μ_i $(i = 1,\ 2,\ 3)$ are unknown yet sufficiently smooth functions, n is an internal normal to the line $x - y = 1$, and the point F has coordinates $F\,(-1/2,-1/2)$.

2 Studying of the Problem

Theorem 2.1. If $\varphi_1,\ \varphi_2 \in C^3[0,1],\ \varphi_3 \in C^2[0,1],\ \psi_1 \in C^3[0,1],\ \psi_2 \in C^3[-1,-1/2],\ \psi_3 \in C^2[0,1]$, and the matching conditions $\varphi_1(0) = \psi_1(1)$ fulfilled, $\psi_2(-1) = \varphi_2(0)$, then Problem 1 is uniquely solvable.

Proof. We will prove the theorem by constructing the solution. To do this, we will rewrite the equation (1) as

$$u_{1xx} - u_{1y} = \omega_1(x+y)\exp(cy), \ \ (x,\,y) \in G_1, \tag{12}$$

$$u_{ixx} - u_{iyy} = \omega_i(x+y)\exp(cy), \ \ (x,\,y) \in G_i, \ \ i = 2,\,3, \tag{13}$$

where the notation $u(x,\,y) = u_i(x,\,y),\ (x,\,y) \in G_i\ \left(i = \overline{1,\,3}\right)$, functions $\omega_i(x+y),\ i = \overline{1,\,3}$ are unknown sufficiently smooth functions to be determined.

The study will be carried out first in the domain G_2. The solution of the equation (13) $(i=2)$ satisfying the conditions (6), (7) can be represented as

$$\begin{aligned} &u_2(x,y) = \tfrac{1}{2}[T(x+y) + T(x-y)] + \\ &+\tfrac{1}{2}\int_{x-y}^{x+y} N(t)dt - \tfrac{1}{2}\int_0^y \exp(c\eta)d\eta \int_{x-y+\eta}^{x+y-\eta} \omega_2(\xi+\eta)d\xi. \end{aligned} \tag{14}$$

Substituting (14) into the condition (5), after some calculations, we find

$$\omega_2(x+y) = -\sqrt{2}\psi_3'\left(\frac{x+y+1}{2}\right)\exp\left[-\frac{c}{2}(x+y-1)\right], \ -1 \le x+y \le 1.$$

Further, substituting (14) into the condition (3), after some simplifications, we obtain the first relation between the unknown functions $T(x)$ and $N(x)$ on the line J_1 of type changing:

$$T'(x) + N(x) = \alpha_1(x), \ -1 \le x \le 1, \tag{15}$$

where $\alpha_1(x) = \psi_1'(\frac{x+1}{2}) + \omega_2(x)\int_0^{(x-1)/2} e^{c\eta}d\eta$.

If we take into account the representation of the function $T(x)$, then for $-1 \le x \le 0$ the equation (15) has the form

$$\tau_2'(x) + \nu_2(x) = \alpha_1(x), \ \ -1 \le x \le 0. \tag{16}$$

Now, substituting (14) into (4), after some transformations, we have

$$\tau_2'(x) - \nu_2(x) = \delta_1(x), \ \ -1 \le x \le 0, \tag{17}$$

where $\delta_1(x) = \psi_2'(\frac{x-1}{2}) + \int_0^{-(x+1)/2} e^{c\eta}\omega_2(x+2\eta)d\eta$.

From (16) and (17) we obtain

$$\tau_2(x) = \frac{1}{2}\int_{-1}^x [\alpha_1(t) + \delta_1(t)]\,dt + \psi_2(-1), \ \ \nu_2(x) = \frac{1}{2}[\alpha_1(x) - \delta_1(x)].$$

For $0 \le x \le 1$ from (15), we have the first relation between the unknown functions $\tau_1(x)$ and $\nu_1(x)$ on the line J_1 of type changing in the following form:

$$\tau_1'(x) + \nu_1(x) = \alpha_1(x), \ \ 0 \le x \le 1. \tag{18}$$

Passing to the limit at $y \to 0$ in the equation (13) $(i=2)$, we will find the second relation between the unknown functions $\tau_1(x)$ and $\mu_1(x)$ on J_1:

$$\tau_1''(x) - \mu_1(x) = \omega_2(x), \ \ 0 \le x \le 1. \tag{19}$$

The equation (1) in the domain G_1 can be rewritten as

$$u_{1xxx} - u_{1xy} - u_{1xxy} + u_{1yy} + cu_{1xx} - cu_{1y} = 0.$$

Passing to the limit at $y \to 0$ in the last equation, we obtain the third relation between the unknown functions $\tau_1(x)$, $\nu_1(x)$ and $\mu_1(x)$ on the line of type changing J_1:

$$\tau_1'''(x) - \nu_1'(x) - \nu_1''(x) + \mu_1(x) + c\tau_1''(x) - c\nu_1(x) = 0, \quad 0 \le x \le 1. \tag{20}$$

Eliminating the functions $\nu_1(x)$ and $\mu_1(x)$ from the equations (18), (19) and (20) and integrating the resulting equation from 0 to x, we arrive at the equation

$$\tau_1''(x) + (1 + \frac{c}{2})\tau_1'(x) + \frac{c}{2}\tau_1(x) = \alpha_2(x) + k_1, \quad 0 \le x \le 1, \tag{21}$$

where $\alpha_2(x) = \frac{1}{2}\alpha_1'(x) + \frac{1}{2}\alpha_1(x) + \frac{1}{2}\int_0^x [\omega_2(t) + c\alpha_1(t)]\, dt$, and k_1 is still unknown constant.

When solving the equation (21), we consider the following cases:

1°. $c \ne 2,\ c \ne 0$;

2°. $c = 2$;

3°. $c = 0$.

In the case 1° it is easy to see that the solution of the equation (21) satisflying the conditions

$$\begin{aligned} &\tau_1(0) = \tfrac{1}{2}\int_{-1}^0 [\alpha_1(t) + \delta_1(t)]\, dt + \psi_2(-1), \\ &\tau_1'(0) = \tfrac{1}{2}[\alpha_1(0) + \delta_1(0)], \\ &\tau_1(1) = \varphi_1(0) \end{aligned} \tag{22}$$

has the form

$$\tau_1(x) = \frac{2}{2-c}\int_0^x \left[e^{\frac{c}{2}(t-x)} - e^{t-x}\right]\alpha_2(t)dt +$$

$$+\frac{2k_1}{2-c}\left[\frac{2}{c}(1 - e^{-\frac{c}{2}x}) - (1 - e^{-x})\right] + k_2 e^{-x} + k_3 e^{-\frac{c}{2}x},$$

where $k_3 = \dfrac{1}{2-c}\left\{\displaystyle\int_{-1}^0 [\alpha_1(t) + \delta_1(t)]\, dt + 2\psi_2(-1) + \alpha_1(0) + \delta_1(0)\right\}$,

$$k_2 = \frac{1}{c-2}\left\{\frac{c}{2}\int_{-1}^0 [\alpha_1(t) + \delta_1(t)]\, dt + c\psi_2(-1) + \alpha_1(0) + \delta_1(0)\right\},$$

$$k_1 = \left[\frac{c}{2}(1 - e^{-\frac{c}{2}}) - (1 - e^{-1})\right]^{-1}\left\{\frac{2-c}{2}\left[\varphi_1(0) - \right.\right.$$

$$\left.\left. - k_2 e^{-1} + k_3 e^{-\frac{c}{2}}\right] - \int_0^1 \left[e^{\frac{c}{2}(t-1)} - e^{t-1}\right]\alpha_2(t)dt\right\}.$$

Also, in the case 2°, one can show that the solution of solving the equation (21) satisfying the conditions (22), has the following form

$$\tau_1(x) = \int_0^x (x-t)e^{t-x}\alpha_2(t)dt + k_1\left[1 - (x+1)e^{-x}\right] + (k_2 + k_3 x)e^{-x},$$

where $k_2 = \frac{1}{2}\int_{-1}^0 [\alpha_1(t) + \delta_1(t)]\, dt + \psi_2(-1)$, $k_3 = k_2 + \frac{1}{2}[\alpha_1(0) + \delta_1(0)]$,

$$k_1 = \frac{1}{e-2}\left[\varphi_1(0)e - k_2 - k_3 - \int_0^1 (1-t)e^t\alpha_2(t)dt\right].$$

Moreover, for the case 3°, the solution of (21) satisflying (22) defined by

$$\tau_1(x) = \int_0^x e^{t-x}\alpha_3(t)dt + k_1(x-1+e^{-x}) + k_2(1-e^{-x}) + k_3e^{-x},$$

where $\alpha_3(x) = \int_0^x \alpha_2(t)dt$, $k_3 = \frac{1}{2}\int_{-1}^0 [\alpha_1(t)+\delta_1(t)]\,dt + \psi_2(-1)$, $k_2 = k_3 + \frac{1}{2}[\alpha_1(0)+\delta_1(0)]$,

$$k_1 = \varphi_1(0)e - k_2(e-1) - k_3 - \int_0^1 e^t\alpha_3(t)dt.$$

Now, we consider the G_3. Let us introduce the notation:

$$\omega_3(x+y) = \begin{cases} \omega_{31}(x+y), & -1 \le x+y \le 0, \\ \omega_{32}(x+y), & 0 \le x+y \le 1\,. \end{cases}$$

Then, passing to the limit at $y \to 0$, in the equations (13) $(i=2)$ and (13) $(i=3)$ due to (6)–(8), we get

$$\omega_{31}(x) = \omega_2(x), \quad -1 \le x \le 0.$$

Now, we consider the following problem:

$$\begin{cases} u_{3xx} - u_{3yy} = \omega_3(x+y)e^{cy}, \\ u_3(x,\,0) = \tau_2(x), \quad u_{3y}(x,\,0) = \nu_2(x), \quad -1 \le x \le 0, \\ u_3(-1,\,y) = \varphi_2(y), \quad u_3(0,\ ,y) = \tau_3(y), \quad 0 \le y \le 1. \end{cases}$$

The solution to this problem will be sought in the form

$$u_3(x,\,y) = u_{31}(x,\,y) + u_{32}(x,\,y) + u_{33}(x,y), \tag{23}$$

where $u_{31}(x,\,y)$ is the solution of the problem

$$\begin{cases} u_{31xx} - u_{31yy} = 0, \\ u_{31}(x,\,0) = \tau_2(x), \quad u_{31y}(x,\,0) = 0, \ -1 \le x \le 0, \\ u_{31}(-1,\,y) = \varphi_2(y), \quad u_{31}(0,\,y) = \tau_3(y), \ 0 \le y \le 1; \end{cases} \tag{24}$$

$u_{32}(x,\,y)$ is the solution of the problem

$$\begin{cases} u_{32xx} - u_{32yy} = 0, \\ u_{32}(x,\,0) = 0, \ u_{32y}(x,\,0) = \nu_2(x), \ -1 \le x \le 0, \\ u_{32}(-1,\,y) = 0, \ u_{32}(0,\,y) = 0, \ 0 \le y \le 1; \end{cases} \tag{25}$$

$u_{33}(x,\,y)$ is the solution of the problem

$$\begin{cases} u_{33xx} - u_{33yy} = \omega_3(x+y)e^{cy}, \\ u_{33}(x,\,0) = 0, \ u_{33y}(x,\,0) = 0, \ -1 \le x \le 0, \\ u_{33}(-1,\,y) = 0, \ u_{33}(0,\,y) = 0, \ 0 \le y \le 1. \end{cases} \tag{26}$$

Using the continuation method, we find solutions to the problems (24)–(26). The solutions can be represented as follows

$$u_{31}(x,\,y) = \frac{1}{2}\left[T_2(x+y) + T_2(x-y)\right], \tag{27}$$

where $T_2(x) = \begin{cases} 2\varphi_2(-1-x) - \tau_2(-2-x), & -2 \le x \le -1, \\ \tau_2(x), & -1 \le x \le 0, \\ 2\tau_3(x) - \tau_2(-x), & 0 \le x \le 1; \end{cases}$

$$u_{32}(x, y) = \frac{1}{2}\int_{x-y}^{x+y} N_2(t)dt, \tag{28}$$

where $N_2(x) = \begin{cases} -\nu_2(-2-x), & -2 \le x \le -1, \\ \nu_2(x), & -1 \le x \le 0, \\ -\nu_2(-x), & 0 \le x \le 1; \end{cases}$

$$u_{33}(x, y) = -\frac{1}{2}\int_0^y e^{c\eta}d\eta \int_{x-y+\eta}^{x+y-\eta} \Omega_3(\xi+\eta)d\xi. \tag{29}$$

Using the condition $u_{33}(-1, y) = 0$, after some transformations, from (29), we obtain

$$\frac{1}{2}\int_{-1-y}^{y-1} e^{\frac{c}{2}(y+1+z)}\Omega_3(z)dz = -\omega_{31}(y-1)\int_0^y e^{c\eta}d\eta. \tag{30}$$

Hence, by differentiating (30), we find

$$\begin{gathered} \Omega_{31}(-1-y) = c\omega_{31}(y-1)\int_0^y e^{c\eta}d\eta - \\ -2\omega_{31}'(y-1)\int_0^y e^{c\eta}d\eta - 3\omega_{31}(y-1)e^{cy}. \end{gathered}$$

Now, using condition $u_{33}(0, y) = 0$, after some transformations, from (30), we have

$$\omega_{32}(y)\int_0^y e^{c\eta}d\eta = -\int_{-y}^y e^{\frac{c}{2}(y+z)}\Omega_3(z)dz.$$

Substituting (27), (28) and (29) into (23), we get

$$\begin{gathered} u_3(x, y) = \tfrac{1}{2}\left[T_2(x+y) + T_2(x-y)\right] + \\ +\tfrac{1}{2}\int_{x-y}^{x+y} N_2(t)dt - \tfrac{1}{2}\int_0^y e^{c\eta}d\eta \int_{x-y+\eta}^{x+y-\eta} \Omega_3(\xi+\eta)d\xi. \end{gathered}$$

Differentiating this solution with respect to x and tending x to zero, and also taking into account the condition (10), after some transformations, we have the following relation:

$$\nu_3(y) = \tau_3'(y) + \tau_2'(-y) - \nu_2(-y) + \frac{1}{2}\int_{-y}^y e^{\frac{c}{2}(y+z)}\Omega_3(z)dz. \tag{31}$$

Passing to the limit at $x \to 0$ in the equations (12) and (13) $(i = 2)$ and taking into account (9) and (11), we obtain

$$\mu_3(y) - \tau_3'(y) = \omega_{11}(y)\exp(-cy), \;\; \mu_3(y) - \tau_3''(y) = \omega_{32}(y)e^{cy}.$$

Eliminating the function $\mu_3(y)$ from these equations, we find

$$\omega_{32}(y) = \omega_{11}(y) - \left[\tau_3''(y) - \tau_3'(y)\right]e^{-cy}. \tag{32}$$

Now, passing to the limit at $y \to 0$ in the equation (13) and taking (6) and (7) into account after replacing x with $x+y$, we obtain

$$\omega_{11}(x+y) = \tau_1''(x+y) - \nu_1(x+y), \;\; 0 \le x+y \le 1, \tag{33}$$

where $\omega_1(x+y)=\begin{cases}\omega_{11}(x+y), & 0\le x+y\le 1,\\ \omega_{12}(x+y), & 1\le x+y\le 2.\end{cases}$

Finally, by substituting (33) into (32), we arrive at the relation

$$\omega_{32}(y)=\tau_1''(y)-\nu_1(y)-\left[\tau_3''(y)-\tau_3'(y)\right]e^{-cy}, \tag{34}$$

and substituting (34) into (31), after some calculations, we get

$$\nu_3(y)=\frac{1}{2}\tau_3'(y)-\frac{c-2}{4}\int_0^y e^{\frac{c}{2}(y-z)}\tau_3'(z)dz+\beta_1(y),$$

where

$$\beta_1(y)=\tau_2'(-y)-\nu_2(-y)+\frac{1}{2}\nu_1(0)e^{\frac{c}{2}y}+$$
$$+\frac{1}{2}\int_{-y}^0 e^{\frac{c}{2}(y+z)}\omega_{31}(z)dz+\frac{1}{2}\int_0^y e^{\frac{c}{2}(y+z)}\left[\tau_1''(z)-\nu_1(z)\right]dz.$$

Now, we consider the domain G_1. The solution of the equation (12) satisfying the conditions (2), (6), (9) has the form

$$u_1(x,\,y)=\left[\int_0^y \tau_3(\eta)G_\xi(x,\,y;\;,0,\,\eta)d\eta-\right.$$
$$-\int_0^y \varphi_1(\eta)G_\xi(x,\,y;\,1,\,\eta)d\eta+\int_0^1 \tau_1(\xi)G(x,\,y;\,\xi,\,0)d\xi-$$
$$-\int_0^y e^{c\eta}d\eta\int_0^{1-\eta}\omega_{11}(\xi+\eta)G(x,\,y;\,\xi,\,\eta)d\xi-$$
$$-\int_0^y e^{c\eta}d\eta\int_{1-\eta}^1\omega_{12}(\xi+\eta)G(x,\,y;\,\xi,\,\eta)d\xi.$$

Differentiating this solution by x and passing x to zero and to one, we obtain the following relations

$$\begin{aligned}
\nu_3(y)=&-\int_0^y \tau_3'(\eta)N(0,y;0,\eta)d\eta+\\
&+\int_0^y \varphi_1'(\eta)N(0,y;1,\eta)d\eta+\int_0^1\tau_1'(\xi)N(0,y;\xi,0)d\xi+\\
&+\int_0^y e^{c\eta}d\eta\int_0^{1-\eta}\left[\tau_1''(\xi+\eta)-\nu_1(\xi+\eta)\right]N_\xi(0,y;\xi,\eta)d\xi+\\
&+\int_0^y e^{c\eta}d\eta\int_{1-\eta}^1\omega_{12}(\xi+\eta)N_\xi(0,\,y;\,\xi,\,\eta)d\xi,
\end{aligned}$$

$$\begin{aligned}
\varphi_3(y)=&-\int_0^y \tau_3'(\eta)N(1,y;0,\eta)d\eta+\\
&+\int_0^y \varphi_1'(\eta)N(1,y;1,\,\eta)d\eta+\int_0^1\tau_1'(\xi)N(1,\,y;\,\xi,\,0)d\xi+\\
&+\int_0^y e^{c\eta}d\eta\int_0^{1-\eta}\left[\tau_1''(\xi+\eta)-\nu_1(\xi+\eta)\right]N_\xi(1,y;\xi,\eta)d\xi+\\
&+\int_0^y e^{c\eta}d\eta\int_{1-\eta}^1\omega_{12}(\xi+\eta)N_\xi(1,y;\xi,\eta)d\xi.
\end{aligned} \tag{35}$$

Here and at the top of the functions $G(x,\,y;\,\xi,\,\eta)$ and $N(x,\,y;\,\xi,\,\eta)$ have the form:

$$\left.\begin{matrix}G(x,y;\xi,\eta)\\ N(x,y;\xi,\eta)\end{matrix}\right\}=\frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{n=-\infty}^{+\infty}\left\{\exp\left[-\frac{(x-\xi-2n)^2}{4(y-\eta)}\right]\mp\exp\left[-\frac{(x+\xi-2n)^2}{4(y-\eta)}\right]\right\}.$$

They are Green's functions of the first and second boundary value problems for the heat equation.
Substituting (32) into (33), after some transformations, we have the equation

$$\tau_3'(y) + \int_0^y \tau_3'(\eta) K_1(y,\eta) d\eta + \int_0^y K_2(y,\eta)\omega_{12}(1+\eta) d\eta = g_1(y). \tag{36}$$

And differentiating the equation (35) after some calculations, we obtain the Volterra integral equation of the second kind with respect to $\omega_{12}(1+y)$:

$$\omega_{12}(1+y) + \int_0^y K_3(y,\eta)\omega_{12}(1+\eta) d\eta + \int_0^y \tau_3'(\eta) K_4(y,\eta) d\eta = g_2(y), \tag{37}$$

where $K_1(y,\eta)$, $K_2(y,\eta)$, $K_3(y,\eta)$, $K_4(y,\eta)$, $g_1(y)$, $g_2(y)$ are known functions, and $K_1(y,\eta)$, $K_3(y,\eta)$ have a weak singularity $(\frac{1}{2})$, and $K_2(y,\eta)$, $K_4(y,\eta)$, $g_1(y)$, $g_2(y)$ are continuous functions.

Solving the system of equations (36), (37), we find the functions $\tau_3'(y)$, $\omega_{12}(1+y)$ and thus, the functions $\nu_3(y)$, $\omega_{32}(y)$, $u_1(x,\, y)$, $u_3(x,\, y)$.

Remark 1. The case when $-1 < \gamma < 0$, the problem is investigated by dividing the domain G_1 into n parts whose heights of the first $n-1$ domains are equal to $-\frac{b}{a}$, and the last – no more than $-\frac{b}{a}$. The problem is solved in each domain sequentially, similar to the case of $\gamma = -1$.

Remark 2. In [4, 11], a number of boundary value problems for more general equations of the third and fourth orders of parabolic-hyperbolic type in a domain with a single line of type change were considered.

Conclusion

This work presents the formulation and comprehensive analysis of a boundary value problem for a third-order parabolic-hyperbolic PDE within a geometrically intricate pentagonal domain. Through the development of a constructive method for the equation's solution, we have established its unique solvability. Our findings enrich the theoretical underpinnings of mixed-type equations and extend the toolkit for addressing boundary value problems in domains with complex geometries. This research not only advances our understanding of parabolic-hyperbolic equations of third order but also has potential implications for their application in modeling multifaceted physical systems and phenomena. Future studies may explore the application of these findings in practical scenarios and the investigation of similar problems in higher-dimensional spaces or with more complex boundary conditions.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Параболалық-гиперболалық типтегі үшінші ретті теңдеу үшін шекаралық шама есебін тұжырымдау және зерттеу туралы

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Мақалада параболалық және гиперболалық аймақтардан тұратын бесбұрышты аймақтағы параболалық-гиперболалық типтегі үшінші ретті ішінара туындылардағы теңдеудің жаңа шекті есебі зерттелген. Мұндай теңдеулер физика, инженерия және қаржы сияқты әртүрлі салалардағы күрделі физикалық құбылыстарды модельдеуде шешуші рөл атқарады, олардың аралас типтегі табиғатына орай динамиканың кең ауқымын инкапсуляциялау қабілетіне байланысты. Шешудің конструктивті тәсілін қолдана отырып, біз қойылған есептің біржақты шешілуін көрсетеміз. Бұл зерттеудің маңыздылығы математикалық физика мен сабақтас пәндердегі теориялық және қолданбалы зерттеулерге жаңа мүмкіндіктер ашатын күрделі геометриялық салалардағы жоғары ретті жартылай дифференциалдық теңдеулерді түсіну және шешу үшін математикалық негізді кеңейту болып табылады.

Кілт сөздер: параболалық-гиперболалық типтегі дифференциалдық теңдеулер, үшінші ретті параболалық-гиперболалық тип.

О постановке и исследовании краевой задачи для уравнения третьего порядка параболо-гиперболического типа

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В статье исследована новая краевая задача для уравнения в частных производных третьего порядка параболо-гиперболического типа в пятиугольной области, состоящей как из параболических, так и из гиперболических областей. Такие уравнения играют решающую роль в моделировании сложных физических явлений в различных областях, таких как физика, инженерия и финансы, благодаря их способности инкапсулировать широкий диапазон динамики из-за своей природы смешанного типа. Используя конструктивный подход к решению, мы демонстрируем однозначную разрешимость поставленной задачи. Значимость этого исследования заключается в расширении математической основы для понимания и решения смешанных уравнений в частных производных высшего порядка в сложных геометрических областях, что открывает новые возможности для теоретических и прикладных исследований в математической физике и смежных дисциплинах.

Ключевые слова: дифференциальные уравнения параболо-гиперболического типа, параболо-гиперболический тип третьего порядка.

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Bulletin of the Karaganda University. Mathematics Series, No. 2(114), 2024, pp. 147–164

https://doi.org/10.31489/2024M2/147-164 *Research article*

The problem with the missing Goursat condition at the boundary of the domain for a degenerate hyperbolic equation with a singular coefficient

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The work is devoted to the formulation and study of the solvability for a problem with missing conditions on the characteristic boundary of the domain and an analogue of the Frankl condition on the segment of the degeneracy for a hyperbolic equation. The difference between this problem and known local and nonlocal problems is that, firstly, a hyperbolic equation is taken with arbitrary positive power degeneracy and singular coefficients on the part of the boundary, and secondly, the characteristic boundary of the domain is arbitrarily divided into two pieces and the value of the desired function is set on the first piece, and the second piece is freed from the boundary condition and this missing Goursat condition is replaced by an analogue of the Frankl condition on the degeneracy segment, and the value of an unknown function on another characteristic boundary of the domain is also considered to be known. The conditions for the coefficients of the equation and the data of the formulated problem, ensuring the validity of the uniqueness theorem are found. The theorem of the existence of a solution to the problem is proved by reducing to the problem of solving a non-standard singular integral equation with a non-Fredholm integral operator in the non-characteristic part of the equation, the kernel of which has an isolated first-order singularity. Applying the Carleman regularization method to the received equation, the Wiener-Hopf integral equation is obtained. It is proved that the index of the Wiener-Hopf equation is zero, therefore it is uniquely reduced to the Fredholm integral equation of the second kind, the solvability of which follows from the uniqueness of the problem's solution.

Keywords: Hyperbolic equation degenerating at the boundary of the domain, missing Goursat condition, Frankl condition, singular coefficient, complete orthogonal system of functions, singular integral equation, Wiener-Hopf equation, index.

2020 Mathematics Subject Classification: 35L80, 35L81, 35L53.

Introduction

Many scientific and practical studies conducted in various fields of mathematics in most cases lead to the study of models of gas dynamics problems, the theory of infinitesimal bends of rotation surfaces, the instantaneous theory of shells and mathematical biology. The study of the fundamental laws of gas dynamics by solving boundary value problems for partial differential equations with singularities in coefficients is an urgent problem.

The development of the theory of degenerate hyperbolic and elliptic equations and mixed type equations originates from the well-known fundamental works of G. Darboux (1894), F. Tricomi (1923), E. Holmgren (1927) and S. Gellerstedt (1938).

After these works, the theory of boundary value problems for degenerate hyperbolic and mixed-type equations began to develop rapidly. E. Holmgren, S. Gellerstedt, F.I. Frankl, M. Keldysh, S.G. Mikhlin,

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The work was carried out with the support of the Fund of the Innovative Development Ministry of the Republic of Uzbekistan (grant No. Ф3-202009211).

Received: 18 December 2023; *Accepted:* 19 February 2024.

A.V. Bitsadze, K.I. Babenko, M.S. Salakhiddinov, A.M. Nakhushev, E.I. Moiseev, and many other scientists made significant contributions to the development of this theory. Degenerate hyperbolic equations with singular coefficients have the peculiarity that the well-posedness of the Cauchy problem does not always take place. The initial problem in the usual formulation may turn out to be unsolvable if the equation degenerates along a line that is both a characteristic or, the coefficients of the hyperbolic equation for the lower terms are singular [1]. Therefore, it is natural to consider a modified Cauchy problem when the initial conditions on the line of parabolic degeneracy are given with weight functions [2, 3].

It is relevant to study the well-posedness of non-standard problems for degenerate hyperbolic equations with singular coefficients. Especially if, in the formulated problems, local and non-local conditions are given in a non-standard form, in particular, the Goursat condition is set on some part of the characteristic; the Bitsadze-Samarsky condition is set on the boundary and parallel internal characteristic, a Frankl type condition is set on the line of degeneracy of the equation, then problems with such non-classical conditions are reduced to previously unexplored new non-standard singular equations, the singular and non-singular parts of the kernel of which are not reduced to each other by means of a fractional linear transformation.

Nowadays, there are many articles and books devoted to the theoretical and applied aspects of degenerate hyperbolic and mixed-type equations [1–14]. It should be noted that the bibliography does not pretend to be complete and mainly concerns issues close to this work.

The work is devoted to the study of the uniqueness and existence of a non-standard problem for a degenerate hyperbolic equation with singular coefficients in a domain bounded by two characteristics of a different family and a segment of the line of degeneration of the equation (characteristic triangle). The peculiarity of this problem is that part of the characteristic boundary of the domain is freed from the Goursat conditions, and Frankl-type conditions are set on the line of the equation's degeneracy.

The purpose of this work is to find conditions for the equation coefficients and the data of the problem, which ensure the validity of the theorems on the existence and uniqueness of the non-standard problem posed.

The work consists of an introduction, three sections and a conclusion.

The first section provides a description of the domain and a restriction on the equation coefficients for a degenerate hyperbolic equation. The statement of the main and auxiliary problems is given.

In the second section, the conditions for the equation coefficients and the data of the problem are found, ensuring the validity of the uniqueness theorem of the problem solution.

In the third section, the existence of a solution to the problem is proved by reducing a non-standard singular integral equation to a solution. Using the Carleman regularization method and the theory of the Wiener-Hopf integral equation, this equation is uniquely reduced to the Fredholm integral equation of the second kind, the solvability of which follows from the uniqueness theorem of the problem solution.

1 Problem formulation A

Let Ω^- be the characteristic triangle of the half-plane $y < 0$ bounded by characteristics AC_1 and BC_1 of the equation

$$-(-y)^m u_{xx} + u_{yy} + \alpha_0(-y)^{(m-2)/2} u_x + (\beta_0/y)u_y = 0,\ y < 0, \tag{1}$$

and segment AB, where $A(-1,0)$, $B(1,0)$, $C_1\left(0, -\left((m+2)/2\right)^{2/(m+2)}\right)$, m, α_0, β_0 are some constants satisfying conditions $m > 0, -m/2 < \beta_0 < 1$, $-(m+2)/2 < \alpha_0 < (m+2)/2$ [4–7].

Correctness of setting boundary value problems for equation (1) significantly depends on its numerical parameters α_0 and β_0 coefficients for the lower terms of the equation, on the parameter plane $\alpha_0 O \beta_0$

consider a triangle $A_0^*B_0^*C_0^*$ bounded by straight lines

$$A_0^*C_0^* : \beta_0 + \alpha_0 = -m/2;\ B_0^*C_0^* : \beta_0 - \alpha_0 = -m/2;\ A_0^*B_0^* : \beta_0 = 1.$$

Let $P(\alpha_0, \beta_0) \in \Delta A_0^*B_0^*C_0^*$, i.e. $0 < \alpha, \beta < 1,\ \alpha + \beta < 1$, where $\alpha = (m + 2(\beta_0 + \alpha_0))/2(m+2)$, $\beta = (m + 2(\beta_0 - \alpha_0))/2(m+2)$.

Denote by A_0 and B_0 intersection points of the characteristics AC_1 and BC_1 respectively with a characteristic coming from a point $E(c, 0)$, where $c \in J_0 = (-1, 1)$ is an interval of the axis $y = 0$.

Let the linear function $p(x) = \delta - kx$, where $k = (1-c)/(1+c),\ \ \delta = 2c/(1+c)$ display the set of points of the segment $[-1, c]$ on the set of points of the segment $[c, 1]$ and $p(-1) = 1,\ p(c) = c$.

In the Goursat problem, the carriers of boundary conditions are boundary characteristics AC_1 and BC_1.

This work is devoted to the study of the correctness of the problem in the domain Ω^-, for hyperbolic equation (1) degenerating at the boundary of the domain, when the boundary characteristic AC_1 of the domain Ω^- is arbitrarily divided into two pieces AA_0 and A_0C_1 and on the first piece $AA_0 \subset AC_1$ the value of the desired function is set, and the second piece $A_0C_1 \subset AC_1$ it is freed from the boundary condition and this missing Goursat condition is replaced by an analogue of the Frankl condition [8–12] on the degeneration segment AB.

Problem A. In the domain Ω^- it is required to find the function $u(x, y) \in C(\bar{\Omega}^-)$ satisfying the following conditions:

1) $u(x, y)$ is generalized solution to the equation (1) from the class R_1 [13].

2)
$$u(x, y)\ |_{BC_1} = \psi_1(x),\ 0 \le x \le 1, \tag{2}$$

3)
$$u(x, y)\ |_{AA_0} = \psi_2(x),\ -1 \le x \le (c-1)/2, \tag{3}$$

4)
$$u(x, 0) - \mu u(p(x), 0) = f(x),\ -1 \le x \le c, \tag{4}$$

where $\mu = const$, $\psi_1(x) \in C[0,1] \cap C^2(0,1)$, $\psi_2(x) \in C[-1,(c-1)/2] \cap C^2(-1,(c-1)/2)$, $f(x) \in C[-1,c] \cap C^2(-1,c)$, and $\psi_1(1) = 0,\ \psi_2(-1) = 0,\ f(c) = 0$.

Condition (3) is an incomplete condition of the Course, since it is set only on AA_0 part of characteristic AC_1.

Condition (4) is an analogue of Frankl condition [14] on the degeneracy segment AB.

By virtue of the designation $u(x, 0) = \tau(x)$ condition (4) we write in the form

$$\tau(x) - \mu\tau(p(x)) = f(x),\ x \in [-1, c]. \tag{4*}$$

Let Ω^+ be a symmetrical domain to the Ω^- with respect to the axis $y = 0$, lying in a half-plane $y > 0$ and let $\Omega = \Omega^- \cup \Omega^+ \cup AB$. The domain Ω^+ is bounded with characteristics AC_2 and BC_2 of the equation

$$-y^m u_{xx} + u_{yy} + \alpha_0 y^{(m-2)/2} u_x + (\beta_0/y) u_y = 0,\ y > 0, \tag{5}$$

where $C_2\left(0, (m+2)/2\right)^{2/(m+2)}$.

Note that if $u(x, y)$ is a solution to equation (1) in a half-plane $y < 0$, then $u(x, -y)$ is a solution to equation (5) in a half-plane $y > 0$. Due to this property of solutions to equations (1) and (5) in a symmetrical domain Ω we consider an auxiliary problem A^*.

Problem formulation A^*. It is required to find in the domain Ω the function $u(x, y) \in C(\bar{\Omega})$ satisfying conditions:

1) $u(x, y)$ is a generalized solution from the class R_1 in domains Ω^- and Ω^+;

2) $u(x,y)$ satisfies the condition

$$u(x,y)\mid_{BC_2}=\psi_1(x),\ 0\le x\le 1, \tag{6}$$

and conditions (3) and (4) of Problem A.

3) on the degeneracy segment $y=0,\ -1<x<1$, a conjugation condition takes place

$$\lim_{y\to -0}(-y)^{\beta_0}\frac{\partial u}{\partial y}=-\lim_{y\to +0}y^{\beta_0}\frac{\partial u}{\partial y}=\nu(x),\ x\in J_0, \tag{7}$$

moreover, these limits at $x\to\pm 1$ may have features of the order less than $1-\alpha-\beta$, where $\alpha+\beta=(m+2\beta_0)/(m+2)\in(0,1)$.

Let $u(x,y)$ is a solution to the problem A^*, we show that $u(x,y)\mid_{BC_1}=\psi(x)$. It is obvious from the form equations (1) and (5) that if $u(x,y)$ is a solution to equation (1) in the half-plane $y<0\,(y>0)$ then $u(x,-y)$ is a solution to equation (5) in the half-plane $y>0\,(y<0)$. Hence from the design of solutions (see below (9)) of equations (1) and (5) it can be seen that for symmetric with respect to the axis $y=0$ points $(x,y)\in\Omega^-$ and $(x,-y)\in\Omega^+$ the equality $u(x,y)=u(x,-y)$ takes place and by virtue of continuity of solutions, this equality is also preserved for points of characteristics BC_1 and BC_2 then by virtue of (2) $u(x,y)\mid_{BC_1}=u(x,-y)\mid_{BC_2}=\psi(x)$, where $y<0$, that is what needed to be shown.

Hence the solution to the problem A^*, in domain Ω^- will also be a solution to the A problem in the same domain Ω^-. Thus, the study of the problem A is reduced to solving the problem A^*.

2 *Uniqueness of the problem solution A^**

The solution to equation (1) in domains Ω^-, Ω^+ satisfying modified Cauchy conditions:

$$\lim_{y\to 0}u(x;y)=\tau(x),\ x\in\bar{J};\ \lim_{y\to\pm 0}|y|^{\beta_0}\frac{\partial u}{\partial y}=\mp\nu(x),\ x\in J_0, \tag{8}$$

has the form [14]

$$\begin{aligned}u(x,y)=\gamma_1\int_{-1}^{1}\tau\left[x+\frac{2t}{m+2}|y|^{\frac{m+2}{2}}\right](1+t)^{\beta-1}(1-t)^{\alpha-1}dt+\gamma_2|y|^{1-\beta_0}\times\\ \times\int_{-1}^{1}\nu\left[x+\frac{2t}{m+2}|y|^{\frac{m+2}{2}}\right](1+t)^{-\alpha}(1-t)^{-\beta}dt,\end{aligned} \tag{9}$$

where

$$\gamma_1=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}2^{1-\alpha-\beta},\quad \gamma_2=-\frac{\Gamma(2-\alpha-\beta)}{(1-\beta_0)\Gamma(1-\alpha)\Gamma(1-\beta)}2^{\alpha+\beta-1}.$$

By virtue of (9) from boundary condition (6) (taking place in domain Ω^+) we have

$$\gamma_1\left(\frac{1-x}{2}\right)^{1-\alpha-\beta}\int_x^1\frac{\tau(s)(1-s)^{\alpha-1}}{(s-x)^{1-\beta}}ds-\gamma_2\left(\frac{m+2}{2}\right)^{1-\alpha-\beta}\times$$

$$\times\int_x^1\frac{\nu(s)(1-s)^{-\beta}}{(s-x)^{\alpha}}ds=\Psi_1(x),\quad x\in(-1,1),$$

or

$$\nu(x)=-\gamma D_{x,1}^{1-\alpha-\beta}\tau(x)+\Psi_1(x),\ x\in(-1,1), \tag{10}$$

where $D^{l}_{x,1}$ is a fractional differentiation operator.

$$\gamma = \frac{2\Gamma(1-\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(1-\alpha-\beta)} \left(\frac{m+2}{4}\right)^{\alpha+\beta},$$

$$\Psi_1(x) = -\frac{(2/(m+2))^{1-\alpha-\beta}}{\gamma_2\Gamma(1-\alpha)}(1-x)^{\beta} D^{1-\alpha}_{x,1}\psi_1\left(\frac{1+x}{2}\right).$$

Relation (10) is the first functional relationship between unknown functions $\tau(x)$ and $\nu(x)$ brought to the interval $(-1,1)$ from the domain Ω^+. Note, that relation (10) is valid for the entire interval $(-1,1)$.

Now by virtue of (9) from boundary condition (3)(taking place in the domain Ω^-) we obtain

$$\nu(x) = \gamma D^{1-\alpha-\beta}_{-1,x}\tau(x) + \Psi_2(x),\ x \in (-1,c), \tag{11}$$

where

$$\Psi_2(x) = \frac{(2/(m+2))^{1-\alpha-\beta}}{\gamma_2\Gamma(1-\beta)}(1+x)^{\alpha} D^{1-\beta}_{-1,x}\psi_2\left(\frac{x-1}{2}\right).$$

Relation (11) is the second functional relationship between unknown functions $\tau(x)$ and $\nu(x)$ brought to the interval $(-1,c)$ from the domain Ω^-.

Theorem 1. The problem A^* when the condition

$$k^{\alpha+\beta} < \mu^2 \tag{12}$$

is met, can have no more than one solution.

Proof. 1^0. Using (10) ($c\ \Psi_1(x) \equiv 0$), we prove that

$$J = \int_{-1}^{1} \tau(x)\nu(x)dx \le 0. \tag{13}$$

We calculate

$$J = -\gamma \int_{-1}^{1} \tau(x)\left(D^{1-\alpha-\beta}_{x,1}\tau(x)\right)dx = \frac{\gamma}{\Gamma(\alpha+\beta)}\int_{-1}^{1}\tau(x)\times \\ \times\left(\frac{d}{dx}\int_x^1 \frac{\tau(t)dt}{(t-x)^{1-\alpha-\beta}}\right)dx. \tag{14}$$

Let

$$\tau(x) = \int_x^1 \frac{\tau_1(s)ds}{(s-x)^{\alpha+\beta}},\ x \in (-1,1), \tag{15}$$

where $\tau_1(x) \in C(\bar{J}) \cap C^2(J), \tau_1(1) = \tau_1^{'}(1) = 0$.

By virtue of (15) equality (14) has the form

$$J = \frac{\gamma}{\Gamma(\alpha+\beta)}\int_{-1}^{1}\tau(x)\left(\frac{d}{dx}\int_x^1 \frac{dt}{(t-x)^{1-\alpha-\beta}}\int_t^1 \frac{\tau_1(s)ds}{(s-t)^{\alpha+\beta}}\right)dx. \tag{16}$$

It is not difficult to prove that

$$\frac{d}{dx}\int_x^1 \frac{dt}{(t-x)^{1-\alpha-\beta}}\int_t^1 \frac{\tau_1(s)ds}{(s-t)^{\alpha+\beta}} = -\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta)\tau_1(x). \tag{17}$$

Therefore, taking into account (17) we write equality (16) in the form

$$J = -\gamma\Gamma(1-\alpha-\beta)\int_{-1}^{1}\tau(x)\tau_1(x)dx. \tag{18}$$

Now by virtue (15) we transform (18) to the form

$$J = -\gamma\Gamma(1-\alpha-\beta)\int_{-1}^{1}\tau_1(x)dx\int_{x}^{1}\frac{\tau_1(s)ds}{(s-x)^{\alpha+\beta}}. \tag{19}$$

Here, changing the order of integration, we have

$$J = -\gamma\Gamma(1-\alpha-\beta)\int_{-1}^{1}\tau_1(s)ds\int_{-1}^{s}\frac{\tau_1(x)dx}{(s-x)^{\alpha+\beta}}. \tag{20}$$

In (19) swapping integration variables s and x, we have

$$J = -\gamma\Gamma(1-\alpha-\beta)\int_{-1}^{1}\tau_1(s)ds\int_{s}^{1}\frac{\tau_1(x)dx}{(x-s)^{\alpha+\beta}}. \tag{21}$$

Now summing up (20) and (21), we obtain

$$J = -\frac{\gamma\Gamma(1-\alpha-\beta)}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{\tau_1(s)\tau_1(x)}{|s-x|^{\alpha+\beta}}dxds. \tag{22}$$

Let us now use the following well-known formula for the function $\Gamma(z)$

$$\int_{0}^{\infty}t^{z-1}cos(kt)dt = \frac{\Gamma(z)}{k^z}cos\left(\frac{z\pi}{2}\right),\ k>0,\ 0<z<1. \tag{23}$$

Let in (23) $k=|s-x|,\ z=\alpha+\beta$ then from (23) we have

$$\frac{1}{|s-x|^{\alpha+\beta}} = \frac{1}{\Gamma(\alpha+\beta)cos\left((\alpha+\beta)\pi/2\right)}\int_{0}^{\infty}\xi^{\alpha+\beta-1}cos\left((s-x)\xi\right)d\xi. \tag{24}$$

By virtue (24) we write equality (22) in the form

$$\begin{gathered} J = -\frac{\gamma\Gamma(1-\alpha-\beta)}{2\Gamma(\alpha+\beta)cos\left((\alpha+\beta)\pi/2\right)}\int_{0}^{\infty}\xi^{\alpha+\beta-1}d\xi\int_{-1}^{1}\int_{-1}^{1}\tau_1(s)\cdot \\ \cdot\tau_1(x)cos\left((s-x)\xi\right)dxds = -\frac{\gamma\Gamma(1-\alpha-\beta)}{2\Gamma(\alpha+\beta)cos\left((\alpha+\beta)\pi/2\right)}\int_{0}^{\infty}\xi^{\alpha+\beta-1}\cdot \\ \cdot\left\{\left[\int_{-1}^{1}\tau_1(t)\cos(t\xi)dt\right]^2+\left[\int_{-1}^{1}\tau_1(t)\sin(t\xi)dt\right]^2\right\}d\xi. \end{gathered} \tag{25}$$

Thus, by virtue (25) we obtain inequality (13).
2^0. Now using (11) ($\Psi_2(x)\equiv 0$) and condition (4*) we show that integral (13) is not negative, i.e.

$$J = \int_{-1}^{1}\tau(x)\nu(x)dx \geq 0. \tag{26}$$

Indeed

$$J = \int_{-1}^{1}\tau(x)\nu(x)dx = \int_{-1}^{c}\tau(x)\nu(x)dx + \int_{c}^{1}\tau(x)\nu(x)dx, \tag{27}$$

here we transform the second integral of the right-hand part (27) i.e.

$$J_1 = \int_c^1 \tau(x)\nu(x)dx. \tag{28}$$

In (28) by replacing the variable integration $x = p(t) = \delta - kt$, we get

$$J_1 = k\int_{-1}^c \tau\left(p(t)\right)\nu\left(p(t)\right)dt. \tag{29}$$

Now we will find $\nu\left(p(x)\right)$, for this purpose, we use relation (10) which is the case for the entire interval $J_0 = (-1, 1)$ in particular for $x \in (-1, 1)$:

$$\nu(x) = -\gamma D_{x,1}^{1-\alpha-\beta}\tau(x) = \frac{\gamma}{\Gamma(\alpha+\beta)}\frac{d}{dx}\int_x^1 \frac{\tau(t)dt}{(t-x)^{1-\alpha-\beta}},\ x \in (c, 1).$$

Here, firstly, performing the integration operation in parts, then, performing the differentiation operation, we have

$$\nu(x) = \frac{\gamma}{\Gamma(\alpha+\beta)}\int_x^1 \frac{\tau'(t)dt}{(t-x)^{1-\alpha-\beta}},\ x \in (c, 1). \tag{30}$$

In (30) by replacing the variable $x \in (c, 1)$ to $p(x)$ (where $p(x) \in (c, 1)$, and an argument $x \in (-1, c)$) we obtain

$$\nu(p(x)) = \frac{\gamma}{\Gamma(\alpha+\beta)}\int_{p(x)}^1 \frac{\tau'(t)dt}{(t-p(x))^{1-\alpha-\beta}},\ x \in (-1, c). \tag{31}$$

Now, in (31) by replacing the variable integration $t = p(s)$, taking into account the condition (4*) ($f(x) \equiv 0$): $\tau(x) = \mu\tau\left(p(x)\right)$, $\tau'(x) = -\mu\tau'\left(p(x)\right)$, we calculate

$$\begin{aligned}\nu(p(x)) = -\frac{\gamma k^{\alpha+\beta-1}}{\mu\Gamma(\alpha+\beta)}\int_{-1}^x \frac{\tau'(t)dt}{(x-t)^{1-\alpha-\beta}} = -\frac{\gamma k^{\alpha+\beta-1}}{\mu\Gamma(\alpha+\beta)}\lim_{\varepsilon\to 0}\int_{-1}^{x-\varepsilon}\frac{d\tau(t)}{(x-t)^{1-\alpha-\beta}} = \\ = -\frac{\gamma k^{\alpha+\beta-1}}{\mu\Gamma(\alpha+\beta)}\lim_{\varepsilon\to 0}\left[\frac{\tau(t)}{(x-t)^{1-\alpha-\beta}}\Big|_{-1}^{x-\varepsilon} - (1-\alpha-\beta)\int_{-1}^{x-\varepsilon}\frac{\tau(t)dt}{(x-t)^{2-\alpha-\beta}}\right] = \\ = -\frac{\gamma k^{\alpha+\beta-1}}{\mu\Gamma(\alpha+\beta)}\lim_{\varepsilon\to 0}\left[\frac{d}{dx}\int_{-1}^{x-\varepsilon}\frac{\tau(t)dt}{(x-t)^{1-\alpha-\beta}}\right].\end{aligned} \tag{32}$$

In (32) moving to the limit at $\varepsilon \to 0$, we will have

$$\nu\left(p(x)\right) = -\frac{\gamma k^{\alpha+\beta-1}}{\mu}D_{-1,x}^{1-\alpha-\beta}\tau(x),\ x \in (-1, c),$$

due to this equality, relation (29) is written as

$$J_1 = -\frac{\gamma k^{\alpha+\beta}}{\mu}\int_{-1}^c \tau\left(p(x)\right)D_{-1,x}^{1-\alpha-\beta}\tau(x)dx. \tag{33}$$

Now, taking into account (4*) (c $f(x) \equiv 0$): $\tau\left(p(x)\right) = \tau(x)/\mu$ and relations (33) equality (27) has the form

$$\begin{aligned}J = \int_{-1}^1 \tau(x)\nu(x)dx = \gamma\left(1 - \frac{k^{\alpha+\beta}}{\mu^2}\right)\int_{-1}^c \tau(x)\left(D_{-1,x}^{1-\alpha-\beta}\tau(x)\right)dx = \\ = \frac{\gamma\Gamma(1-\alpha-\beta)a^{2-(\alpha+\beta)}}{2\Gamma(\alpha+\beta)cos\left((\alpha+\beta)\pi/2\right)}\left(1 - \frac{k^{\alpha+\beta}}{\mu^2}\right)\int_0^\infty \xi^{\alpha+\beta-1}\times \\ \times\left\{\left[\int_{-1}^1 \tau_2(at-b)\cos(t\xi)dt\right]^2 + \left[\int_{-1}^1 \tau_2(at-b)\sin(t\xi)dt\right]^2\right\}d\xi,\end{aligned} \tag{34}$$

where

$$\tau(x) = \int_{-1}^{x} \frac{\tau_2(s)ds}{(x-s)^{\alpha+\beta}},\ x \in (-1, c),$$

$\tau_2(x) \in C[-1, c] \cap C^2(-1, c),\ \tau_2(-1) = \tau_2'(-1) = 0$, taking into account (12) from (34) it follows (26).

Therefore by virtue of inequalities (13) and (26) we have

$$J = \int_{-1}^{1} \tau(x)\nu(x)dx = 0.$$

Thus, the right-hand side of (25) is equal to zero, but both terms of the integral expression in (25) are non-negative, therefore they are also equal to zero:

$$\int_{-1}^{1} \tau_1(t)cos(t\xi)dt \equiv 0,\ \int_{-1}^{1} \tau_1(t)sin(t\xi)dt \equiv 0, \tag{35}$$

for all $\xi \in [0, +\infty]$ and in particular for $\xi = k\pi, k = 0, 1, 2, \ldots$, for such values ξ trigonometric systems of functions $cos(t\xi)$ and $sin(t\xi)$ form a complete orthogonal system of functions in $L_2[-1, 1]$. Therefore, in (35) $\tau_1(t) \equiv 0$ almost everywhere on $[-1, 1]$, but by virtue of continuity of the function $\tau_1(x)$ on $[-1, 1]$ it follows, that $\tau_1(x) \equiv 0$ everywhere $\forall x \in [-1, 1]$, hence by virtue of (15) we conclude that $\tau(x) \equiv 0$, $\forall x \in [-1, 1]$. Hence by virtue of (10) (с $\Psi_1(x) \equiv 0$) and also it follows, that $\nu(x) \equiv 0, \forall x \in (-1, 1)$. Now by virtue of (7), restoring the solution of the problem A^* as solutions of modified Cauchy problem with zero modified initial Cauchy data (8) (с $\tau(x) \equiv 0, \nu(x) \equiv 0$) according to the Darboux formula (9) we obtain $u(x, y) \equiv 0$ в $\bar{\Omega}$. Theorem 1 is proved.

3 The existence of a solution to the problem A^*

Theorem 2. Let for the numerical parameters of problem A^* inequality (12) be valid

$$\frac{k^{(1-2\theta)/2} \sin(\theta\pi) \left|\ln k\right|}{\mu} < 1, \tag{36}$$

where $2\theta = 1 - \alpha - \beta$, then the problem A^* is unambiguously solvable.

Note that the set of numerical parameters of the problem A^*, satisfying inequalities (12) and (36) is non-empty. Indeed, if we suppose $c > 0$, i.e. $k > 1$ and $\mu > 1$ then inequality (12) holds.

By virtue of (12) $\left(k^{\alpha+\beta} < \mu^2\right)$ taking into account $2\theta = 1 - \alpha - \beta$ from (36), we have

$$\frac{k^{(1-2\theta)/2} \sin(\theta\pi) \left|\ln k\right|}{\mu} = \frac{k^{(1-2\theta)/2} \cos(((\alpha+\beta)\pi)/2) \left|\ln k\right|}{\mu} <$$

$$< \frac{\mu \cos(((\alpha+\beta)\pi)/2) \left|\ln k\right|}{\mu} < \left|\ln k\right| = \left|\ln \frac{1-c}{1+c}\right| < 1,$$

from here it is obvious that if $c \in (0, (e-1)/(1+e)$, then inequality (36) holds.

Thus,the set of numerical parameters of the problem A^* is nonempty, since inequalities (12) and (36) holds for the values of numerical parameters $c \in (0, (e-1)/(1+e)$ and $\mu > 1$.

Proof of Theorem 2.

3.1 Derivation of the singular integral equation

From functional relations (10) and (11) excluding $\nu(x)$, we obtain

$$D_{-1,x}^{1-\alpha-\beta}\tau(x)+D_{x,1}^{1-\alpha-\beta}\tau(x)=\frac{1}{\gamma}\left(\Psi_1(x)-\Psi_2(x)\right),\ x\in(-1,c). \tag{37}$$

Applying the fractional integration operator $D_{-1,x}^{\alpha+\beta-1}$ to equality (37) taking into account $\tau(-1)=0$ and identities

$$D_{-1,x}^{1-\alpha-\beta}D_{-1,x}^{\alpha+\beta-1}\tau(x)=\tau(x),$$

$$D_{-1,x}^{1-\alpha-\beta}D_{x,1}^{\alpha+\beta-1}\tau(x)=cos\left((1-\alpha-\beta)\pi\right)\tau(x)-\frac{sin\left((1-\alpha-\beta)\pi\right)}{\pi}\times$$

$$\times\int_{-1}^{1}\left(\frac{1+x}{1+t}\right)^{1-\alpha-\beta}\frac{\tau(t)dt}{t-x},$$

equation (37) is written in the form

$$\tau(x)-\lambda\int_{-1}^{1}\left(\frac{1+x}{1+t}\right)^{1-\alpha-\beta}\frac{\tau(t)dt}{t-x}=F(x),\ x\in(-1,c), \tag{38}$$

where

$$\lambda=\frac{sin((\alpha+\beta)\pi)}{\pi\left(1-cos(\alpha+\beta)\pi\right)},\quad F(x)=\frac{1}{\gamma\left(1-cos(\alpha+\beta)\pi\right)}\left(\Psi_1(x)-\Psi_2(x)\right).$$

Note that in (38) $x\in(-1,c)$, therefore equation (38) has a singular feature only when the integration variable is $t\in(-1,c)$. In order to highlight the singular part of equation (38) integration interval $(-1,1)$ divide it into two intervals $(-1,c)$ and $(c,1)$ and write (38) in the form

$$\tau(x)-\lambda\int_{-1}^{c}\left(\frac{1+x}{1+t}\right)^{1-\alpha-\beta}\frac{\tau(t)dt}{t-x}-\lambda\int_{c}^{1}\left(\frac{1+t}{1+x}\right)^{1-\alpha-\beta}\frac{\tau(t)dt}{t-x}=F(x),\ x\in(-1,c). \tag{39}$$

In the second integral of the left-hand side of (39), by replacing the integration variable $t=p(s)$, $dt=-kds,\ p(-1)=1,\ p(c)=c$, we obtain

$$\begin{aligned}\tau(x)-\lambda\int_{-1}^{c}\left(\frac{1+x}{1+t}\right)^{1-\alpha-\beta}\frac{\tau(t)dt}{t-x}-\lambda k\int_{-1}^{c}\left(\frac{1+p(s)}{1+x}\right)^{1-\alpha-\beta}\times\\ \times\frac{\tau(p(s))ds}{p(s)-x}=F(x),\ x\in(-1,c).\end{aligned} \tag{40}$$

By virtue of condition (4*) equation (40) is written in the form

$$\begin{aligned}\tau(x)-\lambda\int_{-1}^{c}\left(\frac{1+x}{1+t}\right)^{1-\alpha-\beta}\frac{\tau(t)dt}{t-x}=\frac{\lambda k}{\mu}\int_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{1-\alpha-\beta}\\ \frac{\tau(s)ds}{p(s)-x}+F_1(x),\ x\in(-1,c),\end{aligned} \tag{41}$$

$$F_1(x)=F(x)-\frac{\lambda k}{\mu}\int_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{1-\alpha-\beta}\frac{f(s)ds}{p(s)-x}.$$

The singular integral equation (41) is typical that the kernel of the right-hand side of the equation has an isolated first-order singularity for $s=c,x=c$, hence the integral operator of the right-hand side of (41) is not a Fredholm operator.

Temporarily considering the right-hand side of the equation (41) as a known function, we write it as

$$\tau(x)-\lambda\int_{-1}^{c}\left(\frac{1+x}{1+t}\right)^{1-\alpha-\beta}\frac{\tau(t)dt}{t-x}=g_0(x),\ x\in(-1,c), \tag{42}$$

where

$$g_0(x)=\frac{\lambda k}{\mu}\int_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{1-\alpha-\beta}\frac{\tau(s)ds}{p(s)-x}+F_1(x). \tag{43}$$

Theorem 3. If $g_0(x)$ satisfies the Helder condition for $x\in(-1,c)$ and $g_0(x)\in L_p(-1,c), p>1$, then the solution to equation (42) in the class of Helder functions H, in which the function $(1+x)^{\alpha+\beta-1}\tau(x)$ can be unlimited at the left end of the interval $(-1,c)$ and bounded at the right end of the interval $(-1,c)$ expressed by the formula

$$\tau(x)=\frac{g_0(x)}{1+\lambda^2\pi^2}+\frac{\lambda}{1+\lambda^2\pi^2}\int_{-1}^{c}\left(\frac{(c-x)(1+x)}{(c-t)(1+t)}\right)^{\theta}\frac{g_0(t)dt}{t-x}. \tag{44}$$

The proof of Theorem 3 is identical to the proof of a similar theorem in work [13].

3.2 Derivation and investigation of the Wiener-Hopf integral equation

Substituting the expression for $g_0(x)$ from (43) into (44), we have

$$\begin{aligned}\tau(x)=\lambda_1\int_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{2\theta}\frac{\tau(s)ds}{p(s)-x}+\lambda\lambda_1\int_{-1}^{c}\left(\frac{(c-x)(1+x)}{(c-t)(1+t)}\right)^{\theta}\frac{dt}{t-x}\times\\ \times\int_{-1}^{c}\left(\frac{1+t}{1+p(s)}\right)^{2\theta}\frac{\tau(s)ds}{(p(s)-t)}+F_2(x),\end{aligned} \tag{45}$$

where

$$\lambda_1=\frac{\lambda k}{\mu(1+\lambda^2\pi^2)},$$

$$F_2(x)=\frac{F_1(x)}{1+\lambda^2\mu^2}+\frac{\lambda}{1+\lambda^2\pi^2}\int_{-1}^{c}\left(\frac{(c-x)(1+x)}{(c-t)(1+t)}\right)^{\theta}\frac{F_1(t)dt}{t-x}.$$

In the double integral of equation (45) changing the order of integration, we have

$$\begin{aligned}\tau(x)=\lambda_1\int_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{2\theta}\frac{\tau(s)ds}{p(s)-x}+\lambda\lambda_1\times\\ \times\int_{-1}^{c}\frac{(c-x)^{\theta}(1+x)^{\theta}\tau(s)ds}{(1+p(s))^{2\theta}}\int_{-1}^{c}\left(\frac{1+t}{c-t}\right)^{\theta}\frac{dt}{(t-x)(p(s)-t)}+F_2(x),\ x\in(-1,c).\end{aligned} \tag{46}$$

Calculate the internal integral in (46)

$$A(x,s)=\int_{-1}^{c}\left(\frac{1+t}{c-t}\right)^{\theta}\frac{dt}{(t-x)(p(s)-t)}, \tag{47}$$

to do this, we decompose the rational multiplier of the integrand into simple fractions

$$\frac{1}{(t-x)(p(s)-t)}=\left(\frac{1}{t-x}+\frac{1}{p(s)-t}\right)\cdot\frac{1}{p(s)-x},$$

then (47) has the form

$$A(x,s)=\frac{1}{p(s)-x}\left[\int_{-1}^{c}\left(\frac{1+t}{c-t}\right)^{\theta}\frac{dt}{t-x}+\int_{-1}^{c}\left(\frac{1+t}{c-t}\right)^{\theta}\frac{dt}{p(s)-t}\right]= \\ =\frac{1}{p(s)-x}\left[A_1(x)+A_2(s)\right]. \tag{48}$$

We calculate $A_1(x)$ by the formula

$$\int_a^b\frac{(x-a)^{\alpha-1}(b-x)^{\beta-1}}{x-y}dx=\frac{\pi ctg(\beta\pi)}{(y-a)^{1-\alpha}(b-y)^{1-\beta}}- \\ -(b-a)^{\alpha+\beta-2}B(\alpha,\beta-1)F\left(1,2-\alpha-\beta,2-\beta;\frac{b-y}{b-a}\right).$$

Here $a=-1, b=c; \alpha-1=\theta, \alpha=1+\theta; \beta-1=-\theta, \beta=1-\theta$.

Thus,

$$A_1(x)=\int_{-1}^{c}\frac{(1+t)^{\theta}(c-t)^{-\theta}}{t-x}dt=(1+x)^{\theta}(c-x)^{-\theta}\pi ctg(1-\theta)\pi- \\ -(1+c)^{\theta}B(1+\theta,-\theta)F\left(1,\theta,1+\theta;\frac{c-x}{2}\right)= \\ =-\pi ctg(\theta\pi)\left(\frac{1+x}{c-x}\right)^{\theta}+\frac{\pi}{sin(\theta\pi)}. \tag{49}$$

Now we calculate

$$A_2(s)=\int_{-1}^{c}\left(\frac{1+t}{c-t}\right)^{\theta}\frac{dt}{p(s)-t}.$$

Here we will replace the variable integration $t=-1+(1+c)\sigma$ and using the integral representation of the hypergeometric function, we have

$$A_2(s)=\frac{1+c}{1+\delta-ks}\frac{\Gamma(1+\theta)\Gamma(1-\theta)}{\Gamma(2)}F\left(1+\theta,1,2;\frac{1+c}{1-\delta-ks}\right).$$

Here, applying the autotransformation formula

$$F(a,b,c;x)=(1-x)^{c-a-b}F(c-a,c-b,c;x),$$

we have

$$A_2(s)=\frac{(1+c)\Gamma(1+\theta)\Gamma(1-\theta)}{1+\delta-ks}\left(\frac{\delta-ks-c}{1+\delta-ks}\right)^{-\theta}\cdot F\left(1-\theta,1,2;\frac{1+c}{1-\delta-ks}\right).$$

Next, using the formula

$$F(a,b,c;z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}z^{-a}F\left(a,a-c+1,a+b-c+1;\frac{z+1}{z}\right)+ \\ +\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}z^{a-c}(1-z)^{c-a-b}F\left(c-a,1-a,c-a-b+1;\frac{z+1}{z}\right),$$

we have

$$A_2(s)=\frac{(1+c)\Gamma(1+\theta)\Gamma(1-\theta)}{1+\delta-ks}\left(\frac{1+\delta-ks}{\delta-ks-c}\right)^{\theta}\times$$

$$\times\left[\frac{\Gamma(2)\Gamma(\theta)}{\Gamma(1+\theta)\Gamma(1)}\left(\frac{1+c}{1+\delta-ks}\right)^{\theta-1}F\left(1-\theta,-\theta,1-\theta;\frac{c-\delta+ks}{1+c}\right)+\right.$$

$$\left.+\frac{\Gamma(2)\Gamma(-\theta)}{\Gamma(1-\theta)\Gamma(1)}\left(\frac{1+c}{1+\delta-ks}\right)^{-\theta-1}\left(\frac{\delta-ks-c}{1+\delta-ks}\right)^{\theta}\cdot F\left(1+\theta,\theta,1+\theta;\frac{c-\delta+ks}{1+c}\right)\right].$$

Here, taking into account the equality

$$F(a,c,c;x)=(1-x)^{-a},$$

we have

$$A_2(s)=\frac{\pi}{sin(\theta\pi)}\left(\frac{1+\delta-ks}{\delta-ks-c}\right)^{\theta}-\frac{\pi}{sin(\theta\pi)}. \tag{50}$$

Now substituting the expressions for $A_1(x)$ and $A_2(s)$ from (49) and (50) into (48) respectively, we obtain

$$\begin{gathered}A(x,s)=\frac{1}{p(s)-x}\left[A_1(x)+A_2(s)\right]=\frac{1}{p(s)-x}\times\\ \times\left[-\pi ctg(\theta\pi)\left(\frac{1+x}{c-x}\right)^{\theta}+\frac{\pi}{sin(\theta\pi)}\left(\frac{1+\delta-ks}{\delta-ks-c}\right)^{\theta}\right].\end{gathered} \tag{51}$$

By virtue (51) the equation (46) is transformed to the form

$$\begin{gathered}\tau(x)=\lambda_1(1-\lambda\pi ctg(\theta\pi))\int_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{2\theta}\frac{\tau(s)ds}{p(s)-x}+\\ +\lambda\lambda_1\frac{\pi}{sin(\theta\pi)}\int_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{\theta}\left(\frac{c-x}{p(s)-c}\right)^{\theta}\frac{\tau(s)ds}{p(s)-x}+F_2(x).\end{gathered} \tag{52}$$

By virtue of the identity $1-\lambda\pi ctg(\theta\pi)=0$, the equation (52) has the form

$$\tau(x)=\frac{\lambda\lambda_1\pi}{sin(\theta\pi)}\int_{-1}^{c}\left(\frac{1+x}{1+p(s)}\right)^{\theta}\left(\frac{c-x}{p(s)-c}\right)^{\theta}\frac{\tau(s)ds}{p(s)-x}+F_2(x),\ x\in(-1,c). \tag{53}$$

Thus, by virtue of the identities

$$p(s)-c=k(c-s),\ p(s)-x=k(c-s)+c-x,$$

equation (53) is written in the form

$$\tau(x)=\frac{\lambda\lambda_1\pi}{sin(\theta\pi)\cdot k^{\theta}}\int_{-1}^{c}\left(\frac{c-x}{c-s}\right)^{\theta}\frac{\tau(s)ds}{k(c-s)+c-x}+R_1[\tau(x)]+F_2(x),\ x\in(-1,c), \tag{54}$$

where $R_1[\tau(x)]=\frac{\lambda\lambda_1\pi}{sin(\theta\pi)k^{\theta}}\int_{-1}^{c}\left[\left(\frac{1+x}{1+p(s)}\right)^{\theta}-1\right]\frac{\tau(s)ds}{k(c-s)+c-x}$ is a regular operator.

In equation (54) we make substitutions $s=c-(1+c)e^{-t}$, $x=c-(1+c)e^{-y}$, where $t\in[0,+\infty)$, $y\in[0,+\infty)$ and introducing notations

$$\rho(y)=\tau[c-(1+c)e^{-y}]e^{(\theta-\frac{1}{2})y},$$

we write equation (54) in the form

$$\rho(y)=\frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty}K(y-t)\rho(t)dt+R_2[\rho]+F_3(y), \tag{55}$$

where

$$K(x) = \frac{\sqrt{2\pi^3}\lambda\lambda_1}{sin(\theta\pi)k^{\theta}(ke^{x/2}+e^{-x/2})},\ F_3(y) = F_2[c-(1+e)e^{-y}]e^{(\theta-\frac{1}{2})y}.$$

$R_2[\rho] = R_1[\tau]e^{(\theta-\frac{1}{2})y}$ is a regular operator. Note that, since $2\theta = 1-\alpha-\beta$ then the following inequality holds: $\theta - 1/2 < 0$. Equation (55) is the Wiener-Hopf integral equation [15]. Using the Fourier transform, like the well-known characteristic special integral equation with the Cauchy kernel, This equation is reduced to the Riemann boundary value problem and thereby it is solved in quadratures. Fredholm's theorems for integral equations of the convolution type will be valid only in one case, when the index of these equations is equal to zero.

The index χ of equation (55) will be the index of the expression $1-K^{\wedge}(x)$ with the reverse sign, i.e. $\chi = -Ind(1-K^{\wedge}(x))$, here [15]

$$K^{\wedge}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}K(t)e^{ixt}dt =$$

$$= \frac{\lambda\lambda_1\pi}{k^{\theta}sin(\pi\theta)}\int_{-\infty}^{+\infty}\frac{e^{ixt}dt}{ke^{t/2}+e^{-t/2}} = \frac{\lambda\lambda_1\pi}{k^{\theta}sin(\theta\pi)}\cdot\frac{\pi e^{-ix}lnk}{\sqrt{k}\cdot ch(\pi x)} =$$

$$= \frac{\lambda\lambda_1\pi^2 lnx}{k^{1/2+\theta}sin(\theta\pi)}\frac{e^{-ix}}{ch(\pi x)} = A^*(x) - iB^*(x),$$

where

$$A^*(x) = \frac{\lambda\lambda_1\pi^2 lnk}{k^{1/2+\theta}sin(\theta\pi)}\frac{cosx}{ch(\pi x)};\, B^*(x) = i\frac{\lambda\lambda_1\pi^2 lnk}{k^{1/2+\theta}sin(\theta\pi)}\frac{sinx}{ch(\pi x)}. \tag{56}$$

From (56) it can be seen that

$$\left.\begin{matrix}|A^*(x)|\\|B^*(x)|\end{matrix}\right\} \leq \frac{\lambda\lambda_1\pi^2|lnk|}{k^{1/2+\theta}sin(\theta\pi)}\cdot\frac{1}{ch(\pi x)},$$

and $A^*(x) = O\left(1/ch(\pi x)\right), B^*(x) = O\left(1/ch(\pi x)\right)$ for large enough ones $|x|$. Hence, by virtue of condition (36) of Theorem 2 it follows that

$$\left.\begin{matrix}|A^*(x)|\\|B^*(x)|\end{matrix}\right\} \leq \frac{\lambda\lambda_1\pi^2|lnk|}{k^{1/2+\theta}sin|\theta\pi|} = \frac{k^{1/2-\theta}sin(\theta\pi)|lnk|}{\mu} < 1.$$

Hence

$$Re(1-K^{\wedge}(x)) > 0. \tag{57}$$

Changing the argument of a complex-valued function $1-K^{\wedge}(x)$ on the real axis, expressed in full revolutions and taken with the reverse sign [15] taking into account the inequality (57) the index χ of equation (55) is equal to

$$x = -Ind(1-K^{\wedge}(x)) = -\frac{1}{2\pi}[arg(1-K^{\wedge}(x))]_{-\infty}^{+\infty} =$$

$$= -\frac{1}{2\pi}\left[arctg\frac{Im(1-K^{\wedge}(x))}{Re(1-K^{\wedge}(x))}\right]_{-\infty}^{+\infty} = -\frac{1}{2\pi}\left[arctg\frac{B^*(x))}{1-A^*(x)}\right]_{-\infty}^{+\infty} =$$

$$= -\frac{1}{2\pi}\left[arctg\frac{0}{1} - arctg\frac{0}{1}\right] = 0,$$

since $A^*(\pm\infty) = 0, B^*(\pm\infty) = 0$. Consequently, equation (55) is uniquely reduced to the Fredholm integral equation of the second kind, the unambiguous solvability of which follows from the uniqueness of the solution of the problem A^*. Theorem 2 is proved.

Conclusion

The paper investigates the issues of unique solvability for one class of problems in a non-standard formulation for a degenerate hyperbolic equation with singular coefficients (1) in a bounded domain.

For equation (1), when the conditions $0 < \alpha,\ \beta < 1,\ \alpha + \beta < 1$ are hold, a non-classical problem is formulated with missing Goursat conditions (3) on the characteristic boundary of the domain and an analog of Frankl condition (4) on the boundary of degeneracy.

It is shown that the validity of the theorem on the uniqueness of the solution to problem A (1)-(4) significantly depends on the ratio between the coefficient μ in Frankl conditions (4), the location of point c lying on the line of degeneracy and on the coefficients α_0 and β_0 in equation (1).

The theorem on the existence of a solution to problem A (1)–(4) is proved by reducing it to the problem of solving a non-standard singular integral equation with a non-Fredholm integral operator in the non-characteristic part of the equation, the kernel of which has an isolated first-order singularity. Further, using the Carleman regularization method, the theory of Wiener-Hopf equations, the problem is equivalently (in the sense of solvability) reduced to an integral equation of the second kind, the solvability of which follows from the uniqueness of the solution to the problem A.

In conclusion, we note that the constructive properties of solutions to equation (1) significantly depend on the values of the parameters m, α, β.

Issues of setting and studying the solvability of similar non-standard problems for other parameter values when $P\left(\alpha_0,\, \beta_0\right) \notin \Delta A_0^* B_0^* C_0^*$ have not been investigated.

Acknowledgments

The work was carried out with the support of the Fund of the Innovative Development Ministry of the Republic of Uzbekistan (grant No. ФЗ-202009211).

Author contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Аймақтың шекарасында өзгешеленетін сингуляр коэффициентті гиперболалық теңдеу үшін жетіспейтін Гурса шарты бар есеп

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Жұмыс гиперболалық теңдеу үшін аймақтың характеристикалық шекарасында жетіспейтін шарттары мен өзгешеленетін сегментіндегі Франкль шартының аналогы бар есепті қоюға және оның шешілу мәселелерін зерттеуге арналған. Бұл есептің белгілі локалды және локалдыемес есептерден айырмашылығы: біріншіден, гиперболалық типтегі теңдеу еркті оң дәрежелі өзгешеленетін және шекара бөліктеріндегі сингулярлық коэффициенттермен алынады, екіншіден, аймақтың характеристикалық шекарасы еркті түрде екі бөлікке бөлінеді және бірінші бөлікте ізделінді функцияның мәні беріледі, ал екінші бөлік шеттік шарттан босатылған және бұл жетіспейтін Гурса шарты өзгешелену сегментіндегі Франкль шартының аналогымен алмастырылады, сондай-ақ белгісіз функцияның аймақтың басқа характеристикалық шекарасындағы мәні белгілі болып саналады. Қойылған есептің жалғыздық жөніндегі теореманың әділдігін қамтамасыз ететін теңдеудің коэффициенттері мен есептің берілгендері үшін жеткілікті шарттар табылған. Есеп шешімінің бар болу теоремасы ядросы бірінші ретті оқшауланған ерекшелігі бар теңдеудің характеристикалық емес бөлігінде фредгольмдік емес интегралдық операторы бар стандартты емес сингулярлық интегралдық теңдеуді шешу туралы есепке келтіру арқылы дәлелденеді. Алынған теңдеуге Карлеманның регуляризациялау әдісін қолдана отырып, Винер-Хопф интегралдық теңдеуін қосамыз. Винер-Хопф теңдеуінің индексі нөлге тең екендігі дәлелденген, сондықтан ол екінші типтегі Фредгольмнің интегралдық теңдеуіне келтіріледі, ал оның шешімділігі есеп шешімінің жалғыздығынан туындайды.

Кілт сөздер: аймақтың шекарасында өзгешеленген гиперболалық теңдеу, жетіспейтін Гурса шарты, Франкль шарты, сингулярлы коэффициент, функциялардың толық ортогональ жүйесі, сингулярлы интегралдық теңдеу, Винер-Хопф теңдеуі, индекс.

Задача с недостающим условием Гурса для вырождающегося на границе области гиперболического уравнения с сингулярным коэффициентом

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Работа посвящена постановке и изучению вопросов разрешимости задачи с недостающими условиями на характеристической границе области и аналогом условия Франкля на отрезке вырождения для гиперболического уравнения. Отличие данной задачи от известных локальных и нелокальных задач состоит в том, что, во-первых, уравнение гиперболического типа берется с произвольным положительным степенным вырождением и сингулярными коэффициентами на части границы, и, во-вторых, характеристическая граница области произвольным образом разбивается на два куска, и на первом куске задается значение искомой функции, а второй кусок освобожден от краевого условия, и это недостающее условие Гурса заменено аналогом условия Франкля на отрезке вырождения, а также считается известным значение неизвестной функции на другой характеристической границе области. Найдены условия на коэффициенты уравнения и данные сформулированной задачи, обеспечивающие справедливость теоремы единственности. Теорема существования решения задачи доказывается сведением к задаче о решении нестандартного сингулярного интегрального уравнения с нефредгольмовым интегральным оператором в нехарактеристической части уравнения, ядро которого имеет изолированную особенность первого порядка. К полученному уравнению, применяя метод регуляризации Карлемана, получается интегральное уравнение Винера-Хопфа. Доказано, что индекс уравнения Винера-Хопфа равен нулю, следовательно, оно однозначно редуцируется к интегральному уравнению Фредгольма второго рода, разрешимость которого следует из единственности решения задачи.

Ключевые слова: вырождающееся на границе области гиперболическое уравнение, недостающее условие Гурса, условие Франкля, сингулярный коэффициент, полная ортогональная система функций, сингулярное интегральное уравнение, уравнение Винера-Хопфа, индекс.

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https://doi.org/10.31489/2024M2/165-177 *Research article*

Modeling of dynamics processes and dynamics control

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Equations and methods of classical mechanics are used to describe the dynamics of technical systems containing elements of various physical nature, planning and management tasks of production and economic objects. The direct use of known dynamics equations with indefinite multipliers leads to an increase in deviations from the constraint equations in the numerical solution. Common methods of constraint stabilization, known from publications, are not always effective. In the general formulation, the problem of constraint stabilization was considered as an inverse problem of dynamics and it requires the determination of Lagrange multipliers or control actions, in which holonomic and differential constraints are partial integrals of the equations of the dynamics of a closed system. The conditions of stability of the integral manifold determined by the constraint equations and stabilization of the constraint in the numerical solution of the dynamic equations were formulated.

Keywords: constraint stabilization, numerical methods, nonholonomic constraints, Helmholtz conditions.

2020 Mathematics Subject Classification: 65D30.

Introduction

The main task of modeling the dynamics is the construction of differential equations of a closed system, the solutions of which have the required properties. The kinematic properties of the motion of a mechanical system and the required properties of the state change of the controlled system are usually given by the constraint equations. The problem of determining the right-hand sides of the equations of dynamics of controlled systems due to the formation of control functions, in essence, refers to the inverse problems of dynamics [1–9]. Methods of classical mechanics are successfully applied to construct the equations of dynamics of a system consisting of elements of various physical nature [10]. The description of analytical dynamics and systems of differential-algebraic equations is proposed in [11]. The analogy between the dynamics of a point of variable mass and the process of change of the simplest economic object allows us to use the equations of classical mechanics to solve problems of control of economic objects and securities portfolios [12–14]. The works [15–17] are devoted to the study of direct and inverse problems of stochastic differential equations describing the dynamics of mechanical systems subject to random influences. In classical mechanics, contact constraints are used, meaning that the initial state and subsequent motion of the system correspond to the constraint equations [18]. In control systems, the equations of servoconstraints [19] are usually introduced, supported by additional control forces. Additional conditions imposed on the solutions of the dynamics equations corresponding to the motion of the image point along the manifold described by the constraint equations and in its vicinity lead to the need to introduce the concepts of program constraint and equations of perturbations of constraints in control systems [20]. The expressions of the controlling influences that ensure the fulfillment of the constraint equations are determined by the relations between the phase coordinates of the system.

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This work for Mukharlyamov R.G. was supported by the Russian Science Foundation and Moscow city № 23-21-10065, https://rscf.ru/en/project/23-21-10065/.

Received: 20 December 2023; *Accepted:* 04 March 2024.

1 Problem Statement

The dynamics of a controlled system with mechanical constraints, the phase state of which is determined by the vectors $q = (q^1, ..., q^n)$, $v = (v^1, ..., v^n)$, is usually described by a system of differential equations

$$\frac{dq^i}{dt} = a^i\left(q, v, t\right), \;\; \frac{dv^i}{dt} = b^i\left(q, v, t\right) + b^{i\kappa}\left(q, v, t\right) u_\kappa, \tag{1}$$

with initial conditions

$$q^i\left(t_0\right) = q_0^i, v^i\left(t_0\right) = v_0^i, \;\; i = 1, \ldots, n, \;\; \kappa = 1, \ldots, s. \tag{2}$$

In equations (and further) summation is assumed for the repeated indices. Control forces are chosen so they satisfy the consraint equations

$$f^\mu\left(q, t\right) = 0, \;\; \varphi^\nu\left(q, v, t\right) = 0, \;\; \mu = 1, \ldots, m, \;\; \nu = m+1, \ldots, r, \;\; r \le s, \tag{3}$$

along with a given accuracy in the numerical solution of a system of equations (1), (2).

In particular, the dynamics of the mechanical system on which the constraints are imposed is described by the equations

$$\frac{dq^i}{dt} = v^i, \;\; \frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial q^i} + Q_i\left(q, v, t\right) + \frac{\partial \varphi^\kappa}{\partial v^i}\lambda_\kappa, \;\; \varphi^\mu = \frac{\partial f^\mu}{\partial q^i}v^i + \frac{\partial f^\mu}{\partial t}, \;\; \kappa = 1, \ldots, \;\; r \le n. \tag{4}$$

Here $L = T - P(q)$ is the Lagrangian, the doubled kinetic energy $2T = m_{ij}(q)v^i v^j, \; i, j = 1, ..., n$, $P = P(q)$ is the potential energy, $Q_i = Q_i(q, v, t)$ are non-potential generalized forces. Lagrange multipliers λ_k are considered as control functions, which must be selected so that the coordinates q^i and the velocities v^i of the system satisfy the constraint equations (3). The system of equations (4) resolved with respect to derivatives is reduced to the form (1) with notation

$$b^l\left(q, v, t\right) = m^{lk}\left(Q_k\left(q, v, t\right) - \frac{1}{2}\left(\frac{\partial m_{ik}}{\partial q^j} + \frac{\partial m_{jk}}{\partial q^i} - \frac{\partial m_{ij}}{\partial q^k}\right)v^i v^j\right),$$

$$b^{l\kappa}\left(q, v, t\right) = m^{lk}\frac{\partial \varphi^\kappa}{\partial v^k}, \;\; m^{lk}m_{kj} = \delta_j^l,$$

$$\delta_j^l = 1, \;\; l = j, \;\; \delta_j^l = 0, \;\; l \ne j, \;\; i, j, k, l = 1, \ldots, n.$$

2 Formulas and theorems

In the case of contact constraints, the initial conditions are

$$q^i\left(t_0\right) = q_0^i, \;\; v^i\left(t_0\right) = v_0^i \tag{5}$$

satisfy the constraint equations: $f^\mu(q_0, t_0) = 0$, $\varphi^k(q_0, v_0, t_0) = 0$, and the Lagrange multipliers are determined from the conditions

$$\frac{d\varphi^\rho\left(q, v, t\right)}{dt} = 0, \;\; \rho = 1, ..., r. \tag{6}$$

From the equalities (6), taking into account the equations (1), a system of linear algebraic equations follows to determine the expression:

$$\frac{\partial \varphi^\rho}{\partial v^i}\left(b^i\left(q, v, t\right) + b^{i\kappa}\left(q, v, t\right) u_\kappa\right) + \frac{\partial \varphi^\rho}{\partial q^i}a^i + \frac{\partial \varphi^\rho}{\partial t} = 0.$$

If the initial conditions (5) are not consistent with the coupling equations (3):

$$f^{\mu}(q_0, t_0) = f_0^{\mu}, \quad \varphi^{\rho}(q_0, v_0, t_0) = \varphi_0^{\rho}, \quad \rho = 1, \ldots, r, \tag{7}$$

it follows from the equalities (6), (7) that with the numerical solution of the system (1), deviations from the coupling equations increase over time:

$$f^{\mu} = f_0^{\mu} + \varphi_0^{\mu} t, \quad \varphi^{\rho}(q_0, v_0, t_0) = \varphi_0^{\rho}.$$

The problem of constraint stabilization arises, for the solution of which it was proposed [21] to use a linear combination of constraint equations with their derivatives:

$$\frac{d^2 f^{\mu}}{dt^2} + k_1 \frac{df^{\mu}}{dt} + k_0 f^{\mu} = 0, \quad \frac{d\varphi^{\rho}}{dt} = \gamma(q, v, t)\, \varphi^{\rho}. \tag{8}$$

In essence, equalities (8) are equations of perturbations of constraints. Obviously, when the constraints are satisfied $k_1 - const, k_0 - const, k_1 > 0,\ k_0 > 0,\ \gamma(q, v, t) > 0$ trivial solutions of $f^{\mu} = 0$, $\varphi^{\rho} = 0$ of equations (8) are asymptotically stable. So in the simplest case, the equations with respect to perturbations of holonomic constraints of equation (8) can be represented by a linear system with constant coefficients [22]

$$\frac{df^{\mu}}{dt} = \varphi^{\mu}, \quad \frac{d\varphi^{\rho}}{dt} = k^{\rho}_{\mu} f^{\mu} + k^{\rho}_{\kappa} \varphi^{\kappa}, \quad \mu = 1, \ldots, m, \quad \rho, \kappa = 1, \ldots, r.$$

To limit deviations from the coupling equations in the numerical solution of the dynamics equations, additional conditions should be imposed on the coefficients of the equations (8). Various modifications of the J. Baumgarte method were proposed, for example, [22, 23], which were reduced to the selection of numerical methods for solving dynamic equations and recommendations for the selection of coefficients of the equations of the system (8). In [22], a hybrid scheme of integration of a controlled system consisting of a non-rigid mechanical subsystem and a rigid controlled subsystem is described. J. Baumgarte is also used to stabilize constraints in higher-order control systems [23]. To determine the expression of the multiplier λ in the right side of the equation of the system

$$\frac{d^m q}{dt^m} = Q\left(q, \frac{dq}{dt}, \ldots, \frac{d^{m-1} q}{dt^{m-1}}, t\right) + B(q, t)\, \lambda, \qquad f(q, t) = 0$$

a linear combination of the constraint equation with derivatives up to the order of $m \geq 2$ is used, which leads to a differential equation of the constraint perturbations:

$$\alpha_{\mu} y^{(\mu)} = 0, \quad y = f(x, t), \quad y^{(\mu)} = \frac{d^{\mu} y}{dt^{\mu}}, \quad \mu = 0, \ldots, m.$$

The coefficients α_{μ} of the differential equation should be chosen so that the roots of the characteristic equation $\alpha_{\mu} \kappa^{\mu} = 0$ have negative real parts, for which it is proposed to use a polynomial of the form $\alpha_{\mu} \kappa^{\mu} = (\kappa + k)^m,\ k - const$. In this case, the solution of the equation of constraints perturbations is represented by the expression

$$y = (A_{\mu} t^{\mu})\, e^{-kt}, \quad \mu = 0, \ldots, m-1.$$

The integration constants are determined by the choice of the initial conditions $y^{(\mu)}(t_0) = y_0^{\mu}$, $\mu = 0, ..., m-1$, and for small values of t, the value of y may be significant. So, for $m = 2$ and initial conditions (5) corresponding to the equalities

$$f(q_0, t_0) = 0, \quad \left(\frac{\partial f}{\partial q^i}\right)_0 v_0^i + \left(\frac{\partial f}{\partial t}\right)_0 = v_0,$$

change deviation from the constraint equation $f(q,t)=0$ defined by the expression $y=v_0 te^{-kt}$ and it can reach a significant value when numerically solving the dynamical equations (Fig. 1).

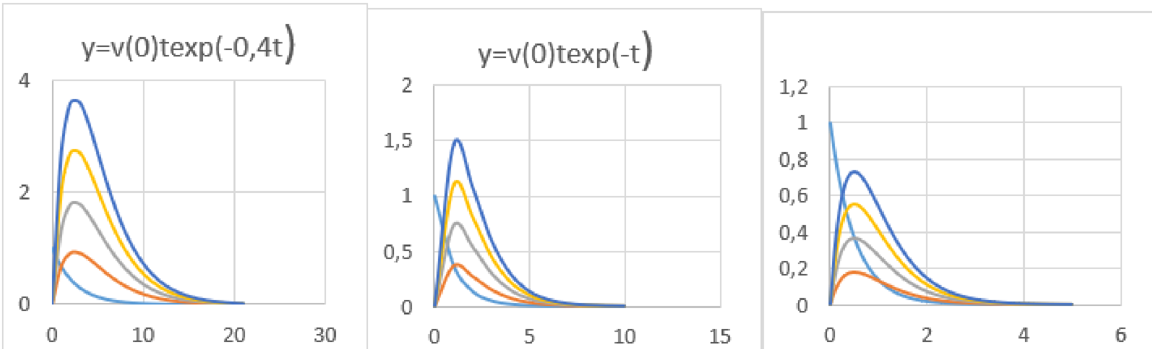

Figure 1. Graphs of the change in the value $y=f(q,t)$ at $v_0=1;2;3;4$ and $k=0,4, k=1, k=2$ correspondingly

3 Construction of systems of differential equations

The concept of program constraints is associated with the construction of systems of differential equations with given partial integrals [24, 25] and to stabilize the constraints it is necessary that the constraints equations constitute partial integrals of the dynamics equations. The behavior of solutions in the vicinity of a set of points determined by the constraint equations must correspond to the operating conditions of a real system.

If the values of control actions u_k are defined as functions $u_k=u_k(q,v,t)$ of variables q,v, then by introducing the phase state vector $x=(q,v)$, the system of equations (1) and constraint equations (3) can be represented equalities

$$\frac{dx^i}{dt}=X^i\ (x,t)\,, \tag{9}$$

$$\varphi^\rho(x,t)=0,\ x=\left(x^1,\ldots,x^{2n}\right),\ \rho=1,\ldots,r\leq 2n,\ i=1,\ldots,2n. \tag{10}$$

Since functions (10) are partial integrals of the system of differential equations (9), the right-hand sides of X^i must satisfy the conditions

$$\frac{\partial\varphi^\rho}{\partial x^i}X^i+\frac{\partial\varphi^\rho}{\partial t}=F^\rho(f,\varphi,x,t)\,,\quad \varphi=\left(\varphi^1,\ldots,\varphi^r\right), \tag{11}$$

where $F^\rho(f,\varphi,x,t)$ are arbitrary functions that satisfy the equalities $F^\rho(0,0,x,t)=0$. From equality (11) it follows that the right-hand sides of the equations of system (9) should have the following structure

$$X^i=\varphi^0X^i_\tau+X^i_\nu,$$

where φ^0 is an arbitrary value, X^i_τ is the corresponding component of the vector product

$$X_\tau=\varphi^0\left[\nabla\varphi^1\ldots\nabla\varphi^r c^{r+1}\ldots c^{2n-1}\right],\quad \nabla\varphi^\rho=\left(\frac{\partial\varphi^\rho}{\partial x^1},\ldots,\frac{\partial\varphi^\rho}{\partial x^{2n}}\right),$$

$$\nabla\varphi^\rho=\left(\varphi^\rho_1,\ldots,\varphi^\rho_{2n}\right),\quad \varphi^\rho_i=\frac{\partial\varphi^\rho}{\partial x^i},$$

$c^{r+1},\ldots,c^{2n-1}$ are arbitrary vectors $c^\sigma=(c^\sigma_1,\ldots,c^\sigma_{2n})$, $\sigma=r+1,\ldots,2n-1$,

$$X^i_\nu=\delta^{ij}\varphi^\alpha_j\omega_{\alpha\rho}F^\rho,\quad \delta^{ij}=0,\quad i\neq j,$$

$$\delta^{ii}=1,\quad (\omega_{\alpha\rho})=\left(\omega^{\rho\gamma}\right)^{-1},\quad \omega^{\rho\gamma}=\varphi^\rho_i\delta^{ij}\varphi^\gamma_j,\quad i=1,\ldots 2n,\quad \alpha,\gamma,\rho=1,\ldots,r.$$

The system of equations represented by equalities (1), (3) constitutes a system of differential algebraic equations. The functions u_k in equations (1) are control actions that ensure the fulfillment of constraint equations (3). To stabilize the constraints (3), we determine possible deviations from the constraint equations (3) by the quantities

$$y^{\mu} = f^{\mu}\left(q,t\right), \quad z^{\rho} = \varphi^{\rho}\left(q,v,t\right), \quad \mu = 1,\ldots,m, \quad \rho = 1,\ldots,r, \quad r \leq s. \tag{12}$$

We define new variables y^{μ}, z^{ρ} as solutions to the system of constraint perturbation equations

$$\frac{dy^{\mu}}{dt} = z^{\mu}, \qquad \frac{dz^{\rho}}{dt} = \quad Z^{\rho}\left(y,z,q,v,t\right), \tag{13}$$

satisfying the equalities $Z^{\rho}(0,0,q,v,t) = 0$ and the initial conditions

$$y_0^{\mu} = f^{\mu}(q_0,t_0), \quad z_0^{\rho} = \varphi^{\rho}\left(q_0,v_0,t_0\right), \quad \mu = 1,\ldots,m, \quad \rho = 1,\ldots,r. \tag{14}$$

Equalities (13) define a system of equations for constraint perturbations, which, when

$$Z^{\mu} = -\omega^2 y^{\mu} - 2\alpha z^{\mu}, \quad Z^{\nu} = -\gamma\left(q,v,t\right)z^{\nu}, \quad \alpha,\omega - const,$$

corresponds to the method of J. Baumgarte [21]. Constraint equations (12), supplemented with conditions (13), (14), constitute the program coupling equations.

From equalities (1), (12), (13) follows a system of equations for determining the control actions u_k:

$$\begin{aligned}
&p^{\rho\kappa}u_{\kappa} = h^{\rho},\\
&p^{\rho\kappa} = \frac{\partial\varphi^{\rho}}{\partial v^i}b^{i\kappa},\ h^{\rho} = Z^{\rho}\left(f,\varphi,q,v,t\right) - \frac{\partial\varphi^{\rho}}{\partial v^i}b^i\left(q,v,t\right) - \frac{\partial\varphi^{\rho}}{\partial q^i}a^i - \frac{\partial\varphi^{\rho}}{\partial t},\\
&f = \left(f^1,\ldots,f^m\right),\ \varphi = \left(\varphi^1,\ldots,\varphi^{\rho}\right),\\
&\rho = 1,\ldots,r,\ \kappa = 1,\ldots,s,\ r \leq s.
\end{aligned} \tag{15}$$

If the rows of the matrix (p_k^{ρ}) are linearly independent, then the expressions of the control actions u_k are determined by solving the system of linear equations (15):

$$u_{\kappa} = c_0\delta_{\kappa}\left[p^1\ldots p^r c^{r+1}\ldots c^{s-1}\right] + \delta_{\beta\kappa}p^{\alpha\beta}\omega_{\alpha\rho}h^{\rho},$$

c_0 is an arbitrary value, $c^{\rho} = (c^{\rho}1,\ldots,c^{\rho}s)$ is an arbitrary vector, $\delta_{\kappa} = (\delta_{\kappa}^1,...,\delta_{\kappa}^s)$,

$$\begin{aligned}
&p^{\rho} = \left(p^{\rho 1},\ldots,p^{\rho s}\right),\ \delta_{\beta\beta} = 1,\ \delta_{\beta\kappa} = 0,\ \beta \neq \kappa,\\
&\omega^{\rho\alpha} = p^{\rho\kappa}\delta_{\kappa\beta}p^{\alpha\beta},\\
&\omega_{\alpha\rho}\omega^{\rho\gamma} = \delta_{\gamma}^{\alpha},\ \delta_{\alpha}^{\alpha} = 1,\ \delta_{\gamma}^{\alpha} = 0,\ \alpha \neq \gamma,\ \alpha,\rho,\gamma = 1,\ldots,r,\ \beta,\kappa = 1,\ldots,s.
\end{aligned}$$

As a result of substituting the resulting expressions into the right-hand sides of the equations, the closed system of equations (1) is written in the following form:

$$\begin{aligned}
&\frac{dq^i}{dt} = a^i\left(q,v,t\right),\ \frac{dv^i}{dt} = b^i\left(q,v,t\right) + b^{i\kappa}\left(q,v,t\right)u_{\kappa}\left(q,v,t\right),\\
&u_{\kappa}\left(q,v,t\right) = u_{\kappa 0}\left(q,v,t\right) + u_{\kappa 1}\left(y,z,q,v,t\right),\\
&u_{\kappa 0}\left(q,v,t\right) = c_0\delta_{\kappa}\left[p^1\ldots p^r c^{r+1}\ldots c^{s-1}\right] - \delta_{\kappa\beta}p^{\alpha\beta}\omega_{\alpha\rho}\left(\frac{\partial\varphi^{\rho}}{\partial v^i}b^i\left(q,v,t\right) + \frac{\partial\varphi^{\rho}}{\partial q^i}a^i + \frac{\partial\varphi^{\rho}}{\partial t}\right),\\
&u_{\kappa 1}\left(y,z,q,v,t\right) = \delta_{\kappa\beta}p^{\alpha\beta}\omega_{\alpha\rho}Z^{\rho}\left(y,z,q,v,t\right),\\
&y = f\left(q,t\right),\ z = \varphi\left(q,v,t\right).
\end{aligned} \tag{16}$$

The system of equations (16) has partial integrals determined by the constraint equations (3).

4 Stability of the integral manifold

Using notation

$$\begin{aligned} & x = \left(x^1, \ldots, x^{2n}\right), \ x^i = q^i, \ x^{n+i} = v^i \\ & \eta = \left(\eta^1, \ldots, \eta^{m+r}\right), \ \eta^\mu = y^\mu, \ \eta^{m+\rho} = z^\rho, \\ & g^\sigma\left(x, t\right) = 0, \ \sigma = 1, \ldots, m+r, \\ & g^\mu = f^\mu, \ g^\rho = \varphi^\rho, \ \mu = 1, \ldots, m, \ \rho = 1, \ldots, r, \end{aligned} \tag{17}$$

let us rewrite the system of equations (12), (13), (16) in a compact form:

$$\eta^\sigma = g^\sigma\left(x, t\right), \tag{18}$$

$$\tfrac{dx^s}{dt} = X^s\left(\eta, x, t\right), \quad s = 1, \ldots, 2n, \tag{19}$$

$$\tfrac{d\eta^\sigma}{dt} = \Upsilon^\sigma\left(\eta, x, t\right), \quad \sigma = 1, \ldots, m+r, \tag{20}$$

$$\begin{aligned} & X^i\left(y, x, t\right) = x^{n+i}, \ X^{n+i}\left(\eta, x, t\right) = X_0^{n+i}\left(x, t\right) + X_1^{n+i}\left(\eta, x, t\right), \\ & X_0^{n+i}\left(x, t\right) = b^i\left(x, t\right) + b^{i\kappa}\left(x, t\right) u_{\kappa 0}\left(x, t\right), \ X_1^{n+i}\left(\eta, x, t\right) = b^{i\kappa}\left(x, t\right) u_{\kappa 1}\left(\eta, x, t\right), \\ & \Upsilon^\mu\left(\eta, x, t\right) = y^{m+\mu}, \quad \Upsilon^{m+\rho}\left(\eta, x, t\right) = Z^\rho\left(\eta, x, t\right). \end{aligned}$$

Setting $x^s(t_0) = x_0^s, \eta^\sigma(t_0) = \eta_0^\sigma \equiv g^\sigma(x_0, t_0)$, we determine the stability conditions [24] of the integral manifold of system (18), given by equalities (17).

Definition 1. The integral manifold of the system of equations (19), defined by the equality $\eta(x,t) = 0$, is stable if for any ϵ there exists a δ such that for all initial conditions $x(t_0) = x_0$ corresponding to the inequalities $|\eta_0| \le \delta$, the value $\eta = \eta(t)$ will satisfy the condition $|\eta(t)| \le \epsilon$ for all $t > t_0$.

The stability of a trivial solution to the system of equations (20) depends on the choice of functions $\Upsilon^{(m+\rho)}(\eta, x, t)$. Stability conditions can be obtained using Lyapunov functions. If the functions $\Upsilon^{(m+\rho)}$ are represented by a linear combination of constraint perturbations, then the system of equations (20) turns out to be linear:

$$\frac{d\eta^\sigma}{dt} = h_\alpha^\sigma(x,t)\,\eta^\alpha, \quad \sigma, \alpha = 1, \ldots, m+r. \tag{21}$$

To study the stability of the trivial solution of system (20), we take as the Lyapunov function a positive definite quadratic form with constant coefficients $V = 0.5 c_{\sigma\alpha}\eta^\sigma\eta^\alpha$. Then there are constants c_1, c_2 corresponding to the constraints $c_1|\eta|^2 \le V \le c_2|\eta|^2$. If the derivative of function V, calculated by virtue of the equations of system (21),

$$\frac{dV}{dt} = p_{\sigma\alpha}\left(x, t\right)\eta^\sigma\eta^\alpha, \quad p_{\sigma\alpha}\left(x, t\right) = c_{\sigma\zeta} h_\alpha^\zeta\left(x, t\right), \quad \sigma, \alpha, \zeta = 1, \ldots, m+r,$$

will be limited:

$$\frac{dV}{dt} \le -a|\eta|^2,$$

then the inequality will be satisfied $|\eta|^2 \le \frac{c_2}{c_1}|\eta_0^2|e^{t-t_0}$, $\lambda = \frac{2a}{c_2}$, and the integral manifold (17) of the system of equations (19) will be stably exponential. If the coefficients h_α^σ of the equations of system (21) are constant, then the stability of the trivial solution is determined by the roots of the characteristic equation.

5 Constraint stabilization of in the numerical solution of dynamic equations

The asymptotic stability of the trivial solution of system (20) is not enough to limit deviations from the constraint equations when numerically solving the dynamic equations

$$\frac{dx^s}{dt} = X^s\left(g\left(x,t\right),x,t\right), \quad x^s\left(t_0\right) = x_0^s. \tag{22}$$

The requirement to stabilize the constraints imposes additional conditions [26] on the right-hand sides of the constraint perturbation equations (19), which are determined by the value of the limitation of deviations from the constraint equations and the choice of the numerical method for solving system (22) [25–28]. Let $|\eta_0| \leq \epsilon$ and let the difference scheme be used to solve system (22)

$$x_{l+1}^s = x_l^s + \left(\Delta x^s\right)_l, \ \left(\Delta x^s\right)_l = \tau X^s\left(x_l, t_l\right), \ \tau = t_{l+1} - t_l, \ l = 1, \ldots, N. \tag{23}$$

Let us represent the functions $\eta_{l+1}^\sigma = g^\sigma(x_{l+1}, t_{l+1})$ by series expansions in powers of τ:

$$g^\sigma\left(x_{l+1}, t_{l+1}\right) = g^\sigma\left(x_l, t_l\right) + \left(\frac{\partial g^\sigma}{\partial x^s}\right)_l \left(\Delta x^s\right)_l + \tau\left(\frac{\partial g^\sigma}{\partial t}\right)_l + \frac{\tau^2}{2}\widetilde{g}_l^{\sigma(2)}, \tag{24}$$

or taking into account equalities (23), (24):

$$\eta_{l+1}^\sigma = \eta_l^\sigma + \tau\Upsilon_l^\sigma\left(y,x,t\right) + \frac{\tau^2}{2}\widetilde{g}_l^{\sigma(2)}.$$

After expanding the function $\Upsilon_l^\sigma = \Upsilon_l^\sigma(\eta, x, t)$ into a series in powers of magnitude η_l^α, the last equality will be rewritten in the following form:

$$\eta_{l+1}^\sigma = \eta_l^\sigma + \tau k_\alpha^\sigma\left(x_l, t_l\right)\eta_l^\alpha + \frac{\tau^2}{2}\widetilde{\Upsilon}_l^{\sigma(2)} + \frac{\tau^2}{2}\widetilde{g}_l^{\sigma(2)}. \tag{25}$$

From equalities (25) the following estimates follow:

$$|\eta_{l+1}^\sigma| \leq |\delta_\sigma^\alpha + \tau k_\alpha^\sigma\left(x_l, t_l\right)\eta_l^\alpha| + \frac{\tau^2}{2}\widetilde{\Upsilon}_l^{\sigma(2)} + \frac{\tau^2}{2}\widetilde{g}_l^{\sigma(2)}, \quad \delta_\alpha^\sigma = 0, \quad \sigma \neq \alpha, \quad \delta_\sigma^\sigma = 1,$$

and statement.

Theorem. If the inequality $|\eta_0| \leq \epsilon$ is satisfied and the functions $\Upsilon^{m+\rho}(\eta, x, t)$, $\eta^\mu = f^\mu(x,t)$, $\eta^{m+\rho}(t) = \varphi^\rho(x,t)$ for all values of x, t corresponding to the solution of system (22), satisfy the conditions $1 + \tau\kappa x$, $t \leq \theta \leq 1$, $\frac{\tau^2}{2}\Upsilon^2 + g^2 \leq 1 - \Theta\epsilon$, then for all $l = 1, \ldots, N$ the inequalities $|\eta_l| \leq \epsilon$ will be satisfied.

Example. Determine the control function $u = u(q^1, q^2)$ for the system

$$\begin{gathered}\frac{dq^1}{dt} = -4q^2 - q^1 b\left(q^1, q^2\right)u, \quad \frac{dq^2}{dt} = q^1 - 4q^2 b\left(q^1, q^2\right)u, \\ b\left(q^1, q^2\right) = \frac{1}{\left(q^1\right)^2 + \left(q^2\right)^2}, \quad q^1\left(0\right) = 2, \quad q^2\left(0\right) = 0,\end{gathered} \tag{26}$$

ensuring the existence of the partial integral $y = 0.5(q^1)^2 + 2(q^2)^2 - 2 = 0$ and its stabilization when solving system (26) numerically using the Euler method with a step $\tau = 0.001$. Constraint perturbation equation

$$\frac{dy}{dt} = -ky, \quad k > 0,$$

has an asymptotically stable trivial solution $y = 0$. Control $u = k((q^1)^2 + 4(q^2)^2 - 4)$ ensures the fulfillment of the constraint equation $0.5(q^1)^2 + 2(q^2)^2 - 2 = 0$ with an accuracy of $\epsilon = 0.001$ at values of k that satisfy the restrictions: $200 < k < 1800$. Figure 2 shows graphs of changes in calculation error values corresponding to the values $k = 50; 300; 2050$.

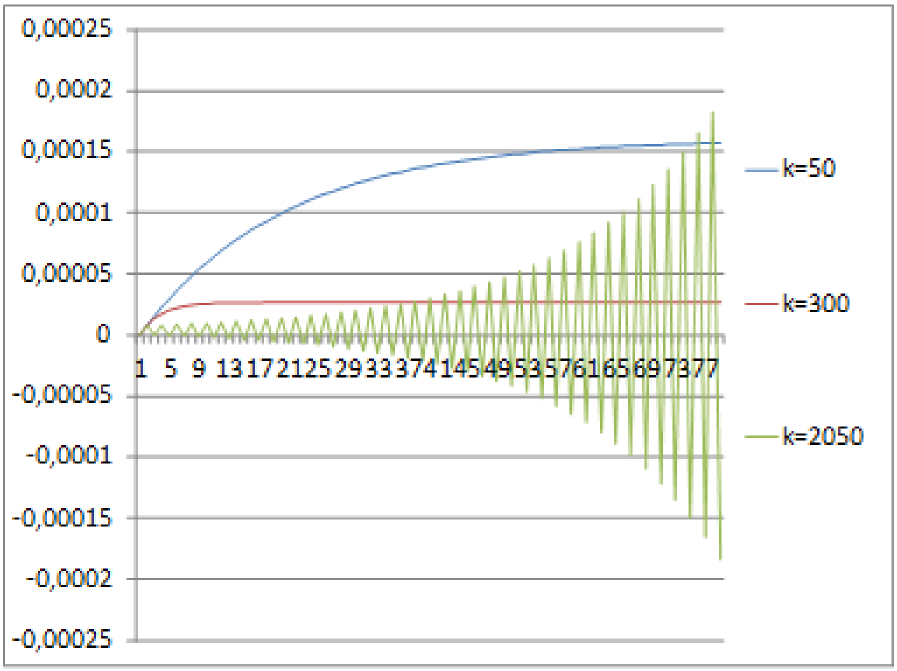

Figure 2. Deviations values on time

6 Conclusion

Methods of constraint stabilization, based on the construction of systems of differential equations with asymptotically stable partial integrals, represent effective ways to model solutions to problems of determining the reactions of constraints and controlling the dynamics of systems for various purposes.

Author Contributions

All authors contributed equally to this work

Conflict of Interest

The authors declare no conflict of interest.

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Динамика процестерін модельдеу және байланыстарды тұрақтандыруды ескере отырып, жүйені басқару синтезі

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Әр түрлі физикалық сипаттағы элементтерден тұратын техникалық басқару жүйелерінің динамикасын, өндіріс пен экономикалық объектілерді жоспарлау және басқару міндеттерін сипаттау үшін классикалық механиканың теңдеулері мен әдістері қолданылады. Анықталмаған факторлары бар белгілі динамикалық теңдеулерді тікелей пайдалану сандық шешімдегі байланыс теңдеулерінен ауытқулардың артуына әкеледі. Басылымдардан белгілі байланыстарды тұрақтандырудың кең таралған әдістері әрдайым тиімді бола бермейді. Жалпы есептің қойылуында байланыстарды тұрақтандыру есебі динамиканың кері есебі ретінде қарастырылған және голономикалық байланыстар мен дифференциалдық байланыстар тұйық жүйе динамикасы теңдеулерінің дербес интегралдары болып табылатын Лагранж факторларын немесе басқару әсерлерін анықтауды талап етеді. Байланыс теңдеулерімен анықталған интегралдық көпбейнеліктердің тұрақтылығы және динамикалық теңдеулерді сандық шешуде байланыстарды тұрақтандыру шарттары тұжырымдалған.

Кілт сөздер: байланыстарды тұрақтандыру, сандық әдістер, голономиялық емес байланыс, Гельмгольц шарттары.

Моделирование процессов динамики и синтез управления системой с учетом стабилизации связей

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Для описания динамики технических систем управления, содержащих элементы различной физической природы, задач планирования и управления производством и экономическими объектами используются уравнения и методы классической механики. Непосредственное применение известных уравнений динамики с неопределенными множителями приводит к возрастанию отклонений от уравнений связей при численном решении. Распространенные методы стабилизации связей, известные по публикациям, оказываются не всегда эффективными. В общей постановке задача стабилизации связей рассмотрена как обратная задача динамики, и она требует определения множителей Лагранжа или управляющих воздействий, при которых голономные связи и дифференциальные связи являются частными интегралами уравнений динамики замкнутой системы. Сформулированы условия устойчивости интегрального многообразия, определяемого уравнениями связей, и стабилизации связей при численном решении уравнений динамики.

Ключевые слова: стабилизация связей, численные методы, неголономная связь, условия Гельмгольца.

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https://doi.org/10.31489/2024M2/178-185 *Research article*

On the existence and coercive estimates of solutions to the Dirichlet problem for a class of third-order differential equations

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As you know, the third order partial differential equation is one of the basic equations of wave theory. For example, in particular, a linearized Korteweg-de Vries type equation with variable coefficients models ion-acoustic waves into plasma and acoustic waves on a crystal lattice. In this paper, the properties of solutions of a class of the third order degenerate partial differential equations with variable coefficients given in a rectangle were studied. Sufficient conditions for the existence and uniqueness of a strong solution have been established. Note that the solution of the degenerate equation does not retain its smoothness, therefore, these difficulties in turn affect the coercive estimates.

Keywords: resolvent, third order differential equation, Dirichlet problem, coercive estimates.

2020 Mathematics Subject Classification: 39A14.

Introduction

In the rectangle $\overline{\Omega} = \{(x,y) : -\pi \le x \le \pi; 0 \le y \le 1\}$, the problem

$$Lu + \mu u = -k(y)\frac{\partial^3 u}{\partial x^3} - \frac{\partial u^2}{\partial y^2} + a(y)\frac{\partial u}{\partial x} + c(y)u + \mu u = f(x,y) \in L_2(\Omega), \tag{1}$$

$$u_x^{(\alpha)}(-\pi, y) = u_x^{(\alpha)}(\pi, y), \quad \alpha = 0,1,2, \tag{2}$$

$$u(x,0) = u(x,1), \tag{3}$$

is considered.

Suppose that the coefficients $k(y)$, $a(y)$, $c(y)$ of equation (1) satisfy the conditions:

1) $k(y) \ge 0$ is a piecewise continuous function on the segment $[0,1]$ and $k(0) = 0$;

2) $a(y) \ge \delta_0$, $c(y) \ge \delta > 0$ are continuous functions on the segment $[0,1]$.

Equation (1) degenerates along the line $y = 0$, i.e. at these points equation (1) changes order. This means that solutions do not retain their smoothness, hence these difficulties in turn affect the coercive estimates of solutions.

Many papers [1–13] and the works cited there are devoted to the study of partial differential equations of the third order. From these works and from a review of literary sources, it follows that previously differential equations without degeneracy were mainly studied.

To present the results obtained regarding this work, we will need the following designations and definitions. By $W_2^1(\Omega)$ we denote the S.L. Sobolev space with norm $\|u\|_{2,1,\Omega} = [\|u_y\|_2^2 + \|u_x\|_2^2 + \|u\|_2^2]^{\frac{1}{2}}$. $C_{0,\pi}^\infty(\overline{\Omega})$ is a set consisting of infinitely differentiable functions and satisfying conditions (2)-(3).

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This paper was supported by the grant AP19676466 of the Ministry of Science and Higher Education of Republic of Kazakhstan.

Received: 25 July 2023; *Accepted:* 29 February 2024.

Definition 1. A function $u(x,y) \in L_2(\Omega)$ is called a strong solution to problem (1)–(3) if there exists a sequence of functions $\{u_n\} \subset C^{\infty}_{0,\pi}(\overline{\Omega})$, such that

$$\|u_n - u\|_{L_2(\Omega)} \to 0, \ \ \|Lu_n - f\|_{L_2(\Omega)} \to 0 as n \to 0.$$

Theorem 1. Let the conditions 1) - 2) be fulfilled. Then for $\mu \geq 0$, for any $f(x,y) \in L_2(\Omega)$ there is a unique strong solution to the problem (1)–(3).

Theorem 2. Let conditions 1) - 2) be fulfilled. Then for $\mu \geq 0$, for any $f(x,y) \in L_2(\Omega)$ there is a unique strong solution to the problem(1)–(3) such that the coercive estimate

$$\|u\|_{1,2,\Omega} \leq C \, \|(L+\mu I)u\|_2$$

is valid for it, where $C > 0$ is a constant, $\|\cdot\|_2$ is the norm of $L_2(\Omega)$.

1 *Proof of Theorems 1-2*

In what follows, we denote by $(L+\mu I)$ the operator corresponding to problem (1)–(3).

Lemma 1. Let the conditions 1) - 2) be fulfilled. Then the following inequality

$$\|(L+\mu I)u\|_2 \geq (\delta_0 + \lambda) \, \|u\|_2 \, , \tag{4}$$

holds for all $u \in D(L)$, where $\delta_0 > 0$, $\mu \geq 0$. $D(L)$ is the domain of definition of the operator L.

Proof. Consider the functionality $< (L+\mu I)u, u >$, $u \in D(L)$, where $< \cdot, \cdot >$ is scalar product in $L_2(\Omega)$. Integrating in parts, we get an estimate (4). Lemma 1 is proved.

Using the Fourier method, we reduce the problem (1)–(3) to the study of the following differential operator with the parameter n $(n = \pm 0, \pm 1, \pm 2, ...)$:

$$(l_n + \mu I)z(y) = -z'(y) + (-ik(y)n^3 + ina(y) + c(y) + \mu)z(y),$$

where $z(y) \in D(l_n)$, $D(l_n)$ is the domain of definition of the operator l_n.

Lemma 2. Let the conditions 1) - 2) be fulfilled. Then the following inequality

$$\|(l_n + \mu I)z\|_2 \geq (\delta_0 + \mu) \, \|z\|_2$$

holds for all $z(y) \in D(l_n + \mu I)$, where $\|\cdot\|_2$ is the norm of the Hilbert space $L_2(0,1)$. $D(l_n)$ is the domain of definition of the operator l_n.

Proof. Let us denote by $C_0^2[0,1]$ the set consisting of doubly differentiable functions and satisfying condition (3). Let $z(y) \in C_0^2[0,1]$ and consider the functional

$$< (l_n + \mu I)z, z >= \int_0^1 [|z'|^2 + (c(y)+\mu)|z|^2 + (in^3 k(y) + ina(y))|z|^2]dy. \tag{5}$$

Hence, using the properties of complex numbers, we find that

$$| < (l_n + \mu I)z, z > | \geq \int_0^1 [|z'|^2 + (c(y)+\mu)|z|^2]dy \geq \int_0^1 (|z'|^2 + (\delta + \mu)|z|^2)dy. \tag{6}$$

From the last inequality, using the Cauchy-Bunyakovsky inequality, we have

$$\|(l_n + \mu I)z\|_2 \geq (\delta_0 + \mu) \, \|z\|_2 \, .$$

Hence, and by virtue of the continuity of the norm in $L_2(0,1)$, we will be convinced of the validity of the last estimate for all $z(y) \in D(l_n)$. Lemma 2 is proved.

Lemma 3. Let the conditions 1) - 2) be fulfilled and $\mu \geq 0$. Then for the operator $(l_n + \mu I)$ there is a bounded inverse operator $(l_n + \mu I)^{-1}$ defined on the whole $L_2(0,1)$.

Proof. Lemma 3 is also proved as Lemma 2.3 of [14, 15].

Lemma 4. Let the conditions 1) - 2) be fulfilled and $\mu \geq 0$. Then the following estimates are valid for operators $(l_n + \mu I)^{-1}$ and $\frac{d}{dy}(l_n + \mu I)^{-1}$:

$$\left\|(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{\delta + \mu}; \tag{7}$$

$$\left\|(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{|n| \cdot \delta_0}, \quad n \neq 0; \tag{8}$$

$$\left\|\frac{d}{dy}(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{(\delta + \mu)^{\frac{1}{2}}}, \tag{9}$$

where $\|\cdot\|_{2\to 2}$ is the norm of the operator from $L_2(\Omega)$ to $L_2(\Omega)$.

Proof. From Lemma 2 we have

$$\left\|(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{\delta + \mu}.$$

Inequality (7) is proved.

Using inequality (5) and properties of complex numbers, we find that

$$< (l_n + \mu I)z, z > \geq |\int_0^1 (in^3 k(y) + ina(y))|z|^2 dy|. \tag{10}$$

Note that by virtue of condition 1) - 2) the functions $k(y)$ and $a(y)$ do not change signs, therefore, from the inequality (10) we find that

$$| < (l_n + \mu I)z, z > | \geq \int_0^1 |in^3 k(y) + ina(y)| \cdot |z|^2 dy. \tag{11}$$

From (11) and given $a(y) \geq \delta_0 > 0$ we have

$$\|(l_n + \mu I)z\|_2 \geq |n|\delta_0 \|z\|_2 .$$

Hence, using the definition of the operator norm, we obtain the following estimate:

$$\left\|(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{|n| \cdot \delta_0}, \quad n \neq 0.$$

Inequality (8) is proved.

Using inequalities (4) and (6) we find that

$$\frac{1}{\delta + \mu} \left\|(l_n + \mu I)^{-1}\right\|_{2\to 2} \geq \left\|z'\right\|_2^2 .$$

Hence, according to the definition of the operator norm, we find

$$\left\|\frac{d}{dy}(l_n + \mu I)^{-1}\right\|_{2\to 2} \leq \frac{1}{(\delta + \mu)^{\frac{1}{2}}}.$$

Inequality (9) is proved. Lemma 4 is proved.

Proof of Theorem 1. Using Lemma 3, we obtain that

$$u_k(x,y) = \sum_{n=-k}^{n=k} (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx}$$

is the solution of the following problem:

$$(L + \mu I) u_k(x,y) = f_k(x,y), \tag{12}$$

$$u_{k,x}^{(\alpha)}(-\pi, y) = u_{k,x}^{(\alpha)}(\pi, y), \quad \alpha = 0, 1, 2, \tag{13}$$

$$u_k(x,0) = u_k(x,1) = 0, \tag{14}$$

where $f_k(x,y) \to f(x,y)$, $f_k(x,y) = \sum\limits_{n=-k}^{k} f_n(y) \cdot e^{inx}$, $i^2 = -1$.

From inequality (4) and using the fundamentality of the sequence $\{f_k(x,y)\}$, we have

$$\|u_k(x,y) + u_m(x,y)\|_2 \leq \frac{1}{\delta + \mu} \|f_k(x,y) - f_m(x,y)\|_2 \to 0, \quad as \;\; k, m \to \infty.$$

From the last inequality and by virtue of the completeness of the Hilbert space $L_2(\Omega)$ we have

$$u_k(x,y) \overset{L_2(\Omega)}{\to} u(x,y). \tag{15}$$

Further, using the equalities (12)–(15) for any $f(x,y) \in L_2(\Omega)$, we obtain that

$$u(x,y) = (L + \mu I)^{-1} f = \sum_{n=-\infty}^{n=\infty} (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx} \tag{16}$$

is a strong solution to the problem (1)–(3). The theorem is proved.

Proof of Theorem 2. From (16) by virtue of the orthonormality of the system $\{e^{inx}\}$

$$\|u\|_2^2 = \left\| \sum_{n=-\infty}^{\infty} (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx} \right\|_{L_2(0,1)}^2 = 2\pi \sum_{n=-\infty}^{\infty} \left\| (l_n + \mu I)^{-1} f_n(y) \cdot e^{inx} \right\|_{L_2(0,1)}^2 \leq$$

$$\leq 2\pi \sum_{n=-\infty}^{\infty} \left\| (l_n + \mu I)^{-1} \right\|_{2\to 2}^2 \cdot \|f_n(y)\|_{L_2(0,1)}^2 \leq \sup_{\{n\}} \left\| (l_n + \mu I)^{-1} \right\|_{2\to 2}^2 2\pi \cdot \sum_{n=-\infty}^{\infty} \|f_n(y)\|_{L_2(0,1)}^2 \leq$$

$$\leq \sup_{\{n\}} \left\| (l_n + \mu I)^{-1} \right\|_{2\to 2}^2 \cdot \|f(x,y)\|_{L_2(\Omega)}^2 . \tag{17}$$

Here we note that we used by virtue of the orthonormality of the system $\{e^{inx}\}$, i.e.

$$\|f(x,y)\|_2^2 = \left\| \sum_{n=-\infty}^{\infty} f_n(y) \cdot e^{inx} \right\|_{L_2(\Omega)}^2 = 2\pi \cdot \sum_{n=-\infty}^{\infty} \|f_n(y)\|_{L_2(0,1)}^2 .$$

From estimates (17), (4) and (7) we obtain that

$$\|u\|_{L_2(\Omega)}^2 \leq (\frac{1}{\delta + \mu})^2 \|f(x,y)\|_{L_2(\Omega)}^2 .$$

From here we finally have

$$\|u\|_2 \le \|f(x,y)\|_{L_2(\Omega)}, \tag{18}$$

where $C_1 = \frac{1}{\delta+\mu}$.

Next, we calculate the norm $\|u_x\|_2$:

$$\|u_x\|^2_{L_2(\Omega)} = \left\| \sum_{n=-\infty}^{\infty} in(l_n + \mu I)^{-1} e^{inx} \right\|^2_{L_2(\Omega)} \le \sup_{\{n\}} \left\| in(l_n+\mu I)^{-1} \right\|^2_{2\to 2} \cdot 2\pi \cdot \sum_{n=-\infty}^{\infty} \|f_n(y)\|^2_{L_2(0,1)} \le$$

$$\le \sup_{\{n\}} |n|^2 \cdot \left\| in(l_n+\mu I)^{-1} \right\|^2_{2\to 2} \cdot \|f(x,y)\|^2_{L_2(\Omega)}.$$

Hence and from inequality (8) we have

$$\|u_x\|^2_{L_2(\Omega)} \le \sup_{\{n\}} |n|^2 \left\| in(l_n+\mu I)^{-1} \right\|^2_{2\to 2} \cdot \|f(x,y)\|^2_{L_2(\Omega)} \le \sup_{\{n\}} |n|^2 \cdot \frac{1}{|n|^2 \cdot \delta_0^2} \|f(x,y)\|^2_{L_2(\Omega)}.$$

Hence

$$\|u_x\|_{L_2(\Omega)} \le C_2 \|f(x,y)\|_{L_2(\Omega)}, \tag{19}$$

where $C_2 = \frac{1}{(\delta+\mu)^{\frac{1}{2}}}$.

Then, repeating the above calculations, we get the following estimate

$$\|u_y\|_{L_2(\Omega)} \le C_3 \|f(x,y)\|_{L_2(\Omega)}, \tag{20}$$

where $C_2 = \frac{1}{\delta_0}$.

Using the equalities (18)–(20), we find that

$$\|u\|_{2,1,\Omega} \le C \|f(x,y)\|_{L_2(\Omega)},$$

where $C = \max\{C_1, C_2, C_3\}$. The theorem is proved.

Acknowledgments

This paper was supported by the grant AP19676466 of the Ministry of Science and Higher Education of Republic of Kazakhstan.

Author Contributions

A.O. Suleimbekova collected and analyzed data, and led manuscript preparation. B.M. Musilimov assisted in data collection and analysis. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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Үшінші ретті дифференциалдық теңдеулердің бір класы үшін Дирихле есебі шешімдерінің бар болуы және коэрцитивті бағалаулары туралы

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Білетініміздей үшінші ретті дербес туындылы дифференциалдық теңдеулер толқындар теориясының негізгі теңдеулерінің бірі. Мысалы, айнымалы коэффициентті сызықталған Кортевег–де Фриз типті теңдеуі иондық акустикалық толқындарды кристалдық тордағы плазмалық және акустикалық толқындарға модельдейді. Жұмыста тіктөртбұрышта берілген айнымалы коэффициентті үшінші ретті дербес туындылы еселенген теңдеулердің бір класының шешімдерінің қасиеттері зерттелген. Күшті шешімнің бар болуы мен жалғыздығына жеткілікті шарттар алынған. Еселенген теңдеудің шешімі өзінің тегістігін сақтамайтынын ескерсек, бұл қиындықтар өз кезегінде коэрцитивті бағалауға әсер етеді.

Кілт сөздер: резольвента, үшінші ретті дифференциалдық теңдеулер, Дирихле есебі, коэрцитивті бағалаулар.

О существовании и коэрцитивных оценках решений задачи Дирихле для одного класса дифференциальных уравнений третьего порядка

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Как известно, уравнения в частных производных третьего порядка являются одним из основных уравнений теории волн. В частности, линеаризованное уравнение типа Кортевега–де Фриза с переменными коэффициентами моделирует ионно-акустические волны в плазменные и акустические волны на кристаллической решетке. В данной работе исследованы свойства решений одного класса вырождающихся уравнений в частных производных третьего порядка с переменными коэффициентами, заданных в прямоугольнике. Установлены достаточные условия существования и единственности сильного решения. Заметим, что решение вырождающегося уравнения не сохраняет свою гладкость, следовательно, эти трудности, в свою очередь, влияют на коэрцитивные оценки.

Ключевые слова: резольвента, дифференциальные уравнения третьего порядка, задача Дирихле, коэрцитивные оценки.

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https://doi.org/10.31489/2024M2/186-196 *Research article*

About unimprovability the embedding theorems for anisotropic Nikol'skii-Besov spaces with dominated mixed derivates and mixed metric and anisotropic Lorentz spaces

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The embedding theory of spaces of differentiable functions of many variables studies important connections and relationships between differential (smoothness) and metric properties of functions and has wide application in various branches of pure mathematics and its applications. Earlier, we obtained the embedding theorems of different metrics for Nikol'skii-Besov spaces with a dominant mixed smoothness and mixed metric, and anisotropic Lorentz spaces. In this work, we showed that the conditions for the parameters of spaces in the above theorems are unimprovable. To do this, we built the extreme functions included in the spaces from the left sides of the embeddings and not included in the "slightly narrowed" spaces from the spaces in the right parts of the embeddings.

Keywords: anisotropic Lorentz spaces, anisotropic Nikol'skii-Besov spaces, generalized mixed smoothness, mixed metric, embedding theorems.

2020 Mathematics Subject Classification: 46E35.

Introduction

One of the first results related to the theory of embedding of spaces of differentiable functions was a result of S.L. Sobolev [1]. This theory studies important relations of differential (smoothness) properties of functions in various metrics. Further development of this theory is associated with new classes of function spaces defined and studied in the works of S.M. Nikol'skii [2], O.V. Besov [3], P.I. Lizorkin [4], H. Triebel [5], J. Bergh and J. Löfström [6], and many others. The development of this research was determined both by its internal problems and by its applications in the theory of boundary value problems of mathematical physics and approximation theory (see, for example, [7–11]).

In the 1960s, in the works of S.M. Nikol'skii [1], A.D. Dzhabrailov [12] and T.I. Amanov [13] begins the study of spaces with a dominant mixed derivative. Further study of spaces with a dominant mixed derivative which is related with the theory of embedding and interpolation and the theory of approximations is associated with the works of A.P. Uninskij, V.N. Temlyakov, E.D. Nursultanov, D.B. Bazarkhanov, A.S. Romanyuk, G.A. Akishev, K.A. Bekmaganbetov, Ye. Toleugazy and others (see, for example, [14–20]).

In a serie of articles [21–23] we studied various properties of Nikol'skii-Besov spaces with a dominant mixed derivative and with a mixed metric. In these articles, we investigated the interpolation properties of these spaces, obtained limit embedding theorems for these spaces and anisotropic Lorentz spaces, and proved theorems on traces and continuations of functions.

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This research of Y. Toleugazy was supported in part by the Committee of Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan (project AP14869553).

Received: 23 December 2023; *Accepted:* 20 February 2024.

In the work of K.A. Bekmaganbetov, K.E. Kervenev, Ye. Toleugazy [22], embeddings for Nikol'skii-Besov spaces with a dominant mixed derivative and a mixed metric and anisotropic Lorentz spaces were studied. In this article we are showing that the conditions in the embedding theorems from the work [22] are unimprovable. We build the extreme functions included in the spaces from the left sides of the embeddings and not included in the "slightly narrowed" spaces from spaces in the right parts of the embeddings.

Preliniminaries and auxiliary results

Let $f(\mathbf{x}) = f(x_1, \dots, x_n)$ be a measurable function defined on $\mathbb{T}^n$. Let multiindexes $\mathbf{1} \le \mathbf{p} = (p_1, \dots, p_n) \le \infty$. A Lebesgue space $L_{\mathbf{p}}(\mathbb{T}^n)$ with mixed metric is the set of functions for which the following quantity is finite

$$\|f\|_{L_{\mathbf{p}}(\mathbb{T}^n)}=\left(\int_{\mathbb{T}}\left(\dots\left(\int_{\mathbb{T}}|f(x_1,\dots,x_n)|^{p_1}dx_1\right)^{p_2/p_1}\dots\right)^{p_n/p_{n-1}}dx_n\right)^{1/p_n}.$$

Here, the expression $\left(\int_{\mathbb{T}}|f(t)|^p dt\right)^{1/p}$ for p=∞ is understood as $\operatorname{esssup}_{t\in\mathbb{T}}|f(t)|$.

For multiple trigonometric series $f(\mathbf{x}) \sim \sum_{\mathbf{k}\in\mathbb{Z}^n} a_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}$ we denote by

$$\Delta_{\mathbf{s}}(f,\mathbf{x}) = \sum_{\mathbf{k}\in\rho(\mathbf{s})} a_{\mathbf{k}}(f)e^{i(\mathbf{k},\mathbf{x})},$$

where $\rho(\mathbf{s}) = \{\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n : 2^{s_i-1} \le |k_i| < 2^{s_i}, i = 1, \dots, n\}$, $(\mathbf{k},\mathbf{x}) = \sum_{j=1}^{n} k_j x_j$ is the inner product of vectors $\mathbf{k}$ and $\mathbf{x}$.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $\mathbf{1} \le \tau = (\tau_1, \dots, \tau_n) \le \infty$. The anisotropic Nikol'skii-Besov space with generalized mixed derivates and mixed metric $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ is a set of the series $f \sim \sum_{\mathbf{k}\in\mathbb{Z}^{\mathbf{n}}} a_{\mathbf{k}} e^{i(\mathbf{k},\mathbf{x})}$ such that

$$\|f\|_{B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)} = \left\|\left\{2^{(\alpha,\mathbf{s})}\|\Delta_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right\}\right\|_{l_\tau} < \infty,$$

where $\|\cdot\|_{l_\tau}$ is the norm of a discrete Lebesgue space with mixed metric l_τ.

We will also need the anisotropic Lorentz spaces which introduced by E.D. Nursultanov in [24].

Let $f(\mathbf{x}) = f(x_1, \dots, x_n)$ be a measurable function defined on $\mathbb{T}^n$. We denote by $f^*(\mathbf{t}) = f^{*_1,\dots,*_n}(t_1, \dots, t_n)$ the function obtained from $f(\mathbf{x}) = f(x_1, \dots, x_n)$ by applying the non-increasing rearrangement successively with respect to each of the variables $x_1, \dots, x_n$ (the other variables are assumed to be fixed).

Let multiindexes $\mathbf{q} = (q_1, \dots, q_n)$, $\theta = (\theta_1, \dots, \theta_n)$ satisfy the conditions: if $0 < q_j < \infty$, then $0 < \theta_j \le \infty$, if $q_j = \infty$, then $\theta_j = \infty$ for every $j = 1, \dots, n$. An anisotropic Lorentz space $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ is the set of functions for which the following quantity is finite

$$\|f\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} =$$

$$= \left(\int_{\mathbb{T}}\dots\left(\int_{\mathbb{T}}\left(t_1^{1/q_1}\dots t_n^{1/q_n} f^{*_1,\dots,*_n}(t_1,\dots,t_n)\right)^{\theta_1}\frac{dt_1}{t_1}\right)^{\theta_2/\theta_1}\dots\frac{dt_n}{t_n}\right)^{1/\theta_n}.$$

The following theorems were obtained in the work [22]:

Theorem A. Let $-\infty < \alpha_{\mathbf{0}} = (\alpha_1^0, \ldots, \alpha_n^0) \leq \alpha_{\mathbf{1}} = (\alpha_1^1, \ldots, \alpha_n^1) < \infty$, $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$ and $\mathbf{1} < \mathbf{p_0} = (p_1^0, \ldots, p_n^0), \mathbf{p_1} = (p_1^1, \ldots, p_n^1) < \infty$. Then the embedding

$$B_{\mathbf{p_1}}^{\alpha_{\mathbf{1}}\tau}(\mathbb{T}^n) \hookrightarrow B_{\mathbf{p_0}}^{\alpha_{\mathbf{0}}\tau}(\mathbb{T}^n)$$

holds for $\alpha_{\mathbf{0}} - \mathbf{1}/\mathbf{p_0} = \alpha_{\mathbf{1}} - \mathbf{1}/\mathbf{p_1}$.

Theorem B. Let $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_n) < \mathbf{q} = (q_1, \ldots, q_n) < \infty$ and $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$. Then the embedding

$$B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{T}^n)$$

holds for $\alpha = \mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}$.

Theorem C. Let $\mathbf{1} < \mathbf{q} = (q_1, \ldots, q_n) < \mathbf{p} = (p_1, \ldots, p_n) < \infty$ and $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$. Then the embedding

$$L_{\mathbf{q}\tau}(\mathbb{T}^n) \hookrightarrow B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$$

holds for $\alpha = \mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}$.

Main results

In this work, we show that the conditions for the parameters providing attachments are unimprovable. The proof of these facts in Theorems A–C we carry out by constructing extreme functions.

The following theorem shows that the condition under which the embedding from Theorem A is valid is unimprovable.

Theorem 1. Let $-\infty < \alpha_{\mathbf{0}} = (\alpha_1^0, \ldots, \alpha_n^0) \leq \alpha_{\mathbf{1}} = (\alpha_1^1, \ldots, \alpha_n^1) < \infty$, $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$, $\mathbf{1} < \mathbf{p_0} = (p_1^0, \ldots, p_n^0)$, $\mathbf{p_1} = (p_1^1, \ldots, p_n^1) < \infty$ and $\alpha_{\mathbf{0}} - \mathbf{1}/\mathbf{p_0} = \alpha_{\mathbf{1}} - \mathbf{1}/\mathbf{p_1}$, then for arbitrary $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) > \mathbf{0}$ and $\delta = (\delta_1, \ldots, \delta_n) > \mathbf{0}$ there is a function $f_\beta^{(1)} \in B_{\mathbf{p_1}}^{\alpha_{\mathbf{1}}\tau}(\mathbb{T}^n)$ such that $f_\beta^{(1)} \notin B_{\mathbf{p_0}}^{(\alpha_{\mathbf{0}}+\varepsilon)\tau}(\mathbb{T}^n) \cup B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\tau}(\mathbb{T}^n)$.

Proof. Taking into account the estimate for the norm of a one-dimensional Dirichlet kernel, we obtain the relation

$$\left\| \sum_{k=2^{\mathbf{s}-1}}^{2^{\mathbf{s}}-1} e^{i(k,\cdot)} \right\|_{L_p(\mathbb{T})} \sim 2^{(1/p',s)}, 1 < p < +\infty.$$

From this relation in the multiple case we have

$$\|\sigma_{\mathbf{s}}(\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} = \left\| \sum_{\mathbf{k}=2^{\mathbf{s}-1}}^{2^{\mathbf{s}}-1} e^{i(\mathbf{k},\cdot)} \right\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \sim 2^{(\mathbf{1}/\mathbf{p}',\mathbf{s})}. \tag{1}$$

Consider the function $f_\beta^{(1)}(\mathbf{x}) = \sum_{\mathbf{s}=\mathbf{0}}^{\infty} 2^{-(\beta,\mathbf{s})}\sigma_{\mathbf{s}}(\mathbf{x})$, where

$$\alpha_{\mathbf{1}} + \frac{\mathbf{1}}{\mathbf{p}_{\mathbf{1}}'} < \beta < \min\left(\alpha_{\mathbf{0}} + \varepsilon + \frac{1}{\mathbf{p}_0'}, \alpha_{\mathbf{0}} + \frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'}\right).$$

According to estimate (1) we have

$$\left\|f_\beta^{(1)}\right\|_{B_{\mathbf{p_1}}^{\alpha_{\mathbf{1}}\tau}(\mathbb{T}^n)} = \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{1}},\mathbf{s})} \left\|\Delta_{\mathbf{s}}(f_\beta^{(1)})\right\|_{L_{\mathbf{p_1}}(\mathbb{T}^n)}\right)^{\tau}\right)^{\mathbf{1}/\tau} =$$

$$= \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{1}},\mathbf{s})} 2^{-(\beta,\mathbf{s})} \left\| \sigma_{\mathbf{s}} \right\|_{L_{\mathbf{p_1}}(\mathbb{T}^n)} \right)^{\tau} \right)^{\mathbf{1}/\tau} =$$

$$= \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{1}}-\beta+\frac{\mathbf{1}}{\mathbf{p}_{\mathbf{1}}'},\mathbf{s})} \right)^{\tau} \right)^{\mathbf{1}/\tau} < +\infty,$$

as $\alpha_{\mathbf{1}} + \frac{\mathbf{1}}{\mathbf{p}_{\mathbf{1}}'} - \beta < \mathbf{0}$.

This means that $f_{\beta}^{(1)} \in B_{\mathbf{p_0}}^{\alpha_{\mathbf{0}}\tau}(\mathbb{T}^n)$.

Similarly, we obtain that

$$\left\| f_{\beta}^{(1)} \right\|_{B_{\mathbf{p_0}}^{(\alpha_{\mathbf{0}}+\varepsilon)\tau}(\mathbb{T}^n)} = \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{0}}+\varepsilon,\mathbf{s})} \left\| \Delta_{\mathbf{s}}(f_{\beta}^{(1)}) \right\|_{L_{\mathbf{p_0}}(\mathbb{T}^n)} \right)^{\tau} \right)^{\mathbf{1}/\tau} \geq$$

$$\geq C_1 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{0}}+\varepsilon,\mathbf{s})} 2^{\left(\frac{\mathbf{1}}{\mathbf{p}_{\mathbf{0}}'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} =$$

$$= C_1 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{\left(\alpha_{\mathbf{0}}+\varepsilon+\frac{\mathbf{1}}{\mathbf{p}_{\mathbf{0}}'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} = +\infty,$$

as $\alpha_{\mathbf{0}} + \varepsilon + \frac{\mathbf{1}}{\mathbf{p}_{\mathbf{0}}'} - \beta > \mathbf{0}$. Therefore $f_{\beta}^{(1)} \notin B_{\mathbf{p_0}}^{(\alpha_{\mathbf{0}}+\varepsilon)\tau}(\mathbb{T}^n)$.

Further, we will show that $f_{\beta}^{(1)} \notin B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\tau}(\mathbb{T}^n)$. We have

$$\left\| f_{\beta}^{(1)} \right\|_{B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\tau}(\mathbb{T}^n)} = \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{0}},\mathbf{s})} \left\| \Delta_{\mathbf{s}}(f_{\beta}^{(1)}) \right\|_{L_{(\mathbf{p_0}+\delta)}(\mathbb{T}^n)} \right)^{\tau} \right)^{\mathbf{1}/\tau} \geq$$

$$\geq C_2 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_{\mathbf{0}},\mathbf{s})} 2^{\left(\frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} =$$

$$= C_2 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{\left(\alpha_{\mathbf{1}}+\frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} = +\infty,$$

considering that $\alpha_{\mathbf{0}} + \frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'} - \beta > 0$. Therefore $f_{\beta}^{(1)} \notin B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\tau}(\mathbb{T}^n)$.

Thus, we have shown that $f_{\beta}^{(1)} \notin B_{\mathbf{p_0}}^{(\alpha_{\mathbf{0}}+\varepsilon)\tau}(\mathbb{T}^n) \cup B_{(\mathbf{p_0}+\delta)}^{\alpha_{\mathbf{0}}\tau}(\mathbb{T}^n)$.

The proof is complete.

The following theorem shows that the condition under which the embedding of Theorem B is valid, is not improved.

Theorem 2. Let $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_n) < \mathbf{q} = (q_1, \ldots, q_n) < \infty$ and $\alpha = \mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}$. Then for an arbitrary $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) > \mathbf{0}$ there is a function $f_{\beta}^{(2)} \in B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ such that $f_{\beta}^{(2)} \notin L_{\mathbf{q}+\varepsilon,\tau}(\mathbb{T}^n)$.

Proof. First, let's show that $f_{\beta}^{(2)} \in B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$. Consider the function $f_{\beta}^{(2)}(\mathbf{x}) = \sum_{\mathbf{s}=\mathbf{0}}^{\infty} 2^{-(\beta,\mathbf{s})} \sigma_{\mathbf{s}}(\mathbf{x})$, where $\alpha + \frac{\mathbf{1}}{\mathbf{p}'} < \beta \leq \frac{\mathbf{1}}{(\mathbf{q}+\varepsilon)'}$.

By analogy with Theorem 1, we have

$$\left\|f_\beta^{(2)}\right\|_{B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)} \sim \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{(\alpha-\beta+\frac{\mathbf{1}}{\mathbf{p}'},\mathbf{s})}\right)^{\tau}\right)^{\mathbf{1}/\tau} < \infty$$

as $\alpha + \frac{\mathbf{1}}{\mathbf{p}'} - \beta < \mathbf{0}$. It means that $f_\beta^{(2)} \in B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$.

In order to show that $f_\beta^{(2)} \notin L_{\mathbf{q}+\varepsilon,\tau}(\mathbb{T}^n)$ we use Theorem C.

We have

$$\left\|f_\beta^{(2)}\right\|_{L_{\mathbf{q}+\varepsilon,\tau}(\mathbb{T}^n)} \geq \left\|f_\beta^{(2)}\right\|_{B_{\mathbf{p}}^{\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}+\varepsilon},\tau}(\mathbb{T}^n)} =$$

$$= \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}+\varepsilon},\mathbf{s}\right)}\left\|\Delta_{\mathbf{s}}(f_\beta^{(2)})\right\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right)^{\tau}\right)^{1/\tau} =$$

$$= C_3\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}+\varepsilon},\mathbf{s}\right)}2^{\left(\frac{\mathbf{1}}{\mathbf{p}'}-\beta,\mathbf{s}\right)}\right)^{\tau}\right)^{1/\tau} =$$

$$= C_3\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}+\frac{\mathbf{1}}{\mathbf{p}'}-\frac{\mathbf{1}}{\mathbf{q}+\epsilon}-\beta,\mathbf{s}\right)}\right)^{\tau}\right)^{1/\tau} =$$

$$= C_3\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{q}+\varepsilon}-\beta,\mathbf{s})}\right)^{\tau}\right)^{1/\tau} = C_3\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{(\mathbf{q}+\varepsilon)'}-\beta,\mathbf{s}\right)}\right)^{\tau}\right)^{1/\tau} = +\infty,$$

as $\frac{1}{(\mathbf{q}+\varepsilon)'} - \beta > 0$. It means that $f_\beta^{(2)} \notin L_{\mathbf{q}+\varepsilon,\tau}(\mathbb{T}^n)$.

The proof is complete.

The following theorem shows that the condition, under which the embedding of Theorem C is valid, is not improved.

Theorem 3. Let $\mathbf{1} < \mathbf{q} = (q_1, \dots, q_n) < \mathbf{p} = (p_1, \dots, p_n) < \infty$ and $\alpha = \mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}$. Then for an arbitrary $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) > \mathbf{0}$ and $\delta = (\delta_1, \dots, \delta_n) > \mathbf{0}$ there is a function $f_\beta^{(3)} \in L_{\mathbf{q},\tau}(\mathbb{T}^n)$ such that $f_\beta^{(3)} \notin B_{\mathbf{p}}^{\alpha+\epsilon,\tau}(\mathbb{T}^n) \cup B_{(\mathbf{p}+\delta)}^{\alpha\tau}(\mathbb{T}^n)$.

Proof. Let's choose a function $f_\beta^{(3)}(\mathbf{x})$ the same as in Theorem 1 with β, satisfying the condition $\frac{\mathbf{1}}{\mathbf{q}'} < \beta \leq \min\left(\alpha + \varepsilon + \frac{1}{\mathbf{p}'}, \alpha + \frac{\mathbf{1}}{(\mathbf{p}+\delta)'}\right)$.

In order to show that $f_\beta^{(3)} \in L_{\mathbf{q},\tau}(\mathbb{T}^n)$ let's use Theorem B. We have

$$\left\|f_\beta^{(3)}\right\|_{L_{\mathbf{q},\tau}(\mathbb{T}^n)} \leq C_4\left\|f_\beta^{(3)}\right\|_{B_{\mathbf{p}}^{\left(\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}}\right)\tau}(\mathbb{T}^n)} =$$

$$= C_4\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}},\mathbf{s}\right)}\left\|\Delta_{\mathbf{s}}(f_\beta^{(3)})\right\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right)^{\tau}\right)^{1/\tau} \leq$$

$$\leq C_5\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{q}},\mathbf{s}\right)}2^{\left(-\beta+\frac{\mathbf{1}}{\mathbf{p}'},\mathbf{s}\right)}\right)^{\tau}\right)^{1/\tau} =$$

$$= C_5\left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty}\left(2^{\left(\frac{\mathbf{1}}{\mathbf{p}}+\frac{\mathbf{1}}{\mathbf{p}'}-\beta-\frac{\mathbf{1}}{\mathbf{q}},\mathbf{s}\right)}\right)^{\tau}\right)^{1/\tau} =$$

$$= C_5 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{q}}-\beta,\mathbf{s})} \right)^{\tau} \right)^{1/\tau} = C_5 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\frac{\mathbf{1}}{\mathbf{q}'}-\beta,\mathbf{s})} \right)^{\tau} \right)^{1/\tau} < +\infty,$$

as $\frac{\mathbf{1}}{\mathbf{q}'} - \beta < \mathbf{0}$, i.e. $f_\beta^{(3)} \in L_{\mathbf{q},\tau}(\mathbb{T}^n)$.

Let's show that $f_\beta \notin B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^n)$.

Let us estimate the norm of this function from below

$$\left\| f_\beta^{(3)} \right\|_{B_{\mathbf{p}}^{(\alpha+\varepsilon)\tau}(\mathbb{T}^n)} = \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha+\varepsilon,\mathbf{s})} \left\| \Delta_{\mathbf{s}}(f_\beta^{(3)}) \right\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right)^{\tau} \right)^{\mathbf{1}/\tau} \geq$$

$$\geq C_6 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha+\varepsilon,\mathbf{s})} 2^{\left(\frac{\mathbf{1}}{\mathbf{p}'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} =$$

$$= C_6 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{\left(\alpha+\varepsilon+\frac{\mathbf{1}}{\mathbf{p}'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} = +\infty,$$

as $\alpha + \varepsilon - \beta + \frac{\mathbf{1}}{\mathbf{p}'} > \mathbf{0}$, i.e. $f_\beta^{(3)} \notin B_{\mathbf{p}}^{(\alpha+\varepsilon)\tau}(\mathbb{T}^n)$.

Further, we will show that $f_\beta^{(3)} \notin B_{(\mathbf{p_0}+\delta)}^{\alpha_\mathbf{0}\ \tau}(\mathbb{T}^n)$.

Considering that $\alpha_\mathbf{0} + \frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'} - \beta > 0$, we have

$$\left\| f_\beta^{(3)} \right\|_{B_{(\mathbf{p_0}+\delta)}^{\alpha_\mathbf{0}\ \tau}(\mathbb{T}^n)} = \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_\mathbf{0},\mathbf{s})} \left\| \Delta_{\mathbf{s}}(f_\beta^{(3)}) \right\|_{L_{(\mathbf{p_0}+\delta)}(\mathbb{T}^n)} \right)^{\tau} \right)^{\mathbf{1}/\tau} \geq$$

$$\geq C_7 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{(\alpha_\mathbf{0},\mathbf{s})} 2^{\left(\frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} =$$

$$= C_7 \left(\sum_{\mathbf{s}=\mathbf{0}}^{\infty} \left(2^{\left(\alpha_\mathbf{1}+\frac{\mathbf{1}}{(\mathbf{p_0}+\delta)'}-\beta,\mathbf{s}\right)} \right)^{\tau} \right)^{\mathbf{1}/\tau} = +\infty,$$

as $f_\beta^{(3)} \notin B_{(\mathbf{p_0}+\delta)}^{\alpha_\mathbf{0}\ \tau}(\mathbb{T}^n)$.

Thus, we have shown that $f_\beta^{(3)} \notin B_{\mathbf{p_0}}^{(\alpha_\mathbf{0}+\varepsilon)\tau}(\mathbb{T}^n) \cup B_{(\mathbf{p_0}+\delta)}^{\alpha_\mathbf{0}\ \tau}(\mathbb{T}^n)$.

The proof is complete.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Үстем аралас туындысы және аралас метрикасы бар анизотропты Никольский-Бесов кеңістіктері және анизотропты Лоренц кеңістіктері үшін ену теоремаларының жетілдірілмейтіндігі туралы

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Дифференциалданатын функциялар кеңістіктерінің енгізу теориясы әртүрлі метрикалардағы функциялардың дифференциалдық (тегістіліктік) қасиеттерінің маңызды байланыстары мен қатынастарын зерттейді. Математикалық физиканың шектік есептер теориясында, жуықтау теориясында және математиканың басқа да салаларында кеңінен колданысқа ие. Мақалада үстем аралас тегістілігі және аралас метрикасы бар Никольский-Бесовтың кеңістіктері үшін және Лоренцтің анизотропты кеңістіктері үшін енгізу теоремалары берілген. Ұсынылған жұмыста жоғарыда көрсетілген теоремалардағы параметрлердің жетілдірілмейтіндігі көрсетілді. Осыны көрсетуге біз сол жақтағы енулердегі кеңістіктер үшін шекті функцияларды құрамыз және олар оң жақтағы енулерде "сәл ғана жіңішкертілген" кеңістіктерде жатпайтындығы көрсетілген.

Кілт сөздер: Лоренцтің анизотропты кеңістіктері, Никольский-Бесов типтес кеңістіктер, үстем аралас туынды, аралас метрика, ену теоремалары.

О неулучшаемости теорем вложения для анизотропных пространств Никольского-Бесова с доминирующей смешанной производной и смешанной метрикой и анизотропных пространств Лоренца

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Теория вложения пространств дифференцируемых функций многих переменных изучает важные связи и соотношения между дифференциальными (гладкостными) и метрическими свойствами функций и имеет широкое применение в различных разделах чистой математики и ее приложениях. Ранее нами были получены предельные теоремы вложения разных метрик для пространств Никольского–Бесова с доминирующей смешанной гладкостью и со смешанной метрикой и для анизотропных пространств Лоренца. В данной работе мы показали, что условия на параметры пространств в отмеченных выше теоремах являются неулучшаемыми. Для этого мы построили крайние функции, входящие в пространства из левых сторон вложений и не входящие в «немного зауженные» пространства, чем пространства, стоящие в правых частях вложений.

Ключевые слова: анизотропные пространства Лоренца, пространства Никольского–Бесова, доминирующая смешанная производная, смешанная метрика, теоремы вложения.

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https://doi.org/10.31489/2024M2/197-210 *Research article*

On conditions for the weighted integrability of the sum of the series with monotonic coefficients with respect to the multiplicative systems

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In this paper, we studied the issues of integrability with the weight of the sum of series with respect to multiplicative systems, provided that the coefficients of the series are monotonic. The conditions for the weight function and the series' coefficients are found; the sum of the series belongs to the weighted Lebesgue space L_p $(1 < p < \infty)$. In addition, the case of $p = 1$ was considered. In this case, other conditions for the weighted integrability of the sum of the series under consideration are found. In the case of the generating sequence's boundedness, the proved theorems imply an analogue of the well-known Hardy-Littlewood theorem on trigonometric series with monotone coefficients.

Keywords: the multiplicative systems, the weighted integrability of the sum of series, generator sequence, monotone coefficients, Hardy-Littlewood theorem.

2020 Mathematics Subject Classification: 42A32, 42C10.

Introduction

The Hardy-Littlewood theorem concerning series with monotone coefficients in the theory of trigonometric series states the following [1, 2]: for the series $\sum_{n=0}^{\infty} a_n \cos nx$, where the coefficients a_n decrease to zero as n approaches infinity, to represent the Fourier series of a function $f(x) \in L_p[0, 2\pi]$, where $1 < p < \infty$, it is both necessary and sufficient that $\sum_{n=1}^{\infty} a_n^p n^{p-2} < \infty$.

An analogue of this theorem for the Walsh system was proved by Moritz F. [3]. For multiplicative systems with bounded generating sequences p $(1 \le \sup_n p_n < c)$ it was proved by Timan M.F., Tukhliev K. [4].

The weighted integrability of the sum of trigonometric series with generalized monotone coefficients was studied in the works of Tikhonov S.Yu., Dyachenko M.I. [5, 6] and others. Weighted integrability for the sum of series for multiplicative systems is considered in the works of Volosivets S.S., Fadeev R.N. [7, 8], Bokayev N.A., Mukanov Zh.B. [9].

In this paper, we consider weight functions with other conditions. This article is a continuation of the article [10].

1 *Notation and Preliminaries*

This paper examines a series characterized by monotonic coefficients concerning the multiplicative systems. We delve into the inquiry: what criteria regarding the weight function and series coefficients ensure that the sum of the series falls within the L_p space with weighting? Before delving into the main discussion, we define multiplicative systems.

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Received: 28 December 2023; *Accepted:* 04 March 2024.

Definition 1. Let $\{p_k\}_{k=1}^{\infty}$ be a sequence of natural numbers $p_k \geq 2,\ k \in \mathbb{N},\ \sup\limits_k p_k = N < \infty$. By definition, let us put

$$m_0 = 1,\ m_n = p_1 p_2 \cdots p_n,\ n \in \mathbb{N}.$$

Then every point $x \in [0,1)$ has decomposition

$$x = \sum_{k=1}^{\infty} \frac{x_k}{m_k}, \quad x_k \in \mathbb{Z} \cap [0, p_k), \tag{1}$$

where $\mathbb{Z}$ is the set of integers. Decomposition (1) is uniquely defined if $x = n/m_k$ takes a decomposition with a finite number of nonzero x_k. If $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is represented as

$$n = \sum_{j=1}^{\infty} \alpha_j m_{j-1}, \quad \alpha_j \in \mathbb{Z} \cap [0, p_j),$$

then for the numbers $x \in [0,1)$ we put by definition

$$\psi_n(x) = exp\left(2\pi i \sum_{j=1}^{\infty} \frac{\alpha_j x_j}{p_j}\right), \quad n \in \mathbb{Z}_+.$$

It is known that the system $\{\psi_n\}_{n=0}^{\infty}$, called the Price system, is an orthonormalized system complete in $L^1(0,1)$ (see [10] or [11]). If all $p_k = 2$, then $\{\psi_n\}_{n=0}^{\infty}$ coincides with Walsh system in the Paley numbering.

Let $L^p(G),\ G := [0,1),\ 1 \leq p < \infty,$ denote Lebesgue space with norm

$$\|f\|_p = \left(\int_G |f(x)|^p dx\right)^{\frac{1}{p}}, \quad \|f\|_\infty = ess \sup_{x \in G} |f(x)|.$$

Definition 2. Let $\varphi(x)$ be a non-negative measurable function on $(0, \infty)$. We say that $\varphi(x)$ satisfies condition B_1, if for all $x \geq 1$

$$\int_x^\infty \frac{\varphi(t)}{t^2} dt \leq C \frac{\varphi(x)}{x},$$

where C is a positive number independent of x.

For example, the function $\varphi(t) = t^\alpha$ $(\alpha < 1)$ satisfies condition B_1.

To prove the main results, we need the following auxiliary assertions.

By

$$D_n(x) = \sum_{k=0}^{n-1} \psi_k(x), \quad n = 1, 2, ...,$$

denote the Dirichlet kernel of the system $\{\psi_n(x)\}$.

Lemma A. (see [11] or [12]) For any $k \in \mathbb{N}$ and $x \in [0,1)$ the Dirichlet kernels have the following properties:

$$D_{m_k} = \begin{cases} m_k, & \text{если } x \in \left[0, \frac{1}{m_k}\right), \\ 0, & \text{если } x \notin \left[0, \frac{1}{m_k}\right). \end{cases} \tag{2}$$

2 Main Results

In this section, under certain conditions imposed on the weight function, a necessary and sufficient condition is given for the function to belong to a Lebesgue space with the weight of the sum of series for the multiplicative systems. The following theorem about integrability with the weight of the sum of series with monotone coefficients is valid.

Theorem 1. [10] Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\sup p_n = N < \infty$

$$f(x) = \sum_{k=0}^{\infty} a_k \psi_k(x), \quad a_k \downarrow 0 \ at\ k \to \infty$$

and let $\varphi(x)$ be a non-negative measurable function on $[1, \infty)$. Then

1^0. If the function $\varphi^p(x)$ satisfies condition B_1 and

$$\sum_{n=1}^{\infty} a_n^p \cdot n^p \int_n^{n+1} \frac{\varphi^p(x)}{x^2} dx < \infty,$$

then $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0, 1)$.

2^0. If $\varphi^{-p'}(x)$ it satisfies the condition B_1 and $\varphi\left(\frac{1}{x}\right) f(x) \in L_p(0, 1)$, then it takes place (6).

Remark. If the weight function $\varphi(x)$ has the form $\varphi(x) = x^{\alpha}$, then in this case $\varphi^p(x)$ and $\varphi^{-p'}(x)$ satisfy condition B_1 at $-\frac{1}{p'} < \alpha < \frac{1}{p}$ and condition (7) has the form

$$\sum_{n=1}^{\infty} a_n^p \cdot n^{p(\alpha+1)-1} < \infty.$$

Let $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\varphi(x) \geq 0$ is some locally integrable on function on $[0, 1]$. Lebesgue measurable on $[0, 1]$ function $f(x)$, belongs to space $L_{p,\varphi}$, if

$$\|f\|_{p,\varphi} = \left(\int_0^1 |f(x)\ \varphi(x)|^p\, dx\right)^{\frac{1}{p}} < \infty.$$

Let us put

$$A_p = \sup_{0 \leq t \leq 1} \left(\int_0^t \varphi^p(x)\, dx\right)^{\frac{1}{p}} \left(\int_t^1 (x\varphi(x))^{-p'} dx\right)^{\frac{1}{p'}},$$

$$B_p = \sup_{0 \leq t \leq 1} \left(\int_t^1 \left(\frac{\varphi(x)}{x}\right)^p dx\right)^{\frac{1}{p}} \left(\int_0^t \varphi^{-p'}(x)\, dx\right)^{\frac{1}{p'}}.$$

The following theorem is true.

Theorem 2. Let $1 \leq p < \infty, \frac{1}{p} + \frac{1}{p'} = 1$, $\sup\limits_n p_n = N < \infty$, and $\varphi(x)$ be some locally integrable function on $[0, 1]$ and

$$f(x) \equiv \sum_{k=0}^{\infty} a_k(f)\, \Psi_k(x),$$

where $a_k(f) \downarrow 0$ at $k \to \infty$. Then

1^0. If $A_p < \infty$ and

$$D_p = \sum_{n=1}^{\infty} a_n^p \cdot n^p \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\, dx < \infty, \tag{3}$$

that $f(x) \in L_{p,\varphi}(0,1)$ and

$$\|f\|_{p,\varphi}^{p} \leq C_p D_p.$$

2^0. If $B_p < \infty$ и $f(x) \in L_{p,\varphi}$, then the series (3) converges and

$$\|f\|_{p,\varphi}^{p} \geq C_p D_p.$$

Combining points 1^0 and 2^0 of this theorem, we can formulate the following statement:

Theorem 3. Let $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\sup\limits_{n} p_n = N < \infty$ and $\varphi(x)$ be a non-negative, locally integrable function such that

$$\max(A_p,\ B_p) < \infty.$$

Then, in order for the function $f(x) \equiv \sum_{k=0}^{\infty} a_k(f)\,\Psi_k(x)$, where $a_k(f) \downarrow 0$ at $k \to \infty$ to belong to the class $L_{p,\varphi}(0,\ 1)$ it is necessary and sufficient to satisfy the condition

$$D_p = \sum_{n=1}^{\infty} a_n^p \cdot n^p \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx < \infty.$$

In this case, there exists a relation

$$\|f\|_{L_{p,\varphi}} \approx D_p^{\frac{1}{p}}.$$

To prove the theorem, we need the following auxiliary statements.

Lemma 1. Let $a_n \downarrow 0$ at $n \to \infty$, $\sup\limits_{n} p_n = N < \infty$ and

$$S_n(x) = \sum_{n=1}^{n-1} a_k \psi_k(x).$$

Then for any integer $n \geq 1$ and for any number $x \in (0,1)$

$$|S_n(x)| \leq C \sum_{k=0}^{\left[\frac{1}{x}\right]} a_k. \tag{4}$$

Proof. At $n < \left[\frac{1}{x}\right]$ inequality (4) is obvious, since $|\psi_k(x)| = 1, \quad k = 0,\ 1,\ 2\ldots$

Let $n > \left[\frac{1}{x}\right]$, Then

$$|S_n(x)| \leq \sum_{k=0}^{\left[\frac{1}{x}\right]} a_k + \left| \sum_{k=\left[\frac{1}{x}\right]+1}^{n-1} a_k \psi_k(x) \right|.$$

Using inequality

$$|D_n(x)| = \left| \sum_{k=0}^{n-1} a_k \psi_k(x) \right| \leq \frac{C}{x}, \quad x \in (0,\ 1),$$

and applying the Abel transform to the last sum, due to the monotonicity of the sequence $\{a_n\}$, we get

$$|S_n(x)| \leq \sum_{k=0}^{\left[\frac{1}{x}\right]} a_k + \frac{C}{x} a_{\left[\frac{1}{x}\right]+1} \leq \sum_{k=0}^{\left[\frac{1}{x}\right]} a_k + C\left(\left[\frac{1}{x}\right] + 1\right) a_{\left[\frac{1}{x}\right]+1} \leq C_1 \sum_{k=0}^{\left[\frac{1}{x}\right]} a_k.$$

Lemma 1 is proved.

Lemma B. [13] Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $u = \{u_n\}_{n=1}^{\infty}$, $v = \{v_n\}_{n=1}^{\infty}$, $u_n \geq 0$, $v_n \geq 0$ and

$$\|a\|_{p,v} = \left(\sum_{n=1}^{\infty} |a_n v_n|^p \right)^{\frac{1}{p}}.$$

Then to satisfy the inequality $\|\sum_{k=1}^{n} a_k\|_{p,v} \leq C \|a\|_{p,u}$ it is necessary and sufficient that

$$A = \sup_l \left(\sum_{n=l+1}^{\infty} u_n^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^{l-1} v_m^{-p'} \right)^{\frac{1}{p'}} < \infty. \tag{5}$$

Lemma C. [14] Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $u(x) \geq 0$, $v(x) \geq 0$, $Pf(x) = \int_0^x f(t)\,dt$. Then, to satisfy the inequality $\|Pf\|_{p,v} \leq C \|f\|_{p,u}$ it s necessary and sufficient that

$$B = \left\{ \int_t^1 (u(x))^p \,dx \right\}^{\frac{1}{p}} \left\{ \int_0^t [v(x)]^{-p'} \,dx \right\}^{\frac{1}{p'}} < \infty.$$

Proof of Theorem 2. 1^0. According to Lemma 1 and by monotony of sequence $a_n(f)$ we have

$$\int_0^1 |f(x)\varphi(x)|^p \,dx = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} |f(x)\varphi(x)|^p \,dx = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \left(\sum_{k=0}^{n+1} a_k \right)^p \varphi^p(x)\,dx \leq \tag{6}$$

$$\leq C \sum_{n=1}^{\infty} \left[\sum_{k=0}^{n} a_k(f) \right]^p \cdot \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p dx.$$

To apply Lemma B, we set

$$u_n = \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx \right)^{\frac{1}{p}}, \quad v_n = n \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx \right)^{\frac{1}{p}}.$$

Condition (5), in this case, has the form:

$$A = \sup_l \left[\sum_{n=l+1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx \right]^{1/P} \left\{ \sum_{n=1}^{l-1} n^{-p'} \left[\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx \right]^{\frac{-p'}{p}} \right\}^{\frac{1}{p'}}. \tag{7}$$

It is clear that

$$\sum_{n=l+1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx \leq \int_0^{\frac{1}{l}} \varphi^p(x)\,dx.$$

Now, we will show that condition (5) of Lemma B is satisfied. To do this, we will show that $A \leq CA_P$ where A_P is from the condition of Theorem 2. Let us transform the second sum in (7).

$$\sum_{n=1}^{l-1} n^{-p'} \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx \right)^{\frac{-p'}{p}} = \sum_{n=1}^{l-1} n^{-p'} \left\{ \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx \right)^{\frac{1}{p}} \cdot \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^{-p'}(x)\,dx \right)^{\frac{1}{p'}} \right\}^{-p'} \times \tag{8}$$

$$\times \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^{-p'}(x)\,dx.$$

By Holder's inequality, we have

$$\frac{1}{n(n+1)} = \int_{\frac{1}{n+1}}^{\frac{1}{n}} dx = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi(x)\varphi(x)^{-1} dx \leq \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^{-p'}(x)\,dx\right)^{\frac{1}{p'}}.$$

Hence,

$$\left\{\left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right)^{\frac{1}{p}} \left(\int \varphi^p(x)\,dx\right)^{\frac{1}{p}}\right\}^{-p'} \leq n^{p'}(n+1)^{p'}.$$

Then from (8) we have

$$\sum_{n=1}^{l-1} n^{-p'} \left(\int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx\right)^{\frac{-p'}{p}} \leq \sum_{n=1}^{l-1} (n+1)^{p'} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^{-p'}(x)\,dx \leq \int_{\frac{1}{l}}^{1} (x\varphi(x))^{-p'}\,dx. \tag{9}$$

Therefore, by (7), (9) we get

$$A = \sup_l \left(\int_0^{\frac{1}{l}} \varphi^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_{\frac{1}{l}}^{1} (x\varphi(x))^{-p'}\,dx\right)^{\frac{1}{p'}} \leq$$

$$\leq \sup_{t\in[0,1]} \left(\int_0^{t} \varphi^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_{t}^{1} (x\varphi(x))^{-p'}\,dx\right)^{\frac{1}{p'}} = A_p < \infty.$$

Therefore, according to Lemma A and conditions $A_p < \infty$ it follows that

$$\int_0^1 |f(x)\varphi(x)|^p\,dx \leq c\sum_{n=1}^{\infty} a_n^p(f)\cdot n^p \int_{\frac{1}{n+1}}^{\frac{1}{n}} \varphi^p(x)\,dx < \infty,$$

that is $f(x)\varphi(x) \in L_p[0,1]$.

2^0. Due to the monotonicity of the sequence $a_n(f)$

$$\sum_{n=1}^{\infty} a_n^p(f)\,n^p \int_{1/(n+1)}^{1/n} \varphi^p(x)\,dx \leq \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} a_k(f)\right)^p \int_{1/(n+1)}^{1/n} \varphi^p(x)\,dx =$$

$$= \sum_{n=0}^{\infty} \left[\sum_{j=m_n}^{m_{n+1}-1} \left[\sum_{k=0}^{j} a_k(f)\right]^p \int_{1/j+1}^{1/j} \varphi^p(x)\,dx\right] \leq$$

$$\leq \sum_{n=0}^{\infty} \sum_{k=0}^{m_{n+1}-1} [a_k(f)]^p \int_{\frac{1}{m_{n+1}}}^{\frac{1}{m_n}} \varphi^p(x)\,dx. \tag{10}$$

By equality (2)

$$\sum_{k=0}^{m_n-1} a_k(f) = \int_0^1 f(x)\,D_{m_n}(x)\,dx = m_n \int_0^{1/m_n} f(x)\,dx \leq m_n F\left[\frac{1}{m_n}\right], \tag{11}$$

where $F(x) = \int_0^x |f(t)|\,dt$.

Now from (10) and (11) follows that

$$\sum_{n=1}^{\infty} a_n^p (f) n^p \int_{1/(n+1)}^{1/n} \varphi^p (x) dx \le \sum_{n=1}^{\infty} a_n^p (f) n^p \left[m_{n+1} F \left[\frac{1}{m_{n+1}} \right] \right]^p \times$$

$$\times \int_{1/m_{n+1}}^{1/m_n} \varphi^p (x) dx \le \sum_{n=0}^{\infty} \int_{1/m_{n+1}}^{1/m_n} \left(\frac{1}{x} F(x) \right)^p \varphi^p (x) dx =$$

$$= \int_0^1 \left(\int_0^x |f(t)| dt \right)^p \left(\frac{\varphi(x)}{x} \right)^p dx. \qquad (12)$$

According to Lemma C from the condition $B_p < \infty$ it follows that

$$\int_0^1 \left(\int_0^x |f(t)| dt \right)^p \left(x^{-1} \varphi(x) \right)^p dx \le C_p \int_0^1 |f(x) \varphi(x)|^p dx.$$

Then from the inequality (12) we have

$$\sum_{n=1}^{\infty} a_n^p (f) n^p \int_{1/n+1}^{1/n} \varphi^p (x) dx \le C_p \int_0^1 |f(x) \varphi(x)|^p dx < \infty.$$

Theorem 2 is proved.

Remark. By direct calculation, it can be shown that the function $\varphi(x) = \left(\frac{x}{(\ln x)^{1+\alpha}} \right)^{\frac{1}{p}}$, $\alpha > 0$ satisfies the condition from 1^0 of Theorem 2 (i.e. $A_p < \infty$) , but does not satisfy the condition from 1^0 of Theorem 1 (i.e. $\varphi^p(x)$ does not satisfy condition B_1).

In addition, the function $\varphi(x) = \left(\frac{x}{(\ln x)^{1+\alpha}} \right)^{-\frac{1}{p'}}$, $\alpha > 0$ satisfies the condition from 2^0 of Theorem 1 (i.e. $B_p < \infty$), but does not satisfy the condition from 2^0 of Theorem 2 (i.e. $\varphi^{-p'}(x)(x)$ does not satisfy condition B_1).

Therefore, the conditions of Theorem 1 and Theorem 2 are, generally speaking, different.

3 About belonging to space $L_1(0, 1)$ with the weight of the sum of series with monotonic coefficients

In the previous paragraphs, we considered the conditions for functions to belong to the class $L_{p,\varphi}(0, 1)$ at $1 < p < \infty$.

In this section, we will consider the case p=1, i.e. questions of belonging of functions to space $L_{1,\varphi}(0, 1)$.

Let $\varphi(x)$ be a non-negative measurable on $(1, \infty)$ function. They say the function $\varphi(x)$ satisfies the condition B_2, if for everyone $x \ge 1$ the following inequality holds

$$\int_1^x \frac{\varphi(t)}{t} dt \le C\varphi(x),$$

where C is a positive number independent of x.

We need the following auxiliary statement.

Lemma D. [15] *If $R_n \downarrow 0$, $0 \le B_n \uparrow$ at $n \to \infty$, then the series*

$$\sum_{n=1}^{\infty} R_n (B_{n+1} - B_n) \ \textit{and} \ \sum_{n=1}^{\infty} B_n (R_{n-1} - R_n)$$

converge or diverge at the same time.

The main goal of this section is to prove the following statement.

Theorem 4. Let $a_k \downarrow 0$ at $k \to \infty$, $\sup\limits_n p_n = N < \infty$ and

$$f(x) = \sum_{k=0}^{\infty} a_k \psi_k(x),$$

and let $\varphi(x) \geq 0$ is measurable on $[1, \infty)$ function such that

$$\varphi\left(\frac{1}{x}\right) \in L_1[0, 1], \quad \frac{1}{x}\varphi\left(\frac{1}{x}\right) \bar{\in} L(0, 1).$$

Then

1^0. If

$$\sum_{k=1}^{\infty} a_k \int_k^{\infty} \frac{\varphi(x)}{x^2} dx < \infty, \tag{13}$$

then

$$\varphi\left(\frac{1}{x}\right) f(x) \in L_1(0,1).$$

2^0. If $\varphi(x)$ satisfies the condition B_2 and $\varphi\left(\frac{1}{x}\right) f(x) \in L_1(0,1)$, that is the case (13) .

3^0. If $\varphi(x) \downarrow$ at $x \geq 1$, positive function and

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{\varphi(t)}{t} dt = \infty, \tag{14}$$

then there is a sequence $a_k \downarrow 0$ at $k \to \infty$, such that the function

$$f(x) = \sum_{k=1}^{\infty} a_k \psi_k(x)$$

integrates on (0,1), $\varphi\left(\frac{1}{x}\right) f(x) \in L_1(0,1)$, however the theries (13) diverges.

Point 3 of this theorem shows that the condition B_2 is essential for fulfilling point 2 of this theorem.

Proof. 1^0. By lemma 1 for any $x \in \left[\frac{1}{m_{\nu+1}}, \frac{1}{m_\nu}\right]$ we have

$$|f(x)| \leq C \sum_{k=0}^{m_{\nu+1}} a_k.$$

Therefore

$$\int_0^1 \varphi\left(\frac{1}{x}\right) |f(x)|\, dx = \sum_{k=1}^{\infty} \int_{1/m_{k+1}}^{1/m_k} \varphi\left(\frac{1}{x}\right) |f(x)|\, dx \leq \sum_{k=1}^{\infty} \sum_{j=0}^{m_{k+1}} a_j \int_{1/m_{k+1}}^{1/m_k} \varphi\left(\frac{1}{x}\right) dx \leq$$

$$\leq C_1 + \sum_{k=1}^{\infty} a_{m_{k+1}} \int_k^{\infty} \frac{\varphi(x)}{x^2} dx \leq C_1 + \sum_{k=1}^{\infty} a_k \int_k^{\infty} \frac{\varphi(x)}{x^2} dx < \infty.$$

2^0. From the conditions of Theorem 4 follows that follows that $f(x) \in L[0, 1]$, hence, $a_n = a_n(f)$ and

$$\sum_{k=0}^{m_n-1} a_k = \int_0^1 f(x) D_{m_n}(x)\, dx = m_n \int_0^{1/m_n} f(x)\, dx.$$

Therefore, due to the monotonicity of the sequence a_n,

$$\int_{1/m_{n+1}}^{1/m_n} f(x)\,dx = \left[(p_{n+1}-1)\sum_{k=0}^{m_n-1} a_k + \sum_{k=m_n}^{m_{n+1}-1} a_k\right] m_{n+1}^{-1} \geq 0. \tag{15}$$

Let us evaluate

$$\sum_{k=1}^{\infty} a_k \int_k^{\infty} \frac{\varphi(t)}{t^2}dt \leq \sum_{n=0}^{\infty}\sum_{k=m_n}^{m_{n+1}-1} a_k \int_k^{\infty} \frac{\varphi(t)}{t^2}dt \leq \sum_{n=0}^{\infty}\int_{m_n}^{\infty}\frac{\varphi(t)}{t^2}dt\left(\sum_{k=m_n}^{m_{n+1}-1} a_k\right) =$$

$$= \sum_{n=0}^{\infty} R_n\left(B_{n+1}-B_n\right), \tag{16}$$

where $R_n = \int_{m_n}^{\infty} \frac{\varphi(t)}{t^2}dt$, $B_n = \sum_{k=0}^{m_n-1} a_k$.

By lemma D for the convergence of the series (16) it is sufficient for the convergence of the series:

$$\sum_{n=0}^{\infty} B_n\left(R_{n-1}-R_n\right) \leq \sum_{n=1}^{\infty}\frac{1}{m_{n-1}}\int_{m_{n-1}}^{m_n}\frac{\varphi(t)}{t}dt\left(\sum_{k=0}^{m_n-1} a_k\right) \leq C\sum_{n=0}^{\infty}\frac{1}{m_{n+1}}\sum_{k=0}^{m_{n+1}-1} a_k\left[D_{n+1}-D_n\right],$$

where

$$D_n = \int_1^{m_n}\frac{\varphi(t)}{t}dt.$$

Applying Lemma D again, taking into account the conditions B_2 and (15), we have

$$\sum_{n=0}^{\infty}\frac{1}{m_{n+1}}\sum_{k=0}^{m_{n+1}-1} a_k\left[D_{n+1}-D_n\right] \leq \sum_{n=1}^{\infty}\int_1^{m_n}\frac{\varphi(t)}{t}dt\int_{1/m_{n+1}}^{1/m_n}|f(x)|\,dx \leq$$

$$\leq \sum_{n=1}^{\infty}\int_{1/m_{n+1}}^{1/m_n}|f(x)|\left(\int_1^{x^{-1}}\frac{\varphi(t)}{t}dt\right) \leq C\sum_{n=1}^{\infty}\int_{1/m_{n+1}}^{1/m_n}|f(x)|\,\varphi\left(\frac{1}{x}\right)dx < \infty.$$

3^0. Let $\varphi(x)\downarrow$ at $x \geq 1$. From the condition $\frac{1}{x}\varphi\left(\frac{1}{x}\right)\bar{\in}L$ follows that

$$\sum_{n=1}^{\infty}\int_n^{\infty}\frac{\varphi(t)}{t^2}dt = \infty.$$

Hence,

$$\sum_{n=0}^{\infty} m_n\int_{m_n}^{\infty}\frac{\varphi(t)}{t^2}dt = \infty. \tag{17}$$

Let's put $a_0 = 1$ and if $m_n \leq k \leq m_{n+1}-1,\ n = 0,\ 1,\ 2,\ldots$, then

$$a_n = \lambda_n = \left[\sum_{j=0}^{n} m_j\int_{m_j}^{\infty}\frac{\varphi(t)}{t^2}dt\right]^{-1}.$$

It means , $a_n \downarrow 0$ at $n\to\infty$. From the sequence definition $\{a_n\}$ using the Abel transform, we have

$$f(x) = \sum_{n=0}^{\infty} a_n\psi_n(x) = \sum_{n=1}^{\infty}\Delta\lambda_{n-1}D_{m_n}(x).$$

Then at $x \in \left[\frac{1}{m_{k+1}},\ \frac{1}{m_k}\right),\ k = 0,\ 1,\ 2, \dots$ based on the property of the Dirichlet kernel by Lemma 1 we have

$$f(x) = \sum_{n=1}^{k} \Delta\lambda_{n-1} m_n,$$

where $f(x) \geq 0,\ f(x) \downarrow 0$ in $(0,\ 1)$.

Further, taking into account the condition $\varphi(x) \downarrow 0,\ x \geq 1$ we get that

$$\int_0^1 \varphi\left(\frac{1}{x}\right) |f(x)|\, dx \leq \varphi(1) \int_0^1 f(x)\, dx \leq \varphi(1) \sum_{k=0}^{\infty} \left(\sum_{n=1}^{k} \Delta\lambda_{n-1} m_n \right) \cdot \frac{1}{m_k} \leq$$

$$\leq C\varphi(1) \sum_{n=1}^{\infty} \Delta\lambda_{n-1} = 2\varphi(1)\lambda_0 < \infty,$$

i.e. $f \in L_{1,\varphi}(0,\ 1)$.

Let us show that relation (13) does not hold, i.e.

$$\sum_{n=1}^{\infty} a_n \int_n^{\infty} \frac{\varphi(t)}{t^2} dt = \infty.$$

Based on property $a_n \downarrow 0$ where this $n \to \infty$ condition is equivalent to the divergence of the series

$$S \equiv \sum_{n=1}^{\infty} m_n a_{m_n} \int_{m_n}^{\infty} \frac{\varphi(t)}{t^2} dt = \sum_{n=1}^{\infty} \lambda_n m_n \int_{m_n}^{\infty} \frac{\varphi(t)}{t^2} dt.$$

According to the famous Kronecker theorem [1; 905] and (14), (17) we have

$$S = \sum_{n=1}^{\infty} m_n \int_{m_n}^{\infty} \frac{\varphi(t)}{t^2} dt \left[\sum_{i=0}^{n} m_i \int_{m_i}^{\infty} \frac{\varphi(t)}{t^2} dt \right]^{-1} = \infty,$$

i.e.

$$\sum_{n=1}^{\infty} \lambda_n \cdot m_n \int_{m_n}^{\infty} \frac{\varphi(t)}{t^2} dt = \infty.$$

Point 3^0 of Theorem 4 proven.

Conclusion

In this paper, we have considered series with respect to the multiplicative systems with monotonic coefficients. Conditions have been obtained for the weight function and for coefficients under which the sum of the series under consideration is L_p $(1 < p < \infty)$ integrable on the interval $[0, 1]$. In addition, the obtained conditions are compared with the previously known conditions for the weight integrability of the sum of such series.

Acknowledgments

The research of the first author was supported by the grant of the Ministry of Science and Higher Education of the Republic of Kazakhstan (project no: AP14869887).

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Коэффициенттері монотонды мультипликативтік жүйелер бойынша қатарлардың қосындысының салмақты интегралдану шарттары туралы

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Мақалада біз қатарлардың коэффициенттері монотонды болған жағдайда мультипликативті жүйелер бойынша қатарлар қосындысының салмағымен интегралдылық мәселелерін қарастырамыз. Салмақ функциясы үшін және қатардың қосындысы салмақпен L_p $(1 < p < \infty)$ Лебег кеңістігіне жататын коэффициенттері үшін шарттар табылды. Сонымен қатар, $p = 1$ жағдайы қарастырылған. Бұл жағдайда қарастырылып отырған қатардың қосындысының салмағымен интегралдылықтың басқа шарттары табылған. Генеративті тізбектің шектелуі жағдайында дәлелденген теоремалар Харди-Литтлвудтың монотонды коэффициенттері бар тригонометриялық қатарлар туралы белгілі теоремасының аналогын білдіреді.

Кілт сөздер: мультипликативті жүйелер, қатарлар қосындысының салмақты интегралдануы, жасаушы тізбек, монотонды коэффициенттер, Харди-Литтлвуд теоремасы.

Об интегрируемости с весом суммы рядов с монотонными коэффициентами по мультипликативным системам

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В статье мы изучили вопросы интегрируемости с весом суммы рядов по мультипликативным системам при условии, что коэффициенты рядов монотонны. Найдены условия для весовой функции и коэффициентов ряда, для которых сумма ряда принадлежит пространству Лебега L_p $(1 < p < \infty)$ с весом. Кроме того, рассмотрен случай $p = 1$. В этом случае найдены другие условия интегрируемости с весом суммы рассматриваемого ряда. В случае ограниченности порождающей последовательности доказанные теоремы подразумевают аналог известной теоремы Харди-Литтлвуда о тригонометрических рядах с монотонными коэффициентами.

Ключевые слова: мультипликативные системы, весовая интегрируемость суммы рядов, образующая последовательность, монотонные коэффициенты, теорема Харди-Литтлвуда.

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Bulletin of the Karaganda University. Mathematics Series, No. 2(114), 2024, pp. 211–220

https://doi.org/10.31489/2024M2/211-220 *Research article*

On closure operators of Jonsson sets

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The work is related to the study of the model-theoretic properties of Jonsson theories, which, generally speaking, are not complete. In the article, on the Boolean of Jonsson subsets of the semantic model of some fixed Jonsson theory, the concept of the Jonsson closure operator Jcl was introduced, defining the J-pregeometry on these subsets, and some results were obtained describing this closure operator.

Keywords: Jonsson theory, semantic model, Jonsson set, closure operator, J-pregeometry.

2020 Mathematics Subject Classification: 03C45, 03C68.

Introduction

This work is related to the study of definable subsets of the semantic model of a fairly wide class of fixed Jonsson theories. The following reasons cause the appeal to definable subsets of the semantic model. The problem of describing the heredity of Jonsson theories is well known. To date, a complete description of this important model-theoretic property is unknown.

The Jonsson theory is called hereditary if the property of being a Jonsson theory is preserved for admissible enrichments of the signature under consideration. Moreover, with admissible enrichment, only those hereditary Jonsson theories are of interest that, in these enrichments, preserve the definability of the type for the stability obtained with such enrichment. If we take into account the fact that the classical version of stability is associated with enrichments only of constants, then the problem of describing heredity is naturally present for complete theories. In particular, if we consider the center of Jonsson theory of fields of a fixed characteristic, then in the case of characteristic 0, the enrichment of this center does not preserve Jonssoness when the enrichment is a one-place predicate interpreted in an algebraically closed field of characteristic 0 as an elementary subfield, i.e. an elementary submodel of an algebraically closed field of characteristic 0 is of sufficiently large power. At the same time, the center of Jonsson theory is always a complete theory by definition. Besides the fact that this is an example of the importance of this problem not only for Jonsson theories but also for complete theories, we also note that in this example, the dimension of the model of the center differs from the dimension of the model of any of the Abelian groups that define a given field over the field itself. This fact is due to the fact that the concept of dimension, which determines the maximum number of independent elements of the model of a given center, is differently connected in these examples with the relation of non-forking of the corresponding types of algebras considered: fields and Abelian groups. The main tool that distinguished these dimensions was once identified by S. Shelah when studying the classification of complete theories, and it was called forking. In the work [1], the basics of forking for Jonsson theories were defined when studying the model-theoretic properties of the semantic model of a fixed class of the Jonsson spectrum of a fixed class of models of the consideration language.

The next interesting and important issue discussed in this article is describing definable subsets of a semantic model using a closure operator that defines some pregeometry on the Boolean of the

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Received: 20 December 2023; *Accepted:* 16 February 2024.

semantic model under consideration. Note that the idea of describing the classification of theories regarding various types of geometries specified by pregeometry was once proposed by B. Zilber [2]. Mustafin T.G. in work [3], within the framework of the study of complete theories, the concept of a pure pair and a semantic triple was proposed using a specific closure operator defined on subsets of a sufficiently large model of a stable complete theory. Theorems were obtained describing the properties of the closure operator defined on special subsets of the monster model of the considered complete stable theory. In contrast, the description of the properties of this operator was closely related to one of the elements of the semantic triple, namely, of the group of automorphisms of this monster model. Note that this approach to describing closure operators differs from the description in the works of B. Zilber.

Yeshkeyev A.R. obtained results on implementing the approach of Mustafin T.G. already within the framework of studying, generally speaking, the incomplete Jonsson stable fixed Jonsson theory [4]. It should also be noted that when determining the syntactic and semantic similarities of Jonsson theories, Yeshkeyev A.R. redefined the concepts of a pure pair and a semantic triple for Jonsson theories and their semantic models [4].

In the work [5], the Jonsson spectrum of a fixed class of models of an arbitrary signature was defined. Note that within the framework of the study of the corresponding invariants of Abelian groups and special types of rings [6, 7], a description of such an important model-theoretic concept as cosemanticness of models of Jonsson theories was obtained. It turned out that the concept of cosemanticness generalizes and clarifies such an important concept as the elementary equivalence of two models. In addition, the concept of cosemanticness of models is related to the syntactic concept of Jonsson theory in that two models are cosemantic with each other if their Jonsson spectra are equal.

Thus, we note the importance of this article in connection with the following circumstance that connects the relevance and novelty of this problem, regarding the fact that any Jonsson theory is a special case of such an important and fruitful concept as the Jonsson spectrum of a fixed class of models of a given signature.

When studying definable subsets of a complete theory, as a rule, one specifies some axioms satisfied by these formulaic subsets of fixed models of this complete theory. In our case, we will do the same for the special case of definable subsets of the semantic model of a fixed Jonsson theory. Namely, these axioms will predetermine the possibility of determining the Morley rank function in its "Jonsson" interpretation, i.e. in conditions where only existentially closed extensions under corresponding monomorphisms, which are not necessarily elementary, are considered.

At the same time, we note that all these arguments will have a positive development for the corresponding types of homomorphisms, i.e. we can transfer the results of this article to positive Jonsson theories. The concept of a positive Jonsson theory and the properties of morphisms of such theories were considered in [8, 9].

Those facts that are not indicated in this article but may be helpful for a deeper understanding of the results of this article can be obtained from the following sources [10–18].

1 Basic concepts and results concerning Jonsson theories

Let us present the necessary definitions and results concerning Jonsson theories.

Definition 1. [19] A theory T is called Jonsson if it has an infinite model, is inductive, and satisfies the joint embedding property (JEP) and amalgamation property (AP).

Note that Jonsson theory, by its definition, is, generally speaking, not complete, i.e. the class of its models can contain both infinite and finite models and, in addition, the definition of JEP and AP considers isomorphic embeddings rather than elementary monomorphisms. There are many examples from classical algebra that satisfy Jonsson theories. These include groups, Abelian groups, rings, fields of fixed characteristic, Boolean algebras, linear orders, vector spaces, modules over a fixed ring and others.

Definition 2. [20] Let $\kappa \geq \omega$. A model M of a theory T is called κ-universal for the theory T if for each model $A \in Mod(T)$ such that $|A| < \kappa$, there is an isomorphism $f\colon A \to M$.

Definition 3. [20] Let $\kappa \geq \omega$. A model M of a theory T is called κ-homogeneous for T if for any two models A, $A_1 \in Mod(T)$, which are submodels of M such that $|A| < \kappa$, $|A_1| < \kappa$, and the isomorphism $f : A \to A_1$, for every extension B of model A that is a submodel of M and model T of cardinality strictly less than κ, there is an extension B_1 of model A_1, which is a submodel of M and an isomorphism $g : B \to B_1$ extending f.

A *homogeneous-universal model* for T is a κ-homogeneous-universal model for T of cardinality $\kappa \geq \omega$.

Definition 4. [20] The semantic model C_T of Jonsson theory T is called the ω^+-homogeneous-universal model of the theory T.

Definition 5. [4] The semantic completion (center) of Jonsson theory T is the elementary theory T^* of the semantic model C_T of the theory T, i.e. $T^* = \mathrm{Th}(C_T)$.

In the case when universally homogeneous models in the Jonsson sense are saturated, a special class of Jonson theories is distinguished, the elements of which are called perfect Jonsson theories.

Definition 6. [4] A Johnson theory T is said to be perfect if every semantic model of the theory T is a saturated model of T^*.

The remarkable property of the existence of a model companion for such theories determines the feature of perfect Jonsson theories.

Theorem 1. [4] Jonsson theory T is perfect if and only if T^* is a model companion of theory T.

An important characteristic of any theory is stability. For complete theories, the concept of stability was introduced by S. Shelah in 1969. In the work [4] Yeshkeyev A.R., the concept of stability in the Jonsson sense was defined. Let us recall the definition of this concept.

Let T be a Jonsson theory. Let $S^J(X)$ denote the set of all existential complete n-types over X that are consistent with T for every finite n.

Definition 7. [4] We say that a Jonsson theory T is a J-λ-stable if for any T-existentially closed model A, for any subset X of the set A since $|X| \leq \lambda$ it follows that $|S^J(X)| \leq \lambda$. The Jonsson theory T is called a J-stable if it is a J-λ-stable for some λ.

In the article [6], a result was obtained showing that stability in the above sense is in good agreement with the classical concept of stability.

Theorem 2. [6] Let T be a perfect Jonsson theory complete for $\exists$-sentences, $\lambda \geq \omega$. Then the following conditions are equivalent:

1 T is a J-λ-stable;

2 T^* is a λ-stable, where T^* is the center of Jonsson theory T.

In the work [1], within the framework of the study of Jonsson theories, the concept of J-pregeometry was introduced.

Let T be some fixed Jonsson theory, $X \subseteq C_T$, $\mathcal{P}(X)$ be the Boolean of the set X and the map $cl\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is some closure operator on the set $\mathcal{P}(X)$. The pair (X, cl) is a *J-pregeometry* if the following conditions are satisfied:

1) if $A \subseteq X$, then $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$;

2) if $A \subseteq B \subseteq X$, then $cl(A) \subseteq cl(B)$;

3) (exchange) $A \subseteq X$, $a, b \in X$ and $a \in cl(A \cup \{b\}) \setminus cl(A)$, then $b \in cl(A \cup \{a\})$;

4) (finite character) If $A \subseteq X$ and $a \in cl(A)$, then there is a finite $A_0 \subseteq A$, such that $a \in cl(A_0)$.

Further, we will assume that the operator cl, which defines the J-pregeometry on a subset of the semantic model of some fixed Jonsson theory, will be the algebraic closure operator, which is equal to the definable closure operator, i.e. $cl = acl = dcl$.

Definition 8. [21] A set X is called Jonsson in the theory T if it satisfies the following properties:

1) X is a definable subset of C_T;

2) $cl(X)$ is the carrier of some existentially closed submodel C_T.

Next, we write down additional axioms to preserve the Morley rank of the above formulaic Jonsson subsets and denote this system of axioms by (*). Due to the result on the equivalence of the Jonsson stability of the Jonsson theory and the corresponding stability of its center for a perfect $\exists$-complete theory (Theorem 2), these axioms not only clarify the boundaries of conservation of the model-theoretic properties of the Morley rank of the considered formulas of a fixed Jonsson theory in a given context but are also correctly related to the closure operator, which specifies the pregeometry on the Boolean of Jonsson subsets of the semantic model of the theory under consideration. Moreover, we note that the semantic model itself is an element of this Boolean due to the fact that the formula $x = x$ is an existential formula. The main results of this article also use the correctness of this axiomatics and the closedness with respect to such subsets.

Let T be some Jonsson theory, let $\mathbb{J}$ denote the set of all Jonsson subsets of the semantic model C_T and let $|\mathbb{J}| = |I|$, where I is index set. It's clear that $\mathbb{J} \subseteq \mathcal{P}(C_T)$. Let us introduce the concept of a family of Jonsson subsets of the semantic model C_T. Let us denote by $\mathbb{J}^n$ the set of all definable Jonsson subsets of the semantic model C_T, the length of whose defining formulas is equal to n.

Let us present a system of axioms, which we denote by (*).

Let $Jset(C_T)$ be the smallest family of Jonsson subsets in $\bigcup\limits_{n \geq 1} C_T^n$ with the following properties:

1) For each $i \in I$, from the fact that $A_i \in \mathbb{J}$ it follows that $A_i \in Jset(C_T)$.

2) The set $Jset(C_T)$ is closed under finite Boolean combinations, i.e. from the fact that $A, B \in \mathbb{J}^n$ it follows that $A, B \in Jset(C_T)$, $A \bigcup B \in Jset(C_T)$, $A \bigcap B \in Jset(C_T)$ and $C_T^n \setminus A \in Jset(C_T)$.

3) The set $Jset(C_T)$ is closed under the Cartesian product, i.e. from the fact that $A, B \in Jset(C_T)$ it follows that $A \times B \in Jset(C_T)$.

4) The set $Jset(C_T)$ is closed under the projection, i.e. if $A \subseteq C_T^{n+m}$, $A \in Jset(C_T)$, $\pi_n(A)$ is the projection of the Jonsson set A onto C_T^n, then $\pi_n(A) \in Jset(C_T)$.

5) The set $Jset(C_T)$ is closed under specialization, i.e. if $A \in Jset(C_T)$, $A \subseteq C_T^{n+k}$ and $\bar{m} \in C_T^n$, then $A(\bar{m}) = \{\bar{b} \in C_T^k \colon (\bar{m}, \bar{b}) \in A\} \in Jset(C_T)$.

6) The set $Jset(C_T)$ is closed under permutation of coordinates, i.e. if $A \in Jset(C_T)$, $A \subseteq C_T^n$, and σ is a permutation of the set $\{1, ..., n\}$, then $\sigma(A) = \{(a_{\sigma(1)}, ..., a_{\sigma(n)}) \colon (a_1, ..., a_n) \in A\} \in Jset(C_T)$.

In work [4], within the framework of the study of Jonsson subsets of the semantic model of given Jonsson theory, the concept of forking was axiomatically introduced, and the equivalence of forking according to Shelah and the axiomatically given forking for existential types over subsets of the semantic model of some Jonsson theory was proven.

Let $\mathcal{X}$ be the class of all Jonsson subsets of the $\exists$-saturated semantic model C_T of some Jonsson theory T, $\mathcal{R}$ be the class of all existential types (not necessarily complete). Let $JNF \subseteq \mathcal{R} \times \mathcal{X}$ be some binary relation. Let us write down in the form of axioms some conditions imposed on JNF (Jonsson nonforking).

Axiom 1. If $(p, A) \in JNF$ and $f : A \to B$ are isomorphic embeddings, then $(f(p), f(A)) \in JNF$.

Axiom 2. If $(p, A) \in JNF$ and $q \subseteq p$, then $(q, A) \in JNF$.

Axiom 3. If $A \subseteq B \subseteq C$ and $p \in S^J(C)$, then $(p, A) \in JNF$ if and only if $(p, B) \in JNF$ and $(p \restriction B, A) \in JNF$.

Axiom 4. If $A \subseteq B$, $dom(p) \subseteq B$ and $(p, A) \in JNF$, then there exists $q \in S^J(B)$ such that $p \subseteq q$ and $(q, A) \in JNF$.

Axiom 5. There is a cardinal μ such that if $A \subseteq B \subseteq C$, $p \in S^J(B)$ and $(p, A) \in JNF$, then $\left|\{q \in S^J(C) : p \subseteq q, (q, A) \in JNF\}\right| < \mu$.

Axiom 6. There is a cardinal $\varkappa$ such that for any $p \in \mathcal{R}$ and for each $A \in \mathcal{X}$, if $(p, A) \in JNF$, then there exists $A_1 \subseteq A$, such that $|A_1| < \varkappa$ and $(p, A_1) \in JNF$.

Axiom 7. If $p \in S^J(A)$, then $(p, A) \in JNF$.

Theorem 3. [4] Let T be a perfect Jonsson theory, complete for $\exists$-sentences. Then the following conditions are equivalent:

1) the relation *JNF* satisfies axioms 1–7 with respect to the theory T;

2) the theory T^* is stable and for any $p \in \mathcal{R}$, $A \in \mathcal{X}$ the pair $(p, A) \in JNF \Leftrightarrow p$ is not forks over A (in the classical sense of S. Shelah [22]).

Next entry $p \not\downarrow^J A$ will mean that $(p, A) \in JNF$. If $tp(\overline{a}, A \cup \overline{b}) \not\downarrow^J A$, then we will write $\overline{a} \underset{A}{\not\downarrow}^J \overline{b}$.

2 Jonsson theories with closure operator

In the work [3] Mustafin T.G., some properties of complete theories admitting a closure operator were considered. In this article, we will consider some properties of the closure operator within the framework of the study of Jonsson theories concerning those additional considerations as the above *Axiomatics (*)* concerning the model-theoretic properties of preserving the Morley rank and the correctness of the definition of Jonsson subsets satisfying given axiomatics and satisfying the properties closure operator defining the pregeometry on the Boolean of the semantic model under consideration.

Recall that a complete theory T admits a closure operator J if on the monster model $\mathcal{C}$ of the theory T one can define a closure operator J so that $J(g(X)) = g(J(X))$ for all $X \in \mathcal{P}(\mathcal{C})$ and $g \in Aut(\mathcal{C})$ [3].

We need the following technical lemma from [3] to prove Theorem 4.

Lemma 1. [3] Let J be some closure operator admitted by the full theory of T, then the following conditions are equivalent:

1) if $M < \mathcal{C}$, then $M = \cup\{J(m)\colon m \in M\}$;

2) $|J(a)| < |\mathcal{C}|$ for any $a \in \mathcal{C}$;

3) $J(a) \subseteq acl(a)$ for any $a \in \mathcal{C}$.

The following definition belongs to A.R. Yeshkeyev. It defines the closure operator for a generally speaking incomplete theory, and it can be used in a broad sense for application in specific algebras, the theory of which is Jonsson.

Let T be a Jonsson theory whose semantic model C_T satisfies Axiomatics (*).

Definition 9. We will say that a Jonsson theory T with a closure operator Jcl if $Jcl(g(X)) = g(Jcl(X))$ for all $X \in \mathcal{P}(C_T)$ and $g \in Aut(C_T)$.

Let T be some Jonsson theory with the closure operator Jcl, X be a Jonsson set, and $Jcl(X) = M \in E_T$, where E_T is the class of all existentially closed models of the theory T. If $a, b \in C_T \backslash M$ then $b \in C_M(a)$ means that there exist $n < \omega$ and the sequence $\langle b_0, ..., b_n\rangle$ elements from $C_T \backslash M$ such that $b_0 = a$, $b_n = b$, $b_i \in cl(b_{i+1})$ or $b_{i+1} \in cl(b_i)$ for all $i < n$. In this case, the sequence $\langle b_0, ..., b_n\rangle$ will be called a Jcl-path outside M between a and b of length n.

Let us consider some conditions imposed on the Jcl operator.

Axiom 1. If $M \in E_T$, then $M = \overline{M}$, where $\overline{M} \leftrightharpoons \cup\{Jcl(m) \mid m \in M\}$.

Axiom 2. If $M \in E_T$ and $M = \overline{M}$, $\bar{a}, \bar{b}$ are tuples of elements from $C_T \setminus M$, $C_M(\bar{a}) \cap C_M(\bar{b}) = \varnothing$, then $\bar{a} \underset{M}{\not\downarrow}^J \bar{b}$.

Further, instead of the closure operator Jcl, we mean the algebraic closure operator acl, which is also the definable closure operator dcl.

It is well-known that ω-stable complete theories are characterized by the fact that any type (respectively, any formula) of a given theory has some Morley rank, i.e., ranked according to Morley. First, J-ω-stability does not coincide with ω-stability in the general case, and in the context of the following Theorems 4 and 5, the condition ω-stability is not assumed even on the center of the Jonson theory under consideration. However, at the same time, earlier in this article, we defined the Axiomatics (*), which is consistent with the Morley rank of definable Jonsson subsets. Therefore, in Theorems 4 and 5, Axiomatics (*) is assumed under the assumption that the closure operator under consideration is related to Morley-ranked subsets of the semantic model under consideration. We also note that due to the perfectness of the theory, the semantic model is saturated in power ω^{+}, which is enough for the rank of formulaic subsets relative to the center of the perfect Jonsson theory to exist.

In connection with the above definitions, we have the following results:

Theorem 4. Let T be a perfect Jonsson J-λ-stable theory, complete for $\exists$-sentences with the closure operator Jcl, whose semantic model C_T satisfies Axiomatics (*). If Jcl satisfies Axioms 1, 2 and $M \in E_T$, then for all $a \in C_T \backslash M$ $C_M(a) = Jcl(C_M(a))$.

Proof. Since by condition the Jonsson theory T is $\exists$-complete, then all $\exists$-types are complete types, i.e., they are all true in C_T. However, C_T is an existentially closed model, so all $\exists$-types are true in M. Due to the fact that $acl = dcl = Jcl$, then Lemma 1 is true for T^*, and therefore for any existentially closed model of the theory T, since T is a perfect Jonsson theory. The inclusion of $C_M(a) \subset cl(C_M(a))$ follows from Lemma 1.

We prove the reverse inclusion by induction on the length of Jcl-paths. Let $\langle b, a\rangle$ be a Jcl-path outside M (of length I), i.e. either $b \in Jcl(a)$ or $a \in Jcl(b)$. In any case, $a \underset{M}{\downarrow} b$ in T^* theory. By Theorem 2, the theory T^* is λ-stable in the classical sense. Then, by Theorem 3 we have $a \underset{M}{\downarrow}^{J} b$ in the Jonsonian sense of J-forking. Hence, by virtue of the definition of the theory with the closure operator Jcl and Axiom 2, we obtain $b \in C_M(a)$. Let now $\langle b_0, ..., b_n\rangle$ Jcl-path outside M between b and a of length n. By induction $b_1 \in C_M(b_n)$ and $b_0 \in C_M(b_1)$. This means $b_0 \in C_M(b_n)$, i.e. $b \in C_M(a)$.

Theorem 5. If T is a perfect Jonsson J-λ-stable theory, complete for $\exists$-sentences with the closure operator Jcl whose semantic model C_T satisfies axiomatics (*), the operator Jcl satisfies the Axioms 1, 2, $M, N \in E_T$, $M \prec_{\exists_1} N$, $a \in N \backslash M$, then:

1) $M \prec_{\exists_1} M \cup (N \cap C_M(a)) \preceq_{\exists_1} N$;

2) $M \preceq_{\exists_1} N \backslash (N \cap C_M(a)) \prec_{\exists_1} N$.

Proof. 1) Let $K \leftrightharpoons M \cup (N \cap C_M(a))$. Let us assume that K is not an elementary submodel of N with respect to $\exists$-formulas, i.e. K is not an existentially closed submodel of N. Then there must exist an element $b \in N$, an existential formula $\theta(x, \bar{y}, \bar{z})$ and tuples $\bar{a} \in N \cap C_M(a)$ and $\bar{m} \in M$ such that $N \models \theta(b, \bar{m}, \bar{a})$, but $N \models \urcorner\theta(c, \bar{m}, \bar{a})$ for all $c \in K$. Hence $b \notin C_M(a)$ and $b \underset{M}{\downarrow} \bar{a}$ in T^* theory. By Theorem 2, the theory T^* is λ-stable in the classical sense. Then, by Theorem 3 we have $a \underset{M}{\downarrow}^{J} b$ in the Jonsson sense of J-forking. Since $\overline{a} \subseteq C_M(a)$, then $C_M(\overline{a}) = C_M(a)$.

Therefore $C_M(b) \cap C_M(\bar{a}) = \oslash$. By Axiom 1 $M = \overline{M}$, and by Axiom 2 $b \underset{M}{\not\downarrow}^{J} \overline{a}$ in the Jonsson sense of J-forking. We have a contradiction.

2) Let $a_i \in N \backslash M, i < \lambda$ such that $N \backslash (N \cap C_M(a)) = M \cup \bigcup\limits_{i<\lambda} (N \cap C_M(a_i))$. Applying point 1) by induction we obtain that $M \preceq_{\exists_1} N \backslash (N \cap C_M(a)) \prec_{\exists_1} N$.

Acknowledgments

The authors express gratitude to their scientific supervisor, Professor A.R. Yeshkeyev, for bringing this topic to our attention and for the valuable comments.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Йонсондық жиындардың тұйықталу операторлары туралы

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Жұмыс жалпы айтқанда толық емес болып табылатын йонсондық теориялардың модельді-теоретикалық қасиеттерін зерттеумен байланысты. Мақала авторлары кейбір бекітілген йонсондық теорияның семантикалық моделінің йонсондық ішкі жиындарының булеанында осы жиындардағы J-алғашқы геометрияны анықтайтын Jcl йонсондық тұйықталу операторы ұғымын енгізді және тұйықталу операторын сипаттайтын негізгі нәтижелер алынды.

Кілт сөздер: йонсондық теория, семантикалық модель, йонсондық жиын, тұйықталу операторы, J-алғашқы геометрия.

Об операторах замыкания йонсоновских множеств

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Данная работа связана с изучением теоретико-модельных свойств йонсоновских теорий, которые, вообще говоря, не являются полными. Авторами статьи на булеане йонсоновских подмножеств семантической модели некоторой фиксированной йонсоновской теории было введено понятие йонсоновского оператора замыкания Jcl, задающего J-предгеометрию на этих подмножествах, и получены некоторые результаты, описывающие указанный выше оператор замыкания.

Ключевые слова: йонсоновская теория, семантическая модель, йонсоновское множество, оператор замыкания, J-предгеометрия.

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ANNIVERSARIES

Dulat Syzdykbekovich Dzhumabaev

Life and scientific activity (dedicated to the 70th birthday anniversary)

Professor Dulat Syzdykbekovich Dzhumabaev, Doctor of Physical and Mathematical Sciences, was a prominent scientist, a well-known specialist in the field of the qualitative theory of differential and integro-differential equations, the theory of nonlinear operator equations, numerical and approximate methods for solving boundary value problems.

Dzhumabaev D.S. was born in Kantagi, Turkistan district, South Kazakhstan region, on April 11, 1954. From 1961 to 1971, he attended secondary school in Turkistan. In 1971, he entered Faculty of Mechanics and Mathematics of Kazakh State University named after S.M. Kirov (now Al-Farabi Kazakh National University). After graduating with honors from the Department of Mathematics in 1976, he continued to pursue postgraduate studies at the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR. His scientific activity began under the guidance of Academician Orymbek Akhmetbekovich Zhautykov, an outstanding scientist and mathematician, who made a huge contribution to the creation and development of the mathematical science in Kazakhstan. After successful completion of postgraduate studies in 1979, Dzhumabaev D.S. joined the Laboratory of Ordinary Differential Equations headed by Academician Zhautykov O.A. He went from being a junior researcher to becoming the head of the Laboratory of Differential Equations, one of the leading divisions of the Institute of Mathematics. He chaired the laboratory from 1996 to 2012.

Dzhumabaev D.S. was a successful scientist and versatile specialist in the field of mathematics and its applications. He devoted his talent and hard work to the study of nonlinear operator equations, to the creation and development of qualitative methods in the theory of boundary value problems for differential equations.

The main research areas and the results obtained by Professor Dzhumabaev can be divided into several groups. The most significant and important scientific results are presented below in chronological order.

1 Boundary value problems for ordinary differential equations with a parameter in a Banach space

During postgraduate studies, his research was focused on nonlinear boundary value problems with parameter for ordinary differential equations of the following form:

$$\frac{dx}{dt} = f(t, x, \lambda), \qquad x(0) = x^0, \tag{1}$$

$$x(T) = x^1, \tag{2}$$

where $f : [0,T]\times B\times B \to B$ is a continuous function satisfying the existence conditions for the Cauchy problem (1) on $[0,T]$ for all values of a parameter λ from some set $G \subset B$; here B is a Banach space.

The problem is to find a pair $(\lambda^*, x^*(t))$, where $\lambda^* \in G$ and $x^*(t)$ is a solution to Cauchy problem (1) with $\lambda = \lambda^*$, satisfying the boundary condition (2).

Let the right-hand part of the differential equation be defined on the set

$$D^0 = \{(t,x,\lambda) : 0 \le t \le T, ||x - x^{(0)}(t)|| \le R(t)\rho, ||\lambda - \lambda^0|| \le \rho\}.$$

Here $\lambda^0 \in G$, $x^{(0)}(t)$ is a solution to Cauchy problem (1) with $\lambda = \lambda^0$, $R(t)$ is a positive function continuously differentiable on $[0,T]$, and ρ is a nonnegative number.

Let $M(f)$ denote a set of triples $(\lambda^0 \in G, R(t) > 0, \rho \ge 0)$ for which the Lipschitz condition

$$||f(t,x,\lambda) - f(t,\tilde{x},\tilde{\lambda})|| \le \alpha(t)\cdot(||x-\tilde{x}|| + ||\lambda - \tilde{\lambda}||)$$

is satisfied on the set D^0, and the inequality

$$(a_1) \qquad \exp\Big\{\int\limits_0^t \alpha(\tau)d\tau\Big\} - 1 \le R(t)$$

holds ($\alpha(t) \in C([0,T])$).

The set $M(f)$ is non-empty if so is the set G.

For a triple $(\lambda^0, R(t), \rho)$, a solution of problem (1), (2) is sought in the set $\alpha^0 = \alpha^0_\lambda \times \alpha^0_{x(t)}$, where $\alpha^0_\lambda = \{\lambda : ||\lambda - \lambda^0|| \le \rho\}$ and $\alpha^0_{x(t)} = \{x(t) : ||x(t) - x^{(0)}(t)|| \le R(t)\rho\}$.

Theorem 1. Problem (1), (2) is solvable if and only if, given some $(\lambda^0, R(t), \rho) \in M(f)$, for any two pairs $(\lambda, x(t))$ and $(\tilde{\lambda}, x(t))$ from the set α^0, there exist an invertible operator $A \in L(B,B)$ and a number $\theta > 0$ satisfying the inequality

$$(a_2) \qquad \Big|\Big|\lambda - \tilde{\lambda} - A\Big[\int_0^T \{f(t,x(t),\lambda) - f(t,x(t),\tilde{\lambda})\}dt\Big]\Big|\Big| \le (1-\theta)||\lambda - \tilde{\lambda}||,$$

and the following inequality is true

$$(a_3) \qquad \frac{1}{\theta}\Big|\Big|A\Big[\int_0^T f(t,x^{(0)}(t),\lambda^0)dt - (x^1 - x^0)\Big]\Big|\Big| \le \rho(1-q),$$

where $q = \frac{||A||}{\theta}\cdot\Big[\exp\Big\{\int\limits_0^T \alpha(t)dt\Big\} - 1 - \int\limits_0^T \alpha(t)dt\Big] < 1$. Here $L(B,B)$ is a space of linear bounded operators mapping B into B.

Under the conditions of Theorem 1, problem (1), (2) is uniquely solvable on the domain α^0.

For the linear boundary value problem

$$\frac{dx}{dt} = Q_1(t)x + Q_2(t)\lambda + f(t), \qquad x(0) = x^0, \qquad x(T) = x^1,$$

the conditions of Theorem 1 are reduced to the bounded invertibility of the operator $\bar{Q} = \int\limits_0^T Q_2(t)dt$.

The inequality (a_3) guarantees the existence and uniqueness of a solution to problem (1), (2) on the domain α^0.

The proposed approach was applied to semi-explicit differential equations with nonlinear boundary conditions:

$$\frac{dx}{dt} = f\Big(t,x,\frac{dx}{dt},\lambda\Big), \qquad x(0) = x^0, \tag{3}$$

$$\Phi[x(T), \dot{x}(T), \lambda] = 0. \tag{4}$$

Here $f : [0, T] \times B \times B \times B \to B$ is a continuous function satisfying the conditions for the existence of a solution to the Cauchy problem (3) on $[0, T]$ for all $\lambda \in G$; $G \subset B$, $\Phi : B \times B \times B \to B$.

Analogously, the right-hand side of the differential equation is considered on the set $\tilde{D}^0 = \{(t, x, y, \lambda) : 0 \le t \le T, ||x - x^{(0)}(t)|| \le R(t)\rho, ||y - \dot{x}^{(0)}(t)|| \le \dot{R}(t)\rho, ||\lambda - \lambda^0|| \le \rho\}$, where $\lambda^0 \in G$, $x^{(0)}(t)$ is a solution to the Cauchy problem (3) with $\lambda = \lambda^0$, $R(t)$ is a positive function continuously differentiable on $[0, T]$, and ρ is a nonnegative number. Let $\tilde{M}(f)$ denote the set of triples $(\lambda^0 \in G, R(t) > 0, \rho \ge 0)$ for which the following inequalities are satisfied:

$$||f(t, x, y, \lambda) - f(t, \tilde{x}, \tilde{y}, \tilde{\lambda})|| \le \alpha_1(t) \cdot (||x - \tilde{x}|| + ||\lambda - \tilde{\lambda}||) + \alpha_2(t) \cdot ||y - \tilde{y}||,$$

$$\alpha_2(t) < 1 \quad \big(\alpha_i(t) \in C([0, T]), i = 1, 2\big); \quad c(t) \exp\Big\{\int\limits_0^t c(\tau) d\tau\Big\} \le \dot{R}(t) \quad \Big(c(t) = \frac{\alpha_1(t)}{1 - \alpha_2(t)}\Big).$$

For a triple $(\lambda^0, R(t), \rho)$, the following sets are introduced:

$$\tilde{\alpha}^0_{x(t)} = \{x(t) : ||x(t) - x^{(0)}(t)|| \le R(t)\rho, ||\dot{x}(t) - \dot{x}^{(0)}(t)|| \le \dot{R}(t)\rho\},$$

$$\tilde{D}^0(T) = \{(u, v, \lambda) : ||u - x^{(0)}(T)|| \le R(T)\rho, ||v - \dot{x}^{(0)}(T)|| \le \dot{R}(T)\rho, ||\lambda - \lambda^0|| \le \rho\}.$$

Let the boundary function in (4) satisfy the Lipschitz condition $||\Phi(u, v, \lambda) - \Phi(\tilde{u}, \tilde{v}, \tilde{\lambda})|| \le \Phi_u||u - \tilde{u}|| + \Phi_v||v - \tilde{v}|| + \Phi_\lambda||\lambda - \tilde{\lambda}||$ on the set $\tilde{D}^0(T)$.

Theorem 2. Problem (3), (4) is solvable if and only if, given some $(\lambda^0, R(t), \rho) \in \tilde{M}(f)$, for any two pairs $(\lambda, x(t))$ and $(\tilde{\lambda}, x(t))$ from the set $\tilde{\alpha}^0 = \alpha^0_\lambda \times \tilde{\alpha}^0_{x(t)}$, there exist an invertible operator $A \in L(B, B)$ and a number $\theta > 0$ satisfying the inequality $||\lambda - \tilde{\lambda} - A\{\tilde{K}_1[\lambda, x(t)] - \tilde{K}_1[\tilde{\lambda}, x(t)]\}|| \le (1 - \theta)||\lambda - \tilde{\lambda}||$, and the following inequality is true:

$$\frac{1}{\theta}||A\tilde{K}_1[\lambda^0, x^{(0)}(t)]|| \le \rho(1 - q),$$

where $q = \frac{||A||}{\theta} \cdot \Big[\Phi_u \cdot \Big\{\exp\Big\{\int\limits_0^T c(t)dt - 1 - \int\limits_0^T \alpha_1(t)dt\Big\} + \Phi_v \cdot \Big\{c(T) \exp\Big\{\int\limits_0^T c(t)dt - \alpha_1(T)\Big\}\Big] < 1$, $\tilde{K}_1[\lambda, x(t)] = \Phi\Big[x^0 + \int\limits_0^T f(t, x(t), \dot{x}(t), \lambda), f(T, x(T), \dot{x}(T), \lambda), \lambda\Big]$.

Conditions for the continuous dependence of a solution on the initial data and a criterion for the existence of an isolated solution to problem (3), (4) were established.

Dzhumabaev D.S. justified a new version of the shooting method for nonlinear two-point boundary value problems of the following form

$$\frac{dz}{dt} = f(t, z), \tag{5}$$

$$g[z(0), z(T)] = 0, \tag{6}$$

where $f : [0, T] \times B \to B$ is continuous in t and z, $g : B \times B \to B$.

Let λ denote the value of $z(t)$ at the point $t = 0$. By the substitution $x(t) = z(t) - \lambda$, problem (5), (6) is reduced to the following boundary value problem with parameter

$$\frac{dx}{dt} = f(t, x + \lambda), \qquad x(0) = 0, \tag{7}$$

$$g[\lambda, \lambda + x(T)] = 0. \tag{8}$$

Assume that in the closed regions $D^0 = \{(t, x, \lambda) : 0 \le t \le T, ||x - x^{(0)}(t)|| \le R(t)\rho, ||\lambda - \lambda^0|| \le \rho\}$ and $D_1^0 = \{(\lambda, u) : ||\lambda - \lambda^0|| \le \rho, ||u - \lambda^0 - x^{(0)}(T)|| \le [1 + R(T)]\rho\}$ (here $x^{(0)}(t)$ is a solution to Cauchy problem (7) for $\lambda = \lambda^0$, $R(t) > 0$ for $t \in [0, T]$, and $\rho > 0$), the following inequalities hold:

$$||f(t, x + \lambda) - f(t, \tilde{x} + \tilde{\lambda})|| \le \alpha(t)(||x - \tilde{x}|| + ||\lambda - \tilde{\lambda}||),$$

$$||g(\lambda, u) - g(\tilde{\lambda}, \tilde{u})|| \le g_\lambda ||\lambda - \tilde{\lambda}|| + g_u ||u - \tilde{u}||,$$

and $\exp\Big\{\int\limits_0^t \alpha(\tau)d\tau\Big\} - 1 \le R(t)$.

Theorem 3. If for any two pairs $(\lambda, x(t))$ and $(\tilde{\lambda}, x(t))$ from the domain $\alpha^0 = \alpha_\lambda^0 \times \alpha_{x(t)}^0$ and for some $N \ge 0$, there exist an invertible operator $A \in L(B, B)$ and a number $\theta > 0$ satisfying the inequality $||\lambda - \tilde{\lambda} - A\{K_N^{(1)}[\lambda, x(t)] - K_N^{(1)}[\tilde{\lambda}, x(t)]\}|| \le (1 - \theta)||\lambda - \tilde{\lambda}||$, and the following inequality holds

$$\frac{1}{\theta}||A\{K_N^{(1)}[\lambda^0, x^{(0)}(t)]\}|| \le \rho(1 - q_N^{(1)}),$$

where $q_N^{(1)} = g_u \cdot \frac{||A||}{\theta} \cdot \Big[\exp\Big\{\int\limits_0^T \alpha(t)dt\Big\} - 1 - \int\limits_0^T \alpha(t)dt - ... - \frac{1}{N!}\Big(\int\limits_0^T \alpha(t)dt\Big)^N\Big] < 1$, then the boundary value problem (7), (8) has a unique solution in α^0.

Here $K_N^{(1)}[\lambda, x(t)] = g\Big[\lambda, \lambda + \int\limits_0^T f(t, \lambda + ... + \int\limits_0^{\tau_{N-3}} f(\tau_{N-2}, \lambda + x(\tau_{N-2}))d\tau_{N-2})...)dt\Big]$, $N = 0, 1, 2, ...$.

For different values of N, various sufficient conditions for the unique solvability to problem (7), (8) can be derived from Theorem 3. The problem of choosing an initial approximation and other replacement versions in problems with parameter were also considered.

Dzhumabaev D.S. also studied nonlinear infinite systems of equations

$$Q_j(\lambda_1, \lambda_2, ..., \lambda_i, ...) = b_j, \qquad j = 1, 2, ..., \tag{9}$$

where $\lambda = (\lambda_1, \lambda_2, ...)$ and $b = (b_1, b_2, ...)$ are elements of l_p $(1 \le p \le \infty)$. It is supposed that in the domain $D' = \{\lambda : ||\lambda - \lambda^0|| < \rho\} \subset l_p$, for all i $(i = 1, 2, ...)$, functions $Q_i(\lambda_1, \lambda_2, ...)$ have continuous partial derivatives with respect to all arguments and

1) $\sum\limits_{j=1}^{\infty} \sup\limits_{\lambda \in D'} \Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_j}\Big| \le k_1 < \infty$; 2) $\sum\limits_{k=1}^{\infty} \sup\limits_{\lambda \in D'} \Big|\frac{\partial Q_k(\lambda)}{\partial \lambda_i}\Big| \le k_2 < \infty$.

Then there exist numbers θ_1 and θ_2 satisfying the inequalities

3) $\Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_i}\Big| \ge \sum\limits_{j \ne i} \Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_j}\Big| + \theta_1$; 4) $\Big|\frac{\partial Q_i(\lambda)}{\partial \lambda_i}\Big| \ge \sum\limits_{k \ne i} \sup\limits_{\lambda \in D'} \Big|\frac{\partial Q_k(\lambda)}{\partial \lambda_i}\Big| + \theta_2$, for all $\lambda \in D'$ and $i = 1, 2, \ldots$

The following definition extends the concept of complete regularity to the case of nonlinear infinite systems in l_p.

Definition 1. An operator $Q = (Q_1, Q_2, ...)$ is called completely regular in the domain D', if it satisfies conditions 1)-4) wherein the numbers θ_1 and θ_2 are such that 5) $\frac{p-1}{p}\theta_1 + \frac{1}{p}\theta_2 = \theta > 0$.

Lemma 1. If Q is a completely regular operator in the domain D' and $\frac{1}{\theta}||Q(\lambda^0) - b|| < \rho$, then the infinite system of nonlinear equations (9) has a unique solution in D'.

Using Lemma 1, the results obtained for problems (1)-(2), (3)-(4), and (5)-(6) were concretized for infinite systems of differential equations. Effective conditions were established for the unique solvability of nonlinear boundary value problems for infinite systems of differential equations in the space l_p.

The findings described in this Section were published in [1–5] and formed the basis of his candidate thesis. In 1980, Dzhumabaev D.S. defended his dissertation “Boundary value problems with a parameter for ordinary differential equations in a Banach space" and earned a degree of Candidate of Physical and Mathematical Sciences in the specialty 01.01.02 – Differential Equations.

The methods and results of [1–5] were applied to nonlinear differential equations of various classes [6–12]. Dzhumabaev's research was then focused on various problems for nonlinear operator equations [13–17].

2 A linearizer and iterative processes for unbounded non-smooth operators

Consider the nonlinear operator equation

$$A(x) = 0, \tag{10}$$

where $x \in B_1$, $A(x) \in B_2$, and each B_i is a Banach space with norm $||\cdot||_i$, $i = 1, 2$. Let $D(A)$ and $R(A)$ denote the domain and range of A, respectively.

For a point $x^0 \in D(A)$, the following sets are constructed: $S(x^0, r) = \{x \in B_1 : ||x - x^0||_1 \leq r\}$, $U^0 = \{x \in D(A) : ||A(x)||_2 \leq ||A(x^0)||_2 = u^0\}$, and $\Omega = S(x^0, r) \cap U^0$. Assume that the operator A is closed on Ω. As is known, iterative methods, that allow one to find a solution under some sufficient conditions for its existence, rely on certain linearizations of the nonlinear operator. Linearization of an unbounded operator naturally leads to unbounded linear operators. This motivated Dzhumabaev D.S. to introduce the concept of a linearizer of an operator A at a point $\hat{x} \in D(A)$ that generalizes the Frechet derivative for unbounded non-smooth operators.

Definition 2. A linear operator $C : B_1 \to B_2$ is called a linearizer of an operator A at a point $\hat{x} \in D(A)$, if $D(A) \subseteq D(C)$ and there exist numbers $\epsilon \geq 0$ and $\delta > 0$ such that

$$||A(x) - A(\hat{x}) - C(x - \hat{x})||_2 \leq \epsilon ||x - \hat{x}||_1$$

for all $x \in D(A)$ satisfying $||x - \hat{x}||_1 < \delta$.

If $C \in L(B_1, B_2)$ is the Frechet derivative of A at a point $\hat{x} \in D(A)$, then it is also a linearizer. However, the definition of a linearizer, unlike that of the Frechet derivative, does not require: a) the boundedness of the operator C and 2) the dependence of ϵ on δ ($\epsilon(\delta) \to 0$ as $\delta \to 0$ for the Frechet derivative).

While the Frechet derivative of an operator A is uniquely determined, there can be infinitely many linearizers of this operator.

Distinctive advantages of linearizers make it possible to extend the domain of application of iterative methods to solving nonlinear operator equations. Dzhumabaev D.S. proposed a method for proving the convergence of iterative processes that takes into account the specificities of unbounded operator equations.

Theorem 4. Suppose that at each point $x \in \Omega$ the operator A has a linearizer C_x with constants ϵ_x and δ_x such that: 1) C_x is a one-to-one mapping of $D(C)$ onto $R(C)$, and $||C_x^{-1}|| \leq \gamma_x \leq \bar{\gamma}$; 2) $\epsilon_x \cdot \delta_x \leq \Theta < 1$; and 3) $\frac{\gamma_x}{\delta_x} \cdot ||A(x)||_2 \leq K$. If $\frac{\bar{\gamma}}{1-\Theta} \cdot ||A(x)||_2 < r$, then (10) has a solution $x^* \in \Omega$, to which the iteration process

$$x^{(n+1)} = x^{(n)} - \frac{1}{\alpha} C_{x^{(n)}}^{-1} \{A(x^{(n)})\}$$

converges, here $\alpha = \max\{1, K\}$, $n = 0, 1, 2, \ldots$

In the case when for a given $\delta > 0$ there exists $\epsilon(\delta)$ independent of x, the following assertion is true.

Theorem 5. Suppose that at each point $x \in \Omega$ and for each $\delta \in (0, h)$ the operator A has a linearizer C_x with constants δ and $\epsilon(\delta) \geq 0$ satisfying the following conditions: 1) C_x^{-1} exists on $R(C)$, and $||C_x^{-1}|| \leq \gamma$, 2) $\lim\limits_{\delta \to 0} \epsilon(\delta) = 0$.

Then (10) has a solution $x^* \in \Omega$, if the following inequality holds: 3) $\gamma \cdot ||A(x)||_2 < r$.

Theorem 5 generalizes the local theorem of Hadamard to unbounded operator equations. This made it possible to extend the well-known Newton-Kantorovich method to unbounded nonsmooth operator equations and apply it to nonlinear boundary value problems for differential equations.

Consider the closed operator equation

$$A(x) \equiv Cx + F(x) = 0, \tag{11}$$

where $C : X \to Y$ is a closed linear operator, $F : X \to Y$ is a continuous operator, and X and Y are Banach spaces with respective norms $||\cdot||_1$ and $||\cdot||_2$.

Assume that F has a Frechet derivative in some domain containing the ball $\bar{S}(x^0, r) = \{x \in X : ||x - x^0||_1 \leq r\}$, $x^0 \in D(C)$, and $R(C + F'(x)) = Y$ for $x \in S(x^0, r)$. Then in $D(A) = D(C) \cap \bar{S}(x^0, r)$ the operator A has the linearizer $C_1(x) = C + F'(x)$, and $D(C_1) = D(C) \cap X = D(C)$.

Theorem 6. Assume that the following conditions hold:
(1) For all $x \in D(A)$, the linearizer $C_1(x)$ has a bounded inverse, and $||C_1^{-1}(x)||_{L(Y,X)} \leq \gamma$;
(2) $||F'(x) - F'(y)||_{L(X,Y)} \leq L \cdot ||x - y||_1$, $x, y \in \bar{S}(x^0, r)$;
(3) $\frac{m}{L\gamma} + \gamma \frac{b_m}{b_0} ||A(x^0)||_2 \sum\limits_{s=0}^{\infty} (b_m)^{2^s - 1} < r$, where $b_0 = \frac{L}{2}\gamma^2 u_0$, $u_0 = ||A(x^0)||_2$, $\beta_k = 1 - \frac{1}{4b_{k-1}}$, $b_k = \beta_k \cdot b_{k-1}$, $k = 1, 2, ..., m$, where m is a nonnegative number such that $b_m < 1$ and $b_{m-1} \geq 1$.

Then the damped Newton-Kantorovich method

$$x^{(k+1)} = x^{(k)} - \frac{1}{\alpha_k}[C + F'(x^k)]^{-1}[Cx^k + F(x^k)], \quad k = 0, 1, 2, ...,$$

where $\alpha_k = 2b_k$ for $k = 0, ..., m-1$ and $\alpha_k = 1$ for $k = m, m+1, ...$, converges to a solution of (11).

Theorem 7. Assume that the following conditions hold:
(1) For all $x \in D(A)$, the linearizer $C_1(x)$ has a bounded inverse, and $||C_1^{-1}(x)||_{L(Y,X)} \leq \gamma$;
(2) The Frechet derivative $F'(x)$ is uniformly continuous in $\bar{S}(x^0, r)$;
(3) $\gamma \cdot ||A(x^0)||_2 < r$.

Then there exist numbers $\alpha_n \geq 1$, $n = 0, 1, ...$, such that the iteration process

$$x^{(m+s+1)} = x^{(m+s)} - [C + F'(x^{m+s})]^{-1}[Cx^{m+s} + F(x^{m+s})], \quad s = 0, 1, 2, ...,$$

converges to an isolated solution $x^* \in D(A)$ of (11). Furthermore, starting with some k^0, we can take α_n ($n \geq k^0$) equal to 1, and the convergence rate becomes superlinear.

These results were published in "News of the Academy of Sciences of Kazakh SSR. Series Physical and Mathematical", 1984 [13, 14], and, at the request of the American Mathematical Society, were translated and published in "American Mathematical Society Translations", 1989 [16, 17], as well as in "Mathematical Notes" [15]. Various aspects of applications of these results were considered in [18–20].

3 The parametrization method for solving boundary value problems

Dzhumabaev D.S. developed the parametrization method for investigation and solving boundary value problems. The method was originally offered in [21, 22] for solving two-point boundary value problems for a linear differential equation of the following form

$$\frac{dx}{dt} = A(t)x + f(t), \qquad x \in \mathbb{R}^n, \tag{12}$$

$$Bx(0) + Cx(T) = d, \tag{13}$$

where $A(t)$ and $f(t)$ are continuous in $(0, T]$, B and C are $n \times n$ matrices, $d \in \mathbb{R}^n$.

Consider a partition dividing the interval $[0,T)$ into N equal parts with step size $h>0$: $[0,T)=\bigcup\limits_{r=1}^{N}[(r-1)h,rh)$. Let $x_r(t)$ denote the restriction of the function $x(t)$ to the r-th subinterval, i.e. $x_r(t)$, $r=\overline{1,N}$, is a vector function of dimension n defined on $[(r-1)h,rh)$ and coinciding there with $x(t)$. Problem (12), (13) is thus transformed into an equivalent multipoint boundary-value problem

$$\frac{dx_r}{dt}=A(t)x_r+f(t),\qquad t\in[(r-1)h,rh),\quad r=1,2,...,N, \tag{14}$$

$$Bx_1(0)+C\lim_{t\to T-0}x_N(t)=d, \tag{15}$$

$$\lim_{t\to sh-0}x_s(t)=x_{s+1}(sh),\qquad s=1,2,...,N-1. \tag{16}$$

Here (16) are the matching conditions for the solution at the interior points of the partition.

Obviously, if $x(t)$ is a solution of problem (12), (13), then the set of restrictions $(x_r(t))$, $r=1,2,...,N$, is a solution of the multi-point problem (14)–(16). Conversely, if a set of vector functions $(x_r(t))$, $r=1,2,....N$, is a solution of problem (14)–(16), then the function $x(t)$ obtained by piecing together $x_r(t)$ is a solution of the original boundary value problem.

On each subinterval $[(r-1)h,rh)$, the substitution $u_r(t)=x_r(t)-\lambda_r$ is made, where λ_r denotes the value of $x_r(t)$ at the point $t=(r-1)h$. Problem (14)–(16) is then reduced to the boundary value problem with parameter

$$\frac{du_r}{dt}=A(t)u_r+A(t)\lambda_r+f(t),\quad t\in[(r-1)h,rh),\quad u_r[(r-1)h]=0,\quad r=1,2,...,N, \tag{17}$$

$$B\lambda_1+C\lambda_N+C\lim_{t\to T-0}u_N(t)=d, \tag{18}$$

$$\lambda_s+\lim_{t\to sh-0}u_s(t)=\lambda_{s+1},\qquad s=1,2,...,N-1. \tag{19}$$

An advantage of problem (17)–(19) is that it involves the initial conditions $u_r[(r-l)h]=0$, so that one can determine $u_r(t)$ from the integral equations

$$u_r(t)=\int\limits_{(r-1)h}^{t}[A(\tau)u_r+A(\tau)\lambda_r]d\tau+\int\limits_{(r-1)h}^{t}f(\tau)d\tau. \tag{20}$$

In (20), replacing $u_r(\tau)$ by the right-hand side of (20) and repeating the process ν ($\nu=1,2,...$) times, one obtains a representation of $u_r(t)$ by a sum of iterated integrals. Letting $t\to rh-0$ and substituting $\lim\limits_{t\to rh-0}u_r(t)$, $r=1,2,\ldots,N$, into (18) and (19) results in a system of nN algebraic equations in the parameters λ_{ri}, $r=1,2,...,N$, $i=1,2,...,n$:

$$Q_\nu(h)\lambda=-F_\nu(h)-G_\nu(u,h),\qquad \lambda\in\mathbb{R}^{Nn}. \tag{21}$$

The basic idea behind the method is to reduce the problem in question to an equivalent problem with a parameter (17)–(19) whose solution is determined as the limit of a sequence of systems of pairs consisting of the parameter λ and the function u. The parameter is found from the system of linear equations (21) determined by the matrices of the differential equation (12) and boundary conditions (13). The functions u_r are solutions of Cauchy problems (17) on the partition subintervals $[(r-1)h,rh)$, $r=1,2,...,N$, for the found values of the parameter. The introduction of parameters made it possible to obtain conditions for the convergence of proposed algorithms and, at the same time, for the existence of a solution, in terms of the input data. This makes the parametrization method different from the shooting method and its modifications, where shooting parameters are found from some equations constructed via general solutions of differential equations, and convergence conditions are also given in terms of general solutions.

Theorem 8. Suppose that for some $h > 0 \quad (Nh = T)$ and $\nu \quad (\nu = 1, 2, \ldots)$ the matrix $Q_\nu(h) : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is invertible and the following inequalities hold:
(a) $||[Q_\nu(h)]^{-1}|| \le \gamma_\nu(h)$;
(b) $q_\nu(h) = \gamma_\nu(h) \max(1, h||C||)[e^{\alpha h} - 1 - \alpha h - ... - \frac{(\alpha h)^\nu}{\nu!}] < 1$, where $\alpha = \max\limits_{t \in [0,T]} ||A(t)||$.

Then the boundary-value problem (12), (13) has a unique solution $x^*(t)$, and the estimate

$$||x^*(t) - x^{(k)}(t)|| \le \gamma_\nu(h) \max(1, h||C||) \frac{(\alpha h)^\nu}{\nu!} e^{\alpha h} \frac{[q_\nu(h)]^\nu}{1 - q_\nu(h)} M(h), \qquad t \in [0, T],$$

holds true, where

$$M(h) = \gamma_\nu(h)[e^{\alpha h} - 1] \max\Big\{1 + h||C|| \sum_{j=0}^{\nu-1} \frac{(\alpha h)^j}{j!}, \sum_{j=0}^{\nu-1} \frac{(\alpha h)^j}{j!}\Big\} \max(||d||, \max_{t \in [0,T]} ||f(t)||)h$$
$$+ e^{\alpha h} \max_{t \in [0,T]} ||f(t)||h,$$

and $x^{(k)}(t)$ is a piecewise-continuously differentiable function on $[0, T]$, for which $\lambda_r^{(k)} + u_r^{(k)}(t)$ is the restriction to $[(r - l)h, rh)$, $r = 1, 2, ..., N$.

It was shown that the conditions of Theorem 8 are also necessary and sufficient for the unique solvability of problem (12), (13).

The parametrization method was then applied to the study of singular problems for which the problem of approximation by regular two-point boundary value problems was solved [23–27]. Necessary and sufficient conditions were obtained for the well-posed solvability of the problem of finding a solution to the system of differential equations (12), that is bounded on the whole axis $\mathbb{R}$. For systems whose matrices and right-hand sides are constant in the limit, approximating regular two-point boundary value problems were constructed. The connection between the well-posed solvability of the original singular problem and that of an approximating problem was established. In the general case, Lyapunov transformations possessing certain properties were used to construct regular two-point boundary value problems as approximations to the problem of determining a solution bounded on the entire real line. The concept of a solution "in the limit as $t \to \infty$" was introduced and the behavior of solutions of linear ordinary differential equations as $t \to \infty$ was investigated. Necessary and sufficient conditions were derived under which a singular boundary value problem with conditions assigned at infinity is uniquely solvable, and an appropriate approximating problem was constructed. These results were developed to the system of differential equations on the real axis:

$$\frac{dx}{dt} = f(t, x), \qquad x \in \mathbb{R}^n. \tag{22}$$

In [28, 29] the results of Section 2 were also extended to system (22) with the nonlinear boundary condition

$$g[x(0), x(T)] = 0.$$

Results of Sections 2 and 3 were included in the doctoral dissertation.

The doctoral dissertation by Dzhumabaev D.S. titled "Singular boundary value problems for ordinary differential equations and their approximation" is a fundamental scientific work that underwent comprehensive approbation in leading scientific centers, such as the Computing Center of the Russian Academy of Sciences (A.A. Abramov, N.B. Konyukhova), the Institute of Applied Mathematics of the Russian Academy of Sciences (K.I. Babenko), Lomonosov Moscow State University (V.M. Millionshchikov, V.A. Kondratiev, N.Kh. Rozov), Institute of Mathematics NAS of Ukraine (Y.A.

Mitropol'skii, A.M. Samoilenko, V.L. Makarov, V.L. Kulik), Voronezh State University (V.I. Perov), I. Vekua Institute of Applied Mathematics of Tbilisi State University (I.T. Kiguradze), Kiev State University named after T. Shevchenko (N.I. Perestyuk). Doctoral dissertation was defended at the Specialized Council of the Institute of Mathematics of the NAS of Ukraine in 1994.

The parametrization method was extended to various linear and nonlinear boundary value problems for ordinary differential equations on a finite interval and on the whole real line; necessary and sufficient solvability conditions for those problems were obtained in [28–49].

4 Nonlocal problems for systems of second-order hyperbolic equations

The results obtained in Sections 2 and 3 provided a basis for solving nonlocal boundary value problems for systems of second-order hyperbolic equations [50–70].

In the domain $\Omega = [0,T] \times [0,\omega]$, consider the following nonlocal boundary value problem for the system of hyperbolic equations with two independent variables:

$$\frac{\partial^2 u}{\partial t \partial x} = A(t,x)\frac{\partial u}{\partial x} + B(t,x)\frac{\partial u}{\partial t} + C(t,x)u + f(t,x), \tag{23}$$

$$P_2(x)\frac{\partial u(t,x)}{\partial x}\Big|_{t=0} + P_1(x)\frac{\partial u(t,x)}{\partial t}\Big|_{t=0} + P_0(x)u(t,x)|_{t=0} +$$

$$+ S_2(x)\frac{\partial u(t,x)}{\partial x}\Big|_{t=T} + S_1(x)\frac{\partial u(t,x)}{\partial t}\Big|_{t=T} + S_0(x)u(t,x)|_{t=T} = \varphi(x), \quad x \in [0,\omega], \tag{24}$$

$$u(t,0) = \psi(t), \qquad t \in [0,T], \tag{25}$$

where $u(t,x) = col(u_1(t,x),...,u_n(t,x))$ is an unknown function, the $n \times n$ matrices $A(t,x)$, $B(t,x)$, $C(t,x)$, $P_i(x)$, $S_i(x)$, $i = \overline{0,2}$, and the n-vector functions $f(t,x)$, $\varphi(x)$ are continuous on Ω and $[0,\omega]$, respectively; the n-vector function $\psi(t)$ is continuously differentiable on $[0,T]$.

Sufficient coefficient conditions for the existence and uniqueness of a classical solution of problem (23)–(25) were established by a modification of the parametrization method [50, 53, 55, 60, 61]. A relationship with the following family of boundary value problems for ordinary differential equations was established:

$$\frac{\partial v}{\partial t} = A(t,x)v + F(t,x), \qquad x \in [0,\omega], \tag{26}$$

$$P_2(x)v(0,x) + S_2(x)v(T,x) = \Phi(x), \tag{27}$$

here n-vector functions $F(t,x)$ and $\Phi(x)$ are continuous on Ω and $[0,\omega]$, respectively.

For fixed $x \in [0,\omega]$ problem (26), (27) is a linear boundary value problem for the system of ordinary differential equations. Suppose the variable x is changed on $[0,\omega]$; then we obtain a family of boundary value problems for ordinary differential equations.

Sufficient and necessary conditions for the well-posedness of nonlocal boundary value problem for the system of hyperbolic equations (25)–(27) were obtained in [59, 64, 66, 67].

Let $C([0,\omega], R^n)$ be a space of continuous on $[0,\omega]$ vector functions $\varphi(x)$ with the norm $||\varphi||_{0,1} = \max\limits_{x\in[0,\omega]} ||\varphi(x)||$;
$C^1([0,T], R^n)$ be a space of continuously differentiable on $[0,T]$ vector functions $\psi(t)$ with the norm $||\psi||_{1,0} = \max\Big(\max\limits_{t\in[0,T]} ||\psi(t)||, \max\limits_{t\in[0,T]} ||\dot{\psi}(t)||\Big)$;
$C^{1,1}(\Omega, R^n)$ be a space of functions $u(t,x) \in C(\Omega, R^n)$ with continuous on Ω partial derivatives $\frac{\partial u(t,x)}{\partial x}$, $\frac{\partial u(t,x)}{\partial t}$, $\frac{\partial^2 u(t,x)}{\partial t \partial x}$ with the norm $||u||_{1,1} = \max\Big(||u||_0, \Big|\Big|\frac{\partial u}{\partial x}\Big|\Big|_0, \Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_0, \Big|\Big|\frac{\partial^2 u}{\partial t \partial x}\Big|\Big|_0\Big)$.

Lemma 2. If problem (26), (27) has a solution for arbitrary $F(t,x) \in C(\Omega, R^n)$ and $\Phi(x) \in C([0,\omega], R^n)$, then this solution is unique.

Definition 3. Problem (26), (27) is called well-posed if for arbitrary $F(t,x) \in C(\Omega, R^n)$ and $\Phi(x) \in C([0,\omega], R^n)$ it has a unique solution $v(t,x) \in C(\Omega, R^n)$ and for it the estimate holds

$$\max_{t\in[0,T]} ||v(t,x)|| \leq K \max\Big(\max_{t\in[0,T]} ||F(t,x)||, ||\Phi(x)||\Big), \tag{28}$$

where the constant K is independent of $F(t,x)$ and $\Phi(x)$, and $x \in [0,\omega]$.

Lemma 3. If $v(t,x)$ is a solution to problem (26), (27), and the estimate holds

$$||v||_0 \leq K \max\Big(||F||_0, ||\Phi||_{0,1}\Big),$$

where K is a constant independent of the functions $F(t,x)$ and $\Phi(x)$, then for every $x \in [0,\omega]$ the inequality (28) is valid.

Denote by $\Omega_\eta = [0,T] \times [0,\eta]$ and $||u||_{0,\eta} = \max\limits_{(t,x)\in\Omega_\eta} ||u(t,x)||$.

Definition 4. Boundary value problem (23)–(25) is called well-posed if for arbitrary $f(t,x) \in C(\Omega, R^n)$ and $\psi(t) \in C^1([0,T], R^n)$ and $\varphi(x) \in C([0,\omega], R^n)$ it has a unique classical solution $u(t,x)$ and this solution satisfies the following estimate

$$\max\Big(||u||_{0,\eta}, \Big|\Big|\frac{\partial u}{\partial x}\Big|\Big|_{0,\eta}, \Big|\Big|\frac{\partial u}{\partial t}\Big|\Big|_{0,\eta}\Big) \leq \tilde{K} \max\Big(||f||_{0,\eta}, ||\psi||_{1,0}, \max_{x\in[0,\eta]} ||\varphi(x)||\Big),$$

where constant $\tilde{K}$ is independent of $f(t,x)$ and $\psi(t)$ and $\varphi(x)$ and $\eta \in [0,\omega]$.

Theorem 9. The boundary value problem (23)–(25) is well-posed if and only if so is problem (26), (27).

From Theorem 9 it follows that the well-posedness of problem (23)–(25) are equivalent to the well-posedness of problem (26), (27).

These results were extended to a nonlocal problem with an integral condition for system (25) (see [71]).

The problem of finding bounded solutions of system (23) and the families of systems (26) was solved in [54, 56–58, 61–63, 65, 72].

The parametrization method was further developed to nonlinear nonlocal problems for a system of hyperbolic equations [68–70, 73].

5 Boundary value problems for loaded and integro-differential equations

On the basis of the parametrization method, constructive algorithms were developed for finding solutions to various boundary value problems for integro-differential and loaded equations [72, 74–82].

In the interval $[0,T]$, consider the following linear two-point boundary value problem for an integro-differential equation:

$$\frac{dx}{dt} = A(t)x + \int\limits_0^T K(t,s)x(s)ds + f(t), \qquad x \in \mathbb{R}^n, \tag{29}$$

$$Bx(0) + Cx(T) = d, \qquad d \in \mathbb{R}^n, \tag{30}$$

where $A(t)$ and $K(t,s)$ are continuous matrices on $[0,T]$ and $[0,T] \times [0,T]$, respectively; $f(t)$ is continuous on $[0,T]$.

It is well known that the basic techniques for analysis and solving boundary value problems for integro-differential equations are the Nekrasov method and the Green's function method. Nekrasov's method applies to problem (29), (30), if we assume the unique solvability of the second-kind Fredholm integral equation

$$x(t) = \int_0^T M(t,s)x(s)ds + F(t), \qquad t \in [0,T],$$

with the kernel $M(t,s) = \int_0^t X(t)X^{-1}(\tau)K(\tau,s)d\tau$, where $X(t)$ is the fundamental matrix of the differential part of equation (29) and $F(t) \in C([0,T],\mathbb{R}^n)$. The Green's function method applies to problem (29), (30) under assumption that the boundary value problem for the differential part of (29) is uniquely solvable; i.e., this method assumes the unique solvability of problem (29), (30) with $K(t,s) = 0$.

However, the assumptions of neither Nekrasov's method nor Green's function method are necessary conditions for the solvability of problem (29), (30).

In [83], a coefficient criterion for the well-posedness of problem (29), (30) was established in terms of approximating boundary value problems for the loaded differential equation

$$\frac{dx}{dt} = A(t)x + \sum_{i=1}^{m} K_i(t)x(\theta_i) + f(t), \qquad x \in \mathbb{R}^n,$$

subject to condition (30), by the parametrization method.

In [84], Dulat Dzhumabaev proposed a method for solving the problem (29), (30) that is based on the parametrization method and properties of a fundamental matrix of the differential part of (29). The interval $[0,T]$ is divided into N equal parts with step size $h > 0$: $[0,T) = \bigcup_{r=1}^{N} [(r-1)h, rh)$. Let $x_r(t)$ be the restriction of $x(t)$ to the rth subinterval $[(r-1)h, rh)$. The values of the solution at the left-endpoints of the subintervals are assumed as additional parameters $\lambda_r = x_r[(r-1)h]$. By the substitution $u_r(t) = x_r(t) - \lambda_r$ at every rth subinterval, the problem (29), (30) is reduced to the multi-point boundary value problem for a system of integro-differential equations with parameters

$$\frac{du_r}{dt} = A(t)u_r + A(t)\lambda_r + \sum_{j=1}^{N} \int_{(j-1)h}^{jh} K(t,s)[u_j(s) + \lambda_j]ds + f(t), \qquad t \in [(r-1)h, rh), \tag{31}$$

$$u_r[(r-1)h] = 0, \qquad r = 1,2,...,N, \tag{32}$$

$$B\lambda_1 + C\lambda_N + C \lim_{t \to T-0} u_N(t) = d, \tag{33}$$

$$\lambda_p + \lim_{t \to ph-0} u_p(t) - \lambda_{p+1} = 0, \qquad p = 1,2,...,N-1. \tag{34}$$

The introduction of additional parameters resulted in the emergence of the initial data (32) for the unknown functions $u_r(t)$, $r = 1,2,...,N$. For fixed parameter values $\lambda \in \mathbb{R}^{nN}$, the system of functions $u[t] = (u_1(t), u_2(t), ..., u_N(t))$ is determined from problem (31), (32), which is a special Cauchy problem for the system of integro-differential equations. Problem (31), (32) is equivalent to the system of integral equations

$$u_r(t) = X(t) \int_{(r-1)h}^{t} X^{-1}(\tau)A(\tau)d\tau\lambda_r + X(t) \int_{(r-1)h}^{t} X^{-1}(\tau) \sum_{j=1}^{N} \int_{(j-1)h}^{jh} K(\tau,s)[u_j(s) + \lambda_j]dsd\tau +$$

$$+ X(t) \int\limits_{(r-1)h}^{t} X^{-1}(\tau)f(\tau)d\tau, \qquad t \in [(r-1)h, rh), \qquad r = 1, 2, ..., N. \tag{35}$$

By solving (35), one can find the representations of $u_r(t)$ in terms of $\lambda \in \mathbb{R}^{nN}$ and $f(t)$. Substituting them into (33) and (34) yields a system of equations for finding the unknown parameters. Thus, when applying the parametrization method to problem (29), (30), one has to solve an auxiliary problem, namely, the special Cauchy problem (31), (32), or the equivalent system of integral equations (35). However, unlike the auxiliary problem of Nekrasov's method, the special Cauchy problem is uniquely solvable for any sufficiently small partition step size $h > 0$. Let a number $h_0 > 0$ satisfy the inequality

$$\sigma(h_0) = \beta T h_0 e^{\alpha h_0} < 1,$$

where $\beta = \max\limits_{(t,s)\in[0,T]\times[0,T]} ||K(t,s)||$ and $\alpha = \max\limits_{t\in[0,T]} ||A(t)||$. It was shown that, for any $h \in (0, h_0]$: $Nh = T$, system (35) is uniquely solvable. This property of the auxiliary problem of the parametrization method made it possible to establish solvability criteria for the boundary value problem considered.

Necessary and sufficient conditions for the solvability, including the unique solvability, of problem (29), (30) were obtained in terms of a matrix $Q_{*,*}(h)$ constructed via the fundamental matrix of the differential part of system (29), the matrices of boundary conditions (30), and the resolvent of an auxiliary Fredholm integral equation of the second kind.

In [85], a family of algorithms was proposed for solving problem (29), (30). The numerical parameters of the family are the partition step $h > 0 : Nh = T$, the number $\nu \in \mathbb{R}^n$ of iterated integrals used in the algorithm, and a nonnegative integer m specifying how many terms of the resolvent of the corresponding Fredholm integral equation of the second kind are used in the algorithm. The basic condition for the feasibility and convergence of the algorithm is that the matrix $Q_\nu^m(h)$ is invertible for chosen numerical parameters. The unknown parameters are found at the first stage of each step in the algorithm by using the invertibility of this matrix. The special Cauchy problem (31), (32) with the found parameter values is solved at the second stage of the algorithm. Necessary and sufficient conditions for the well-posedness of problem (29), (30) were established in terms of the input data without using the fundamental matrix or the resolvent.

In [86], the method and results of [84] were generalized to the case of an arbitrary partition. Let Δ_N denote a partition of $[0,T]$ into N parts: $t_0 = 0 < t_1 < \ldots < t_N = T$; the case of no partitioning is denoted by Δ_1. Each partition Δ_N is associated with a homogeneous Fredholm integral equation of the second kind. The partition Δ_N is called regular if the corresponding equation has only the trivial solution. The regularity of Δ_N leads to a unique solvability of the special Cauchy problem mentioned above. The solvability criteria for linear two-point boundary value problem for Equation (29) obtained in [86] are applicable for arbitrary regular partition Δ_N. The algorithms of the parameterization method for solving linear boundary value problems for Fredholm integro-differential equations were offered in [70].

These results were extended to boundary value problems for impulsive integro-differential equations in [87].

6 New general solutions to linear Fredholm integro-differential equations and their applications in solving boundary value problems

It is known that Volterra integro-differential equations are solvable for any right-hand side and have classical general solutions. However, there exist linear loaded differential equations and Fredholm integro-differential equations that do not admit classical general solutions. The question arises as to whether it is possible to construct such general solutions that exist for all differential and integro-differential equations and would allow solving boundary value problems for these equations.

Dzhumabaev D.S. proposed a novel approach to the concept of the general solution for a linear ordinary Fredholm integro-differential equation based on the parametrization method in [88]. The domain interval is partitioned and the values of the solution at the left endpoints of the subintervals are considered as additional parameters. By introducing new unknown functions on the partition subintervals, a special Cauchy problem for a system of integro-differential equations with parameters is obtained. Using the solution of this problem, a new general solution of the linear Fredholm integro-differential equation was constructed.

Suppose Δ_N is a partition $t_0 = 0 < t_1 < \ldots < t_N = T$. Let $x(t)$ be a function, piecewise continuous on $[0,T]$ with the possible points of discontinuity: $t = t_p$, $p = 1,2,...,N-1$. Let $x_r(t)$ be the restriction of $x(t)$ to the rth subinterval $[t_{r-1}, t_r)$, i.e. $x_r(t) = x(t)$, $t \in [t_{r-1}, t_r)$, $r = 1,2,\ldots,N$. For definiteness, assume that $x_r(t_{r-1}) = \lim\limits_{t \to t_{r-1}+0} x_r(t)$, $r = 1,2,...,N$. If $x(t)$ is piecewise continuously differentiable on $(0,T)$ and satisfies the Fredholm integro-differential equation (29) for each $t \in (0,T) \backslash \{t_p, p = 1,2,...,N-1\}$, then the system of its restrictions $x[t] = (x_1(t),...,x_N(t))$ satisfies the following system of integro-differential equations:

$$\frac{dx_r}{dt} = A(t)x_r + \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} K(t,\tau)x_j(\tau)d\tau + f(t), \qquad t \in [t_{r-1}, t_r), \quad r = 1,2,...,N. \tag{36}$$

Let $C([0,T], \Delta_N, \mathbb{R}^{nN})$ denote the space of function systems $x[t] = (x_1(t), x_2(t), ..., x_N(t))$, where $x_r : [t_{r-1}, t_r) \to \mathbb{R}^n$ is continuous and has the finite left-sided limit $\lim\limits_{t \to t_r - 0} x_r(t)$ for any $r = 1,2,...,N$, with the norm $x[\Delta]_2 = \max\limits_{r=1,2,...,N} \sup\limits_{t \in [t_{r-1}, t_r)} ||x_r(t)||$.

A function system $x[t] = (x_1(t), x_2(t), ..., x_N(t)) \in C([0,T], \Delta_N, \mathbb{R}^{nN})$ is called a solution to the system of integro-differential equations (35) if the functions $x_r(t)$, $r = 1,2,...,N$, are continuously differentiable on (t_{r-1}, t_r) and satisfy equations (36).

Suppose that the function system $x^*[t] = (x_1^*(t), x_2^*(t), ..., x_N^*(t))$ is a solution to (36). Then the function $x^*(t)$, defined as $x^*(t) = x_r^*(t)$ for $t \in [t_{r-1}, t_r)$, $r = 1,2,...,N$, and $x^*(T) = \lim\limits_{t \to T-0} x_N^*(t)$, is piecewise continuously differentiable and consistent with Eq. (29) for $t \in (0,T) \backslash \{t_p, p = 1,2,...,N-1\}$. The introduction of the parameters $\lambda_r = x_r(t_{r-1})$, $r = 1,2,...,N$, and substituting new unknown functions $u_r(t) = x_r(t) - \lambda_r$ on each subinterval $[t_{r-1}, t_r)$, yields the system of integro-differential equations with parameters

$$\frac{du_r}{dt} = A(t)u_r + A(t)\lambda_r + \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} K(t,\tau)[u_j(\tau) + \lambda_j]d\tau + f(t), \quad t \in [t_{r-1}, t_r), \; r = 1,...,N, \tag{37}$$

subject to the initial conditions

$$u_r(t_{r-1}) = 0, \qquad r = 1,2,...,N. \tag{38}$$

Problem (37), (38) is called a special Cauchy problem for the system of integro-differential equations with parameters. Without the interval's partition, problem (37), (38) is the Cauchy problem with the initial condition at $t = 0$ for the Fredholm integro-differential equation with parameter.

A solution to the special Cauchy problem (37), (38) with fixed values of parameters $\lambda_r^* \in \mathbb{R}^n$, $r = 1,...,N$, is a function system $u[t, \lambda^*] = (u_1(t, \lambda^*), u_2(t, \lambda^*), ..., u_N(t, \lambda^*)) \in C([0,T], \Delta_N, \mathbb{R}^{nN})$, which satisfies the system of integro-differential equations (37) with $\lambda = \lambda^*$ and initial conditions (38).

Let $X_r(t)$ be a fundamental matrix of the differential equation $\frac{dx}{dt} = A(t)x$ on the interval $[t_{r-1}, t_r]$. Then problem (37), (38) is equivalent to the system of integral equations

$$u_r(t) = X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau_1)A(\tau_1)d\tau_1\lambda_r + X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau_1) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} K(\tau_1, \tau)[u_j(\tau) + \lambda_j]d\tau d\tau_1 +$$

$$+X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau_1)f(\tau_1)d\tau_1, \qquad t \in [t_{r-1}, t_r), \quad r = 1, 2, ..., N.$$

Take an arbitrary partition Δ_N and consider the corresponding homogeneous Fredholm integral equation of the second kind

$$y(t) = \int_0^T M(\Delta_N, t, \tau)y(\tau)d\tau, \qquad t \in [0, T], \tag{39}$$

where $M(\Delta_N, t, \tau) = \int\limits_{\tau}^{t_1} K(t, \tau_1)X_1(\tau_1)d\tau_1 X_1^{-1}(\tau)$, $t \in [0, T]$, $\tau \in [0, t_1]$,

$M(\Delta_N, t, \tau) = \int\limits_{\tau}^{t_j} K(t, \tau_1)X_j(\tau_1)d\tau_1 X_j^{-1}(\tau)$, $t \in [0, T]$, $\tau \in (t_{j-1}, t_j]$, $j = 2, ..., N$.

Definition 5. A partition Δ_N is called regular for Equation (29) if the integral equation (39) has only the trivial solution.

Let $\sigma([0, T])$ denote the set of regular partitions of the interval $[0, T]$. The set $\sigma([0, T])$ is not empty.

Definition 6. The special Cauchy problem (37), (38) is called uniquely solvable if it has a unique solution for any pair $(f(t), \lambda)$ with $f(t) \in C([0, T], \mathbb{R}^n)$ and $\lambda \in \mathbb{R}^{nN}$.

Definition 7. Suppose that $\Delta_N \in \sigma([0, T])$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N) \in \mathbb{R}^{nN}$, and the function system $u[t, \lambda] = (u_1(t, \lambda), u_2(t, \lambda), ..., u_N(t, \lambda))$ is a solution to the special Cauchy problem for the system of integro-differential equations with parameters (37), (38). Then the function $x(\Delta_N, t, \lambda)$ defined by the equalities $x(\Delta_N, t, \lambda) = \lambda_r + u_r(t, \lambda)$, $t \in [t_{r-1}, t_r)$, $r = 1, 2, ..., N$, and $x(\Delta_N, T, \lambda) = \lambda_N + \lim\limits_{t \to T-0} u_N(t, \lambda)$ is called the Δ_N general solution to the integro-differential equation (29).

Theorem 10. For any $\Delta_N \in \sigma([0, T])$, there exists a unique Δ_N general solution to the linear Fredholm integro-differential equation (29).

In contrast to the classical general solution, the Δ_N general solution exists for all linear nonhomogeneous Fredholm integro-differential equations and contains N arbitrary parameters $\lambda_r \in \mathbb{R}^n$.

The concept of new general solution, introduced by Dzhumabaev, made it possible to derive the solvability criteria for the linear Fredholm integro-differential equations and boundary value problems for this equation. The proposed method consists of the construction of Δ_N general solutions and solving linear algebraic equations with respect to parameters of those solutions. The Cauchy problems for ordinary differential equations and problems of evaluation of the definite integrals on the subintervals are used as auxiliary problems. Depending on the choice of methods for solving auxiliary problems, either numerical or approximate methods were obtained in order to solve the linear boundary value problems for Fredholm integro-differential equations [89–92].

The new general solution made it possible to propose new numerical and approximate methods for solving boundary value problems with and without parameter for nonlinear ordinary differential equations [93–98]. These methods are based on the construction and solving a system of algebraic equations for arbitrary vectors of the new general solution. The coefficients and the right-hand sides of this system are determined using solutions of the Cauchy problems for ordinary differential equations on the subintervals. Using the new general solution, solvability criteria were established for boundary value problems for nonlinear ordinary differential equations.

The results and methods were extended to linear nonlocal boundary value problems for systems of loaded hyperbolic equations and Fredholm hyperbolic integro-differential equations [99, 100].

The new approach to the general solution became the basis of methods for research and solving nonlinear boundary value problems for loaded differential and integro-differential equations [101–111]. The methods are based on the construction and solving systems of nonlinear algebraic equations for arbitrary vectors of new general solutions. To solve nonlocal boundary value problems for nonlinear partial differential and integro-differential equations, a modification of Euler's broken lines method was developed.

These results were further extended to multi-point problems, periodic problems with impulse effects, and control problems for various classes of differential, loaded differential, integro-differential, and partial differential equations [112–114].

Conclusion

Dzhumabaev D.S. was a highly qualified expert in the theory of differential, integral and nonlinear operator equations, computer and mathematical modeling of applied problems. He has published over 300 papers in scientific journals, including authoritative periodicals like *Journal of Mathematical Analysis and Applications, Journal of Computational and Applied Mathematics, Mathematical Methods in Applied Sciences, Mathematical Notes, Computational Mathematics and Mathematical Physics, Differential Equations, Ukrainian Mathematical Journal, Journal of Integral Equations and Applications, Journal of Mathematical Sciences, Eurasian Mathematical Journal*, etc. The list of his major publications is given below.

The research findings were presented and discussed at many international symposia and conferences. His scientific results were widely recognized in Kazakhstan and at the international level by experts in the field of differential equations and computational mathematics. The scientific direction formed by Dzhumabaev D.S. has been further developed by his students, who successfully work at the Institute of Mathematics and Mathematical Modeling and leading universities in Kazakhstan.

In 1998, Dzhumabaev D.S. was awarded the title of professor (specialty 01.01.00 – Mathematics). Under his supervision, two doctoral, twenty candidate dissertations, and one PhD thesis were defended. He supervised five PhD students. In 2004-2005, Dzhumabaev D.S. was the chair of the Expert Commission on Mathematics and Computer Science of the Committee on Supervision and Certification in Education and Science of the Ministry Education and science of the Republic of Kazakhstan.

Professor Dzhumabaev made a great contribution to academic community. He led a scientific seminar on the qualitative theory of differential equations at the Institute of Mathematics and Mathematical Modeling. He was a scientific expert of the State Expertise of the Ministry of Education and Science of the Republic of Kazakhstan. For many years, Dzhumabaev D.S. was a member of Dissertation Councils at the Institute of Mathematics, Al-Farabi Kazakh National University, Abai Kazakh National Pedagogical University, K. Zhubanov Aktobe Regional State University.

In 2014, at the invitation of the university authorities, Professor Dzhumabaev began to deliver lectures at the International University of Information Technology. He taught such courses as “Mathematical Analysis”, “Methods of solving linear and nonlinear boundary value problems for ordinary differential equations”, “Problems for integro-differential equations of processes with consequences”, “Boundary value problems, their applications and methods for solving”. It should be noted that his scientific results of recent years were obtained under the influence of teaching at the

International University of Information Technology. While giving lectures and conducting practical classes, he realized with great clarity the importance of developing numerical methods for solving applied problems. Having set himself the goal of bringing to the final numerical implementation the theoretical results and algorithms of the parameterization method, he made a breakthrough in the field of mathematical and computer modeling. Under scientific supervision of Professor Dzhumabaev, master students and undergraduates of the International University of Information Technology carried out research in the area of numerical methods for solving boundary value problems for differential and integro-differential equations.

Professor Dzhumabaev chaired the Mathematics Section of Academic Council of the Institute of Mathematics and Mathematical Modeling. He was a member of the editorial board of the scientific journals *News of NAS RK. Series: Physics and Mathematics, Kazakh Mathematical Journal, Bulletin of Karaganda State University. Series: Mathematics.*

Dzhumabaev D.S. was awarded the lapel badge "For Contribution to the Development of Science and Technology" and the Certificate of Merit of the Ministry of Education and Science of the Republic of Kazakhstan (2014).

Since 2018, Dzhumabaev D.S. headed the Department of Mathematical Physics and Mathematical Modeling at the Institute of Mathematics and Mathematical Modeling. In 2019, his research team, together with mathematicians from Ukraine, Uzbekistan, Azerbaijan, Germany, and the Czech Republic, received funding from the European Union's Horizon 2020 research and innovation programme under EU grant agreement 873071-H2020-MSCA-PISE-2019 (Marie Sklodowska-Curie Research and Innovation Staff Exchange), project titled "Spectral Optimization: From Mathematics to Physics and Advanced Technology" (SOMPATY).

The first publication in the framework of this project is devoted to the application of the parameterization method to multipoint problems for Fredholm integro-differential equations and was published in *Kazakh Mathematical Journal* (2020, Vol. 20, No. 1).

At the end of 2019, having applied for the competition from the International University of Information Technology, Professor Dzhumabaev became the owner of the grant "The Best University Teacher 2019" of the Ministry of Education and Science of the Republic of Kazakhstan.

A prominent scientist, an outstanding teacher, and a talented organizer, Dulat Syzdykbekovich Dzhumabaev passed away on February 20, 2020. He will be lovingly remembered by his wife Klara Kabdygalymovna, daughters Dana and Damira, son Anuar, and fours grandchildren. His memory will live in the hearts of his friends, colleagues, as well as generations of grateful and adoring students. His research, scientific ideas and plans will be continued and implemented by his students.

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