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CONTENT

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Approximate solutions of the Riemann problem for a two-phase flow of immiscible liquids based on the Buckley–Leverett model

The article proposes an approximate method based on the "vanishing viscosity"method, which ensures the smoothness of the solution without taking into account the capillary pressure. We will consider the vanishing viscosity solution to the Riemann problem and to the boundary Riemann problem. It is not a weak solution, unless the system is conservative. One can prove that it is a viscosity solution actually meaning the extension of the semigroup of the vanishing viscosity solution to piecewise constant initial and boundary data. It is known that without taking into account the capillary pressure, the Buckley–Leverett model is the main one. Typically, from a computational point of view, approximate models are required for time slicing when creating computational algorithms. Analysis of the flow of a mixture of two immiscible liquids, the viscosity of which depends on pressure, leads to a further extension of the classical Buckley–Leverett model. Some two-phase flow models based on the expansion of Darcy's law include the effect of capillary pressure. This is motivated by the fact that some fluids, e.g., crude oil, have a pressure-dependent viscosity and are noticeably sensitive to pressure fluctuations. Results confirm the insignificant influence of cross-coupling terms compared to the classical Darcy approach.

Keywords: Darcy's law, two-phase flows, phases coupling, Buckley–Leverett theory, isothermal filtration, capillary pressure.

Introduction

S. Bianchini and A. Bressan [1] show that the solutions of the viscous approximations $u_t+A(u)u_x = \varepsilon u_{xx}$ are defined globally in time and satisfy uniform BV estimates, independent of ε. Letting $\varepsilon \to 0$, these viscous solutions converge to a unique limit. In the conservative case where $A = Df$ is the Jacobian of some flux function $f : R^n \to R^n$, the vanishing viscosity limits are the unique entropy-weak solutions to the system of conservation laws $u_t + f(u)u_x = 0$.

Buckley and Leverett proposed and calculated a model of fluid behavior in a porous medium in 1942. In the process of further development, many different calculation methods were proposed, in particular [2–5] and many others.

To read scientific studies on the relationships between phases in multiphase flow modeling, we refer to [6] for an analysis and links to these papers.

We will provide a clear description of the displacement of an incompressible fluid during the formation of a porous medium given by Dominique Guerillot et al. [6]. This is the mass conservation equation for two phases (oil and water):

$$\frac{\partial \rho_o S_o \phi}{\partial t} + \nabla \cdot (\rho_o \nu_o) = 0,$$
$$\frac{\partial \rho_\varpi S_\varpi \phi}{\partial t} + \nabla \cdot (\rho_\varpi \nu_\varpi) = 0$$

with the natural physical constraint $S_o + S_w = 1$, where ϕ — the effective porosity of the reservoir;

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ρ_o, S_o and ρ_ϖ, S_ϖ — the density and saturations of oil and water, respectively; ν_o, p_o and ν_ϖ, p_ϖ — the superficial velocity and the pressure of the oil and the water phases, respectively.

Fluid flow through a porous medium is common in many areas of technology and science. At the same time, the problem of single-phase flow has been well studied both from an engineering and mathematical point of view [7]. The classical Darcy's law, widely used for practical purposes, can be obtained by modeling a sluggishly current incompressible flow. In practice, a porous medium is considered a periodic array of cells filled with a Newtonian fluid. The problem is formulated at the cell scale (microscale), and then scaled by homogenization in the entire area, providing the classical Darcy's law.

According to Darcy's equation, a porous solid has a resistance to the liquid in the pores, which is directly proportional to the speed of the liquid relative to the solid, usually called the drag coefficient.

Oil production in most cases occurs when it is displaced in the pore space of the productive reservoir by water or gas. This process is used in natural operating modes and in artificial methods of maintaining reservoir pressure by flooding or gas injection. The theory of isothermal filtration serves as the basis for calculating such processes [8–10].

Simulation can be without taking into account nonlinear effects [11], assuming that the flow of immiscible two fluids is separated by a smooth boundary layer [12]. In such a formulation of the problem, a solid matrix is considered an impenetrable rigid body, and the classical no-slip condition is imposed on its boundary. The result is a system of equations for saturation and pressure. Such a system is reduced to the classical Buckley–Leverett equation, when the viscosities of both fluids are independent of pressure. It is found that the relative permeabilities depend on the pressure of the liquid and when the solid matrix is considered rigid.

Some models consider the exchange of momentum between the phases of flows of two immiscible fluids in a porous medium. Sometimes creeping flow models are used that include an explicit relationship between two phases by adding cross-terms to the generalized Darcy's law [13]. These models show that cross-terms in macroscale models can significantly affect flow compared to results obtained using generalized Darcy's laws without cross-terms. Investigations with the availability of experimental data for analytical solutions suggest that the influence of this dependence on the dynamics of saturation fronts and stationary profiles is very sensitive to gravitational effects, the ratio of viscosity between two phases and permeability. These results indicate that the effects of momentum exchange on two-phase flow can increase with increasing porous medium permeability when the effect of liquid-liquid interfaces becomes similar to the effect of solid-liquid interfaces.

In the parabolic case, solvability has been sufficiently studied by S.N. Antontsev, V.N. Monakhov, O.B. Bocharov [14–16], and others. It should be noted that equations in the form (1) are the simplest mathematical models of many natural phenomena, sometimes reflecting the essence of these phenomena. In particular, the Leverett function is determined experimentally according to the materials of Kern. This approach does not give the desired results in problems of filtration theory.

The Cauchy problem for a system of conservation laws in one space dimension takes the form [7]:

$$u_t + f(u)_x = 0. \tag{1}$$

Here $u\,(0,x) = \tilde{u}\,(x)$ is initial conditions, $u = (u_o, u_\varpi)$ is the vector of conserved quantities (oil and water, respectively), while the components of $f = (f_o, f_\varpi)$ are the fluxes of oil and water, respectively. We assume that the flux function $f : \mathbf{R}^2 \to \mathbf{R}^2$ is smooth and that the system is strictly hyperbolic; i.e., at each point u the Jacobian matrix $A = Df\,(u)$ has u real, distinct eigenvalues

$$\lambda_1\,(u) < ... < \lambda_n\,(u)\,.$$

One can then select bases of right t and left eigenvectors $r_i\,(u)\,, l_i\,(u)$ normalized so that

$$|r_i| \equiv 1, l_i \cdot r_i = \begin{cases} 1 & \text{if} \;\; i = j, \\ 0 & \text{if} \;\; i \neq j. \end{cases}$$

Several fundamental laws of physics take the form of a conservation equation.

The lack of regularity is a major source of complexity since most of the standard differential calculus tools are not applicable. Special methods are needed, in particular, the main building block is the so-called Riemann problem [1, 17], in which the initial data are piecewise constant with one jump at the origin:

$$u\left(0,x\right)=\begin{cases} u^{-} & \text{if} \quad x<0, \\ u^{+} & \text{if} \quad x>0. \end{cases}$$

The viscosity solution of a Cauchy problem is unique and coincides with the limit of Glimm, and the front-tracking approximations for a strictly hyperbolic system of conservation laws satisfy the standard assumptions.

For each $i \in \{1,...,n\}$, the i-th characteristic field is either linearly degenerate, so that

$$D\lambda_i\left(u\right)\cdot r_i\left(u\right)=0$$

for all u, or else it is genuinely nonlinear, i.e.,

$$D\lambda_i\left(u\right)\cdot r_i\left(u\right)>0,$$

0 for all u.

The definition given in [7] was motivated by a natural conjecture. Namely, the viscosity solutions (characterized in terms of local integral estimates) should coincide precisely with the limits of vanishing viscosity approximations. In the present paper, we adopt a similar definition of viscosity solutions and prove that the above conjecture is indeed true. Our results apply to the more general case of (possibly nonconservative) quasilinear strictly hyperbolic systems. In particular, we obtain the uniqueness of the vanishing viscosity limit.

For a comprehensive account of the recent uniqueness and stability theory, we refer to [7, 8].

A long-standing conjecture is that the entropic solutions of the hyperbolic system (1) actually coincide with the limits of solutions to the parabolic system

$$u_t+f(u)_x=\varepsilon u_{xx}$$

when the viscosity coefficient $\varepsilon \to 0$. In view of the recent uniqueness results, it looks indeed plausible that the vanishing viscosity limit should single out the unique "good" solution of the Cauchy problem, satisfying the appropriate entropy conditions. In earlier literature, results in this direction were based on three main techniques [7]: Comparison principles for parabolic equations; Singular perturbations; Compensated compactness.

In our point of view, to develop a satisfactory theory of vanishing viscosity limits, the heart of the matter is to establish a priori BV bounds on solutions $u\left(t,\cdot\right)$ of (1.8) ε, uniformly valid for all $t\in[0,\infty)$ and $\varepsilon>0$. This is indeed what we will accomplish in the present paper. Our results apply, more generally, to strictly hyperbolic systems with viscosity, not necessarily in conservation form:

$$u_t+A\left(u\right)u_x=\varepsilon u_{xx}.$$

The modeling multiphase flows in porous media is of major importance in many fields of applications. In particularly in enhanced oil recovery applications of petroleum engineering. The classical mathematical models for multiphase flows are based on a straightforward generalization of Darcy's law for a single-phase flow [9]. A natural question arises: How important is the influence of one phase on the other phase? In some applications, it is shown that the coupling effects are small, and therefore negligible.

In [18–20], it was developed a mathematical model to apply Buckley–Leverett frontal advance theory to immiscible displacement in non-communicating stratified reservoirs. The influence of the coefficient of viscosity and coefficient of variation of the Dykstra-Parsons permeability (VDP) on productivity has been investigated. The introduction of pseudo-relative permeability functions is discussed. It was shown in [21] that mass conservation in several layers unconnected with each other can be used to derive the interlayer ratios of various emerging fronts. The ratio of flow areas in an immiscible two-phase flow in a porous medium was studied in [22, 23]. They used dynamic pore network modeling that uses an interface model used to simulate steady state two-phase flow.

Numerical modeling of the flow of immiscible fluids is of great importance in many areas for the proper management of underground resources, in particular water. Recently presented high-resolution numerical model that simulates a three-phase immiscible fluid flow in both unsaturated and saturated zones in a porous aquifer [22] is relevant.

In the theory of immiscible two-phase flow presented in [23], the conservation of mass is provided by general equations, which require some additions for a porous medium. The basic equation can be derived from the relative permeability data. It turns out that it has a surprisingly simple form when expressed in the correct variables [24]. The resulting system of equations can then be solved for a structured porous medium. However, the question remains what happens when the porous medium has a nontrivial structure along its entire length.

1 Mathematical modelling

Consider filtration of a two-phase liquid in a porous medium in water-pressure mode. The field is covered by a network of wells and their location schemes can be different. The oil-bearing formation is considered unlimited, of constant thickness, the porous medium is non-deformable, and the ratio of capillary pressure to the total hydrodynamic pressure drop is small, which allows to consider the problem obeying the classical Buckley–Leverett model.

High precision modeling of immiscible two-phase flows in porous media is paramount. Nevertheless even with such high-precision numerical modeling, the lack of information or its fuzziness, for example, on the relative permeability and functions of capillary pressure in them does not allow a detailed comparison with experiments [25].

Without taking into account gravity, two-phase filtration for the case of straight-parallel displacement was considered by S. Buckley and M. Leverett in 1942, and later independently by A.M. Pirverdyan, who also studied the case of a more general filtration law for two-phase flow [10].

In the case of one-dimensional flow of incompressible immiscible liquids under conditions where capillary pressure and the influence of gravity can be ignored, the displacement process allows a simple mathematical description.

It should be noted that the equation of the form (1):

$$u_t + f(u)_x = 0$$

one-dimensional space variables have been considered by many researchers. A significant contribution to the non-local theory of the Cauchy problem for this equation was made by O.A. Oleinik, A.N. Tikhonov, A.A. Samarsky, and I.M. Gelfand. The Buckley–Leverett mathematical model belongs to equation (1).

A detailed specification allows to define a porous medium as either hydrophilic or hydrophobic.

It is known that if $\varepsilon > 0$ is the coefficient of viscosity, then the viscous friction force acting on each particle of the porous medium $x\,(t)$ and related to the unit of mass can be assumed to be equal to $\varepsilon \cdot u_{xx}$. Then returning to the mathematical model of Buckley–Leverett (then instead of $u\,(t,x)$ we will write $s\,(t,x)$ - water saturation)

$$s_t + s \cdot s_x = \varepsilon \cdot s_{xx} \tag{2}$$

where $F'\,(s) = \frac{1}{2}s$ is the Leverett function.

The equation of the form (2) was studied by O.B. Bocharov, B.N. Monakhov, I.L. Telegin [14, 15, 17].

If the viscous term tends to zero, the uniqueness of the vanishing viscosity limit is proved based on comparative estimates for the solutions of the corresponding Hamilton-Jacobi equation, as stated in [26, 27]. As an application, they obtained the existence and uniqueness of solutions for the class of triangular systems of 2×2 conservation laws with hyperbolic degeneracy. However, the forecast calculations did not give the desired results.

The assumed method at $\varepsilon \to 0$ is called the "vanishing viscosity"method. Given that

$$s_t = \left(\varepsilon \cdot s_x - \frac{s^2}{2}\right)_x$$

we introduce the potential $u\,(x,t)$ defined by the equality

$$du + \left(\varepsilon \cdot s_x - \frac{s^2}{2}\right)_x dt.$$

In this case

$$u_x = s,$$
$$u_t = \varepsilon \cdot s_x - \tfrac{s^2}{2} = \varepsilon \cdot u_{xx} - \tfrac{u^2 x}{2},$$

that is, the function $u\,(x,t)$ satisfies the equation

$$u_t + \frac{1}{2}u_x^2 = \varepsilon \cdot u_{xx}. \tag{3}$$

Make a replacement in (3)

$$u = -2\varepsilon \cdot \ln z.$$

Then

$$u_t = -2\varepsilon \cdot \tfrac{z_t}{z},$$
$$u_x = -2\varepsilon \cdot \tfrac{z_x}{z},$$
$$u_{xx} = -2\varepsilon \cdot \tfrac{z_{xx}}{z} + 2\varepsilon \cdot \tfrac{z_x^2}{z^2},$$

Equation (3) will take the form

$$-2\varepsilon \cdot \frac{z_t}{z} + 2\varepsilon^2 \cdot \frac{z_x^2}{z^2} = -2\varepsilon^2 \cdot \frac{z_{xx}}{z} + 2\varepsilon^2 \cdot \frac{z_x^2}{z^2},$$

in other words, the thermal conductivity equation is obtained regarding to $z\,(x,t)$:

$$z_t = \varepsilon \cdot z_{xx}. \tag{4}$$

This method is often called the Florin–Hopf–Cole transformation. From the made, substitutions it follows that the solution to equation (2) has the form:

$$s = u_x = -2\varepsilon \cdot \frac{z_x}{z}$$

where $z\,(x,t)$ is the solution (4).

Suppose that a wave of the form propagates through an injection well:

$$s\,(x,t) = s_- + \frac{s_+ - s_-}{2} \cdot (1 + sign(x - \omega t)) = \begin{cases} s_-, ats < \omega t \\ s_+, ats > \omega t \end{cases} \tag{5}$$

where $\omega = const$. Suppose that there is a generalized solution of the equation of the form (1) in the sense of fulfilling the integral identity. To do this, it is necessary and sufficient that the condition is met on the break line $\omega = const$

$$\omega = \frac{dx}{dt} = \frac{F(s_+) - F(s_-)}{s_+ - s_-}. \tag{6}$$

The idea of the "vanishing viscosity"method in this case is that this solution (discontinuous) of the form (5), (6) is acceptable. That is, for $x \neq \omega$ solutions of of $s^\varepsilon(x,t)$ the equation

$$s_t^\varepsilon = +(F(s^\varepsilon))_x = \varepsilon \cdot s_{xx}^\varepsilon \tag{7}$$

for $\varepsilon \to 0$, it is obtained as a pointwise limit.

Below, the proposed method by I.M. Gelfand has the desired result in applied problems.

Given the structure of the solution we will look for a solution $s(x,t)$ to equation (7) and (8) in the form:

$$s^\varepsilon(x,t) = u(\xi), \xi = \frac{x - \omega t}{\varepsilon}. \tag{8}$$

Substituting a solution of this type in (7), we get that the function $U(\xi)$ is the solution of the equation

$$-\omega \cdot v' + (F(v))' = v''. \tag{9}$$

At $x \neq \omega t$, the function $s^\varepsilon = v\left(\frac{x-\omega t}{\varepsilon}\right)$ pointwise approximates for $\varepsilon \to 0$ function $s(x,t)$ of the form (5) if and only if the function $v(\xi)$ satisfies the boundary conditions:

$$s(-n,t) = s_-, \quad s(n,t) = s_+ \tag{10}$$

where n is a sufficiently large distance from the well.

It should be noted that $v(t)$ is not the only solution, i.e., there can be $\tilde{v} = v(\xi - \xi_0)$, for any $\xi_0 \in R$.

Integrating (9) and (10), we get

$$\begin{array}{l} v' = -\omega \cdot v + \Phi(v) + C = \tilde{\Phi}(v) + C, \\ C = const. \end{array} \tag{11}$$

If these conditions are met, the solutions of equation (9) that interest us are given by the formula.

Following the method of I.M. Gelfand, in order for an autonomous equation (11) with a smooth right part of $\tilde{\Phi}(v) + C$ to have a solution that tends to the constants s_- at $n \to -\infty$ and s_+ at $n \to +\infty$, it is necessary and sufficient to meet the following conditions:

a) s_- and s_+ -special points of the original equation, i.e., zero the right side of the equation (11):

$$\Phi(v) + C = \tilde{\Phi}(v) + C = 0,$$

that is, as a result, we have

$$\tilde{\Phi}(s_-) = \tilde{\Phi}(s_+) = -C;$$

b) another option between s_- and s_+ there are no other special points and the right part (11) on the specified interval:

1) positive at $s_- < s_+$ the solution increases, i.e.,

$$\tilde{\Phi}(\upsilon) - \tilde{\Phi}(s_-) > 0, \forall \upsilon \in (s_-, s_+) \tag{12}$$

2) negative at $s_- > s_+$, i.e., the solution decreases:

$$\tilde{\Phi}(\upsilon) - \tilde{\Phi}(s_+) < 0, \forall \upsilon \in (s_+, s_-). \tag{13}$$

If these conditions are met, the solutions of equation (9) that interest us are given by the formula

$$\int_{\upsilon_0}^{\upsilon} \frac{d\upsilon}{\tilde{\Phi}(\upsilon) - \tilde{\Phi}(s_-)} = \xi - \xi_0$$

where $\upsilon_0 = \frac{s_+ + s_-}{2}$ — location of wells.

The given conditions (12), (13) are an analytical record of the tolerance condition.

By varying s_-, s_+, and $F(s)$, various converging sequences of valid generalized solutions can be constructed. At the same time, any point-to-point limits of acceptable solutions are also considered acceptable.

2 Numerical Results

As a result, we get that the solution $s(x,t)$ can jump from s_- to s_+ (in the direction of increasing x). That is, in fact, this jump occurs during the transition from the water phase to the oil phase. In this case, the conditions for an acceptable gap are met (Fig. 1):

1) for $s_- < s_+$, the graph of the function $F(s)$ on the segment $[s_-, s_+]$ must be located below the chord with the ends $(s_-, F(s_-))$ and $(s_+, F(s_+))$;

2) in the case of $s_- > s_+$, the graph of the function $F(s)$ on the segment $[s_+, s_-]$ must be located no higher than the chord with the ends $(s_-, F(s_-))$ and $(s_+, F(s_+))$.

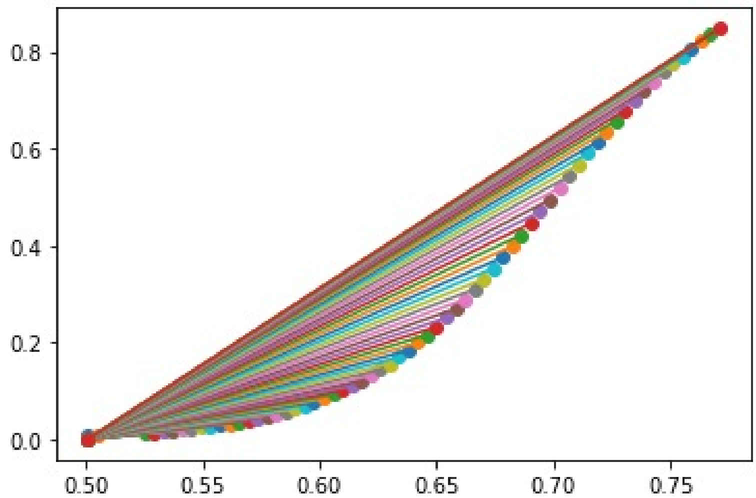

Figure 1. Construction of chord (s, F(s)) front saturation

The obtained conditions make it possible to regulate filtration processes in the bottomhole formation zone taking into account the initial information, in particular, some data from Table 1.

Table 1

Initial data used in modeling with the one-dimensional Buckley-Leverett problem

Parameter	Value
Porosity	0.28
Oil viscosity	1.e-4 kg/ms
Water viscosity	0.5e-4 kg/ms
Oil density	881 kg/m^3
Water density	1000 kg/m^3
Water relative perm calculation for a given water saturation	11.174
Oil relative perm calculation for a given water saturation	3.326
Cross-sectional area	0.4 m^2

The gap tolerance conditions obtained by the "vanishing viscosity"method are in perfect agreement with the forecast calculations. Indeed, the convexity property of the function $F(s)$ in the Buckley-Laverett mathematical model (up) down by definition means that any chord connecting points in a straight line shows the validity of the Buckley-Laverett mathematical model itself.

Figure 2 presents water saturation profile.

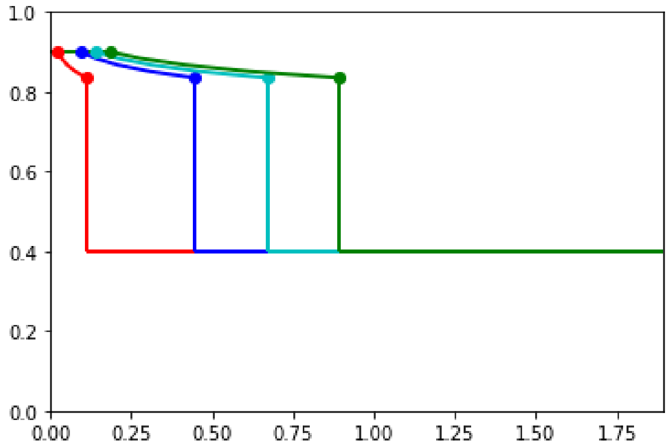

Figure 2. Water saturation profile as a function of time “t” and distance “x”

Figure 3 illustrates derivative of fractional flow curve.

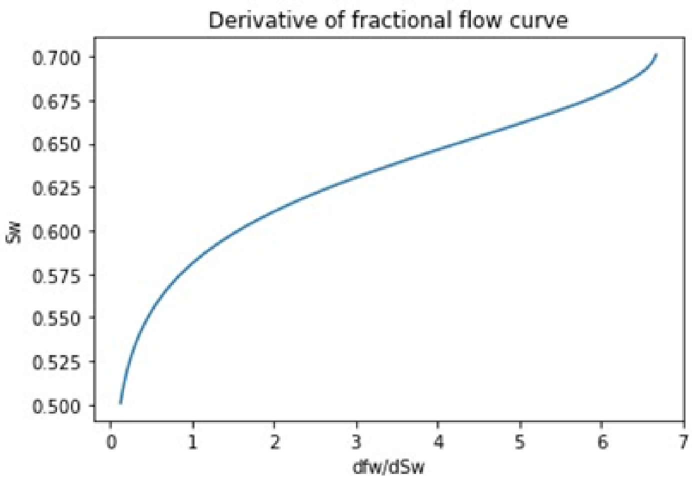

Figure 3. Derivative of fractional flow dF/ds

Table 2 shows the results of derivative of the fractional flow rate curve calculation.

Table 2

Results of numerical calculation of the derivative of the fractional flow rate curve

S_w	$\frac{dF_s}{ds_w}$
0.500	0.152
0.525	0.313
0.550	0.487
0.575	0.889
0.600	1.519
0.625	2.721
0.650	4.219
0.675	5.817
0.700	6.613

Presentation of the model of nonlinear wave propagation and how the use of the method allows one to cope with sharp fronts (or discontinuities) and develop them correctly, as well as to follow the formation of a jump and rarefaction (Fig. 4a, 4b). The formation of an abrupt jump (jump) is observed.

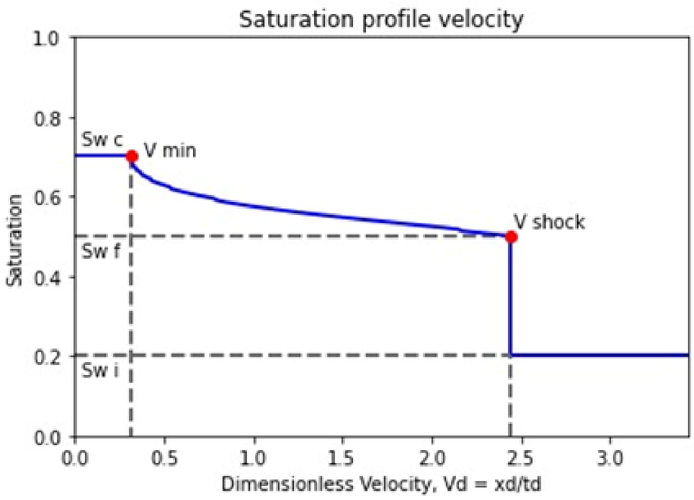

Figure 4. a) Shock and rarefaction formation

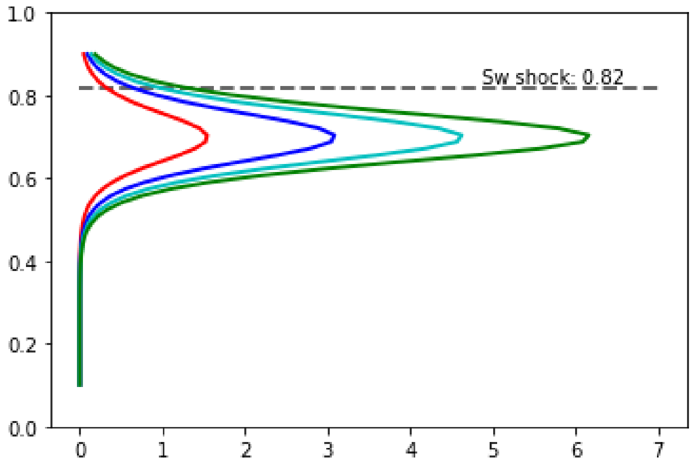

Figure 4. b) Shock and rarefaction formation

Conclusions

The article proposes one of the methods for solving the problem of filtration of a two-phase incompressible fluid. The problem of mathematical filtering is posed on the basis of the classical Buckley-Leverett model and an approximate solution is constructed. For the effective use of the described method, relevant data are needed, such as the coefficient of fluid viscosity, the density of formation fluids, etc., to plot the curves of the viscosity ratio. The considered method, based on the Buckley-Leverett theory, uses vanishing viscosity for frontal advance, but, in general, it can be applied to various systems that use different technological approaches and open the way for further research. In particular, stochastic analysis of two-phase flow in stratified porous media seems promising [28, 29]. Stochastic models, which include some assumptions about porous media, simplify and stabilize fuzzy information. In the future, we plan to use stochastic data and analyze them.

References

1 Bianchini S. Vanishing viscosity solutions of nonlinear hyperbolic systems / S. Bianchini, A. Bressan // Annals of Mathematics. — 2005. — 161. — P. 223–342.

2 Welge H.J. A simplified method for computing oil recovery by gas or water drive / H.J. Welge // J. Pet. Technol. — 1952. — 4. — P. 91–98.

3 Sheldon J.W. One-dimensional incompressible noncapillary two-phase fluid flow in a porous medium / J.W. Sheldon, W.T. Cardwell // Pet. Trans. AIME. — 1959. — 216. — P. 290–296.

4 McWhorter D.B. Exact integral solutions for two-phase flow / D.B. McWhorter, D.K. Sunada // Water Resour. Res. — 1990. — 26. — P. 399–413.

5 Muskat M. The Flow of Homogeneous Fluids through Porous Media / M. Muskat. — New York: The Maple Press Company, 1946.

6 Guerillot D. Buckley–Leverett Theory for Two-Phase Immiscible Fluids Flow Model with Explicit Phase-Coupling Terms / D. Guerillot, M. Kadiri, S. Trabelsi // Water, 2020. — 12(11). — P. 3041.

7 Buckley S.E. Mechanism of fluid displacement in sands / S.E. Buckley, M.C. Leverett // Trans. AIME. — 1942. — 146. — P. 107–116.

8 Басниев К.С. Подземная гидравлика: учеб. для вузов / К.С. Басниев, А.М. Власов, И.Н. Кочина, В.М. Максимов. — М.: Недра, 1986. — 303 с.

9 Коновалов А.Н. Задачи фильтрации многофазной несжимаемой жидкости / А.Н. Коновалов. — Новосибирск: Наука, 1988. — 166 с. — ISBN 5-02-028569 -2.

10 Пирвердян А.М. Физика и гидравлика нефтяного пласта / А.М. Пирвердян. — М.: Недра, 1982. — 192 с.

11 Alimhan K. Further Results on Output Tracking for a Class of Uncertain High-Order Nonlinear Time-Delay Systems / K. Alimhan, M.N. Kalimoldayev, A.A. Adamov, O.Zh. Mamyrbayev, N. Tasbolatuly, A. Smolarz // Przeglad Elektrotechniczny. — 2019. — 95(5). — P. 88–91.

12 Fusi L. Buckley–Leverett Equation with viscosities and Relative Permeabilities Dependending on Pressure / L. Fusi, A. Farina, G. Saccomandi // SIAM Journal on Applied Mathematics. — 2015. — 75(5). — P. 1983–2000.

13 Pasquier S. Modeling two-phase flow of immiscible fluids in porous media: Buckley-Leverett theory with explicit coupling terms / S. Pasquier, M. Quintard, Y. Davit // Physical Review Fluids. — 2017. — 2. — P. 104101/1-104101/19.

14 Антонцев С.Н. Краевые задачи механики неоднородных жидкостей / С.Н. Антонцев, А.В. Кажихов, В.Н. Монахов. — Новосибирск: Наука, 1983. — 320 с.

15 Bocharov O.B. Numerical investigation of discontinuous solutions of conjugation of two non-isothermal filtration models / O.B. Bocharov, I.G. Telegin // Computational Technologies. — 2008. — 13(1). — P. 20–28.

16 Bocharov O.B. Boundary value problems of non-isothermal two-phase filtration in porous media / O.B. Bocharov, V.N. Monakhov // Dynamics of a continuous medium: Collection of scientific works, AS USSR Sibir Department. Institute of hydrodynamics. — 1988. — 86. — P. 47–69.

17 Daripa P. Polymer Floods: A Case Study of Nonlinear Wave Analysis And of Instability Control In Tertiary Oil Recovery / P. Daripa, J. Glimm, B. Lindquis, O. McBrayan // Society for Industrial and Applied Mathematics. — 1988. — 48(2). — P. 353–374.

18 El-Khatib N.A.F. The Application of Buckley-Leverett Displacement to Waterflooding in Non-Communicating Stratified Reservoirs / N.A.F. El-Khatib // In Proceedings of the SPE Middle East Oil Show. — 2001. — P. 91–102.

19 Prince A.O. Buckley-Leverett Displacement Theory for Waterflooding Performance in Stratified Reservoir / A.O. Prince // Petroleum and Coal. — 2014. — 56(3). — P. 267–281.

20 Zhao L. Analytical Model of Waterflood Sweep Efficiency in Vertical Heterogeneous Reservoirs under Constant Pressure / L. Zhao, L. Li, Zh. Wu, Ch. Zhang // Mathematical Problems in Engineering. — 2016. — Article ID 6273492. — P. 9.

21 Alisawi M. In Reservoir Engineering Method Buckley-Leverett method in water Flooding approach prepared by Mahdi Mohsen / M. Alisawi. — Research Proposal, 2020.

22 Feo A. High resolution shock capturing numerical simulations of three phase immiscible fluids from the unsaturated to the saturated zone / A. Feo, F. Celico // Scientific Reports. — 2011. — P. 1–11.

23 Roy S. Flow-Area Relations in Immiscible Two-Phase Flow in Porous Media / S. Roy, S. Sinha // Frontiers in Physics. — 2020. — 8.

24 Roy S. The Co-Moving Velocity in Immiscible Two-Phase Flow in Porous Media / S. Roy, H. Pedersen, S. Sinha, A. Hansen // arXiv:2108.10187v1 [physics.flu-dyn]. — 2021. — P. 1–27.

25 Riaz A. Numerical simulation of immiscible two-phase flow in porous media / A. Riaz, H.A. Tchelepi // Phys. Fluids. — 2006. — 18. — P. 1–27.

26 Spinola L.V. Vanishing Viscosity Solutions of a 2x2 Triangular Hyperbolic System with Dirichlet Conditions on Two Boundaries / L.V. Spinola // Indiana University Mathematics Journals. — 2007. — 56(1). — P. 279–364.

27 Bressan A. The unique limit of the Glimm scheme / A. Bressan // Arch. Rational Mech. Anal. — 1995. — 130. — P. 205–230.

28 Bressan A. Vanishing Viscosity Solutions for Conservation Laws with Regulated Flux / A. Bressan, G. Guerra, W. Shen // Journal of Differential Equations. — 2019. — 266(1). — P. 312–351

29 Dias M.M. Network models for two-phase flow in porous media Part 1. Immiscible microdisplacement of non-wetting fluids / M.M. Dias, C.A. Payatakes // Journal of Fluid Mechanics. — 1986. — 164. — P. 305–336.

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Бакли–Леверетт моделінің негізінде екіфазалы араласпайтын сұйықтар ағыны үшін Риман есебінің жуық шешімдері

Мақалада капиллярлық қысымды есепке алмастан, шешімнің тұтастығын қамтамасыз ететін «жойылатын тұтқырлық» әдісіне негізделген жуықтау әдісі ұсынылған. Риман есебі мен шекаралық есебінің тұтқырлығы жойылған шешімі қарастырылған. Жүйе консервативті болмаса, бұл әлсіз шешім емес екенін ескеру керек, бірақ оның тұтқырлық шешім екенін дәлелдеуге болады, бұл шын мәнінде жойылып кететін тұтқырлық шешімнің жарты группаларды бөлшек-тұрақты бастапқы және шекаралық шарттарға дейін кеңейтуді білдіреді. Капиллярлық қысымды есепке алмағанда, Бакли-Леверетт моделі негізгі болып табылатыны белгілі. Нақты болжамдық есептеулерге сүйене отырып, модель көптеген салаларда өзін дәлелдеді. Әдетте, есептеу тұрғысынан алғанда, есептеу алгоритмдерін құру кезінде уақытты кванттау үшін жуықтау әдістері қажет. Тұтқырлығы қысымға тәуелді екі араласпайтын сұйықтықтардың ағынын талдау Бакли-Леверетт классикалық үлгісін одан әрі кеңейтуге әкеледі. Дарси заңының кеңеюіне негізделген кейбір екіфазалы ағын модельдері капиллярлық қысымның әсерін қосады. Бұл кейбір сұйықтықтардың, мысалы, шикі мұнайдың, қысымға тәуелді тұтқырлығына және қысымның ауытқуына айтарлықтай сезімталдығына негізделген. Нәтижелер классикалық Дарси әдісімен салыстырғанда кросс-байланыс жағдайларының елеусіз әсерін растайды.

Кілт сөздер: Дарси заңы, екіфазалы ағындар, фазалық байланыс, Бакли–Леверетт теориясы, изотермиялық сүзу, капиллярлық қысым.

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Приближенные решения задачи Римана для двухфазного потока несмешивающихся жидкостей на основе модели Бакли–Леверетта

В статье предложен приближенный метод, основанный на «исчезающей вязкости», которая обеспечивает гладкость раствора без учета капиллярного давления. Мы будем рассматривать решение задачи Римана и краевой задачи Римана с исчезающей вязкостью. Обратите внимание на то, что это не слабое решение, если система не является консервативной, то можно доказать, что это вязкостное решение, фактически означающее расширение полугруппы решения исчезающей вязкости до кусочно-постоянных начальных и граничных данных. Известно, что, без учета капиллярного давления, модель Бакли–Леверетта является основной. Основанная на реальных прогнозных расчетах модель положительно зарекомендовала себя во многих сферах. Обычно, с вычислительной точки зрения, приближенные модели требуются для квантования времени при создании вычислительных алгоритмов. Анализ потока из двух несмешивающихся жидкостей, вязкость которых зависит от давления, приводит к дальнейшему расширению классической модели Бакли–Леверетта. Некоторые модели двухфазного потока, основанные на расширении закона Дарси, включают эффект капиллярного давления. Это мотивировано тем фактом, что некоторые жидкости, например, сырая нефть, имеют вязкость, зависящую от давления, и заметно чувствительны к колебаниям давления. Результаты подтвердили незначительное влияние условий кросс-связывания по сравнению с классическим подходом Дарси.

Ключевые слова: закон Дарси, двухфазные потоки, связь фаз, теория Бакли–Леверетта, изотермическая фильтрация, капиллярное давление.

References

1 Bianchini, S., & Bressan, A. (2005). Vanishing viscosity solutions of nonlinear hyperbolic systems. *Annals of Mathematics*, 161, 223–342.

2 Welge, H.J. (1952). A simplified method for computing oil recovery by gas or water drive. *J. Pet. Technol.*, 4, 91–98.

3 Sheldon, J.W., & Cardwell, W.T. (1959). One-dimensional incompressible noncapillary two-phase fluid flow in a porous medium *Pet. Trans. AIME.*, 216, 290–296.

4 McWhorter, D.B., & Sunada, D.K. (1990). Exact integral solutions for two-phase flow. *Water Resour. Res.*, 26, 399–413.

5 Muskat, M. (1946). *The Flow of Homogeneous Fluids through Porous Media.* New York: The Maple Press Company.

6 Guerillot, D., Kadiri, M., & Trabelsi, S. (2020). Buckley–Leverett Theory for Two-Phase Immiscible Fluids Flow Model with Explicit Phase-Coupling Terms. *Water*, 12(11), 3041.

7 Buckley, S.E., & Leverett, M.C. (1942). Mechanism of fluid displacement in sands. *Trans. AIME*, 146, 107–116.

8 Basniev, K.S., Vlasov, A.M., Kochina, I.N., & Maksimov, V.M. (1986). *Podzemnaia gidravlika [Underground Hydraulics].* Moscow: Nedra [in Russian].

9 Konovalov, A.N. (1988). *Zadachi filtratsii mnogofaznoi neszhimaemoi zhidkosti [Problems of filtering a multiphase incompressible fluid].* Novosibirsk: Nauka [in Russian]. ISBN 5-02-028569-2

10 Pirverdian, A.M. (1982). *Fizika i gidravlika neftianogo plasta [Physics and hydraulics of the oil reservoir].* Moscow: Nedra [in Russian].

11 Alimhan, K., Kalimoldayev, M.N., Adamov, A.A., Mamyrbayev, O.Zh., Tasbolatuly, N., & Smolarz, A. (2019). Further Results on Output Tracking for a Class of Uncertain High-Order Nonlinear Time-Delay Systems. *Przeglad Elektrotechniczny*, 95(5), 88–91.

12 Fusi, L., Farina, A., & Saccomandi, G. (2015). Buckley–Leverett Equation with viscosities and Relative Permeabilities Dependending on Pressure. *SIAM Journal on Applied Mathematics,* 75(5), 1983–2000.

13 Pasquier, S., Quintard, M., & Davit, Y. (2017). Modeling two-phase flow of immiscible fluids in porous media: Buckley-Leverett theory with explicit coupling terms. *Physical Review Fluids*, 2, 104101/1-104101/19.

14 Antontsev, S.N., Kazhikhov, A.V., & Monakhov, V.N. (1983). *Kraevye zadachi mekhaniki neodnorodnykh zhidkostei [Boundary value problems in the mechanics of inhomogeneous fluids].* Novosibirsk: Nauka [in Russian].

15 Bocharov, O.B., & Telegin, I.G. (2008). Numerical investigation of discontinuous solutions of conjugation of two non-isothermal filtration models. *Computational Technologies*, 13(1), 20–28.

16 Bocharov, O.B., & Monakhov, V.N. (1988). Boundary value problems of non-isothermal two-phase filtration in porous media. *Dynamics of a continuous medium: Collection of scientific works, AS USSR Sibir Department. Institute of hydrodynamics*, 86, 47–69.

17 Daripa, P., Glimm, J., Lindquis, B., & McBrayan, O. (1988). Polymer Floods: A Case Study of Nonlinear Wave Analysis And of Instability Control In Tertiary Oil Recovery. *Society for Industrial and Applied Mathematics*, 48(2), 353–374.

18 El-Khatib, N.A.F. (2001). The Application of Buckley-Leverett Displacement to Waterflooding in Non-Communicating Stratified Reservoirs. *In Proceedings of the SPE Middle East Oil Show*, 91–102.

19 Prince, A.O. (2014). Buckley-Leverett Displacement Theory for Waterflooding Performance in Stratified Reservoir. *Petroleum and Coal*, 56(3), 267–281.

20 Zhao, L., Li, L., Wu, Zh., & Zhang, Ch. (2016). Analytical Model of Waterflood Sweep Efficiency in Vertical Heterogeneous Reservoirs under Constant Pressure. *Mathematical Problems in Engineering*, Article ID 6273492, 9.

21 Alisawi, M. (2020). In Reservoir Engineering Method Buckley-Leverett method in water Flooding approach prepared by Mahdi Mohsen. *Research Proposal*, DOI: 10.13140/RG.2.2.33588.48007

22 Feo, A., & Celico, F. (2011). High resolution shock capturing numerical simulations of three phase immiscible fluids from the unsaturated to the saturated zone. *Scientific Reports*, 1–11.

23 Roy, S., & Sinha, S. (2020). Flow-Area Relations in Immiscible Two-Phase Flow in Porous Media. *Frontiers in Physics*, 8.

24 Roy, S., Pedersen, H., Sinha, S., & Hansen, A. (2021). The Co-Moving Velocity in Immiscible Two-Phase Flow in Porous Media. *arXiv:2108.10187v1 [physics.flu-dyn]*, 1–27.

25 Riaz, A., & Tchelepi, H.A. (2006). Numerical simulation of immiscible two-phase flow in porous media. *Phys. Fluids.*, 18, 1–27.

26 Spinola, L.V. (2007). Vanishing Viscosity Solutions of a 2x2 Triangular Hyperbolic System with Dirichlet Conditions on Two Boundaries. *Indiana University Mathematics Journals*, 56(1), 279–364.

27 Bressan, A. (1995). The unique limit of the Glimm scheme. *Arch. Rational Mech. Anal.*, 130, 205–230.

28 Bressan, A., Guerra, G., & Shen, W. (2019). Vanishing Viscosity Solutions for Conservation Laws with Regulated Flux. *Journal of Differential Equations*, 266(1), 312–351.

29 Dias, M.M., & Payatakes, C.A. (1986). Network models for two-phase flow in porous media Part 1. Immiscible microdisplacement of non-wetting fluids. *Journal of Fluid Mechanics*, 164, 305–336.

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On the nonlocal problems in time for subdiffusion equations with the Riemann-Liouville derivatives

Initial boundary value problems with a time-nonlocal condition for a subdiffusion equation with the Riemann-Liouville time-fractional derivatives are considered. The elliptical part of the equation is the Laplace operator, defined in an arbitrary $N-$ dimensional domain Ω with a sufficiently smooth boundary $\partial\Omega$. The existence and uniqueness of the solution to the considered problems are proved. Inverse problems are studied for determining the right-hand side of the equation and a function in a time-nonlocal condition. The main research tool is the Fourier method, so the obtained results can be extended to subdiffusion equations with a more general elliptic operator.

Keywords: time-nonlocal problems, Riemann-Liouville derivatives, subdiffusion equation, inverse problems.

Introduction

Let $\beta < 0$, $0 < \rho < 1$ and a function $q(t)$ be defined on $[0, \infty)$. Denote by $J_t^\beta q(t)$ and $\partial_t^\rho q(t)$ the fractional integrals and the Riemann-Liouville derivatives, respectively, defined as (see, e.g. [1; 14]):

$$J_t^\beta q(t) = \frac{1}{\Gamma(-\beta)} \int_0^t \frac{q(\xi)}{(t-\xi)^{\beta+1}} d\xi, \qquad \partial_t^\rho q(t) = \frac{d}{dt} J_t^{\rho-1} q(t), \quad t > 0.$$

Let Ω be an arbitrary N – dimensional domain with a sufficiently smooth boundary $\partial\Omega$.

Consider the following time-nonlocal problem:

$$\partial_t^\rho u(x,t) - \triangle u(x,t) = f(x,t), \quad x \in \Omega, \quad 0 < t \le T; \tag{1}$$

$$u(x,t)\big|_{\partial\Omega} = 0; \tag{2}$$

$$J_t^{\rho-1} u(x,t)\big|_{t=\xi} = \alpha \lim_{t\to 0} J_t^{\rho-1} u(x,t) + \varphi(x), \quad 0 < \xi \le T, \quad x \in \overline{\Omega}, \tag{3}$$

where $f(x,t)$, $\varphi(x)$ are given functions, α is a constant, ξ is a fixed point and $\triangle = \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator. This problem is also called *the forward problem.*

We note the following property of the Riemann-Liouville integrals, which simplifies the verification of the initial condition (3) (see, e.g. [1; 104]):

$$\lim_{t\to+0} J_t^{\rho-1} u(x,t) = \Gamma(\rho) \lim_{t\to+0} t^{1-\rho} u(x,t).$$

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From here, in particular, it follows that the solution to the forward problem can have a singularity at zero $t = 0$ of order $t^{\rho-1}$.

When solving the forward problem, we will first solve various auxiliary problems for equations with the Riemann-Liouville derivative. We will also consider inverse problems. The definition of a classical solution in all these cases is exactly the same. As an example, we present the definition of the classical solution to the forward problem (1)–(3).

Definition 1. A function $u(x,t)$ with the properties

1 $t^{1-\rho}u(x,t) \in C(\overline{\Omega} \times [0,T])$,

2 $\partial_t^\rho u(x,t),\ \triangle u(x,t) \in C(\overline{\Omega} \times (0,T])$

and satisfying conditions (1)–(3) is called *the solution* to the forward problem.

The main goal of this work is to study the influence of parameter α on the correctness of problem (1)–(3). In this regard, we will apply the Fourier method, which ensures the consideration of the following spectral problem

$$\begin{cases} -\triangle v(x) = \lambda v(x), \quad x \in \Omega; \\ v(x)\big|_{\partial\Omega} = 0. \end{cases} \tag{4}$$

Since the boundary $\partial\Omega$ is sufficiently smooth, this problem has a complete in $L_2(\Omega)$ set of orthonormal eigenfunctions $\{v_k(x)\}$, $k \geq 1$, and a countable set of positive eigenvalues $\{\lambda_k\}$, (see, e.g., [2–4]). It is convenient to assume that $0 < \lambda_1 \leq \lambda_2 \cdots \to +\infty$.

We note that the method proposed here, based on the Fourier method, is applicable to equation (1) with an arbitrary elliptic differential operator $A(x, D)$, if only the corresponding spectral problem has a complete system of orthonormal eigenfunctions in $L_2(\Omega)$.

We also note that if $\alpha = 0$, then the considering forward problem passes to *the backward problem*, which is well-studied in work [5]. The backward problem for equation (1) with the Caputo derivative was studied in [6–8]. Therefore, further we assume that $\alpha \neq 0$. About backward problems, we note only the following: These problems are not well-posed in the sense of Hadamard, i.e., a small change in function φ in condition (3) leads to a large change in the solution.

As will be shown below, if $\alpha \notin [0,1)$, then, under standard conditions on the given functions f and φ, problem (1)–(3) is unconditionally solvable and has a unique solution. If $\alpha \in (0,1)$, then the solvability of the problem depends on whether there exists an eigenvalue λ_{k_0} of the spectral problem (4) such that $E_\rho(-\lambda_{k_0}t^\rho) = \alpha$ and what is the multiplicity p_0 of this eigenvalue λ_{k_0} (here E_ρ is the Mittag-Leffler function, see the definition below). If such an eigenvalue exists, then for the solution to the problem to exist, it is necessary that each function f and φ satisfy p_0 additional orthogonality conditions. Moreover, the solution of the problem will not be unique. If there is no eigenvalue λ_{k_0} for which $E_\rho(-\lambda_{k_0}t^\rho) = \alpha$, then problem (1)–(3) is again unconditionally solvable.

We will also study two inverse problems for determining the right-hand side of the equation and function φ in the nonlocal condition (3), respectively. In this case, for both inverse problems, as an additional condition, we take the condition

$$u(x,\theta) = \Psi(x), \quad 0 < \theta \leq T, \quad \theta \neq \xi, \quad x \in \overline{\Omega}. \tag{5}$$

Here, to avoid the uniqueness problem, we will assume that $\alpha \geq 1$. In the case of the inverse problem of determining the right-hand side of the equation, we will assume that f depends only on the spatial variables x: $f = f(x)$.

Note that all these problems for equation (1) with the fractional Caputo derivative were considered in [9]. However, in this work, the existence of a generalized solution to the problems is proved. The convenience of studying the generalized solution by the Fourier method lies in the fact that when proving the convergence of the corresponding series, one can use the Parseval equality and reduce the question of the convergence of functional series to the study of the convergence of numerical series. When proving uniform convergence, this approach does not work. Therefore, in the present paper, we apply the lemma of Krasnoselskii et al. [10], which reduces the study of uniform convergence to the study of convergence in $L_2(\Omega)$.

Usually, to determine the solution of non-stationary differential equations uniquely, an initial condition is specified. However, in some cases, non-local conditions are used, for example, in the form of an integral over time (see [11], in the case of diffusion equations, [12] for fractional-order equations), or in the form of a relationship between the value of the solution at the initial and final times (see [13], [14]). We also note papers [15], [16], where boundary value problems given with fractional derivatives are studied.

As for the inverse problem of determining the function φ, we point that such a problem was studied only in the work [17] (with the exception of work [9], which was mentioned above). The authors of [17] considered this problem for the subdiffusion equation, which includes the fractional Caputo derivative, the elliptic part of which is a differential expression with two variables and constant coefficients.

The inverse problems of determining the right-hand side (the heat source density) of various subdiffusion equations have been considered by many researchers (see, e.g., [18]). We note that the inverse problem of determining the right-hand side of the equation given in an abstract form $f(x,t)$ has not yet been studied. The obtained results deal with the separated source term $s(t)f(x)$. The appropriate choice of the overdetermination depends on the choice whether the unknown is $s(t)$ or $f(x)$. It should be noted that studies of inverse problems, where the function $s(t)$ is the unknown, are relatively few (see, e.g., [18] in the case of fractional order equations and [19]–[21] in the case of equations of integer order).

Many authors have considered an equation in which $s(t) \equiv 1$ and $f(x)$ is unknown (see, e.g., [22]–[40]). Let us mention just a few of these works. The case of subdiffusion equations whose elliptic part is an ordinary differential expression is considered in [22]–[28]. The authors of the papers [29]–[33] studied subdiffusion equations in which the elliptic part is either a Laplace operator or a second-order operator. The article [34] examined the inverse problem for an abstract subdiffusion equation with the Cauchy condition. In article [34] and in most other articles, including [29]–[32], the Caputo derivative is used as a fractional derivative. Recent articles [35]–[36] are devoted to the inverse problem for the subdiffusion equation with the Riemann-Liouville derivative.

In [33], [38], [39], non-self-adjoint differential operators (with non-local boundary conditions) were taken as the elliptical part of a subdiffusion equation, and solutions to the inverse problem were found in the form of bioorthagonal series.

In our previous work [40], we examined the inverse problem for the simultaneous determination of the order of the Riemann-Liouville fractional derivative and the source function in the subdiffusion equations. Using the classical Fourier method, the authors proved the uniqueness and existence of a solution to this inverse problem.

We also note works [41]–[44] close to the given topic in which inverse problems of determining boundary functions in problems of control of heat propagation processes are studied.

1 Preliminaries

In this section, we formulate the lemma noted above from the study by Krasnoselskii et al. [10], the fundamental result of V.A. Il'in [3] on the convergence of the Fourier coefficients and recall some properties of the Mittag-Leffler function.

Let A stand for the operator acting in $L_2(\Omega)$ as $Ag(x) = -\triangle g(x)$ with the domain of definition $D(A) = \{g \in C^2(\overline{\Omega}) : g(x) = 0, x \in \partial\Omega\}$. We denote the self-adjoint extension of A in $L_2(\Omega)$ by $\hat{A}$.

To formulate the indicated lemma, it is necessary to introduce the power of operator $\hat{A}$.

Let σ be an arbitrary real number. The power of operator A, acting in $L_2(\Omega)$ is defined as:

$$\hat{A}^\sigma g(x) = \sum_{k=1}^{\infty} \lambda_k^\sigma\, g_k\, v_k(x), \quad g_k = (g, v_k),$$

and the domain of definition has the form

$$D(\hat{A}^\sigma) = \{g \in L_2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |g_k|^2 < \infty\}.$$

For elements of $D(\hat{A}^\sigma)$ we introduce the norm

$$||g||_\sigma^2 = \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |g_k|^2 = ||\hat{A}^\sigma g||^2.$$

The following lemma plays an essential role in our reasoning (see, e.g., [10; 453]).

Lemma 1 . Let $\sigma > \frac{N}{4}$. Then operator $\hat{A}^{-\sigma}$ continuously maps the space $L_2(\Omega)$ into $C(\overline{\Omega})$, and moreover, the following estimate holds

$$\|\hat{A}^{-\sigma} g\|_{C(\Omega)} \leq C\|g\|_{L_2(\Omega)}.$$

When proving the existence of solutions to forward and inverse problems, it is necessary to study the convergence of series of the form

$$\sum_{k=1}^{\infty} \lambda_k^\tau |h_k|^2, \; \tau > \frac{N}{2}, \tag{6}$$

where h_k is the Fourier coefficient of function $h(x)$. In the case of integers τ, the conditions for the convergence of such series in terms of the membership of the function $h(x)$ in classical Sobolev spaces $W_2^k(\Omega)$ were obtained in the work of V.A. Il'in [3]. To formulate these conditions, we introduce the class $\dot{W}_2^1(\Omega)$ as the closure in the $W_2^1(\Omega)$ norm of the set of all functions that are continuously differentiable in Ω and vanish near the boundary of Ω.

So, if function $h(x)$ satisfies the conditions

$$h(x) \in W_2^{\left[\frac{N}{2}\right]+1}(\Omega), \quad \text{and} \quad h(x), \triangle h(x), \cdots, \triangle^{\left[\frac{N}{4}\right]} h(x) \in \dot{W}_2^1(\Omega), \tag{7}$$

then the number series (6) (we can take $\tau = \frac{N}{2} + 1$ if N is even, and $\tau = \frac{N+1}{2}$ if N is odd) converges.

Similarly, if in (6) we replace τ by $\tau + 2$, then the convergence conditions will have the form:

$$h(x) \in W_2^{\left[\frac{N}{2}\right]+3}(\Omega), \quad \text{and} \quad h(x), \triangle h(x), \cdots, \triangle^{\left[\frac{N}{4}\right]+1} h(x) \in \dot{W}_2^1(\Omega). \tag{8}$$

Next, let us remind some properties of the Mittag-Leffler functions. For $0 < \rho < 1$ and an arbitrary complex number μ, by $E_{\rho,\mu}(z)$ we denote the Mittag-Leffler function with two parameters (see, e.g. [1; 12]):

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \mu)}.$$

If the parameter $\mu = 1$, then we have the classical Mittag-Leffler function: $E_\rho(z) = E_{\rho,1}(z)$.

In what follows, we need the asymptotic estimate of the Mittag-Leffler function with a sufficiently large negative argument. The estimate has the form (see, e.g. [45; 136])

$$|E_{\rho,\mu}(-t)| \le \frac{C}{1+t}, \quad t>0, \tag{9}$$

where μ is an arbitrary complex number. This estimate essentially follows from the following asymptotic estimate (see, e.g. [45; 134]):

$$E_{\rho,\mu}(-t) = \frac{t^{-1}}{\Gamma(\mu-\rho)} + O\big(t^{-2}\big). \tag{10}$$

We will also use a coarser estimate with a positive number λ and $0<\varepsilon<1$:

$$|t^{\rho-1}E_{\rho,\rho}(-\lambda t^{\rho})| \le \frac{Ct^{\rho-1}}{1+\lambda t^{\rho}} \le C\lambda^{\varepsilon-1}t^{\varepsilon\rho-1}, \quad t>0, \tag{11}$$

which is easy to verify. Indeed, let $t^{\rho}\lambda<1$, then $t<\lambda^{-1/\rho}$ and

$$t^{\rho-1} = t^{\rho-\varepsilon\rho}t^{\varepsilon\rho-1} < \lambda^{\varepsilon-1}t^{\varepsilon\rho-1}.$$

If $t^{\rho}\lambda\ge 1$, then $\lambda^{-\varepsilon}\le t^{\varepsilon\rho}$ and

$$\lambda^{-1}t^{-1} = \lambda^{-1+\varepsilon}\lambda^{-\varepsilon}t^{-1} \le \lambda^{\varepsilon-1}t^{\varepsilon\rho-1}.$$

Proposition 1. The Mittag-Leffler function of negative argument $E_{\rho}(-x)$ is monotonically decreasing function for all $0<\rho<1$ and

$$0<E_{\rho}(-x)<1. \tag{12}$$

Proof of this proposition can be found, for example, in [9].

Proposition 2. Let $\rho>0$ and $\lambda\in\mathbb{C}$. Then for all positive t one has

$$\int\limits_0^t \eta^{\rho-1}E_{\rho,\rho}(\lambda\eta^{\rho})d\eta = t^{\rho}E_{\rho,\rho+1}(\lambda t^{\rho}), \tag{13}$$

and

$$J_t^{\rho-1}\Big(t^{\rho-1}E_{\rho,\rho}(\lambda t^{\rho})\Big) = E_{\rho}(\lambda t^{\rho}). \tag{14}$$

Proof of this proposition can be found, for example, in [45; 120].

2 Well-posedness of the forward problem

First, we consider the problem for the homogeneous equation:

$$\begin{cases} \partial_t^{\rho}w(x,t) - \triangle w(x,t) = 0, \quad x\in\Omega \quad 0<t\le T; \\ w(x,t)\big|_{\partial\Omega} = 0; \\ J_t^{\rho-1}w(x,t)\big|_{t=\xi} = \alpha\lim\limits_{t\to 0}J_t^{\rho-1}w(x,t) + \psi(x), \quad 0<\xi\le T, \quad x\in\overline{\Omega}, \end{cases} \tag{15}$$

where $\psi(x)$ is a given function.

Theorem 1. Let function $\psi(x)$ satisfy conditions (7).

If $\alpha\notin[0,1)$ or $\alpha\in(0,1)$, but $E_{\rho}(-\lambda_k\xi^{\rho})\ne\alpha$ for all $k\ge 1$, then problem (15) has a unique solution, which has the form

$$w(x,t) = \sum_{k=1}^{\infty}\frac{\psi_k}{E_{\rho}(-\lambda_k\xi^{\rho})-\alpha}t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^{\rho})v_k(x), \tag{16}$$

where ψ_k is the Fourier coefficient of function $\psi(x)$.

If $\alpha \in (0,1)$ and $E_\rho(-\lambda_{k_0}\xi^\rho) = \alpha$ for some eigenvalue λ_{k_0} with the multiplicity p_0, then we assume that the orthogonality conditions

$$\psi_k = (\psi, v_k) = 0, \;\; k \in K_0 = \{k_0, k_0+1, ..., k_0+p_0-1\} \tag{17}$$

are satisfied. Then solutions to problem (15) have the form

$$w(x,t) = \sum_{k \notin K_0} \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha}\, t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)v_k(x) + \sum_{k\in K_0} b_k t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)v_k(x), \tag{18}$$

with arbitrary coefficients b_k, $k \in K_0$.

Proof. In accordance with the Fourier method, we will look for a solution to problem (15) in the form of a series:

$$w(x,t) = \sum_{k=1}^{\infty} T_k(t)v_k(x),$$

where $T_k(t)$, $k \geq 1$, are solutions to the nonlocal problems:

$$\partial_t^\rho T_k(t) + \lambda_k T_k(t) = 0, \quad 0 < t \leq T, \tag{19}$$

$$J_t^{\rho-1}T_k(t)\big|_{t=\xi} = \alpha \lim_{t\to 0} J_t^{\rho-1}T_k(t) + \psi_k. \tag{20}$$

Let us denote

$$\lim_{t\to 0} J_t^{\rho-1}T_k(t) = b_k. \tag{21}$$

Then, the unique solution of the equation (19), that satisfies the condition (21) has the form $T_k(t) = b_k t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)$ (see, e.g. [46; 173], [1; 16], and [47]).

Equality (14) implies

$$J_t^{\rho-1}T_k(t)\bigg|_{t=\xi} = b_k E_\rho(-\lambda_k\xi^\rho).$$

Therefore, from the nonlocal condition (20) we obtain

$$b_k\big(E_\rho(-\lambda_k\xi^\rho) - \alpha\big) = \psi_k. \tag{22}$$

By virtue of property (12) of the Mittag-Leffler function, $E_\rho(-\lambda_k\xi^\rho) \neq \alpha$ for all $\alpha \geq 1$ and $\alpha < 0$ (note, $\xi > 0$ and $\lambda_k > 0$). Therefore, from (22) we have

$$b_k = \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha}, \quad |b_k| \leq C_\alpha|\psi_k|, \quad k \geq 1, \tag{23}$$

where C_α is a constant.

Let $0 < \alpha < 1$. Then according to Proposition 1, there is a unique $\lambda_0 > 0$ such that $E_\rho(-\lambda_0\xi^\rho) = \alpha$. If there is no eigenvalue equal to λ_0 , then the estimate in (23) holds with some constant $C_\alpha > 0$.

Thus, if $\alpha \notin [0,1)$ or $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all $k \geq 1$, then the formal solution of problem (15) has the form (16).

Finally, let $0 < \alpha < 1$ and there is an eigenvalue equal to λ_0, having the multiplicity p_0: $\lambda_k = \lambda_0$ for $k = k_0, k_0+1, ..., k_0+p_0-1$. Then the nonlocal problem (19), (20) has a solution if the boundary function $\psi(x)$ satisfies the orthogonality conditions (17). Since $\psi_k = 0$, then arbitrary numbers b_k are solutions of equation (22). For all other k we have

$$b_k = \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha}, \quad |b_k| \leq C_\alpha|\psi_k|, \quad k \notin K_0.$$

Thus, the formal solution to problem (15) in this case has the form (18).

Let us show that the operators $A = -\triangle$ and ∂_t^ρ can be applied term-by-term to series (16) and the resulting series converges uniformly in $(x,t) \in \overline{\Omega} \times (0,T]$; for series (18), this question is considered in a similar way.

Let $S_j(x,t)$ be the partial sum of series (16). Then

$$-\triangle S_j(x,t) = \sum_{k=1}^{j} \lambda_k \frac{\psi_k}{E_\rho(-\lambda_k \xi^\rho) - \alpha}\, t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k(x).$$

Using the equality

$$\hat{A}^{-\sigma} v_k(x) = \lambda_k^{-\sigma} v_k(x),$$

with $\sigma > \frac{N}{4}$ and applying Lemma 1 for $g(x) = -\triangle S_j(x,t)$, we have

$$|| -\triangle S_j(x,t)||^2_{C(\Omega)} \le C \sum_{k=1}^{j} \lambda_k^{2(\sigma+1)} \left| \frac{\psi_k}{E_\rho(-\lambda_k \xi^\rho) - \alpha}\, t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) \right|^2, \ t > 0.$$

Here, to estimate the $L_2(\Omega)$ norm, we applied the Parseval's equality.

Apply estimates (9) and (23) to obtain

$$|| -\triangle S_j(x,t)||^2_{C(\Omega)} \le C_\alpha t^{2\rho-2} \sum_{k=1}^{j} \lambda_k^{2(\sigma+1)} \left| \frac{\psi_k}{1+\lambda_k t^\rho} \right|^2 \le C_\alpha t^{-2} \sum_{k=1}^{j} \lambda_k^\tau |\psi_k|^2, \ \tau = 2\sigma > \frac{N}{2}.$$

Therefore, if $\psi(x)$ satisfies conditions (7), then $-\triangle u(x,t) \in C(\overline{\Omega} \times (0,T])$. From equation (15) one has $\partial_t^\rho u(x,t) = \triangle u(x,t)$, $t > 0$, and the above estimates imply

$$||\partial_t^\rho w(x,t)||^2_{C(\Omega)} \le C_\alpha t^{-2} \sum_{k=1}^{j} \lambda_k^\tau |\psi_k|^2, \ t > 0,$$

which means $\partial_t^\rho w(x,t) \in C(\overline{\Omega} \times (0,T])$.

For $S_j(x,t)$, taking into account estimate (9), we obtain

$$||t^{1-\rho} S_j(x,t)||^2_{C(\Omega)} \le C_\alpha \sum_{k=1}^{j} \lambda_k^\tau |\psi_k|^2, \ \tau > \frac{N}{2}.$$

Hence $t^{1-\rho} w(x,t) \in C(\overline{\Omega} \times [0,T])$, which was required by the definition of the solution to problem (15).

The uniqueness of the solution to problem (15) is proved in exactly the same way as in work [9]. For the convenience of the reader, we present this proof.

It is sufficient to show that the solution to the problem:

$$\begin{cases} \partial_t^\rho w(x,t) - \triangle w(x,t) = 0, \quad x \in \Omega, \quad 0 < t \le T; \\ w(x,t)\big|_{\partial\Omega} = 0; \\ J_t^{\rho-1} w(x,t)\big|_{t=\xi} = \alpha \lim_{t\to 0} J_t^{\rho-1} w(x,t), \quad 0 < \xi \le T, \quad x \in \overline{\Omega}, \end{cases}$$

is identically equal to zero.

Let $w_k(t) = (w(x,t), v_k(x))$. Since operator $A = -\triangle$ is self-adjoint, one has

$$\partial_t^\rho w_k(t) = (\partial_t^\rho w(x,t), v_k(x)) = (Aw(x,t), v_k(x)) = (w(x,t), Av_k(x)) = -\lambda_k w_k(t)$$

or

$$\partial_t^\rho w_k(t) = -\lambda_k w_k(t) \tag{24}$$

and the nonlocal condition implies

$$J_t^{\rho-1} w_k(t)\big|_{t=\xi} = \alpha \lim_{t\to 0} J_t^{\rho-1} w_k(t). \tag{25}$$

Let us denote $\lim\limits_{t\to 0} J_t^{\rho-1} w_k(t) = b_k$. Then the unique solution to the differential equation (24) with this initial condition has the form $w_k(t) = b_k t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)$ (see, e.g.[46; 174]). From equality (14) and the nonlocal conditions of (25) we obtain the following equation to find the unknown numbers b_k:

$$b_k\big(E_\rho(-\lambda_k \xi^\rho) - \alpha\big) = 0. \tag{26}$$

If $\alpha \notin [0,1)$, then by virtue of the Proposition 1 we obtain $b_k = 0$ for all $k \geq 1$. If $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all k, then $E_\rho(-\lambda_k \xi^\rho) \neq \alpha$ and therefore $b_k = 0$. Hence, if $\alpha \notin [0,1)$ or $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all k, we have all b_k are equal to zero, therefore $w_k(t) = 0$. By virtue of completeness of the set of eigenfunctions $\{v_k(x)\}$, we conclude that $w(x,t) \equiv 0$. Thus, problem (15) in this case has a unique solution.

Now, suppose that $\alpha \in (0,1)$ and $\lambda_k = \lambda_0$, $k \in K_0$. Then $E_\rho(-\lambda_k \xi^\rho) = \alpha$, $k \in K_0$ and therefore equation (26) has the following solution: $b_k = 0$ if $k \notin K_0$ and b_k is an arbitrary number for $k \in K_0$. Thus, in this case, there is no uniqueness of the solution to problem (15). Theorem 1 is completely proved.

Now consider the following auxiliary initial-boundary value problem:

$$\begin{cases} \partial_t^\rho \omega(x,t) - \triangle \omega(x,t) = f(x,t), \quad x \in \Omega, \quad 0 < t \leq T; \\ \omega(x,t)\big|_{\partial\Omega} = 0; \\ \lim\limits_{t\to 0} J_t^{\rho-1} \omega(x,t) = 0, \quad x \in \overline{\Omega}. \end{cases} \tag{27}$$

We have the following theorem for this problem:

Theorem 2. Let $t^{1-\rho} f(x,t)$ as a function of x satisfy conditions (7) for all $t \in [0,T]$. Then problem (27) has a unique solution and this solution has the representation

$$\omega(x,t) = \sum_{k=1}^{\infty} \left[\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right] v_k(x), \tag{28}$$

where $f_k(t)$ are the Fourier coefficients of function $f(x,t)$: $f_k(t) = (f(\cdot,t), v_k)$.

Proof. It is known that the formal solution of the problem (27) has the form (28) (see, e.g. [46; 173], [47]). In order to prove that function (28) is actually a solution to the problem, it remains to substantiate this formal statement, i.e., to show that the operators $A = -\triangle$ and ∂_t^ρ can be applied term-by-term to series (28) and the resulting series converges uniformly in $(x,t) \in \overline{\Omega} \times (0;T]$.

Let $S_j(x,t)$ be the partial sum of series (28). Then

$$-\triangle S_j(x,t) = \sum_{k=1}^{j} \left[\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right] \lambda_k v_k(x).$$

Let $\sigma > \frac{N}{4}$. Repeating the above reasoning based on Lemma 1, we arrive at

$$|| -\triangle S_j(x,t)||^2_{C(\Omega)} \leq \left\| \hat{A}^{-\sigma} \sum_{k=1}^{j} \lambda_k^{\sigma+1} v_k(x) \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right\|^2_{C(\Omega)} \leq$$

$$\leq \left\| \sum_{k=1}^{j} \lambda_k^{\sigma+1} v_k(x) \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right\|^2_{L_2(\Omega)} \leq$$

(apply Parseval's equality to obtain)

$$\leq C \sum_{k=1}^{j} \lambda_k^{2(\sigma+1)} \left| \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right|^2, \ t > 0.$$

Then, by inequality (11) with $0 < \varepsilon < 1$ one has

$$|| - \triangle S_j(x,t)||^2_{C(\Omega)} \leq C \sum_{k=1}^{j} \left[\int_0^t \eta^{\varepsilon\rho-1} (t-\eta)^{\rho-1} \lambda_k^{\sigma+\varepsilon} |(t-\eta)^{1-\rho} f_k(t-\eta)| d\eta \right]^2,$$

or, by the generalized Minkowski inequality,

$$|| - \triangle S_j(x,t)||^2_{C(\Omega)} \leq C \left[\int_0^t \eta^{\varepsilon\rho-1} (t-\eta)^{\rho-1} \left(\sum_{k=1}^{j} |\lambda_k^{\tau} (t-\eta)^{1-\rho} f_k(t-\eta)|^2 \right)^{\frac{1}{2}} d\eta \right]^2, \ \tau = \sigma + \varepsilon > \frac{N}{2}.$$

Since $t^{1-\rho} f(x,t)$ as a function of x satisfies conditions (7) for all $t \in [0,T]$, then

$$|| - \triangle S_j(x,t)||^2_{C(\Omega)} \leq C, \ t \geq 0.$$

Hence $-\triangle \omega(x,t) \in C(\overline{\Omega} \times [0,T])$ and in particular $\omega(x,t) \in C(\overline{\Omega} \times [0,T])$.

Further, from equation (1) one has $\partial_t^\rho S_j(t) = \triangle S_j(x,t) + \sum\limits_{k=1}^{j} f_k(t) v_k(x)$, $t > 0$. Therefore, from the above reasoning, we have $\partial_t^\rho \omega(x,t) \in C(\overline{\Omega} \times (0,T])$.

The uniqueness of the solution can be proved by the standard technique based on completeness in $L_2(\Omega)$ of the set of eigenfunctions $\{v_k(x)\}$ (see, e.g. [5]).

Theorem 2 is completely proved.

Now let us move on to solving the main problem (1)–(3). Let $\varphi(x)$ and $t^{1-\rho} f(x,t)$ (for all $t \in [0,T]$) satisfy conditions (7). If we put $\psi(x) = \varphi(x) - J_t^{\rho-1} \omega(x,t)\Big|_{t=\xi}$ and $\omega(x,t)$ and $w(x,t)$ are the solutions of problems (27) and (15) correspondingly, then function $u(x,t) = \omega(x,t) + w(x,t)$ is a solution to problem (1)–(3). Therefore, we can use the already proven assertions.

Thus, if $\alpha \notin [0,1)$ or $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all $k \geq 1$, then

$$u(x,t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k - \omega_k(\xi)}{E_\rho(-\lambda_k \xi^\rho) - \alpha} t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) + \omega_k(t) \right] v_k(x), \tag{29}$$

where

$$\omega_k(t) = \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta.$$

The uniqueness of the function $u(x,t)$ follows from the uniqueness of the solutions $\omega(x,t)$ and $w(x,t)$.

If $\alpha \in (0,1)$ and $\lambda_k = \lambda_0$, $k \in K_0$, then

$$u(x,t) = \sum_{k \notin K_0} \left[\frac{\varphi_k - \omega_k(\xi)}{E_\rho(-\lambda_k \xi^\rho) - \alpha} t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) + \omega_k(t) \right] v_k(x) + \sum_{k \in K_0} b_k t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k(x). \tag{30}$$

The orthogonality conditions (17) have the form

$$(\varphi, v_k) = (\, J_t^{\rho-1}\omega(x,t)\Big|_{t=\xi}, v_k),\ k\in K_0;\ K_0=\{k_0,k_0+1,....,k_0+p_0-1\}.$$

Instead of these conditions, we can take orthogonality conditions that is easy to verify:

$$(\varphi, v_k)=0,\ (f(\cdot,t),v_k)=0,\ \text{for all } t\in[0,T],\ k\in K_0;\ K_0=\{k_0,k_0+1,....,k_0+p_0-1\}. \tag{31}$$

Thus, we have proved the main result of this section:

Theorem 3. Let $\varphi(x)$ and $t^{1-\rho}f(x,t)$ (for all $t\in[0,T]$) satisfy conditions (7). If $\alpha\notin[0,1)$ or $\alpha\in(0,1)$, but $\lambda_k\neq\lambda_0$ for all $k\geq1$, then problem (1)–(3) has a unique solution and this solution has the form (29).

If $\alpha\in(0,1)$ and $\lambda_k=\lambda_0$, $k\in K_0$, then we assume that the orthogonality conditions (31) are satisfied. The solution of problem (1)–(3) has the form (30) with arbitrary coefficients b_k, $k\in K_0$.

3 Inverse problem of determining the right-hand side of the equation

Let us consider *the inverse problem*

$$\begin{cases} \partial_t^\rho u(x,t)-\triangle u(x,t)=f(x), \quad 0<t\leq T; \quad x\in\Omega;\\ u(x,t)\big|_{\partial\Omega}=0;\\ J_t^{\rho-1}u(x,t)\big|_{t=\xi}=\alpha\lim\limits_{t\to0}J_t^{\rho-1}u(x,t)+\varphi(x), \quad 0<\xi\leq T, \quad x\in\overline{\Omega}, \end{cases} \tag{32}$$

with the additional condition

$$u(x,\theta)=\Psi(x), \quad 0<\theta\leq T, \quad \theta\neq\xi, \quad x\in\overline{\Omega}, \tag{33}$$

where the unknown function $f(x)$, characterizing the action of heat sources, does not depend on t and $\Psi(x),\varphi(x)$ are given functions, $\alpha\geq1$, ξ and θ are fixed points of $(0,T]$.

Note that if $\theta=\xi$, then the nonlocal condition in (32) coincides with the Cauchy condition $\lim\limits_{t\to0}J_t^{\rho-1}u(x,t)=\varphi_1$ with some φ_1. In this case, this inverse problem was studied in [35].

Theorem 4. Let functions $\varphi(x),\Psi(x)$ satisfy conditions (8). Then the inverse problem (32), (33) has a unique solution $\{u(x,t),f(x)\}$ and this solution has the following form

$$f(x)=\sum_{k=1}^{\infty}\Bigg[\frac{\alpha-E_\rho(-\lambda_k\xi^\rho)}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)+\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha-E_\rho(-\lambda_k\xi^\rho)]}\,\Psi_k+$$
$$+\frac{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)+\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha-E_\rho(-\lambda_k\xi^\rho)]}\,\varphi_k\Bigg]v_k(x), \tag{34}$$

$$u(x,t)=\sum_{k=1}^{\infty}\left[\frac{E_{\rho,\rho}(-\lambda_k t^\rho)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,t^{\rho-1}[\varphi_k-f_k\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)]+f_kt^\rho E_{\rho,\rho+1}(-\lambda_kt^\rho)\right]v_k(x). \tag{35}$$

Proof. Let us first show that the series (34) and (35) are formal solutions to the inverse problem. Then we show the uniform convergence and differentiability of these series.

Suppose $f(x)$ is known. Then the unique solution to problem (32) has the form (29). Since $f(x)$ does not depend on t, then, owing to formulas

$$\omega_k(t)=f_k\int\limits_0^t\eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)d\eta$$

and (13), it is easy to verify that the formal solution of problem (32) has the form of (35).

Due to the additional condition (33) and completeness of the system $\{v_k(x)\}$ we obtain:

$$\frac{E_{\rho,\rho}(-\lambda_k\theta^\rho)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,\theta^{\rho-1}[\varphi_k - f_k\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)] + f_k\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho) = \Psi_k.$$

After simple calculations, we get

$$f_k = \frac{\alpha - E_\rho(-\lambda_k\xi^\rho)}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)+\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha - E_\rho(-\lambda_k\xi^\rho)]}\,\Psi_k+$$

$$+\frac{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)+\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha - E_\rho(-\lambda_k\xi^\rho)]}\,\varphi_k \equiv f_{k,1}+f_{k,2}.$$

Therefore, series (34) is a formal solution of the inverse problem.

Let us prove the convergence of this series uniformly in $x \in \overline{\Omega}$.

If $F_j(x)$ is the partial sums of series (34), then by applying Lemma 1 as above, we have

$$||F_j(x)||^2_{C(\Omega)} \le \sum_{k=1}^{j}\lambda_k^{2\sigma}[f_{k,1}+f_{k,2}]^2 \le 2\sum_{k=1}^{j}\lambda_k^{2\sigma}f_{k,1}^2 + 2\sum_{k=1}^{j}\lambda_k^{2\sigma}f_{k,2}^2 \equiv 2I_{1,j}+2I_{2,j}, \tag{36}$$

where $\sigma > \frac{N}{4}$. Since $\xi > 0$, then $\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho) > 0$. Therefore,

$$I_{1,j} \le \sum_{k=1}^{j}\left|\frac{\alpha - E_\rho(-\lambda_k\xi^\rho)}{\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha - E_\rho(-\lambda_k\xi^\rho)]}\right|^2 \lambda_k^{2\sigma}|\Psi_k|^2 = \sum_{k=1}^{j}\frac{\lambda_k^{2\sigma}|\Psi_k|^2}{|\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)|^2}.$$

Apply the asymptotic estimate (10) to get

$$I_{1,j} \le \sum_{k=1}^{j}\frac{\lambda_k^{2(\sigma+1)}|\Psi_k|^2}{\big(1+O\big((-\lambda_k\theta^\rho)^{-1}\big)\big)^2} \le C\sum_{k=1}^{j}\lambda_k^{\tau+2}|\Psi_k|^2,\ \tau = 2\sigma > \frac{N}{2}.$$

Since $\theta > 0$ and $\alpha \ge 1$, then $\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha - E_\rho(-\lambda_k\xi^\rho)] > 0$. Therefore,

$$I_{2,j} \le \sum_{k=1}^{j}\left|\frac{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)}\right|^2 \lambda_k^{2\sigma}|\varphi_k|^2 = \sum_{k=1}^{j}\frac{\lambda_k^{2(\sigma+1)}|\varphi_k|^2}{|\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)|^2}.$$

By virtue of (10),

$$I_{2,j} \le \sum_{k=1}^{j}\frac{\lambda_k^{2(\sigma+1)}|\varphi_k|^2}{\big(1+O\big((-\lambda_k\xi^\rho)^{-1}\big)\big)^2} \le C\sum_{k=1}^{j}\lambda_k^{\tau+2}|\varphi_k|^2,\ \tau > \frac{N}{2}.$$

Thus, if $\varphi(x), \Psi(x)$ satisfy conditions (8), then from estimates of $I_{i,j}$ and (36) we obtain $f(x) \in C(\overline{\Omega})$.

Further, the fact that function $u(x,t)$ given by the series (35) is a solution to the inverse problem is proved exactly as in Theorem 1.

The uniqueness of the solution follows from the completeness of the systems of eigenfunctions $\{v_k(x)\}$ (see [9]).

4 The inverse problem of determining function φ from the nonlocal condition

Let us assume that in forward problem (1)–(3) not only function $u(x,t)$, but also function $\varphi(x)$ from nonlocal condition (3) is unknown. As an additional condition for this inverse problem, we again take condition (5). We note that if $\theta=\xi$ in this condition, then the nonlocal condition $J_t^{\rho-1}u(x,t)\big|_{t=\xi}=\alpha\lim\limits_{t\to 0}J_t^{\rho-1}u(x,t)+\varphi(x)$ passes to the Cauchy condition $\lim\limits_{t\to 0}J_t^{\rho-1}u(x,t)=\varphi_1(x)$ (with some $\varphi_1(x)$), which is investigated, for instance, in [35].

Theorem 5. Let $t^{1-\rho}f(x,t)$ as a function of x satisfy conditions (7) for all $t\in[0,T]$ and let function $\Psi(x)$ satisfy conditions (8). Then the inverse problem (1)–(3), (5) has a unique solution $\{u(x,t),\varphi(x)\}$ and this solution has the form

$$\varphi(x)=\sum_{k=1}^{\infty}\left[\frac{E_\rho(-\lambda_k\xi^\rho)-\alpha}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}[\Psi_k-\omega_k(\theta)]+\omega_k(\xi)\right]v_k(x), \tag{37}$$

$$u(x,t)=\sum_{k=1}^{\infty}\left[\frac{\varphi_k-\omega_k(\xi)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)+\omega_k(t)\right]v_k(x), \tag{38}$$

where

$$\omega_k(t)=\int\limits_0^t\eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)f_k(t-\eta)d\eta.$$

Proof. The solution to problem (1)–(3) has the form (38) (see Theorem 3). Therefore, condition (5) implies:

$$u(x,\theta)=\sum_{k=1}^{\infty}\left[\frac{\varphi_k-\omega_k(\xi)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)+\omega_k(\theta)\right]v_k(x)=\Psi(x).$$

Passing to the Fourier coefficients, we have

$$\frac{\varphi_k-\omega_k(\xi)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)+\omega_k(\theta)=\Psi_k,\ k\geq 1,$$

or

$$\varphi_k=\frac{E_\rho(-\lambda_k\xi^\rho)-\alpha}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}[\Psi_k-\omega_k(\theta)]+\omega_k(\xi).$$

Thus, equality (37) is formally established. Now, we show that series (37) converges uniformly in $x\in\overline{\Omega}$.

Let $\Phi_j(x)$ be the partial sum of series (37). Then applying Lemma 1 as above, we arrive at

$$||\Phi_j(x)||^2_{C(\Omega)}\leq\sum_{k=1}^{j}\lambda_k^{2\sigma}\left|\frac{E_\rho(-\lambda_k\xi^\rho)-\alpha}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}\Big[\Psi_k-\omega_k(\theta)\Big]+\omega_k(\xi)\right|^2\leq$$

$$\leq 3\sum_{k=1}^{j}\lambda_k^{2\sigma}\left[\left|\frac{E_\rho(-\lambda_k\xi^\rho)-\alpha}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}\right|^2\Big[|\Psi_k|^2+|\omega_k(\theta)|^2\Big]+|\omega_k(\xi)|^2\right]\equiv\Phi_j^1+\Phi_j^2+\Phi_j^3, \tag{39}$$

where $\sigma>\frac{N}{4}$. Since $|E_\rho(-\lambda_k\xi^\rho)-\alpha|\leq C$, then by virtue of the asymptotic estimate (10) we obtain

$$\Phi_j^1\leq C\sum_{k=1}^{j}\frac{\lambda_k^{2(\sigma+1)}\theta^2\Gamma^2(1-\rho)}{\big(1+O\big((-\lambda_k\theta^\rho)^{-1}\big)\big)^2}|\Psi_k|^2\leq C_1\sum_{k=1}^{j}\lambda_k^{\tau+2}|\Psi_k|^2,\ \tau=2\sigma>\frac{N}{2}.$$

Similarly, by estimates (10) and (11) we have

$$\Phi_j^2 \le C \sum_{k=1}^{j} \frac{\lambda_k^{2\sigma+2}\theta^2\Gamma^2(1-\rho)}{\big(1+O\big((-\lambda_k\theta^\rho)^{-1}\big)\big)^2} \left| \int_0^\theta \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k\eta^\rho) f_k(\theta-\eta) d\eta \right|^2 \le$$

$$\le \sum_{k=1}^{j} \frac{C_\varepsilon \lambda_k^{2\sigma+2}}{\big(1+O\big((-\lambda_k\theta^\rho)^{-1}\big)\big)^2} \left| \int_0^\theta \eta^{\varepsilon\rho-1}(\theta-\eta)^{\rho-1}\lambda_k^{\varepsilon-1}|(\theta-\eta)^{1-\rho} f_k(\theta-\eta)| d\eta \right|^2$$

(by the generalized Minkowski inequality)

$$\le C_\varepsilon \left[\int_0^\theta \eta^{\varepsilon\rho-1}(\theta-\eta)^{\rho-1} \left(\sum_{k=1}^{j} \lambda_k^{\tau+2\varepsilon} |(\theta-\eta)^{1-\rho} f_k(\theta-\eta)|^2 \right)^{\frac{1}{2}} d\eta \right]^2 \le$$

$$\le C_\varepsilon \max_{t\in[0,T]} \sum_{k=1}^{j} \lambda_k^{\tau+2\varepsilon} |t^{1-\rho} f_k(t)|^2.$$

For Φ_j^3, one has

$$\Phi_j^3 \le \sum_{k=1}^{j} \lambda_k^{2\sigma} \left| \int_0^\xi \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k\eta^\rho) f_k(\xi-\eta) d\eta \right|^2 \le$$

$$\le C \left[\int_0^\theta \eta^{\rho-1}(\theta-\eta)^{\rho-1} \left(\sum_{k=1}^{j} \lambda_k^{\tau} |(\theta-\eta)^{1-\rho} f_k(\theta-\eta)|^2 \right)^{\frac{1}{2}} d\eta \right]^2 \le C \max_{t\in[0,T]} \sum_{k=1}^{j} \lambda_k^{\tau} |t^{1-\rho} f_k(t)|^2.$$

Since functions $\Psi(x)$, $f(x,t)$ satisfy conditions of the theorem, then by virtue of estimate (39), we have $\varphi(x) \in C(\overline{\Omega})$.

The fact that the function defined by equality (38) is a solution to problem (1)–(3) is proved similarly to Theorem 3.

The uniqueness of the solution of the inverse problem follows from the completeness of the system of eigenfunctions $\{v_k(x)\}$ in the space $L_2(\Omega)$ in the standard way.

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References

1 Псху А.В. Уравнения в частных производных дробного порядка / А.В. Псху. — М.: Наука, 2005. — 199 с.

2 Ильин В.А. О фундаментальных системах функций оператора Лапласа в произвольной области и об асимптотической оценке фундаментальных чисел / В.А. Ильин // Докл. АН СССР. — 1967. — 177. — № 1. — С. 25–28.

3 Ильин В.А. О разрешимости смешанных задач для гиперболического и параболического уравнений / В.А. Ильин // Успехи математических наук. —1960. — 15. — № 2(92). — С. 97–154.

4 Ладыженская О.А. Смешанная задача для гиперболического уравнения / О.А. Ладыженская. — М.: Гос. изд-во техн.-теор. лит., 1953. — 280 с.

5 Alimov Sh.A. On the backward problems in time for time-fractional subdiffusion equations / Sh.A. Alimov, R.R. Ashurov // Fractional Differential Calculus. — 2021. — 11. — No. 2. — P. 203–217. DOI: 10.7153/fdc-2021-11-14.

6 Liu J. A backward problem for the time-fractional diffusion equation / J. Liu, M. Yamamoto // Appl. Anal. — 2010. — 89. — P. 1769–1788.

7 Sakamoto K. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems / K. Sakamoto, M. Yamamoto // J. Math. Anal. Appl. — 2011. — 382. — P. 426–447.

8 Floridia G. Well-posedness for the backward problems in time for general time-fractional diffusion equation / G. Floridia, Z. Li, M. Yamamoto // Rend. Lincei Mat. Appl. — 2020. — 31. — P. 593–610.

9 Ashurov R. On the nonlocal problems in time for time-fractional subdiffusion equations / R.R. Ashurov, Yu. Fayziev // Fractal Fract. — 2022. — 6. — No. 41. https://doi.org/10.3390/fractalfract6010041

10 Красносельский М.А. Интегральные операторы в пространствах суммируемых функций / М.А. Красносельский, П.П. Забрейко, Е.И. Пустыльник, П.Е. Соболевский. — М.: Наука, 1966.

11 Pao C.V. Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions/ C.V. Pao // J. Math. Anal. Appl. — 1995. — 195. — P. 702–718.

12 Tuan N.H. On a time fractional diffusion with nonlocal in time conditions / N.H. Tuan, N.A. Triet, N.H. Luc, N.D. Phuong // Advances in Difference Equations. — 2021. — 204. https://doi.org/10.1186/s13662-021-03365-1

13 Ashyralyev A.O. Coercive stability of a multidimensional difference elliptic equation of 2m-th order with variable coefficients / A.O. Ashyralyev, P.E. Sobolevskii // Investigations in the Theory of Differential Equations. Minvuz Turkmen. SSR, Ashkhabad. — 1987. — P. 31–43.

14 Ashyralyev A.O. Coercive solvability of nonlocal boundary value problem for parabolic equations / A.O. Ashyralyev, A. Hanalyev, P.E. Sobolevskii // Abstract and Applied Analysis. — 2001. — 6. — No. 1. — P. 53–61.

15 Умаров С.Р. О некоторых краевых задачах для эллиптических уравнений с граничным оператором дробного порядка / С.Р. Умаров // Докл. РАН. — 1993. — 333. — № 6. — С. 708–710.

16 Ashurov R.R. On some boundary value problems for equations with boundary operators of fractional order / R.R. Ashurov, Yu.E. Fayziev // International Journal of Applied Mathematics. — 2021. — 34. — No. 2. — P. 283–295. DOI: 10.12732/ijam.v34i2.6

17 Yuldashev T.K. Inverse problem for a partial differential equation with Gerasimova-Caputo-type operator and degeneration / T.K. Yuldashev, B.J. Kadirkulov // Fractal Fract. — 2021. — 5. — No. 58. https://doi.org/10.3390/fractalfract5020058

18 Liu Y. Inverse problems of determining sources of the fractional partial differential equations / Liu Y., Li Z., Yamamoto M. // Handbook of Fractional Calculus with Applications. — J.A.T. Marchado Ed. DeGruyter. — 2019. — 2. — P. 411–430.

19 Ashyralyev A. Time-dependent source identification problem for the Schrodinger equation with nonlocal boundary conditions / A. Ashyralyev, M. Urun // AIP Conf. Proc. — 2019. — 2183. — Art. 070016.

20 Ashyralyev A. On the Crank-Nicolson difference scheme for the time-dependent source identification problem / A. Ashyralyev, M. Urun // Bulletin of the Karaganda University. Mathematics. — 2021. — 102(2). — P. 35–44.

21 Ashyralyev A. Time-dependent source identification Schrodinger type problem / A. Ashyralyev, M. Urun // International Journal of Applied Mathematics. — 2021. — 34(2). — P. 297–310.

22 Zhang Y. Inverse source problem for a fractional differential equations / Y. Zhang, X. Xu // Inverse Prob. — 2011. — 27. — No. 3. — P. 31–42.

23 Furati K.M. An inverse problem for a generalized fractional diffusion / K.M. Furati, O.S. Iyiola, M. Kirane // Applied Mathematics and Computation. — 2014. — 249. — P. 24–31.

24 Ismailov M.I. Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions / M.I. Ismailov, M. Cicek // Applied Mathematical Modelling. — 2016. — 40. — P. 4891–4899.

25 Kirane M. Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time / M. Kirane, A.S. Malik // Applied Mathematics and Computation. — 2011. — 218. — P. 163–170.

26 Kirane M. Determination of an unknown source term and the temperature distribution for the subdiffusion equation at the initial and final data / M. Kirane, B. Samet, B.T. Torebek // Electronic Journal of Differential Equations. — 2017. — 217. — P. 1–13.

27 Nguyen H.T. Regularized solution of an inverse source problem for a time fractional diffusion equation / H.T. Nguyen, D.L. Le, V.T. Nguyen // Applied Mathematical Modelling. — 2016. — 40. — P. 8244–8264.

28 Torebek B.T. Some inverse problems for the nonlocal heat equation with Caputo fractional derivative / B.T. Torebek, R. Tapdigoglu // Mathematical Methods in Applied Sciences. — 2017. — 40. — P. 6468–6479.

29 Li Z. Initial-boundary value problem for multi-term time-fractional diffusion equation with positive constant coefficients / Z. Li, Y. Liu, M. Yamamoto // Applied Mathematica and Computation. — 2015. — 257. — P. 381–397.

30 Rundell W. Recovering an unknown source in a fractional diffusion problem / W. Rundell, Z. Zhang // Journal of Computational Physics. — 2018. — 368. — P. 299–314.

31 Asl N.A. Identifying an unknown time-dependent boundary source ib time-fractional diffusion equation with a non-local boundary condition / N.A. Asl, D. Rostamy // Journal of Computational and Applied Mathematics. — 2019. — 335. — P. 36–50.

32 Sun L. Recovering the time-dependent potential function in a multi-term time-fractional diffusion equation / Sun L., Y. Zhang, T. Wei // Applied Numerical Mathematics. — 2019. — 135. — P. 228–245.

33 Malik S.A. An inverse source problem for a two-parameter anomalous diffusion equation with nonlocal boundary conditions / S.A. Malik, S. Aziz // Computers and Mathematics with applications. — 2017. — 3. — P. 7–19.

34 Ruzhansky M. Inverse source problems for positive operators / M. Ruzhansky, N. Tokmagambetov, B.T. Torebek // I: Hypoelliptic diffusion and subdiffusion equations. J. Inverse Ill-Possed Probl. — 2019. — 27. — P. 891–911.

35 Ashurov R. Inverse problem of determining the heat source density for the subdiffusion equation / R. Ashurov, O. Muhiddinova // Differential equations. — 2020. — 56. — No. 12. P. 1550–1563.

36 Ashurov R. Initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary elliptic differential operator / R. Ashurov, O. Muhiddinova // Lobachevskii Journal of Mathematics. — 2021. — 42. — No. 3.

37 Shuang Zh. Initial boundary value problems for a multi-term time fractional diffusion equation with generalized fractional derivatives in time / Zh. Shuang, R. Saima, R. Asia, K. Khadija, M.A. Abdullah // AIMS Mathematics. — 2021. — 6. — No. 11. — P. 12114–12132. DOI: 10.3934/math.2021703

38 Kirane M. An inverse source problem for a two-dimensional time fractional diffusion equation with nonlocal boundary conditions / Kirane M., Salman A.M., Mohammed A. Al-Gwaiz // Math. Meth. Appl. Sci. — 2012, doi: 10.1002/mma.2661

39 Muhammad A. An inverse problem for a family of time fractional diffusion equations /A. Muhammad, A.M. Salman, // Inverse Problems in Science and Engineering. — 2016. — 25(9). — P. 1299–1322. DOI: 10.1080/17415977.2016.1255738

40 Ashurov R.R. Determination of fractional order and source term in a fractional subdiffusion equation [Electronic resource] / R.R. Ashurov, Yu.E. Fayziev. — Access mode: https://www.researchgate.net/publication/354997348

41 Алимов Ш.А. Об одной задаче управления процессом теплообмена / Ш.А. Алимов // Докл. АН РАН. — 2008. — 421. — № 5. — С. 583–585.

42 Alimov Sh.O. On a time-optimal control problem associated with the heat exchange process / Sh.O. Alimov // Appl. Math. Optim. — 2008. — 57. — P. 58–68.

43 Fayziev Yu.E. On one control problem related to the process of heat transfer into rectangles / Yu.E. Fayziev, A. Kuchkarov, D. Nosirova // Acta NUUz. — 2017. — No. 2/2. — P. 239–244.

44 Fayziev Yu.E. On the control of the heat conduction / Yu.E. Fayziev // IIUM Engineering Journal. — 2018. — 19. — No. 1. — P. 168–177.

45 Джрбашян М.М. Интегральные преобразования и представления функций в комплексной области / М.М. Джрбашян. — М.: Наука, 1966.

46 Gorenflo R. Mittag-Leffler functions, related topics and applications / R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogozin. Springer, 2014.

47 Ашуров Р.Р. О построении решений линейного дифференциального уравнения с дробными производными и постоянными коэффициентами / Р.Р. Ашуров, Ю.Э. Файзиев // УзМЖ. — 2017. — № 3. — С. 23–41.

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Риман-Лиувилль туындысы бар субдиффузия теңдеулері үшін уақыт бойынша локальдыемес есептер туралы

Уақыт бойынша бөлшек ретті Риман-Лиувилль туындылары бар субдиффузия теңдеулері үшін уақыт бойынша локальдыемес шарты бар бастапқы-шеттік есептер қарастырылған. Теңдеудің эллипстік бөлігі $\partial\Omega$ жеткілікті тегіс шекарасы бар кез келген $N-$ өлшемді Ω облысында анықталған Лаплас операторын береді. Қарастырылып отырған есептердің шешімінің бар болуы мен жалғыздығы дәлелденді. Теңдеудің оң жағын және уақыт бойынша локальдыемес шартты функцияны анықтау үшін кері есептер зерттелді. Фурье әдісі зерттеудің негізгі құралы болып табылады, сондықтан алынған нәтижелер анағұрлым жалпы эллипстік операторы бар субдиффузия теңдеулеріне таралуы мүмкін.

Кілт сөздер: уақыт бойынша локальдыемес есептер, Риман-Лиувилль туындылары, субдиффузия теңдеуі, кері есептер.

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О нелокальных задачах по времени для уравнений субдиффузии с производными Римана–Лиувилля

Рассмотрены начально-краевые задачи с нелокальным по времени условием для уравнения субдиффузии с дробными по времени производными Римана–Лиувилля. Эллиптическая часть уравнения представляет собой оператор Лапласа, определенный в произвольной N-размерной области Ω с достаточно гладкой границей $\partial\Omega$. Доказаны существование и единственность решения рассматриваемых задач. Исследованы обратные задачи для определения правой части уравнения и функции в нелокальном во времени условии. Основным инструментом исследования является метод Фурье, поэтому полученные результаты могут быть распространены на уравнения субдиффузии с более общим эллиптическим оператором.

Ключевые слова: нелокальные по времени задачи, производные Римана-Лиувилля, уравнение субдиффузии, обратные задачи.

References

1 Pskhu, A.V. (2005). *Uravneniia v chastnykh proizvodnykh drobnogo poriadka [Fractional partial differential equations].* Moscow: Nauka [in Russian].

2 Il'in, V.A. (1967). O fundamentalnykh sistemakh funktsii operatora Laplasa v proizvolnoi oblasti i ob asimptoticheskoi otsenke fundamentalnykh chisel [Fundamental systems of functions of the Laplace operator in an arbitrary domain and asymptotic estimates for fundamental numbers]. *Doklady Akademii nauk SSSR – Proceedings of the USSR Academy of Sciences, 177*, 1, 25–28 [in Russian].

3 Il'in, V.A. (1960). O razreshimosti smeshannykh zadach dlia giperbolicheskogo i parabolicheskogo uravnenii [On the solvability of mixed problems for hyperbolic and parabolic equations]. *Uspekhi matematicheskikh nauk – Russian Math. Surveys, 15*, 2, 97–154 [in Russian].

4 Ladyzhenskaya, O.A. (1953). *Smeshannaia zadacha dlia giperbolicheskogo uravneniia [Mixed problem for a hyperbolic equation].* Moscow: Gosudarstvennoe izdatelstvo tekhniko-teoreticheskoi literatury [in Russian].

5 Alimov, Sh.A., & Ashurov, R.R. (2021). On the backward problems in time for time-fractional subdiffusion equations. *Fractional Differential Calculus, 11*, 2, 203–217. DOI: 10.7153/fdc-2021-11-14.

6 Liu, J., & Yamamoto, M. (2010). A backward problem for the time-fractional diffusion equation. *Appl. Anal., 89,* 1769–1788.

7 Sakamoto, K., & Yamamoto, M. (2011). Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl., 382,* 426–447.

8 Floridia, G., Li, Z., & Yamamoto, M. (2020). Well-posedness for the backward problems in time for general time-fractional diffusion equation. *Rend. Lincei Mat. Appl., 31,* 593–610.

9 Ashurov, R., & Fayziev, Yu. (2022). On the nonlocal problems in time for time-fractional subdiffusion equations. *Fractal Fract., 6,* 41. https://doi.org/10.3390/ fractalfract6010041

10 Krasnoselskii, M.A., Zabreiko, P.P., Pustilnik, E.I., & Sobolevskii, P.S. (1966). *Integralnye operatory v prostranstvakh summiruemykh funktsii [Integral operators in the spaces of integrable functions].* Moscow: Nauka [in Russian].

11 Pao, C.V. (1995). Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions. *J. Math. Anal. Appl., 195,* 702–718.

12 Tuan, N.H., Triet, N.A., Luc, N.H., & Phuong, N.D. (2021). On a time fractional diffusion with nonlocal in time conditions. *Advances in Difference Equations, 204.* https://doi.org/10.1186/ s13662-021-03365-1

13 Ashyralyev, A.O., & Sobolevskii, P.E. (1987). Coercive stability of a multidimensional difference elliptic equation of 2m-th order with variable coefficients. *Investigations in the Theory of Differential Equations. Minvuz Turkmen. SSR,* 31–43.

14 Ashyralyev, A.O., Hanalyev, A., & Sobolevskii, P.E. (2001). Coercive solvability of nonlocal boundary value problem for parabolic equations. *Abstract and Applied Analysis, 6*, 1, 53–61.

15 Umarov, S.R. (1993). O nekotorykh kraevykh zadachakh dlia ellipticheskikh uravnenii s granichnym operatorom drobnogo poriadka [On some boundary value problems for elliptic equations with a boundary operator of fractional order]. *Doklady Rossiskoi akademi nauk – Proceeding of the Russian Academy of Sciences, 6,* 708–710 [in Russian].

16 Ashurov, R.R., & Fayziev, Yu.E. (2021). On some boundary value problems for equations with boundary operators of fractional order. *International Journal of Applied Mathematics, 34*, 2, 283–295. http://dx.doi.org/10.12732/ijam.v34i2.6b

17 Yuldashev, T.K., & Kadirkulov, B.J. (2021). Inverse problem for a partial differential equation with Gerasimova-Caputo-type operator and degeneration. *Fractal Fract., 5*, 58. https://doi.org/10.3390/fractalfract5020058

18 Liu, Y., Li Z., & Yamamoto, M. (2019). Inverse problems of determining sources of the fractional partial differential equations. *Handbook of Fractional Calculus with Applications. J.A.T. Marchado Ed. DeGruyter, 2,* 411–430.

19 Ashyralyev, A., & Urun, M. (2019). Time-dependent source identification problem for the Schrodinger equation with nonlocal boundary conditions. *AIP Conf. Proc., 2183,* art. 070016.

20 Ashyralyev, A., & Urun, M. (2021). On the Crank-Nicolson difference scheme for the time-dependent source identification problem. *Bulletin of the Karaganda university. Mathematics, 102*, 2, 35–44.

21 Ashyralyev, A., & Urun, M. (2021). Time-dependent source identification Schrodinger type problem. *International Journal of Applied Mathematics, 34*, 2, 297–310.

22 Zhang, Y., & Xu, X. (2011). Inverse source problem for a fractional differential equations. *Inverse Prob., 27*, 3, 31–42.

23 Furati, K.M., Iyiola, O.S., & Kirane, M. (2014). An inverse problem for a generalized fractional diffusion. *Applied Mathematics and Computation, 249,* 24–31.

24 Ismailov, M.I., & Cicek, M. (2016). Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions. *Applied Mathematical Modelling, 40,* 4891–4899.

25 Kirane, M., & Malik, A.S. (2011). Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time. *Applied Mathematics and Computation, 218,* 163–170.

26 Kirane, M., Samet, B., & Torebek, B.T. (2017). Determination of an unknown source term and the temperature distribution for the subdiffusion equation at the initial and final data. *Electronic Journal of Differential Equations, 217,* 1–13.

27 Nguyen, H.T., Le, D.L., & Nguyen, V.T. (2016). Regularized solution of an inverse source problem for a time fractional diffusion equation. *Applied Mathematical Modelling, 40,* 8244–8264.

28 Torebek, B.T., & Tapdigoglu, R. (2017). Some inverse problems for the nonlocal heat equation with Caputo fractional derivative. *Mathematical Methods in Applied Sciences, 40,* 6468–6479.

29 Li, Z., Liu, Y., & Yamamoto, M. (2015). Initial-boundary value problem for multi-term time-fractional diffusion equation with positive constant coefficients. *Applied Mathematica and Computation, 257,* 381–397.

30 Rundell, W., & Zhang, Z. (2018). Recovering an unknown source in a fractional diffusion problem. *Journal of Computational Physics, 368,* 299–314.

31 Asl, N.A., & Rostamy, D. (2019). Identifying an unknown time-dependent boundary source in time-fractional diffusion equation with a non-local boundary condition. *Journal of Computational and Applied Mathematics, 335,* 36–50.

32 Sun, L., Zhang, Y., & Wei, T. (2019). Recovering the time-dependent potential function in a multi-term time-fractional diffusion equation. *Applied Numerical Mathematics, 135,* 228–245.

33 Malik, S.A., & Aziz, S. (2017). An inverse source problem for a two-parameter anomalous diffusion equation with nonlocal boundary conditions. *Computers and Mathematics with applications, 3,* 7–19.

34 Ruzhansky, M., Tokmagambetov, N., & Torebek B.T. (2019). Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations. *J. Inverse Ill-Possed Probl., 27,* 891–911.

35 Ashurov, R., & Muhiddinova, O. (2020). Inverse problem of determining the heat source density for the subdiffusion equation. *Differential equations, 56,* 12, 1550–1563.

36 Ashurov, R,. & Muhiddinova, O. (2021). Initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary elliptic differential operator. *Lobachevskii Journal of Mathematics, 42*, 3.

37 Shuang, Zh., Saima, R., Asia, R., Khadija, K., & Abdullah M.A. (2021). Initial boundary value problems for a multi-term time fractional diffusion equation with generalized fractional derivatives in time. *AIMS Mathematics, 6*, 11, 12114–12132. DOI: 10.3934/math.2021703

38 Kirane, M., Salman, A.M., & Mohammed, A. Al-Gwaiz (2012). An inverse source problem for a two-dimensional time fractional diffusion equation with nonlocal boundary conditions. *Math. Meth. Appl. Sci.* DOI: 10.1002/mma.2661

39 Muhammad, A., & Salman, A.M. (2016). An inverse problem for a family of time fractional diffusion equations. *Inverse Problems in Science and Engineering, 25*, 9, 1299–1322. DOI: 10.1080/17415977.2016.1255738

40 Ashurov, R.R., & Fayziev, Yu.E. Determination of fractional order and source term in a fractional subdiffusion equation. https://www.researchgate.net/publication/354997348

41 Alimov, Sh.O. (2008). Ob odnoi zadache upravleniia protsessom teploobmena [On a control problem associated with the heat exchange process]. *Doklady Rossiiskoi akademii nauk – Proceeding of the Russian Academy of Sciences, 421*, 5, 583–585 [in Russian].

42 Alimov, Sh.O., & Albeverio, S. (2008). On a time-optimal control problem associated with the heat exchange process. *Appl. Math. Optim., 57,* 58–68.

43 Fayziev, Yu.E., Kuchkarov, A., & Nosirova, D. (2017). On one control problem related to the process of heat transfer into rectangles. *Acta NUUz, 2,* 239–244.

44 Fayziev, Yu.E. (2018). On the control of the heat conduction. *IIUM Engineering Journal, 19*, 1, 168–177.

45 Djerbashian, M.M. (1966). *Integralnye preobrazovaniia i predstavleniia funktsii v kompleksnoi oblasti [Integral transforms and representation of functions in the complex domain].* Moscow: Nauka [in Russian].

46 Gorenflo, R., Kilbas, A.A., Mainardi, F., & Rogozin, S.V. (2014). Mittag-Leffler functions, related topics and applications. Springer.

47 Ashurov, R.R., & Fayziev, Yu.E. (2017). O postroenii reshenii lineinogo differentsialnogo uravneniia s drobnymi proizvodnymi i postoiannymi koeffitsientami [On construction of solutions of linear fractional differentional equations with constant coefficients and the fractional derivatives]. *Uzbekskii matematicheskii zhurnal – Uzbek Mathematical Journal, 3,* 3–21 [in Russian].

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On a second-order integro-differential equation with difference kernels and power nonlinearity

The article studies a second-order integro-differential equation with difference kernels and power nonlinearity. A connection is established between this equation and an integral equation of the convolution type, which arises when describing the processes of liquid infiltration from a cylindrical reservoir into an isotropic homogeneous porous medium, the propagation of shock waves in pipes filled with gas and others. Since non-negative continuous solutions of this integral equation are of particular interest from an applied point of view, solutions of the corresponding integro-differential equation are sought in the cone of the space of continuously differentiable functions. Two-sided a priori estimates are obtained for any solution of the indicated integral equation, based on which the global theorem of existence and uniqueness of the solution is proved by the method of weighted metrics. It is shown that any solution of this integro-differential equation is simultaneously a solution of the integral equation and vice versa, under the additional condition on the kernel that any solution of this integral equation is a solution of this integro-differential equation. Using these results, a global theorem on the existence, uniqueness and method of finding a solution to an integro-differential equation is proved. It is shown that this solution can be found by the method of successive approximations of the Picard type and an estimate for the rate of their convergence is established. Examples are given to illustrate the obtained results.

Keywords: integro-differential equation, power nonlinearity, difference kernels, weight metrics method.

Introduction

In this paper, we study the second-order nonlinear integro-differential equation

$$u^{\alpha}(x)=\int\limits_0^x h(x-t)u'(t)\,dt+\int\limits_0^x k(x-t)u''(t)\,dt,\quad x>0,\quad \alpha>1, \tag{1}$$

with initial conditions:

$$u\,(0)=0,\quad u'(0)=0.$$

On the kernels $h(x)$ and $k(x)$ of equation (1) the conditions:

$$h\in C^2\,[0,\infty)\,,\ h''(x)\text{ does not decrease on }\ [0,\infty)\,,\ h(0)=h'(0)=0\text{ and }h''(0)\ge 0, \tag{2}$$

$$k\in C^3\,[0,\infty)\,,\ k'''(x)\text{ does not decrease on }\ [0,\infty)\,,\ k(0)=k'(0)=k''(0)=0\ \text{ and }\ k'''(0)>0 \tag{3}$$

are imposed.

The integro-differential equation (1) is closely related to the convolution type nonlinear integral equation

$$u^{\alpha}(x)=\int\limits_0^x K\,(x-t)\,u\,(t)\,dt,\quad x>0,\quad \alpha>1, \tag{4}$$

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at $K(x) = h'(x) + k''(x)$, arising when describing the processes of fluid infiltration from a cylindrical reservoir into an isotropic homogeneous porous medium [1, 2], the shock wave propagation in pipes filled with gas [3, 4] and others (see [5–7]).

Equations (1) and (4) have the trivial solution $u(x) \equiv 0$. From the theoretical and applied points of view, nontrivial nonnegative continuous solutions of these equations are of special interest. Since, for $0 < \alpha \leq 1$, equations (1) and (4) can only have the trivial solution $u(x) \equiv 0$ in the cone Q of the space $C[0, \infty)$ consisting of all nonnegative continuous functions on $[0, \infty)$, then it makes sense to study them only for $\alpha > 1$. Any solution of equations (1) and (4) in the cone Q, including nontrivial ones, satisfies the condition $u(0) = 0$. In addition, if $u(x) \in Q$ is a nontrivial solution to equation (1) or equation (4), then for any $\delta > 0$ its shifts:

$$u_\delta(x) = u\left(x - \delta\right) \text{ at } x > \delta; \quad u_\delta(x) = 0 \text{ at } x \leq \delta, \text{ and } u_{-\delta}(x) = u\left(x + \delta\right) \text{ at } x > 0$$

are also solutions to these equations, which is verified by direct substitution. Consequently, equations (1) and (4) can have a continuum of nontrivial solutions in the cone Q. Therefore, to make the problem of finding non-trivial solutions of equations (1) and (4) correct and since continuous positive solutions for $x > 0$ are of interest from the applied point of view, we will look for solutions to the integro-differential equation (1) in the cone

$$Q_0^2 = \left\{u(x) : \quad u(x) \in C\left[0, \infty\right) \cap C^2\left(0, \infty\right), \quad u(0) = u'(0) = 0 \text{ and } u(x) > 0 \text{ at } x > 0\right\},$$

and solutions of the integral equation (4) will be sought in the cone

$$Q_0 = \left\{u(x) : \quad u(x) \in C\left[0, \infty\right), \quad u\left(0\right) = 0 \text{ and } u(x) > 0 \text{ at } x > 0\right\}.$$

Conditions (2), (3) imply that the kernel $K(x) = h'(x) + k''(x)$ of equation (4) satisfies the condition:

$$K(x) \in C^1\left[0, \infty\right), \quad K'(x) \text{ does not decrease on } [0, \infty), \quad K(0) = 0 \text{ and } K'(0) > 0. \tag{5}$$

We consider equation (4) based on the two-sided a priori estimates and the weighted metrics method, an analogue of the Bielecki method (see [8; 218]).

In contrast to the Bielecki's method, during the construction of the metric, this study uses an exact a priori estimate from below of the solution to equation (4) as a weight function, which allows us to prove the global existence and uniqueness theorem for the solution to equation (4) without restrictions on the domain of its existence.

For the first time, in works [1, 2], the method of weight metrics was applied to equation (4) under the condition that $K(0) > 0$. In addition, in [1, 2], when constructing the metric, the role of the weight function is played by the difference between the upper and lower a priori estimates, and for the correctness of this metric (so that the denominator does not vanish), a specially overestimated a priori estimate from above is used. As a result, such a metric led to additional restrictions and rather cumbersome calculations in [1, 2].

This paper shows that any solution to equation (1) in Q_0^2 under conditions (2), (3) is simultaneously a solution of equation (4) and vice versa, under the additional condition imposed on the kernel $K(x) = h'(x) + k''(x)$ any solution of equation (4) from Q_0 belongs to the class Q_0^2 and is a solution of equation (1). The main result of the paper is that, using the above relationship between equations (1) and (4), the global existence, uniqueness theorem is proved and the method for solving equation (1) is found. The Picard successive approximation method is applied to solve the considered equation. The convergence rate estimates are established. Examples are provided to illustrate the obtained results.

Main part

Before proceeding to the study of equation (1), we first consider equation (4). The next two lemmas contain information on the properties of non-negative solutions (if they exist) for equation (4).

Lemma 1. Let the condition (5) hold. If $u \in Q_0$ is a solution to the integral equation (4), then the function $u(x)$ does not decrease on $[0,\infty)$ and is twice continuously differentiable for $x > 0$ i.e., $u \in C^2(0,\infty)$.

Proof. Let us first prove that the function $u(x)$ does not decrease on the entire semiaxis $[0,\infty)$, if $u \in Q_0$ and is a solution to equation (4). Let $x_1, x_2 \in [0,\infty)$ be any number, and $x_1 < x_2$. Since by virtue of condition (5), $K'(x) \geq K'(0) > 0$ for any $x \in [0,\infty)$, i.e., the kernel $K(x)$ increases on $[0,\infty)$, then

$$u^\alpha(x_2) - u^\alpha(x_1) \equiv \int\limits_0^{x_1} [K(x_2 - t) - K(x_1 - t)]\, u(t)\, dt + \int\limits_{x_1}^{x_2} K(x_2 - t)\, u(t)\, dt \geq 0$$

consequently, $u(x_2) \geq u(x_1)$, which is required.

Finally, we prove that $u \in C^2(0,\infty)$. Once both parts of identity (4) have been differentiated taking into account $K(0) = 0$, obtain

$$u'(x) = \frac{1}{\alpha} u^{1-\alpha}(x) \int\limits_0^x K'(x-t) u(t)\, dt. \tag{6}$$

This means that $u'(x)$ is continuous at $x > 0$. However, then $u''(x)$ exists and is continuous as the product of two continuously differentiable functions for any $x > 0$. Accordingly, $u \in C^2(0,\infty)$ and the lemma is completely proved.

Lemma 2. Let the condition (5) hold. If a function $u \in Q_0$ and is a solution to the integral equation (4), then for any $x \geq 0$ the inequalities

$$F(x) \equiv c(\alpha) \cdot x^{2/(\alpha-1)} \leq u(x) \leq \left(\frac{\alpha - 1}{\alpha} \int\limits_0^x K(t)\, dt \right)^{1/(\alpha-1)} \equiv G(x), \tag{7}$$

where

$$c(\alpha) = \left(\frac{K'(0) \cdot (\alpha - 1)^2}{2\alpha \cdot (\alpha + 1)} \right)^{1/(\alpha-1)},$$

are valid.

Proof. Let $u(x) \in Q_0$ be a solution of equation (4). Lemma 1 implies that the function $u(x)$ does not decrease on $[0,\infty)$ and $u \in C^2(0,\infty)$.

Prove the estimate $F(x) \leq u(x)$. By differentiating identity (4) twice, in view of condition (5), obtain:

$$(u^\alpha(x))'' = \int\limits_0^x K''(x-t) u(t) dt + K'(0) u(x) \geq K'(0) u(x).$$

Introduce the new function $v(x)$, denoting $u^\alpha(x) = v(x)$. The result is the second-order non-linear differential inequality $v'' \geq K'(0) v^{1/\alpha}$ that does not contain an explicitly independent variable x. By substituting this inequality $v' = p$, $p = p(v)$ (then $v'' = p \cdot p'$) we get $p \cdot p' \geq K'(0) v^{1/\alpha}$. Since

$$v(x) \equiv \int_0^x K(x - t)\, v^{1/\alpha}(t)\, dt \quad \textit{and} \quad K(0) = 0,$$

then

$$v'(x) \equiv \int_0^x K'(x-t)\, v^{1/\alpha}(t)\, dt.$$

Hence, $v(0) = v'(0) = 0$ and $v'(x) \geq 0$. Therefore, writing the previous inequality as $pdp \geq K'(0)v^{1/\alpha}dv$ and integrating from 0 to x, we obtain

$$\frac{[v'(x)]^2}{2} \geq \frac{K'(0))\cdot\alpha}{\alpha+1}[v(x)]^{(\alpha+1)/\alpha} \quad or \quad v'(x) \geq \sqrt{\frac{2K'(0)\cdot\alpha}{\alpha+1}}\cdot[v(x)]^{(\alpha+1)/(2\alpha)}.$$

Now separate the variables and integrate again from 0 to x and obtain

$$\frac{2\alpha}{\alpha-1}\cdot[v(x)]^{(\alpha-1)/(2\alpha)} \geq \sqrt{\frac{2K'(0)\cdot\alpha}{\alpha+1}}\cdot x$$

or

$$[v(x)]^{1/\alpha} \geq \left[\left(\frac{\alpha-1}{2\alpha}\right)^2 \cdot \frac{2K'(0)\cdot\alpha}{\alpha+1}\cdot x^2\right]^{1/(\alpha-1)} \equiv F(x).$$

Recalling that $u^\alpha(x) = v(x)$, from the last inequality we obtain the provable lower bound: $u(x) \geq F(x)$.

It remains to prove the upper estimate, i.e. $u(x) \leq G(x)$. Since $K(x)$ and $u(x)$ are nondecreasing functions, by applying the Chebyshev inequality (17.6) [6] in (4) obtain:

$$u(x) \leq \left(\int_0^x K(t)\, u(t)\, dt\right)^{\frac{1}{\alpha}} \qquad for\ any\ \ x>0. \tag{8}$$

Hence,

$$K(x)u(x)\left(\int_0^x K(t)\, u(t)\, dt\right)^{-1/\alpha} \leq K(x).$$

Therefore, once the integration has taken place, get:

$$\left(\int_0^x K(t)\, u(t)\, dt\right)^{1/\alpha} \leq \left(\frac{\alpha-1}{\alpha}\right)^{1/(\alpha-1)}\left(\int_0^x K(t)\, dt\right)^{1/(\alpha-1)} \equiv G(x). \tag{9}$$

Using estimate (9) by inequality (8) obtain: $u(x) \leq G(x)$, which is required.

Example 1. The function

$$u^*(x) = \left(\frac{(\alpha-1)^2}{2\alpha\cdot(\alpha+1)}\right)^{1/(\alpha-1)} x^{2/(\alpha-1)}$$

is a solution to the equation (4) for $K(x) = x$.

Example 1 shows that $F(x) \equiv u^*(x)$ at $K(x) = x$, i.e., a priori lower bound of the solution to the equation (4) is unimprovable.

Obviously, Lemma 2 implies that the solution to equation (4) should be sought in the class

$$P = \{u(x):\quad u(x) \in C\,[0,\infty) \quad and \quad F(x) \leq u(x) \leq G(x)\},$$

as $F(0) = G(0) = 0$ and $F(x) > 0$ at $x > 0$.

Now consider the operator T:

$$(Tu)(x) = \left(\int_0^x K(x-t)u(t)dt \right)^{1/\alpha}, \qquad x > 0.$$

Lemma 3. The operator T transforms the class P into itself.

Proof. Assume $u(x) \in P$ is an arbitrary function. Consequently, we have to prove that $(Tu)(x) \in P$. By Theorem 17.9 [6] $(Tu)(x) \in C[0,\infty)$. It remains to prove that $F(x) \le (Tu)(x) \le G(x)$. As $u(x) \ge F(x)$ and by condition (5) $K'(x) \ge K'(0) > 0$, $K(0) = 0$, then

$$[(Tu)(x)]^\alpha \ge \int_0^x K(x-t)c(\alpha)t^{2/(\alpha-1)}dt = c(\alpha)\frac{\alpha-1}{\alpha+1}\int_0^x K(x-t)dt^{(\alpha+1)/(\alpha-1)} =$$

$$= c(\alpha)\frac{\alpha-1}{\alpha+1}\int_0^x K'(x-t)t^{(\alpha+1)/(\alpha-1)}dt \ge c(\alpha)\frac{\alpha-1}{\alpha+1}K'(0)\int_0^x t^{(\alpha+1)/(\alpha-1)}dt \equiv [F(x)]^\alpha,$$

that is $(Tu)(x) \ge F(x)$.

On the other hand, since $u(x) \le G(x)$ then taking into account condition (6) and the Chebyshev integral inequality (17.6) [6], where the role of function $u(x)$ is already played by the function $G(x)$, which is non-decreasing either (see the proof of Theorem 17.12 [6]) we get:

$$[(Tu)(x)]^\alpha \le \int_0^x K(t)\,G(t) \equiv [G(x)]^\alpha, \qquad i.e. \qquad (Tu)(x) \le G(x).$$

Lemma 3 is proved.

Now consider the class

$$P_b = \{u(x): \quad u(x) \in C[0,b] \ \text{ and } \ F(x) \le u(x) \le G(x)\},$$

where $b > 0$ is any number, and introduce the metric in it ρ_b by imposing $\forall u(x), v(x) \in P_b$:

$$\rho_b(u,v) = \sup_{0<x\le b} \frac{|u(x)-v(x)|}{x^{2/(\alpha-1)}e^{\beta x}}, \quad \text{where} \quad \beta > 0 \ \text{ is any number.}$$

It is proved directly in view of the equalities $F(0) = G(0) = 0$ and the complete metric space $C[0,b]$ with Chebyshev metric that the pair (P_b, ρ_b) forms a complete metric space (see Theorem 17.13 [6]).

Choose a number $\mu \in (0,b)$ so that condition

$$K'(\mu) < \alpha \cdot K'(0) \tag{10}$$

is satisfied and sets

$$\beta = \frac{1}{K'(0)} \sup_{\mu \le x \le b} \frac{K'(x) - K'(0)}{x}.$$

Then, by lemma 18.5 [6], we obtain that the inequality

$$K(x)e^{-\beta x} \le x \cdot K'(\mu) \tag{11}$$

holds.

Theorem 1. If the kernel $K(x)$ satisfies condition (6), then equation (4) has a unique solution $u^*(x)$ in the cone Q_0 (and in P_b for any $b > 0$). This solution can be found by the successive approximations method using the formula $u_n = Tu_{n-1}$, $n \in \mathbb{N}$, which converge to it according to the metric ρ_b at any $b < \infty$, and the convergence rate estimate

$$\rho_b\left(u_n, u^*\right) \leq \frac{\gamma^n}{1-\gamma} \rho_b\left(Tu_0, u_0\right) \tag{12}$$

is valid, where $\gamma = K'(\mu) / \left[\alpha K'(0)\right] < 1$, and $u_0(x) \in P_b$ the initial approximation (on arbitrary function).

Proof. Write equation (4) as the operator equation $u = Tu$. First, show that the operator T acting according to Lemma 3 from P_b to P_b is contractive.

Let $u, v \in P_b$ be arbitrary functions. It is clear that

$$|u(x) - v(x)| \leq x^{2/(\alpha-1)} e^{\beta x} \rho_b\left(u, v\right).$$

Therefore, using inequality (11) get

$$\left| \int\limits_0^x K(x-t)\left[u\left(t\right) - v(t)\right] dt \right| \leq \rho_b(u, v) \int\limits_0^x K(x-t) e^{-\beta(x-t)} e^{\beta x} t^{2/(\alpha-1)} dt \leq$$

$$\leq e^{\beta x} K'(\mu) \rho_b(u, v) \int\limits_0^x \left(x-t\right) t^{2/(\alpha-1)} dt = \frac{(\alpha-1)^2 K'(\mu)}{2\alpha\left(\alpha+1\right)} e^{\beta x} \rho_b(u, v) x^{2\alpha/(\alpha-1)}.$$

Next employing the Lagrange theorem (finite-increments formula) in view of the later estimate (see the proof of Theorem 17.14 [6]) obtain

$$\left|\left(Tu\right)(x) - \left(Tv\right)(x)\right| \leq \frac{1}{\alpha} \frac{\left| \int\limits_0^x K(x-t)\left[u\left(t\right) - v\left(t\right)\right] dt \right|}{\left[F(x)\right]^{\alpha-1}} \leq \frac{K'(\mu)}{\alpha K'(0)} e^{\beta x} x^{2/(\alpha-1)} \rho_b\left(u, v\right),$$

whence

$$\rho_b\left(Tu, Tv\right) \leq \frac{K'(\mu)}{\alpha \cdot K'(0)} \cdot \rho_b\left(u, v\right), \tag{13}$$

i.e., the operator T, by condition (10) is a contractive operator. Hence, based on the contraction mapping principle, the equation $u = Tu$ has the unique solution $u^*(x) \in P_b$, which can be found by the formula $u_n = Tu_{n-1}$, $n \in \mathbb{N}$, and the estimate (12) is valid.

The only thing left to show is that equation (4) has a unique solution in the cone Q_0. Suppose $P_\infty = \cup_{b>0} P_b$. Since equation (4) has the unique solution in P_b at any $b > 0$ and the contraction coefficient in (13) does not depend on b, equation (4) has the unique solution $u^*(x)$ in P_∞. Since any solution of equation (4) in Q_0 satisfies a priori estimates (7), this solution will also be the only one in Q_0.

Theorem 1 is proved.

Let us finally proceed to the study of integro-differential equation (1).

The following lemma establishes the relationship between integro-differential equation (1) and integral equation (4).

Lemma 4. Let conditions (2) and (3) be satisfied. Then any solution of equation (1) in the cone Q_0^2 is a solution to integral equation (4) in the cone Q_0. Conversely, if conditions (2), (3), and the additional condition

$$\lim_{x\to 0}\frac{\int\limits_0^x K'(x-t)\left[\int\limits_0^t K\,(s)\,ds\right]^{1/(\alpha-1)}dt}{\left[\int\limits_0^x K\,(x-t)\cdot t^{2/(\alpha-1)}dt\right]^{(\alpha-1)/\alpha}}=0, \tag{14}$$

are satisfied, then any solution of integral equation (4) in the cone Q_0 belongs to the cone Q_0^2 and is a solution of equation (1).

Proof. Initially, prove the first part of the lemma. Assume $u(x)\in Q_0^2$ and is a solution to equation (1). Then, applying the integration by parts formula twice by identity (1) taking into account conditions (2) and (3), obtain:

$$u^\alpha(x)=\int\limits_0^x h(x-t)\,du(t)+\int\limits_0^x k(x-t)\,du'(t)=\int\limits_0^x u(t)h'(x-t)dt+\int\limits_0^x u'(t)k'(x-t)dt=$$

$$=\int\limits_0^x h'(x-t)u(t)dt+\int\limits_0^x u(t)k''(x-t)dt=\int\limits_0^x K(x-t)u(t)dt,$$

i.e. $u(x)\in Q_0$ and is a solution of integral equation (4).

Next, prove the second part of the lemma. Let $u(x)\in Q_0$ be the solution of integral equation (4). Therefore by lemma 1, $u(x)$ does not decrease on $[0,\infty)$ and is twice continuously differentiable on $(0,\infty)$, i.e. $u\in C^2\,(0,\infty)$ and satisfies the inequalities $F(x)\le u(x)\le G(x)$. Prove that $u'(0)=0$. By identity (4) in view of condition (5) get

$$\alpha u^{\alpha-1}(x)u'(x)=\int\limits_0^x K'(x-t)u(t)dt+K(0)u(x)=\int\limits_0^x K'(x-t)u(t)\,dt,$$

whence

$$u'(x)=\frac{\int\limits_0^x K'(x-t)u(t)\,dt}{\alpha\cdot[u^\alpha(x)]^{(\alpha-1)/\alpha}}=\frac{\int\limits_0^x K'(x-t)u(t)\,dt}{\alpha\cdot\left[\int\limits_0^x K(x-t)u(t)dt\right]^{(\alpha-1)/\alpha}}\ge 0. \tag{15}$$

Employing a priori estimates (7), by (15) obtain:

$$0\le u'(x)\le\frac{\int\limits_0^x K'(x-t)G(t)\,dt}{\alpha\cdot\left[\int\limits_0^x K(x-t)F(t)\,dt\right]^{(\alpha-1)/\alpha}}=\frac{\int\limits_0^x K'(x-t)\left[\frac{\alpha-1}{\alpha}\cdot\int\limits_0^t K(s)\,ds\right]^{1/(\alpha-1)}dt}{\alpha\cdot\left[\int\limits_0^x K(x-t)c(\alpha)t^{2/(\alpha-1)}dt\right]^{(\alpha-1)/\alpha}}=$$

$$=\left[\frac{\alpha-1}{\alpha}\right]^{1/(\alpha-1)}\frac{1}{\alpha\cdot[c(\alpha)]^{(\alpha-1)/\alpha}}\cdot\frac{\int\limits_0^x K'(x-t)\left[\int\limits_0^t K(s)\,ds\right]^{1/(\alpha-1)}dt}{\left[\int\limits_0^x K(x-t)\cdot t^{2/(\alpha-1)}dt\right]^{(\alpha-1)/\alpha}}\to 0\quad\text{at}\quad x\to 0,$$

by virtue of condition (14). Therefore $u'(0)=0$.

Thus, $u\in C^2\,[0,\infty)$, $u\,(0)=u'(0)=0$ and $u(x)>0$ at $x>0$, i.e. $u\in Q_0^2$. All that remains is to prove that $u(x)$ is a solution of equation (1). Employing the equality $K(x)=h'(x)+k''(x)$, commutative property of convolution and applying the integration-by-parts formula twice, taking into account conditions (2) and (3), by identity (4) we obtain:

$$u^{\alpha}(x) = \int\limits_0^x [h'(t) + k''(t)]u(x-t)\,dt = \int\limits_0^x u(x-t)dh(t) + \int\limits_0^x u(x-t)dk'(t) =$$

$$= \int\limits_0^x h(t)u'(x-t)dt + \int\limits_0^x k'(t)u'(x-t)\,dt = \int\limits_0^x h(x-t)u'(t)\,dt + \int\limits_0^x u'(x-t)\,dk(t) =$$

$$= \int\limits_0^x h(x-t)u'(t)dt + \int\limits_0^x k(t)u''(x-t)dt = \int\limits_0^x h(x-t)u'(t)\,dt + \int\limits_0^x k(x-t)u''(t)\,dt,$$

i.e. $u(x)$ is a solution of equation (1).

Lemma 4 is proved.

Lemma 4 implies that under conditions (2), (3), and (14) integro-differential equation (1) and integral equation (4) are simultaneously solvable or not, while they have the same solutions. Therefore, based on Theorem 1, the following fundamental theorem is true.

Theorem 2. If conditions (2), (3), and (14) are satisfied, then equation (1) has a unique solution u^* in Q_0^2 (and in the space P_b at any $b > 0$). This solution can be found in the space P_b using the Picard successive approximation method $u_n = Tu_{n-1}$, $n \in \mathbb{N}$, which converge to it according to the metric ρ_b at any $b < \infty$, and estimate convergence rate (12) is valid.

Example 2. When $\alpha > 1$, $h(x) = x^2$ and $k(x) = x^3$, i.e., at $K(x) = 8x$, in the cone Q_0 integral equation (4) has the unique solution

$$u^*(x) = \left[\frac{4(\alpha-1)^2}{\alpha(\alpha+1)}\right]^{1/(\alpha-1)} x^{2/(\alpha-1)}.$$

When $K(x) = 8x$ condition (14) takes the form

$$A(\alpha) \cdot \lim_{x \to 0} x^{(3-\alpha)/(\alpha-1)} = 0, \quad \text{where} \quad A(\alpha) = \frac{8 \cdot 4^{\frac{1}{\alpha-1}}(\alpha-1)}{\alpha+1}\left[\frac{\alpha(\alpha+1)}{4(\alpha-1)^2}\right]^{(\alpha-1)/\alpha}.$$

Hence, for $1 < \alpha < 3$, the function $u^*(x)$ is also the unique solution of integro-differential equation (1) in the cone Q_0^2.

In particular, when $\alpha = 2$, $h(x) = x^2$ and $k(x) = x^3$, equations (1) and (4) have the unique solution $u(x) = \frac{2}{3}x^2$ in the cones Q_0^2 and Q_0, respectively.

Note also that $u^* \in Q_0^2$ only if $1 < \alpha < 3$, since

$$(u^*(x))' = \left[\frac{4(\alpha-1)^2}{\alpha(\alpha+1)}\right]^{1/(\alpha-1)} \frac{2}{\alpha-1} x^{(3-\alpha)/(\alpha-1)}$$

and therefore $u^* \notin Q_0^2$ at $\alpha \geq 3$. This shows that condition (14) is essential to the validity of Lemma 4 and Theorem 2.

Following the monograph [6; 211] it can be proved that for $0 < \alpha < 1$, as in the case of the corresponding linear equations obtained for $\alpha = 1$, equations (1) and (4) have only a trivial solution $u(x) \equiv 0$ in the cone of the space of functions continuous on $C \in [0, \infty)$ consisting of non-negative functions continuous on the half-axis $[0, \infty)$.

Consequently, based on the results obtained non-linear homogeneous integral and integro-differential equations type (1) and (4) except for the trivial solution $u(x) \equiv 0$ can have the non-trivial solution

$u(x) \neq 0$ at $\alpha > 1$ strictly positive at $x > 0$. This is the fundamental difference between the theory of the considered nonlinear equations and the well-developed so far theory of the corresponding linear homogeneous integral and integro-differential equations, which have only the trivial solution $u(x) \equiv 0$. In addition, the theory of nonlinear equations differs from the theory of the corresponding linear equations not only in the obtained results but also in research methods related to the choice of space and nonlinearity properties.

In conclusion, following the works [9–12], it is possible to study integro-differential equations of the form (1) with variable coefficients and inhomogeneities in the linear part, as well as systems of such equations. Other methods for studying nonlinear equations of the convolution type are given in many research works, such as [13], [14].

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References

1 Okrasinski W. On the existence and uniqueness of nonnegative solutions of a certain non-linear convolution equation / W. Okrasinski // Annal. Polon. Math. — 1979. — 36. — No. 1. — P. 61–72.

2 Okrasinski W. On a non-linear convolution equation occurring in the theory of water percolation / W. Okrasinski // Annal. Polon. Math. — 1980. — 37. — No. 3. — P. 223–229.

3 Keller J.J. Propagation of simple non-linear waves in gas filled tubes with friction / J.J. Keller // Z. Angew. Math. Phys. — 1981. — 32. — No. 2. — P. 170–181.

4 Schneider W.R. The general solution of a nonlinear integral equation of the convolution type / W.R. Schneider // Z. Angew. Math. Phys. — 1982. — 33. — No. 1. — P. 140–142.

5 Askhabov S.N. Nonlinear convolution type equations / S.N. Askhabov, M.A. Betilgiriev // Seminar Analysis Operat. Equat. Numer. Anal. 1989/90. Karl-Weierstras-Institut fur Mathematik. Berlin. — 1990. — P. 1–30.

6 Асхабов С.Н. Нелинейные уравнения типа свертки / С.Н. Асхабов. — М.: Физматлит, 2009. — 304 с.

7 Brunner H. Volterra integral equations: an introduction to the theory and applications / H. Brunner. — Cambridge: Cambridge Univ. Press, 2017. — 387 p.

8 Эдвардс Р. Функциональный анализ / Р. Эдвардс. — М.: Мир, 1969. — 1072 с.

9 Асхабов С.Н. Интегро-дифференциальное уравнение типа свёртки со степенной нелинейностью и неоднородностью в линейной части / С.Н. Асхабов // Дифференц. уравнения. — 2020. — 56. — № 6. — С. 786–795.

10 Асхабов С.Н. Интегро-дифференциальное уравнение типа свертки со степенной нелинейностью и переменным коэффициентом / С.Н. Асхабов // Дифференц. уравнения. — 2021. — 57. — № 3. — С. 387–398.

11 Askhabov S.N. Nonlinear convolution integro-differential equation with variable coefficient / S.N. Askhabov // Fractional Calculus and Applied Analysis. — 2021. — 24. — No. 3. — P. 848–864.

12 Асхабов С.Н. Система интегро-дифференциальных уравнений типа свертки со степенной нелинейностью / С.Н. Асхабов // Сиб. журн. индустриальной математики. — 2021. — 24. — № 3. — С. 5–18.

13 Gripenberg G. Volterra integral and functional equations / G. Gripenberg, S.-O. Londen, O. Staffans. — Cambridge: Cambridge University Press, 1990. — 701 p.

14 Асхабов С.Н. Об одном классе нелинейных интегральных уравнений типа свертки / С.Н. Асхабов, Х.Ш. Мухтаров // Дифференц. уравнения. — 1987. — 23. — № 3. — С. 512–514.

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Айырымдық ядролары және дәрежелік сызықтық еместігі бар екінші ретті интегро-дифференциалдық теңдеу туралы

Мақалада айырымдық ядролары және дәрежелік сызықтық еместігі бар екінші ретті интегро-дифференциалдық теңдеу зерттелген. Осы теңдеудің цилиндрлік резервуардан изотропты біртекті кеуекті ортаға сұйықтықтың инфильтрациясы, газ толтырылған құбырлардағы соққы толқындарының таралуы және т.б. процестерін сипаттау кезінде туындайтын үйірткі түріндегі интегралдық теңдеумен байланысы анықталды. Қолданбалы жағынан осы интегралдық теңдеудің теріс емес үзіліссіз шешімдері ерекше қызығушылық тудырады, сондықтан интегро-дифференциалдық теңдеудің сәйкес шешімдері үзіліссіз-дифференциалданатын кеңістік конусында ізделінеді. Көрсетілген интегралдық теңдеудің кез келген шешімі үшін екі жақты априорлы бағалар алынған, оның негізінде шешімнің бар болуы мен бірегейлігінің ғаламдық теоремасы салмақты метрика әдісімен дәлелденген. Берілген интегралдық-дифференциалдық теңдеудің кез келген шешімі бір мезгілде интегралдық теңдеудің шешімі болатыны және керісінше ядроға қосымша шарт қойылған кезде осы интегралдық теңдеудің кез келген шешімі осы интегралдық-дифференциалдық теңдеудің шешімі болатыны көрсетілген. Осы нәтижелерді пайдалана отырып, интегро-дифференциалдық теңдеудің бар болуы, бірегейлігі және шешімін табу әдісі туралы ғаламдық теорема дәлелденді. Бұл шешімді дәйекті Пикард типті жуықтаулар әдісімен табуға болатыны көрсетіліп, олардың жинақтылық жылдамдығына баға белгіленген. Алынған нәтижелерді көрсету үшін мысалдар келтірілген.

Кілт сөздер: интегро-дифференциалдық теңдеу, дәрежелік сызықтық еместік, айырымдық ядролар, салмақты метрика әдісі.

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Об интегро-дифференциальном уравнении второго порядка с разностными ядрами и степенной нелинейностью

В статье рассмотрено интегро-дифференциальное уравнение второго порядка с разностными ядрами и степенной нелинейностью. Установлена связь этого уравнения с интегральным уравнением типа свёртки, возникающим при описании процессов инфильтрации жидкости из цилиндрического резервуара в изотропную однородную пористую среду, распространения ударных волн в трубах, наполненных газом, и других. Поскольку, с прикладной точки зрения, особый интерес представляют неотрицательные непрерывные решения этого интегрального уравнения, решения соответствующего интегро-дифференциального уравнения разыскиваются в конусе пространства непрерывно-дифференцируемых функций. Получены двусторонние априорные оценки для любого решения указанного интегрального уравнения, на основе которых методом весовых метрик доказана глобальная теорема существования и единственности решения. Показано, что любое решение данного интегро-дифференциального уравнения является одновременно и решением интегрального уравнения, и, обратно, при дополнительном условии на ядро, что любое решение этого интегрального уравнения является решением данного интегро-дифференциального уравнения. Используя указанные результаты, доказана глобальная теорема о существовании, единственности и способе нахождения решения интегро-дифференциального уравнения. Показано, что это решение можно найти методом последовательных приближений пикаровского типа, при этом и установлена оценка скорости их сходимости. Приведены примеры, иллюстрирующие полученные результаты.

Ключевые слова: интегро-дифференциальное уравнение, степенная нелинейность, разностные ядра, метод весовых метрик.

References

1 Okrasinski, W. (1979). On the existence and uniqueness of nonnegative solutions of a certain non-linear convolution equation. *Annales Polonici Mathematici, 36*(1), 61–72.

2 Okrasinski, W. (1980). On a non-linear convolution equation occurring in the theory of water percolation. *Annales Polonici Mathematici, 37*(3), 223–229.

3 Keller, J.J. (1981). Propagation of simple non-linear waves in gas filled tubes with friction. *Zeitschrift für angewandte Mathematik und Physik, 32*(2), 170–181.

4 Schneider, W.R. (1982). The general solution of a nonlinear integral equation of the convolution type. *Zeitschrift für angewandte Mathematik und Physik, 33*(1), 140–142.

5 Askhabov, S.N., & Betilgiriev, M.A. (1990). Nonlinear convolution type equations. *Seminar Analysis Operat. Equat. Numer. Anal.* 1989/90. Karl-Weierstras-Institut fur Mathematik, Berlin.

6 Askhabov, S.N. (2009). *Nelineinye uravneniia tipa svertki [Nonlinear equations of convolution type].* Moscow: FIZMATLIT, 304 [in Russian].

7 Brunner, H. (2017). *Volterra integral equations: an introduction to the theory and applications.* Cambridge: Cambridge Univ. Press, 387.

8 Edwards, R. (1969). *Funktsionalnyi analiz [Functional analysis].* Moscow: Mir, 1072 [in Russian].

9 Askhabov, S.N. (2020). Integro-differentsialnoe uravnenie tipa svertki so stepennoi nelineinostiu i neodnorodnostiu v lineinoi chasti [Integro-differential equation of the convolution type with a power nonlinearity and an inhomogeneity in the linear part]. *Differentsialnye uravneniia — Differential Equations, 56*(6), 775–784 [in Russian].

10 Askhabov, S.N. (2020). Integro-differentsialnoe uravnenie tipa svertki so stepennoi nelineinostiu i peremennym koeffitsientom [Integro-differential equation of the convolution type with a power nonlinearity and a nonlinear coefficient]. *Differentsialnye uravneniia — Differential Equations, 57*(3), 366–378 [in Russian].

11 Askhabov, S.N. (2021). [Nonlinear convolution integro-differential equation with variable coefficient. *Fractional Calculus and Applied Analysis, 24*(3), 848–864.

12 Askhabov, S.N. (2021). Sistema integro-differentsialnykh uravnenii tipa svertki so stepennoi nelineinostiu [System of integro-differential equations of convolution type with power nonlinearity]. *Sibirskii zhurnal industrialnoi matematiki — Journal of Applied and Industrial Mathematics, 24*(3), 5–18 [in Russian].

13 Gripenberg, G., Londen, S.-O., & Staffans, O. (1990). *Volterra integral and functional equations.* Cambridge: Cambridge University Press, 701.

14 Askhabov, S.N., & Mukhtarov, Kh.Sh. (1987). Ob odnom klasse nelineinykh integralnykh uravnenii tipa svertki [On a class of nonlinear integral equations of convolution type. *Differentsialnye uravneniia — Differential Equations, 23*(3), 512–514 [in Russian].

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Boundary control problem for a hyperbolic equation loaded along one of its characteristics

This paper investigates the unique solvability of the boundary control problem for a one-dimensional wave equation loaded along one of its characteristic curves in terms of a regular solution. The solution method is based on an analogue of the d'Alembert formula constructed for this equation. We point out that the domain of definition for the solution of DE, when the initial and final Cauchy data given on intervals of the same length is a square. The side of the squire is equal to the interval length. The boundary controls are established by the components of an analogue of the d'Alembert formula, which, in turn, are uniquely established by the initial and final Cauchy data. It should be noted that the normalized distribution and centering are employed in the final formulas of sought boundary controls, which is not typical for initial and boundary value problems initiated by equations of hyperbolic type.

Keywords: hyperbolic equation, distributed oscillatory system, damping problem, gas/liquid flows, loaded equation, initial conditions, boundary conditions, analogue of the d'Alembert formula, boundary controls, normal distribution, distribution function.

Introduction

Let an oscillatory system be described by equation

$$u_{xx} - u_{tt} = \lambda\, u\left(\frac{x+t}{2},\, \frac{x+t}{2}\right) \tag{1}$$

with the following initial and boundary conditions

$$u(x,\,0) = \varphi_0(x), \quad u_t(x,\,0) = \psi_0(x), \qquad 0 \leqslant x \leqslant l, \tag{2}$$

$$u(0,\,t) = \mu(t), \quad u(l,\,t) = \nu(t), \qquad 0 \leqslant t \leqslant T, \tag{3}$$

where λ is an arbitrary real number.

The boundary control problem involves searching admissible boundary values $\mu(t)$ and $\nu(t)$, that in a minimum time interval move the oscillatory system from the initial (2) to a predetermined final phase

$$u(x,\,T) = \varphi_1(x), \quad u_t(x,\,T) = \psi_1(x), \qquad 0 \leqslant x \leqslant l. \tag{4}$$

Control (1) belongs to the class of loaded differential equations [1]. The point $\left(\frac{x_0+t_0}{2},\, \frac{x_0+t_0}{2}\right)$ lies on the characteristic curve $x - y = 0$ of equation (1), for an arbitrary point $(x_0,\, t_0) \in \mathbb{R}^2$. The point $(x_0,\, t_0)$ also moves to the point $\left(\frac{x_0+t_0}{2},\, \frac{x_0+t_0}{2}\right)$ along the characteristic curve $x + y = x_0 + y_0$ of (1). The boundary control problem for the equation (1) with the right-hand side of the form $\lambda\, u_{tt}(x_0,\, t)$, which, according to [2], is called an essentially loaded equation, was studied in [3], [4]. Boundary value problems for hyperbolic equations with a load along one of the characteristic curves are studied in [5], [6]. For $\lambda = 0$, the formulated problem is fully investigated in [7]. Here important special cases

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of the same problem were investigated, namely, the problem of complete oscillation excitation and damping. The boundary control problem for equation (1) for $\lambda = 0$ with various nonlocal, including integral form conditions, were studied in [8–12].

A number of specific formulations for distributed control problems, with oscillatory nature of the movement, are described in detail in the monograph [13], and moreover various solution methods are proposed. For example, the damping fluid flow pulsation problem in automated long main pipelines design and pipeline irrigation systems [14], [15]. As earliest works devoted to the study of boundary control problems (1) at $\lambda = 0$ the woks [16–18] are worth mentioning.

There are following main results in the work:

1. Necessary and sufficient conditions are established for the functions $\varphi_0(x)$, $\psi_0(x)$, $\varphi_1(x)$, $\psi_1(x)$, which ensure the existence of the desired boundary values.

$$\psi_0(l) + \varphi_0'(l) - \psi_1(0) - \varphi_1'(0) = 0, \tag{5}$$

$$\psi_0(0) - \varphi_0'(0) - \psi_1(l) + \varphi_1'(l) = 0, \tag{6}$$

$$\lambda\left[\psi_0(l) + \varphi_0'(l) + \psi_1(0) + \varphi_1'(0)\right] = 0, \tag{7}$$

$$\varphi_0''(l) + \psi_0'(l) - \varphi_1''(0) - \varphi_1'(0) = 0, \tag{8}$$

$$\varphi_0''(0) - \psi_0'(0) - \varphi_1''(l) + \varphi_1'(l) = 0, \tag{9}$$

$$\varphi_0(0) - \varphi_1(l) + [\varphi_0(l) - \varphi_1(0)]\, e^{\frac{\lambda l^2}{8}} +$$
$$+\frac{1}{2}\int_0^l e^{\frac{\lambda t^2}{8}}\,[\psi_0(t) - \psi_1(l-t)]\,dt - \frac{\lambda}{4}\int_0^l t\, e^{\frac{\lambda t^2}{8}}\,[\varphi_0(t) - \varphi_1(l-t)]\,dt = 0. \tag{10}$$

2. Under conditions (5)–(10), an explicit analytical form of the sought boundary controls is found

$$\mu(t) = \varphi_1(l-t) + F_1(t,\,\lambda) - \frac{\lambda}{4}\,t\,F_2(t,\,\lambda), \tag{11}$$

$$\nu(t) = \varphi_0(l-t) + F_1(l-t,\,\lambda) - \frac{\lambda}{4}\,t\,F_2(l-t,\,\lambda), \tag{12}$$

where

$$F_1(t,\,\lambda) = \frac{1}{2}\,e^{-\frac{\lambda t^2}{8}}\,[\varphi_0(0) - \varphi_1(l)] + \frac{1}{2}\,[\varphi_0(t) - \varphi_1(l-t)] -$$
$$-\frac{\lambda}{8}\int_0^t \xi\, e^{\frac{\lambda\left(t^2-\xi^2\right)}{8}}\,[\varphi_0(\xi) - \varphi_1(l-\xi)]\,d\xi - \frac{1}{8}\int_0^t e^{\frac{\lambda\left(t^2-\xi^2\right)}{8}}\,[\psi_0(\xi) - \psi_1(l-\xi)]\,d\xi,$$

$$F_2(t,\,\lambda) = \sqrt{\frac{2\,\pi}{\lambda}}\,\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,t\right)[\varphi_0(0) - \varphi_1(l)] +$$
$$+\frac{1}{2}\int_0^t [\varphi_0(\xi) - \varphi_1(l-\xi)]\,d\xi -$$
$$-\sqrt{\frac{\lambda\,\pi}{8}}\int_0^t \left[\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,t\right) - \Phi_0\left(\frac{\sqrt{\lambda}}{2}\,\xi\right)\right]\xi\, e^{\frac{\lambda \xi^2}{8}}\,[\varphi_0(\xi) - \varphi_1(l-\xi)]\,d\xi +$$

$$+\sqrt{\frac{2\,\pi}{\lambda}}\int_0^t\left[\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,t\right)-\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,\xi\right)\right]e^{\frac{\lambda\xi^2}{8}}\left[\psi_0(\xi)-\psi_1(l-\xi)\right]d\xi,$$

where $\Phi_0(x)=\frac{1}{\sqrt{2\,\pi}}\int_0^x e^{-\frac{\xi^2}{2}}\,d\xi$ is the distribution function for normalizing and centering processes.

1 Main part

1.1 An analogue of the d'Alembert formula

Using the characteristic variables $\xi=x-t$, $\eta=x+t$ equation (1) is written out as

$$v_{\xi\eta}=\frac{\lambda}{4}\,v(0,\,\eta), \tag{13}$$

where $v(\xi,\,\eta)=u\Big(\frac{\xi+\eta}{2},\,\frac{\eta-\xi}{2}\Big)$.

Any solution to equation (13) is a solution to the following loaded integral equation.

$$v(\xi,\,\eta)-\frac{\lambda}{4}\,\xi\int_a^\eta v(0,\,t)\,dt=f(\xi)+g(\eta), \tag{14}$$

where $f(\xi)$ and $g(\eta)$ are arbitrary twice continuously differentiable functions. Since for $\xi=0$ by (14) it follows that $v(0,\,\eta)=f(0)+g(\eta)$, then solution (14) takes the form

$$v(\xi,\,\eta)=f(\xi)+g(\eta)+\frac{\lambda}{4}\,\xi\int_a^\eta\left[f(0)+g(t)\right]dt.$$

Replacing $P(\eta)=\int_a^\eta\left[f(0)+g(t)\right]dt$ in the last formula and then renaming $P(\eta)$ by $g(\eta)$, we obtain

$$v(\xi,\,\eta)=f(\xi)-f(0)+g'(\eta)+\frac{\lambda}{4}\,\xi\,g(\eta).$$

Or when using the old coordinates, get

$$u(x,\,t)=f(x-t)-f(0)+g'(x+t)+\frac{\lambda}{4}\,(x-t)\,g(x+t). \tag{15}$$

We call formula (15) an analogue of the d'Alembert formula for equation (1).

Formula (15) below is more convenient for further use. It should be noted that in (15) the functions $f(t)$ and $g'(t)$ are twice continuously differentiable.

1.2 Searching algorithm for $\mu(t)$ and $\nu(t)$

Before searching for boundary controls, let us remember that when $\lambda=0$ the minimum time t required for the desired control is uniquely equal to l. As noted in [12], this time is determined by the characteristics of the initial equation that simulates the oscillation process. From a mathematical point of view, problem (2), (4) for equation (1) with $\lambda=0$ in the rectangle $(0,\,l)\times(0,\,T)$ has a unique solution if and only if $T=l$. If $T<l$, the problem is overdetermined, $T>l$, the problem is underdetermined. In our case when $\lambda\neq 0$ the equation commands the condition $T=l$. This is due to the fact that for any arbitrary point $(x,\,t)\in(0,\,l)\times(0,\,T)$, belonging to the point $\left(\frac{x+t}{2},\,\frac{x+t}{2}\right)\in(0,\,l)\times(0,\,T)$ is possible if and only if $T=l$. Therefore, further we assume that $T=l$.

We introduce the following equation for the functions $f(x)$ and $g(x)$ in formula (15).

$$f(x)=\begin{cases} f_0(x), & x\in[0,\,l]\,,\\ f_1(x), & x\in[-l,\,0]\,,\end{cases} \tag{16}$$

$$g(x)=\begin{cases} g_0(x), & x\in[0,\,l]\,,\\ g_1(x), & x\in[l,\,2\,l]\,.\end{cases} \tag{17}$$

We call the function $u(x,\,t)\in C^1\left(\overline{\Omega}\right)\cap C^2\left(\Omega\right)$ a regular solution to equation (1) where $\Omega=(0,\,l)\times(0,\,l)$.

Letting (15) satisfy (3) and taking into account (16), (17), we obtain

$$\mu(t)=f_1(-t)-f_0(0)+g_0'(t)-\frac{\lambda}{4}\,t\,g_0(t),\quad t\in[0,\,l]\,, \tag{18}$$

$$\nu(t)=f_0(l-t)-f_0(0)+g_0'(l+t)+\frac{\lambda}{4}\,(l-t)\,g_0(l+t),\quad t\in[0,\,l]\,. \tag{19}$$

Letting (15) satisfy (2) and (4) for $T=l$ we obtain

$$f_0(x)-f_0(0)+g_0'(x)+\frac{\lambda}{4}\,x\,g_0(x)=\varphi_0(x), \tag{20}$$

$$-f_0'(x)+g_0''(x)-\frac{\lambda}{4}g_0(x)+\frac{\lambda}{4}\,x\,g_0'(x)=\psi_0(x), \tag{21}$$

$$f_1(x-l)-f_0(0)+g_1'(x+l)+\frac{\lambda}{4}\,(x-l)\,g_1(x+l)=\varphi_1(x), \tag{22}$$

$$-f_1'(x-l)+g_1''(x+l)-\frac{\lambda}{4}\,g_1(x+l)+\frac{\lambda}{4}\,(x-l)\,g_1'(x+l)=\psi_1(x), \tag{23}$$

where $x\in[0,\,l]$.

Differentiating (20), (22), subtracting and adding term by term with (21), (23), respectively, obtain

$$\begin{aligned} f_0'(x)+\frac{\lambda}{4}\,g_0(x)&=\frac{1}{2}\,\varphi_0'(x)-\frac{1}{2}\,\psi_0(x),\\ g_0''(x)+\frac{\lambda}{4}\,x\,g_0'(x)&=\frac{1}{2}\,\varphi_0'(x)+\frac{1}{2}\,\psi_0(x),\end{aligned} \tag{24}$$

$$\begin{aligned} f_1'(x-l)+\frac{\lambda}{4}\,g_1(x+l)&=\frac{1}{2}\,\varphi_1'(x)-\frac{1}{2}\,\psi_1(x),\\ g_1''(x+l)+\frac{\lambda}{4}\,(x-l)\,g_1'(x+l)&=\frac{1}{2}\,\varphi_1'(x)+\frac{1}{2}\,\psi_1(x).\end{aligned} \tag{25}$$

Employing the second equation of (24) we obtain

$$\left[g_0'(x)\,e^{\frac{\lambda\,x^2}{8}}\right]'=\frac{1}{2}\,e^{\frac{\lambda\,x^2}{8}}\,\left[\varphi_0'(x)+\psi_0(x)\right],$$

$$g_0'(x)=g_0'(0)\,e^{-\frac{\lambda\,x^2}{8}}+\frac{1}{2}\,e^{-\frac{\lambda\,x^2}{8}}\int_0^x e^{\frac{\lambda\,t^2}{8}}\,\varphi_0'(t)\,dt+\frac{1}{2}\,e^{-\frac{\lambda\,x^2}{8}}\int_0^x e^{\frac{\lambda\,t^2}{8}}\,\psi_0(t)\,dt.$$

Using integration by parts for the first integral and taking into account that $g_0'(0)=\varphi_0(0)$, obtain

$$g_0'(x)=\frac{1}{2}\,\varphi_0(0)\,e^{-\frac{\lambda\,x^2}{8}}+\frac{1}{2}\,\varphi_0(x)-\frac{\lambda}{8}\,e^{-\frac{\lambda\,x^2}{8}}\int_0^x t\,e^{\frac{\lambda\,t^2}{8}}\,\varphi_0(t)\,dt+\frac{1}{2}\,e^{-\frac{\lambda\,x^2}{8}}\int_0^x e^{\frac{\lambda\,t^2}{8}}\,\psi_0(t)\,dt. \tag{26}$$

Hence

$$g_0(x)=g_0(0)+\frac{1}{2}\,\varphi_0(0)\int_0^x t\,e^{-\frac{\lambda\,t^2}{8}}\,dt+\frac{1}{2}\int_0^x\varphi_0(t)\,dt-$$

$$-\frac{\lambda}{8}\int_0^x e^{-\frac{\lambda\,\xi^2}{8}}\int_0^\xi t\,e^{\frac{\lambda\,t^2}{8}}\,\varphi_0(t)\,dt\,d\xi+\frac{1}{2}\int_0^x e^{-\frac{\lambda\,\xi^2}{8}}\int_0^\xi e^{\frac{\lambda\,t^2}{8}}\,\psi_0(t)\,dt\,d\xi.$$

Changing the order of integration in the last two integrals, get

$$g_0(x)=g_0(0)+\sqrt{\frac{2\,\pi}{\lambda}}\,\varphi_0(0)\,\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,x\right)+\frac{1}{2}\int_0^x\varphi_0(t)\,dt-$$

$$-\sqrt{\frac{\lambda\,\pi}{8}}\int_0^x\left[\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,x\right)-\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,t\right)\right]t\,e^{\frac{\lambda\,t^2}{8}}\,\varphi_0(t)\,dt+$$

$$+\sqrt{\frac{2\,\pi}{\lambda}}\int_0^x\left[\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,x\right)-\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,t\right)\right]e^{\frac{\lambda\,t^2}{8}}\,\psi_0(t)\,dt. \tag{27}$$

By (20) we have

$$f_0(x)=\varphi_0(x)-g_0'(x)-\frac{\lambda}{4}\,x\,g_0(x)+f_0(0),$$

$$f_0(l-t)=\varphi_0(l-t)-g_0'(l-t)-\frac{\lambda}{4}\,(l-t)\,g_0(l-t)+f_0(0). \tag{28}$$

Employing the second relation of (25) we obtain

$$\left(g_1'(x+l)\,e^{\frac{\lambda\,(x-l)^2}{8}}\right)'=\frac{1}{2}\,e^{\frac{\lambda}{8}(x-l)^2}\,\varphi_1'(x)+\frac{1}{2}\,e^{\frac{\lambda}{8}(x-l)^2}\,\psi_1(x),$$

$$g_1'(x+l)=g_1'(2\,l)\,e^{\frac{\lambda}{8}(x-l)^2}+\frac{1}{2}\,e^{\frac{\lambda}{8}(x-l)^2}\int_l^x e^{\frac{\lambda}{8}(t-l)^2}\,\varphi_1'(t)\,dt+\frac{1}{2}\,e^{-\frac{\lambda}{8}(x-l)}\int_l^x e^{\frac{\lambda}{8}(x-l)^2}\,\psi_1(t)\,dt.$$

Taking into account $g_1'(2\,l)=\varphi_1(l)$, integrating by parts the first integral, substituting x by $l-x$ and then substituting in both integrals $t=l-t$ get

$$g_1'(2\,l-x)=\frac{1}{2}\,\varphi_1(l)\,e^{-\frac{\lambda\,x^2}{8}}+\frac{1}{2}\,\varphi_1(l-x)+\frac{\lambda}{8}\,e^{-\frac{\lambda\,x^2}{8}}\int_0^x t\,e^{-\frac{\lambda\,t^2}{8}}\,\varphi_1(l-t)\,dt+$$

$$+\frac{1}{2}\,e^{-\frac{\lambda\,x^2}{8}}\int_0^x t\,e^{-\frac{\lambda\,t^2}{8}}\,\psi_1(l-t)\,dt. \tag{29}$$

Hence

$$g_1(2\,l-x)=g_1(2\,l)+\frac{1}{2}\,\varphi_1(l)\int_0^x e^{-\frac{\lambda\,t^2}{8}}\,dt+\frac{1}{2}\int_0^x\varphi_1(l-t)\,dt-$$

$$-\frac{\lambda}{8}\int_0^x e^{-\frac{\lambda\,\xi^2}{8}}\int_0^\xi t\,e^{\frac{\lambda\,t^2}{8}}\,\varphi_1(l-t)\,dt\,d\xi+\frac{1}{2}\int_0^x e^{-\frac{\lambda\,\xi^2}{8}}\int_0^\xi e^{\frac{\lambda\,t^2}{8}}\,\psi_1(l-t)\,dt\,d\xi.$$

Changing the integration order in the last two integrals we get

$$g_1(2\,l-x)=g_1(2\,l)+\sqrt{\frac{2\,\pi}{\lambda}}\,\varphi_1(l)\,\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,x\right)+\frac{1}{2}\int_0^x\varphi_1(l-t)\,dt-$$

$$-\sqrt{\frac{\lambda\,\pi}{8}}\int_0^x\left[\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,x\right)-\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,t\right)\right]t\,e^{\frac{\lambda\,t^2}{8}}\,\varphi_1(l-t)\,dt+$$

$$+\sqrt{\frac{2\,\pi}{\lambda}}\int_0^x\left[\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,x\right)-\Phi_0\left(\frac{\sqrt{\lambda}}{2}\,t\right)\right]e^{-\frac{\lambda\,t^2}{8}}\,\psi_1(l-t)\,dt. \tag{30}$$

By (22) we have

$$f_1(x-l)=\varphi_1(x)+f_0(0)-g_1'(x+l)-\frac{\lambda}{4}\,(x-l)\,g_1(x+l)$$

or

$$f_1(-t)=\varphi_1(l-t)+f_0(0)-g_1'(2\,l-t)+\frac{\lambda}{4}\,t\,g_1(2\,l-t). \tag{31}$$

Substituting $f_1(-t)$ by (31) into (18), and $f_0(l-t)$ by (28) into (19), we obtain

$$\mu(t)=\varphi_1(l-t)+g_0'(t)-g_1'(2\,l-t)-\frac{\lambda}{4}\,t\left[g_0(t)-g_1(2\,l-t)\right], \tag{32}$$

$$\nu(t)=\varphi_0(l-t)-\left[g_0'(l-t)-g_1'(l+t)\right]-\frac{\lambda}{4}\,(l-t)\left[g_0(l-t)-g_1(l+t)\right]. \tag{33}$$

Substituting $g_0'(t)$, $g_0(t)$, $g_1'(2\,l-t)$, $g_1(2\,l-t)$ calculated by (26), (27), (29), (30) into formulas (32), (33), and taking into account, due to the assumed continuity of $f'(x)$, that

$$\frac{\lambda}{2}\left[g_0(0)-g_1(2\,l)\right]=\varphi_0'-\psi_0(0)-\varphi_1'(l)+\psi_0(l)$$

obtain the sought result (11), (12).

In order for the solution to be regular, sufficient conditions must be satisfied that ensure the existence of a boundary control. It is therefore only natural these conditions provide the functions $f(x)$ and $g(x)$ with required smoothness determined by rule (16), (17) respectively, and by the initial and boundary conditions

$$f_0(0)=f_1(0),\quad f_0'(0)=f_1'(0),\quad f_0''(0)=f_1''(0),$$

$$g_0(l)=g_1(l),\quad g_0'(l)=g_1'(l),\quad g_0''(l)=g_1''(l),\quad g_0'''(l)=g_1'''(l),$$

$$\mu(0)=\varphi_0(0),\quad \mu(l)=\varphi_1(0),\quad \nu(0)=\varphi_0(l),\quad \nu(l)=\varphi_1(l),$$

$$\mu'(0)=\psi_0(0),\quad \mu'(l)=\psi_1(0),\quad \nu'(0)=\psi_0(l),\quad \nu'(l)=\psi_1(l).$$

Satisfying the expression for the function $f_0(x)$, $f_1(x)$, $g_0(x)$, $g_1(x)$, $\mu(x)$, $\nu(x)$ represented by formulas (28), (31), (27), (30), (32), (33) according to the conditions above and after some simple transformations obtain (5)–(10). It should be noted that for $\lambda=0$ the obtained results in this work coincide with the results obtained in [7].

References

1 Нахушев А.М. О задаче Дарбу для одного вырождающегося нагруженного интегро-дифференциального уравнения второго порядка / А.М. Нахушев // Дифференц. уравнения. — 1976. — 12. — № 1. — С. 103–108.

2 Дженалиев М.Т. Нагруженные уравнения как возмущения дифференциальных уравнений / М.Т. Дженалиев, М.И. Рамазанов. — Алматы: Ғылым, 2010. — 334 с.

3 Аттаев А.Х. Задача граничного управления для существенно нагруженного уравнения колебания струны / А.Х. Аттаев // Докл. Адыгской (Черкесской) Междунар. акад. наук. — 2011. — 13. — № 1. — С. 30–35.

4 Аттаев А.Х. Задача граничного управления для нагруженного уравнения колебания струны / А.Х. Аттаев // Дифференц. уравнения. — 2020. — 56. — № 5. — С. 646–651.

5 Хубиев К.У. О задаче Коши для одного нагруженного волнового уравнения / К.У. Хубиев // Докл. Адыгской (Черкесской) Междунар. акад. наук. — 2020. — 20. — № 4. — С. 9–13.

6 Khubiev K.U. Analogue of Tricomi problem for characteristically loaded hyperbolic-parabolic equation with variable coefficients / K.U. Khubiev // Ufa Mathematical Journal. — 2017. — 9. — No. 2. — P. 92–101.

7 Ильин В.А. Волновое уравнение с граничным управлением на двух концах и о полном успокоении колебательного процесса / В.А. Ильин, В.В. Тихомиров // Дифференц. уравнения. — 1999. — 35. — № 5. — С. 692–704.

8 Ильин В.А. Граничное управление смещением на одном конце струны при наличии нелокального граничного условия одного из четырех типов и его оптимизация / В.А. Ильин // Дифференц. уравнения. — 2008. — 44. — № 11. — С. 1487–1498.

9 Моисеев Е.И. Оптимальное граничное управление смещенными колебаниямим струны с нелокальным условием нечетности первого рода / Е.И. Моисеев, А.А. Холомеева // Дифференц. уравнения. — 2010. — 46. — № 11. — С. 1623–1630.

10 Моисеев Е.И. Оптимальное граничное управление смещенными колебаниями струны с нелокальным условием четности второго рода / Е.И. Моисеев, А.А. Холомеева // Дифференц. уравнения. — 2011. — 47. — № 1. — С. 127–134.

11 Attaev A.Kh. Boundary control by displacement at one end of string and the integral condition on the other / A.Kh. Attaev // Int. Conf. "Functional Analysis in Interdisciplinary Applications" (FAIA 2017), Melville, New York: Am. Inst. Phys. — 2017. — P. 040005.

12 Attaev A.Kh. On Problem of Internal Boundary Control for String Vibration Equation / A.Kh. Attaev // Bulletin of the Karaganda University. Mathematics series. — 2021. — 1. — No. 101. — P. 4–10.

13 Бутковский А.Г. Методы управления системами с распределенными параметрами / А.Г. Бутковский — М.: Наука, 1975. — 568 с.

14 Байрактаров Б.Р. Оптимизация гидравлических систем с переменными параметрами / Б.Р. Байрактаров // Мат. моделирование. — 1998. — 10. — № 2. — С. 41—47.

15 Bayraktar B. Determining optimum hydraulic parameters in branched water supply system / B. Bayraktar // Journal of the Balkan Tribological Association. — 2012. — 18. — No. 2. — P. 174–183.

16 Бутковский А.Г. Оптимальное управление двумерной колебательной системой / А.Г. Бутковский, Л.И. Полтавский // Автоматика и телемеханика. — 1966. — № 4. — С. 32–41.

17 Lions J.L. Exact controllability, stabilization, and perturbations for distributed systems / J.L. Lions // SIAM Rev. — 1988. — 30. — No. 2. — P. 1–68.

18 Васильев Ф.П. О двойственности в линейных задачах управления и наблюдения / Ф.П. Васильев // Дифференц. уравнения. — 1995. — 31. — № 11. — С. 1893–1900.

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Гиперболалық теңдеу үшін оның сипаттамаларының бірі бойынша жүктелген шекаралық бақылау есебі

Мақалада регулярлы шешім тұрғысынан бір өлшемді жолдың тербеліс теңдеуі үшін сипаттамаларының бірі бойынша жүктелген шекаралық бақылау есебінің бірегей шешімі зерттелген. Шешу әдісі осы теңдеу үшін құрылған Даламбер формуласының аналогына негізделген. Ұзындығы бірдей кесінділер бойынша бастапқы және соңғы Коши деректері берілгенде бұл теңдеудің шешімінің анықталу облысы квадрат болатыны атап өтілген. Квадраттың қабырғасы берілген кесіндінің ұзындығына тең. Шекаралық бақылаудың өзі Даламбер формуласының аналогының құрамдас бөліктері бойынша анықталған, олар өз кезегінде Кошидің бастапқы және соңғы деректері бойынша бірегей түрде анықталады. Ізделінді шекаралық бақылауларға арналған соңғы формулаларда нормаланған және орталықтандырылған үлестірімдердің таралу функциясы қатысатынын атап өткен жөн, бұл жалпы айтқанда гиперболалық типті теңдеулермен басталатын бастапқы және шекаралық есептер үшін тән емес.

Кілт сөздер: гиперболалық теңдеу, таралған тербелмелі жүйе, газ немесе сұйық ағындарының пульсациясын бәсеңдету есебі, жүктелген теңдеу, бастапқы шарттар, шекаралық шарттар, Даламбер формуласының аналогы, шекаралық бақылау, қалыпты үлестірім, үлестірім функциясы.

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Задача граничного управления для нагруженного вдоль одной из своих характеристик гиперболического уравнения

В статье исследована однозначная разрешимость задачи граничного управления для нагруженного вдоль одной из своих характеристик одномерного уравнения колебания струны в терминах регулярного решения. Метод решения основан на аналоге формулы Даламбера, построенного для данного уравнения. Отмечено, что областью определения решения данного уравнения, когда начальные и финальные данные Коши задаются на отрезках одинаковой длины, является квадрат. Сторона квадрата равна длине данного отрезка. Сами граничные управления определены через компоненты аналога формулы Даламбера, которые, в свою очередь, однозначно определяются через начальные и финальные данные Коши. Следует отметить, что в окончательных формулах для искомых граничных управлений участвует функция распределения нормированного и центрированного распределения, что, вообще говоря, не характерно для начальных и граничных задач инициированных уравнениями гиперболического типа.

Ключевые слова: гиперболическое уравнение, распределённая колебательная система, задача гашения пульсации потоков газа или жидкости, нагруженное уравнение, начальные условия, граничные условия, аналог формулы Даламбера, граничные управления, нормальное распределение, функция распределения.

References

1 Nakhushev, A.M. (1976). O zadache Darbu dlia odnogo vyrozhdaiushchegosia nagruzhennogo integro-differentsialnogo uravneniia vtorogo poriadka [On the Darboux problem for a degenerate loaded integro-differential equation of the second order]. *Differentsialnye uravneniia — Differ. Equations, 12*(1), 103–108 [in Russian].

2 Dzhenaliev, M.T., & Ramazanov, M.I. (2010). *Nagruzhennye uravneniia kak vozmushcheniia differentsialnykh uravnenii [Loaded equations as perturbations of differential equations]*. Almaty: Gylym [in Russian].

3 Attaev, A.Kh. (2011). Zadacha granichnogo upravleniia dlia sushchestvenno nagruzhennogo uravneniia kolebaniia struny [Boundary control problem for essentially loaded string vibration equation]. *Doklady Adygskoi (Cherkesskoi) Mezhdunarodnoi akademii nauk — Reports of Adyghe (Circassian) International Academy of Sciences*, 13 (1), 30–35 [in Russian].

4 Attaev, A.Kh. (2020). Zadacha granichnogo upravleniia dlia nagruzhennogo uravneniia kolebaniia struny [Boundary control problem for a loaded wave equation]. *Differentsialnye uravneniia — Differ. Equations, 56*(5), 646–651 [in Russian].

5 Khubiev, K.U. (2020). O zadache Koshi dlia odnogo nagruzhennogo volnovogo uravneniia [On the Cauchy problem for one-loaded wave equation]. *Doklady Adygskoi (Cherkesskoi) Mezhdunarodnoi akademii nauk — Reports of Adyghe (Circassian) International Academy of Sciences, 20*(4), 9–13 [in Russian].

6 Khubiev, K.U. (2017). Analogue of Tricomi problem for characteristically loaded hyperbolic-parabolic equation with variable coefficients. *Ufa Mathematical Journal, 9*(2), 92–101.

7 Ilin, V.A., & Tikhomirov, V.V. (1999). Volnovoe uravnenie s granichnym upravleniem na dvukh kontsakh i o polnom uspokoenii kolebatelnogo protsessa [The wave equation with boundary control at two ends and the problem of the complete oscillation damping].*Differentsialnye uravneniia — Differ. Equations, 35*(5), 692–704 [in Russian].

8 Ilin, V.A. (2008). Granichnoe upravlenie smeshcheniem na odnom kontse struny pri nalichii nelokalnogo granichnogo usloviia odnogo iz chetyrekh tipov i ego optimizatsiia [Boundary control by displacement at one end of a string in the presence of a nonlocal boundary condition of one of four types and its optimization]. *Differentsialnye uravneniia — Differ. Equations, 44*(11), 1487–1498 [in Russian].

9 Moiseev, E.I., & Kholomeeva, A.A. (2010). Optimalnoe granichnoe upravlenie smeshchennymi kolebaniiami struny s nelokalnym usloviem nechetnosti pervogo roda [Optimal boundary displacement control of string vibrations with a nonlocal oddness condition of the first kind]. *Differentsialnye uravneniia — Differ. Equations, 46*(11), 1623–1630 [in Russian].

10 Moiseev, E.I., & Kholomeeva, A.A. (2011). Optimalnoe granichnoe upravlenie smeshchennymi kolebaniiami struny s nelokalnym usloviem chetnosti vtorogo roda [Optimal boundary displacement control of string vibrations with a nonlocal evenness condition of the second kind]. *Differentsialnye uravneniia — Differ. Equations, 47*(1), 127–134 [in Russian].

11 Attaev, A.Kh. (2017). Boundary control by displacement at one end of string and the integral condition on the other. Int. Conf. "*Analysis in Interdisciplinary Applications*" (FAIA 2017), Melville, New York: Am. Inst. Phys., 040005.

12 Attaev, A.Kh. (2021). On Problem of Internal Boundary Control for String Vibration Equation. *Bulletin of the Karaganda University. Mathematics series, 1*(101), 4–10.

13 Butkovskii, A.G. (1975). *Metody upravleniia sistemami s raspredelennymi parametrami [Methods of control systems with distributed parameters]*. Moscow: Nauka [in Russian].

14 Bayraktarov, B.R. (1998). Optimizatsiia gidravlicheskikh sistem s peremennymi parametrami [Optimization of hydraulic systems with variable parameters]. *Matematicheskoe modelirovanie — Mathematical Models and Computer Experiment,* 41—47 [in Russian].

15 Bayraktar, B. (2012). Determining optimum hydraulic parameters in branched water supply system. *Journal of the Balkan Tribological Association, 18*(2), 174–183.

16 Butkovskiy, A.G., & Poltavskiy, L.I. (1966). Optimalnoe upravlenie dvumernoi kolebatelnoi sistemoi [Optimal control over a two-dimensional vibrational system]. *Avtomatika i telemekhanika — Automation and Remote Control, 4*, 32–41 [in Russian].

17 Lions, J.L. (1988). Exact controllability, stabilization, and perturbations for distributed systems. *SIAM Rev, 30*(2), 1–68.

18 Vasilev, F.P. (1995). O dvoistvennosti v lineinykh zadachakh upravleniia i nabliudeniia [On duality in linear control and observation problems]. *Differentsialnye uravneniia — Differ. Equations, 31*(11), 1893–1900 [in Russian].

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Inner boundary value problem with displacement for a second order mixed parabolic-hyperbolic equation

This paper investigates inner boundary value problems with a shift for a second-order mixed-hyperbolic equation consisting of a wave operator in one part of the domain and a degenerate hyperbolic operator of the first kind in the other part. We find sufficient conditions for the given functions to ensure the existence of a unique regular solution to the problems under study. In some special cases, solutions are obtained explicitly.

Keywords: wave equation, degenerate hyperbolic equation of the first kind, Volterra integral equation, Fredholm integral equation, Tricomi method, method of integral equations, methods of fractional calculus theory.

Introduction. Notation. Formulation of the problem

In the Euclidean plane with independent variables x and y consider the equation

$$0 = \begin{cases} (-y)^m u_{xx} - u_{yy} + \lambda (-y)^{\frac{m-2}{2}} u_x, & y < 0, \\ u_{xx} - u_{yy} + f, & y > 0, \end{cases} \qquad (1)$$

where λ, m are given numbers; $m > 0$, $|\lambda| \le \frac{m}{2}$; $f = f(x,y)$ is a given function; $u = u(x,y)$ is an unknown function.

When $y < 0$ equation (1) is a degenerate hyperbolic equation of the first kind [1]

$$(-y)^m u_{xx} - u_{yy} + \lambda (-y)^{\frac{m-2}{2}} u_x = 0, \qquad (2)$$

but when $y < 0$ coincides with the inhomogeneous wave equation

$$u_{xx} - u_{yy} + f(x,y) = 0. \qquad (3)$$

Equation (2) belongs to the class of the first kind degenerate hyperbolic equations [1; 21], that is, at no point of the degenerate line $y = 0$ the tangent line does coincide with the characteristic direction of the equation (2). An important property of equation (2) is the fact that when $|\lambda| \le \frac{m}{2}$ the Cauchy problem is correct for it in the usual formulation with data on the parabolic degeneracy line $y = 0$ despite that the Protter condition [2] is violated. When $m = 2$ equation (2) turns into the Bitsadze-Lykov equation [3; 37], [4], [5; 234], and for $\lambda = 0$ from equation (2) we come to the Gellerstedt equation, which, as shown in the monograph [6; 234], finds application in the problem of determining the shape of the dam slot. Apart from that as well the particular case for equation (2) is the Tricomi equation, which presents the theoretical basis for transonic gas dynamics [7; 38], [8; 280].

Equation (1) is considered in the domain $\Omega = \Omega_1 \cup \Omega_2 \cup I$, where Ω_1 is the domain restricted by characteristics $\sigma_1 = AC : x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0$ and $\sigma_2 = CB : x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = r$ of equation

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(2) for $y<0$, outgoing from the point $C=(r/2,y_c)$, $y_c=-\left[\frac{(m+2)r}{4}\right]^{\frac{2}{m+2}}$, passing through the points $A=(0,0)$ and $B=(r,0)$, and the segment $I=AB$ of the strait line $y=0$; Ω_2 is the domain restricted by characteristics $\sigma_3=AD:\ x-y=0$, $\sigma_4=BD:\ x+y=r$ of equation (3), outgoing from the points A and B intersecting at the point $D=\left(\frac{r}{2},\frac{r}{2}\right)$ and the segment $I=AB$.

Let us introduce the following notation:

$$\varepsilon_1=\frac{m-2\lambda}{2\,(m+2)},\ \varepsilon_2=\frac{m+2\lambda}{2\,(m+2)},\ \varepsilon=\varepsilon_1+\varepsilon_2=\frac{m}{m+2},$$

$$\gamma_1=\frac{\Gamma\,(\varepsilon)}{\Gamma\,(\varepsilon_1)},\ \gamma_2=\frac{\Gamma\,(2-\varepsilon)\,(2-2\varepsilon)^{\varepsilon-1}}{\Gamma\,(1-\varepsilon_2)};$$

$$a(x)=\frac{\alpha_2(x)+\gamma_1\alpha_1(x)}{\alpha_3(x)-\gamma_1\alpha_1(x)},\ b(x)=\frac{\gamma_1\beta_1(x)+\beta_3(x)}{\gamma_2\beta_1(x)-\beta_2(x)};$$

$$K(x,t)=\frac{a(x)}{(t-x)^{1-\varepsilon}}+\int\limits_x^t\frac{a'\,(s)}{(t-s)^{1-\varepsilon}}ds,\ L(x,t)=\begin{cases}K\,(r,\,t)\,, & 0\le x<t,\\ K\,(r,t)-K(x,t), & t<x\le r,\end{cases}$$

$$F_1(x)=2\psi_1(x)-\psi_1\,(r)+\int\limits_x^r\frac{\psi_2(t)}{\alpha_3(t)-\gamma_2\alpha_1(t)}dt-\int\limits_x^r\int\limits_0^{\frac{r-t}{2}}f\,(t+s,s)\,dsdt,$$

$$F_2(x)=b(x)\left[2\,\varphi_1(x)-\varphi_1\,(0)-\int\limits_0^{x/2}\int\limits_t^{x-t}f\,(s,t)\,dsdt\right]-\frac{\varphi_2(x)}{\gamma_2\beta_1(x)-\beta_2(x)};$$

$$\theta_{00}(x)=\left(\frac{x}{2},-\left(\frac{m+2}{4}\right)^{2/(m+2)}x^{2/(m+2)}\right)=\left(\frac{x}{2},-\,(2-2\varepsilon)^{\varepsilon-1}\,x^{1-\varepsilon}\right),$$

$$\theta_{r0}(x)=\left(\frac{r+x}{2},-\left(\frac{m+2}{4}\right)^{2/(m+2)}(r-x)^{2/(m+2)}\right)=\left(\frac{r+x}{2},-\,(2-2\varepsilon)^{\varepsilon-1}\,(r-x)^{1-\varepsilon}\right)$$

– affixes of characteristics intersection points that leave the point $(x,0)$ with characteristics of AC and BC of equation (3), correspondingly;

$$\theta_{01}(x)=\left(\frac{x}{2},\frac{x}{2}\right),\ \theta_{r1}(x)=\left(\frac{r+x}{2},\frac{r-x}{2}\right)$$

– affixes of characteristics intersection points that leave the point $(x,0)$ with characteristics AD and BD of equation (3), correspondingly;

$$B\,(p,\,q)=\int_0^1 t^{p-1}\,(1-t)^{q-1}\,dt,\ \Gamma(p)=\int_0^\infty\exp\,(-t)\ t^{p-1}dt,\ B\,(p,q)=\frac{\Gamma(p)\,\Gamma\,(q)}{\Gamma\,(p+q)}$$

– Euler integrals of the first and second kind and their relationship;

$$D^{\alpha}_{cx}g(t)=\begin{cases}\frac{sgn(x-c)}{\Gamma(-\alpha)}\int\limits_c^x\frac{g(t)}{|x-t|^{1+\alpha}}dt, & \alpha<0,\\ sgn^{[\alpha]+1}\,(x-c)\ \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}}D^{\alpha-[\alpha]-1}_{cx}g(t), & \alpha>0\end{cases}$$

– fractional integro-differentiation operator (in the Riemann-Liouville sense) of order $|\alpha|$ with starting point c [5], [6], [9]; the regularized fractional derivative (Caputo derivative) is defined using the equality [10]

$$\partial_{cx}^{\alpha} g(t) = sgn^n\,(x-c)\; D_{cx}^{\alpha-n} g^{(n)}(t),\;\; n-1<\alpha\le n,\;\; n\in N;$$

and it is related to the Riemann-Liouville derivative by the relation [10]

$$\partial_{cx}^{\alpha} g(t) = D_{cx}^{\alpha} g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}\,(c)}{\Gamma\,(k-\alpha+1)},$$

where $n-1<\alpha\le n$, $n\in N$;

$$E_{\rho}\,(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\,(1+n\rho)},\;\; E_{\rho}\,(z,\,\mu) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\,(\mu+n\rho^{-1})},\;\; E_{\rho}\,(z,\,1) = E_{1/\rho}\,(z)$$

– the Mittag-Leffler function and the function of Mittag-Leffler type [11].

Assume the function $u=u(x,y)$ of class $C\left(\bar{\Omega}\right)\cap C^1\,(\Omega)\cap C^2\,(\Omega_1\cup\Omega_2)$, $u_x,\,u_y\;\in L_1\,(I)$ in the domain Ω is a regular solution, which by substitution transforms equation (1) into identity.

Problem 1. Find a regular solution of equation (1) in the domain Ω satisfying the conditions

$$u\,[\theta_{r1}(x)] = \psi_1(x),\quad 0\le x\le r, \tag{4}$$

$$\alpha_1(x)\;(r-x)^{\varepsilon_2}\,D_{rx}^{1-\varepsilon_1}\;\{u\,[\theta_{r0}(t)]\} + \alpha_2(x)\,D_{rx}^{1-\varepsilon} u\,(t,\,0) + \alpha_3(x)\,u_y(x,0) = \psi_2(x),\quad 0<x<r, \tag{5}$$

where $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ are given functions on the line segment $0\le x\le r$, moreover $\alpha_1^2(x)+\alpha_2^2(x)+\alpha_3^2(x)\ne 0\quad\forall\, x\in[0,r]$.

Problem 2. Find a regular solution of equation (1) in the domain Ω satisfying the conditions

$$u\,[\theta_{01}(x)] = \varphi_1(x),\quad 0\le x\le r, \tag{6}$$

$$\beta_1(x)\;(r-x)^{1-\varepsilon_1}\,D_{rx}^{\varepsilon_2}\left\{(r-t)^{\varepsilon-1}\,u\,[\theta_{r0}(t)]\right\} +$$

$$+\beta_2(x)\,D_{rx}^{\varepsilon-1} u_y\,(t,\,0) + \beta_3(x)\,u(x,0) = \varphi_2(x),\quad 0<x<r, \tag{7}$$

where $\beta_1(x)$, $\beta_2(x)$, $\beta_3(x)$, $\varphi_1(x)$, $\varphi_2(x)$ are given functions on the line segment $0\le x\le r$, moreover $\beta_1^2(x)+\beta_2^2(x)+\beta_3^2(x)\ne 0\quad\forall\, x\in[0,r]$.

Earlier, the Goursat problem for the first kind degenerate hyperbolic equations was investigated in [12], [13]. The criterion of continuity for the Goursat problem for equation (2) is investigated in [12], and the solution of the Goursat problem for a model equation degenerating inside the domain is written in explicit form in [13]. The first boundary value problem for the hyperbolic equation degenerating inside the domain is considered in [14]. Boundary value problems for degenerate hyperbolic equations in a characteristic quadrangle with data on opposite characteristics were investigated in [15–17].

Inner boundary value problems 1 and 2 considered in this paper belong to the class of boundary value problems with a displacement of the Zhegalov-Nakhushev [18–20] and are generalization of the Goursat problem and problems with data on opposite characteristic lines for an equation of the type (1). The displacement problems for the first kind hyperbolic equations degenerating inside the domain were previously studied in [21–24]. The displacement problems for the first kind degenerate hyperbolic equation of the type (2) were investigated in [25], presented as generalization of the first and second Darboux problems. A rather complete bibliography of works devoted to the formulation and study of the displacement problems for various types of partial differential equations is provided in [26–32]. In this paper, sufficient conditions are found for the given functions $\alpha_i(x)$, $\beta_i(x)$, $i=\overline{1,3}$; $\varphi_j(x)$, $\psi_j(x)$, $j=\overline{1,2}$; $f(x,y)$ *that insure a unique regular solution of investigated problems 1 and 2*. In particular cases, when the relation $a(x)=\frac{\alpha_2(x)+\gamma_1\alpha_1(x)}{\alpha_3(x)-\gamma_1\alpha_1(x)}=a=const$ or $b(x)=\frac{\gamma_1\beta_1(x)+\beta_3(x)}{\gamma_2\beta_1(x)-\beta_2(x)}=b=const$ the regular solutions of *problems 1 and 2* are written explicitly.

Research task 1

The following Theorem holds.

Theorem 1. Let the given functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ and $f(x,y)$ be such that

$$\alpha_1(x), \alpha_2(x), \alpha_3(x), \psi_2(x) \in C\,[0,\,r] \cap C^2\,(0,\,r)\,, \tag{8}$$

$$\psi_1(x) \in C^1\,[0,\,r] \cap C^3\,(0,\,r)\,, \tag{9}$$

$$f(x,y) \in C^1\left(\bar{\Omega}_2\right), \tag{10}$$

$$\alpha_3(x) - \gamma_1\alpha_1(x) \neq 0 \quad \forall\, x \in [0,r]\,. \tag{11}$$

Then there is a unique regular solution of Problem 1 in the domain Ω.

Proof. Assume there is a solution of problem (1), (4), (5) and assume that

$$u(x,0) = \tau(x), \quad 0 \le x \le r, \tag{12}$$

$$\lim_{y\to 0} u_y(x,y) = u_y(x,0) = \nu(x), \quad 0 < x < r. \tag{13}$$

Find the relations between the functions $\tau(x)$ and $\nu(x)$ brought from Ω_1 and Ω_2 of the domain Ω onto the line I. The solution to problem (12), (13), when $|\lambda| \le \frac{m}{2}$ for equation (2), is written out according to one of the formulas [33]:

$$u(x,y) = \frac{1}{B\left(\varepsilon_1,\varepsilon_2\right)}\int\limits_0^1 \tau\left[x + (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\,(2t-1)\right]\, t^{\varepsilon_2-1}\,(1-t)^{\varepsilon_1-1}\,dt +$$

$$+\frac{y}{B\left(1-\varepsilon_1,\,1-\varepsilon_2\right)}\int\limits_0^1 \nu\left[x + (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\,(2t-1)\right]\, t^{-\varepsilon_1}\,(1-t)^{-\varepsilon_2}\,dt,\;\; |\lambda| < \frac{m}{2}, \tag{14}$$

$$u(x,y) = \tau\left[x + (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\right] +$$

$$+\,(1-\varepsilon)\;y\int\limits_0^1 \nu\left[x + (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\,(2t-1)\right]\,(1-t)^{-\varepsilon}\,dt,\;\; \lambda = \frac{m}{2}, \tag{15}$$

$$u(x,y) = \tau\left[x - (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\right] +$$

$$+\,(1-\varepsilon)\;y\int\limits_0^1 \nu^-\left[x + (1-\varepsilon)\,(-y)^{\frac{1}{1-\varepsilon}}\,(1-2t)\right]\,(1-t)^{-\varepsilon}\,dt,\;\; \lambda = -\frac{m}{2}. \tag{16}$$

First, consider the case for $|\lambda| < \frac{m}{2}$. In this instance, employing (14) we get

$$u\left[\theta_{r0}(x)\right] = u\left(\frac{r+x}{2},\,-\left(2-2\varepsilon\right)^{\varepsilon-1}(r-x)^{1-\varepsilon}\right) = \frac{1}{B\left(\varepsilon_1,\,\varepsilon_2\right)}\int\limits_0^1 \tau\left[x+(r-x)\,t\right]\,t^{\varepsilon_2-1}\,(1-t)^{\varepsilon_1-1}\,dt -$$

$$-\frac{1}{B\left(1-\varepsilon_1,\,1-\varepsilon_2\right)}\left(2-2\varepsilon\right)^{\varepsilon-1}(r-x)^{1-\varepsilon}\int\limits_0^1 \nu\left[x+(r-x)\,t\right]\,t^{-\varepsilon_1}\,(1-t)^{-\varepsilon_2}\,dt.$$

Introducing a new variable $z = x + (r - x)\, t$ we rewrite the last equality as

$$u\left[\theta_{r0}(x)\right] = \frac{(r-x)^{1-\varepsilon}}{B\left(\varepsilon_1,\, \varepsilon_2\right)} \int\limits_x^r \frac{\tau\,(z)\ (r-z)^{\varepsilon_1 - 1}}{(z-x)^{1-\varepsilon_2}}\, dz - \frac{(2-2\varepsilon)^{\varepsilon-1}}{B\left(1-\varepsilon_1,\, 1-\varepsilon_2\right)} \int\limits_x^r \frac{\nu\,(z)\ (r-z)^{-\varepsilon_2}}{(z-x)^{\varepsilon_1}}\, dz.$$

In terms of fractional (in the sense of Riemann-Liouville) integro-differentiation operator, the previous equality can be rewritten as

$$u\left[\theta_{r0}(x)\right] = \frac{\Gamma\left(\varepsilon\right)}{\Gamma\left(\varepsilon_1\right)} (r-x)^{1-\varepsilon}\ D_{rx}^{-\varepsilon_2} \left[\tau(t)\,(r-t)^{\varepsilon_1 - 1}\right] -$$

$$- \frac{\Gamma\left(2-\varepsilon\right)}{\Gamma\left(1-\varepsilon_2\right)} (2-2\varepsilon)^{\varepsilon-1}\ D_{rx}^{\varepsilon_1 - 1} \left[\nu(t)\,(r-t)^{-\varepsilon_2}\right]. \tag{17}$$

Further we use the following properties of the weighted composition of operators for fractional differentiation and integration with the same origins [5], [6], [9]:

$$D_{cx}^{-\beta}\, D_{ct}^{\beta} \varphi\,(s) = \varphi(x), \tag{18}$$

$$D_{cx}^{\alpha}\, |t-c|^{\alpha+\beta}\ D_{ct}^{\beta} \varphi\,(s) = |x-c|^{\beta}\ D_{cx}^{\alpha+\beta}\, |t-c|^{\alpha}\, \varphi(t), \tag{19}$$

where $0 < \alpha \leq 1$, $\beta < 0$, $\alpha + \beta > -1$; $\varphi(x) \in L\,[a,\, b]$, and when $\alpha + \beta > 0$ the function $\varphi(x)$ contains the fractional derivative $D_{cx}^{\alpha+\beta} \varphi(t)$.

Applying to both sides of equality (17) the operator $D_{rx}^{1-\varepsilon_1}$ and using the above composition properties (18) and (19) we find

$$D_{rx}^{1-\varepsilon_1} u\left[\theta_{r0}(t)\right] = \gamma_1\ (r-x)^{-\varepsilon_2}\ D_{rx}^{1-\varepsilon} \tau(t) - \gamma_2\ (r-x)^{-\varepsilon_2}\, \nu(x). \tag{20}$$

Substituting the value $D_{rx}^{1-\varepsilon_1}\, u\left[\theta_{r0}(t)\right]$ from (20) into (5) we come to the ratio

$$\left[\alpha_2(x) + \gamma_1 \alpha_1(x)\right]\ D_{rx}^{1-\varepsilon} \tau(t) + \left[\alpha_3(x) - \gamma_1 \alpha_1(x)\right]\, \nu(x) = \psi_2(x). \tag{21}$$

The obtained relation (21) is the first fundamental relation between the functions $\tau(x)$ and $\nu(x)$ taken from the domain Ω_1 onto the line I.

Next, we find the fundamental relationship between the functions $\tau(x)$ and $\nu(x)$ taken from the domain Ω_2 onto the line I. The solution of problem (12), (13) for equation (3) is written by the d'Alembert formula [34]:

$$u(x,y) = \frac{\tau\,(x+y) + \tau\,(x-y)}{2} + \frac{1}{2} \int\limits_{x-y}^{x+y} \nu(t) dt + \frac{1}{2} \int\limits_0^y \int\limits_{x-y+t}^{x+y-t} f\,(t,\, s)\ ds dt. \tag{22}$$

Satisfying (22) to condition (4), we obtain

$$u\left[\theta_{r1}(x)\right] = u\left(\frac{r+x}{2},\, \frac{r-x}{2}\right) = \frac{\tau\,(r) + \tau(x)}{2} + \frac{1}{2} \int\limits_x^r \nu(t) dt + \frac{1}{2} \int\limits_0^{(r-x)/2} \int\limits_{x+t}^{r-t} f\,(s,\, t)\ ds dt = \psi_1(x). \tag{23}$$

Differentiating (23) we arrive at the relation

$$\nu(x) = \tau'(x) - 2\psi_1'(x) - \int\limits_0^{(r-x)/2} f\,(x+t,\, t)\ dt. \tag{24}$$

Relation (24) is the second fundamental relation between the functions $\tau(x)$ and $\nu(x)$ taken from the domain Ω_2 onto the line I.

Excluding the sought function $\nu(x)$ from (21) and (24) in view of the matching condition $\tau(r) = \psi_1(r)$, with respect to $\tau(x)$, we arrive at the following problem for the first-order ordinary differential equation with a fractional derivative in the lower terms

$$[\alpha_3(x) - \gamma_1\alpha_1(x)]\,\tau'(x) + [\alpha_2(x) + \gamma_1\alpha_1(x)]\,D_{rx}^{1-\varepsilon}\tau(t) =$$

$$= 2\,[\alpha_3(x) - \gamma_1\alpha_1(x)]\,\psi_1'(x) - \psi_2(x) + [\alpha_3(x) - \gamma_1\alpha_1(x)]\int\limits_0^{(r-x)/2} f\,(x+t,t)\,dt, \tag{25}$$

$$\tau\,(r) = \psi_1\,(r)\,. \tag{26}$$

If condition (11) of Theorem 1 is satisfied, then by dividing each term in equation (25) by $\alpha_3(x) - \gamma_1\,\alpha_1(x)$ with the subsequent integration of the resulting equation over x ranging from x to r we come to the integral equation

$$\tau(x) - \frac{1}{\Gamma\,(\varepsilon)}\int\limits_x^r K(x,t)\,\tau(t)dt = F_1(x), \tag{27}$$

which corresponds to problem (25), (26).

It follows from properties (8), (9), (10) that equation (27) is a Volterra integral equation of the second kind with the kernel $K(x,t) \in L_2\,(\,[0,\,r] \times [0,\,r]\,)$ and with the right-hand side $F_1(x) = C^1\,[0,\,r] \cap C^3\,(0,\,r)$. According to the general theory on Volterra integral equations the solution of equation (27) exists, is unique and can be written out by the formula:

$$\tau(x) = F_1(x) + \frac{1}{\Gamma\,(\varepsilon)}\int\limits_x^r R(x,t)F_1(t)dt, \tag{28}$$

where $R(x,t) = \sum\limits_{n=0}^{\infty}\Gamma^{-n}\,(\varepsilon)\;K_n(x,t)$ is a kernel resolvent $K(x,t)$; $K_0(x,t) = K(x,t)$, $K_{n+1}(x,t) = \int\limits_x^t K\,(x,s)\;K_n\,(s,t)\,ds$ are iterated kernels of the basic kernel $K(x,t)$; moreover, the resolvent $R(x,t)$, as well as the basic kernel $K(x,t)$ of equation (27), will belong to the class $R(x,t) \in L_2\,([0,r] \times [0,r])$, and the solution $\tau(x)$ of equation (27), as well as its right side $F_1(x)$, will belong to the class $\tau(x) \in C^1\,[0,\,r] \cap C^3\,(0,\,r)$.

The solution of equation (27) for $a(x) = a = const$ is written explicitly by the formula:

$$\tau(x) = F_1(x) + a\int\limits_x^r (t-x)^{\varepsilon-1}\,E_{1/\varepsilon}\left[a\,(t-x)^{\varepsilon}\,;\varepsilon\right]F_1(t)dt. \tag{29}$$

The sought function $\tau(x)$ for $\lambda = \pm\frac{m}{2}$ is found again employing formulas (28) or (29), but $\varepsilon_2 = 0$, $\varepsilon = \varepsilon_1 = \frac{m}{m+2}$, $\gamma_1 = 1$, $\gamma_2 = 2^{\varepsilon-1}\,(1-\varepsilon)^{\varepsilon}\,\Gamma\,(1-\varepsilon)$ at $\lambda = -\frac{m}{2}$ and $\varepsilon_1 = 0$, $\varepsilon = \varepsilon_2 = \frac{m}{m+2}$, $\gamma_1 = 0$, $\gamma_2 = 2^{\varepsilon-1}\,(1-\varepsilon)^{\varepsilon}$ at $\lambda = \frac{m}{2}$.

Once the function $\tau(x)$ has been found, the second sought function $\nu(x)$ is found employing formulas (21) or (24). Then the solution of the studied problem 1 in the domain Ω_1 is written out according to one of the (14), (15) or (16) formulas and in the domain Ω_2 the problem (12), (13) for equation (3) is solved by formula (22).

Research task 2

Similarly as above, satisfying (14) to condition (7) we find the first fundamental relation between the sought functions $\tau(x)$ and $\nu(x)$ taken in the domain Ω_1 onto the line I:

$$[\gamma_1\beta_1(x)+\beta_3(x)]\,\tau(x)=[\gamma_2\beta_1(x)-\beta_2(x)]\,D_{rx}^{\varepsilon-1}\nu(t)+\varphi_2(x). \tag{30}$$

Employing (22) under condition (6), we find the second fundamental relation between $\tau(x)$ and $\nu(x)$ taken in the domain Ω_2 onto the line I:

$$\tau(x)=2\varphi_1(x)-\tau(0)-\int\limits_0^x\nu(t)dt-\int\limits_0^{x/2}\int\limits_t^{x-t}f(s,t)\,dsdt. \tag{31}$$

Excluding in (30) and (31) the sought function $\tau(x)$ in view of the matching condition $\tau(0)=\varphi_1(0)$ with respect to $\nu(x)$ we obtain the equation

$$[\gamma_2\beta_1(x)-\beta_2(x)]\,D_{rx}^{\varepsilon-1}\nu(t)+[\gamma_1\beta_1(x)+\beta_3(x)]\int\limits_0^x\nu(t)dt=$$

$$=2\,[\gamma_1\beta_1(x)+\beta_3(x)]\,[\varphi_1(x)-\varphi_1(0)]-\varphi_2(x)-[\gamma_1\beta_1(x)+\beta_3(x)]\int\limits_0^{x/2}\int\limits_t^{x-t}f(s,t)\,dsdt.$$

Denoting $\upsilon(x)=\int\limits_0^x\nu(t)dt$ provided that $\gamma_2\beta_1(x)-\beta_2(x)\neq0\;\forall\,x\in[0,r]$ the recent equality is rewritten as follows

$$D_{rx}^{\varepsilon-1}\upsilon'(t)+b(x)\upsilon(x)=F_2(x),\quad 0<x<r, \tag{32}$$

while

$$\upsilon(0)=0. \tag{33}$$

Once the operator $D_{rx}^{1-\varepsilon}$ has been applied to both sides of the equation (32) it could be represented as follows

$$\upsilon'(x)+D_{rx}^{1-\varepsilon}b(t)\upsilon(t)=D_{rx}^{1-\varepsilon}F_2(t),\quad 0<x<r. \tag{34}$$

Integrating equation (34) over x ranging from x to r taking into account (33) we arrive at the equation of the form

$$\upsilon(x)+\frac{1}{\Gamma(\varepsilon)}\int\limits_0^r b(t)K(x,t)\upsilon(t)dt=-\frac{1}{\Gamma(\varepsilon)}\int\limits_0^r K(x,t)F_2(t)dt, \tag{35}$$

equivalent to problem (32)–(33), where $K(x,t)=\begin{cases} t^{\varepsilon-1}-(t-x)^{\varepsilon-1}, & 0\le t<x,\\ t^{\varepsilon-1}, & x<t\le r.\end{cases}$

If $b(x)\in C^1[0,r]\cap C^3(0,r)$ is a positive non-decreasing function, then there is a unique regular solution of equation (35) [6; 133]. Then $\nu(x)=\upsilon'(x)$ and $\tau(x)$ are found by one of the formulas (30) or (31).

In the case, when $b(x)=b=const$ the solution of equation (34) is written explicitly by the formula:

$$\upsilon(x)=-\frac{1+b\int\limits_x^r(t-x)^{\varepsilon-1}\,E_{1/\varepsilon}\left[b\,(t-x)^{\varepsilon};\,\varepsilon\right]dt}{1+b\int\limits_0^r t^{\varepsilon-1}E_{1/\varepsilon}\left(b\,t^{\varepsilon};\,\varepsilon\right)dt}\left[D_{0r}^{-\varepsilon}F_2(t)+b\int\limits_0^r t^{\varepsilon-1}\,E_{1/\varepsilon}\left(b\,t^{\varepsilon};\,\varepsilon\right)\,D_{rt}^{-\varepsilon}F_2\left(s\right)dt\right]+$$

$$+D_{rx}^{-\varepsilon}F_2(t)+b\int\limits_x^r (t-x)^{\varepsilon-1}\ E_{1/\varepsilon}\left[b\ (t-x)^{\varepsilon}\,;\,\varepsilon\right]D_{rt}^{-\varepsilon}F_2\left(s\right)dt$$

provided that $1+b\int\limits_0^r t^{\varepsilon-1}\,E_{1/\varepsilon}\left(b\,t^{\varepsilon};\,\varepsilon\right)dt\neq 0$. If $b\geq 0$, then the fulfillment of this inequality is obvious.

Once the functions $\tau(x)$ and $\nu(x)$ have been found similarly as for the previous problem 1, the solution to problem 2 in the domain Ω_1 is written employing one of the formulas (14), (15) or (16), and in the domain Ω_2 the problem (12)–(13) for equation (3) is solved by the formula (22).

References

1 Смирнов М.М. Уравнения смешанного типа / М.М. Смирнов. – М.: Наука, 1970. – 296 с.

2 Protter M.H. The Cauchy problem for a hyperbolic second-order equation with data on the parabolic line / M.H. Protter // Canad. J. of Math. – 1954. – 6. – P. 542-553.

3 Бицадзе А.В. Уравнения смешанного типа / А.В. Бицадзе. – М.: Изд-во АН СССР, 1959. – 164 с.

4 Лыков А.В. Применение методов термодинамики необратимых процессов к исследованию тепло- и массообмена / А.В. Лыков // Инж.-физ. журн. – 1955. – 9. – №3. – С. 287-304.

5 Нахушев А.М. Уравнения математической биологии / А.М. Нахушев. – М.: Высш. шк., 1995. – 301 с.

6 Нахушев А.М. Дробное исчисление и его применение / А.М. Нахушев. – М.: Физматлит, 2003. – 272 с.

7 Берс Л. Математические вопросы дозвуковой и околозвуковой газовой динамики / Л. Берс. – М.: Иностранная литература, 1961. – 208 с.

8 Франкль Ф.И. Избранные труды по газовой динамике / Ф.И. Франкль. – М.: Наука, 1973. – 771 с.

9 Самко С.Г. Интегралы и производные дробного порядка и некоторые их приложения / С.Г. Самко, А.А. Килбас, О.И. Маричев. – Минск: Наука и техника, 1987. – 688 с.

10 Псху А.В. Уравнения в частных производных дробного порядка / А.В. Псху. – М.: Наука, 2005. – 199 с.

11 Джрбашян М.М. Интегральные преобразования и представления функций в комплексной плоскости / М.М. Джрбашян. – М.: Наука, 1966. – 672 с.

12 Кальменов Т.Ш. Критерий единственности решения задачи Дарбу для одного вырождающегося гиперболического уравнения / Т.Ш. Кальменов // Дифференц. уравнения. – 1971. – 7. – №1. – С. 178-181.

13 Балкизов Ж.А. Краевая задача для вырождающегося внутри области гиперболического уравнения / Ж.А. Балкизов // Изв. высш. учеб. зав. Северо-Кавказский регион. Сер. Естественные науки. – 2016. – №1(189). – С. 5-10.

14 Балкизов Ж.А. Первая краевая задача для вырождающегося внутри области гиперболического уравнения / Ж.А. Балкизов // Владикавказ. мат. журн. – 2016. – 18. – № 2. – С. 19-30.

15 Кумыкова С.К. Об одной краевой задаче для гиперболического уравнения, вырождающегося внутри области / С.К. Кумыкова, Ф.Б. Нахушева // Дифференц. уравнения. – 1978. – 14. – № 1. – С. 50–65.

16 Балкизов Ж.А. Краевые задачи с данными на противоположных характеристиках для смешанно-гиперболического уравнения второго порядка // Докл. Адыгской (Черкесской) Междунар. акад. наук. – 2020. – 20. – № 3. – С. 6–13.

17 Балкизов Ж.А. Краевые задачи для смешанно-гиперболического уравнения / Ж.А. Балкизов // Вестн. Дагестан. гос. ун-та. Сер. 1: Естественные науки. – 2021. – 36. – № 1. – С. 7–14.

18 Жегалов В.И. Краевая задача для уравнения смешанного типа с граничными условиями на обеих характеристиках и с разрывами на линии перехода / В.И. Жегалов // Ученые записки Казанского университета. – 1962. – 122. – 3. – С. 3–16.

19 Нахушев А.М. Новая краевая задача для одного вырождающегося гиперболического уравнения / А.М. Нахушев // Докл. Акад. наук СССР. – 1969. – 187. – № 4. – С. 736–739.

20 Нахушев А.М. О некоторых краевых задачах для гиперболических уравнений и уравнений смешанного типа / А.М. Нахушев // Дифференц. уравнения. – 1969. – 5. – № 1. – С. 44–59.

21 Салахитдинов М.С. О некоторых краевых задачах для гиперболического уравнения, вырождающегося внутри области / М.С. Салахитдинов, М. Мирсабуров // Дифференц. уравнения. – 1981. – 17. – № 1. – С. 129–136.

22 Салахитдинов М.С. О двух нелокальных краевых задачах для вырождающегося гиперболического уравнения / М.С. Салахитдинов, М. Мирсабуров // Дифференц. уравнения. – 1982. – 18. – № 1. – С. 116–127.

23 Репин О.А. Задача с нелокальными условиями на характеристиках для уравнения влагопереноса / С.В. Ефимова, О.А. Репин // Дифференц. уравнения. – 2004. – 40. – № 1. – С. 116–127.

24 Репин О.А. О задаче с операторами М. Сайго на характеристиках для вырождающегося внутри области гиперболического уравнения / О.А. Репин // Вестн. Самар. гос. техн. ун-та. Сер. физ.-мат. науки. – 2006. – Вып. 43. – С. 10–14.

25 Балкизов Ж.А. Задача со смещением для вырождающегося гиперболического уравнения первого рода / Ж.А. Балкизов // Вестн. Самар. гос. техн. ун-та. Сер. физ.-мат. науки. – 2021. – 25. – №1. – С. 21–34.

26 Салахитдинов М.С. Уравнения смешанно-составного типа / М.С. Салахитдинов. – Ташкент: ФАН, 1974. – 165 с.

27 Репин О.А. Краевые задачи со смещением для уравнений гиперболического и смешанного типов / О.А. Репин. – Самара: Изд-во Самар. филиала Саратов. ун-та, 1992. – 162 с.

28 Кальменов Т.Ш. Краевые задачи для линейных уравнений в частных производных гиперболического типа / Т.Ш. Кальменов. – Шымкент: Гылая, 1993. – 328 с.

29 Салахитдинов М.С. Краевые задачи для уравнений смешанного типа со спектральным параметром / М.С. Салахитдинов, А.К. Уринов. – Ташкент: ФАН, 1997. – 165 с.

30 Нахушев А.М. Задачи со смещением для уравнений в частных производных / А.М. Нахушев. – М.: Наука, 2006. – 287 с.

31 Нахушева З.А. Нелокальные краевые задачи для основного и смешанного типов дифференциальных уравнений / З.А. Нахушева. – Нальчик: КБНЦ РАН, 2011. – 196 с.

32 Сабитов К.Б. К теории уравнений смешанного типа / К.Б. Сабитов. – М.: Физматлит, 2014. – 304 с.

33 Смирнов М.М. Вырождающиеся гиперболические уравнения / М.М. Смирнов. – Минск: Выш. шк., 1977. – 160 с.

34 Тихонов А.Н. Уравнения математической физики / А.Н. Тихонов, А.А. Самарский. – М.: МГУ; Наука, 2004. – 798 с.

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Екінші ретті аралас-гиперболалық теңдеу үшін ығысуы бар ішкі-шеттік есептер

Мақалада облыстың бір бөлігінде толқындық оператордан, ал басқасында бірінші ретті өзгеше гиперболалық оператордан тұратын екінші ретті аралас-гиперболалық теңдеу үшін ығысуы бар ішкі-шеттік есептер зерттелген. Берілген функциялар бойынша зерттелетін есептердің шешімінің бар болуын, бірегейлігін қамтамасыз ететін жеткілікті шарттар анықталды. Кейбір дербес жағдайларда зерттелетін есептердің шешімдері айқын түрде жазылған.

Кілт сөздер: толқын теңдеуі, бірінші ретті өзгеше гиперболалық теңдеу, Вольтерраның интегралдық теңдеуі, екінші типті Фредгольм интегралдық теңдеуі, Трикоми әдісі, интегралдық теңдеулер әдісі, бөлшек есептеу теориясының әдісі.

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Внутренне-краевые задачи со смещением для смешанно-гиперболического уравнения второго порядка

В статье исследованы внутренне-краевые задачи со смещением для смешанно-гиперболического уравнения второго порядка, состоящего из волнового оператора в одной части области и вырождающегося гиперболического оператора первого рода — в другой. Найдены достаточные условия на заданные функции, обеспечивающие существование единственного регулярного решения исследуемых задач. В некоторых частных случаях решения исследуемых задач выписаны в явном виде.

Ключевые слова: волновое уравнение, вырождающееся гиперболическое уравнение первого рода, интегральное уравнение Вольтерра, интегральное уравнение Фредгольма второго рода, метод Трикоми, метод интегральных уравнений, методы теории дробного исчисления.

References

1 Smirnov, M.M. (1970). *Uravneniia smeshannogo tipa [Mixed type equations].* Moscow: Nauka [in Russian].

2 Protter, M.H. (1954). The Cauchy problem for a hyperbolic second-order equation with data on the parabolic line. *Canad. J. of Math., 6,* 542–553.

3 Bitsadze, A.V. (1959). *Uravneniia smeshannogo tipa [Mixed type equations].* Moscow: Izdatelstvo AN SSSR [in Russian].

4 Lykov, A.V. (1955). Primenenie metodov termodinamiki neobratimykh protsessov k issledovaniiu teplo- i massoobmena [Application of methods of thermodynamics of irreversible processes to the study of heat and mass transfer]. *Inzhenerno-fizicheskii zhurnal – Journal of Engineering Physics and Thermophysics, 9, 3,* 287–304 [in Russian].

5 Nakhushev, A.M. (1995). *Uravneniia matematicheskoi biologii [Equations of mathematical biology].* Moscow: Vysshaia shkola [in Russian].

6 Nakhushev, A.M. (2003). *Drobnoe ischislenie i ego primenenie [Fractional calculus and its application].* Moscow: Fizmatlit [in Russian].

7 Bers, L. (1961). *Matematicheskie voprosy dozvukovoi i okolozvukovoi gazovoi dinamiki [Mathematical problems of subsonic and transonic gas dynamics].* Moscow: Inostrannaia literatura [in Russian].

8 Frankl, F.I. (1973). *Izbrannye trudy po gazovoi dinamike [Selected papers on gas dynamics].* Moscow: Nauka [in Russian].

9 Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1987). *Integraly i proizvodnye drobnogo poriadka i nekotorye ikh prilozheniia [Integrals and derivatives of fractional order and some of their applications].* Minsk: Nauka i tekhnika [in Russian].

10 Pskhu, A.V. (2005). *Uravneniia v chastnykh proizvodnykh drobnogo poriadka [Fractional partial differential equations].* Moscow: Nauka [in Russian].

11 Dzhrbashian, M.M. (1966). *Integralnye preobrazovaniia i predstavleniia funktsii v kompleksnoi ploskosti [Integral transformations and representations of functions in the complex plane].* Moscow: Nauka [in Russian].

12 Kalmenov, T.Sh. (1971). *Kriterii edinstvennosti resheniia zadachi Darbu dlia odnogo vyrozhdaiushchegosia giperbolicheskogo uravneniia [A criterion for unique solvability for Darboux problem for a degenerate hyperbolic equation]. Differentsialnye uravneniia – Differential Equations, 7, 1,* 178–181 [in Russian].

13 Balkizov, Zh.A. (2016). Kraevaia zadacha dlia vyrozhdaiushchegosia vnutri oblasti giperbolicheskogo uravneniia [A boundary value problem for a hyperbolic equation degenerating inside the domain]. *Izvestiia vysshikh uchebnykh zavedenii. Severo-Kavkazskii region. Seriia Yestestvennye nauki – Bulletin Of Higher Education Institutes North Caucasus Region. Natural Sciences, 1(189),* 5–10 [in Russian].

14 Balkizov, Zh.A. (2016). Pervaia kraevaia zadacha dlia vyrozhdaiushchegosia vnutri oblasti giperbolicheskogo uravneniia [The first boundary value problem for a hyperbolic equation degenerating inside the domain]. *Vladikavkazskii matematicheskii zhurnal – Vladikavkaz Mathematical Journal, 18, 2,* 19–30 [in Russian].

15 Kumykova, S.K., & Nakhusheva, F.B. (1978). Ob odnoi kraevoi zadache dlia giperbolicheskogo uravneniia, vyrozhdaiushchegosia vnutri oblasti [On a boundary value problem for a hyperbolic equation degenerating inside a domain]. *Differentsialnye uravneniia – Differential Equations, 14, 1,* 50–65 [in Russian].

16 Balkizov, Zh.A. (2020). Kraevye zadachi s dannymi na protivopolozhnykh kharakteristikakh dlia smeshanno-giperbolicheskogo uravneniia vtorogo poriadka [Boundary value problems with data on opposite characteristics for a second-order mixed-hyperbolic equation]. *Doklady Adygskoi (Cherkesskoi) Mezhdunarodnoi akademii nauk – Reports of Adyghe (Circassian) International Academy of Sciences, 20, 3,* 6–13 [in Russian].

17 Balkizov, Zh.A. (2021). Kraevye zadachi dlia smeshanno-giperbolicheskogo uravneniia [Boundary value problems for a mixed-hyperbolic equation]. *Vestnik Dagestanskogo gosudarstvennogo universiteta. Seriia 1: Yestestvennye nauki – Herald of Dagestan State University. Series 1. Natural Sciences, 36, 1,* 7–14 [in Russian].

18 Zhegalov, V.I. (1962). Kraevaia zadacha dlia uravneniia smeshannogo tipa s granichnymi usloviiami na obeikh kharakteristikakh i s razryvami na linii perekhoda [Boundary value problem for a mixed-type equation with boundary conditions on both characteristics and with discontinuities on the transition line]. *Uchenye zapiski Kazanskogo universiteta – Proceedings of Kazan University, 122, 3,* 3–16 [in Russian].

19 Nakhushev, A.M. (1969). *Novaia kraevaia zadacha dlia odnogo vyrozhdaiushchegosia giperbolicheskogo uravneniia [A new boundary value problem for a degenerate hyperbolic equation]. Doklady Akademii nauk SSSR – Proceedings of the USSR Academy of Sciences, 187, 4,* 736–739 [in Russian].

20 Nakhushev, A.M. (1969). O nekotorykh kraevykh zadachakh dlia giperbolicheskikh uravnenii i uravnenii smeshannogo tipa [On some boundary value problems for hyperbolic equations and equations of mixed type]. *Differentsialnye uravneniia – Differential Equations, 5, 1,* 44–59 [in Russian].

21 Salakhitdinov, M.S., & Mirsaburov, M. (1981). O nekotorykh kraevykh zadachakh dlia giperbolicheskogo uravneniia, vyrozhdaiushchegosia vnutri oblasti [On some boundary value problems for a hyperbolic equation degenerating inside a domain]. *Differentsialnye uravneniia – Differential Equations, 17, 1,* 129–136 [in Russian].

22 Salakhitdinov, M.S., & Mirsaburov, M. (1982). O dvukh nelokalnykh kraevykh zadachakh dlia vyrozhdaiushchegosia giperbolicheskogo uravneniia [On two nonlocal boundary value problems for a degenerate hyperbolic equation]. *Differentsialnye uravneniia – Differential Equations, 18, 1,* 116–127 [in Russian].

23 Efimova, S.V., & Repin, O.A. (2004). Zadacha s nelokalnymi usloviiami na kharakteristikakh dlia uravneniia vlagoperenosa [Problem with nonlocal conditions on characteristics for the moisture transfer equation]. *Differentsialnye uravneniia – Differential Equations, 40, 1,* 116–127 [in Russian].

24 Repin, O.A. (2006). O zadache s operatorami M. Saigo na kharakteristikakh dlia vyrozhdaiushchegosia vnutri oblasti giperbolicheskogo uravneniia [On the problem with M. Saigo's operators on characteristics for a hyperbolic equation degenerating inside the domain]. *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Seriia Fiziko-matematicheskie nauki – Journal of Samara State Technical University, Ser. Physical and Mathematical Sciences, 43,* 10–14 [in Russian].

25 Balkizov, Zh.A. (2021). Zadacha so smeshcheniem dlia vyrozhdaiushchegosia giperbolicheskogo uravneniia pervogo roda [Problem with displacement for first kind degenerate hyperbolic equation]. *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Seriia Fiziko-matematicheskie nauki – Journal of Samara State Technical University, Ser. Physical and Mathematical Sciences, 25, 1,* 21–34 [in Russian].

26 Salakhitdinov, M.S. (1974). *Uravneniia smeshanno-sostavnogo tipa [Mixed-compound equations].* Tashkent: FAN [in Russian].

27 Repin, O.A. (1992). *Kraevye zadachi so smeshcheniem dlia uravnenii giperbolicheskogo i smeshannogo tipov [Boundary value problems with displacement for hyperbolic and mixed type equations].* Samara: Izdatelstvo Samarskogo filiala Saratovskogo universiteta [in Russian].

28 Kalmenov, T.Sh. (1993). *Kraevye zadachi dlia lineinykh uravnenii v chastnykh proizvodnykh giperbolicheskogo tipa [Boundary value problems for hyperbolic type linear partial differential equations].* Shymkent: Gylaia [in Russian].

29 Salakhitdinov, M.S., & Urinov, A.K. (1997). *Kraevye zadachi dlia uravnenii smeshannogo tipa so spektralnym parametrom [Boundary value problems for equations of mixed type with a spectral parameter].* Tashkent: FAN [in Russian].

30 Nakhushev, A.M. (2006). *Zadachi so smeshcheniem dlia uravnenii v chastnykh proizvodnykh [Boundary value problems with shift for partial differential equations].* Moscow: Nauka [in Russian].

31 Nakhusheva, Z.A. (2011). *Nelokalnye kraevye zadachi dlia osnovnogo i smeshannogo tipov differentsialnykh uravnenii [Nonlocal boundary value problems for differential equations of basic and mixed types].* Nalchik: KBNTS RAN [in Russian].

32 Sabitov, K.B. (2014). *K teorii uravnenii smeshannogo tipa [On the theory of mixed type equations].* Moscow: Fizmatlit [in Russian].

33 Smirnov, M.M. (1977). *Vyrozhdaiushchiesia giperbolicheskie uravneniia [Degenerate hyperbolic equations].* Minsk: Vysheishaia shkola [in Russian].

34 Tikhonov, A.N., & Samarskii, A.A. (2004). *Uravneniia matematicheskoi fiziki [Equations of mathematical physics].* Moscow: Moskovskii gosudarstvennyi universitet; Nauka [in Russian].

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Examples of weakly compact sets in Orlicz spaces

This paper provides a number of examples of relatively weakly compact sets in Orlicz spaces. We show some results arising from these examples. Particularly, we provide a criterion which ensures that some Orlicz function is increasing more rapidly than another (in a sense of T. Ando). In addition, we point out that if a bounded subset K of the Orlicz space L_Φ is not bounded by the modular $\boldsymbol{\Phi}$, then it is possible for a set K to remain unbounded under any modular Ψ increasing more rapidly than Φ.

Keywords: conjugate (complementary) functions, relative weak compactness, Orlicz spaces, $N-$functions.

Introduction

We provide a number of examples of relatively weakly compact sets in Orlicz spaces based mainly on criteria obtained by a classical work of T. Ando from 1962 (see [1]). It should be noted that there is a shortage of such examples in the literature. Some (maybe the most important) examples may be found in the classical book by M.M. Rao and Z.D. Ren [2]. Another paper, devoted to the study of weak compactness in Orlicz spaces that we use extensively in this paper is by J. Alexopoulos [3].

On the contrary, weak compactness criteria in both Orlicz function and sequence spaces have been stidied by many researchers, see, for example [1–14], and references therein.

In particular, T. Ando see [1] obtained weak compactness criteria in Orlicz (function) spaces from the perspective of Köthe duality. The study results of T. Ando were extended (with some restrictive condition) from the setting of finite measure spaces to the setting of σ-finite measure spaces in the work of M. Nowak in 1986 [11]. The objective of this paper is to study such criteria and provide examples that satisfy these criteria. We also prove some related propositions (see Propositions 2.13 and 2.17).

1 Preliminaries

Initially, the study provides the definition of an N-function (as in [1]), which will be used throughout the text.

Definition 1.1. A convex function $\Phi : [0, \infty) \to [0, \infty)$ is called an N-function if
(i) $\Phi(0) = 0$,
(ii) $\frac{\Phi(\lambda)}{\lambda} \to \infty$ as $\lambda \to \infty$.

We note that in the above definition by T. Ando, it is not necessarily true that $\frac{\Phi(\lambda)}{\lambda} \to 0$ as $\lambda \to 0+$, unlike in many other classical works (e.g., [15, formulae 1.12 and 1.15], [2; 13], [3, Proposition 1.1]).

The following two definitions specify some important classes of N-functions.

Definition 1.2. ([2, Definition 1] and [3, Definition 1.5]) An $N-$function Φ is said to satisfy the $\triangle_2$ condition ($\Phi \in \triangle_2$) if $\limsup_{x\to\infty} \frac{\Phi(2x)}{\Phi(x)} < \infty$. That is, there is a $K > 0$ so that $\Phi(2x) \le K \cdot \Phi(x)$ for large values of x.

Definition 1.3. ([2, Definition 2] and [3, Definition 1.8]) An $N-$function Φ is said to satisfy the ∇_2 condition ($\Phi \in \nabla_2$) if there is a $K > 0$ so that $(\Phi(x))^2 \le \Phi(Kx)$ for large values of x.

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1.1 Decreasing rearrangement

Let (I, m) denote the measure space, where $I = (0, \infty)$ (resp. $(0, 1)$), equipped with Lebesgue measure m. Let $L(I, m)$ be the space of all measurable real-valued functions on I equipped with Lebesgue measure m. Define $S(I, m)$ to be the subset of $L(I, m)$, which consists of all functions f such that $m(\{t : |f(t)| > s\}) < \infty$ for some $s > 0$. Note that if $I = (0, 1)$, then $S(I, m) = L(I, m)$.

For $f \in S(I, m)$, we denote by $\mu(f)$ the decreasing rearrangement of the function $|f|$. That is,

$$\mu(t, f) = \inf\{s \geq 0 : \; m(\{|f| > s\}) \leq t\}, \quad t > 0.$$

1.2 Orlicz spaces

Definition 1.4. A function $G : [0, \infty) \to [0, \infty]$ is said to be an Orlicz function if [9; 258]
(i) $G(0) = 0$,
(ii) G is not identically equal to zero,
(iii) G is convex,
(iv) G is continuous at zero.

It follows from the definitions that not every $N-$function is an Orlicz function (e.g., an N-function may be discontinuous at zero). The converse also does not hold. For example, the function $G(t) = t$ is an Orlicz function but not an $N-$function. For an Orlicz function (or N-function) G we shall consider an (extended) real-valued function $\mathbf{G}(f)$ (also called the modular defined by an N-function G) defined on the class of all measurable functions f on I, by

$$\mathbf{G}(f) = \int_I G(|f(t)|) dt.$$

The set

$$L_G = \{f \in S(I, m) : \quad \|f\|_{L_G} < \infty\},$$

where

$$\|f\|_{L_G} = \inf\left\{c > 0 : \quad \int_I G\left(\frac{|f|}{c}\right) dm \leq 1\right\}$$

is called an Orlicz space defined by the Orlicz function (or N-function) G (equipped with Orlicz norm).

In fact, we have the following ([2, Chapter 3.5, Theorem 1]):

Proposition 1.5. If an N-function $\Phi \in \triangle_2$, then L_Φ is separable (provided the measure space is separable).

It should be stated that notions of N-functions and Orlicz functions used interchangeably in many situations. However, in this text we will denote N-functions by Greek letters Φ, Ψ and Orlicz functions by Latin letters G, F to distinguish between them.

Using various (partial) order relations on Orlicz functions one may define the corresponding relations in the Orlicz spaces. We also note that since (in this paper) Orlicz (function) spaces are defined on finite measure spaces we only need local order relations. However, some results will be also stated for Orlicz spaces on positive half-line (with small differences on local relations).

We define the notion of majorization for Orlicz functions (for σ-finite measure spaces). Let G_1 and G_2 be two Orlicz functions.

Definition 1.6. (e.g., [16, Definition 16.1.1]) We say that
(1) G_1 majorises G_2 at 0 ($G_1 \succ_0 G_2$) if there exist positive numbers a, b, x_0 such that

$$G_2(x) \leq bG_1(ax) \quad \text{for all} \quad 0 \leq x \leq x_0.$$

(2) G_1 majorises G_2 at ∞ ($G_1 \succ_\infty G_2$) if there exist positive numbers a, b, x_0 such that

$$G_2(x) \leq bG_1(ax) \quad \text{for all} \quad x \geq x_0.$$

(3) G_1 majorises G_2 ($G_1 \succ G_2$) if $G_1 \succ_0 G_2$ and $G_1 \succ_\infty G_2$.

Moreover, one can set $b = 1$ in the above definition (see [16, Proposition 16.1.2]). Also, the condition $G_1 \succ G_2$ may be checked via the following (see [16, Proposition 16.1.3]):

Proposition 1.7. $G_1 \succ G_2$ if and only if

$$G_2(x) \leq bG_1(ax), \quad x \geq 0$$

for some $b > 0$ and $a > 0$.

Also, we provide a definition of equivalent Orlicz functions on $\sigma-$finite measure space (see [16, Definition 16.3.1]):

Definition 1.8. Two Orlicz functions G_1 and G_2 are called equivalent, denoted $G_1 \approx G_2$, if $G_1 \succ G_2$ and $G_2 \succ G_1$.

The following definition for equivalence of N-functions on finite measure space may be found in [3, Definition 1.3]:

Definition 1.9. For $N-$functions Φ_1, Φ_2 we write $\Phi_1 \prec \Phi_2$ if there is a $K > 0$ so that $\Phi_1(x) \leq \Phi_2(Kx)$ for large values of x. If $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$ then we say that Φ_1 and Φ_2 are equivalent.

Note for finite measure space, the notion of majorisation is slightly different as we do not care about majorisation at zero.

We will denote by Ψ the function complementary (or conjugate) to an $N-$function Φ in the sense of Young (with the condition $\frac{\Phi(t)}{t} \to 0+$ as $t \to 0$), defined by ([15; 11])

$$\Psi(t) = \sup\{s|t| - \Phi(s) : \quad s \geq 0\}.$$

We notice that Ψ is again an $N-$function (see [9; 258]).

2 Weakly compact sets in Orlicz spaces

In this section, we recall known criteria of relative weak compactness in Orlicz spaces and provide examples of such sets. We will also state some concluding remarks and prove related propositions.

The following theorem was proved by T. Ando in [1, Theorem 1].

Theorem 2.1. Let Φ be an N-function and let (Ω, Σ, μ) be a finite measure space. A subset K of L_Φ is relatively $\sigma(L_\Phi, L_\Psi)-$compact if and only if

$$\frac{\mathbf{\Phi}(\lambda f)}{\lambda} \to \alpha \int_\Omega |f(t)| d\mu \quad \text{as} \quad \lambda \downarrow 0$$

uniformly with respect to $f(t) \in K$, where $\alpha = \lim_{\lambda \to 0+} \Phi(\lambda)/\lambda$.

It is worthwhile to note that Theorem 2.1 (unlike many others) is valid for any $N-$function (in the sense of Ando), that is, α is not necessarily zero by definition. We note that the extension of this Theorem to the σ-finite case was given by M. Nowak [14, Theorem 1.1] only in the case $\alpha = 0$.

Below we provide examples of relatively weakly compact sets in $L_\Phi[0,1]$ by using Theorem 2.1. Later we will provide another criteria of weak compactness in Orlicz spaces and apply these criteria to the following examples.

Example 2.2. (i) Let $\Phi(x) = e^x - 1$. Then the subset $K = \{f_p(x) = x^p : \quad p \geq 1\}$ of $L_\Phi[0,1]$ is relatively weakly compact. Indeed, K is bounded and since $\alpha = \lim_{\lambda \to 0} \frac{e^\lambda - 1}{\lambda} = 1$, it is enough to show (by Theorem 2.1) that

$$\frac{\int_0^1 e^{\lambda t^p} dt - 1}{\lambda} \to \int_0^1 t^p dt = \frac{1}{p+1}$$

uniformly with respect to $p \geq 1$ as $\lambda \downarrow 0$.

Applying the L'Hopital's rule, we get

$$0 \leq \lim_{\lambda \downarrow 0} \frac{\int_0^1 \left(e^{\lambda t^p} - \lambda t^p - 1\right) dt}{\lambda} = \lim_{\lambda \downarrow 0} \frac{\int_0^1 \left(t^p e^{\lambda t^p} - t^p\right) dt}{1} =$$

$$\leq \lim_{\lambda \downarrow 0} \int_0^1 \left(e^{\lambda t^p} - 1\right) dt \leq \lim_{\lambda \downarrow 0} \int_0^1 \left(e^{\lambda t} - 1\right) dt = \frac{e^\lambda - 1}{\lambda} - 1 \to 0$$

as $\lambda \to 0$ uniformly with respect to $f \in K$ (independent of p).

For example, when $p = 1$, then

$$\frac{\int_0^1 e^{\lambda t} dt - 1}{\lambda} = \frac{e^\lambda - 1 - \lambda}{\lambda^2} \to \frac{1}{2}$$

as $\lambda \downarrow 0$ as desired.

However, when $p > 1$, the integral $\int_0^1 e^{\lambda t^p} dt$ is not expressed in terms of elementary functions.

(ii) Let $\Phi(x) = e^x - x - 1$ and the subset K as in (i). Note in this case $\alpha = 0$ (so Φ is an $N-$function). Obviously

$$\frac{\int_0^1 \left(e^{\lambda t^p} - \lambda t^p - 1\right) dt}{\lambda} \to 0$$

uniformly with respect to p as $\lambda \downarrow 0$ as it is reduced to case (i). For example, when $p = 1$

$$\frac{\mathbf{\Phi}(\lambda f)}{\lambda} = \frac{\int_0^1 (e^{\lambda t} - \lambda \cdot t - 1) dt}{\lambda} = \frac{2e^\lambda - 2 - \lambda^2 - 2\lambda}{2\lambda^2} \to 0$$

as $\lambda \downarrow 0$. As in (i), The uniform convergence holds for every $p > 1$, however, as in example (i), the integral is not expressed in terms of elementary functions either.

We note that in the statement of Theorem 2.1 the uniform convergence is crucial as the following example illustrates, i.e., pointwise convergent is not sufficient.

Example 2.3. Let $\Phi(x) = x \cdot \ln(x+1)$ and $K = \{f_p(x) = e^{px} : \ p > 0\}$ be a subset of $L_\Phi[0,1]$. Note that Φ is an $N-$function with $\alpha = 0$. Now we check the condition of uniform convergence as in Theorem 2.1:

$$\frac{\mathbf{\Phi}(\lambda f)}{\lambda} = \frac{\int_0^1 \lambda e^{px} \cdot \ln(\lambda e^{px} + 1) dx}{\lambda} =$$

$$= \frac{\lambda e^p \cdot \ln(\lambda e^p + 1) - \lambda \cdot \ln(\lambda + 1) - \lambda e^p + \lambda + \ln(\lambda e^p + 1) - \ln(\lambda + 1)}{p \cdot \lambda}$$

$$\approx \frac{(\lambda e^p)^2 - \lambda^2}{p \cdot \lambda} = \frac{\lambda(e^{2p} - 1)}{p} \to 0$$

as $\lambda \downarrow 0$. However, it is easy to see that the convergence is not uniform with respect to p since

$$\sup_{p>0} \frac{\mathbf{\Phi}(\lambda f)}{\lambda} = \infty$$

for all $\lambda > 0$. Hence, by Theorem 2.1, the subset K is not relatively weakly compact in $L_\Phi[0,1]$.

Note, however, if $0 < p \leq 1$, that is $K = \{f_p(x) = e^{px} : \ 0 < p \leq 1\}$, then K is relatively weakly compact in $L_\Phi[0,1]$. We also note that the set K in Example 2.3 is not bounded (in norm) in $L_\Phi[0,1]$. Hence, this fact clearly implies that K is not relatively weakly compact. In general, norm boundedness does not imply weak compactness.

Also note that $L_\Phi[0,1]$ is separable since $\Phi \in \triangle_2$ by Proposition 1.5. Indeed, $\Phi(2x) \leq K \cdot \Phi(x)$ for large x since $\ln(2x+1) \leq 3\ln(x+1) = \ln(x+1)^3$ or equivalently, $2x+1 \leq (x+1)^3$ for large x.

Below we state another two criteria of weak compactness criteria in Orlicz spaces due to T. Ando.

Lemma 2.4. (see [1; 171]) A subset K of $L_\Phi[0,1]$ is (relatively) weakly compact if and only if it is weakly bounded and equi continuous in the following sense:

$$\sup_{f\in K} \int_E |f(t) \cdot g(t)| d\mu \to 0 \ \text{ as } \ \mu(E) \to 0, \quad E \subset [0,1], \quad g(t) \in L_\Psi[0,1].$$

Lemma 2.5. ([1; 172]) Let B be a σ-algebra of subsets of $(0,1)$. When B is atomless, boundedness by modular $\mathbf{\Phi}(f)$ implies (relative) weak compactness, if and only if $\Phi(\lambda)$ has (∇_2), i.e.

$$\liminf_{\lambda\to\infty} \frac{\Phi(\eta\lambda)}{\Phi(\lambda)} \geq 2\eta \ \text{ for some } \ \eta > 0.$$

Remark. Note that Lemma 2.5 may also be applied to show that the set K in Example 2.2 (both (i) and (ii)) is relatively weakly compact in $L_\Phi[0,1]$. Indeed, as for (i), the boundedness by the modular $\mathbf{\Phi}(f)$ is obvious. Also, (with $\eta = 2$)

$$\liminf_{\lambda\to\infty} \frac{\Phi(2\lambda)}{\Phi(\lambda)} = \liminf_{\lambda\to\infty} \frac{e^{2\lambda}-1}{e^\lambda - 1} \geq 2 \cdot 2 = 4.$$

As for (ii), we have

$$\int_0^1 \Phi(x^p)dx = \int_0^1 (e^{x^p} - x^p - 1)dx \leq \int_0^1 e^{x^p} dx \leq 3, \quad \text{for all } \ p > 0.$$

Taking supremum over all $p > 0$, we obtain boundedness of a set K by modular $\mathbf{\Phi}$. Also, by setting $\eta = 2$, we obtain

$$\liminf_{\lambda\to\infty} \frac{e^{2\lambda} - 2\lambda - 1}{e^\lambda - \lambda - 1} \geq 2 \cdot 2 = 4.$$

Recall that a set K in Example 2.3 is not relatively weakly compact, which may not be proved via using Lemma 2.5 since a set K is not bounded by the modular $\mathbf{\Phi}(f)$. Indeed,

$$\mathbf{\Phi}(f) = \int_0^1 \Phi(e^{px})dx = \int_0^1 e^{px} \ln(e^{px}+1)dx =$$

$$= \frac{1}{p}[e^p \ln(e^p+1) - 2\ln 2 - e^p + 1 + \ln(e^p+1)] \to \infty \quad \text{as } \ p \to \infty.$$

Though $\Phi(\lambda)$ fails (∇_2),

$$\liminf_{\lambda\to\infty} \frac{\Phi(\eta\lambda)}{\Phi(\lambda)} = \liminf_{\lambda\to\infty} \frac{\eta\lambda\ln(\eta\lambda+1)}{\lambda\ln(\lambda+1)} = \eta < 2\eta \quad \text{for all } \ \eta > 0.$$

Note that in general, if K is not relatively weakly compact, then it is not necessarily true that K is not bounded by the modular $\mathbf{\Phi}(f)$.

Now we provide an example of a set K such that K is bounded by modular $\mathbf{\Phi}$ and $\Phi(\lambda)$ that does not have (∇_2).

Example 2.6. Let $\Phi(x) = x \cdot \ln(x+1)$ and $K = \{f_p(x) = x^p : \quad p > 0\}$. Note K is bounded by the modular $\mathbf{\Phi}(f)$. Indeed,

$$\sup_{p>0} \int_0^1 x^p \cdot \ln(x^p + 1) dx \le 1.$$

However, since $\Phi(x)$ fails (∇_2), we conclude that K is not relatively weakly compact by Lemma 2.5.

Now we state the relation between conjugate N-functions in terms of $\triangle_2, \nabla_2$ relations (see [2, Chapter 2.3, Theorem 3]).

Remark 2.7. $\Phi(x)$ has ∇_2 if and only if its conjugate $\Psi(x)$ has $\triangle_2$.

For example, let $\Phi(x) = e^x - x - 1, \ x \ge 0$. Then it is easy to see that $\Phi \in \nabla_2$. Its conjugate function $\Psi(x) = x \cdot \ln(x+1) - x + \ln(x+1), \ x \ge 0$ has $\triangle_2$.

The following definition may be found in [17, Definition 53.1], [2, Chapter 1.3].

Definition 2.8. A function $\Phi : \mathbb{R} \to \mathbb{R}$ is called a *Young function* if and only if:

(i) $\Phi(x) = \int_0^{|x|} \phi(s) ds$ for all $x \in \mathbb{R}$;

(ii) $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and strictly increasing;

(iii) $\phi(0) = 0$ and $\phi(s) \to \infty$ as $s \to \infty$.

Throughout this paper, however, we restrict our attention to Young functions $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. It is noted that every Young function is not an $N-$function (in the sense of Ando). However, in most papers the definition of a Young function coincides with the definition of an $N-$function (with the condition $\lim_{x\to 0+} \frac{\Phi(x)}{x} = 0$), for example, [3, Definition 1.1]. Clearly, not every Orlicz function is a Young (or N-)function. For example, $\Phi(x) = x \arctan x$ is such a function.

The notion of Orlicz functions (as well as of $N-$functions or Young functions) is known since 1940s. Nonetheless, for the sake of convenience, we provide a list of Orlicz functions below. We note that some of them are not $N-$functions, and some are not Young functions.

Examples of Orlicz functions $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ are:

$\Phi(x) = x^p, \quad p \ge 1$ (corresponding to Lebesgue spaces L_p. If $p > 1$, then it is also both an $N-$function and a Young function);

$\Phi(x) = \begin{cases} 0 & \text{if} \quad 0 \le x \le 1, \\ \infty & \text{if} \quad x > 1; \end{cases}$ (corresponding to the Lebesgue space L_∞, neither Young nor N-function).

$\Phi(x) = e^x - 1$ (neither Young nor $N-$function);

$\Phi(x) = e^x - x - 1$ (both Young and $N-$function);

$\Phi(x) = e^x - \frac{x^2}{2} - x - 1$, and in general $e^x - \sum_{k=0}^n \frac{x^k}{k!}$ for any $n \in \mathbf{N}$;

$\Phi(x) = x \ln(x+1)$ (not Young but $N-$function);

$\Phi(x) = (x+1)\ln(x+1)$ (neither Young nor $N-$function);

$\Phi(x) = x \ln(x^2+1)$ (both Young and $N-$function);

$\Phi(x) = xe^{x^p}$, where $p \ge 1$ (neither Young nor $N-$function);

$\Phi(x) = (x+e)\ln(x+e) - (x+e)$ (neither Young nor $N-$function);

$\Phi(x) = x \arctan x$ (not Young but $N-$function);

$\Phi(x) = \begin{cases} x^2 & \text{if} \quad 0 \le x < 1, \\ x^3 & \text{if} \quad x \ge 1; \end{cases}$

$\Phi(x) = \begin{cases} x^2 & \text{if} \quad 0 \le x < 1/2, \\ x - 1/4 & \text{if} \quad x \ge 1/2; \end{cases}$ etc.

While discussing relative weak compactness in Orlicz space L_{Φ_1}, it is natural to ask whether there is another Orlicz function $\Phi_2 \ne \Phi_1$ such that L_{Φ_2} coincides with L_{Φ_1}. The following theorem shows equivalent conditions when two Orlicz spaces L_{Φ_1} and L_{Φ_2} coincide on $(0, \infty)$ (as sets), see, for example [16, Theorem 16.3.2].

Theorem 2.9. Let Φ_1 and Φ_2 be two Orlicz functions. The following are equivalent:
(1) $\Phi_1 \approx \Phi_2$ (i.e., $\Phi_1 \succ \Phi_2$ and $\Phi_2 \succ \Phi_1$ as in Definition 1.6) ;
(2) $L_{\Phi_1}(0,\infty) = L_{\Phi_2}(0,\infty)$ as sets;
(3) $||\cdot||_{L_{\Phi_1}}$ and $||\cdot||_{L_{\Phi_2}}$ are equivalent, i.e.,

$$a_1||f||_{L_{\Phi_1}} \leq ||f||_{L_{\Phi_2}} \leq a_2||f||_{L_{\Phi_1}}$$

for all f and some $a_1 > 0,\ a_2 > 0$;
(4) $a_1\varphi_{L_{\Phi_1}}(x) \leq \varphi_{L_{\Phi_2}}(x) \leq a_2\varphi_{L_{\Phi_1}}(x)$ for all $x \geq 0$ and some $a_1 > 0,\ a_2 > 0$;
(5) $\Phi_1(a_1x) \leq \Phi_2(x) \leq \Phi_1(a_2x)$ for all $x \geq 0$ and some $a_1 > 0,\ a_2 > 0$.

In condition (4) above, $\varphi_{L_\Phi}(x)$ stands for the fundamental function of an Orlicz space L_Φ, and is defined as follows:

$$\varphi_{L_\Phi}(x) = \|\mathbf{1_{[0,x]}}\|_{\mathbf{L_\Phi}}.$$

Note that constants a_1 and a_2 in the above conditions (3), (4), and (5) may be chosen the same.

However, if we consider L_Φ on a finite measure space, the notion of equivalent Orlicz functions is slightly different (compare Definitions 1.6 and 1.9), which entails the corresponding changes in Theorem 2.9. For example, let

$$\Phi(x) = \begin{cases} x^2 & \text{if} \quad 0 \leq x < 1/2, \\ x - 1/4 & \text{if} \quad x \geq 1/2. \end{cases}$$

Then $L_\Phi(0,1) = L_1(0,1)$ while $L_\Phi(0,\infty) \neq L_1(0,\infty)$. Indeed, $\Phi(x) \approx x$ as in Definition 1.9, thus $L_\Phi(0,1) = L_1(0,1)$. To show that $L_\Phi(0,\infty) \neq L_1(0,\infty)$, one may consider a function $f(x) = x^{-1/2}\chi_{(0,1/2)}(x)$, which belongs to $L_1(0,\infty)$ and does not belong to $L_\Phi(0,\infty)$. Or, on the other hand, this is easily checked since one cannot have $x \leq b\Phi(ax)$ for all large x and some positive a and b. Therefore, $\Phi(x)$ is not equivalent to x on $(0,\infty)$.

Now we state Theorem 2.9 for $L_\Phi(0,1)$.

Theorem 2.10. Let Φ_1 and Φ_2 be two Orlicz functions. The following are equivalent:
(1) $\Phi_1 \approx \Phi_2$ (i.e., $\Phi_1 \succ \Phi_2$ and $\Phi_2 \succ \Phi_1$ as in Definition 1.9);
(2) $L_{\Phi_1}(0,1) = L_{\Phi_2}(0,1)$ as sets;
(3) $||\cdot||_{L_{\Phi_1}}$ and $||\cdot||_{L_{\Phi_2}}$ are equivalent, i.e.,

$$a_1||f||_{L_{\Phi_1}} \leq ||f||_{L_{\Phi_2}} \leq a_2||f||_{L_{\Phi_1}}$$

for all f and some $a_1 > 0,\ a_2 > 0$;
(4) $a_1\varphi_{L_{\Phi_1}}(x) \leq \varphi_{L_{\Phi_2}}(x) \leq a_2\varphi_{L_{\Phi_1}}(x)$ for all $x \geq x_0$ and some $a_1 > 0,\ a_2 > 0,\ x_0 > 0$;
(5) $\Phi_1(a_1x) \leq \Phi_2(x) \leq \Phi_1(a_2x)$ for all $x \geq x_0$ and some $a_1 > 0,\ a_2 > 0,\ x_0 > 0$.

The following definition will be needed to state another weak compactness criterion in Orlicz spaces.

Definition 2.11. (see [1; 173]) We say that $\Psi(x)$ is increasing more rapidly than $\Phi(x)$, if for any $\eta > 0$ there exist $\rho, x_0 > 0$ such that

$$\Psi(\rho x) \geq \rho\cdot\eta\cdot\Phi(x) \quad \text{for} \quad x \geq x_0.$$

Sometimes it is convenient to use the following equivalent definition.

Definition 2.12. (see [1; 173]) We say that $\Psi(x)$ is increasing more rapidly than $\Phi(x)$, if for any $\varepsilon > 0$ there exist $\delta, x_0 > 0$ such that

$$\varepsilon\Psi(x) \geq \frac{\Phi(\delta x)}{\delta} \quad \text{for} \quad x \geq x_0.$$

We note that $\Psi(\lambda)$ has (∇_2) if and only if Ψ is increasing more rapidly than itself [1; 173]. If $\Psi(x) \geq \Phi(x)$ for all $x \geq 0$, then it is not necessarily true that Ψ is increasing more rapidly than Φ. Now we prove a result, which allows one to check whether one Orlicz function is increasing more rapidly than another.

Proposition 2.13. Let Φ and Ψ be two Orlicz functions. If $\lim_{x\to\infty} \frac{\Psi(x)}{\Phi(x)} = \infty$, then Ψ is increasing more rapidly than Φ.

Proof. If $\lim_{x\to\infty} \frac{\Psi(x)}{\Phi(x)} = \infty$, then for any $\eta > 1$ there exists $x_1 > 0$ such that $\Psi(x) \geq \eta \cdot \Phi(x)$ for all $x \geq x_1$. Since Ψ is convex there exist $\rho, x_2 > 0$ such that $\Psi(\rho x) \geq \rho \cdot \Psi(x)$ for all $x \geq x_2$. Hence,

$$\Psi(x) \geq \rho \cdot \Psi(x) \geq \rho \cdot \eta \cdot \Phi(x)$$

for all $x \geq x_0$, where $x_0 = \max\{x_1, x_2\}$ (this is even stronger statement than required).

The following theorem is also due to T. Ando [1, Theorem 2].

Theorem 2.14. A subset K of $L_\Phi[0,1]$ is relatively weakly compact if and only if it is bounded by the modular defined by an $N-$function (depending on K) $\Psi(x)$ increasing more rapidly than $\Phi(x)$.

Example 2.15. Let $\Phi(x) = e^x - 1$, then the Orlicz function $\Psi(x) = e^{x^2} - 1$ is increasing more rapidly than $\Phi(x)$.

Indeed, fix any $\varepsilon > 0$ and choose $\delta = 1$. Then we need to show that there exists $x_0 > 0$ such that $\varepsilon \cdot (e^{x^2} - 1) \geq e^x - 1$ for all $x > x_0$. It is obvious that for any $\varepsilon > 0$ one can find such $x_0 > 0$ since $\liminf_{x\to\infty} \frac{e^{x^2}-1}{e^x-1} = \infty$. Note $\Phi(x) = e^x - 1$ is an $N-$function (in the sense of Ando) with $\alpha = \lim_{x\to 0} \frac{\Phi(x)}{x} = 1$, while Ψ is an $N-$function with $\alpha = 0$.

Now using the Theorem 2.14 we prove that a set K in Example 2.2 (both (i) and (ii)) is relatively weakly compact in $L_\Phi[0,1]$.

Example 2.16. Recall in Example 2.2 (i), $\Phi(x) = e^x - 1$. It has been shown that the subset $K = \{f_p(x) = x^p : \quad p \geq 1\}$ is relatively weakly compact in $L_\Phi[0,1]$.

Alternatively, by Theorem 2.14 and Example 2.15 it remains to show that a set K is bounded by the modular defined by an $N-$function $\Psi(x) = e^{x^2} - 1$, that is, to show that $\sup_{p\geq 1} \int_0^1 \Psi(x^p)dx < \infty$. Indeed,

$$\int_0^1 (e^{x^{2p}} - 1)dx = \int_0^1 e^{x^{2p}} dx - 1 \leq \int_0^1 e^{x^2} dx - 1 < 1/2.$$

Taking supremum over all $p \geq 1$, we obtain the desired result.

As for (ii), we note that $\Psi(x) = e^{x^2} - 1$ is also increasing more rapidly than $\Phi(x) = e^x - x - 1$, since $e^x - x - 1 \leq e^x - 1$ for all $x \geq 0$. Thus, by the previous argument we may conclude that the set K in Example 2.2 (ii) is also relatively weakly compact.

Now we show the following proposition.

Proposition 2.17. If a set K is not bounded by a modular $\mathbf{\Phi}$, defined by an Orlicz function Φ, then it is not necessarily true that K is not bounded by the modular $\mathbf{\Psi}$, defined by an Orlicz function Ψ, increasing more rapidly than Φ.

Proof. Indeed, it suffices to find an N-function function Φ, another Orlicz function Ψ, which increases more rapidly than Φ and a function f for which the inequality $\int_0^1 \Phi(f(x))dx \leq \int_0^1 \Psi(f(x))dx$ fails. For such purposes, one may choose $\Phi(x) = x$, $\Psi(x) = x^{100}$ and $f(x) = x$.

Thus, recall that a set K in Example 2.3 was not bounded by the modular $\mathbf{\Phi}$, hence by Remark 2.17 it is not necessarily true that K is not bounded by the modular $\mathbf{\Psi}$, defined by some Orlicz function Ψ, increasing more rapidly than Φ. However, since a set K is not relatively weakly compact in $L_\Phi[0,1]$ we conclude (by Theorem 2.14) that there is no such function Ψ such that K is bounded by the modular $\mathbf{\Psi}$.

Recall that the complementary (or conjugate) function Ψ to Φ in the sense of Young, is defined by (see [15; 11])

$$\Psi(t) = \sup\{s|t| - \Phi(s) : \quad s \geq 0\}. \tag{1}$$

Since in this paper we work on positive half-line $\mathbb{R}_+$ (that is $t \geq 0$), we may omit the modulus sign in the formula (1).

The following constructive way of identifying a conjugate function to a given Young function is given in [2, Theorem 3, Formula (14) and Corollary 2, p. 10].

Theorem 2.18. Let $\Phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a Young function, that is

$$\Phi(x) = \int_0^x \phi(s)ds, \quad x \geq 0,$$

where $\phi(0) = 0$, $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ is nondecreasing and left continuous. Let $\psi(\cdot)$ be the (generalized) inverse of ϕ. Then the conjugate function Ψ to Φ may be defined as follows:

$$\Psi(x) = \int_0^x \psi(s)ds, \quad x \geq 0.$$

Now we provide examples of pairs of mutually conjugate Orlicz functions.

Example 2.19. Let $\Phi(x) = e^x - x - 1, \quad x \geq 0$, then it is easy to find its conjugate function (via Theorem 2.18) $\Psi(x) = x \cdot \ln(x+1) - x + \ln(x+1), \quad x \geq 0$, which, by definition, is also an Orlicz (moreover, both of them are $N-$functions) function.

Indeed, $\Phi'(x) = e^x - 1$ whose inverse is $\Psi'(x) = \ln(x+1)$. Thus, integrating by parts we obtain $\Psi(x) = \int_0^x \ln(t+1)dt = x \cdot \ln(x+1) - x + \ln(x+1)$. It is easy to see that this function coincides with the one defined by formula (1).

We note that Φ is not equivalent to Ψ on $(0, \infty)$ (there is no $C > 0$ such that $\Phi(x) \leq \Psi(Cx)$).

Example 2.20. Let $\Phi(x) = \begin{cases} x^2 & \text{if} \quad 0 \leq x < 1, \\ x^3 & \text{if} \quad x \geq 1. \end{cases}$

We note that Φ is an $N-$function and $L_\Phi(0, \infty) \neq L_p(0, \infty)$ for any $p \geq 1$. However, $L_\Phi(0, 1) = L_3(0, 1)$. Its conjugate function is given by

$$\Psi(x) = \begin{cases} \frac{x^2}{4} & \text{if} \quad 0 \leq x < 1, \\ \frac{2}{3\sqrt{3}}x^{3/2} + \frac{1}{4} - \frac{2}{3\sqrt{3}} & \text{if} \quad x \geq 1. \end{cases}$$

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References

1 Ando, T. (1962). Weakly compact sets in Orlicz spaces. *Canad.J.Math., 14,* 170–176.

2 Rao, M.M., & Ren, Z.D. (1991). *The Theory of Orlicz spaces.* New York: Marcel Dekker,

3 Alexopoulos, J. (1994). De La Vallée Poussin's theorem and weakly compact sets in Orlicz spaces. *Quaestiones Mathematicae*, 231–248.

4 Astashkin, S.V. (2019). Rearrangement invariant spaces satisfying Dunford-Pettis criterion of weak compactness. *Contemporary Mathematics, 733*, 45–59.

5 Barcenas, D., & Finol, C.E. (2009). On Vector Measures, Uniform Integrability and Orlicz Spaces. *Operator theory: Advances and Applications, 201*, 51–57.

6 Chong, K.M. (1976). Doubly stochastic operators and rearrangement theorems. *J. Math. Anal. Appl., 56*, 309–316.

7 Chong, K.M. (1974). Spectral orders, uniform integrability and Lebesgue's dominated convergence theorem. *Trans. Amer. Math. Soc., 191*, 395–404.

8 Dodds, P.G., Sukochev, F., & Schlüchtermann, G. (2001). Weak compactness criteria in symmetric spaces of measurable operators. *Math.Proc.Camb.Phil.Soc, 131*, 363–384.

9 Kamińska, A., & Mastylo, M. (2002). The Schur and (weak) Dunford-Pettis properties in Banach Lattices. *J.Austral.Math.Soc., 73*, 251–278.

10 Lefévre, P., Li, D., Queffèlec, H., & Rodriguez-Piazza, L. (2008). Weak compactness and Orlicz spaces. *Colloquium Math., 112*(1), 23–32.

11 Nowak, M. (1986). A characterization of the Mackey topology $\tau(L^{\varphi}, L^{\varphi^*})$ on Orlicz spaces. *Bulletin of the Polish Academy of Sciences, Mathematics, 34*, 9-10, 577–583.

12 Orlicz, W. (1936). Über Raüme (L^M). *Bull. Acad. Polon. Sci. Ser. A*, 93–107.

13 Zhang, C., & Hou, Y.L. (2013). Convergence of weighted averages of martingales. *Sci China Math, 56*, 823–830.

14 Zhang, C., & Hou, Y.L. (2012). Convergence of weighted averages of martingales in noncommutative Banach function spaces. *Acta Mathematica Scientia, 32B*(2), 735–744.

15 Krasnoselskii, M.A., & Rutickii, Ya.B. (1961). *Convex functions and Orlicz spaces*. Translated by Leo F.Boron, Noorhoff Ltd., Groningen.

16 Rubshtein, B.A., Grabarnik, G.Ya., Muratov, M.A., & Pashkova, Y.S. (2016). *Foundations of Symmetric Spaces of Measurable Functions: Lorentz, Marcinkiewicz and Orlicz spaces.* Developments in Mathematics, Volume 45, Springer.

17 Zeidler, E. (1985). *Nonlinear Functional Analysis and its Applications III, Variational methods and Optimization*, Springer-Verlag, New York Inc.

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Орлич кеңістіктеріндегі әлсіз жинақы жиындардың мысалдары

Мақалада Орлич кеңістіктеріндегі салыстырмалы әлсіз жинақы жиындардың кейбір мысалдары келтірілген. Сондай-ақ осы мысалдардан туындайтын кейбір нәтижелер көрсетілген. Атап айтқанда, кейбір Орлич функциясының екіншісіне қарағанда жылдамырақ өсетінін қамтамасыз ететін критерийлер берілген (Т. Андо мағынасында). Сонымен қатар, егер L_Φ Орлич кеңістігінің K шектелген ішкі жиыны модуляр **Ф**-мен шектелмеген болса, онда K жиынының Ф-ға үшін қарағанда жылдам өсетін кез келген Ψ модуляры шекелмеген күйінде қалуы мүмкін екені анықталған.

Кілт сөздер: түйіндес (толықтырғыш) функциялар, салыстырмалы әлсіз жинақылық, Орлич кеңістіктері, $N-$функциялар.

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Примеры слабо компактных множеств в пространствах Орлича

В статье мы приводим ряд примеров относительно слабо компактных множеств в пространствах Орлича. Кроме того, получены некоторые результаты, вытекающие из этих примеров. В частности, получен критерий, который гарантирует, что одна функция Орлича возрастает быстрее, чем другая (в смысле Т. Андо). Кроме того, показано, что если ограниченное подмножество K пространства Орлича L_Φ не ограничено модуляром $\mathbf{\Phi}$, то множество K может оставаться неограниченным для любого модуляра Ψ, растущим быстрее, чем Φ.

Ключевые слова: сопряженные (дополнительные) функции, относительная слабая компактность, пространства Орлича, N-функции.

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An analogue of the Lyapunov inequality for an ordinary second-order differential equation with a fractional derivative and a variable coefficient

This paper studies an ordinary second-order differential equation with a fractional differentiation operator in the sense of Riemann-Liouville with a variable coefficient. We use the Green's function's method to find a representation of the solution of the Dirichlet problem for the equation under consideration when the solvability condition is satisfied. Green's function to the problem is constructed in terms of the fundamental solution of the equation under study and its properties are proved. The necessary integral condition for the existence of a nontrivial solution to the homogeneous Dirichlet problem, called an analogue of the Lyapunov inequality, is found.

Keywords: fractional Riemann–Liouville integral, fractional Riemann–Liouville derivative, Gerasimov–Caputo fractional derivative, Dirichlet problem, Green's function, analogue of Lyapunov inequality.

Introduction

In the interval $0 < x < l$, consider the equation

$$\mathbf{L}u \equiv u''(x) + q(x)D^{\alpha}_{0x}u(x) = f(x), \qquad 0 < \alpha < 1, \tag{1}$$

where

$$D^{\alpha}_{0x}u(x) = \frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int\limits_0^x \frac{u(t)dt}{(x-t)^{\alpha}}$$

is the operator of fractional (in the sense of Riemann-Liouville) differentiation of order α [1], $\Gamma(z)$ is the Euler gamma function, $q(x)$ and $f(x)$ are given functions, $u(x)$ is the desired function.

In [2] (see Theorem 3), the unconditional and unambiguous solvability of the Dirichlet problem $u(0) = 0$, $u(l) = 0$ for the equation (1) is proved for $q(x) \le 0$. Also in [2], for $q(x) = \lambda$, where $\lambda = \text{const}$, the question of the spectrum of the homogeneous Dirichlet problem for the homogeneous equation (1) is investigated, in particular, it is shown that the numbers $\lambda \le 0$ cannot be eigenvalues of the operator $\mathbf{L}$, and the number $\lambda > 0$ is an eigenvalue of the operator $\mathbf{L}$ if and only if $E_{2-\alpha,2}(-\lambda) = 0$, where

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k + \mu)}$$

– Mittag-Leffler type function [3; 117].

Lyapunov's inequality plays an important role in the study of spectral properties of ordinary differential equations. More detailed information can be found in [4–6]. Here we give the classical Lyapunov inequality.

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If $u(x)$ is a nontrivial solution to the problem

$$u''(x) + q(x)u(x) = 0, \qquad u(a) = u(b) = 0,$$

where $q(x)$ is a real, continuous function, then it holds true the inequality

$$\int_a^b |q(x)|dx > \frac{4}{b-a}. \tag{2}$$

There are works where various generalizations of the Lyapunov inequality (2) are constructed. For example, in [7], for an ordinary fractional differential equation containing a composition of fractional derivatives with different beginnings, the necessary integral condition for the existence of a nontrivial solution to the homogeneous Dirichlet problem is found, namely: *if $u(x)$ is a nontrivial solution to the problem*

$$D_{0x}^{\alpha}\partial_{1x}^{\alpha}u(x) - q(x)u(x) = 0, \qquad u(0) = u(1) = 0, \quad \frac{1}{2} < \alpha < 1,$$

where $q(x)$ is a real continuous function, then the following inequality is true:

$$\int_0^1 |q(x)|dx > (2\alpha - 1)\frac{\Gamma^2(\alpha)}{h}, \qquad h = \sup_{0<x<1}\left[(1-x)^{2\alpha-1} - (1-x^{2\alpha-1})^2\right].$$

The work [8] shows that for the existence of a nontrivial solution to the homogeneous Dirichlet problem for an ordinary second-order differential equation with a distributed integration operator

$$u''(x) + q(x)\int_0^\beta \mu(\alpha)D_{0x}^{-\alpha}u(x)d\alpha = 0, \qquad u(0) = u(l) = 0,$$

the condition must be fulfilled

$$\int_0^\beta |\mu(\alpha)|\frac{l^\alpha d\alpha}{\Gamma(\alpha+1)}\int_0^l |q(x)|dx \geq \frac{4}{l},$$

which is an analogue of the Lyapunov inequality.

In this paper, a representation of the solution to the Dirichlet problem for the equation (1), using the Green function, is found in the case when $q(x) \leq 0$, and an analogue of the Lyapunov inequality is proved.

Problem statement

We call a regular solution a function $u(x)$ that belongs to the class $C[0,l] \cap C^2]0,l[$ and satisfies the equation (1) for all $x \in]0,l[$.

Problem. Find a regular solution $u(x)$ to the equation (1) in the interval $]0,l[$ satisfying the conditions

$$u(0) = u_0, \quad u(l) = u_l, \tag{3}$$

where u_0, u_l are the specified constants.

Supporting statements

Let $q(x)$ be absolutely continuous on the segment $[0,l]$. Consider two functions defined in the compact $\overline{\Omega}=[0,l]\times[0,l]$

$$W(x,t)=x-t-\int\limits_t^x (x-s)R(s,t)ds, \tag{4}$$

$$G(x,t)=H(x-t)W(x,t)-\frac{1}{W(l,0)}\Big[W(x,0)W(l,t)\Big]. \tag{5}$$

Here

$$R(x,t)=\sum_{n=1}^{\infty}(-1)^{n+1}K_n(x,t), \qquad K_1(x,t)=\partial_{xt}^{\alpha}[(x-t)q(t)],$$

$$K_{n+1}(x,t)=\int\limits_t^x K_n(x,s)K_1(s,t)dt, \ \ n\in\mathbb{N}, \qquad \partial_{0x}^{\alpha}u(x)=\frac{1}{\Gamma(1-\alpha)}\int\limits_0^x \frac{u'(t)dt}{(x-t)^{\alpha}}, \ \ \alpha\in]0,1[$$

– the operator of fractional (in the sense of Gerasimov–Caputo) differentiation of order α, $H(x)$ – Heaviside function.

Lemma 1. The function $W(x,t)$ with respect to the variable x is the solution of the problem

$$W_{xx}(x,t)+q(x)D_{tx}^{\alpha}W(x,t)=0, \tag{6}$$

$$W(t,t)=0, \qquad W_x(t,t)=1, \qquad \forall t\in[0,x], \tag{7}$$

and according to the variable t is the solution of the problem

$$W_{tt}(x,t)+\partial_{xt}^{\alpha}[q(t)W(x,t)]=0, \tag{8}$$

$$W(x,x)=0, \qquad W_t(x,x)=-1, \qquad \forall x\in[0,l]. \tag{9}$$

Lemma 1 is proved by directly substituting formula (4) into the equalities (6)–(9).

Definition. The Green function of the Dirichlet problem (3) for equation (1) is called the function $v(x,t)$, having the following properties:
1. $v(x,t)$ is continuous in $\overline{\Omega}$.
2. $v(x,t)$ as a function of the variable x is the solution of the problem

$$v_{xx}(x,t)+q(x)D_{0x}^{\alpha}v(x,t)=0, \qquad v(0,t)=0, \quad v(l,t)=0, \tag{10}$$

by the variable t is the solution of the problem

$$v_{tt}(x,t)+\partial_{lt}^{\alpha}[q(t)v(x,t)]=0, \qquad v(x,0)=0, \quad v(x,l)=0. \tag{11}$$

3. For $t=x$, the derivatives $v_x(x,t)$ and $v_t(x,t)$ have a jump equal to one, that is

$$v_x(x,x+0)-v_x(x,x-0)=-1, \tag{12}$$

$$v_t(x,x+0)-v_t(x,x-0)=1. \tag{13}$$

Lemma 2. Let the condition $W(l,0)\neq 0$ be fulfilled. Then the function $G(x,t)$, defined by formula (5), is the Green function of the Dirichlet problem (3) for equation (1).

Proof. The continuity of the Green function $G(x,t)$ in the compact $\overline{\Omega}$ follows from the continuity of the function $W(x,t)$ in this compact $\overline{\Omega}$.

The second property is proved by direct substitution of equality (5) in formulas (10), (11) and taking into account the relations (6) – (9)

$$G_{xx}(x,t)+q(x)D^{\alpha}_{0x}G(x,t)=H(x-t)\Big[W_{xx}(x,t)+q(x)D^{\alpha}_{tx}W(x,t)\Big]-$$
$$-\frac{W(l,t)}{W(l,0)}\Big[W_{xx}(x,0)+q(x)D^{\alpha}_{0x}W(x,0)\Big]=0, \tag{14}$$
$$G_{tt}(x,t)+\partial^{\alpha}_{lt}[q(t)G(x,t)]=H(x-t)\Big[W_{tt}(x,t)+\partial^{\alpha}_{xt}[q(t)W(x,t)]\Big]+$$
$$+\frac{W(x,0)}{W(l,0)}\Big[W_{tt}(l,t)+\partial^{\alpha}_{lt}[q(t)W(l,t)]\Big]=0. \tag{15}$$

From the representation (5), by virtue of the relations $W(0,0)=0$, $W(l,l)=0$, we have the equality

$$G(0,t)=0,\quad G(l,t)=0,\quad G(x,0)=0,\quad G(x,l)=0. \tag{16}$$

Differentiating equality (5) by x and by t

$$G_x(x,t)=H(x-t)W_x(x,t)-\frac{1}{W(l,0)}\Big[W_x(x,0)W(l,t)\Big], \tag{17}$$
$$G_t(x,t)=H(x-t)W_t(x,t)-\frac{1}{W(l,0)}\Big[W(x,0)W_t(l,t)\Big], \tag{18}$$

and substituting formulas (17), (18) into relations (12), (13), taking into account that $W(x)$ is a continuous function, $H(x)$ is a function discontinuous at zero, and taking into account the equalities $W_x(x,x)=1$, $W_t(x,x)=-1$, we get

$$G_x(x,x+0)-G_x(x,x-0)=\lim_{\varepsilon\to-0}H(\varepsilon)W_x(x,x)-\frac{1}{W(l,0)}\Big[W_x(x,0)W(l,x)\Big]-$$
$$-\lim_{\varepsilon\to+0}H(\varepsilon)W_x(x,x)+\frac{1}{W(l,0)}\Big[W_x(x,0)W(l,x)\Big]=-1, \tag{19}$$
$$G_t(x,x+0)-G_t(x,x-0)=\lim_{\varepsilon\to-0}H(\varepsilon)W_t(x,x)-\frac{1}{W(l,0)}\Big[W(x,0)W_t(l,x)\Big]-$$
$$-\lim_{\varepsilon\to+0}H(\varepsilon)W_t(x,x)+\frac{1}{W(l,0)}\Big[W(x,0)W_t(l,x)\Big]=1, \tag{20}$$

which proves the validity of formulas (12) and (13). Lemma 2 is proved.

It follows from the relations (17), (18), by virtue of formulas (7) and (9), the equalities

$$G_x(x,0)=0,\qquad G_x(x,l)=0, \tag{21}$$
$$G_t(x,0)=W_t(x,0)-\frac{W(x,0)W_t(l,0)}{W(l,0)},\qquad G_t(x,l)=\frac{W(x,0)}{W(l,0)}. \tag{22}$$

Presentation of the solution

At this point, we will find a representation of the solution to the problem (1), (3).

Theorem 1. Let $q(x)$ be absolutely continuous on the segment $[0,l]$, and $q(x)\le 0$, $f(x)\in L[0,l]\cap C]0,l[$. Then if the condition $W(l,0)\neq 0$ is satisfied, there is a unique regular solution to the problem (1), (3). The solution has the form

$$u(x)=-u_0G_t(x,0)+u_lG_t(x,l)+\int\limits_0^l G(x,t)f(t)dt. \tag{23}$$

Proof. Let $u(x)$ be a regular solution of the equation (1). Multiply both sides of the equation (1) by the function $G(x,t)$ after changing the variable x into t, and integrate the resulting equality by t in the range from 0 to l. Then we have

$$\int_0^l G(x,t)u''(t)dt + \int_0^l G(x,t)q(t)D_{0t}^{\alpha}u(t)dt = \int_0^l G(x,t)f(t)dt. \tag{24}$$

Integrating in parts the first term of the left side of equality (24), given that the function $G_t(x,t)$ has a jump, having previously split the integration interval into two intervals from 0 to x and from x to l, we will have

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{l-\varepsilon} G(x,t)u''(t)dt = \lim_{\varepsilon\to 0}\left[u'(l-\varepsilon)G(x,l-\varepsilon) - u'(\varepsilon)G(x,\varepsilon) - \int_{\varepsilon}^{x} G_t(x,t)u'(t)dt - \right.$$

$$\left. - \int_{x}^{l-\varepsilon} G_t(x,t)u'(t)dt\right] = u'(l)G(x,l) - u'(0)G(x,0) - u(l)G_t(x,l) + u(0)G_t(x,0) +$$

$$+u(x)\big[G_t(x,x+0) - G_t(x,x-0)\big] + \int_0^l G_{tt}(x,t)u(t)dt. \tag{25}$$

According to the formula of fractional integration by parts and equalities (16), the second term of the left part of the formula (24) can be rewritten as

$$\int_0^l G(x,t)q(t)D_{0t}^{\alpha}u(t)dt = q(l)G(x,l)D_{0l}^{\alpha-1}u(t) - q(0)G(x,0)D_{00}^{\alpha-1}u(t) -$$

$$-\int_0^l \frac{\partial}{\partial t}[q(t)G(x,t)]D_{0t}^{\alpha-1}u(t)dt = -\int_0^l u(t)D_{lt}^{\alpha-1}\frac{\partial}{\partial t}[q(t)G(x,t)]dt =$$

$$= \int_0^l u(t)\partial_{lt}^{\alpha}[q(t)G(x,t)]dt. \tag{26}$$

Considering formulas (20), (25), and (26) by equality (24) we obtain

$$u(x) + \int_0^l u(t)\Big[G_{tt}(x,t) + \partial_{lt}^{\alpha}[q(t)G(x,t)]\Big]dt =$$

$$= -u'(l)G(x,l) + u'(0)G(x,0) + u(l)G_t(x,l) - u(0)G_t(x,0) + \int_0^l G(x,t)f(t)dt,$$

from which, by virtue of the relations (15) and (16), we obtain the formula (23).

Let us now show that the function $u(x)$, defined by formula (23), is indeed the solution of problem (1), (3). Differentiating equality (23) twice, taking into account the formula (19), we will have

$$u'(x) = -u_0\big[G_t(x,0)\big]' + u_l\big[G_t(x,l)\big]' + \int_0^l G_x(x,t)f(t)dt. \tag{27}$$

From formula (27) we get

$$u''(x) = -u_0\big[G_t(x,0)\big]'' + u_l\big[G_t(x,l)\big]'' + \frac{d}{dx}\left[\int_0^x G_x(x,t)f(t)dt + \int_x^l G_x(x,t)f(t)dt\right] =$$

$$= -u_0\big[G_t(x,0)\big]'' + u_l\big[G_t(x,l)\big]'' + \int_0^l G_{xx}(x,t)f(t)dt + f(x). \tag{28}$$

Further, from formula (23) we have

$$D_{0x}^{\alpha}u(x) = -u_0 D_{0x}^{\alpha}G_t(x,0) + u_l D_{0x}^{\alpha}G_t(x,l) + \int_0^l f(t)D_{0x}^{\alpha}G(x,t)dt. \tag{29}$$

Substituting formulas (28) and (29) into equation (1), by virtue of equality (14), we obtain that the function defined by relation (23) is indeed the solution of equation (1). Taking into account formulas (16), (21), (22), the direct substitution of function (23) into equality (3) gives the correct identities. Theorem 1 is proved.

An analogue of the Lyapunov inequality

At this point, we reduce the homogeneous problem

$$u''(x) + q(x)D_{0x}^{\alpha}u(x) = 0, \qquad u(0) = 0, \quad u(l) = 0 \tag{30}$$

to the Fredholm integral equation of the second kind, with the help of which we obtain an analogue of the Lyapunov inequality.

Since by the condition $u(0) = 0$, then through the property of the fractional differentiation operator we have the equality

$$D_{0x}^{\alpha}u(x) = D_{0x}^{\alpha-1}u'(x).$$

Given the last formula, we will act on both parts of the first equality (30) with the operator D_{0x}^{-1}. Then, with respect to the function $u'(x)$, we obtain the loaded integral equation

$$u'(x) + \int_0^x q(t)D_{0t}^{\alpha-1}u'(t)dt = u'(0). \tag{31}$$

To determine the unknown constant $u'(0)$ in formula (31), we will act on both parts of equality (31) with the operator D_{lx}^{-1}. Then we will have

$$u(l) - u(x) + \int_x^l \int_0^t q(s)D_{0s}^{\alpha-1}u'(s)dsdt = u'(0)(l-x). \tag{32}$$

Letting x tend to zero in equation (32) and taking into account the equalities $u(0) = 0$, $u(l) = 0$, we get that

$$u'(0) = \frac{1}{l}\int_0^l \int_0^t q(s) D_{0s}^{\alpha-1} u'(s) ds dt. \tag{33}$$

Substituting now formula (33) into equality (31), after simple transformations, we obtain the Fredholm integral equation of the second kind with respect to the function $u'(x)$

$$u'(x) = \int_0^l q(t)\left[\frac{l-t}{l} - H(x-t)\right] D_{0t}^{\alpha-1} u'(t) dt. \tag{34}$$

Theorem 2. Let $q(x)$ be continuous on the segment $[0, l]$, the homogeneous problem (30) has a nontrivial solution $u(x)$. Then there is an inequality

$$\int_0^l |q(x)| dx > \frac{\Gamma(2-\alpha)}{l^{1-\alpha}}. \tag{35}$$

Proof. First, we note that if $u(x)$ is a nontrivial solution of equation (30) satisfying the conditions $u(0) = 0$, $u(l) = 0$, then and the function $u'(x) \neq 0$. It is valid if $u'(x) = 0$, then $u(x) = \text{const}$, and by the condition $u(0) = 0$, therefore, in this case, we have $u(x) = 0$, which contradicts the condition of Theorem 2.

Suppose that

$$\overline{u} = \max_{x\in[0,l]} |u'(x)|.$$

Then from equation (34) we have the inequality

$$\overline{u} \leq \overline{u} \cdot \max_{x\in[0,l]} \int_0^l |q(t)| \left|\frac{l-t}{l} - H(x-t)\right| \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} dt. \tag{36}$$

The function

$$F(x,t) = \left|\frac{l-t}{l} - H(x-t)\right| \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$$

takes the largest value at $x = t = l$, therefore, given the equality

$$F_{\max} = F(l,l) = \frac{l^{1-\alpha}}{\Gamma(2-\alpha)}$$

from the relation (36) we have the inequality

$$\frac{l^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^l |q(t)| dt \geq 1,$$

that is equivalent to (35). Let us call inequality (35) an analogue of Lyapunov's inequality.

References

1 Нахушев А.М. О положительности операторов непрерывного и дискретного дифференцирования и интегрирования весьма важных в дробном исчислении и в теории уравнений смешанного типа / А.М. Нахушев // Дифференц. уравнения. — 1998. — 34. — № 1. — С. 101–109.

2 Нахушев А.М. Задача Штурма–Лиувилля для обыкновенного дифференциального уравнения второго порядка с дробными производными в младших членах / А.М. Нахушев // Докл. АН СССР. — 1977. — 234. — № 2. — С. 308–311.

3 Джрбашян М.М. Интегральные преобразования и представления функций в комплексной области / М.М. Джрбашян — М.: Наука, 1966. — 672 с.

4 Ляпунов А.М. Объ одном вопросе, касающемся линейных дифференціальных уравненій второго порядка съ періодическими коэффиціентами / А.М. Ляпунов // Сообщ. Харьків. мат. общества. Вторая сер. — 1897. — 5. — С. 190–254.

5 Brown R.C. Lyapunov Inequalities and their Applications / R.C. Brown, D.B. Hinton // Survey on Classical Inequalities. Mathematics and Its Applications. — 2000. — 517. — P. 1–25.

6 Wintner A. On the nonexistence of conjugate points / A. Wintner // American Journal Mathematics. — 1951. — 73. — P. 368–380.

7 Энеева Л.М. Неравенство Ляпунова для уравнения с производными дробного порядка с различными началами / Л.М. Энеева // Вестн. КРАУНЦ. Физ.-мат. науки. — 2019. — 28. — № 3. — С. 32–39.

8 Эфендиев Б.И. Неравенство Ляпунова для обыкновенного дифференциального уравнения второго порядка с оператором распределенного интегрирования / Б.И. Эфендиев // Итоги науки и техники. Сер. Современная математика и ее приложения. Тематические обзоры. Дифференциальные уравнения и математическая физика // ВИНИТИ РАН. — 2021. — С. 133–137.

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Бөлшек туындылы және айнымалы коэффициентті екінші ретті қарапайым дифференциалдық теңдеу үшін Ляпунов теңсіздігінің аналогы

Мақалада айнымалы коэффициентті Риман–Лиувиль мағынасындағы бөлшек дифференциалдау операторы бар екінші ретті қарапайым дифференциалдық теңдеу зерттелген. Грин функциясының әдісімен қарастырылған теңдеудің шешімділік шартын орындауда Дирихле есебінің шешімінің мәні табылған. Зерттелетін теңдеудің іргелі шешімі бойынша Гриннің тиісті функциясы құрылды және оның қасиеттері дәлелденді. Ляпунов теңсіздігінің аналогы деп аталатын біртекті Дирихле есебінің тривиал емес шешімі болуының қажетті интегралдық шарты табылды.

Кілт сөздер: Риман–Лиувиль бөлшек интегралы, Риман–Лиувиль бөлшек туындысы, Герасимов–Капуто бөлшек туындысы, Дирихле есебі, Грин функциясы, Ляпунов теңсіздігінің аналогы.

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Аналог неравенства Ляпунова для обыкновенного дифференциального уравнения второго порядка с дробной производной и с переменным коэффициентом

В статье исследовано обыкновенное дифференциальное уравнение второго порядка с оператором дробного дифференцирования в смысле Римана–Лиувилля с переменным коэффициентом. Методом функции Грина найдено представление решения задачи Дирихле для рассматриваемого уравнения при выполнении условия разрешимости. Построена соответствующая функция Грина в терминах фундаментального решения исследуемого уравнения и доказаны ее свойства. Найдено необходимое интегральное условие существования нетривиального решения однородной задачи Дирихле, названное аналогом неравенства Ляпунова.

Ключевые слова: дробный интеграл Римана–Лиувилля, дробная производная Римана–Лиувилля, дробная производная Герасимова–Капуто, задача Дирихле, функция Грина, аналог неравенства Ляпунова.

References

1 Nakhushev, A.M. (1998). O polozhitelnosti operatorov nepreryvnogo i diskretnogo differentsirovaniia i integrirovaniia vesma vazhnykh v drobnom ischislenii i v teorii uravnenii smeshannogo tipa [On the positivity of the operators of continuous and discrete differentiation and integration, which are very important in fractional calculus and in the theory of mixed-type equations]. *Differentsialnye uravneniia – Differential equations, 34*(1), 101–109 [in Russian].

2 Nakhushev, A.M. (1977). Zadacha Shturma–Liuvillia dlia obyknovennogo differentsialnogo uravneniia vtorogo poriadka s drobnymi proizvodnymi v mladshikh chlenakh [The Sturm-Liouville problem for an ordinary second-order differential equation with fractional derivatives in the lower terms]. *Doklady AN SSSR – Proceedings of the USSR Academy of Sciences, 234*(2), 308–311 [in Russian].

3 Dzhrbashyan, M.M. (1966). *Integralnye preobrazovaniia i predstavleniia funktsii v kompleksnoi oblasti [Integral transformations and representations of functions in the complex domain]*. Moscow: Nauka [in Russian].

4 Lyapunov, A.M. (1897). Ob odnom voprose, kasaiushchemsia lineinykh differentsialnykh uravnenii vtorogo poriadka s periodicheskimi koeffitsientami [About a question concerning second-order linear differential equations with periodic coefficients]. *Soobsheniia Kharkovskogo matematicheskogo obshchestva. Vtoraia seriia – Reports of the Kharkiv Mathematical Society. The second edition, 5,* 190–254 [in Ukrainian].

5 Brown, R.C., & Hinton, D.B. (2000). Lyapunov Inequalities and their Applications. *Survey on Classical Inequalities. Mathematics and Its Applications, 517,* 1–25.

6 Wintner, A. (1951). On the nonexistence of conjugate points. *American Journal Mathematics. 517,* 368–380.

7 Eneeva, L.M. (2019). Neravenstvo Liapunova dlia uravneniia s proizvodnymi drobnogo poriadka s razlichnymi nachalami [Lyapunov inequality for an equation with fractional order derivatives with different beginnings]. *Vestnik KRAUNTs. Fiziko-matematicheskie nauki – Bulletin KRASEC. Physical and Mathematical Sciences, 28*(3), 32–39 [in Russian].

8 Efendiev, B.I. (2021). Neravenstvo Liapunova dlia obyknovennogo differentsialnogo uravneniia vtorogo poriadka s operatorom raspredelennogo integrirovaniia [Lyapunov inequality for an ordinary second-order differential equation with a distributed integration operator]. *Itogi nauki i tekhniki. Seriia Sovremennaia matematika i ee prilozheniia. Tematicheskie obzory. Differentsialnye uravneniia i matematicheskaia fizika – Results of science and technology. The series modern mathematics and its applications. Thematic reviews. Differential equations and mathematical physics, Moscow: VINITI RAS,* 133–137 [in Russian].

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Integro-differential equations with bounded operators in Banach spaces

The paper investigates integro-differential equations in Banach spaces with operators, which are a composition of convolution and differentiation operators. Depending on the order of action of these two operators, we talk about integro-differential operators of the Riemann—Liouville type, when the convolution operator acts first, and integro-differential operators of the Gerasimov type otherwise. Special cases of the operators under consideration are the fractional derivatives of Riemann—Liouville and Gerasimov, respectively. The classes of integro-differential operators under study also include those in which the convolution has an integral kernel without singularities. The conditions of the unique solvability of the Cauchy type problem for a linear integro-differential equation of the Riemann—Liouville type and the Cauchy problem for a linear integro-differential equation of the Gerasimov type with a bounded operator at the unknown function are obtained. These results are used in the study of similar equations with a degenerate operator at an integro-differential operator under the condition of relative boundedness of the pair of operators from the equation. Abstract results are applied to the study of initial boundary value problems for partial differential equations with an integro-differential operator, the convolution in which is given by the Mittag-Leffler function multiplied by a power function.

Keywords: integro-differential equation, integro-differential operator, convolution, Cauchy problem, Cauchy type problem, degenerate evolution integro-differential equation, initial boundary value problem.

Introduction

In recent decades, the importance of fractional integro-differential calculus has grown markedly in solving both theoretical and applied problems in many areas of mathematical modeling: In continuum mechanics, in mathematical biology, in finance theory, etc. [1–4]. At the same time, over the past few years, works have appeared containing the construction of new fractional derivatives, which in most cases are compositions of a convolution operator and the operator of an integer order differentiation, but unlike classical fractional derivatives, the kernel in the convolution operator has no singularities [5, 6].

This paper considerers abstract integro-differential operators of the form of composition of a convolution and an integer order differentiation and equations in Banach spaces with them. Using the methods of the Laplace transform theory, we investigate the initial problems for such equations are formulated and the issues of the unique solvability of such problems are investigated. If $m-1<\alpha\leq m\in\mathbb{N}$, the kernel in the convolution is a power function $s^{m-\alpha}/\Gamma(\alpha)$ at the differentiation operator of the order m, the integro-differential operator is the Riemann—Liouville or Gerasimov fractional derivative, depending on the order of action of the convolution and the integer order differentiation. In other cases, we obtain other integro-differential operators of Riemann—Liouville or Gerasimov type. Note also that the kernel in the convolution is supposed to be operator-valued. This makes it possible to study some systems of equations within the framework of the studied equations in Banach spaces, for example, with fractional derivatives of various orders.

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The first section contains the Cauchy type problem for the linear equation in a Banach space with an integro-differential operator of Riemann—Liouville type, when the convolution operator acts on the function first, and with a bounded operator at the unknown function. A unique solvability theorem was proved for the problem, the solution is presented in the form of a sum of the Dunford–Taylor integrals. In the second section, the Cauchy problem is studied for the equation with an integro-differential operator of Gerasimov type, when the convolution operator acts after the differentiation operator. We show that there exists a unique solution to such problem, and present the solution in the similar form as in the previous section. In the third and fourth sections, initial problems for analogous linear equations with a degenerate operator at an integro-differential operator are studied under the condition of relative boudedness of the pair of operators from the equation. The last section contains an application of abstract results to initial boundary value problems with an integro-differential operator of Atangana–Baleanu type [6] with singular kernel (with the Mittag-Leffler function multiplied by a negative power as the kernel of the convolution) with respect to time and with some differential operators in spatial variables.

Note that, by similar methods, various fractional differential equations in Banach spaces, including degenerate ones, were researched in the works [7–10], see the references therein also. In this sense, it is necessary to mention the monograph by J. Prüss [11] on evolution integral equations in Banach spaces.

1 Integro-differential equation of Riemann—Liouville type

Let $\mathcal{X}$ be a Banach space, $\mathcal{L}(\mathcal{X})$ be the Banach space of all linear bounded operators on $\mathcal{X}$, $A \in \mathcal{L}(\mathcal{X})$, $\mathbb{R}_+ = \{a \in \mathbb{R} : a > 0\}$, $\overline{\mathbb{R}}_+ := \{0\} \cup \mathbb{R}_+$, $K \in C(\mathbb{R}_+; \mathcal{L}(\mathcal{X}))$. Define the convolution

$$(J^K x)(t) := \int_0^t K(t-s)x(s)ds$$

and integro-differential operator of the Riemann—Liouville type

$$(D^{m,K}x)(t) := D^m (J^K x)(t) := D^m \int_0^t K(t-s)x(s)ds,$$

where D^m is a usual derivative of the order m. Consider the Cauchy type problem

$$(J^K x)^{(k)}(0) = x_k \in \mathcal{X}, \quad k = 0, 1, \dots, m-1, \tag{1}$$

for the equation

$$(D^{m,K}x)(t) = Ax(t), \ t > 0. \tag{2}$$

A solution of problem (1), (2) is called a function $x : \mathbb{R}_+ \to \mathcal{X}$, such that $J^K x \in C^{m-1}(\overline{\mathbb{R}}_+; \mathcal{X}) \cap C^m(\mathbb{R}_+; \mathcal{X})$, conditions (1) and equality (2) for $t \in \mathbb{R}_+$ are satisfied.

For a function $h : \mathbb{R}_+ \to \mathcal{X}$ we denote its Laplace transform by $\widehat{h}$, or $\mathfrak{L}[h]$, if the expression for h is too long.

Suppose that $\widehat{K}$ is a single-valued analytic operator-function in the region

$$\Omega_{R_0} := \{\mu \in \mathbb{C} : |\mu| > R_0, \ |\arg \mu| < \pi\}$$

for some $R_0 > 0$ and define the operators

$$X_k(t) = \frac{1}{2\pi i} \int_{\gamma} (\lambda^m \widehat{K}(\lambda) - A)^{-1} \lambda^{m-1-k} e^{\lambda t} d\lambda, \quad t > 0, \quad k = 0, 1, \dots, m-1,$$

where $\gamma := \gamma_R \cup \gamma_{R,+} \cup \gamma_{R,-}$ is a positively oriented contour, $\gamma_R := \{Re^{i\varphi} : \varphi \in (-\pi, \pi)\}$, $\gamma_{R,+} := \{re^{i\pi} : r \in [R, \infty)\}$, $\gamma_{R,-} := \{re^{-i\pi} : r \in [R, \infty)\}$, $R > R_0$.

Theorem 1. Let $A \in \mathcal{L}(\mathcal{X})$, $K \in C(\mathbb{R}_+; \mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and

$$\exists \chi > 0 \quad \exists c > 0 \quad \forall \lambda \in \Omega_{R_0} \quad \|\widehat{K}(\lambda)^{-1}\|^{-1}_{\mathcal{L}(\mathcal{X})} > c|\lambda|^{\chi-1}. \tag{3}$$

Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$. Then for all $x_0, x_1, \dots, x_{m-1} \in \mathcal{X}$ there exists an unique solution to problem (1), (2). It has the form

$$x(t) = \sum_{k=0}^{m-1} X_k(t) x_k.$$

Proof. Due to condition (3) there exists $\delta \geq R_0 > 0$ such that for all $\lambda \in \Omega_\delta$ $\|\lambda^{-m}\widehat{K}(\lambda)^{-1}\|_{\mathcal{L}(\mathcal{X})} < c^{-1}|\lambda|^{1-\chi-m} < (2\|A\|_{\mathcal{L}(\mathcal{X})})^{-1}$. Hence, there exists the inverse operator $(\lambda^m \widehat{K}(\lambda) - A)^{-1}$ and

$$\|(\lambda^m \widehat{K}(\lambda) - A)^{-1}\|_{\mathcal{L}(\mathcal{X})} = \sum_{n=0}^{\infty} |\lambda|^{-m(n+1)} \|\widehat{K}(\lambda)^{-1}\|^{n+1}_{\mathcal{L}(\mathcal{X})} \|A\|^n_{\mathcal{L}(\mathcal{X})} <$$

$$< \sum_{n=0}^{\infty} c^{-n-1} |\lambda|^{(1-\chi-m)(n+1)} \|A\|^n_{\mathcal{L}(\mathcal{X})} < \frac{2}{c|\lambda|^{\chi-1+m}}.$$

Here we obtain the inequality $\|(I - \lambda^{-m} A \widehat{K}(\lambda)^{-1})^{-1}\|_{\mathcal{L}(\mathcal{X})} < 2$ also. Besides,

$$\|(\lambda^m \widehat{K}(\lambda) - A)^{-1} \lambda^{m-1-k}\|_{\mathcal{L}(\mathcal{X})} = \|(I - \lambda^{-m} \widehat{K}(\lambda)^{-1} A)^{-1} \widehat{K}(\lambda)^{-1} \lambda^{-1-k}\|_{\mathcal{L}(\mathcal{X})} < 2c^{-1}|\lambda|^{-k-\chi}$$

and there exists the Laplace transform $\widehat{X}_k$ for $k = 1, 2, \dots, m-1$ and for $k = 0$, if $\chi > 1$. For $k = 0$, $\chi \in (0, 1)$ we have by the definition

$$\|X_0(t)\| \leq \frac{2R^{1-\chi} e^{Rt}}{c} + \frac{2}{\pi c} \int_R^\infty r^{-\chi} e^{-rt} dr = \frac{2R^{1-\chi} e^{Rt}}{c} + \frac{2\Gamma(1-\chi) t^{\chi-1}}{\pi c} \leq C t^{\chi-1} e^{Rt};$$

for $k = 0$, $\chi = 1$, choosing $R > 1$, obtain

$$\|X_0(t)\| \leq \frac{2e^{Rt}}{c} + \frac{2}{\pi c} \int_R^\infty r^{-1/2} e^{-rt} dr = \frac{2e^{Rt}}{c} + \frac{2\Gamma(1/2) t^{-1/2}}{\pi c} \leq C t^{-1/2} e^{Rt}, \ t > 0.$$

There exists the Laplace transform $\widehat{X}_0$.

Take $R > \delta$ in the definition of γ. We have for $l \in \{0, 1, \dots, m-1\}$

$$\widehat{J^K X_l}(\lambda) = \widehat{K}(\lambda) \widehat{X}_l(\lambda) = \widehat{K}(\lambda)(\lambda^m \widehat{K}(\lambda) - A)^{-1} \lambda^{m-1-l} = \lambda^{-1-l}(I - \lambda^{-m} A \widehat{K}(\lambda)^{-1})^{-1},$$

consequently,

$$J^K X_l(t) = \frac{1}{2\pi i} \int_\gamma \lambda^{-1-l} (I - \lambda^{-m} A \widehat{K}(\lambda)^{-1})^{-1} e^{\lambda t} d\lambda, \quad t > 0,$$

$$(J^K X_l)^{(k)}(t) = \frac{1}{2\pi i}\int\limits_\gamma \lambda^{k-1-l}(I-\lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}e^{\lambda t}d\lambda, \quad t>0,$$

for $k,l=0,1,\dots,m-1$. For every $k=0,1,\dots,l-1$

$$\|\lambda^{k-1-l}(I-\lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}\|_{\mathcal{L}(\mathcal{Z})} < \frac{2}{|\lambda|^2},$$

hence, $(J^K X_l)^{(k)}(0)=0$. For $k=l$

$$(J^K X_l)^{(l)}(t) = \frac{1}{2\pi i}\int\limits_\gamma \lambda^{-1}(I-\lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}e^{\lambda t}d\lambda =$$

$$= \frac{1}{2\pi i}\int\limits_\gamma \lambda^{-1}\sum_{n=0}^{\infty}\lambda^{-mn}(A\widehat{K}(\lambda)^{-1})^n e^{\lambda t}d\lambda = I + \frac{1}{2\pi i}\int\limits_\gamma \lambda^{-1}\sum_{n=1}^{\infty}\lambda^{-mn}(A\widehat{K}(\lambda)^{-1})^n e^{\lambda t}d\lambda,$$

$$\left\|\lambda^{-1}\sum_{n=1}^{\infty}\lambda^{-mn}(A\widehat{K}(\lambda)^{-1})^n\right\|_{\mathcal{L}(\mathcal{X})} \le |\lambda|^{-1}\sum_{n=1}^{\infty}c^{-n}|\lambda|^{(1-\chi-m)n}\|A\|^n_{\mathcal{L}(\mathcal{X})} =$$

$$= \frac{c^{-1}|\lambda|^{1-\chi-m}\|A\|_{\mathcal{L}(\mathcal{X})}}{|\lambda|(1-c^{-1}|\lambda|^{1-\chi-m}\|A\|_{\mathcal{L}(\mathcal{X})})} < \frac{2\|A\|_{\mathcal{L}(\mathcal{X})}}{c|\lambda|^{m+\chi}}.$$

Therefore, $(J^K X_l)^{(l)}(0)=I$.

Now let $k=l+1,l+2,\dots,m-1$, then

$$(J^K X_l)^{(k)}(t) = \frac{1}{2\pi i}\int\limits_\gamma \lambda^{k-1-l-m}(I-\lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}A\widehat{K}(\lambda)^{-1}e^{\lambda t}d\lambda, \quad t>0,$$

$$\|\lambda^{k-1-l-m}(I-\lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}A\widehat{K}(\lambda)^{-1}\|_{\mathcal{L}(\mathcal{X})} \le \frac{2\|A\|_{\mathcal{L}(\mathcal{X})}}{c|\lambda|^{\chi+1}}$$

due to (3). Hence, $(J^K X_l)^{(k)}(0)=0$ and all conditions (1) are satisfied.

We have

$$D^m(J^K X_l)(t) = \frac{1}{2\pi i}\int\limits_\gamma \lambda^{m-1-l}(I-\lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}e^{\lambda t}d\lambda =$$

$$= \frac{1}{2\pi i}\int\limits_\gamma \lambda^{2m-1-l}\widehat{K}(\lambda)(\lambda^m\widehat{K}(\lambda)-A)^{-1}e^{\lambda t}d\lambda = AX_l(t), \quad t>0,$$

hence, equality (2) holds.

If there exist two solutions y_1 and y_2 to problem (1), (2), then $y:=y_1-y_2$ is a solution to the same problem with $x_0=x_1=\dots=x_{m-1}=0$. Define y on $(T,+\infty)$ at some $T>0$ by zero. Then there exists $\widehat{y}$, and due to (1), (2) $(\lambda^m\widehat{K}(\lambda)-A)\widehat{y}(\lambda)=0$ for $\mathrm{Re}\lambda>0$. Under the conditions of this theorem $\widehat{y}(\lambda)\equiv 0$, therefore, $y(t)=0$ for $t\in(0,T)$. Since we can choose an arbitrary $T>0$, then $y(t)=0$ and $y_1(t)=y_2(t)$ for all $t>0$.

Consider the inhomogeneous equation

$$(D^{m,K}x)(t) = Ax(t)+f(t),\ t\in(0,T], \tag{4}$$

with $f : (0,T] \to \mathcal{X}$.

Lemma 1. Let $A \in \mathcal{L}(\mathcal{X})$, $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and condition (3) hold. Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$, $f \in C((0,T];\mathcal{X}) \cap L_1(0,T;\mathcal{X})$. Then there exists an unique solution to problem (1), (4) with $x_0 = x_1 = \ldots = x_{m-1} = 0$. It has the form

$$x_f(t) = \int\limits_0^t X_{m-1}(t-s)f(s)ds.$$

Proof. We have $\widehat{x}_f(\lambda) = \widehat{X}_{m-1}(\lambda)\widehat{f}(\lambda) = (\lambda^m\widehat{K}(\lambda) - A)^{-1}\widehat{f}(\lambda)$, therefore,

$$\widehat{J^K x_f}(\lambda) = \widehat{K}(\lambda)(\lambda^m\widehat{K}(\lambda) - A)^{-1}\widehat{f}(\lambda), \quad J^K x_f(t) = \int\limits_0^t X(t-s)f(s)ds,$$

where

$$X(t) = \frac{1}{2\pi i}\int\limits_\gamma \widehat{K}(\lambda)(\lambda^m\widehat{K}(\lambda) - A)^{-1}e^{\lambda t}d\lambda = \frac{1}{2\pi i}\int\limits_\gamma \lambda^{-m}(I - \lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}e^{\lambda t}d\lambda.$$

Hence, $\|X^{(k)}(t)\|_{\mathcal{L}(\mathcal{X})} \leq Ct^{m-k-1}$ for all $t \in (0,T]$, $k = 0,1,\ldots,m-1$; $X^{(k)}(0) = 0$, $k = 0,1,\ldots,m-2$, and

$$(J^K x_f)^{(k)}(t) = \int\limits_0^t X^{(k)}(t-s)f(s)ds, \quad k = 0,1,\ldots,m-1,$$

$$\|(J^K x_f)^{(k)}(t)\|_{\mathcal{L}(\mathcal{X})} \leq C_1\int\limits_0^t \|f(s)\|_{\mathcal{L}(\mathcal{X})}ds, \quad (J^K x_f)^{(k)}(0) = 0, \quad k = 0,1,\ldots,m-1.$$

Finally,

$$\mathfrak{L}[(J^K x_f)^{(m)}] = \lambda^m\widehat{K}(\lambda)(\lambda^m\widehat{K}(\lambda) - A)^{-1}\widehat{f}(\lambda) = A(\lambda^m\widehat{K}(\lambda) - A)^{-1}\widehat{f}(\lambda) + \widehat{f}(\lambda),$$

therefore, equality (4) is fulfilled. Hence, x_f is a solution to problem (1), (4). The uniqueness of a solution can be proved in the same way, as for the homogeneous equation.

The assertions follow immediately from Theorem 1 and Lemma 1 due to the linearity of equation (4).

Theorem 2. Let $A \in \mathcal{L}(\mathcal{X})$, $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and condition (3) hold. Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$, $f \in C((0,T];\mathcal{X}) \cap L_1(0,T;\mathcal{X})$. Then for all $x_0, x_1, \ldots, x_{m-1} \in \mathcal{X}$ there exists an unique solution of problem (1), (4). It has the form

$$x(t) = \sum_{k=0}^{m-1} X_k(t)x_k + \int\limits_0^t X_{m-1}(t-s)f(s)ds.$$

2 Integro-differential equation of Gerasimov type

Consider the integro-differential operator of Gerasimov type

$$(D^{K,m}x)(t) := J^K(D^m x)(t) := \int_0^t K(t-s)x^{(m)}(s)ds.$$

Consider the Cauchy problem

$$x^{(k)}(0) = x_k \in \mathcal{X}, \ \ k = 0, 1, \dots, m-1, \tag{5}$$

for the equation

$$(D^{K,m}x)(t) = Ax(t), \ t \geq 0. \tag{6}$$

A solution to problem (5), (6) is called a function $x \in C^{m-1}(\overline{\mathbb{R}}_+;\mathcal{X}) \cap C^m(\mathbb{R}_+;\mathcal{X})$, such that $J^K x^{(m)} \in C(\overline{\mathbb{R}}_+;\mathcal{X})$, conditions (5) and equality (6) for $t \in \overline{\mathbb{R}}_+$ are satisfied.

Theorem 3. Let $A \in \mathcal{L}(\mathcal{X})$, $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and condition (3) hold. Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$. Then for all $x_0, x_1, \dots, x_{m-1} \in \mathcal{X}$ there exists an unique solution to problem (5), (6). It has the form

$$x(t) = \sum_{k=0}^{m-1} Y_k(t)x_k,$$

where

$$Y_k(t) = \frac{1}{2\pi i}\int_\gamma (\lambda^m \widehat{K}(\lambda) - A)^{-1}\widehat{K}(\lambda)\lambda^{m-1-k}e^{\lambda t}d\lambda, \quad k = 0, 1, \dots, m-1.$$

The contour γ is defined as in the previous section.

Proof. We have

$$\|(\lambda^m \widehat{K}(\lambda) - A)^{-1}\widehat{K}(\lambda)\lambda^{m-1-k}\|_{\mathcal{L}(\mathcal{X})} = \|(I - \lambda^{-m}\widehat{K}(\lambda)^{-1}A)^{-1}\lambda^{-1-k}\|_{\mathcal{L}(\mathcal{X})} < 2|\lambda|^{-k-1}.$$

So, there exists the Laplace transform $\widehat{Y}_k$ for $k = 1, 2, \dots, m-1$. For $k = 0$

$$Y_0(t) = I + \frac{1}{2\pi i}\int_\gamma \lambda^{-1}\sum_{n=1}^{\infty}\lambda^{-mn}(\widehat{K}(\lambda)^{-1}A)^n e^{\lambda t}d\lambda,$$

$$\left\|\lambda^{-1}\sum_{n=1}^{\infty}\lambda^{-m}(\widehat{K}(\lambda)^{-1}A)^n\right\|_{\mathcal{L}(\mathcal{X})} \leq |\lambda|^{-1}\sum_{n=1}^{\infty}c^{-n}|\lambda|^{(1-\chi-m)n}\|A\|^n_{\mathcal{L}(\mathcal{X})} =$$

$$= \frac{c^{-1}|\lambda|^{1-\chi-m}\|A\|_{\mathcal{L}(\mathcal{X})}}{|\lambda|(1 - c^{-1}|\lambda|^{1-\chi-m}\|A\|_{\mathcal{L}(\mathcal{X})})} < \frac{2\|A\|_{\mathcal{L}(\mathcal{X})}}{c|\lambda|^{\chi+m}}.$$

Thus, there exists the Laplace transform $\widehat{X}_0$.

For large enough $R > 0$ in the definition of γ, $k, l \in \{0, 1, \dots, m-1\}$

$$Y_l^{(k)}(t) = \frac{1}{2\pi i}\int_\gamma \lambda^{k-1-l}(I - \lambda^{-m}\widehat{K}(\lambda)^{-1}A)^{-1}e^{\lambda t}d\lambda, \quad t > 0.$$

By repeating the reasoning from Theorem 1, we get the fulfillment of conditions (5) with arbitrary $x_l \in \mathcal{X}$, $x_k = 0$ for every $k \in \{0, 1, \dots, m-1\} \setminus \{l\}$.

Further, we have

$$\widehat{J^K Y_l^{(m)}}(\lambda) = \widehat{K}(\lambda)\widehat{Y_l^{(m)}}(\lambda) = \widehat{K}(\lambda)(\lambda^m \widehat{Y}_l(\lambda) - \lambda^{m-l-1}I),$$

$$J^K Y_l^{(m)}(t) = \frac{1}{2\pi i}\int_\gamma \lambda^{m-1-l}\widehat{K}(\lambda)[(I - \lambda^{-m}\widehat{K}(\lambda)^{-1}A)^{-1} - I]e^{\lambda t}d\lambda =$$

$$= A\frac{1}{2\pi i}\int\limits_{\gamma} \lambda^{-1-l}(I - \lambda^{-m}\widehat{K}(\lambda)^{-1}A)^{-1}e^{\lambda t}d\lambda = AY_l(t), \quad t > 0.$$

The uniqueness of a solution can be proved in the same way as in Theorem 1.

Consider the inhomogeneous equation with $f : [0,T] \to \mathcal{X}$

$$(D^{K,m}x)(t) = Ax(t) + f(t),\ t \in [0,T]. \tag{7}$$

Lemma 2. Let $A \in \mathcal{L}(\mathcal{X})$, $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and condition (3) hold. Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$, $f \in C([0,T];\mathcal{X})$. Then there exists an unique solution to problem (5), (7) with $x_0 = x_1 = \ldots = x_{m-1} = 0$. It has the form

$$x_f(t) = \int\limits_0^t X_{m-1}(t-s)f(s)ds.$$

Proof. For $k = 0,1,\ldots,m-2$ we have $X_{m-1}^{(k)}(0) = 0$, hence,

$$x_f^{(k)}(t) = \int\limits_0^t X_{m-1}^{(k)}(t-s)f(s)ds, \quad k = 0,1,\ldots,m-1.$$

Since

$$\|\lambda^k(\lambda^m\widehat{K}(\lambda) - A)^{-1}\|_{\mathcal{L}(\mathcal{X})} = \|\lambda^{k-m}\widehat{K}(\lambda)^{-1}(I - \lambda^{-m}A\widehat{K}(\lambda)^{-1})^{-1}\|_{\mathcal{L}(\mathcal{X})} < \frac{2}{c|\lambda|^{m-k-1+\chi}},$$

we have $\|X_{m-1}^{(k)}(t)\|_{\mathcal{L}(\mathcal{X})} \leq Ct^{m-k-2+\chi}$, consequently,

$$\|x_f^{(k)}(t)\|_{\mathcal{L}(\mathcal{X})} \leq C_1\|f\|_{C([0,T];\mathcal{X})}t^{m-k-1+\chi}, \quad x_f^{(k)}(0) = 0, \quad k = 0,1,\ldots,m-1.$$

Further,

$$\mathfrak{L}[J^K x_f^{(m)}] = \lambda^m\widehat{K}(\lambda)(\lambda^m\widehat{K}(\lambda) - A)^{-1}\widehat{f}(\lambda) = A(\lambda^m\widehat{K}(\lambda) - A)^{-1}\widehat{f}(\lambda) + \widehat{f}(\lambda),$$

hence, x_f is a solution to problem (5), (7). The proof of the uniqueness of a solution is the same as in Theorem 1.

Theorem 4. Let $A \in \mathcal{L}(\mathcal{X})$, $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and condition (3) hold. Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$, $f \in C([0,T];\mathcal{X})$. Then for all $x_0, x_1, \ldots, x_{m-1} \in \mathcal{X}$ there exists an unique solution to problem (5), (7). It has the form

$$x(t) = \sum_{k=0}^{m-1} Y_k(t)x_k + \int\limits_0^t X_{m-1}(t-s)f(s)ds.$$

Example 1. Take $m - 1 < \alpha \leq m \in \mathbb{N}$, $K_\alpha(s) := \frac{s^{\alpha-1}}{\Gamma(\alpha)}I$, then $J^{K_\alpha} := J^\alpha$ is the operator of the fractional Riemann—Liouville integration of the order α, $D^{m,K_{m-\alpha}} := {}^{RL}D^\alpha$ is the operator of the fractional Riemann—Liouville differentiation of the order α, $D^{K_{m-\alpha},m} := {}^{GC}D^\alpha$ is the operator of the fractional Gerasimov—Caputo differentiation of the order α.

Example 2. Take $\mathcal{X} = \mathbb{R}^2$, $a_{ij}, b_{ij} \in \mathbb{R}$, $m_{ij} - 1 < \alpha_{ij} < m_{ij} \in \mathbb{N}$, $i, j = 1, 2$, $m := \max\limits_{i,j=1,2} m_{ij}$,

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad K(s) := \begin{pmatrix} b_{11}\frac{s^{m-\alpha_{11}-1}}{\Gamma(m-\alpha_{11})} & b_{12}\frac{s^{m-\alpha_{12}-1}}{\Gamma(m-\alpha_{12})} \\ b_{21}\frac{s^{m-\alpha_{21}-1}}{\Gamma(m-\alpha_{21})} & b_{22}\frac{s^{m-\alpha_{22}-1}}{\Gamma(m-\alpha_{22})} \end{pmatrix}$$

then

$$D^{m,K} = \begin{pmatrix} b_{11}{}^{RL}D^{\alpha_{11}} & b_{12}{}^{RL}D^{\alpha_{12}} \\ b_{21}{}^{RL}D^{\alpha_{21}} & b_{22}{}^{RL}D^{\alpha_{22}} \end{pmatrix},$$

for Gerasimov—Caputo derivatives similar construction case is possible in the general, if $m_{11} = m_{12} = m_{21} = m_{22}$. So, equation (2) has the form of the system of equations

$$b_{11}{}^{RL}D^{\alpha_{11}}x_1(t) + b_{12}{}^{RL}D^{\alpha_{12}}x_2(t) = a_{11}x_1(t) + a_{12}x_2(t),$$
$$b_{21}{}^{RL}D^{\alpha_{21}}x_1(t) + b_{22}{}^{RL}D^{\alpha_{22}}x_2(t) = a_{21}x_1(t) + a_{22}x_2(t).$$

Note that

$$\widehat{K}(\lambda) := \begin{pmatrix} b_{11}\lambda^{\alpha_{11}-m} & b_{12}\lambda^{\alpha_{12}-m} \\ b_{21}\lambda^{\alpha_{21}-m} & b_{22}\lambda^{\alpha_{22}-m} \end{pmatrix},$$

therefore, condition (3) is fulfilled with some $\chi \in (0, \alpha + 1 - m)$, and the condition of reversibility of $\widehat{K}(\lambda)$ for large enough $|\lambda|$ is not too restrictive. Indeed, $\widehat{K}(\lambda)$ is invertible, only if the matrix, consisting of b_{ij}, does not contain zero rows and zero columns, and $\alpha_{11}\alpha_{22} \neq \alpha_{12}\alpha_{21}$, or $b_{11}b_{22} \neq b_{12}b_{21}$ in the case $\alpha_{11}\alpha_{22} = \alpha_{12}\alpha_{21}$.

3 Degenerate equation of Riemann—Liouville type

Assume that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $L \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, i.e., it is a linear bounded operator from $\mathcal{X}$ to $\mathcal{Y}$, $M \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$, i.e., it is a linear closed operator with a dense domain D_M in $\mathcal{X}$, acting to $\mathcal{Y}$. Introduce the denotations $\rho^L(M) := \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})\}$, $R^L_\mu(M) := (\mu L - M)^{-1}L$, $L^L_\mu := L(\mu L - M)^{-1}$. We will suppose that $\ker L \neq \{0\}$, in other words, the operator L is degenerate.

An operator M is called (L, σ)-*bounded*, if

$$\exists a > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)).$$

In [12; 89, 90], it was shown that if an operator M is (L, σ)-bounded, $\gamma_r := \{\mu \in \mathbb{C} : |\mu| = r > a\}$, then the operators

$$P = \frac{1}{2\pi i}\int\limits_{\gamma_r} R^L_\mu(M)\, d\mu \in \mathcal{L}(\mathcal{X}), \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma_r} L^L_\mu(M)\, d\mu \in \mathcal{L}(\mathcal{Y})$$

are projections. Put $\mathcal{X}^0 := \ker P$, $\mathcal{X}^1 := \mathrm{im} P$, $\mathcal{Y}^0 := \ker Q$, $\mathcal{Y}^1 := \mathrm{im} Q$. Denote by L_k (M_k) the restriction of the operator L (M) on $\mathcal{X}^k$ $(D_{M_k} = D_M \cap \mathcal{X}^k)$, $k = 0, 1$.

Theorem 5 [12; 91]. Let an operator M be (L, σ)-bounded. Then

(i) $M_1 \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y}^1)$, $M_0 \in \mathcal{C}l(\mathcal{X}^0; \mathcal{Y}^0)$, $L_k \in \mathcal{L}(\mathcal{X}^k; \mathcal{Y}^k)$, $k = 0, 1$;

(ii) there exist operators $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$, $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$.

Denote $G := M_0^{-1}L_0$. For $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ operator M is called (L, p)-bounded, if it is (L, σ)-bounded, $G^p \neq 0$, $G^{p+1} = 0$.

Consider the initial problem

$$D^{k,K}(Px)(0) = x_k, \quad k = 0, 1, \ldots, m - 1, \tag{8}$$

for a linear inhomogeneous integro-differential equation of Riemann—Liouville type

$$LD^{m,K}x(t) = Mx(t) + g(t), \quad t \in (0, T], \tag{9}$$

in which $g \in C((0,T];\mathcal{Y})$. This equation is called degenerate, since it contains degenerate operator L at the integro-differential operator.

A solution to problem (8), (9) is called a function $x : (0,T] \to D_M$, for which $Mx \in C((0,T];\mathcal{Y})$, $J^K Px \in C^{m-1}([0,T];\mathcal{Y})$, $J^K x \in C^m((0,T];\mathcal{Y})$, equality (9) is valid for all $t \in (0,T]$ and conditions (8) are true.

Lemma 3. Let $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, $H \in \mathcal{L}(\mathcal{X})$ be a nilpotent operator with a power $p \in \mathbb{N}_0$, a function $h : (0,T] \to \mathcal{X}$ be such that for $l = 0, 1, \dots, p$ $(D^{m,K}H)^l h \in C((0,T];\mathcal{X})$, $D^{m,K}(D^{m,K}H)^l h \in C((0,T];\mathcal{X})$. Then there exists a unique solution to the equation

$$D^{m,K}Hx(t) = x(t) + h(t). \tag{10}$$

It has the form

$$x(t) = -\sum_{l=0}^{p}(D^{m,K}H)^l h(t). \tag{11}$$

Proof. Let $z = z(t)$ be a solution of (10). Act by the operator H on the both parts of (10) and obtain the equality $HD^{m,K}Hz(t) = Hz(t) + Hh(t)$. Under the theorem conditions there exists a continuous derivative $D^{m,K}$ for the the right-hand side of this equality. Acting by $D^{m,K}$ on the both parts of this equality, we will get

$$(D^{m,K}H)^2 z = D^{m,K}Hz + D_t^{m,K}Hh = z + h + D^{m,K}Hh.$$

Continuing such arguing, we obtain that

$$z + \sum_{l=0}^{p}(D^{m,K}H)^l h = (D^{m,K}H)^{p+1}z = (D^{m,K})^{p+1}H^{p+1}z \equiv 0$$

due to the continuity and nilpotency of the operator H. The existence of a solution can be checked by the substitution of (11) into (10).

The difference of two solutions is a solution of equation (10) with $h \equiv 0$, then (11) implies that the difference is identically equal to zero.

Define

$$U_k(t) = \frac{1}{2\pi i}\int_\gamma (\lambda^m \widehat{K}(\lambda) - L_1^{-1}M_1)^{-1}\lambda^{m-1-k}e^{\lambda t}d\lambda, \quad t > 0, \quad k = 0, 1, \dots, m-1.$$

Theorem 6. Let an operator M be (L,p)-bounded, $K \in C(\mathbb{R}_+;\mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and condition (3) hold. Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$, $g \in C((0,T];\mathcal{Y}) \cap L_1(0,T;\mathcal{Y})$, $(D^{m,K}G)^l M_0^{-1}(I-Q)g$, $D^{m,K}(D^{m,K}G)^l M_0^{-1}(I-Q)g \in C((0,T];\mathcal{X})$ for $l = 0, 1, \dots, p$, $x_k \in \mathcal{X}^1$, $k = 0, 1, \dots, m-1$. Then there exists a unique solution to problem (8), (9), it has the form

$$x(t) = \sum_{k=0}^{m-1}U_k(t)x_k + \int_0^t U_{m-1}(t-s)L_1^{-1}Qg(s)ds - \sum_{l=0}^{p}(D^{m,K}G)^l M_0^{-1}(I-Q)g(t).$$

Proof. Acting on the both sides of (9) by $L_1^{-1}Q \in \mathcal{L}(\mathcal{Y}^1;\mathcal{X}^1)$, obtain

$$D^{m,K}v(t) = L_1^{-1}M_1 v(t) + L_1^{-1}Qg(t), \tag{12}$$

where $v(t) = Px(t)$. Act by the operator $M_0^{-1}(I-Q) \in \mathcal{L}(\mathcal{Y}^0;\mathcal{X}^0)$ on (9) and get

$$D^{m,K}Gw(t) = w(t) + M_0^{-1}(I-Q)g(t), \tag{13}$$

$w(t) = (I-P)x(t)$. Here we use the evident equalities $LP = QL$, $MP = QM$ and Theorem 5. Conditions (8) can be rewritten in the form

$$D^{k,K}v(0) = x_k,\ k = 0,1,\ldots,m-1. \tag{14}$$

By Theorem 2, problem (12), (14) has an unique solution, and it has the form

$$v(t) = \sum_{k=0}^{m-1} U_k(t)x_k + \int_0^t U_{m-1}(t-s)L_1^{-1}Qg(s)ds.$$

Due to Lemma 3, equation (13) has an unique solution

$$w(t) = -\sum_{l=0}^{p}(D^{m,K}G)^l M_0^{-1}(I-Q)g(t).$$

Remark 1. It is not difficult to make sure that for $p = 0$ we have $L_0 = 0$, hence, initial conditions (8) are equivalent to the conditions

$$D^{m,K}Lx(0) = y_k, \quad k = 0,1,\ldots,m-1, \tag{15}$$

where $y_k = Lx_k$, or $x_k = L_1^{-1}y_k$, $k = 0,1,\ldots,m-1$.

Remark 2. It follows from the proof of Theorem 6 that if we consider the Cauchy type problem

$$D^{m,K}x(0) = x_k, \quad k = 0,1,\ldots,m-1,$$

for equation (9), we obtain the necessity of conditions

$$(I-P)x_k = -\sum_{l=0}^{p}(D^{m,K}G)^l M_0^{-1}(I-Q)g(0), \quad k = 0,1,\ldots,m-1,$$

for the problem solvability.

4 Degenerate equation of Gerasimov type

Now consider the initial problem

$$(Px)^{(k)}(0) = x_k, \quad k = 0,1,\ldots,m-1, \tag{16}$$

for a degenerate linear inhomogeneous integro-differential equation of Gerasimov type

$$LD^{K,m}x(t) = Mx(t) + g(t), \quad t \in [0,T], \tag{17}$$

in which $g \in C([0,T];\mathcal{Y})$.

A solution to problem (16), (17) is called a function $x : [0,T] \to D_M$, for which $Mx \in C([0,T];\mathcal{Y})$, $Px \in C^{m-1}([0,T];\mathcal{Y})$, $LJ^K x^{(m)} \in C([0,T];\mathcal{Y})$, equality (17) is valid for all $t \in [0,T]$ and conditions (16) are fulfilled.

Analogously to Lemma 3 the next assertion can be proved.

Lemma 4. Let $K \in C(\mathbb{R}_+; \mathcal{L}(\mathcal{X}))$, $H \in \mathcal{L}(\mathcal{X})$ be a nilpotent operator with a power $p \in \mathbb{N}_0$, a function $h : [0,T] \to \mathcal{X}$ be such that for $l = 0, 1, \dots, p$ $(D^{K,m}H)^l h \in C([0,T]; \mathcal{X})$, $D^{K,m}(D^{K,m}H)^l h \in C([0,T]; \mathcal{X})$. Then there exists an unique solution to the equation

$$D^{K,m}Hx(t) = x(t) + h(t).$$

And it has the form

$$x(t) = -\sum_{l=0}^{p} (D^{K,m}H)^l h(t).$$

Define

$$V_k(t) = \frac{1}{2\pi i} \int_{\gamma} (\lambda^m \widehat{K}(\lambda) - L_1^{-1}M_1)^{-1} \widehat{K}(\lambda) \lambda^{m-1-k} e^{\lambda t} d\lambda, \quad t > 0, \quad k = 0, 1, \dots, m-1,$$

Theorem 7. Let an operator M be (L,p)-bounded, $K \in C(\mathbb{R}_+; \mathcal{L}(\mathcal{X}))$, there exist $\widehat{K}$, which be single-valued analytic operator-function in Ω_{R_0} for some $R_0 > 0$, and condition (3) hold. Suppose that for all $\lambda \in \Omega_{R_0}$ there exists $\widehat{K}(\lambda)^{-1} \in \mathcal{L}(\mathcal{X})$, $g \in C([0,T]; \mathcal{Y})$, $(D^{K,m}G)^l M_0^{-1}(I-Q)g \in C([0,T]; \mathcal{X})$, $D^{K,m}(D^{K,m}G)^l M_0^{-1}(I-Q)g \in C([0,T]; \mathcal{X})$ for $l = 0, 1, \dots, p$, $x_k \in \mathcal{X}^1$, $k = 0, 1, \dots, m-1$. Then there exists an unique solution to problem (16), (17), it has the form

$$x(t) = \sum_{k=0}^{m-1} V_k(t) x_k + \int_0^t U_{m-1}(t-s) L_1^{-1} Q g(s) ds - \sum_{l=0}^{p} (D^{K,m}G)^l M_0^{-1}(I-Q) g(t).$$

Proof. As in the proof of Theorem 6, reduce the problem to the system

$$D^{K,m}v(t) = L_1^{-1}Mv(t) + L_1^{-1}Qg(t), \quad D^{K,m}Gw(t) = w(t) + M_0^{-1}(I-Q)g(t),$$

where $v(t) = Px(t)$, $w(t) = (I-P)x(t)$, endowed by the initial conditions

$$v^{(k)}(0) = x_k, \ k = 0, 1, \dots, m-1.$$

By Theorem 4 and Lemma 4 we get the required.

Remark 3. For $p = 0$ initial conditions (16) are equivalent to the conditions

$$D^{m,K}Lx(0) = y_k, \quad k = 0, 1, \dots, m-1,$$

where $y_k = Lx_k$, $k = 0, 1, \dots, m-1$.

Remark 4. For the Cauchy problem

$$x^{(k)}(0) = x_k, \quad k = 0, 1, \dots, m-1,$$

to equation (17) the conditions

$$(I-P)x_k = -\sum_{l=0}^{p} (D^{K,m}G)^l M_0^{-1}(I-Q) g(0), \quad k = 0, 1, \dots, m-1,$$

are necessary for the problem solvability.

5 Application to initial boundary value problems

Take $a \in \mathbb{R}$, $\alpha > 0$, $\beta \in (0,1)$, $K(s) = s^{-\beta} E_{\alpha,1-\beta}(as^{\alpha})I$, then

$$\widehat{K}(\lambda) = \frac{\lambda^{\alpha+\beta-1}}{\lambda^{\alpha} - a} I$$

satisfies condition (3) with $\chi \in (0,\beta)$, and it is invertible for all $|\lambda| > a^{1/\alpha}$. Here $E_{\alpha,\delta}$ is the Mittag-Leffler function. Note that the kernel $K(s)$ is singular at zero.

Let $P_{\varrho}(\lambda) = \sum\limits_{j=0}^{\varrho} c_j \lambda^j$, $Q_{\varrho}(\lambda) = \sum\limits_{j=0}^{\varrho} d_j \lambda^j$, $c_j, d_j \in \mathbb{C}$, $j = 0,1,\dots,\varrho \in \mathbb{N}$, $c_{\varrho} \neq 0$. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded region with a smooth boundary $\partial\Omega$,

$$(\Lambda u)(s) := \sum_{|q| \le 2r} a_q(s) \frac{\partial^{|q|} u(s)}{\partial s_1^{q_1} \partial s_2^{q_2} \dots \partial s_d^{q_d}}, \quad a_q \in C^{\infty}(\overline{\Omega}),$$

$$(B_l u)(s) := \sum_{|q| \le r_l} b_{lq}(s) \frac{\partial^{|q|} u(s)}{\partial s_1^{q_1} \partial s_2^{q_2} \dots \partial s_d^{q_d}}, \quad b_{lq} \in C^{\infty}(\partial\Omega),\ l = 1,2,\dots,r,$$

$q = (q_1, q_2, \dots, q_d) \in \mathbb{N}_0^d$, $|q| = q_1 + \dots + q_d$, the operator pencil $\Lambda, B_1, B_2, \dots, B_r$ is regularly elliptical [13]. Define an operator $\Lambda_1 \in \mathcal{C}l(L_2(\Omega))$, acting on the domain

$$D_{\Lambda_1} = H^{2r}_{\{B_l\}}(\Omega) := \{v \in H^{2r}(\Omega) : B_l v(s) = 0,\ l = 1,2,\dots,r,\ s \in \partial\Omega\}$$

by the rule $\Lambda_1 u := \Lambda u$. Let Λ_1 be a self-adjoint operator, then the spectrum $\sigma(\Lambda_1)$ of the operator Λ_1 is real, discrete, with finite multiplicity [13]. Suppose, in addition, that the spectrum $\sigma(\Lambda_1)$ is bounded from the right and does not contain zero, denote by $\{\varphi_k : k \in \mathbb{N}\}$ an orthonormal in $L_2(\Omega)$ system of eigenfunctions of the operator Λ_1, numbered in the order of non-increasing of the corresponding eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$, taking into account their multiplicity.

Consider the initial boundary value problem

$$\frac{\partial^k}{\partial t^k} \int\limits_0^t (t-s)^{-\beta} E_{\alpha,1-\beta}(a(t-s)^{\alpha}) u(\xi,s) ds|_{t=0} = u_k(\xi),\ k = 0,1,\dots,m-1,\ \xi \in \Omega, \tag{18}$$

$$B_l \Lambda^k u(\xi,t) = 0, \quad k = 0,1,\dots,\varrho-1, \quad l = 1,2,\dots,r, \quad (\xi,t) \in \partial\Omega \times (0,T], \tag{19}$$

$$P_{\varrho}(\Lambda) \frac{\partial^m}{\partial t^m} \int\limits_0^t (t-s)^{-\beta} E_{\alpha,1-\beta}(a(t-s)^{\alpha}) u(\xi,s) ds = Q_{\varrho}(\Lambda) u(\xi,t) + h(\xi,t) \tag{20}$$

in $\Omega \times (0,T]$. Here

$$J^K u(\xi,t) = \int\limits_0^t (t-s)^{-\beta} E_{\alpha,1-\beta}(a(t-s)^{\alpha}) u(\xi,s) ds$$

is the Atangana—Baleanu type integral [6], but with a singular kernel, $h : \Omega \times [0,T] \to \mathbb{R}$. Take $\mathcal{X} = \{v \in H^{2r\varrho}(\Omega) : B_l \Lambda^k v(s) = 0,\ k = 0,1,\dots,\varrho-1,\ l = 1,2,\dots,r,\ s \in \partial\Omega\}$, $\mathcal{Y} = L_2(\Omega)$, $L = P_{\varrho}(\Lambda)$, $M = Q_{\varrho}(\Lambda) \in \mathcal{L}(\mathcal{X};\mathcal{Y})$.

Let $P_{\varrho}(\lambda_k) \neq 0$ for all $k \in \mathbb{N}$, then there exists an inverse operator $L^{-1} \in \mathcal{L}(\mathcal{Y};\mathcal{X})$ and problem (18)–(20) is representable as problem (1), (4), where $A = L^{-1}M \in \mathcal{L}(\mathcal{Z})$, $x_k = u_k(\cdot)$, $k = 0,1,\dots,m-1$,

$f(t) = L^{-1}h(\cdot, t)$. By Theorem 2 there exists a unique solution to problem (18)–(20) for any $u_k \in \mathcal{X}$, $k = 0, 1, \ldots, m-1$, if $h \in C((0,T]; L_2(\Omega)) \cap L_1(0,T;\mathcal{X})$.

Now assume that $P_\varrho(\lambda_k) = 0$ for some $k \in \mathbb{N}$. If the polynomials P_ϱ and Q_ϱ have no common roots on the set $\{\lambda_k\}$, the operator M is $(L, 0)$-bounded (see [14]), the projectors have the form

$$P = \sum_{P_\varrho(\lambda_k)\neq 0} \langle \cdot, \varphi_k\rangle \varphi_k, \quad Q = \sum_{P_\varrho(\lambda_k)\neq 0} \langle \cdot, \varphi_k\rangle \varphi_k,$$

where $\langle \cdot, \varphi_k\rangle$ is the inner product in $L_2(\Omega)$. The initial conditions, taking into account Remark 1, can be given in the form

$$P_\varrho(\Lambda)\frac{\partial^k}{\partial t^k}\int_0^t (t-s)^{-\beta}E_{\alpha,1-\beta}(a(t-s)^\alpha)u(\xi,s)ds|_{t=0} = y_k(s),\ k = 0,1,\ldots,m-1,\ s \in \Omega. \tag{21}$$

Then problem (19)–(21) is represented as (9), (15) with the spaces $\mathcal{X}$, $\mathcal{Y}$ and the operators L, M selected above. Theorem 6 implies the unique solvability of problem (19)–(21), if $h \in C([0,T]; L_2(\Omega))$ and $y_k \in L_2(\Omega)$, $k = 0, 1, \ldots, m-1$, such that $\langle y_k, \varphi_l\rangle = 0$ for all $l \in \mathbb{N}$, for which $P_\varrho(\lambda_l) = 0$.

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References

1 Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional integrals and derivatives. Theory and applications.* Philadelphia: Gordon and Breach Science Publishers.

2 Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and applications of fractional differential equations.* Amsterdam, Boston, Heidelberg: Elsevier Science Publishing.

3 Tarasov, V.E. (2011). *Fractional dynamics: Applications of fractional calculus to dynamics of particles, fields and media.* New York: Springer.

4 Kostić, M. (2015). *Abstract Volterra integro-differential equations.* Boca Raton: CRC Press.

5 Caputo, M., & Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation and Applications, 1*(2), 1–13.

6 Atangana, A., & Baleanu, D. (2016). New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Thermal Science, 20*, 763–769.

7 Fedorov, V.E., Phuong, T.D., Kien, B.T., Boyko, K.V., & Izhberdeeva, E.M. (2020). A class of distributed order semilinear equations in Banach spaces. *Chelyabinsk Physical and Mathematical Journal, 5*(3), 342–351.

8 Fedorov, V.E., & Avilovich, A.S. (2021). Semilinear fractional-order evolution equations of Sobolev type in the sectorial case. *Complex Variables and Elliptic Equations, 66*(6–7), 1108–1121.

9 Fedorov, V.E., Plekhanova, M.V., & Izhberdeeva, E.M. (2021). Initial value problems of linear equations with the Dzhrbashyan — Nersesyan derivative in Banach spaces. *Symmetry, 13*, 1058.

10 Fedorov, V.E., & Turov, M.M. (2021). The Defect of a Cauchy Type Problem for Linear Equations with Several Riemann — Liouville Derivatives. *Siberian Mathematical Journal, 62*(5), 925–942.

11 Prüss, J. (1993). *Evolutionary integral equations and applications.* Basel: Springer.

12 Sviridyuk, G.A., & Fedorov, V.E. (2003). *Linear Sobolev type equations and degenerate semigroups of operators.* Utrecht, Boston: VSP.

13 Triebel, H. (1978). *Interpolation theory. Function spaces. Differential operators.* Amsterdam: North-Holland Publishing Company.

14 Fedorov, V.E. (2004). Strongly holomorphic groups of linear equations of Sobolev type in locally convex spaces. *Differential Equations, 40*(5), 753–765.

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Банах кеңістіктеріндегі шектелген операторлары бар интегро-дифференциалдық теңдеулер

Мақалада және дифференциалдау және үйірткі операторларының құрамдары болып табылатын операторлармен Банах кеңістігіндегі интегралдық-дифференциалдық теңдеулер зерттелген. Осы екі оператордың әрекет ету ретіне байланысты үйірткі операторы бірінші әрекет еткенде Риман-Лиувиль типіндегі интегро-дифференциалдық операторлар, ал басқаша Герасимов типті интегро-дифференциалдық операторлар туралы айтылады. Қарастырылып отырған операторлардың дербес жағдайлары сәйкесінше Риман-Лиувиль және Герасимов бөлшек туындылары болып табылады. Зерттелетін интегро-дифференциалдық операторлардың кластарына үйірткісі сингулярлықсыз интегралдық ядросы барлар да кіреді. Риман-Лиувилль типті сызықтық интегро-дифференциалдық теңдеу үшін Коши типтес есептің және ізделінді функция үшін шектелген операторы бар Герасимов типті сызықтық интегро-дифференциалдық теңдеу үшін Коши есебінің бірегей шешімін табу шарттары алынды. Бұл нәтижелер теңдеуден операторлар жұбының салыстырмалы шектелуі шартында интегро-дифференциалдық оператор үшін өзгеше операторы бар ұқсас теңдеулерді зерттеуде қолданылды. Абстрактілі нәтижелер Миттаг-Леффлер функциясымен берілген, яғни ерекшеліктері жоқ, интегро-дифференциалдық үйірткі операторы бар дербес туындылы теңдеулер үшін бастапқы-шектік есептерді зерттеуде пайдаланылды.

Кілт сөздер: интегро-дифференциалдық теңдеу, интегро-дифференциалдық оператор, үйірткі, Коши есебі, Коши типтес есеп, өзгеше интегро-дифференциалдық теңдеу, бастапқы-шекаралық есеп.

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Интегро-дифференциальные уравнения с ограниченными операторами в банаховых пространствах

В статье исследованы интегрально-дифференциальные уравнения в банаховых пространствах с операторами, представляющими собой композицию операторов свертки и дифференцирования. В зависимости от порядка действия этих двух операторов говорится об интегро-дифференциальных операторах типа Римана-Лиувилля, когда первым действует оператор свертки, и интегро-дифференциальных операторах типа Герасимова в противном случае. Частными случаями рассматриваемых операторов являются дробные производные Римана—Лиувилля и Герасимова соответственно. В исследуемые классы интегро-дифференциальных операторов входят и такие, в которых свертка имеет интегральное ядро без сингулярностей. Получены условия однозначной разрешимости задачи типа Коши для линейного интегро-дифференциального уравнения типа Римана-Лиувилля и задачи Коши для линейного интегро-дифференциального уравнения типа Герасимова с ограниченным оператором при искомой функции. Эти результаты использованы при исследовании аналогичных уравнений с вырожденным оператором при интегро-дифференциальном операторе при условии относительной ограниченности пары операторов из уравнения. Абстрактные результаты использованы при исследовании начально-краевых задач для уравнений в частных производных с интегро-дифференциальным оператором, свертка в котором задается функцией Миттаг-Леффлера, то есть не имеет особенностей.

Ключевые слова: интегро-дифференциальное уравнение, интегро-дифференциальный оператор, свертка, задача Коши, задача типа Коши, вырожденное интегро-дифференциальное уравнение, начально-краевая задача.

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Generalized boundary value problem for a linear ordinary differential equation with a discretely distributed fractional differentiation operator

This paper formulates and solves a generalized boundary value problem for a linear ordinary differential equation with a discretely distributed fractional differentiation operator. The fractional derivative is understood as the Gerasimov–Caputo derivative. The boundary conditions are given in the form of linear functionals, which makes it possible to cover a wide class of linear local and non-local conditions. A representation of the solution is found in terms of special functions. A necessary and sufficient condition for the solvability of the problem under study is obtained, as well as conditions under which the solvability condition is certainly satisfied. The theorem of existence and uniqueness of the solution is proved.

Keywords: fractional differentiation operator, Caputo derivative, boundary value problem, functional, Wright function.

Introduction and statement of the problem

In the interval $0 < x < 1$, let us consider the equation

$$\sum_{j=1}^{m} \beta_j \partial_{0x}^{\alpha_j} u(x) + \lambda u(x) = f(x), \tag{1}$$

where $\alpha_j \in (1,2)$, λ, $\beta_j \in \mathbb{R}$, $\beta_1 > 0$, $\alpha_1 > \alpha_2 > ... > \alpha_m$, $\partial_{0x}^{\gamma} u(x)$ is the Caputo derivative [1; 11]:

$$\partial_{sx}^{\gamma} u(x) = \operatorname{sign}^n (x - s) D_{sx}^{\gamma - n} u^{(n)}(x), \quad n - 1 < \gamma \le n, \, n \in \mathbb{N}, \tag{2}$$

and D_{sx}^{γ} is the Riemann–Liouville fractional integro-differentiation operator of order γ with respect to the variable x [1; 9], which is defined by the formula

$$D_{sx}^{\gamma} u(x) = \frac{\operatorname{sign}(x-s)}{\Gamma(-\gamma)} \int_{s}^{x} \frac{u(t)dt}{|x-t|^{\gamma+1}}, \quad \gamma < 0,$$

$$D_{sx}^{\gamma} u(x) = u(x), \quad \gamma = 0,$$

$$D_{sx}^{\gamma} u(x) = \operatorname{sign}^n (x - s) \frac{d^n}{dx^n} D_{sx}^{\gamma - n} u(x), \quad n - 1 < \gamma \le n, \, n \in \mathbb{N}.$$

Operator (2) is also known in the literature as the Gerasimov–Caputo operator [2, 3].

At present, differential equations of fractional order are being extensively studied in connection with practical applications in various areas of physics and mathematical modeling. The theory of fractional differential equations has proven itself well in the study of «classical» viscoelastic models. All this is supported by new applied problems [4–8].

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One of the first works devoted to fractional calculus and its applications is the monograph [9]. The main theoretical results and solution methods are reflected in the works [7], [6], and [10].

Linear fractional ordinary differential equations were studied by many authors; a detailed bibliography on this subject can be found in [1, 5, 6, 11]. A significant contribution to the study of fractional differential equations was made by the authors of [12–15].

Differential equations with discretely distributed differentiation operator

$$\sum_{k=1}^{m} \lambda_k \frac{\partial^{\sigma_k}}{\partial y^{\sigma_k}}$$

can be treated as an operator

$$\int_{\alpha}^{\beta} \left(\lambda(y,t) \frac{\partial^t}{\partial y^t} \right) d\mu(t)$$

with a measure concentrated on a discrete set [16, 17].

Differential equations with discretely distributed differentiation operators and continuously distributed differentiation operators were studied in [18–20], where equations with discretely distributed differentiation operators were used to search for approximate solutions of equations with continuously distributed differentiation operators. We also note the papers [16], [17], [21], [22], where equations with fractional discretely distributed differentiation operators were studied.

In this paper, we investigate a generalized boundary value problem (in the terminology of M.A. Naimark) for equation (1) [22; 16]. An explicit representation of the solution of the problem under study is constructed, a condition for unique solvability is found, and a uniqueness theorem for the solution is proved. We specified boundary conditions in the form of linear functionals, which makes it possible to cover a wide class of linear local and nonlocal conditions. Various boundary value problems for equation (1) were studied in the works [23–25]. Note also that in work [26] a generalized boundary value problem for an ordinary differential equation of fractional order with general conditions was investigated.

A *regular solution* to equation (1) is said to be a function $u = u(x)$ that has an absolutely continuous first-order derivative on the closed interval $[0,1]$ and satisfies equation (1) for all $x \in (0,1)$.

Problem. Find a regular solution to equation (1) in the interval $(0,1)$, which satisfies the conditions

$$\ell_0[u] = u_0, \tag{3}$$

$$\ell_1[u] = u_1, \tag{4}$$

where u_0, u_1 are given real numbers, ℓ_0, ℓ_1 are linear bounded functionals in $C^1[0,1]$.

Notation and auxiliary statements

We use the following notation (see [27])

$$G_m^{\mu}(x) = G_m^{\mu}(x;\nu_1,...,\nu_m;\gamma_1,...,\gamma_m) \equiv \int_0^{\infty} e^{-t} S_m^{\mu}(x;\nu_1 t,...,\nu_m t;\gamma_1,...,\gamma_m)dt,$$

$$\nu_1 = -\frac{\lambda}{\beta_1}, \quad \nu_j = -\frac{\beta_j}{\beta_1}, \quad \gamma_1 = \alpha_1, \quad \gamma_j = \alpha_1 - \alpha_j, \quad (j = \overline{2,m}),$$

$$S_m^{\mu}(x;z_1,...,z_m;\gamma_1,...,\gamma_m) = (h_1 * h_2 * ... * h_m)(x),$$

by

$$(g*h)(x)=\int\limits_0^x g(x-t)h(t)dt$$

we denote the Laplace convolution of the functions $g(x)$ and $h(x)$,

$$h_j=h_j(x)\equiv x^{\mu_j-1}\phi(\gamma_j,\mu_j;z_jx^{\gamma_j}),$$

where

$$\phi(\rho,\zeta;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\rho k+\zeta)}$$

is the Wright function (see [28]).

Further, we assume that the parameters $G_m^{\mu}(x)$ range over

$$x>0,\quad z_i\in\mathbb{R},\quad \gamma_j>0,\quad \mu_j>0.$$

We note that the function $G_m^{\mu}(x)$ is independent of the distribution of the numbers $\mu_j>0$, but only depends on their sum $\mu=\sum\limits_{j=1}^{m}\mu_j$.

The following equalities for the function $G_m^{\mu}(x)$ (see [24]) hold:

$$G_m^{\mu}(x)=O(x^{\mu-1})\quad \text{при}\quad x\to 0,$$

$$D_{0x}^{\nu}G_m^{\mu}(x)=G_m^{\mu-\nu}(x),\quad \text{если}\quad \mu>\nu, \tag{5}$$

$$G_m^{\mu}(x)-\sum_{j=1}^{m}\nu_jD_{0x}^{-\gamma_j}G_m^{\mu}(x)=\frac{x^{\mu-1}}{\Gamma(\mu)},\quad \mu>0. \tag{6}$$

In particular, from equalities (5) and (6) we obtain the formula (see [25])

$$\sum_{j=1}^{m}\beta_jG_m^{\mu-\gamma_j}(x)+\lambda G_m^{\mu}(x)=\frac{\beta_1x^{\mu-\alpha_1-1}}{\Gamma(\mu-\alpha_1)},\quad \mu>\alpha_1.$$

We also need the following auxiliary statement proved in (see [26]).

It should be noted that hereinafter the ℓ functionality is applied to the function depending on x.

Lemma. Let $K(x,t)\in C([0,1]\times[0,1])$ end $\frac{\partial}{\partial x}K(x,t)\in C([0,1]\times[0,1])$, $\ell-$ linear bounded functional in space $C^1[0,1]$. Then the following relation is true

$$\ell\left[\int\limits_0^1 K(x,t)dt\right]=\int\limits_0^1 \ell[K(x,t)]dt. \tag{7}$$

Main result

Theorem. Let a function $f(x)$ satisfy the conditions

$$x^{1-\mu}f(x)\in C[0,1],\quad f(x)=D_{0x}^{\alpha_1-2}g(x),\quad g(x)\in L[0,1],\quad \mu>0.$$

and the inequality

$$\det A=\ell_0[\mathcal{W}_2(x)]\ell_1[\mathcal{W}_3(x)]-\ell_0[\mathcal{W}_3(x)]\ell_1[\mathcal{W}_2(x)]\neq 0 \tag{8}$$

be fulfilled. Then a function $u(x)$ defined by the relation

$$u(x) = \int_0^1 f(t)\mathcal{T}(x,t)dt + \overline{\mathcal{W}}(x)\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \tag{9}$$

$$\mathcal{T}(x,t) = \left[1 - \overline{\mathcal{W}}(x)\mathcal{H}\overline{\ell}\right]\mathcal{W}_1(x-t), \quad \overline{\mathcal{W}}(x) = (\mathcal{W}_2(x), \mathcal{W}_3(x)), \quad \overline{\ell} = \begin{pmatrix} \ell_0 \\ \ell_1 \end{pmatrix},$$

$$\mathcal{W}_1(x) = \frac{1}{\beta}G_m^{\alpha_1}(x); \quad \mathcal{W}_2(x) = x + \nu_1 G_m^{\alpha_1+2}(x); \quad \mathcal{W}_3(x) = 1 + \nu_1 G_m^{\alpha_1+1}(x),$$

$$A = \overline{\ell}\,[\overline{\mathcal{W}}(x)] = \begin{pmatrix} \ell_0[\mathcal{W}_2(x)] & \ell_0[\mathcal{W}_3(x)] \\ \ell_1[\mathcal{W}_2(x)] & \ell_1[\mathcal{W}_3(x)] \end{pmatrix}, \quad \mathcal{H} = A^{-1} = \frac{1}{\det A}\begin{pmatrix} \ell_1[\mathcal{W}_3(x)] & -\ell_0[\mathcal{W}_3(x)] \\ -\ell_1[\mathcal{W}_2(x)] & \ell_0[\mathcal{W}_2(x)] \end{pmatrix},$$

is a regular solution to problem (1), (3), (4). The solution to problem (1), (3), (4) is unique if and only if the condition (8) is satisfied.

Proof. Let $u(x)$ be a regular solution to the problem (1), (3), (4). To find a solution to the problem (1), (3), (4) we use the solution of the Cauchy problem for the equation (1), which can be represented as [24, 25]:

$$u(x) = \int_0^1 f(t)\mathcal{W}_1(x-t)dt + \overline{\mathcal{W}}(x)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{10}$$

Further, taking into account the introduced notation and equalities (7), we satisfy (10) the boundary conditions

$$\int_0^1 f(t)\overline{\ell}[\mathcal{W}_1(x-t)]dt + A\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

From this we find

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \int_0^1 f(t)\mathcal{H}\overline{\ell}[\mathcal{W}_1(x-t)]dt.$$

After elementary transformations, substituting the found value into (10), we obtain a representation of the solution to problem (1), (3), (4) in the form (8). This, in particular, implies the uniqueness of the solution.

Let us now check the fulfillment of the boundary conditions (3), (4).

$$\overline{\ell}\left[\int_0^1 f(t)\mathcal{T}(x,t)dt + \overline{\mathcal{W}}(x)\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right] = I_1 + I_2,$$

where

$$I_1 = \overline{\ell}\left[\int_0^1 f(t)\mathcal{T}(x,t)dt\right] \quad \text{end} \quad I_2 = \overline{\ell}\left[\overline{\mathcal{W}}(x)\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right].$$

Taking into account (7), we have

$$I_1 = \int_0^1 \overline{\ell}\,[\mathcal{T}(x,t)]\,f(t)dt = \int_0^1 \left(\overline{\ell}[\mathcal{W}_1(x,t)] - \overline{\ell}[\overline{\mathcal{W}}(x)]\mathcal{H}\overline{\ell}[\mathcal{W}_1(x,t)]\right)f(t)dt.$$

By virtue of the equality $\overline{\ell}[\overline{\mathcal{W}}(x)] = \mathcal{H}^{-1}$, we obtain that $I_1 = 0$. Similarly, for I_2 we have

$$I_2 = \overline{\ell}[\overline{\mathcal{W}}(x)]\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \mathcal{H}^{-1}\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Let us prove that the function u(x), given by equality (9), is a solution to problem (3), (4) for equation (1).

$$\left[\sum_{j=1}^{m}\beta_j\partial_{0x}^{\alpha_j} + \lambda\right] y_i = f(x), \quad i = \overline{1,3},$$

$$y_1 = \int_0^x f(t)\mathcal{W}_1(x-t)dt, \quad y_2 = \int_0^x f(t)\overline{\mathcal{W}}(x)\mathcal{H}\overline{\ell}\mathcal{W}_1(x-t)dt, \quad y_3 = \overline{\mathcal{W}}(x)\mathcal{H}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Taking into account relations (5) – (8) and since the functions y_2 and y_3 is a linear combinations of the functions $\mathcal{W}_i(x), i = \overline{1,3}$ we have

$$\left[\sum_{j=1}^{m}\beta_j\partial_{0x}^{\alpha_j} + \lambda\right] y_i = 0, \quad i = \overline{2,3}.$$

Considering that (see [25])

$$\sum_{j=1}^{m}\beta_j\partial_{0x}^{\alpha_j}\int_0^x f(t)\mathcal{W}_1(x-t)dt = -\frac{\lambda}{\beta_1}\int_0^x f(t)G_m^{\alpha}(x-t)dt + f(x),$$

we get

$$\left[\sum_{j=1}^{m}\beta_j\partial_{0x}^{\alpha_j} + \lambda\right] y_1 = f(x).$$

It means that the solution satisfies equation (1).

Let us show that if condition (8) is satisfied, that is,

$$\ell_0[\mathcal{W}_2(x)]\ell_1[\mathcal{W}_3(x)] - \ell_0[\mathcal{W}_3(x)]\ell_1[\mathcal{W}_2(x)] = 0, \tag{11}$$

then the solution to the homogeneous problem is not unique.
Consider the function

$$\widetilde{u}(x) = (k_1(x), k_2(x))\begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

where C_1 and C_2 are arbitrary constants,

$$k_1(x) = \overline{\mathcal{W}}(x)\begin{pmatrix} \ell_0[\mathcal{W}_2(x)] \\ -\ell_0[\mathcal{W}_3(x)] \end{pmatrix}, \quad k_2(x) = \overline{\mathcal{W}}(x)\begin{pmatrix} -\ell_1[W_2(x)] \\ \ell_1[W_3(x)] \end{pmatrix}.$$

Then it follows from (11) that the function $\widetilde{u}(x)$ is a solution to a homogeneous problem

$$\sum_{j=1}^{m}\beta_j\partial_{0x}^{\alpha_j}\widetilde{u}(x) + \lambda\widetilde{u}(x) = 0, \quad \ell_0[\widetilde{u}] = 0, \quad \ell_1[\widetilde{u}] = 0.$$

References

1 Нахушев А.М. Дробное исчисление и его применение / А.М. Нахушев. — М.: Физматлит, 2003. — 272 с.

2 Килбас А.А. Теория и приложения дифференциальных уравнений дробного порядка (курс лекций). Методологическая школа-конференция «Математическая физика и нанотехнологии» / А.А. Килбас. — Самара, 2009. — 121 с.

3 Новоженова О.Г. Биография и научные труды Алексея Никифоровича Герасимова. О линейных операторах, упруго-вязкости, элевтерозе и дробных производных / О.Г. Новоженова. — М.: Перо, 2018. — 235 с.

4 Джрбашян М.М. Интегральные преобразования и представления функций в комплексной области / М.М. Джрбашян. — М.: Наука, 1966. — 672 с.

5 Самко С. Г. Интегралы и производные дробного порядка и некоторые их приложения / С. Г. Самко, А. А. Килбас, О. И. Маричев. — Минск: Наука и техника, 1987. — 688 с.

6 Нахушев А. М. О континуальных аналогах реологических уравнений состояния и логистическом законе изменения вязкоупругих свойств полимера / А. М. Нахушев, Р. Б. Тхакахов // Докл. Адыгской (Черкесской) Междунар. акад. наук. — 1995. — 1. — № 2. — С. 6–11.

7 Podlubny I. Fractional Differential Equations / I. Podlubny. — ACADEMIC PRESS, 1999. — 198. — 340 p.

8 Kilbas A. A. Theory and applications of fractional differential equations / A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. — North-Holland Math. Stud., Elsevier, Amsterdam, 2006. — 204. — 540 p.

9 Oldham K.B. The fractional calculus / K. B. Oldham, J. Spanier. — N.-Y.; L.: Acad. press, 1974. — 234 p.

10 Учайкин В.В. Метод дробных производных / В.В. Учайкин. — Ульяновск: Изд-во «Артишок», 2008. — 512 с.

11 Miller K.S. An introduction to the fractional calculus and fractional differential equations / K.S. Miller, B. Ross. — New York: Jon Wiley & Sons. Inc., 1993. — 366 p.

12 Псху А. В. Уравнения в частных производных дробного порядка / А. В. Псху. — М.: Наука, 2005. — 199 с.

13 Barrett J. H. Differential equations of non-integer order / J. H. Barrett // Canadian J. Math. — 1954. — 6. — No. 4. — P. 529–541.

14 Джрбашян М. М. Дробные производные и задача Коши для дифференциальных уравнений дробного порядка / М. М. Джрбашян, А. Б. Нерсесян // Изв. АН АрмянскойССР. Матем. — 1968. — 3. — № 1. — С. 3–28.

15 Джрбашян М. М. Краевая задача для дифференциального оператора дробного порядка типа Штурма-Лиувилля / М. М. Джрбашян // Изв. АН АрмянССР. — 1970. — 5. — № 2. — С. 71–96.

16 Pskhu A. V. Boundary Value Problem for a First-Order Partial Differential Equation with a Fractional Discretely Distributed Differentiation Operator / A. V. Pskhu // Differential Equations. — 2016. — 52:12. — P. 1610-1623.

17 Псху А. В. Уравнение дробной диффузии с оператором дискретно распределенного дифференцирования / А. В. Псху // Sibirean electr. Math. Reports. — 2016. — 13. — С. 1078–1098.

18 Caputo M. Diffusion with space memory modelled with distributed order space fractional differential equations / M. Caputo // Annals of Geophysics. — 2003. — 46. — No. 2. — P. 223–234.

19 Diethelm K. Numerical analysis for distributed-order differential equations / K. Diethelm, N.J. Ford // J. of Comp. and Appl. Math. — 2009. — 225. — P. 96–104.

20 Jiao Z. Distributed-Order Dynamic Systems: Stability, Simulation, Applications and Perspectives / Z. Jiao, Y. Qu. Chen, I. Podlubny. — London, 2012. — 103 p.

21 Ozturk I. On the theory of fractional differential equation / I. Ozturk // Reports of Adyghe (Circassian) International Academy of Sciences. — 1998. — 3. — No. 2. — P. 35–39.

22 Наймарк М.А. Линейные дифференциальные операторы / М.А. Наймарк. — М.: Наука, 1969. — 528 с.

23 Gadzova L. Kh. Dirichlet and Neumann problems for a fractional ordinary differential equation with constant coefficients / L.Kh. Gadzova // Differential Equations. — 2015. — 51:12. — P. 1556–1562.

24 Gadzova L. Kh. Boundary Value Problem for a Linear Ordinary Differential Equation with a Fractional Discretely Distributed Differentiation Operator / L.Kh. Gadzova // Differential Equations. — 2018. — 54:2. — P. 178–184.

25 Gadzova L. Kh. Nonlocal Boundary-Value Problem for a Linear Ordinary Differential Equation with Fractional Discretely Distributed Differentiation Operator / L.Kh. Gadzova // Mathematical Notes. — 2019. — 106. — No. 5–6. — P. 904–908.

26 Гадзова Л.Х. Обобщённая краевая задача для обыкновенного дифференциального уравнения дробного порядка / Л.Х. Гадзова // Челяб. физ.-мат. журн. — 2022. — 7. — № 1. — С. 20-29. https://doi.org/10.47475/2500-0101-2022-17102

27 Pskhu, A.V. Initial-value problem for a linear ordinary differential equation of noninteger order / A.V. Pskhu // Sbornik: Mathematics. — 2011. — 202. — No. 4. — P. 571–582.

28 Wright E.M. On the coefficients of power series having exponential singularities / E.M. Wright // J. London Math. Soc. — 1933. — 8. — No.29. — P. 71–79.

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Бөлшек дискретті үлестірілген дифференциалдау операторы бар сызықтық қарапайым дифференциалдық теңдеу үшін жалпыланған шеттік есеп

Мақалада бөлшек дискретті үлестірілген дифференциалдау операторы бар сызықтық қарапайым дифференциалдық теңдеу үшін жалпыланған шеттік есеп құрастырылып және шешілген. Бөлшек туынды Герасимов-Капуто туындысы мағынасында түсініледі. Шеттік шарттар сызықтық функционалдар түрінде берілген, бұл сызықтық жергілікті және жергілікті емес жағдайлардың жеткілікті кең класын қамтуға мүмкіндік береді. Шешімнің мәні арнайы функциялар арқылы табылды. Зерттелетін есептің шешілу мүмкіндігінің қажетті және жеткілікті шарты, сондай-ақ шешілу шарты сөзсіз орындалатын шарттар алынды. Шешімнің бар болуы және бірегейлігі теоремасы дәлелденді.

Кілт сөздер: бөлшек дифференциалдау операторы, Капуто туындысы, шеттік есеп, функционал, Райт функциясы.

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Обобщенная краевая задача для линейного обыкновенного дифференциального уравнения с оператором дробного дискретно распределенного дифференцирования

В статье сформулирована и решена обобщенная краевая задача для линейного обыкновенного дифференциального уравнения с оператором дробного дискретно распределенного дифференцирования. Дробная производная понимается в смысле производной Герасимова–Капуто. Краевые условия задаются в форме линейных функционалов, это позволяет охватить достаточно широкий класс линейных локальных и нелокальных условий. В терминах специальных функций найдено представление решения. Получено необходимое и достаточное условие разрешимости исследуемой задачи, а также условия, при которых условие разрешимости заведомо выполняется. Доказана теорема существования и единственности решения.

Ключевые слова: оператор дробного дифференцирования, производная Капуто, краевая задача, функционал, функция Райта.

References

1 Nakhushev, A.M. (2003). *Drobnoe ischislenie i ego primenenie [Fractional calculus and its application].* Moscow: Fizmatlit [in Russian].

2 Kilbas, A.A. (2009). *Teoriia i prilozheniia differentsialnykh uravnenii drobnogo poriadka (kurs lektsii) [Theory and applications of fractional differential equations (course of lectures).]* Metodologicheskaia shkola-konferentsiia «Matematicheskaia fizika i nanotekhnologii» — "Mathematical Physics and Nanotechnology"Methodological School-Conference [in Russian].

3 Novozhenova, O.G. (2018). *Biografiia i nauchnye trudy Alekseia Nikiforovicha Gerasimova. O lineinykh operatorakh, uprugo-viazkosti, elevteroze i drobnykh proizvodnykh [Biography and scientific works of Alexey Nikiforovich Gerasimov. On linear operators, elastic viscosity, eleutherosis and fractional derivatives].* Moscow: Pero [in Russian].

4 Dzhrbashian, M.M. (1966). *Integralnye preobrazovaniia i predstavleniia funktsii v kompleksnoi oblasti [Integral transformations and representations of functions in the complex domain].* Moscow: Nauka [in Russian].

5 Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1987). *Integraly i proizvodnye drobnogo poriadka i nekotorye ikh prilozheniia [Integrals and derivatives of fractional order and some of their applications]* Minsk: Nauka i tekhnika [in Russian].

6 Nakhushev, A. M., & Tkhakakhov, R. B. (1995). O kontinualnykh analogakh reologicheskikh uravnenii sostoianiia i logisticheskom zakone izmeneniia viazkouprugikh svoistv polimera [On continual analogs of rheological equations of state and the logistic law of change in the viscoelastic properties of a polymer] *Doklady Adygskoi (Cherkesskoi) Mezhdunarodnoi akademii nauk – Reports of Adyghe (Circassian) International Academy of Sciences. 1,* 2, 6–11 [in Russian].

7 Podlubny, I. (1999). *Fractional Differential Equations.* ACADEMIC PRESS, New York.

8 Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations* North-Holland Math. Stud., Elsevier, Amsterdam.

9 Oldham, K. B., & Spanier J. (1974). *The fractional calculus* N.-Y.; L.: Acad. press.

10 Uchaikin, V. V. (2008). *Metod drobnykh proizvodnykh [The method of fractional derivatives].* Ulianovsk: Izdatelstvo «Artishok» [in Russian].

11 Miller, K.S., & Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations.* New York.: Jon Wiley & Sons. Inc.

12 Pskhu, A.V. (2005). Uravneniia v chastnykh proizvodnykh drobnogo poriadka [Partial differential equations of fractional order]. Moscow: Nauka [in Russian].

13 Barrett, J.H. (1954). Differential equation of non-integer order. *Canad. J. Math., 6,* 4, 529—541.

14 Dzhrbashyan, M.M., & Nersesyan, A.B. (1968). Drobnye proizvodnye i zadacha Koshi dlia differentsialnykh uravnenii drobnogo poriadka [Fractional derivatives and the Cauchy problem for differential equations of fractional order]. *Izvestiia Akademii nauk Armianskoi SSR – Proceedings of the Academy of Sciences of the Armenian SSR, 3,* 1, 3–28 [in Russian].

15 Dzhrbashyan, M.M. (1970). Kraevaia zadacha dlia differentsialnogo operatora drobnogo poriadka tipa Shturma-Liuvillia [Boundary problem for fractional differential operator of Sturm-Liouville type]. *Izvestiia Akademii nauk Armianskoi SSR – Proceedings of the Academy of Sciences of the Armenian SSR, 5,* 2, 71–96 [in Russian].

16 Pskhu, A. V. (2016). Boundary value problem for a first-order partial differential equation with a fractional discretely distributed differentiation operator. *Differential Equations, 52,* 12, 1610–1623.

17 Pskhu, A. V. (2016). Uravnenie drobnoi diffuzii s operatorom diskretno raspredelennogo differentsirovaniia [Fractional diffusion equation with discretely distributed differentiation operator]. *Siberian Electronic Mathematical Reports, 13,* 1078–1098.

18 Caputo, M. (2003). Diffusion with space memory modelled with distributed order space fractional differential equations. *Annals of Geophysics, 46,* 2, 223–234.

19 Diethelm, K., & Ford, N. J. (2009). Numerical analysis for distributed-order differential equations. *J. of Comp. and Appl. Math, 225,* 96–104.

20 Jiao, Z., Chen, Y. Qu., & Podlubny, I. (2012). Distributed-Order Dynamic Systems: Stability, Simulation, Applications and Perspectives. London.

21 Ozturk, I. (1998). On the theory of fractional differential equation. *Reports of Adyghe (Circassian) International Academy of Scienses, 3,* 2, 35–39.

22 Naimark, M.A. (1969). *Lineinye differentsialnye operatory [Linear differential operators].* Moscow: Nauka [in Russian].

23 Gadzova, L. Kh. (2015). Dirichlet and Neumann problems for a fractional ordinary differential equation with constant coefficients. *Differential Equations, 51,* 12, 1556–1562.

24 Gadzova, L. Kh. (2018). Boundary Value Problem for a Linear Ordinary Differential Equation with a Fractional Discretely Distributed Differentiation Operator. *Differential Equations, 54* 2, 178–184.

25 Gadzova, L. Kh. (2019). Nonlocal Boundary-Value Problem for a Linear Ordinary Differential Equation with Fractional Discretely Distributed Differentiation Operator. *Mathematical Notes, 106,* 5-6, 904–908.

26 Gadzova, L. Kh. (2022). Obobshchennaia kraevaia zadacha dlia obyknovennogo differentsialnogo uravneniia drobnogo poriadka [Generalized boundary value problem for an ordinary differential equation of fractional order]. *Cheliabinskii fiziko-matematicheskii zhurnal — Chelyabinsk Physical and Mathematical Journal, 7*(1), 20-29. [in Russian]. https://doi.org/10.47475/2500-0101-2022-17102

27 Pskhu, A.V. (2011). Initial-value problem for a linear ordinary differential equation of noninteger order *Sbornik: Mathematics, 202,* 4, 571–582.

28 Wright, E.M. (1933). On the coefficients of power series having exponential singularities *J. London Math. Soc., 8,* 29, 71–79.

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An initial boundary value problem for the Boussinesq equation in a Trapezoid

This paper considers an initial boundary value problem for a one-dimensional Boussinesq-type equation in a domain, that is, a trapezoid. Using the methods of the theory of monotone operators, we establish theorems on their unique weak solvability in Sobolev classes.

Keywords: Boussinesq-type equation, boundary value problem, trapezoid, theory of monotone operators.

Introduction

The theory of the Boussinesq equations and its modifications always attracts the attention of both mathematicians and applied scientists. The Boussinesq equation, as well as its modifications, occupies an important place in describing the motion of liquids and gas, including in the theory of unsteady filtration in porous media. Here we note only the works [1–6]. In recent years, boundary problems for these equations have been actively studied, since they model processes in porous media. The processes occurring in porous media acquire special importance for deep understanding in the tasks of exploration and effective development of oil and gas fields.

In this paper, we study the issues of the correct formulation of initial boundary value problems for a one-dimensional Boussinesq-type equation in a domain with a movable boundary. The domain is represented by a trapezoid. Using the method of monotone operators, we prove theorems on the unique weak solvability of the considered boundary value problems.

1 Statement of the initial boundary problem and the main result

Let $\Omega_t = \{0 < x < t\}$, and $\partial\Omega_t$ be the boundary of Ω_t, $0 < t_0 < T < \infty$. In domain $Q_{xt} = \Omega_t \times (t_0, T)$, i.e., a trapezoid, we consider the initial boundary problem for the Boussinesq-type equation

$$\partial_t u - \partial_x \left(|u|\partial_x u\right) = f, \;\; \{x, t\} \in Q_{xt}, \tag{1.1}$$

with boundary

$$u = 0, \;\; \{x, t\} \in \Sigma_{xt} = \partial\Omega_t \times (t_0, T), \tag{1.2}$$

and initial conditions

$$u = u_0, \;\; x \in \Omega_{t_0} = (0, t_0), \tag{1.3}$$

where $f(x, t)$, $u_0(x)$ are given functions.

We have established the following theorems.

Theorem 1.1 (Main result). Let

$$f \in L_{3/2}((t_0, T); W^{-1}_{3/2}(\Omega_t)), \;\; u_0 \in H^{-1}(\Omega_{t_0}).$$

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Then initial boundary problem (1.1)–(1.3) has a unique solution

$$u \in L_3(Q_{xt}).$$

Theorem 1.2 (On smoothness). Let

$$f \in L_{3/2}(Q_{xt}),\ \ u_0 \in L_2(\Omega_{t_0}).$$

Then initial boundary problem (1.1)–(1.3) has a unique solution

$$u \in L_\infty((t_0,T);L_2(\Omega_t)), \quad |u|^{1/2}u \in L_2((t_0,T);H_0^1(\Omega_t)), \quad \partial_t u \in L_{3/2}((t_0,T);W_{3/2}^{-1}(\Omega_t)).$$

2 Auxiliary initial boundary problem in a rectangle

To prove Theorem 1.1, we first consider an auxiliary initial boundary value problem. For this purpose, we pass from variables $\{x,t\}$ to $\{y,t\}$ by formulas $y=\frac{x}{t}$, $t=t$ and transform the trapezoid Q_{xt} into the rectangular domain $Q_{yt}=\Omega\times(t_0,T)$, $0<t_0<T<\infty$, where $y\in\Omega=$ $=(0,1)$, $\partial\Omega=\{0\}\cup\{1\}$, $\Sigma_{xt}=\partial\Omega\times(t_0,T)$. This transformation is one-to-one. Introducing the notation $w(y,t)=u(yt,t)=w(\frac{x}{t},t)$, $w_0(y)=u_0(yt_0,t_0)$ and $g(y,t)=f(yt,t)$, we write the auxiliary initial boundary value problem for (1.1)–(1.3) in the following form:

$$\partial_t w - \frac{1}{t^2}\partial_y\left(|w|\partial_y w\right) - \frac{y}{t}\partial_y w = g,\ \ \{y,t\}\in Q_{yt}, \tag{2.1}$$

$$w=0,\ \ \{y,t\}\in\Sigma_{yt}, \tag{2.2}$$

$$w=w_0,\ \ y\in\Omega. \tag{2.3}$$

By virtue of the one-to-one transformation of independent variables $\{x,t\}\to\{y,t\}$ the given functions in problem (2.1)–(2.3) obviously satisfy the conditions:

$$g\in L_{3/2}((t_0,T);W_{3/2}^{-1}(0,1)),\ \ w_0\in H^{-1}(0,1). \tag{2.4}$$

The following theorems are true.

Theorem 2.1 Under conditions (2.4) initial boundary value problem (2.1)–(2.3) is uniquely solvable

$$w\in L_3(Q_{yt}).$$

Theorem 2.2 (On smoothness). Let

$$g\in L_{3/2}(Q_{yt}),\ \ w_0\in L_2(\Omega).$$

Then initial boundary problem (2.1)–(2.3) has a unique solution

$$w \in L_\infty((t_0,T);L_2(\Omega)), \quad |w|^{1/2}w \in L_2((t_0,T);H_0^1(\Omega)), \quad \partial_t w \in L_{3/2}((t_0,T);W_{3/2}^{-1}(\Omega)).$$

3 Auxiliary statements

To prove Theorem 2.1, we first establish a number of auxiliary statements. Denote by A the operator of problem (2.1)–(2.3)

$$A(t,w)=\frac{1}{t^2}A_1(w)+\frac{1}{t}A_{21}(w), \quad \text{where}\ \ A_1(w)=-\partial_y(|w|\partial_y w),\ \ A_2(w)=-y\partial_y w, \tag{3.1}$$

and the operator $A_2(w)$ can be represented as:

$$A_2(w)=A_{21}(w)+A_{22}(w),\ \ \text{where}\ \ A_{21}(w)=w,\ \ A_{22}(w)=-\partial_y(yw). \tag{3.2}$$

Let us show that the operator $A_1(w)+A_{21}(w)$ will have the monotonicity property if we introduce the scalar product in an appropriate way. For this purpose, we take as a scalar product

$$\langle\varphi,\psi\rangle=\int_0^1\varphi\left[\left(-d_y^{\,2}\right)^{-1}\psi\right]dy,\ \ \forall\,\varphi,\psi\in H^{-1}(\Omega), \tag{3.3}$$

where $d_y^{\,2}=\frac{d^2}{dy^2}$, $\tilde\psi=\left(-d_y^{\,2}\right)^{-1}\psi:\ -d_y^{\,2}\tilde\psi=\psi,\ \tilde\psi(0)=\tilde\psi(1)=0,\forall\,\psi\in H^{-1}(\Omega)$.

Let us show the validity of the following lemma.

Lemma 3.1. The operator A_1+A_{21} is monotone in the sense of the scalar product (3.3) in the space $H^{-1}(0,1)$, i.e., the following inequality is true:

$$\langle(A_1+A_{21})(w_1)-(A_1+A_{21})(w_2),w_1-w_2\rangle\geq 0,\ \forall\,w_1,w_2\in\mathfrak{D}(\Omega). \tag{3.4}$$

To the proof of Lemma 3.1. It suffices for us to show that the operator A_1 is monotone and condition (3.4) will be satisfied (according to [7], chap. 2, s. 3.1). Indeed, on the one hand, we have

$$\langle A_1(\varphi)-A_1(\psi),\varphi-\psi\rangle=\frac{1}{2}\int_0^1\left(-d_y^2\right)\left(|\varphi|\varphi-|\psi|\psi\right)\left(-d_y^2\right)^{-1}(\varphi-\psi)\,d\,y=$$

$$=\frac{1}{2}\int_0^1(|\varphi|\varphi-|\psi|\psi)(\varphi-\psi)\,d\,y,\ \ \forall\,\varphi,\,\psi\in\mathfrak{D}(\Omega).$$

On the other hand, the convexity condition of the functional $J_1(\varphi)=\frac{1}{3}\int_0^1|\varphi(y)|^3\,d\,y,\ \ \varphi\in\mathfrak{D}(\Omega)$, implies

$$\langle J_1'(\varphi)-J_1'(\psi),\varphi-\psi\rangle\ \geq 0,\ \ \forall\,\varphi,\,\psi\in\mathfrak{D}(\Omega).$$

Thus, we get

$$\int_0^1(|\varphi|\varphi-|\psi|\psi)(\varphi-\psi)\,d\,y\ \geq 0,\ \ \forall\,\varphi,\,\psi\in\mathfrak{D}(\Omega).$$

For the operator A_{21} according to scalar product (3.3) we have:

$$\langle A_{21}(\varphi),\psi\rangle=\int_0^1\varphi\tilde\psi\,dy=$$

$$=\int_0^1\varphi\left(-d_y^2\right)^{-1}\psi\,dy=\int_0^1\left(\left(-d_y^2\right)^{-1}\varphi\right)\psi\,dy,\ \ \forall\,\varphi,\,\psi\in\mathfrak{D}(\Omega), \tag{3.5}$$

where $\tilde\psi$ is the solution to the following problem: $-d_y^{\,2}\tilde\psi=\psi,\ \ \tilde\psi(0)=\tilde\psi(1)=0$.

Let us introduce the convex functional

$$J_{21}(u)=\frac{1}{2}\int_0^1\left[\left(-d_y^{\,2}\right)^{-\frac{1}{2}}u\right]^2dy. \tag{3.6}$$

For the Gateaux derivative of functional (3.6) we have

$$J_{21}'(u)=\left(-d_y^{\,2}\right)^{-1}u, \tag{3.7}$$

that is, taking into account (3.7), we obtain the following convexity conditions of functional (3.6):

$$\langle J_{21}'(u)-J_{21}'(v),u-v\rangle=\int_0^1\left[\left(-d_y^{\,2}\right)^{-1}(u-v)\right](u-v)\,dy\geq 0\ \ \forall\,u,v\in\mathfrak{D}(\Omega). \tag{3.8}$$

Remark 3.1. On the other hand, inequality (3.8) is a consequence of the positivity of the operator $\left(-d_y^2\right)^{-1}$. Further, based on relations (3.5) and (3.8), we establish that the following monotonicity condition holds for operator A_{21}:

$$\langle A_{21}(t,u) - A_{21}(t,v), u - v\rangle \geq 0 \ \forall\, u, v \in \mathfrak{D}(\Omega), \ \ \forall\, t \in (t_0, T).$$

Thus, we have shown the validity of statement (3.4) of Lemma 3.1.

4 To the proof of Theorem 2.1

Let us preliminarily note that the nonlinear operator $A(t,v) \equiv (0.5\,t^{-2}A_1 + t^{-1}A_{21})v : L_3(\Omega) \to L_{3/2}(\Omega)$ (3.1)–(3.2) of boundary value problem (2.1)–(2.2) has the following properties:

$$A(t,v) : L_3(\Omega) \to L_{3/2}(\Omega) \text{ is a hemicontinuous operator,} \tag{4.1}$$

$$\|A(t_0,v)\|_{L_{3/2}(\Omega)} \leq c\|v\|^2_{L_3(\Omega)}, \ \ c > 0, \ \ \forall\, v \in L_3(\Omega), \tag{4.2}$$

$$\langle A(T,v), v\rangle \geq \alpha\|v\|^3_{L_3(\Omega)}, \ \ \alpha > 0, \ \ \forall\, v \in L_3(\Omega). \tag{4.3}$$

This follows directly from Lemma 4.1, as well as from ([7], Chap. 2, Proposition 1.1).

Recall the definition of a hemicontinuous operator.

Definition 4.1. Every operator $B:\ V \to V'$, having the following property:

$$\forall\, u, v, w \in V \ \text{ function } \ \lambda \to \langle B(u + \lambda v), w\rangle \ \text{ is continuous as a function from } \mathbb{R} \text{ to } \mathbb{R},$$

is called hemicontinuous.

Now we take as the main space:

$$H = H^{-1}(\Omega), \ \ (u,v)_H = \left(u, (-d_y^2)^{-1}v\right), \tag{4.4}$$

where $(-d_y^2)^{-1}v = \tilde{v}$ is the solution to problem

$$-d_y^2\tilde{v} = v, \ \ \tilde{v}(0) = \tilde{v}(1) = 0, \ \ v \in H^{-1}(\Omega). \tag{4.5}$$

Further, we have

$$V = L_3(\Omega), \ \ V \subset H \subset V', \tag{4.6}$$

where each embedding is dense and continuous. In notation (4.4)–(4.6), we introduce a linear continuous functional

$$L(v) = \langle g, v\rangle = (g, \tilde{v}), \ \text{ i.e. the element } \ g \in L_{3/2}(\Omega) \text{ is defined.}$$

Finally, we introduce

$$a(t,u,v) = \langle A(t,u), v\rangle = \int\limits_0^1 \left[\frac{1}{2\,t^2}|u|uv + \frac{1}{t}\left(-d_y^2\right)^{-1} uv\right] dy, \ \ \forall\, u, v \in L_3(\Omega).$$

We have

$$a(t,u,u) = \langle A(t,u), u\rangle = \frac{1}{2\,t^2}\|u\|^3_{L_3(\Omega)} + \frac{1}{t}\left\|\left(-d_y^2\right)^{-1/2} u\right\|^2_{L_2(\Omega)},$$

and

$$a(t,u,u-v) - a(t,v,u-v) \geq 0 \ \ \forall\, t \in (t_0, T), \tag{4.7}$$

where the form $a(t,u,v)$ corresponds to variational inequalities (3.4) and (3.8). Now, using (4.7), we obtain the following variational formulation for initial boundary problem (2.1)–(2.3):

$$(w'(t),v)_H + a(t,w(t),v) - b(t,w(t),v) = (g(t),v) \ \ \forall\, v \in L_3(\Omega) \subset H^{-1}(\Omega), \tag{4.8}$$

$$w(0) = w_0, \tag{4.9}$$

where $b(t,w(t),v) = t^{-1}\langle A_{22}(w), v\rangle$.

We show that relations (4.8), (4.9) admit unique solvability.

4.1 Existence of the solution

Let us show that variational problem (4.8) and (4.9) has a solution. We will use the Faedo-Galerkin method. Let $v_1,...,v_m,...$ be a "basis" in the space $L_3(\Omega)$. According to relations (4.8) and (4.9), we define an approximate solution $w_m(t)$ of initial boundary value problem (2.1)–(2.3) on a subspace $[v_1,...,v_m]$ spanned by $v_1,...,v_m$:

$$(w_m'(t),v_j) + a(t,w_m(t),v_j) - b(t,w_m(t),v_j) = (g(t),v_j), \ \ 1 \le j \le m, \tag{4.10}$$

$$w_m(0) = w_{0m} \in [v_1,...,v_m], \ \ w_{0m} \to w_0 \ \text{ in } \ H^{-1}(\Omega). \tag{4.11}$$

From equations (4.10)–(4.11), $w_m(t)$ is determined on the interval $[t_0,t_m],\ t_m > t_0$. However, due to the validity of inequality (4.3) $\langle A(t,v),v\rangle \ge \alpha\|v\|^3_{L_3(\Omega)},\ \ \alpha>0$, from (4.10)–(4.11) we obtain

$$\frac{1}{2}\|w_m(t)\|^2_{H^{-1}(0,1)} + \alpha\int\limits_{t_0}^{t}\|w_m(\tau)\|^3_{L_3(\Omega)}\,d\tau \le \frac{C_2}{t_0}\int\limits_{t_0}^{t}\|w_m(\tau)\|^3_{L_{3/2}(\Omega)}\|w_m(\tau)\|_{L_3(\Omega)}\,d\tau +$$

$$+\int\limits_{t_0}^{t}\|g(\tau)\|_{L_{3/2}(\Omega)}\|w_m(\tau)\|_{L_3(\Omega)}\,d\tau + \frac{1}{2}\|w_{0m}\|^2_{H^{-1}(\Omega)}, \tag{4.12}$$

since

$$|b(t,w_m(t),w_m(t))| \le \frac{1}{t_0}\,\|A_{22}w_m(t)\|_{L_{3/2}(\Omega)}\,\|w_m(t)\|_{L_3(\Omega)},$$

$$\|A_{22}w_m(t)\|_{L_{3/2}(\Omega)} \le C_2\|w_m(t)\|_{L_{3/2}(\Omega)},$$

$$\frac{C_2}{t_0}\|w_m(t)\|_{L_{3/2}(\Omega)}\|w_m(t)\|_{L_3(\Omega)} \le \frac{8}{9\sqrt{3\,\alpha}}\left(\frac{C_2}{t_0}\right)^{3/2}\|w_m(t)\|^{3/2}_{L_{3/2}(\Omega)} + \frac{\alpha}{4}\|w_m(t)\|^3_{L_3(\Omega)} \le$$

$$\le \frac{8}{9\sqrt{3\,\alpha}}K^{3/2}\left(\frac{C_2}{t_0}\right)^{3/2}\left[\|w_m(t)\|^2_{H^{-1}(\Omega)}\right]^{3/4} + \frac{\alpha}{4}\|w_m(t)\|^3_{L_3(\Omega)},$$

where K is the embedding constant of $\left(H^{-1}(\Omega)\right)' \hookrightarrow L_{3/2}(\Omega)$, since by assumptions (4.4) and (4.6): $L_3(\Omega) \subset H^{-1}(\Omega) \equiv \left(H^{-1}(\Omega)\right)' \subset L_{3/2}(\Omega) \equiv \left(L_3(\Omega)\right)'$. Here we also use Young's inequality ($p^{-1} + q^{-1} = 1$) :

$$|AB| = \left|\left(d^{1/p}A\right)\left(d^{1/q}\frac{B}{d}\right)\right| \le \frac{d}{p}\,|A|^p + \frac{d}{qd^q}\,|B|^q\,,$$

where

$$A = \frac{C_2}{t_0}\,\|w_m(t)\|_{L_{3/2}(\Omega)}\,, \ \ B = \|w_m(t)\|_{L_3(\Omega)}\,, \ \ d = \frac{2}{\sqrt{4\,\alpha}}, \ \ p = 3/2, \ \ q = 3.$$

We have similar calculations for the expression from (4.12):

$$\|g(t)\|_{L_{3/2}(\Omega)}\|w_m(t)\|_{L_3(\Omega)} \leq \frac{8}{9\sqrt{3}\,\alpha}K^{3/2}\left[\|g(t)\|^2_{H^{-1}(\Omega)}\right]^{3/4} + \frac{\alpha}{4}\|w_m(t)\|^3_{L_3(\Omega)}.$$

Now, using a variant of Bihari's lemma from ([8], Chapter 1, p.1.3, Example 1.3.1; it is important here that $3/4 < 1$), it follows from (4.12) that $t_m = T$ and that

$$w_m(t) \text{ are bounded in } L_\infty((t_0,T);H^{-1}(\Omega)) \cap L_3(Q_{yt}).$$

Hence, we can extract such a subsequence of $w_\mu(t)$ that

$$w_\mu \to w \ \ *\text{-weak in } L_\infty((t_0,T);H^{-1}(\Omega)),$$

$$w_\mu \to w \text{ weak in } L_3(Q_{yt}),$$

$$w_\mu(T) \to \xi \text{ weak in } H^{-1}(\Omega),$$

$$A(t,w_\mu) \to \chi(t) \text{ weak for almost every } t \in (t_0,T) \text{ in } L_{3/2}(Q_{yt}),$$

due to condition (4.2) $\|A(t,v)\|_{L_{3/2}(\Omega)} \leq c\|v\|^2_{L_3(\Omega)}$, $c > 0$, and hence $A(t,w_\mu)$ are bounded in $L_{3/2}(Q_{yt})$.

We extend $w_m(t)$, $A(t,w_m(t)),\ldots$ on the real axis with zero outside the interval $[t_0,T]$, and denote the corresponding continuations by $\tilde{w}_m(t)$, $\widetilde{A(t,w_m(t))},\ldots$ It follows from (4.10)–(4.11) that

$$\left(\tilde{w}'_m(t),v_j\right)_H + \langle\widetilde{A(t,w_m(t))},v_j\rangle - t^{-1}\langle\widetilde{A_{22}w_m(t)},v_j\rangle =$$

$$= (\tilde{g}(t),v_j) + (w_{0m},v_j)\delta(t-t_0) - (w_m(T),v_j)\delta(t-T). \tag{4.13}$$

Now we can pass to the limit in (4.13) at $m=\mu$ and fixed j, whence we have

$$\left(\tilde{w}'(t),v_j\right)_H + \langle\tilde{\chi}(t) - t^{-1}\widetilde{A_{22}}w(t),v_j\rangle = (\tilde{g}(t),v_j) + (w_0,v_j)\delta(t-t_0) - (\xi,v_j)\delta(t-T) \ \ \forall\, j$$

and hence

$$\tilde{w}'(t) + \tilde{\chi}(t) - t^{-1}\widetilde{A_{22}}w(t) = \tilde{g}(t) + w_0\delta(t-0) - \xi\delta(t-T). \tag{4.14}$$

By restricting (4.14) (t_0,T), we get that

$$w'(t) + \chi(t) - t^{-1}A_{22}w(t) = g(t), \tag{4.15}$$

from where $w'(t) \in L_{3/2}(Q_{yt})$, hence $w(t_0)$ and $w(T)$ make sense, and comparing with (4.14), we get that $w(t_0) = w_0$ and $w(T) = \xi$. So, we will prove the existence of a solution if we show that

$$\chi(t) = A(t,w). \tag{4.16}$$

From property (3.4), i.e., (4.7), it follows that

$$X_\mu \equiv \int\limits_{t_0}^{T} \langle A(t,w_\mu(t)) - A(t,v(t)), w_\mu(t) - v(t)\rangle\, dt \geq 0 \ \ \forall\, v \in L_3(Q_{yt}). \tag{4.17}$$

According to (4.10)–(4.11),

$$\int_{t_0}^{T} \langle A(t, w_\mu), w_\mu \rangle \, dt = \int_{t_0}^{T} t^{-1} \langle A_{22} w_\mu(t), w_\mu(t) \rangle \, dt + \int_{t_0}^{T} (g, w_\mu) \, dt + \frac{1}{2} \|w_{0\mu}\|^2_{H^{-1}(\Omega)} - \frac{1}{2} \|w_\mu(T)\|^2_{H^{-1}(\Omega)} \tag{4.18}$$

and, therefore,

$$X_\mu = \int_{t_0}^{T} (g, w_\mu) \, dt + \frac{1}{2} \|u_{0\mu}\|^2_{H^{-1}(\Omega)} - \frac{1}{2} \|w_\mu(T)\|^2_{H^{-1}(\Omega)} +$$

$$+ \int_{t_0}^{T} t^{-1} \langle A_{22} w_\mu, w_\mu \rangle \, dt - \int_{t_0}^{T} \langle A(t, w_\mu), v \rangle \, dt - \int_{t_0}^{T} \langle A(t, v), w_\mu - v) \rangle \, dt,$$

whence (since lim inf $\|w_\mu(T)\|^2_{H^{-1}(\Omega)} \geq \|w(T)\|^2_{H^{-1}(\Omega)}$):

$$\limsup X_\mu \leq \int_{t_0}^{T} (g, w) \, dt + \frac{1}{2} \|u_0\|^2_{H^{-1}(\Omega)} - \frac{1}{2} \|w(T)\|^2_{H^{-1}(\Omega)} +$$

$$+ \int_{t_0}^{T} t^{-1} \langle A_{22} w, w \rangle \, dt - \int_{t_0}^{T} \langle \chi(t), v \rangle \, dt - \int_{t_0}^{T} \langle A(t, v), w - v) \rangle \, dt. \tag{4.19}$$

From (4.15) we can conclude, since integration by parts is legal, that

$$\int_{t_0}^{T} t^{-1} \langle A_{22} w, w \rangle \, dt + \int_{t_0}^{T} (g, w) \, dt + \frac{1}{2} \|u_0\|^2_{H^{-1}(\Omega)} - \frac{1}{2} \|w(T)\|^2_{H^{-1}(\Omega)} = \int_{t_0}^{T} \langle \chi, w \rangle \, dt.$$

Comparing this equality with (4.17) and (4.19), and also considering (4.18), we get

$$\int_{t_0}^{T} \langle \chi(t) - A(t, v), w - v \rangle \, dt \geq 0. \tag{4.20}$$

Now we use the hemicontinuity property (4.1) of the operator $A(t, w)$ to prove that (4.20) implies (4.16). Let $v = w - \lambda u, \ \lambda > 0, \ u \in L_3(Q_{yt})$; then it follows from (4.20) that

$$\lambda \int_{t_0}^{T} \langle \chi(t) - A(t, w - \lambda u), u \rangle \, dt \geq 0,$$

whence

$$\int_{t_0}^{T} \langle \chi(t) - A(t, w - \lambda u), u \rangle \, dt \geq 0; \tag{4.21}$$

when $\lambda \to 0$ in (4.21), then we get that

$$\int_{t_0}^{T} \langle \chi(t) - A(t, w), u \rangle \, dt \geq 0 \ \ \forall \, u.$$

Therefore, $\chi(t) = A(t, w)$. The existence of a solution to problem (2.1) and (2.3) is proved.

4.2 Uniqueness of the solution

Let $w_1(t)$ and $w_2(t)$ be two solutions to problem (4.8)-(4.9). Then their difference $w(t) = w_1(t) - w_2(t)$ satisfies the homogeneous problem:

$$w'(t) + A(t, w_1(t)) - A(t, w_2(t)) - t^{-1}A_{22}w(t) = 0, \;\; w(0) = 0,$$

$$\big(w'(t), w(t)\big) + \langle (A(t, w_1(t)) - A(t, w_2(t)), w_1(t) - w_2(t)\rangle - t^{-1}\langle (A_{22}w(t), w(t)\rangle = 0$$

and, due to (4.12) and the monotonicity property of the operator $A(t, w)$, we have:

$$\big(w'(t), w(t)\big) = \frac{d}{2\,dt}\|w(t)\|^2_{H^{-1}(\Omega)} \le \frac{C_2K}{t_0}\|w(t)\|^2_{H^{-1}(\Omega)}, \;\text{ i.e. }\; w(t) \equiv 0,$$

where K is the norm of the operator $\left(-d_y^2\right)^{-1/2} : H^{-1}(\Omega) \to \left[H_0^1(\Omega); H^{-1}(\Omega)\right]_{1/2}$, $\left[H_0^1(\Omega); H^{-1}(\Omega)\right]_{1/2}$ is an intermediate space [9].

Remark 4.1. Let us give the interpretation of the solution to problem (4.8)–(4.9) as the solution to problem (2.1)–(2.3). By introducing $\tilde v$ in (4.8), we obtain

$$\int\limits_0^1 \partial_t w\, \tilde v\, dy + \int\limits_0^1 \left[\frac{1}{2t^2}|w|w + \frac{1}{t}\left(-\partial_y^{\,2}\right)^{-1} w + \frac{1}{t}\left(-\partial_y^{\,2}\right)^{-\frac12}(y\,w)\right]\left(-\partial_y^2\tilde v\right)\, dy =$$

$$= \int\limits_0^1 g(t)\tilde v\, dy, \;\; \forall\, \tilde v \in H_0^1(\Omega).$$

Hence, from here we have

$$\int\limits_0^1 \left(\partial_t w - \partial_y^2\left[\frac{1}{2\,t^2}|w|w + \frac{1}{t}\left(-\partial_y^{\,2}\right)^{-1} w + \frac{1}{t}\left(-\partial_y^{\,2}\right)^{-\frac12}(y\,w)\right]\right)\tilde v\, dy = \int\limits_0^1 g(t)\tilde v\, dy +$$

$$+ \left[\frac{1}{2\,t^2}|w|w + \frac{1}{t}\left(-\partial_y^{\,2}\right)^{-1} w + \frac{1}{t}\left(-\partial_y^{\,2}\right)^{-\frac12}(y\,w)\right]\partial_y\tilde v\Big|_{y=0}^{y=1} \;\forall\, \tilde v \in H_0^1(\Omega). \tag{4.22}$$

Or, taking into account equality (3.5), the last identity can be written in the following form

$$\int\limits_0^1 \left(\partial_t w - \frac{1}{t^2}\partial_y\left(|w|\partial_y w\right) - \frac{y}{t}\,\partial_y w - g(t)\right)\tilde v\, dy = 0 \;\; \forall\, \tilde v \in \mathfrak{D}(\Omega), \tag{4.23}$$

that is, the function $w(y, t)$ satisfies a Boussinesq type equation (2.1). Now, returning to (4.22) and taking into account (4.23), we get

$$\left[\frac{1}{2t^2}|w|w + \frac{1}{t}\left(-\partial_y^{\,2}\right)^{-1} w + \frac{1}{t}\left(-\partial_y^{\,2}\right)^{-\frac12}(y\,w)\right]\partial_y\tilde v\big|_{|y=0} = 0 \;\; \forall\, \tilde v \in H_0^1(\Omega),$$

$$\left[\frac{1}{2t^2}|w|w + \frac{1}{t}\left(-\partial_y^{\,2}\right)^{-1} w + \frac{1}{t}\left(-\partial_y^{\,2}\right)^{-\frac12}(y\,w)\right]\partial_y\tilde v\big|_{|y=1} = 0 \;\; \forall\, \tilde v \in H_0^1(\Omega).$$

The last equalities imply the fulfillment of boundary conditions (2.2). Finally, from the continuity of the function $w : [t_0, T] \to H$ we get that initial condition (2.3) makes sense. This completes the proof of Theorem 4.2.

5 To the proof of Theorem

Since the transformation of independent variables $\{x,t\} \to \{y,t\}$ is one-to-one, there is a mutual correspondence of functional classes defining the given functions and solutions of initial boundary value problems. Therefore, from Theorem 2.1 we obtain the validity of the statement of Theorem 1.1 in terms of the existence of a solution to initial boundary value problem (1.1)–(1.3). Let us show the validity of the assertion of Theorem 1.1 in terms of the uniqueness of the solution to problem (1.1)–(1.3).

We show that the operator $A_1(t,u)$ in problem (1.1)–(1.3) will have the monotonicity property if a scalar product is introduced accordingly. For this purpose, we take as the scalar product

$$\langle \varphi, \psi \rangle = \int_0^t \varphi \left[\left(-d_x^{\,2}\right)^{-1} \psi \right] dx, \ \ \forall \varphi, \psi \in H^{-1}(\Omega_t), \ \ \forall t \in (t_0, T), \tag{5.1}$$

where $d_x^{\,2} = \frac{d^2}{dx^2}$, $\tilde{\psi} = \left(-d_x^{\,2}\right)^{-1} \psi : \ -d_x^{\,2}\tilde{\psi} = \psi$, $\tilde{\psi}(0) = \tilde{\psi}(t) = 0, \forall \psi \in H^{-1}(\Omega_t), \ \forall t \in (t_0, T)$.

The following lemma is valid.

Lemma 5.1. Operator $A_1(t,u)$ is monotone in the sense of the scalar product (5.1) in the space $H^{-1}(\Omega_t)$, i.e., the following inequalities hold:

$$\langle A_1(t,u_1) - A_1(t,u_2), u_1 - u_2 \rangle \geq 0, \ \forall u_1, u_2 \in \mathfrak{D}(\Omega_t), \ \ \forall t \in (t_0, T). \tag{5.2}$$

To the proof of Lemma 5.1. For each $t \in (t_0, T)$ operator A_1 is monotone and condition (5.2) is satisfied (according to [7], chap. 2, p. 3.1). Indeed, on the one hand, we have

$$\langle A_1(t,\varphi) - A_1(t,\psi), \varphi - \psi \rangle = \frac{1}{2} \int_{\Omega_t} \left(-d_x^2\right) \left(|\varphi|\varphi - |\psi|\psi\right) \left(-d_x^2\right)^{-1} (\varphi - \psi)\, d\,x =$$

$$= \frac{1}{2} \int_{\Omega_t} (|\varphi|\varphi - |\psi|\psi)(\varphi - \psi)\, d\,x, \ \ \forall \varphi, \, \psi \in \mathfrak{D}(\Omega_t), \ \ \forall t \in (t_0, T).$$

On the other hand, the convexity condition of the functional $J(t,\varphi) = \frac{1}{3} \int_{\Omega_t} |\varphi(x)|^3 \, d\,x, \ \varphi \in \mathfrak{D}(\Omega_t), \ \ \forall t \in (t_0, T)$, implies

$$\langle J'(t,\varphi) - J'(t,\psi), \varphi - \psi \rangle \ \geq 0, \ \ \forall \varphi, \, \psi \in \mathfrak{D}(\Omega_t), \ \ \forall t \in (t_0, T).$$

Thus, we get

$$\int\limits_{\Omega_t} (|\varphi|\varphi - |\psi|\psi)(\varphi - \psi)\, d\,x \ \geq 0, \ \ \forall \varphi, \, \psi \in \mathfrak{D}(\Omega_t), \ \ \forall t \in (t_0, T),$$

that is, inequality 5.2 is established. Lemma 5.1 is proved.

Now we are ready to show the uniqueness of the solution to problem (1.1)–(1.3). To do this, using inequality (5.2), we obtain the following variational formulation for initial boundary problem (1.1)–(1.3):

$$(u'(t), v)_{H^{-1}(0,t)} + a_0(t, u(t), v) = (f(t), v)_{H^{-1}(\Omega_t)} \ \ \forall v \in L_3(0,t) \subset H^{-1}(\Omega_t), \ \ \forall t \in (t_0, T), \tag{5.3}$$

$$u(t_0) = u_0, \tag{5.4}$$

where

$$a_0(t,u,v) = \langle A_1(t,u(t)), v \rangle = \int\limits_{\Omega_t} |u(x,t)|\, u(x,t)\, v(x)\, dx, \ \ \forall t \in (t_0, T).$$

Let $u_1(t)$ and $u_2(t)$ be two solutions to problem (5.3)–(5.4). Then their difference $u(t) = u_1(t) - u_2(t)$ satisfies the homogeneous problem:

$$\big(u'(t), u(t)\big)_{H^{-1}(\Omega_t)} + \langle A_1(t,u_1(t)) - A_1(t,u_2(t)), u(t) \rangle = 0, \ \ \forall t \in (t_0, T); \ \ u(0) = 0,$$

and, due to the monotonicity property of the operator $A_1(t,u)$ (5.2), we have:

$$\big(u'(t),u(t)\big)_{H^{-1}(\Omega_t)}=\frac{d}{2\,dt}\|u(t)\|^2_{H^{-1}(\Omega_t)}\leq 0,\ \text{ i.e. } u(t)\equiv 0.$$

Thus, Theorem (1.1) is completely proved.

Conclusions

The initial boundary value problems for a one-dimensional Boussinesq type equation in a trapezoid domain are studied. Theorems on their unique weak solvability in Sobolev classes are proved by methods of the theory of monotone operators.

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References

1 Zhong X. Strong solutions to the nonhomogeneous Boussinesq equations for magnetohydrodynamics convection without thermal diffusion / X.Zhong // Electronic Journal of Differential Equations. — 2020. — 24. — P. 1–23.

2 Zhang H. Global existence, asymptotic stability and blow-up of solutions for the generalized Boussinesq equation with nonlinear boundary condition / H. Zhang, Q. Hu, G. Liu // Mathematische Nachrichten. — 2020. — 293(2). — P. 386–404.

3 Oruc G. Existence and uniqueness of solutions to initial boundary value problem for the higher order Boussinesq equation / G.Oruc, G.M. Muslu // Nonlinear Analysis – Real World Applications. — 2019. — 47. — P. 436–445.

4 Ding W. Global existence and asymptotic behavier of the Boussinesq-Burgers system / W.Ding, Zh.-A. Wang // Journal of Mathematical Analysis and Applications. — 2015. — 424(1). — P. 584–597.

5 Zhu N. On the Boussinesq-Burgers equations driven by dynamic boundary condition / N.Zhu, Zh.Liu, K.Zhao // Journal of Differential Equations. — 2018. — 264(3). — P. 2287–2309.

6 Crank J. Free and Moving Boundary Problems / J.Crank. — Oxford University Press, 1984. — P. 424.

7 Lions J.-L. Quelques methodes de resolution des problemes aux limites non lineaires / J.-L.Lions. — Dunod Gauthier-Villars, Paris, 1969. — P. 554.

8 Лакшмикантам В. Устойчивость движения: метод сравнения / В. Лакшмикантам, С. Лила, А.А. Мартынюк. — Киев: Наук. думка, 1991. — 248 с.

9 Lions J.-L. Non-Homogeneous Boundary Value Problems and Applications / J.-L.Lions, E.Magenes. — Vol. 1. — Springer Verlag, Berlin, 1972.

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Трапециядағы Буссинеск типті теңдеу үшін бастапқы-шекаралық есеп

Мақалада трапеция облысындағы бір өлшемді Буссинеск типті теңдеу үшін бастапқы-шекаралық есеп қарастырылған. Соболев кластарындағы олардың бірегей әлсіз шешілетіндігі туралы теоремалар монотонды операторлар теориясының әдістерімен анықталған.

Кілт сөздер: Буссинеск типті теңдеу, шекаралық есеп, трапеция, монотонды операторлар теориясы.

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Начально-граничная задача для уравнения типа Буссинеска в трапеции

В статье рассмотрена начально-граничная задача для одномерного уравнения типа Буссинеска в области, представляющей собой трапецию. Методами теории монотонных операторов установлены теоремы об их однозначной слабой разрешимости в соболевских классах.

Ключевые слова: уравнение типа Буссинеска, граничная задача, трапеция, теории монотонных операторов.

References

1 Zhong, X. (2020). Strong solutions to the nonhomogeneous Boussinesq equations for magnetohydrodynamics convection without thermal diffusion. *Electronic Journal of Differential Equations, 24*, 1–23.

2 Zhang, H., Hu, Q., & Liu, G.(2020). Global existence, asymptotic stability and blow-up of solutions for the generalized Boussinesq equation with nonlinear boundary condition. *Mathematische Nachrichten, 293*(2), 386–404.

3 Oruc, G., & Muslu, G.M. (2019). Existence and uniqueness of solutions to initial boundary value problem for the higher order Boussinesq equation. *Nonlinear Analysis – Real World Applications, 47*, 436–445.

4 Ding, W., & Wang, Zh.-A. (2015). Global existence and asymptotic behavier of the Boussinesq-Burgers system. *Journal of Mathematical Analysis and Applications, 424*(1), 584–597.

5 Zhu, N., Liu, Zh., & Zhao, K. (2018). On the Boussinesq-Burgers equations driven by dynamic boundary condition. *Journal of Differential Equations, 264*(3), 2287–2309.

6 Crank, J. (1984). *Free and Moving Boundary Problems.* Oxford University Press.

7 Lions, J.-L. (1969). *Quelques methodes de resolution des problemes aux limites non lineaires.* Dunod Gauthier-Villars, Paris.

8 Lakshmikantam, V., Leela, S., & Martynyuk, A.A. (1991). *Ustoichivost dvizheniia: metod sravneniia [Motion Stability: Comparison Method].* Kiev: Naukova dumka [in Russian].

9 Lions, J.-L., & Magenes, E. (1972). Non-Homogeneous Boundary Value Problems and Applications. V. 1. *Springer Verlag, Berlin.*

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Inverse problems of determining coefficients of time type in a degenerate parabolic equation

The paper is devoted to the study of the solvability of inverse coefficient problems for degenerate parabolic equations of the second order. We study both linear inverse problems – the problems of determining an unknown right-hand side (external influence), and nonlinear problems of determining an unknown coefficient of the equation itself. The peculiarity of the studied work is that its unknown coefficients are functions of a time variable only. The work aims to prove the existence and uniqueness of regular solutions to the studied problems (having all the generalized in the sense of S.L. Sobolev derivatives entering the equation).

Keywords: degenerate parabolic equations, linear inverse problems, non-linear inverse problems, regular solutions, existence.

Introduction

The paper studies the solvability of some inverse problems of finding the solution to a degenerate parabolic equation and a certain coefficient of the equation itself. If the unknown coefficient determines the free term (external influence) in the equation, then such an inverse problem will be linear, but if the unknown coefficient is a multiplier for one or another derivative of the solution, then it will be nonlinear. In this paper, both linear and nonlinear inverse problems will be studied.

The problems studied in the work will have two features.The first of them is that inverse coefficient problems for time-variable degenerate parabolic equations will be studied. The second feature is that the unknown coefficient in our problems will also be a function of the time variable only.

Inverse problems for parabolic equations without degeneracy and with unknown coefficients depending only on the time variable seem to be thoroughly studied (see [1–12]. As for similar problems for time-variable degenerate parabolic equations, there are few works here – only works [13–15] can be named, and in these works either the nature of degeneracy is different, or the problem itself is completely distinct.

Note the following. The presence of degeneracy in parabolic equations means that the well-posed boundary value problems for them may differ significantly from the classical initial boundary value problems for non-degenerate equations (see [16–19]). This is the situation that will be studied in this paper – a situation in which the boundary conditions in linear problems will be different than in natural initial boundary value problems.

All constructions and reasoning in the work will be conducted based on the Lebesgue L_p and SobolevW_p^l spaces. The necessary definitions and description of the properties of functions from these spaces can be found in monographs [20–22].

The purpose of this work is to prove the existence and uniqueness of regular solutions to the problem, i.e., solutions having all the generalized in the sense of S.L. Sobolev derivatives, included in the corresponding equation.

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The presence of additional unknown functions in inverse problems requires that, in addition to the boundary conditions natural for a particular class of differential equations, some additional conditions are also set – overdetermination conditions. In this paper, overdetermination conditions, called integral overdetermination conditions in the literature, will be used. Inverse coefficient problems, linear and nonlinear, with integral overdetermination conditions are well-studied for both classical (elliptic, parabolic and hyperbolic) and non-classical differential equations. However, for time-variable degenerate parabolic equations, inverse coefficient problems with integral overdetermination have not been studied before.

Overall, the content of the work consists of four parts. The first part provides studied linear and nonlinear problems statement. The second part investigates the solvability of linear inverse problems for degenerate parabolic equations of the second order. The third part studies the solvability of some nonlinear inverse coefficient problem for degenerate parabolic equations of the second order. Finally, the fourth part describes some generalizations and amplifications of the results obtained in the second and third parts of the work, discusses their possible development.

Problem statement

Let Ω be a bounded domain from the space R^n with a smooth (for simplicity – infinitely differentiable) boundary Γ, Q be a cylinder $\Omega \times (0,T)$ variables (x,t) of finite height T, $S = \Gamma \times (0,T)$ be the lateral boundary of Q.

Next, let $\varphi(t), c(x,t), f(x,t), N(x), h(x,t), \mu(t)$ and $u_0(x)$ be the given functions defined at $x \in \bar{\Omega}$, $t \in [0,T]$, respectively. L is a differential operator whose action on a given function $v(x,t)$ is determined by the equality

$$Lv = \varphi(t)v_t - \Delta v + c(x,t)v$$

(Δ is the Laplace operator for variables $x_1, x_2, ..., x_n$).

Inverse problem I: Find the functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation

$$Lu = f(x,t) + q(t)h(x,t), \tag{1}$$

when the conditions for the function $u(x,t)$ are met

$$u(x,t)|_S = 0, \tag{2}$$

$$\int_\Omega N(x)u(x,t)dx = 0, t \in (0,T). \tag{3}$$

Inverse problem II: Find the functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation (1), when the conditions (2) and (3) are met for the function $u(x,t)$, as well as the conditions

$$u(x,0) = u(x,T) = 0, x \in \Omega. \tag{4}$$

The inverse problems I and II are linear inverse problems for the parabolic equation $Lu = F$. Note that in the problem I there are no boundary conditions for the variable t, in the problem II, on the contrary, two boundary conditions are set for the variable t. Both of these situations do not seem to be characteristic of first-order differential equations (with respect to a time variable), nevertheless, sufficient conditions will be specified for each of them to ensure the existence and uniqueness of regular solutions to the corresponding inverse problems.

Along with the inverse problems I and II, is easy to study linear inverse problems for equation (1) with the setting of one boundary condition for the variable t at $t = 0$ or at $t = T$. The sufficient conditions for the existence and uniqueness of regular solutions to such problems is presented in the fourth part of the work.

Let us consider the problem statement of a nonlinear inverse problem for degenerate parabolic equations.

Inverse problem III: Find the functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation

$$Lu + q(t)u = f(x,t), \tag{5}$$

when the function $u(x,t)$ fulfills the condition (2), as well as the conditions

$$u(x,0) = u_0(x), x \in \Omega, \tag{6}$$

$$\int_\Omega N(x)u(x,t)dx = \mu(t), t \in (0,T). \tag{7}$$

The inverse problem III corresponds to the usual first initial-boundary value problem for parabolic equations of the second order, the inhomogeneity of conditions (2) and (7) is explained by the nonlinearity of the problem.

Solvability of inverse problems I and II

Let us put

$$h_0(t) = \int_\Omega N(x)h(x,t)dx,$$

$$h_1(x,t) = \frac{h(x,t)}{h_0(t)},$$

$$f_0(t) = \int_\Omega N(x)f(x,t)dx,$$

$$f_1(x,t) = f(x,t) - h_1(x,t)f_0(t).$$

Next, by the given function $v(x,t)$, we define the functions $A_1(t;v)$ and $A_2(t;v)$:

$$A_1(t;v) = \int_\Omega N(x)\Delta v(x,t)dx,$$

$$A_2(t;v) = \int_\Omega c(x,t)N(x)v(x,t)dx.$$

For the function $\omega(x)$ from the space $W_2^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega)$ there are inequalities

$$\int_\Omega \omega^2(x)dx \le d_0 \sum_{i=1}^n \int_\Omega \omega_{x_i}^2(x)dx \le d_0^2 \int_\Omega [\Delta\omega(x)]^2 dx \tag{8}$$

with the number d_0 defined only by the domain Ω (see [20–22]). We will need these inequalities and the actual number d_0 below.

In addition to the number d_0, we will also need the following numbers:

$$\overline{h}_1 = \max_{\overline{Q}} |h_1(x,t)|,$$

$$N_1 = \overline{h}_1 ||N||_{L_2(\Omega)} mes^{1/2}\Omega,$$

$$N_2 = d_0\overline{h}_1 \max_{0\le t\le T} [\int_\Omega c^2(x,t)N^2(x)dx]^{1/2} mes^{1/2}\Omega.$$

Theorem 1. Let the conditions be satisfied

$$\varphi(t) \in C^1([0;T]), \varphi(0) \leq 0, \varphi(T) \geq 0; \tag{9}$$

$$\begin{gathered} c(x,t) = c_1(x,t) + c_0, \;\; c_1(x,t) \in C^2(\overline{Q}) \\ c_1(x,t) \geq 0, \quad \Delta c_1(x,t) \leq 0 \;\text{ at }\; (x,t) \in \overline{Q}, c_0 = const > 0; \end{gathered} \tag{10}$$

$$2c_0 - \varphi^{'}(t) \geq \overline{c}_0 > 0, \quad 2c_0 + \varphi^{'}(t) \geq \overline{c}_1 > 0 \;\text{ at }\; (x,t) \in \overline{Q}; \tag{11}$$

$$N(x) \in W^1_\infty(\Omega), h(x,t) \in L_\infty(Q), h_t(x,t) \in L_2(Q); \tag{12}$$

$$|h_0(t)| \geq \overline{h}_0 > 0 \;\text{ at }\; t \in [0,T]; \tag{13}$$

$$N_1 + N_2 < 1. \tag{14}$$

Then for any function $f(x,t)$ such that $f(x,t) \in L_2(Q), f_t(x,t) \in L_2(Q)$, the inverse problem has the solution $(u(x,t),\, q(t))$ such that $u(x,t) \in W_2^{2,1}(Q)$, $q(t) \in L_2(Q)$.

Proof. Consider the boundary value problem: Find the function $u(x,t)$, which is the solution to the equation in the cylinder Q.

$$Lu = f_1(x,t) - h_1(x,t)[A_1(t;u) - A_2(t;u)] \tag{15}$$

and such that the condition (2) is satisfied for it. In this problem, equation (15) is a degenerate parabolic integro-differential equation (similar equations are also called "loaded" [23], [24]). We will prove its solvability in the space $W_2^{2,1}(Q)$ using the regularization method and the continuation method by parameter.

Let ε be a positive number. Consider the boundary value problem: Find the function $u(x,t)$, which in the cylinder Q is the solution to the equation

$$-\varepsilon u_{tt} + Lu = f_1(x,t) - h_1(x,t)[A_1(t;u) - A_2(t;u)] \tag{16}$$

and such that condition (2) is met for it, as well as the condition

$$u_t(x,0) = u_t(x,T) = 0, \;\; x \in \Omega. \tag{17}$$

This problem is a mixed boundary value problem for an elliptic (non-degenerate) "loaded" equation (16); its solvability in the space $W_2^2(Q)$ is not difficult to show using the continuation method by parameter [25].

Let λ be a number from the segment [0;1]. Consider a family of problems: Find the function $u(x,t)$, which in the cylinder Q is the solution to the equation

$$-\varepsilon u_{tt} + Lu = f_1(x,t) - \lambda h_1(x,t)[A_1(t;u) - A_2(t;u)] \tag{18}$$

and such that conditions (2) and (17) are met for it.

Boundary value problem (18), (2), (17) in the case of $\lambda = 0$ with a fixed ε and if the conditions of the theorem are met, it is solvable in the space $W_2^2(Q)$ for any function $f(x,t)$ belonging to the space $L_2(Q)$ (see [21]). Further, integrating by parts in equality (19)

$$\varepsilon \int_Q u_{tt} \Delta u dx dt - \int_Q Lu \Delta u dx dt = - \int_Q \{f_1(x,t) - \lambda h_1(x,t)[A_1(t;u) - A_2(t;u)]\} \Delta u dx dt \tag{19}$$

(which is a consequence of equation (18)), using conditions (9)–(14) and applying the Helder and Young inequalities, it is easy to obtain that for all possible solutions $u(x,t)$ to the boundary value problem (18), (2), (17) we take an estimate

$$\varepsilon \sum_{i=1}^{n} \int_Q u_{x_i t}^2 dxdt + \int_Q (\Delta u)^2 dxdt \le M_1 \int_Q f^2 dxdt \tag{20}$$

with a constant M_1, defined only by the functions $\varphi(t), c(x,t), h(x,t), N(x)$, as well as the domain Ω.

Consider now the equality

$$\varepsilon \int_Q u_{tt}^2 dxdt - \int_Q Luu_{tt} dxdt = -\int_Q \{f_1(x,t) - \lambda h_1(x,t)[A_1(t;u) - A_2(t;u)]\} u_{tt} dxdt. \tag{21}$$

Integrating again by parts, using conditions (9)–(14) and applying the Helder and Young inequalities, we obtain that for all possible solutions $u(x,t)$ to the boundary value problem (18), (2), (17) a priori estimate is performed

$$\varepsilon \int_Q u_{tt}^2 dxdt + \sum_{i=1}^{n} \int_Q u_{x_i t}^2 dxdt \le M_2 \int_Q f^2 dxdt, \tag{22}$$

where the constant M_2 is defined by the functions $\varphi(t), c(x,t), h(x,t)$ and $N(x)$, the domain Ω, and the number ε.

From estimates (20) and (22), as well as from the second basic inequality for elliptic operators [21], it follows that for solutions $u(x,t)$ to the boundary value problem (18), (2), (17) the next estimate is true

$$||u||_{W_2^2(Q)} \le M_3 ||f||_{L_2(Q)}, \tag{23}$$

where the constant M_3 is defined by the functions $\varphi(t), c(x,t), h(x,t)$ and $N(x)$, the domain Ω, and the number ε.

From estimate (23), from the solvability in the space $W_2^2(Q)$ of the problem (18), (2), (17) in the case of $\lambda = 0$, as well as from the theorem on the continuation method by parameter [25], it follows that for a fixed ε, for an arbitrary λ from the segment [0,1] and if conditions (9)–(14) are met, the boundary value problem (18), (2), (17) will be solvable in the space $W_2^2(Q)$ for any function $f(x,t)$ from the space $L_2(Q)$.

Let $\{\varepsilon_m\}_{m=1}$ be a sequence of positive numbers converging to zero. According to the above, the boundary value problem (18), (2), (17) in the case of $\varepsilon = \varepsilon_m$ and $\lambda = 1$, there is a solution $u_m(x,t)$ belonging to the space $W_2^2(Q)$. For the family $\{u_m(x,t)\}_{m=1}^{\infty}$, there is a priori estimate (20) which is uniform by ε. Next, on the right side of the equality (21) with $\varepsilon = \varepsilon_m$, we will integrate by parts with respect to the variable t. Further, using the conditions of the theorem and applying the Helder and Young inequalities, we obtain that for the functions $u_m(x,t)$ there is a true estimate

$$\varepsilon_m \int_Q u_{mtt}^2 dxdt + \sum_{i=1}^{n} \int_Q u_{mx_i t}^2 dxdt \le M_4 \int_Q (f^2 + f_t^2) dxdt, \tag{24}$$

the constant M_4 where is defined only by the functions $\varphi(t), c(x,t), h(x,t)$ and $N(x)$, as well as the domain Ω.

Estimates (20) and (24) for functions $u_m(x,t)$, the reflexivity property of the Hilbert space [25], as well as embedding theorems [20–22] mean that there are functions $u_{m_k}(x,t), k = 1,2,...,$ and $u(x,t)$ such that for $k \to \infty$ there are convergences

$$u_{m_k}(x,t) \to u(x,t) \text{ weakly in } W_2^{2,1}(Q),$$
$$u_{m_k x_i}(x,t) \to u_{x_i}(x,t) \text{ strongly in } L_2(Q) \text{ for } ß = 1,...,n,$$
$$u_{m_k x_i}(x,t) \to u_{x_i}(x,t) \text{ strongly in } L_2(S) \text{ for } ß = 1,...,n,$$
$$\varepsilon_{m_k} u_{m_k tt}(x,t) \to 0 \text{ weakly in } L_2(Q).$$

From these convergences, as well as from the representation

$$A_1(t;u_{m_k}) = -\sum_{i=1}^{n} \int_\Omega N_{x_i}(x) u_{m_k x_i}(x,t)dx - \int_\Gamma N(x)\frac{\partial u_{m_k}}{\partial \nu} ds$$

it follows that for the limit function $u(x,t)$, equation (15) will be fulfilled. The function $u(x,t)$ belongs to the space $W_2^{2,1}(Q)$.

Let us put

$$q(t) = \frac{1}{h_0(t)}[A_2(t;u) - A_1(t;u) - f_0(t)].$$

The functions $u(x,t)$ and $q(t)$ are connected in the cylinder Q by equation (1). We show that the condition (3) holds for the function $u(x,t)$.

We multiply equation (1) with the function $q(t)$ defined above by the function $N(x)$ and integrate over the domain Ω. Considering the form of the functions $h_0(t), f_0(t), h_1(x,t), A_1(t;u)$ and $A_2(t;u)$, we obtain that for the function $\omega(t)$ which is defined by equality

$$\omega(t) = \int_\Omega N(x)u(x,t)dx,$$

the equation is performed

$$\varphi(t)\omega_t + c_0\omega = 0.$$

Multiplying this equation by the function ω and integrating over the segment $[0,T]$, we get

$$\omega(t) \equiv 0 \quad \text{at} \quad t \in [0,T].$$

Hence, it follows that for the function $u(x,t)$, which is the solution to the boundary value problem (15), (2) the overdetermination condition (3) is satisfied.

All of the above means that the found functions $u(x,t)$ and $q(t)$ give the desired solution to the inverse problem I.

The theorem is proved.

There is a similar result to the above for the inverse problem II.

Theorem 2. Let the condition (25) be satisfied

$$\varphi(t) \in C^1([0;T]), \varphi(0) > 0, \varphi(T) < 0; \tag{25}$$

as well as conditions (10)–(14). Then for any function $f(x,t)$ such that $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$, $f(x,0) = f(x,T) = 0$ for $x \in \Omega$ the inverse problem II has a solution $(u(x,t), q(t))$ such that $u(x,t) \in W_2^{2,1}(Q), q(t) \in L_2([0,T])$.

The proof of this theorem is carried out in general analogous to the proof of Theorem 1, the only difference is that in the boundary value problem for equation (16), conditions are not (2) and (17), they are (2) and (4).

Solvability of the inverse problem III

The study of the solvability of the nonlinear inverse problem III will also be carried out by using the transition to integro-differential (loaded) equations. For simplicity, we will limit ourselves to analyzing the case of $c(x,t) \equiv 0$; the general case will differ from the one considered only by the greater cumbersomeness of conditions and calculations.

Let us put

$$M_1 = d_0 \int_Q \frac{f^2(x,t)}{\varphi(t)} dxdt + \int_\Omega u_0^2(x)dx.$$

Theorem 3. Let the conditions be satisfied

$$\varphi(t) \in C([0;T]), [\varphi(t)]^{-1} \in L_2([0;T]), \varphi(t) \geq 0, \text{ when } t \in [0;T]; \tag{26}$$

$$c(x,t) \equiv 0 \text{ at } (x,t) \in \overline{Q}; \tag{27}$$

$$\begin{gathered} N(x) \in W_\infty^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega),\ \mu(t) \in W_\infty^1([0,T]),\ u_0(x) \in W_2^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega), \\ f(x,t) \in L_\infty(0,T;\overset{0}{W}{}_2^1(\Omega)); \end{gathered} \tag{28}$$

$$\mu(t) \geq \mu_0 > 0,\ f_0(t) - \varphi(t)\mu'(t) \geq \mu_1 > 0 \text{ at } t \in [0,T]; \tag{29}$$

$$M_1^{1/2} ||\Delta N||_{L_2(\Omega)} \leq \mu_1; \tag{30}$$

$$\int_\Omega N(x)u_0(x)dx = \mu(0), \tag{31}$$

Then the inverse problem III has a solution $(u(x,t), q(t))$ such that $u(x,t) \in L_\infty(0,T; W_2^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega))$, $u_t(x,t) \in L_2(Q)$, $q(t) \in L_\infty([0,T])$, $q(t) \geq 0$ at $t \in [0,T]$.

Proof. Let $\{\varepsilon_m\}_{m=1}^\infty$ be a sequence of positive numbers converging to 0. Denote $\varphi_m(t) = \varphi(t) + \varepsilon_m$. Next, we define the cutting function $G_M(\xi), \xi \in R:$

$$G_M(\xi) = \begin{cases} \xi, & \text{if } \|\xi| < M, \\ M, & \text{if } \xi \geq M, \\ -M, & \text{if } \xi \leq -M. \end{cases}$$

Let M_0 be a number from the interval $(0, \mu_1]$. Consider the boundary value problem: Find the function $u(x,t)$, which is the solution to the equation in the cylinder Q

$$\varphi_m(t)u_t - \Delta u + \varepsilon_m \varphi_m(t)\Delta^2 u + \frac{1}{\mu(t)}[f_0(t) - \varphi(t)\mu'(t) + G_{M_0}(A_1(t;u))]u = f(x,t) \tag{32}$$

and such that conditions (2) and (6) are met for it, as well as the condition

$$\Delta u(x,t)|_S = 0. \tag{33}$$

In this problem, equation (32) for a fixed m is a non-degenerate parabolic equation of the fourth order with bounded nonlinearity in the lower term. Using standard energy estimates for parabolic equations [26], the Galerkin method or the fixed-point method, it is easy to establish that the problem (32), (2), (6), (33) has a solution $u_m(x,t)$ belonging to the space $W_2^{4,1}$. We show that using the functions $u_m(x,t)$ it is possible to find a solution to the inverse problem III.

Multiply equation (32) by the function $[\varphi(t)]^{-1}u_m(x,t)$ and integrate over the domain Ω and over the time variable from 0 to the current point. After integrating by parts and reassigning variables, we get equality

$$\begin{gathered}\int_\Omega u_m^2(x,t)dx+\sum_{i=1}^n\int_0^t\int_\Omega\frac{u^2_{mx_i}(x,\tau)}{\varphi_m(\tau)}dxd\tau+\varepsilon_m\int_0^t\int_\Omega[\Delta u_m(x,\tau)]^2dxd\tau+\\ +\int_0^t\int_\Omega\frac{u_m^2(x,\tau)}{\mu(\tau)}[f_0(\tau)-\varphi(\tau)\mu'(\tau)+G_{M_0}(A_1(\tau;u_m))]dxd\tau=\\ =\int_0^t\int_\Omega\frac{f(x,\tau)}{\varphi_m(\tau)}u_m(x,\tau)dxd\tau+\frac{1}{2}\int_\Omega u_0^2(x)dx.\end{gathered}\tag{34}$$

Due to conditions (26) and (29), all the terms of the left side of this equality are non-negative. Applying the Young's inequality and inequality (8) to the first term of the right part (34), taking into account also condition (28), we obtain that for functions $u_m(x,t)$ for $t\in[0,T]$, the evaluation is performed

$$\int_\Omega u_m^2(x,t)dx\le M_1.\tag{35}$$

Analyzing the equalities obtained after multiplying equation (32) by the functions $-[\varphi_m(t)]^{-1}\Delta u_m(x,t)$, $[\varphi_m(t)]^{-1}\Delta^2u_m(x,t)$ with subsequent integration over the domain Ω and over the time variable from 0 to the current point, we obtain by using conditions (26), (28) and (29), and the Helder inequality and the inequality (35) that for the functions $u_m(x,t)$ the evaluation is performed

$$\begin{gathered}\sum_{i=1}^n\int_\Omega u^2_{mx_i}(x,t)dx+\int_\Omega[\Delta u_m(x,t)]^2dx+\varepsilon_m\sum_{i=1}^n\int_0^t\int_\Omega(\Delta u_{mx_i}(x,t))^2dxd\tau+\\ +\varepsilon_m\int_0^t\int_\Omega(\Delta^2u_m(x,t))^2dxd\tau\le M_2\end{gathered}\tag{36}$$

with a constant M_2 defined only by the functions $\varphi(t),\mu(t),N(x),f(x,t)$ and $u_0(x)$, as well as the domain Ω and the number T. To obtain the last necessary estimate, multiply equation (32) by the function $[\varphi_m(t)]^{-1}u_{mt}(x,t)$ and integrate over the cylinder Q. After simple transformations using the conditions of the theorem, the Gelder inequality and estimates (35) and (36), we obtain that for the function $u_m(x,t)$ the inequality holds

$$\int_Q u^2_{mt}(x,t)dxdt\le M_3\tag{37}$$

with a constant M_3 defined only by the functions $\varphi(t),\mu(t),N(x),f(x,t)$ and $u_0(x)$, as well as the domain Ω and the number T. Let's clarify the value of the number M_0: $M_0=\mu_1$. With this choice of the number M_0, it follows from the estimate (34) and condition (30) that $G_{M_0}(A_1(t;u_m))=A_1(t;u_m)$ is satisfied in equations (32). Further, from the estimates (34)–(37) and from the reflexivity property of the Hilbert space, as well as from the embedding theorems, it follows that there exists a subsequence $\{u_{mk}(x,t)\}_{m=1}^\infty$ from the sequence of solutions to the boundary value problem (32), (2), (6), (33), and the function $u(x,t)$ such that for $k\to\infty$ there are convergences:

$$u_{m_k}(x,t)\to u(x,t)\ \text{ weak in }\ W_2^{2,1}(Q)\ \text{and strong in}\ L_2(Q),$$

$$\varepsilon_{m_k}\Delta^2u_{m_k}(x,t)\to 0\ \text{ weak in }\ L_2(Q).$$

Let us put

$$q(t)=\frac{1}{\mu(t)}[f_0(t)-\varphi(t)\mu'(t)+A_1(t;u)],\tag{38}$$

$$\omega(t)=\int_\Omega N(x)u(x,t)dx-\mu(t).\tag{39}$$

For the function $u(x,t)$ and for the function $q(t)$ defined by equality (38) in the cylinder Q, equation

(5) is fulfilled. Further, for the function $u(x,t)$, conditions (2) and (6) are fulfilled. We show that the overdetermination condition (7) is satisfied for the function $u(x,t)$. Multiply equation (5) by the function $N(x)$ and integrate over the domain Ω. Comparing the obtained equality with equality (38), we come to the equation for the function $\omega(t)$:

$$\varphi(t)\omega'(t) + q(t)\omega(t) = 0. \tag{40}$$

Since the function $\omega(t)$ is bounded on the segment $[0,T]$, the function $[\varphi(t)]^{-1}$ belongs to the space $L_2([0,T])$, then (40) can be written as

$$\omega'(t) + \tfrac{q(t)}{\varphi(t)}\omega(t) = 0.$$

Multiplying the last equality by the function $\omega(t)$ and integrating, we come to equality

$$\frac{1}{2}\omega^2(t) + \int_0^t \frac{q(\tau)}{\varphi(\tau)}\omega^2(\tau)d\tau = \frac{1}{2}\omega^2(0). \tag{41}$$

Since the function $q(t)$ is non-negative on the segment $[0,T]$ and $\omega(0) = 0$ (due to condition (31)), then from (41) it follows that $\omega(t)$ is an identically zero function on the segment $[0,T]$.

The equality to zero of the function $\omega(t)$ and the formula (39) mean that the overdetermination condition (7) is satisfied for the found function $u(x,t)$.

So, for the functions $u(x,t)$ and $q(t)$ defined above, equation (5) is fulfilled, boundary conditions (2) and (6) are fulfilled, as well as the overdetermination condition (7). Belonging of the functions $u(x,t)$ and $q(t)$ to the required classes follows from a priori estimates (34)–(37). Consequently, these functions will give the desired solution to the inverse problem III.

The theorem is proved.

Comments and additions

1. Throughout the work, it is assumed that certain inequalities or conditions for functions from the Lebesgue or Sobolev spaces (conditions (13), (14), etc.) are fulfilled in the sense of their truth almost everywhere on the corresponding set, that is, truth everywhere except, perhaps, for some set of zero Lebesgue measure.

2. The approaches to proving the solvability of the corresponding inverse problems in clause 3 and clause 4 are significantly different. First of all, we note that the statement of problem III does not imply, despite the possible reversal of the function $\varphi(t)$ to zero at $t = 0$, the liberation of the set $\{x \in \Omega,\ t = 0\}$ from carrying the initial condition. Further, the conditions of Theorem 3 do not imply differentiability of the function $\varphi(t)$, which is required in Theorems 1 and 2. All this is explained by the fact that the conditions of Theorem 3 allow only weak degeneracy at $t = 0$, with weak degeneracy and the nondifferentiability of the function $\varphi(t)$ at the points of its vanishing, the liberation of the initial manifold of the initial data does not occur.

Note also that the conditions on the right side of $f(x,t)$ in Theorems 1, 2, and 3 differ significantly.

3. The paper studies the solvability of some inverse problems for model parabolic equations. Similar results (with minor changes) can be obtained for more general equations, for example, with the replacement of the Laplace operator by an elliptic operator

$$\sum_{i=1}^{n} \tfrac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j}),$$

or for equations with first derivatives in variables $x_1, ..., x_n$,, etc.

4. Let one of the conditions be satisfied in equation (1)

$$\varphi(0) > 0, \;\; \varphi(T) \geq 0,$$

or

$$\varphi(0) \leq 0, \;\; \varphi(T) < 0.$$

In these cases, the inverse problems of finding the functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by equation (1) will be correct when conditions (2) and (3) are set, as well as the initial or final conditions:

$$u(x,0) = 0, \;\; x \in \Omega,$$

or

$$u(x,T) = 0, \;\; x \in \Omega.$$

The proof of the corresponding existence theorems is carried out analogously to the proof of Theorem 1, the only difference is that in regularized problems either such conditions are given

$$u(x,0) = u_t(x,T) = 0, \;\; x \in \Omega,$$

or else such

$$u_t(x,0) = u(x,T) = 0, \;\; x \in \Omega,$$

and for the function $f(x,t)$, $f(x,0) = 0$ or $f(x,T) = 0$ must be executed for $x \in \Omega$.

5. The condition of turning the function $N(x)$ to zero at $x \in \Gamma$ (see condition (28)) can be abandoned. Let the condition be true for the function $f(x,t)$

$$f(x,t) \in L_\infty(0,T; W_2^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega)).$$

Consider the problem: Find the function $\overline{u}(x,t)$, which is the solution to the equation in the cylinder Q

$$\varphi(t)\overline{u}_t + \tfrac{1}{\mu(t)}[f_0(t) - \varphi(t)\mu'(t) + \textstyle\int_\Omega N(y)\overline{u}(y,t)dy]\overline{u} = \Delta f(x,t)$$

and such that the condition (2) and the condition are fulfilled for it

$$\overline{u}(x,0) = \Delta u_0(x), \;\; x \in \Omega.$$

The existence of regular solutions to this problem (under conditions similar to conditions (26)–(31)) is easy to prove by the method by which Theorem 3 was proved. Finding the function $\overline{u}(x,t)$, it will not be difficult to further find the desired solution $(u(x,t), q(t))$ to the inverse problem III.

6. On the contrary, if in the inverse problems I and II the function $N(x)$ vanishes at $x \in \Gamma$ and belongs to the space $W_2^2(\Omega)$, then using the representation

$$A_1(t;u) = \textstyle\int_\Omega \Delta N(y)u(y,t)dy$$

it is not difficult to obtain a condition other than (14) for the solvability of inverse problems I and II.

7. The conditions of theorems 1 and 2 are satisfied if the measure of the domain Ω is small, the functions $c(x,t)$ and $\varphi(t)$ are small.

We show that in the inverse problem III, the set of initial data for which all the conditions of Theorem 3 are satisfied is not empty.

Let $n = 1$, $\Omega = (0;1)$, $N(x) = x(1-x)$, $u_0(x)$ and $\widetilde{f}(x)$ be functions from the space $W_2^2(\Omega)\cap \overset{0}{W}{}_2^1(\Omega)$ positive in Ω. Next, let γ be a positive number, a number for which the inequality holds

$$2\Big(\int_\Omega u_0^2(x)dx\Big)^{\frac{1}{2}} < \gamma \int_\Omega \widetilde{f}(x)N(x)dx.$$

If now $\mu(t)$ is an arbitrary decreasing on the segment $[0,T]$ continuously differentiable function such that

$$\mu(0) = \int_\Omega N(x)u_0(x)dx,$$

$f(x,t)$ and $\varphi(t)$ are functions of $\gamma\widetilde{f}(x)$ and t^α, $0 < \alpha < \frac{1}{2}$, then all the conditions of the Theorem 3 will be executed for sufficiently small numbers T.

Other examples can be given for the inverse problems I, II, and III.

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References

1 Cannon I.R. Determination of a Parameter $p(t)$ in some Quasi-Linear Parabolic Equations / I.R. Cannon // Inverse problems. — 1988. — 4. — 35–45.

2 Камынин В.Л. Об одной обратной задаче для параболического уравнения высокого порядка / В.Л. Камынин, Э. Франчини // Математические заметки. — 1998. — 64. — 5. — С. 680–691.

3 Камынин В.Л. Нелинейная обратная задача для параболического уравнения высокого порядка / В.Л. Камынин, М. Саролди // Журн. вычислит. мат. и мат. физ. — 1998. — 38. — № 10. — С. 1683–1691.

4 Prilepko A.I. Methods for Solving Inverse Problems in Mathematical Physics / A.I. Prilepko, D.G. Oilovsky, I.A. Vasin. — Marcel Dekker: New York, USA, 1999.

5 Anikonov Yu.E. Inverse Problems for Kinetic and Other Evolution Equation / Yu.E. Anikonov. — VSP: Utrecht, The Netherlands, 2001.

6 Belov Yu.Ya. Inverse Problems for Partial Differential Equations / Yu.Ya. Belov. — VSP: Vtrecht, The Netherlands, 2002.

7 Ivanchov M. Inverse Problems for Equations of Parabolic Type / M. Ivanchov. — WNTI Publishers: Lviv, Ukraine, 2003.

8 Пятков С.Г. О некоторых классах линейных обратных задач для параболических систем уравнений / С.Г. Пятков, Е.И. Сафонов // Сиб. электрон. мат. изв. — 2014. — 11. — С. 777–799.

9 Slodicka M. Determination of a Solely Time-Dependent Source in a Semilinear Parabolic Problem by Means of a Boundary Measurements / M. Slodicka // J. Comput. Appl. — 2015. — 289. — P. 433–440.

10 Кожанов А.И. Параболические уравнения с неизвестными коэффициентами, зависящими от времени / А.И. Кожанов // Журн. вычислит. мат. и мат. физ. — 2017. — 57.— № 6. — С. 961–972.

11 Barbu V. An Identification Problem for a linear Evolution Equation in a Banach Space / V. Barbu, G. Marinoshi // Discret. Contin. Dyn. — 2020. — 13. — P. 337–352.

12 Pyatkov S.G. On some Inverse Problems with Pointwise Overdetermination / S.G. Pyatkov, V.A. Baranchuk // Journal of Siberian Federal University. Mathematics and Physics. — 2021. — 14. — No. 4. — P. 463–474.

13 Ivanchov M. An inverse problem for strongly degenerate heat equation / M. Ivanchov, N. Saldina // J. of Inverse and Ill-Posed Problems. — 2006. — 14. — No. 5. — P. 465–98.

14 Huzyk N. Inverse problem of determing the coefficients in a degenerate parabolic equation / N. Huzyk // Electronic Journal of Differential equations. — 2014. — No. 172. — P. 1–11.

15 Kawamoto A. Inverse problem for degenerate parabolic equations by 'time-like' Carleman estimate / A. Kawamoto // J. of Inverse and Ill-Posed Problems. — 2015. — 23. — No. 1. — P. 1–21.

16 Fichera G. On a united theory of boundary value problems for elliptic-parabolic equations of second order / G. Fichera // Boundary problems in differential equations. — WI: Univ. Press, Madison, 1960. — P. 97–120.

17 Олейник О.А. Уравнения второго порядка с неотрицательной харакеристической формой / О.А. Олейник, Е.В. Радкевич // Итоги науки и техники. Сер. мат. — 1971.

18 Егоров И.Е. Неклассические уравнения математической физики высокого порядка / И.Е. Егоров, В.Е. Федоров. — Новосибирск: Вычислительный центр СО РАН, 1995.

19 Кожанов А.И. Вырождающиеся параболические уравнения с переменным направлением эволюции / А.И. Кожанов, Е.Е. Мациевская // Сиби. электрон. мат. изв. — 2019. — С. 718–731.

20 Соболев С.Л. Некоторые применения функционального анализа в математической физике / С.Л. Соболев. — М.: Наука, 1988.

21 Ладыженская О.А. Линейные и квазилинейные уравнения эллиптического типа / О.А. Ладыженская, Н.Н. Уральцева. — М.: Наука, 1973.

22 Triebel H. Interpolation theory. Functional spaces Differential operators / H. Triebel. — Berlin: Veb Deutsher. Verl. Wiss, 1978.

23 Нахушев А.М. Нагруженные уравнения и их применения / А.М. Нахушев. — М.: Наука, 2012.

24 Дженалиев М.Т. К теории линейных краевых задач для нагруженных дифференциальных уравнений / М.Т. Дженалиев. — Алматы: Ин-т теор. и приклад. мат., 1995.

25 Треногин В.А. Функциональный анализ / В.А. Треногин. — М.: Наука, 1980.

26 Ладыженская О.А. Линейные и квазилинейные уравнения параболического типа / О.А. Ладыженская, В.А. Солонников, Н.Н. Уральцева. — М.: Наука, 1967.

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Өзгешеленген псевдопараболалық теңдеуге қойылған коэффициентті кері есептер

Мақала өзгешеленген екінші ретті параболалық теңдеулер үшін коэффициентті кері есептердің шешілімділігін зерттеуге арналған. Сызықтық кері есептер ретінде теңдеудің белгісіз оң жағын (сыртқы әсер) анықтау есептері және белгісіз бір коэффициентті анықтаудың сызықтық емес есебі қарастырылған. Зерттелетін жұмыстардың ерекшелігі – олардағы белгісіз коэффициенттер тек уақыт айнымалысының функциялары болып табылады. Жұмыстың мақсаты – зерттелетін есептердің тұрақты шешімдерінің (теңдеуге қатысатын функциялардың С.Л. Соболев мағынасында барлық жалпылама туындылары бар) бар және жалғыздығын дәлелдеу.

Кілт сөздер: өзгешеленген параболалық теңдеулер, сызықтық кері есептер, сызықты емес кері есептер, регуляр шешімдер, шешімнің бар болуы.

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Обратные задачи определения коэффициентов временного типа в вырождающемся параболическом уравнении

Статья посвящена исследованию разрешимости обратных коэффициентных задач для вырождающихся параболических уравнений второго порядка. Изучены как линейные обратные задачи — задачи определения неизвестной правой части (внешнего воздействия), так и нелинейные задачи определения некоторого коэффициента самого уравнения. Особенностью изучаемых работ является то, что неизвестные коэффициенты в них являются функциями лишь от временной переменной. Цель работы — доказательство существования и единственности регулярных решений изучаемых задач (решений, имеющих все обобщенные, по С.Л. Соболеву, производные, входящие в уравнение).

Ключевые слова: вырождающиеся параболические уравнения, линейные обратные задачи, нелинейные обратные задачи, регулярные решения, существование.

References

1 Cannon, I.R. (1988). Determination of a Parameter $p(t)$ in some Quasi-Linear Parabolic Equations. *Inverse problems, 4*, 35–45.

2 Kamynin, V.L., & Franchini, E. (1998). Ob odnoi obratnoi zadache dlia parabolicheskogo uravneniia vysokogo poriadka [On an inverse problem for a higher order parabolic equation]. *Matematicheskie zametki — Math notes, 64*, 5, 680–691 [in Russian].

3 Kamynin, V.L., & Saroldi, M. (1998). Nelineinaia obratnaia zadacha dlia parabolicheskogo uravneniia vysokogo poriadka [Nonlinear inverse problem for a higher order parabolic equation]. *Zhurnal vychislitelnoi matematiki i matematicheskoi fiziki — Journal of Computational Mathematics and Mathematical Physics, 38*, 10, 1683–1691 [in Russian].

4 Prilepko, A.I., Oilovsky, D.G., & Vasin, I.A. (1999). *Methods for Solving Inverse Problems in Mathematical Physics.* Marcel Dekker: New York, USA.

5 Anikonov, Yu.E. (2001). *Inverse Problems for Kinetic and Other Evolution Equation.* VSP: Utrecht, The Netherlands.

6 Belov, Yu.Ya. (2002). *Inverse Problems for Partial Differential Equations.* VSP: Vtrecht, The Netherlands.

7 Ivanchov, M. (2003). *Inverse Problems for Equations of Parabolic Type.* WNTI Publishers: Lviv, Ukraine.

8 Piatkov, S.G., & Safonov, E.I. (2014). O nekotorykh klassakh lineinykh obratnykh zadach dlia parabolicheskikh sistem uravnenii [On some classes of linear inverse problems for parabolic systems of equations]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Reports, 11*, 777–799 [in Russian].

9 Slodicka, M. (2015). Determination of a Solely Time-Dependent Source in a Semilinear Parabolic Problem by Means of a Boundary Measurements. *Journal of Computational and Applied Mathematics, 289*, 433–440.

10 Kozhanov, A.I. (2017). Parabolicheskie uravneniia s neizvestnymi koeffitsientami, zavisiashchimi ot vremeni [Parabolic equations with unknown coefficients depending on time]. *Zhurnal vychislitelnoi matematiki i matematicheskoi fiziki — Journal of Computational Mathematics and Mathematical Physics, 57*(6), 961–972 [in Russian].

11 Barbu, V., & Marinoshi, G. (2020). An Identification Problem for a linear Evoluation Equation in a Banach Space. *Discret. Contin. Dyn., 13*, 337–352.

12 Pyatkov, S.G., & Baranchuk, V.A. (2021). On some Inverse Problems with Pointwise Overdetermination. *Journal of Siberian Federal University. Mathematics and Physics, 14*(4), 463–474.

13 Ivanchov, M., & Saldina, N. (2006). An inverse problem for strongly degenerate heat equation. *J. of Inverse and Ill-Posed Problems, 14*, 5, 465–98.

14 Huzyk, N. (2014). Inverse problem of determing the coefficients in a degenerate parabolic equation. *Electronic Journal of Differential equations, 2014*, 172, 1–11.

15 Kawamoto, A. (2015). Inverse problem for degenerate parabolic equations by 'time-like' Carleman estimate. *J. of Inverse and Ill-Posed Problems, 23*, 1, 1–21.

16 Fichera, G. (1960). On a united theory of boundary value problems for elliptic-parabolic equations of second order. *Boundary problems in differential equations. WI: Univ. Press. Madison*, 97–120.

17 Oleinik, O.A., & Radkevich, E.V. (1971). Uravneniia vtorogo poriadka s neotritsatelnoi kharakteristicheskoi formoi [Equations of the second order with a non-negative characteristic form]. *Itogi nauki i tekhniki. Seriia matematicheskaia — Results of science and technology. Math. Series.* Moscow: VINITI [in Russian].

18 Egorov, I.E., & Fedorov, V.E. (1995). *Neklassicheskie uravneniia matematicheskoi fiziki vysokogo poriadka [Non-classical equations of mathematical physics of high order].* Novosibirsk: Vychislitelnyi tsentr SO RAN [in Russian].

19 Kozhanov, A.I., & Matsievskaia, E.E. (2019). Vyrozhdaiushchiesia parabolicheskie uravneniia s peremennym napravleniem evoliutsii [Degenerate parabolic equations with a variable direction of evolution]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Reports*, 718–731 [in Russian].

20 Sobolev, S.L. (1988). *Nekotorye primeneniia funktsionalnogo analiza v matematicheskoi fizike [Some applications of functional analysis in mathematical physics].* Moscow: Nauka [in Russian].

21 Ladyzhenskaia, O.A. (1973). *Lineinye i kvazilineinye uravneniia ellipticheskogo tipa [Linear and quasilinear equations of elliptic type].* Moscow: Nauka [in Russian].

22 Triebel, H. (1978). *Interpolation theory. Functional spaces Differential operators.* Berlin: Veb Deutsher. Verl. Wiss.

23 Nakhushev, A.M. (2012). *Nagruzhennye uravneniia i ikh primeneniia [Loaded equations and their applications].* Moscow: Nauka [in Russian].

24 Dzhenaliev, M.T. (1995). *K teorii lineinykh kraevykh zadach dlia nagruzhennykh differentsialnykh uravnenii [On the theory of linear boundary value problems for loaded differential equations].* Almaty: Institut teoreticheskoi i prikladoi matematiki [in Russian].

25 Trenogin, V. (1980). *Funktsionalnyi analiz [Functional analysis].* Moscow: Nauka [in Russian].

26 Ladyzhenskaia, O.A., Solonnikov, V.A., & Uraltseva, N.N. (1967). *Lineinye i kvazilineinye uravneniia parabolicheskogo tipa [Linear and quasilinear equations of parabolic type].* Moscow: Nauka [in Russian].

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Boundary value problem for a system of partial differential equations with the Dzhrbashyan–Nersesyan fractional differentiation operators

A boundary value problem in a rectangular domain for a system of partial differential equations with the Dzhrbashyan–Nersesyan fractional differentiation operators with constant coefficients is studied in the case when the matrix coefficients of the system have complex eigenvalues. Existence and uniqueness theorems for the solution to the boundary value problem under study are proved. The solution is constructed explicitly in terms of the Wright function of the matrix argument.

Keywords: system of partial diffrential equations, fractional derivatives, Dzhrbashyan–Nersesyan operator, boundary value problem, fundamental solution, Wright matrix function.

Introduction

Consider the system of differential equations

$$Lu(x,y) \equiv D_{0x}^{\{\alpha_0,\alpha_1,...,\alpha_k\}}u(x,y) + AD_{0y}^{\{\beta_0,\beta_1,...,\beta_m\}}u(x,y) = Bu(x,y) + f(x,y), \quad (1)$$

in the domain $\Omega = \{(x,y):\ 0 < x < a,\ 0 < y < b\}$, $a, b < \infty$, where $D_{0x}^{\{\alpha_0,\alpha_1,...,\alpha_k\}}$ and $D_{0y}^{\{\beta_0,\beta_1,...,\beta_m\}}$ are the Dzhrbashyan–Nersesyan fractional differentiation operators [1] of orders $\alpha = \sum\limits_{i=0}^{k}\alpha_i - 1 > 0$ and $\beta = \sum\limits_{i=0}^{m}\beta_i - 1 > 0$, respectivly, $\alpha_i, \beta_j \in (0,1]$, $(i = \overline{0,k}, j = \overline{0,m})$; $f(x,y) = ||f_1(x,y),...,f_n(x,y)||$ and $u(x,y) = ||u_1(x,y),...,u_n(x,y)||$ are given and desired n-dimensional vectors, respectively, A and B are given constant real square matrices of order n.

The Dzhrbashyan–Nersesyan fractional differentiation operator $D_{0t}^{\{\gamma_0,\gamma_1,...,\gamma_k\}}$ of the order $\gamma = \sum\limits_{i=0}^{k}\gamma_i - 1 > 0$, $\gamma_i \in (0,1]$, $(i = \overline{0,k})$, associated with the sequence $\{\gamma_0,\gamma_1,...,\gamma_k\}$, is determined by the relation [1]

$$D_{0t}^{\{\gamma_0,\gamma_1,...,\gamma_k\}}v(t) = D_{0t}^{\gamma_k - 1}D_{0t}^{\gamma_{k-1}}...D_{0t}^{\gamma_1}D_{0t}^{\gamma_0}v(t),$$

where D_{0t}^{γ} is the Riemann–Liouville fractional integro-differentiation operator [2; 9].

The operator $D_{0t}^{\{\gamma_0,\gamma_1,...,\gamma_k\}}$ was introduced in [1], where the form of the initial conditions for the ordinary differential equations with such an operator, and the Cauchy problem was studied. The Dzhrbashyan-Nersesyan operator generalizes a number of definitions of fractional derivatives, including the Riemann–Liouville and Gerasimov–Caputo derivatives.

A review of works related to the study of the equation (1) with Riemann–Liouville and Gerasimov–Caputo derivatives, including in the scalar case $n = 1$, can be found in [3] and [4].

Equations of the order not higher than one containing operators of the form $D_{0t}^{\{\gamma_0,\gamma_1,...,\gamma_k\}}$ are studied in [5–10]. In [5], for a linear partial differential equation of fractional order with many independent variables a fundamental solution is constructed and a boundary problem is solved.

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In papers [6], [7], boundary value problems in a rectangular domain for first-order partial differential equations with variable coefficients are studied. In papers [8] and [9], a boundary value problem with an integral condition and a boundary value problem in a rectangle, respectively, are studied for equations with constant coefficients. The study [10] considers an equation containing the Dzhrbashyan–Nersesyan operators in two independent variables, and in one of the variables the equation includes a linear combination of two Dzhrbashyan–Nersesyan operators, the orders of which are associated with the sequences $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$. The question of the influence of the distribution of the values of these parameters on the setting of the initial conditions is studied.

We also note the papers [11, 12] where the unique solvability of initial problems for some classes of linear equations with operator coefficients in the Banach spaces are studied.

In papers [3], [4], [13–15], boundary value problems in rectangular domains and the Cauchy problem for systems with sign-definite eigenvalues of matrix coefficients in the main part, with Riemann–Liouville partial derivatives whose the order does not exceed one, are considered. For these systems, the situation with the formulation of boundary value problems is similar to the case of a single equation. In this paper, we extend the class of such systems to include systems with eigenvalues of the coefficients of the main part lying in some corner of the complex plane, and with more general Dzhrbashyan–Nersesyan operators of fractional differentiation.

1 Auxiliary assertions

The Riemann-Liouville fractional integro-differentiation operator D^{ν}_{ay} of order ν is defined as follows [2; 9]:

$$D^{\nu}_{ay}g(y) = \frac{\operatorname{sign}(y-a)}{\Gamma(-\nu)} \int\limits_a^y \frac{g(s)ds}{|y-s|^{\nu+1}}, \quad \nu < 0,$$

for $\nu \geq 0$ the operator D^{ν}_{ay} can be determined by recursive relation

$$D^{\nu}_{ay}g(y) = \operatorname{sign}(y-a)\frac{d}{dy}D^{\nu-1}_{ay}g(y), \quad \nu \geq 0.$$

The following series

$$\phi(\rho, \mu; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\rho k + \mu)}, \quad \rho > -1, \quad \mu \in \mathbb{C}$$

defines the Wright function [16], [17] which depends on two parameters ρ and μ.

The following relation holds [4]

$$\phi(\rho, \mu; z)\big|_{z=0} = \frac{1}{\Gamma(\mu)}. \tag{2}$$

If $\beta \in (0,1)$, $\mu \in \mathbb{R}$, then the following estimate is valid [18]

$$|\phi(-\beta, \mu; -z)| \leq C \exp\left(-\sigma |z|^{\frac{1}{1-\beta}}\right), \tag{3}$$

where $C = C(\beta, \mu, \sigma)$ and

$$\sigma < (1-\beta)\beta^{\frac{\beta}{1-\beta}} \cos\frac{\arg z}{1-\beta}, \quad 0 \leq |\arg z| < \frac{1-\beta}{2}\pi.$$

Under the condition

$$\beta \in (0,1), \quad 0 \leq |\arg \lambda| < \frac{1-\beta}{2}\pi, \tag{4}$$

the inequality

$$|y^{\mu-1}\phi(-\beta,\mu;-\lambda xy^{-\beta})| \leq Cx^{-\theta}y^{\mu+\beta\theta-1}, \quad x>0, \quad y>0, \tag{5}$$

holds [18], where $C=C(\mu,\beta,\theta,\lambda)$, $\theta \geq 0$ for $\mu \neq 0,-1,-2,...$, and $\theta \geq -1$ for $\mu=0,-1,-2,....$

The following differentiation formula is valid [17]:

$$\frac{d}{dz}\phi(\rho,\mu;z)=\phi(\rho,\mu+\rho;z), \quad \rho>-1. \tag{6}$$

Let $\beta \in (0,1)$, $\mu,\nu \in \mathbb{R}$, and the inequality (4) holds, then the formula [18]

$$D_{0y}^{\nu}y^{\mu-1}\phi(-\beta,\mu;-\lambda y^{-\beta})=y^{\mu-\nu-1}\phi(-\beta,\mu-\nu;-\lambda y^{-\beta}) \tag{7}$$

is true. By (7), we have

$$D_{0y}^{\{\gamma_0,...,\gamma_j\}}y^{\mu-1}\phi(-\beta,\mu;-\lambda y^{-\beta})=y^{\mu-\mu_j}\phi(-\beta,\mu-\mu_j+1;-\lambda y^{-\beta}), \tag{8}$$

where $\mu_j=\sum\limits_{i=0}^{j}\gamma_i$.

Formulas (6)–(8) give the equality

$$\left(\frac{\partial}{\partial x}+\lambda D_{0y}^{\{\gamma_0,...,\gamma_j\}}\right)y^{\mu-1}\phi(-\beta,\mu;-\lambda xy^{-\beta})=0, \quad \beta=\sum_{i=0}^{j}\gamma_i-1<1. \tag{9}$$

Using the integration by parts formula and the relations (2) and (3), one can show that the equality

$$\int\limits_0^\infty t^n\phi(-\beta,\mu;-\lambda t)dt=\frac{n!}{\lambda^{n+1}\Gamma(\mu+(n+1)\beta)}, \quad n=0,1,.... \tag{10}$$

holds under the condition (4).

For $\lambda=1$, the equality (10) was obtained in [19].

2 Special solutions

2.1 Wright matrix function

In papers [3], [4] the Wright matrix function was defined

$$\phi(\rho,\mu;A)=\sum_{k=0}^{\infty}\frac{A^k}{k!\Gamma(\rho k+\mu)}, \quad \rho>-1, \quad \mu \in \mathbb{C}$$

and its following properties were established.

1. Let the matrix A be reduced with the help of the matrix H to the Jordan normal form $J(\lambda)$, i.e.

$$A=HJ(\lambda)H^{-1},$$

where $J(\lambda)=\mathrm{diag}[J_1(\lambda_1),...,J_p(\lambda_p)]$ is the quasidiagonal matrix with cells of the form

$$J_k \equiv J_k(\lambda_k)=\begin{Vmatrix} \lambda_k & 1 & \ldots & 0 \\ & \lambda_k & \ldots & 0 \\ & 0 & \ddots & \vdots \\ & & & \lambda_k \end{Vmatrix}, \quad k=1,...,p,$$

$\lambda_1, ..., \lambda_p$ are the eigenvalues of the matrix A, $J_k(\lambda_k)$ are the square matrices of order $r_k + 1$, $\sum\limits_{k=1}^{p} r_k + p = n$. Then the function $\phi(\rho, \mu; Az)$ can be represented as

$$\phi(\rho, \mu; Az) = H\phi(\rho, \mu; J(\lambda)z)H^{-1}, \tag{11}$$

where

$$\phi(\rho, \mu; J(\lambda)z) = \mathrm{diag}[\phi(\rho, \mu; J_1(\lambda_1)z), ..., \phi(\rho, \mu; J_p(\lambda_p)z)],$$

$$\phi(\rho, \mu; J_k(\lambda_k)z) = \begin{Vmatrix} \phi^0_{\rho,\mu}(\lambda_k z) & \phi^1_{\rho,\mu}(\lambda_k z) & \dots & \phi^{r_k}_{\rho,\mu}(\lambda_k z) \\ & \phi^0_{\rho,\mu}(\lambda_k z) & \dots & \phi^{r_k-1}_{\rho,\mu}(\lambda_i z) \\ & 0 & \ddots & \vdots \\ & & & \phi^0_{\rho,\mu}(\lambda_k z) \end{Vmatrix},$$

$$\phi^m_{\rho,\mu}(\lambda z) = \frac{1}{m!}\frac{\partial^m}{\partial\lambda^m}\phi(\rho, \mu; \lambda z) = \frac{z^m}{m!}\phi(\rho, \mu + \rho m; \lambda z).$$

2. Using the representation (11) and equality (2), we obtain

$$\phi(\rho, \mu; Az)\big|_{z=0} = \frac{1}{\Gamma(\mu)}I, \tag{12}$$

where I is the identity matrix of order n.

3. The following differentiation formula is valid

$$\frac{d}{dz}\phi(\rho, \mu; Az) = A\phi(\rho, \rho + \mu; Az). \tag{13}$$

Further, we assume that all eigenvalues $\lambda_1, ..., \lambda_p$ of the matrix A satisfy the condition

$$0 \le |\arg\lambda_i| < \frac{1-\beta}{2}\pi, \quad i = \overline{1,p}. \tag{14}$$

Due to the relations (2), (3), (5), (6), (7), (10), the following properties proved in [3], [4] remain valid under the condition (14).

4. Due to (7) and (11), for $\beta \in (0, 1)$, $\mu, \nu \in \mathbb{R}$, we have

$$D^{\nu}_{0y} y^{\mu-1}\phi(-\beta, \mu; -A\tau y^{-\beta}) = y^{\mu-\nu-1}\phi(-\beta, \mu - \nu; -A\tau y^{-\beta}). \tag{15}$$

From (15) it follows

$$D^{\{\gamma_0,...,\gamma_j\}}_{0y} y^{\mu-1}\phi(-\beta, \mu; -A\tau y^{-\beta}) = y^{\mu-\mu_j}\phi(-\beta, \mu - \mu_j + 1; -A\tau y^{-\beta}), \quad \mu_j = \sum_{i=0}^{j}\gamma_i. \tag{16}$$

5. The equalities (13), (15) and (16) imply the equality

$$\left(\frac{\partial}{\partial\tau} + AD^{\{\gamma_0,...,\gamma_j\}}_{0y}\right) y^{\mu-1}\phi(-\beta, \mu; -A\tau y^{-\beta}) = 0, \quad \beta = \sum_{i=0}^{j}\gamma_i - 1 < 1. \tag{17}$$

6. By virtue of (10) and (11) it follows

$$\int\limits_0^\infty \phi(-\beta, \mu; -Az)dz = \frac{1}{\Gamma(\mu+\beta)}A^{-1}. \tag{18}$$

7. Let $A(x,y)$ be the matrix with entries $a_{ij}(x,y)$. By $|A(x,y)|_*$ we denote a scalar function taking the maximum absolute value of entries of the matrix $A(x,y)$ for each (x,y), i.e., $|A(x,y)|_* = \max_{i,j}|a_{ij}(x,y)|$. Likewise, for a vector $b(x,y)$ with components $b_i(x,y)$, we set $|b(x,y)|_* = \max_i |b_i(x,y)|$.

From the estimate (5) it follows that

$$|y^{\nu-1}\phi(-\beta,\nu;-A\tau y^{-\beta})|_* \leq C\tau^{-\theta}y^{\nu+\beta\theta-1}, \quad \tau>0, \quad y>0, \tag{19}$$

where $\beta\in(0,1)$ and $\theta\geq 0$ for $\nu\neq 0,-1,-2,...$; and $\theta\geq -1$ for $\nu=0,-1,-2,...$.

8. Formulas (3) and (11) yields the estimate

$$|\phi(-\delta,\varepsilon;-Az)|_* \leq C\exp\left(-\sigma|z|^{\frac{1}{1-\delta}}\right), \quad z\geq 0, \tag{20}$$

where $\delta\in(0,1)$, $\varepsilon\in\mathbb{R}$, $\sigma<(1-\delta)\delta^{\frac{1}{1-\delta}}\lambda_0^{\frac{\delta}{1-\delta}}$, $\lambda_0=\min\limits_{1\leq i\leq p}\{|\lambda_i|\}$, $\lambda_1,...,\lambda_p$ are the eigenvalues of the matrix A.

2.2 Properties of the function $\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)$

In [3], the following function is defined

$$\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)\equiv\int\limits_0^\infty e^{B\tau}x^{\mu-1}\phi(-\alpha,\mu;-\tau x^{-\alpha})y^{\nu-1}\phi(-\beta,\nu;-A\tau y^{-\beta})d\tau. \tag{21}$$

The estimates (3) and (5) imply the convergence of the integral (21) for any $\mu,\nu\in\mathbb{R}$, and $x^2+y^2\neq 0$.

The following assertions are true.

Lemma 2.1. For all $\mu,\nu\in\mathbb{R}$ the following equalities hold:

$$D^{\varepsilon}_{0x}\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)=\Phi^{\mu-\varepsilon,\nu}_{\alpha,\beta}(x,y), \quad \alpha+\mu>0, \tag{22}$$

$$D^{\delta}_{0y}\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)=\Phi^{\mu,\nu-\delta}_{\alpha,\beta}(x,y), \quad \beta+\nu>0. \tag{23}$$

Lemma 2.1 follows from the formulas (7), (15), (21).

Lemma 2.1 implies the equalities

$$D^{\{\varepsilon_0,...,\varepsilon_j\}}_{0x}\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)=\Phi^{\mu-\mu_j+1,\nu}_{\alpha,\beta}(x,y), \quad \alpha+\mu_i>0, \quad \mu_j=\sum_{i=0}^{j}\varepsilon_i, \tag{24}$$

$$D^{\{\delta_0,...,\delta_j\}}_{0y}\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)=\Phi^{\mu,\nu-\nu_j+1}_{\alpha,\beta}(x,y), \quad \beta+\nu_j>0, \quad \nu_j=\sum_{i=0}^{j}\delta_i. \tag{25}$$

Lemma 2.2. The estimate

$$\left|\Phi^{\mu,\nu}_{\alpha,\beta}(x,y)\right|_* \leq Cx^{\alpha+\mu-\alpha\theta-1}y^{\nu+\beta\theta-1}, \quad \theta\in[\theta_1,\theta_2) \tag{26}$$

holds for all $x\in[0;x_0]$, where $\theta_1=\begin{cases} 0, & -\nu\notin\mathbb{N}_0,\\ -1, & -\nu\in\mathbb{N}_0,\end{cases}$ $\theta_2=\begin{cases} 1, & \mu\neq 0,\\ 2, & \mu=0,\end{cases}$ $\mathbb{N}_0=\{0,1,2,...\}$, and the constant C depends on x_0.

The validity of Lemma 2.2, under the condition (14), follows from the formulas (3) and (19), similarly to the case when all eigenvalues of the matrix A are positive [3].

Lemma 2.3. Let $AB = BA$, $\sum\limits_{i=0}^{k_1} \varepsilon_i = \alpha + 1$, $\sum\limits_{i=0}^{m_1} \delta_i = \beta + 1$, then the following equality holds:

$$\left(D_{0x}^{\{\varepsilon_0,\dots,\varepsilon_{k_1}\}} + A D_{0y}^{\{\delta_0,\dots,\delta_{m_1}\}} - B \right) \Phi_{\alpha,\beta}^{\mu,\nu}(x,y) = \frac{x^{\mu-1} y^{\nu-1}}{\Gamma(\mu)\Gamma(\nu)} I. \tag{27}$$

Proof. Let us denote

$$h_\alpha^\mu(x,\tau) = x^{\mu-1}\phi(-\alpha,\mu;-\tau x^{-\alpha}), \quad h_\beta^\nu(y,\tau) = y^{\nu-1}\phi(-\beta,\nu;-A\tau y^{-\beta}).$$

Using the fact that due to (9)

$$\left(D_{0x}^{\{\varepsilon_0,\dots,\varepsilon_{k_1}\}} + \frac{\partial}{\partial \tau} - B \right) e^{B\tau} h_\alpha^\mu(x,\tau) = 0,$$

the integration by parts formula and relations (2), (3), (12) and (20), we obtain

$$D_{0x}^{\{\varepsilon_0,\dots,\varepsilon_{k_1}\}} \Phi_{\alpha,\beta}^{\mu,\nu}(x,y) = \int\limits_0^\infty e^{B\tau} D_{0x}^{\{\varepsilon_0,\dots,\varepsilon_{k_1}\}} h_\alpha^\mu(x,\tau) h_\beta^\nu(y,\tau) d\tau =$$

$$= \frac{x^{\mu-1} y^{\nu-1}}{\Gamma(\mu)\Gamma(\nu)} I + B\Phi_{\alpha,\beta}^{\mu,\nu}(x,y) + \int\limits_0^\infty e^{B\tau} h_\alpha^\mu(x,\tau) \frac{\partial}{\partial \tau} h_\beta^\nu(y,\tau) d\tau. \tag{28}$$

By virtue of (25), we get

$$D_{0y}^{\{\delta_0,\dots,\delta_{m_1}\}} \Phi_{\alpha,\beta}^{\mu,\nu}(x,y) = \int\limits_0^\infty e^{B\tau} h_\alpha^\mu(x,\tau) D_{0y}^{\{\delta_0,\dots,\delta_{m_1}\}} h_\beta^\nu(y,\tau) d\tau. \tag{29}$$

By (28) and (29), taking into account the equality

$$\left(A D_{0y}^{\{\delta_0,\dots,\delta_{m_1}\}} + \frac{\partial}{\partial \tau} - B \right) e^{B\tau} h_\beta^\nu(y,\tau) = 0,$$

which follows from (17), we get (27). Lemma 2.3 is proven.

3 Problem statement and main theorem

Let all eigenvalues $\lambda_1, \dots, \lambda_p$ of the matrix A satisfy the condition (14). We formulate a boundary value problem for the system (1).

Problem 3.1. Find a solution $u(x,y)$ of system (1) with the boundary conditions

$$\lim_{x\to 0} D_{0x}^{\{\alpha_0,\alpha_2,\dots,\alpha_i\}} u = \varphi_i(y), \quad 0 \le i \le k-1, \quad 0 < y < b, \tag{30}$$

$$\lim_{y\to 0} D_{0y}^{\{\beta_0,\beta_1,\dots,\beta_j\}} u = \psi_j(x), \quad 0 \le j \le m-1, \quad 0 < x < a, \tag{31}$$

where $\varphi_i(y)$ and $\psi_j(x)$ are given n-vectors functions.

A regular solution of system (1) in the domain Ω is defined as a vector function $u(x,y)$ satisfying at all points $(x,y)\in\Omega$ the system (1) and the inclusions

$$D_{0x}^{\{\alpha_0,\dots,\alpha_k\}}u, D_{0y}^{\{\beta_0,\dots,\beta_m\}}u\in C(\Omega); \tag{32}$$

$$\begin{array}{c} D_{0x}^{\{\alpha_0,\dots,\alpha_i\}}u\in C(\Omega\cup\{x=0\}),\quad D_{0y}^{\{\beta_0,\dots,\beta_j\}}u\in C(\Omega\cup\{y=0\}),\\ \frac{\partial}{\partial x}D_{0x}^{\{\alpha_0,\dots,\alpha_i\}}u, \frac{\partial}{\partial y}D_{0y}^{\{\beta_0,\dots,\beta_j\}}u\in C(\Omega)\cap L(\Omega),\quad (i=\overline{0,k-1}, j=\overline{0,m-1}); \end{array} \tag{33}$$

$x^{1-\varepsilon}y^{1-\delta}u(x,y)\in C(\overline{\Omega})$, for some $\varepsilon>0$ and $\delta>0$.

We accept the following notation:

$$\mu_j=\sum_{p=0}^{j}\alpha_p,\quad \bar{\mu}_j=\sum_{p=j}^{k}\alpha_p,\quad \mu_j^i=\sum_{p=j}^{i}\alpha_p;\quad \nu_i=\sum_{p=0}^{i}\beta_p,\quad \bar{\nu}_i=\sum_{p=i}^{m}\beta_p,\quad \nu_i^j=\sum_{p=i}^{j}\beta_p.$$

Theorem 3.1. Let $AB=BA$, all eigenvalues $\lambda_1,...,\lambda_p$ of the matrix A satisfy the condition (14), $\alpha_0+\alpha_k>1$, $\beta_0+\beta_m>1$,

$$\begin{array}{c} \varphi_j(y)=D_{0y}^{-\rho_j}\varphi_j^*(y),\quad y^{1-\eta_j}\varphi_j^*(y),\, y^{1-\nu}\varphi_{k-1}(y)\in C[0,b],\\ \eta_j>0,\quad \rho_j>\max\left\{\mu_{j+1}^{k-1}\frac{\beta}{\alpha},1-\beta_m\right\},\quad j=\overline{0,k-2}; \end{array} \tag{34}$$

$$\begin{array}{c} \psi_i(y)=D_{0x}^{-\sigma_i}\psi_i^*(x),\quad x^{1-\xi_i}\psi_i^*(x),\, x^{1-\mu}\psi_{m-1}(x)\in C[0,a],\\ \xi_i>0,\quad \sigma_i>\max\left\{\nu_{i+1}^{m-1}\frac{\alpha}{\beta},1-\alpha_k\right\},\quad i=\overline{0,m-2}; \end{array} \tag{35}$$

$$\begin{array}{c} f(x,y)=D_{0x}^{-\sigma}D_{0y}^{-\rho}f^*(x,y),\quad x^{1-\xi}y^{1-\eta}f^*(x,y)\in C(\overline{\Omega}),\\ \sigma>1-\alpha_k,\quad \rho>1-\beta_m,\quad \xi>0,\quad \eta>0; \end{array} \tag{36}$$

whereinto $\varepsilon<\min\{\alpha_0,\sigma_i+\xi_i,\sigma+\xi,\mu\}$, $\delta<\min\{\beta_0,\rho_i+\eta_i,\rho+\eta,\nu\}$. Then there exists a unique regular solution to problem (1), (30), (31) in Ω. The solution has the form

$$u(x,y)=\sum_{j=0}^{k-1}\int_0^y D_{0x}^{\{\alpha_k,\alpha_{k-1},\dots,\alpha_{j+1}\}}G(x,y-s)\varphi_j(s)ds+$$

$$+\sum_{i=0}^{m-1}\int_0^x D_{0y}^{\{\beta_m,\beta_{m-1},\dots,\beta_{i+1}\}}G(x-t,y)A\psi_i(t)dt+\int_0^y\int_0^x G(x-t,y-s)f(t,s)dtds, \tag{37}$$

where

$$G(x,y)=\Phi_{\alpha,\beta}^{0,0}(x,y).$$

Remark 3.1. If we put

$$\sigma_i=\sigma=\mu=\alpha_0,\quad \rho_j=\rho=\nu=\beta_0,\quad \xi_i=\eta_j=\xi=\eta=1,$$

then $\varepsilon<\alpha_0$, $\delta<\beta_0$, and the conditions (34)–(36) will take the form

$$\begin{array}{c} \varphi_j(y)=D_{0y}^{-\beta_0}\varphi_j^*(y),\quad \varphi_j^*(y),\, y^{1-\beta_0}\varphi_{k-1}(y)\in C[0,b],\\ \beta_0>\max\left\{\beta-\frac{\alpha_0+\alpha_k-1}{\alpha}\beta,1-\beta_m\right\},\quad j=\overline{0,k-2}; \end{array} \tag{38}$$

$$\begin{array}{c} \psi_i(y)=D_{0x}^{-\alpha_0}\psi_i^*(x),\quad \psi_i^*(x),\, x^{1-\alpha_0}\psi_{m-1}(x)\in C[0,a],\\ \alpha_0>\max\left\{\alpha-\frac{\beta_0+\beta_m-1}{\beta}\alpha,1-\alpha_k\right\},\quad i=\overline{0,m-2}; \end{array} \tag{39}$$

$$f(x,y)=D_{0x}^{-\alpha_0}D_{0y}^{-\beta_0}f^*(x,y),\quad f^*(x,y)\in C(\overline{\Omega}). \tag{40}$$

Remark 3.2. In the case of a system with Riemann–Liouville derivatives, i.e., when $k=m=1$, $D_{0x}^{\{\alpha_0,1\}}=D_{0x}^{\alpha_0}$, $D_{0y}^{\{\beta_0,1\}}=D_{0y}^{\beta_0}$, the conditions (38)–(40) will take the form

$$y^{1-\beta_0}\varphi_0(y)\in C[0,b],\quad x^{1-\alpha_0}\psi_0(x)\in C[0,a],\quad f(x,y)=D_{0x}^{-\alpha_0}D_{0y}^{-\beta_0}f^*(x,y),\quad f^*(x,y)\in C(\overline{\Omega}).$$

The solution has the form

$$u(x,y)=\int\limits_0^y G(x,y-s)\varphi_0(s)ds+\int\limits_0^x G(x-t,y)A\psi_0(t)dt+$$

$$+\int\limits_0^y\int\limits_0^x G(x-t,y-s)f(t,s)dtds.$$

Remark 3.3. In the case of a system with Gerasimov–Caputo derivatives, i.e., when $k=m=1$, $D_{0x}^{\{1,\alpha_1\}}=\partial_{0x}^{\alpha_1}$, $D_{0y}^{\{1,\beta_1\}}=\partial_{0y}^{\beta_1}$, the conditions (38)–(40) will take the form

$$\varphi_0(y)\in C[0,b],\quad \psi_0(x)\in C[0,a],\quad f(x,y)=D_{0x}^{-1}D_{0y}^{-1}f^*(x,y),\quad f^*(x,y)\in C(\overline{\Omega}).$$

The solution has the form

$$u(x,y)=\int\limits_0^y D_{0x}^{\alpha_1-1}G(x,y-s)\varphi_0(s)ds+\int\limits_0^x D_{0y}^{\beta_1-1}G(x-t,y)A\psi_0(t)dt+$$

$$+\int\limits_0^y\int\limits_0^x G(x-t,y-s)f(t,s)dtds.$$

In what follows, for brevity, we will denote

$$u_{\varphi_j}(x,y)=\int\limits_0^y D_{0x}^{\{\alpha_k,\ldots,\alpha_{j+1}\}}G(x,y-s)\varphi_j(s)ds,$$

$$u_{\psi_i}(x,y)=\int\limits_0^x D_{0y}^{\{\beta_m,\ldots,\beta_{i+1}\}}G(x-t,y)A\psi_i(t)dt,$$

$$u_f(x,y)=\int\limits_0^x\int\limits_0^y G(x-t,y-s)f(t,s)dsdt.$$

3.1 Representation of solutions

Lemma 3.1. Every regular solution $u(x,y)$ to problem (1), (30), (31) in Ω can be represented in the form (37).

Proof. Let $u(x,y)$ be a solution to problem (1), (30), (31), and matrix $V\equiv V(x-t,y-s)$ be a solution to the equation

$$L^*V\equiv D_{xt}^{\{\alpha_k,\alpha_{k-1},\ldots,\alpha_0\}}V+D_{ys}^{\{\beta_m,\beta_{m-1},\ldots,\beta_0\}}VA=VB+I, \tag{41}$$

satisfying the conditions

$$\lim_{t\to x} D_{xt}^{\{\alpha_k,\alpha_{k-1},\dots,\alpha_i\}}V = 0, \quad 1 \le i \le k, \tag{42}$$

$$\lim_{s\to y} D_{ys}^{\{\beta_m,\beta_{m-1},\dots,\beta_j\}}V = 0, \quad 1 \le j \le m, \tag{43}$$

where I is the identity matrix.

Lemmas 2.2 and 2.3 show that $V(x-t,y-s) = \Phi_{\alpha,\beta}^{1,1}(x-t,y-s)$ is the solution to the problem (41)–(43). From (22) and (23) we see that

$$V_{xy}(x,y) = G(x,y). \tag{44}$$

We have the following formula [20]

$$\int_0^x \left[h(x,t)D_{0t}^{\{\alpha_0,\alpha_1,\dots,\alpha_m\}}g(t) - D_{xt}^{\{\alpha_m,\alpha_{m-1},\dots,\alpha_0\}}h(x,t)\cdot g(t)\right]dt =$$

$$= \sum_{i=1}^{m} D_{xt}^{\{\alpha_m,\alpha_{m-1},\dots,\alpha_{m+1-i}\}}h(x,t)\cdot D_{0t}^{\{\alpha_0,\alpha_1,\dots,\alpha_{m-i}\}}g(t)\Big|_{t=0}^{t=x}. \tag{45}$$

By (1) and (41) we get

$$V(x-t,y-s)Lu(t,s) - L^*V(x-t,y-s)\cdot u(t,s) = V(x-t,y-s)f(t,s) - u(t,s),$$

or

$$\left(VD_{0t}^{\{\alpha_0,\dots,\alpha_k\}}u - D_{xt}^{\{\alpha_k,\dots,\alpha_0\}}V\cdot u\right) + \left(VAD_{0s}^{\{\beta_0,\dots,\beta_m\}}u - D_{ys}^{\{\beta_m,\dots,\beta_0\}}VA\cdot u\right) = Vf - u.$$

Integrating the last equality, taking into account the formula (45), we obtain

$$\int_0^x\int_0^y u(t,s)dsdt = \int_0^x\int_0^y V(x-t,y-s)u(t,s)dsdt -$$

$$-\sum_{i=1}^{k}\int_0^y D_{xt}^{\{\alpha_k,\alpha_{k-1},\dots,\alpha_{k+1-i}\}}V(x,y-s)D_{0t}^{\{\alpha_0,\alpha_1,\dots,\alpha_{k-i}\}}u(t,s)\Big|_{t=0}^{t=x}ds -$$

$$-\sum_{j=1}^{m}\int_0^x D_{ys}^{\{\beta_m,\beta_{m-1},\dots,\beta_{m+1-j}\}}V(x-t,y)AD_{0s}^{\{\beta_0,\beta_1,\dots,\beta_{m-j}\}}u(t,s)\Big|_{s=0}^{s=y}dt. \tag{46}$$

Therefore, differentiating (46) with respect to x and with respect to y, taking into account (30), (31), (42), (43) and (44), and then changing the order of summation, we get (37). Lemma 3.1 is proved.

3.2 Properties of the fundamental solution

Lemma 3.2. [3] Let $AB = BA$, then the equality

$$\left(D_{0x}^{\alpha} + AD_{0y}^{\beta} - B\right)G(x,y) = 0 \tag{47}$$

holds.

Lemma 3.3. Let the vectors $\psi_i(x)$ $(i = \overline{0, k-1})$, and $\varphi_j(y)$ $(j = \overline{0, m-1})$, satisfy the conditions of Theorem 3.1, then the relations

$$\lim_{x\to 0} D_{0x}^{\{\alpha_0,\dots,\alpha_i\}} u_{\varphi_j}(x,y) = \begin{cases} 0, & i \neq j, \\ \varphi_j(y), & i = j, \end{cases}, \quad y > \varepsilon > 0, \tag{48}$$

$$\lim_{y\to 0} D_{0y}^{\{\beta_0,\dots,\beta_j\}} u_{\psi_j}(x,y) = \begin{cases} 0, & i \neq j, \\ \psi_i(y), & i = j, \end{cases}, \quad x > \varepsilon > 0, \tag{49}$$

$$\lim_{y\to 0} D_{0y}^{\{\beta_0,\dots,\beta_j\}} u_{\varphi_j}(x,y) = 0, \quad x > \varepsilon > 0, \tag{50}$$

$$\lim_{x\to 0} D_{0x}^{\{\alpha_0,\dots,\alpha_i\}} u_{\psi_j}(x,y) = 0, \quad y > \varepsilon > 0, \tag{51}$$

hold, where the limits (48) and (51) are uniform on any closed subset $(0; b)$, and the limits (49) and (50) on any closed subset $(0; a)$.

Proof. Using (22), we write

$$u_{\varphi_j}(x,y) = \int_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},0}(x, y-s)\varphi_j(s)ds.$$

By virtue of the formula (24) and the estimate (26), we obtain

$$D_{0x}^{\{\alpha_0,\dots,\alpha_i\}} \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},0}(x,y) = \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1}+1-\mu_i,0}(x,y),$$

$$\left|\Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1}+1-\mu_i,0}(x,y)\right|_* \leq C x^{1+\alpha-\bar{\mu}_{j+1}-\mu_i-\alpha\theta} y^{\beta\theta-1}, \quad \theta \in [-1,1].$$

Let $i < j$, then, taking into account the fact that $\sum\limits_{s=i+1}^{j} \alpha_s < \alpha$ for $\alpha_0 + \alpha_k > 1$, we get that there exists $\theta \in (0,1)$, such that

$$1 + \alpha - \bar{\mu}_{j+1} - \mu_i - \alpha\theta = \sum_{s=0}^{k} \alpha_s - \sum_{s=0}^{i} \alpha_s - \sum_{s=j+1}^{k} \alpha_s - \alpha\theta = \sum_{s=i+1}^{j} \alpha_s - \alpha\theta > 0.$$

By virtue of the last relations, for $i < j$, we obtain

$$D_{0x}^{\{\alpha_0,\dots,\alpha_i\}} u_{\varphi_j}(x,y) = \int_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1}+1-\mu_i,0}(x, y-s)\varphi_j(s)ds \in C(\Omega \cup \{x = 0\}), \tag{52}$$

$$\lim_{x\to 0} D_{0x}^{\{\alpha_0,\dots,\alpha_i\}} u_{\varphi_j}(x,y) = 0. \tag{53}$$

Consider now the case $i = j$. Taking into account that $1 - \bar{\mu}_{j+1} + 1 - \mu_j = 1 - \alpha$ we obtain

$$D_{0x}^{\{\alpha_0,\dots,\alpha_j\}} u_{\varphi_j}(x,y) = \int_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1}+1-\mu_j,0}(x, y-s)\varphi_j(s)ds = \int_0^y \Phi_{\alpha,\beta}^{1-\alpha,0}(x, y-s)\varphi_j(s)ds.$$

Hence, in view of the equality [3]

$$\lim_{x\to 0} \int_0^y \Phi_{\alpha,\beta}^{1-\alpha,0}(x, y-s)q(s)ds = q(y),$$

which under condition (14) is proved using equality (18), we get

$$\lim_{x\to 0} D_{0x}^{\{\alpha_0,\dots,\alpha_j\}} u_{\varphi_j}(x,y) = \varphi_j(y). \tag{54}$$

For $i > j$, i.e. $1 \le i \le k-1$, and $0 \le j \le k-2$, due to (24) and equality (47) we obtain

$$D_{0x}^{\{\alpha_0,\dots,\alpha_i\}} u_{\varphi_j}(x,y) = D_{0x}^{\{\alpha_{j+1},\dots,\alpha_i\}} \int\limits_0^y D_{0x}^{\alpha} G(x,y-s)\varphi_j(s)ds =$$

$$= D_{0x}^{\{\alpha_{j+1},\dots,\alpha_i\}} \int\limits_0^y \left(B - AD_{ys}^{\beta}\right) G(x,y-s)\varphi_j(s)ds = I_1(x,y) - I_2(x,y). \tag{55}$$

It follows from the relation $\alpha > \alpha_j + 1 + ... + \alpha_i$ that there exists $\theta \in (0,1)$, such that $\alpha(1-\theta) > \alpha_j + 1 + ... + \alpha_i$. Therefore, by the estimate

$$\left| \Phi_{\alpha,\beta}^{1-\mu_{j+1}^i,0}(x,y-s) \right|_* \le Cx^{\alpha-\alpha\theta-\mu_{j+1}^i} y^{\beta\theta-1}, \quad \theta \in [-1,1),$$

we get the relations

$$I_1(x,y) = B \int\limits_0^y \Phi_{\alpha,\beta}^{1-\mu_{j+1}^i,0}(x,y-s)\varphi_j(s)ds \in C(\Omega \cup \{x=0\}), \tag{56}$$

$$\lim_{x\to 0} I_1(x,y) = 0. \tag{57}$$

Let us consider the second term

$$I_2(x,y) = A \int\limits_0^y D_{0x}^{\mu_{j+1}^i - 1} D_{ys}^{\beta} G(x,y-s) D_{0s}^{-\rho_j}\varphi_j(s)ds = A \int\limits_0^y \Phi_{\alpha,\beta}^{1-\mu_{j+1}^i,\rho_j-\beta}(x,y-s)\varphi_j^*(s)ds.$$

In view of the estimate

$$\left| \Phi_{\alpha,\beta}^{1-\mu_{j+1}^i,\rho_j-\beta}(x,y) \right|_* \le Cx^{\alpha-\alpha\theta-\mu_{j+1}^i} y^{\rho_j-\beta+\beta\theta-1}, \quad \theta \in [0,1),$$

we get that the integral $I_2(x,y)$ converges under the condition $\begin{cases} \alpha - \alpha\theta - \mu_{j+1}^i > 0, \\ \rho_j - \beta + \beta\theta > 0, \end{cases}$ i.e., when θ satisfies the condition $\frac{\mu_{j+1}^i}{\alpha} < 1 - \theta < \frac{\rho_j}{\beta}$, at that

$$\lim_{x\to 0} I_2(x,y) = 0. \tag{58}$$

By (53), (54), (55), (57) and (58) we get (48).

The relation (49) is proved similarly.

Let us prove the relation (50). Formulas (25) and (26) give the equalities

$$\frac{\partial^s}{\partial y^s} D_{0y}^{\{\beta_0,\dots,\beta_i\}} \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j}(x,y) = \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\nu_i+1-s}(x,y), \quad s = 0,1,$$

and

$$\left| \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\nu_i+1-s}(x,y) \right|_* \le Cx^{\alpha-\alpha\theta-\bar{\mu}_{j+1}} y^{\rho_j-\nu_i+\beta\theta-s}, \quad s = 0,1, \quad \theta \in [0,1).$$

By the last estimate, taking into account that due to $\rho_j > 1 - \beta_m > 1 - \bar{\nu}_{i+1}$ one can choose θ sufficiently close to 1, so that $\rho_j - \nu_i + \beta\theta = \rho_j + \bar{\nu}_{i+1} - 1 + \beta(\theta - 1) > 0$, we get

$$D_{0y}^{\{\beta_0,...,\beta_i\}} u_{\varphi_j}(x, y) = \int_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j+1-\nu_i}(x, y - s)\varphi_j^*(s)ds \in C(\Omega \cup \{y = 0\}), \tag{59}$$

$$\frac{\partial}{\partial y} D_{0y}^{\{\beta_0,...,\beta_i\}} u_{\varphi_j}(x, y) = \int_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\nu_i}(x, y - s)\varphi_j^*(s)ds \in C(\Omega) \cup L(\Omega), \tag{60}$$

and

$$\lim_{y\to 0} D_{0y}^{\{\beta_0,...,\beta_i\}} u_{\varphi_j}(x, y) = 0, \quad 0 \le i \le m - 1.$$

The relation (51) is proved similarly. Lemma 3.3 is proved.

Lemma 3.4. The function (37) is a solution (1) satisfying the inclusions (32) and (33).

Proof. Using estimates

$$\left|\Phi_{\alpha,\beta}^{-\mu_{j+1}^{k-1},0}(x, y)\right|_* \le Cx^{\alpha-\alpha\theta-\mu_{j+1}^{k-1}-1}y^{\beta\theta-1}, \quad \theta \in [0, 1),$$

$$\left|\Phi_{\alpha,\beta}^{-\mu_{j+1}^{k-1},\rho_j-\beta}(x, y)\right|_* \le Cx^{\alpha-\alpha\theta-\mu_{j+1}^{k-1}-1}y^{\rho_j-\beta+\beta\theta-1}, \quad \theta \in [0, 1),$$

and inequalities $\alpha - \alpha\theta - \mu_{j+1}^{k-1} > 0$, $\rho_j - \beta + \beta\theta > 0$, by (55) with $i = k - 1$ we get

$$\frac{\partial}{\partial x} D_{0x}^{\{\alpha_0,...,\alpha_{k-1}\}} u_{\varphi_j}(x, y) = B \int_0^y \Phi_{\alpha,\beta}^{-\mu_{j+1}^{k-1},0}(x, y - s)\varphi_j(s)ds-$$

$$-A \int_0^y \Phi_{\alpha,\beta}^{-\mu_{j+1}^{k-1},\rho_j-\beta}(x, y - s)\varphi_j^*(s)ds \in C(\Omega) \cap L(\Omega). \tag{61}$$

By (61) it follows that

$$D_{0x}^{\{\alpha_0,...,\alpha_k\}} u_{\varphi_j}(x, y) = B \int_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},0}(x, y - s)\varphi_j(s)ds-$$

$$-A \int_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\beta}(x, y - s)\varphi_j^*(s)ds \in C(\Omega \cup \{x = 0\}). \tag{62}$$

Due to the estimate

$$\left|\Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\nu_i}(x, y)\right|_* \le Cx^{\alpha-\alpha\theta-\bar{\mu}_{j+1}}y^{\rho_j-\nu_i+\beta\theta-1}, \quad \theta \in [0, 1),$$

and the inequality $\rho_j + \beta_m > 1$, one can always choose θ sufficiently close to 1, so that

$$\rho_j - \nu_i + \beta\theta > \beta_{i+1} + ... + \beta_{m-1} + (\theta - 1)\beta > 0,$$

so

$$\frac{\partial}{\partial y} D_{0y}^{\{\beta_0,...,\beta_i\}} u_{\varphi_j}(x, y) = \int_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\nu_i}(x, y - s)\varphi_j^*(s)ds \in C(\Omega) \cap L(\Omega). \tag{63}$$

From (63) for $i=m-1$ it follows that

$$D_{0y}^{\{\beta_0,\ldots,\beta_m\}}u_{\varphi_j}(x,y)=\int\limits_0^y \Phi_{\alpha,\beta}^{1-\bar{\mu}_{j+1},\rho_j-\beta}(x,y-s)\varphi_j^*(s)ds\in C(\Omega). \tag{64}$$

It can be seen from (62) and (64) that $u_{\varphi_j}(x,y)$ are solutions of the homogeneous system

$$D_{0x}^{\{\alpha_0,\ldots,\alpha_k\}}u_{\varphi_j}(x,y)+AD_{0y}^{\{\beta_0,\ldots,\beta_m\}}u_{\varphi_j}(x,y)=Bu_{\varphi_j}(x,y).$$

The proof for $u_{\psi_i}(x,y)$ is similar.

Let us show that

$$u_f(x,y)=\int\limits_0^x\int\limits_0^y \Phi_{\alpha,\beta}^{\sigma,\rho}(x-t,y-s)f^*(t,s)dsdt$$

system solution (1). In view of

$$\left|\Phi_{\alpha,\beta}^{\sigma-\mu_j+1,\rho}(x,y)\right|_*\le Cx^{\bar{\mu}_{j+1}-\alpha\theta+\sigma-1}y^{\rho+\beta\theta-1},$$

we get

$$\left|D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}}u_f(x,y)\right|_*\le Cx^{\sigma+\xi-(1-\bar{\mu}_{j+1})-\alpha\theta}y^{\rho+\eta+\beta\theta-1}.$$

The inequality $\sigma+\xi>1-\alpha_k$, implies $\sigma+\xi-(1-\bar{\mu}_{j+1})-\alpha\theta>0$ and the relations

$$D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}}u_f(x,y)=\int\limits_0^x\int\limits_0^y \Phi_{\alpha,\beta}^{\sigma-\mu_j+1,\rho}(x-t,y-s)f^*(t,s)dsdt\in C(\Omega\cup\{x=0\}), \tag{65}$$

$$\lim_{x\to 0}D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}}u_f(x,y)=0,\quad 0\le j\le k-1. \tag{66}$$

It follows from (48)–(51) and (66) that the boundary conditions (30) and (31) hold.

By the estimate

$$\left|\Phi_{\alpha,\beta}^{\sigma-\mu_j,\rho}(x,y)\right|_*\le Cx^{\alpha-\mu_j-\alpha\theta+\sigma-1}y^{\rho+\beta\theta-1},$$

and inequalities $\alpha-\mu_j-\alpha\theta+\sigma=\bar{\mu}_{j+1}+\sigma-1-\alpha\theta>0$, which follow from the inequality $\sigma>1-\alpha_k$, we get

$$\frac{\partial}{\partial x}D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}}u_f(x,y)=\int\limits_0^x\int\limits_0^y \Phi_{\alpha,\beta}^{\sigma-\mu_j,\rho}(x-t,y-s)f^*(t,s)dsdt. \tag{67}$$

By (67), estimate

$$\left|\frac{\partial}{\partial x}D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}}u_f(x,y)\right|_*\le Cx^{\sigma+\xi+\bar{\mu}_{j+1}-1-\alpha\theta}y^{\rho+\eta+\beta\theta-1},$$

due to the inequality $\sigma>1-\alpha_k$, we obtain

$$\frac{\partial}{\partial x}D_{0x}^{\{\alpha_0,\ldots,\alpha_j\}}u_f(x,y)\in C(\Omega)\cap L(\Omega),\quad 0\le j\le k-1. \tag{68}$$

Therefore

$$D_{0x}^{\{\alpha_0,\ldots,\alpha_k\}}u_f(x,y)=\int\limits_0^x\int\limits_0^y \Phi_{\alpha,\beta}^{\sigma-\alpha,\rho}(x-t,y-s)f^*(t,s)dsdt\in C(\Omega). \tag{69}$$

Similarly, we get

$$D_{0x}^{\{\beta_0,\dots,\beta_m\}} u_f(x,y) = \int\limits_0^x \int\limits_0^y \Phi_{\alpha,\beta}^{\sigma,\rho-\beta}(x-t,y-s) f^*(t,s)dsdt \in C(\Omega). \quad (70)$$

By (69) and (70), taking into account Lemma 2.3, we obtain

$$\left(D_{0x}^{\{\alpha_0,\dots,\alpha_k\}} + AD_{0x}^{\{\beta_0,\dots,\beta_m\}} - B\right) u_f(x,y) =$$

$$= \int\limits_0^x \int\limits_0^y \left(D_{0x}^{\{\alpha_0,\dots,\alpha_k\}} + AD_{0x}^{\{\beta_0,\dots,\beta_m\}} - B\right) \Phi_{\alpha,\beta}^{\sigma,\rho}(x-t,y-s) f^*(t,s)dsdt =$$

$$= \int\limits_0^x \int\limits_0^y \frac{(x-t)^{\sigma-1}(y-s)^{\rho-1}}{\Gamma(\sigma)\Gamma(\rho)} f^*(t,s)dsdt = D_{0x}^{-\sigma} D_{0y}^{-\rho} f^*(t,s) = f(t,s).$$

From (52), (56), (59)–(65), (68)–(70), it follows that (37) satisfies the inclusions (32), (33). Lemma 3.4 is proved.

3.3 Proof of the main theorem

Using the estimate (26) and the conditions of Theorem 3.1 on the functions $\psi_i(x)$, $\varphi_j(y)$ and $f(x,y)$, we obtain the estimates

$$x^{1-\varepsilon}y^{1-\delta}|u_{\varphi_j}(x,y)|_* \leq Cx^{\alpha-\alpha\theta-\bar{\mu}_{j+1}+1-\varepsilon}y^{\rho_j+\eta_j+\beta\theta-\delta}, \quad \theta \in (0,1), \quad (71)$$

$$x^{1-\varepsilon}y^{1-\delta}|u_{\psi_i}(x,y)|_* \leq Cx^{\alpha-\alpha\theta+\sigma_i+\xi_i-\varepsilon}y^{\beta\theta-\bar{\nu}_{i+1}+1-\delta}, \quad \theta \in (0,1), \quad (72)$$

$$x^{1-\varepsilon}y^{1-\delta}|u_f(x,y)|_* \leq Cx^{\alpha-\alpha\theta+\sigma+\xi-\varepsilon}y^{\beta\theta+\rho+\eta-\delta}, \quad \theta \in (0;1). \quad (73)$$

Considering that $\alpha + 1 - \bar{\mu}_{j+1} = \mu_j > \varepsilon$, due to $\alpha_0 > \varepsilon$, and the fact that $\rho_j + \eta_j > \varepsilon$, by (71) we get $x^{1-\varepsilon}y^{1-\delta}u_{\varphi_j} \in C(\overline{\Omega})$. Taking into account the inequalities $\sigma_i + \xi_i > \varepsilon$, and the fact that $\beta\theta - \bar{\nu}_{i+1} + 1 = \beta(\theta-1) + \nu_i > \delta$, due to $\beta_0 > \delta$, by (72) we get $x^{1-\varepsilon}y^{1-\delta}u_{\psi_i} \in C(\overline{\Omega})$. It follows from (73) and the inequalities $\sigma + \xi > \varepsilon$ and $\rho + \eta > \delta$ that the inclusion $x^{1-\varepsilon}y^{1-\delta}u_f \in C(\overline{\Omega})$.

The above together with Lemmas 3.2, 3.3, and 3.4 proves the existence of a regular solution to the problem (1), (30), (31). The uniqueness of the solution to the problem follows by Lemma 3.1. Theorem 3.1 is proved.

References

1 Джрбашян М.М. Дробные производные и задача Коши для дифференциальных уравнений дробного порядка / М.М. Джрбашян, А.Б. Нерсесян // Изв. АН АрмянССР. Матем. — 1968. — С. 3–28.

2 Нахушев А.М. Дробное исчисление и его применение / А.М. Нахушев. — М.: Физматлит, 2003. — 272 с.

3 Мамчуев М.О. Краевая задача для линейной системы уравнений с частными производными дробного порядка / М.О. Мамчуев // Челяб. физ.-мат. журн. — 2017. — 2. — № 3. — С. 295–311.

4 Мамчуев М.О. Краевая задача для многомерной системы уравнений с дробными производными Римана–Лиувилля / М.О. Мамчуев // Сиб. электрон. мат. изв. — 2019. — 16. — С. 732–747.

5 Псху А.В. Краевая задача для многомерного дифференциального уравнения дробного порядка / А.В. Псху // Дифференц. уравнения. — 2011. — 17. — № 5. — С. 385–395.

6 Богатырева Ф.Т. Краевая задача для уравнения в частных производных первого порядка с оператором Джрбашяна–Нерсесяна / Ф.Т. Богатырева // Докл. Адыгской (Черкесской) Междунар. акад. наук. — 2015. — 15. — № 1. — С. 9–16.

7 Богатырева Ф.Т. Краевая задача для уравнения в частных производных с оператором дробного дифференцирования Джрбашяна–Нерсесяна / Ф.Т. Богатырева // Докл. Адыгской (Черкесской) Междунар. акад. наук. — 2015. — 15. — № 2. — С. 17–24.

8 Богатырева Ф.Т. Краевая задача с интегральным условием Самарского для уравнения в частных производных первого порядка / Ф.Т. Богатырева // Изв. Кабардино-Балкарского научного центра РАН. — 2017. — № 6 (80). — С. 10–14.

9 Богатырева Ф.Т. Краевая задача для уравнения в частных производных с оператором дробного дифференцирования Джрбашяна–Нерсесяна / Ф.Т. Богатырева // Изв. Кабардино-Балкарского научного центра РАН. — 2018. — № 6 (86). — С. 10–14.

10 Богатырева Ф.Т. Краевые задачи для уравнения в частных производных первого порядка с операторами Джрбашяна–Нерсесяна / Ф.Т. Богатырева // Челяб. физ.-мат. журн. — 2021. — 6. — № 4. — С. 403–416.

11 Fedorov V.E. Initial value problems of linear equations with the Dzhrbashyan–Nersesyan derivative in Banach spaces / V.E. Fedorov, M.V. Plekhanova, E.M. Izhberdeeva // Symmetry. — 2021. — 13. — P. 1058.

12 Волкова А.Р. Начальные задачи для уравнений с композицией дробных производных / А.Р. Волкова, Е.М. Ижбердеева, В.Е. Федоров // Челяб. физ.-мат. журн. — 2021. — 6. — № 3. — С. 269–277.

13 Мамчуев М.О. Краевая задача для системы уравнений с частными производными дробного порядка / М.О. Мамчуев // Дифференц. уравнения. — 2008. — 44. — № 12. — С. 1674–1686.

14 Мамчуев М.О. Краевая задача для системы многомерных дифференциальных уравнений дробного порядка / М.О. Мамчуев // Вестн. Самар. гос. ун-та. Ест.-науч. сер. — 2008. — 8(2). — № 67. — С. 164–175.

15 Мамчуев М.О. Задача Коши для системы уравнний с частными производными дробного порядка / М.О. Мамчуев // Вестн. КРАУНЦ. Физ.-мат. науки. — 2018. — № 3 (23). — С. 76–82.

16 Wright E.M. On the coefficients of power series having exponential singularities / E.M. Wright // J. London Math. Soc. — 1933. — 8. — 29. — P. 71–79.

17 Wright E.M. The asymptotic expansion of the generalized Bessel function / E.M. Wright // Proc. London Math. Soc. Ser. II. — 1934. — 38. — P. 257–270.

18 Pskhu A. Fundamental solutions and Cauchy problems for an odd-order partial differential equation with fractional derevative / A. Pskhu // Electronic Journal of Differential Equations. — 2019. — 21. — P. 1–13.

19 Luchko Y. An operational method for solving fractional differential equations with the Caputo derivatives / Y. Luchko, R. Gorenflo // Acta Math. Vietnam. — 1999. — 24. — P. 207–233.

20 Mamchuev M. Cauchy problem for a linear system of ordinary differential equations of the fractional order / M. Mamchuev// Mathematics. — 2020. — 8. — 1475.

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Джрбашян–Нерсесян бөлшек дифференциалдау операторы бар дербес туындылы теңдеулер жүйесіне арналған шеттік есеп

Тікбұрышты облыста жүйенің матрицалық коэффициенттері күрделі меншікті мәндерге ие болған жағдайда тұрақты коэффициенттері бар Джрбашян-Нерсесян бөлшек дифференциалдау операторы бар дербес туындылы теңдеулер жүйесіне арналған шеттік есеп зерттелді. Зерттелетін шеттік есептердің шешімінің бар болуы және жалғыздық теоремалары дәлелденді. Шешім матрицалық аргументтің Райт функциясы тұрғысынан анық түрде құрастырылған.

Кілт сөздер: дербес туындылы теңдеулер жүйесі, бөлшек ретті туынды, Джрбашян-Нерсесян операторы, шеттік есеп, іргелі шешім, матрицалық аргументтің Райт фукнциясы.

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Краевая задача для системы уравнений в частных производных с операторами дробного дифференцирования Джрбашяна–Нерсесяна

Исследована краевая задача в прямоугольной области для линейной системы уравнений с частными операторами дробного дифференцирования Джрбашяна–Нерсесяна с постоянными коэффициентами в случае, когда матричные коэффициенты системы имеют комплексные собственные значения. Доказаны теоремы существования и единственности решения исследуемой краевой задачи. Решение построено в явном виде в терминах функции Райта матричного аргумента.

Ключевые слова: система уравнений с частными производными, производные дробного порядка, оператор Джрбашяна–Нерсесяна, краевая задача, фундаментальное решение, функция Райта матричного аргумента.

References

1 Dzhrbashian, M.M., & Nersesian, A.B. (1968). Drobnye proizvodnye i zadacha Koshi dlia differentsialnykh uravnenii drobnogo poriadka [Fractional derivatives and the Cauchy problem for fractional differential equations]. *Izvestiia AN Armianskoi SSR. Matematika – Reports of Academy of Sciences of the Armenian SSR. Math.,* 3–28 [in Russian].

2 Nakhushev, A.M. (2003). *Drobnoe ischislenie i ego primenenie [Fractional calculus and its application].* Moscow: Fizmatlit [in Russian].

3 Mamchuev, M.O. (2017). Kraevaia zadacha dlia lineinoi sistemy uravnenii s chastnymi proizvodnymi drobnogo poriadka [Boundary value problem for a linear system of equations with partial derivatives of fractional order]. *Cheliabinskii fiziko-matematicheskii zhurnal – Chelyabinsk Physical and Mathematical Journal, 2, 3,* 295–311 [in Russian].

4 Mamchuev, M.O. (2019). Kraevaia zadacha dlia mnogomernoi sistemy uravnenii s drobnymi proizvodnymi Rimana–Liuvillia [Boundary value problem for a multidimensional system of equations with Riemann–Liouville fractional derivatives]. *Sibirskie elektronnye matematicheskie izvestiia – Sibererian Electronic Mathematical Reports, 16,* 732–747 [in Russian].

5 Pskhu, A.V. (2011). Kraevaia zadacha dlia mnogomernogo differentsialnogo uravneniia drobnogo poriadka [Boundary value problem for a multidimensional differential equation of fractional order]. *Differentsialnye uravneniia – Differential Equations, 17, 5,* 385–395 [in Russian].

6 Bogatyreva, F.T. (2015). Kraevaia zadacha dlia uravneniia v chastnykh proizvodnykh pervogo poriadka s operatorom Dzhrbashiana–Nersesiana [Boundary value problem for a partial differential equation of first order with Dzhrbashyan–Nersesyan operator]. *Doklady Adygskoi (Cherkesskoi) Mezhdunarodnoi akademii nauk – Reports of Adyghe (Circassian) International Academy of Sciences, 15, 1,* 9–16 [in Russian].

7 Bogatyreva, F.T. (2015). Kraevaia zadacha dlia uravneniia v chastnykh proizvodnykh s operatorom drobnogo differentsirovaniia Dzhrbashiana–Nersesiana [Boundary value problem for a partial differential equation with Dzhrbashyan–Nersesyan operator]. *Doklady Adygskoi (Cherkesskoi) Mezhdunarodnoi academii nauk – Reports of Adyghe (Circassian) International Academy of Sciences, 15, 2,* 17–24 [in Russian].

8 Bogatyreva, F.T. (2017). Kraevaia zadacha s integralnym usloviem Samarskogo dlia uravneniia v chastnykh proizvodnykh pervogo poriadka [Boundary value problem with Samarskii integral condition for a first-order partial differential equation]. *Izvestiia Kabardino-Balkarskogo nauchnogo tsentra RAN – Reports of Kabardino-Balkarian Scientific Centre of RAS, 6 (80),* 10–14 [in Russian].

9 Bogatyreva, F.T. (2018). Kraevaia zadacha dlia uravneniia v chastnykh proizvodnykh s operatorom drobnogo differentsirovaniia Dzhrbashiana–Nersesiana [Boundary value problem for a partial differential equation with Dzhrbashyan–Nersesyan fractional differentiation operator]. *Izvestiia Kabardino-Balkarskogo nauchnogo tsentra RAN – Reports of Kabardino-Balkarian Scientific Centre of RAS, 6 (86),* 10–14 [in Russian].

10 Bogatyreva, F.T. (2021). Kraevye zadachi dlia uravneniia v chastnykh proizvodnykh pervogo poriadka s operatorami Dzhrbashiana–Nersesiana [Boundary value problems for a first-order partial differential equation with the Dzhrbashyan–Nersesyan operators]. *Cheliabinskii fiziko-matematicheskii zhurnal – Chelyabinsk Physical and Mathematical Journal, 6, 4,* 403–416 [in Russian].

11 Fedorov, V.E., Plekhanova, M.V., & Izhberdeeva, E.M. (2021). Initial value problems of linear equations with the Dzhrbashyan–Nersesyan derivative in Banach spaces. *Symmetry, 13,* 1058.

12 Volkova, A.R., Izhberdeeva, E.M., & Fedorov, V.E. (2021). Nachalnye zadachi dlia uravnenii s kompozitsiei drobnykh proizvodnykh [Initial problems for equations with a composition of fractional derivatives]. *Cheliabinskii fiziko-matematicheskii zhurnal – Chelyabinsk Physical and Mathematical Journal, 6, 3,* 269–277 [in Russian].

13 Mamchuev, M.O. (2008). Kraevaia zadacha dlia sistemy uravnenii s chastnymi proizvodnymi drobnogo poriadka [Boundary value problem for a system of equations with partial derivatives of fractional order]. *Differentsialnye uravneniia – Differential Equations, 44, 12,* 1674–1686 [in Russian].

14 Mamchuev, M.O. (2008). Kraevaia zadacha dlia sistemy mnogomernykh differentsialnykh uravnenii drobnogo poriadka [Boundary value problem for a system of multidimensional equations of fractional order]. *Vestnik Samarskogo gosudarstvennogo universiteta. Estestvennonauchnaia seriia – Vestnik of Samara University. Natural Science Series, 8(2), 67,* 164–175 [in Russian].

15 Mamchuev, M.O. (2018). Zadacha Koshi dlia sistemy uravnnii s chastnymi proizvodnymi drobnogo poriadka [Cauchy problem for a system of partial differential equations of fractional order]. *Vestnik KRAUNTs. Fiziko-matematicheskie nauki – Bulletin KRASEC. Physicsal and Mathematicsal Sciences, 3 (23),* 76–82 [in Russian].

16 Wright, E.M. (1933). On the coefficients of power series having exponential singularities. *J. London Math. Soc., 8, 29,* 71–79.

17 Wright, E.M. (1934). The asymptotic expansion of the generalized Bessel function. *Proc. London Math. Soc. Ser. II, 38,* 257–270.

18 Pskhu, A. (2019). Fundamental solutions and Cauchy problems for an odd-order partial differential equation with fractional derevative. *Electronic Journal of Differential Equations, 21,* 1–13.

19 Luchko, Y., & Gorenflo, R. (1999). An operational method for solving fractional differential equations with the Caputo derivatives. *Acta Mathematica Vietnamica, 24,* 207–233.

20 Mamchuev, M. (2020). Cauchy problem for a linear system of ordinary differential equations of the fractional order. *Mathematics, 2020, 8,* 1475.

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Steklov problem for a linear ordinary fractional delay differential equation with the Riemann-Liouville derivative

This paper studies a nonlocal boundary value problem with Steklov's conditions of the first type for a linear ordinary delay differential equation of a fractional order with constant coefficients. The Green's function of the problem with its properties is found. The solution to the problem is obtained explicitly in terms of the Green's function. A condition for the unique solvability of the problem is found, as well as the conditions under which the solvability condition is satisfied. The existence and uniqueness theorem is proved using the representation of the Green's function and its properties, as well as the representation of the fundamental solution to the equation and its properties. The question of eigenvalues is investigated. The theorem on the finiteness of the number of eigenvalues is proved using the notation of the solution in terms of the generalized Wright function, as well as the asymptotic properties of the generalized Wright function as $\lambda \to \infty$ and $\lambda \to -\infty$.

Keywords: fractional differential equation, delay differential equation, Steklov's boundary value problem, Green function, generalized Mittag-Leffler function, generalized Wright function.

Introduction

In this paper, we consider the equation

$$D_{0t}^{\alpha} u(t) - \lambda u(t) - \mu H(t-\tau) u(t-\tau) = f(t), \quad 0 < t < 1, \tag{1}$$

where D_{0t}^{α} is the Riemann–Liouville fractional derivative [1], $1 < \alpha \leq 2$, λ, μ are the arbitrary constants, τ is the fixed positive number, $H(t)$ denotes the Heaviside function.

In [1–6], the theory of fractional calculus is studied (see also the references in these works). Barrett [7] investigated a linear ordinary differential equation of fractional order. For a fractional order differential equation the existence and uniqueness theorem is proved in [8], and the boundary value problem with the Sturm-Liouville type conditions was considered in [9]. In paper [10], the initial value problem for a linear ordinary differential equation of fractional order was studied.

To the theory of delay differential equations were devoted the following works [11–15].

The Cauchy problem for Eq.(1) was solved in [16], and the solutions to the Dirichlet and the Neumann problems were obtained in [17]. The boundary value problem with Sturm-Liouville type conditions was founded in [18].

The papers [19], [20] are devoted to the study of the Steklov problem for a fractional order differential equation. In this paper, we construct the solution to the first-type Steklov boundary value for Eq.(1) and prove the existence and uniqueness theorem and the finiteness theorem for the number of real eigenvalues of the problem under study.

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Main results

A function $u(t)$ is called *a regular solution* of equation (1) if $D_{0t}^{\alpha-2}u(t) \in C^2(0,1)$, $u(t) \in L(0,1)$ and $u(t)$ satisfies Eq. (1) for all $0 < t < 1$.

The problem we solve here is to find the regular solution to equation (1) satisfying the conditions

$$\begin{aligned} a_1 \lim_{t\to 0} D_{0t}^{\alpha-2}u(t) + a_2 \lim_{t\to 0} D_{0t}^{\alpha-1}u(t) + a_3 \lim_{t\to 1} D_{0t}^{\alpha-2}u(t) + a_4 \lim_{t\to 1} D_{0t}^{\alpha-1}u(t) = 0, \\ b_1 \lim_{t\to 0} D_{0t}^{\alpha-2}u(t) + b_2 \lim_{t\to 0} D_{0t}^{\alpha-1}u(t) + b_3 \lim_{t\to 1} D_{0t}^{\alpha-2}u(t) + b_4 \lim_{t\to 1} D_{0t}^{\alpha-1}u(t) = 0. \end{aligned} \tag{2}$$

In the case $a_2b_4 - a_4b_2 \neq 0$ the conditions (2) can be write out in the form

$$\begin{aligned} D_{0t}^{\alpha-1}u(t)\big|_{t=1} = c_1 D_{0t}^{\alpha-2}u(t)\big|_{t=0} + c_2 D_{0t}^{\alpha-2}u(t)\big|_{t=1}, \\ D_{0t}^{\alpha-1}u(t)\big|_{t=0} = c_3 D_{0t}^{\alpha-2}u(t)\big|_{t=0} + c_4 D_{0t}^{\alpha-2}u(t)\big|_{t=1}, \end{aligned}$$

where

$$c_1 = \frac{a_1b_2 - a_2b_1}{a_2b_4 - a_4b_2}, \quad c_2 = \frac{-a_2b_3 + a_3b_2}{a_2b_4 - a_4b_2}, \quad c_3 = \frac{-a_1b_4 + a_4b_1}{a_2b_4 - a_4b_2}, \quad c_4 = \frac{-a_3b_4 + a_4b_3}{a_2b_4 - a_4b_2}. \tag{3}$$

Previously, in work [21], it was defined the function

$$W_\nu(t) = W_\nu(t,\tau;\lambda,\mu) = \sum_{m=0}^{\infty} \mu^m (t - m\tau)_+^{\alpha m+\nu-1} E_{\alpha,\alpha m+\nu}^{m+1}(\lambda(t - m\tau)_+^{\alpha}), \nu \in \mathbb{R}, \tag{4}$$

where

$$E_{\alpha,\beta}^{\rho}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{\Gamma(\alpha k + \beta) k!}$$

is the generalized Mittag-Leffler function [22], $\Gamma(z)$ is the Gamma function, $(\rho)_k = \frac{\Gamma(\rho+k)}{\Gamma(\rho)}$ is the Pochhammer symbol,

$$(t - m\tau)_+ = \begin{cases} t - m\tau, & t - m\tau > 0, \\ 0, & t - m\tau \leq 0. \end{cases}$$

Function (4) satisfies the following properties [21]:

1) *for some m the expression $t - m\tau < 0$, therefore the series in (4) contains a finite number of the terms $N \leq [\frac{t}{\tau}] + 1$;*

2) it follows from (4) that

$$W_k^{(i)}(0) = \begin{cases} 0, k \neq i+1, \\ 1, k = i+1; \end{cases}$$

3) it holds true the integrodifferentional formula

$$D_{0t}^{\alpha} W_\nu(t) = W_{\nu-\alpha}(t), \quad \alpha \in \mathbb{R}, \quad \nu > 0 \tag{5}$$

and the autotransformation formula

$$W_\nu(t) = \lambda W_{\nu+\alpha}(t) + \mu W_{\nu+\alpha}(t - \tau) + \frac{t^{\nu-1}}{\Gamma(\nu)}, \qquad \alpha > 0, \quad \nu \in \mathbb{R}. \tag{6}$$

The solution to the Cauchy problem to the equation (1) was found in the paper [16] and has the form

$$u(t) = \int_0^t f(\xi) W_\alpha(t - \xi) d\xi + D_{0t}^{\alpha-1}u(t)\big|_{t=0} W_\alpha(t) + D_{0t}^{\alpha-2}u(t)\big|_{t=0} W_{\alpha-1}(t). \tag{7}$$

Using formula (7) we can define $D_{0t}^{\alpha-1}u(t)\big|_{t=1}$ and $D_{0t}^{\alpha-2}u(t)\big|_{t=1}$:

$$D_{0t}^{\alpha-1}u(t)\big|_{t=1}=\int\limits_0^1 f(\xi)W_1(1-\xi)d\xi+D_{0t}^{\alpha-1}u(t)\big|_{t=0}W_1(1)+D_{0t}^{\alpha-2}u(t)\big|_{t=0}\left[\lambda W_\alpha(1)+\mu W_\alpha(1-\tau)\right], \quad (8)$$

$$D_{0t}^{\alpha-2}u(t)\big|_{t=1}=\int\limits_0^1 f(\xi)W_2(1-\xi)d\xi+D_{0t}^{\alpha-1}u(t)\big|_{t=0}W_2(1)+D_{0t}^{\alpha-2}u(t)\big|_{t=0}W_1(1). \quad (9)$$

Inserting (8) and (9) into the first formula of the system (2), we have

$$D_{0t}^{\alpha-2}u(t)\big|_{t=0}\Big[a_1+a_3W_1(1)+a_4\Big(\lambda W_\alpha(1)+\mu W_\alpha(1-\tau)\Big)\Big]+D_{0t}^{\alpha-1}u(t)\big|_{t=0}\Big[a_2+a_3W_2(1)+a_4W_1(1)\Big]+$$

$$+\int\limits_0^1 f(\xi)\Big[a_3W_2(1-\xi)+a_4W_1(1-\xi)\Big]d\xi=0,$$

or

$$A_1D_{0t}^{\alpha-2}u(t)\big|_{t\to0}+A_2D_{0t}^{\alpha-1}u(t)\big|_{t\to0}+F_1=0,$$

where

$$A_1=a_1+a_3W_1(1)+a_4\Big(\lambda W_\alpha(1)+\mu W_\alpha(1-\tau)\Big),\quad A_2=a_2+a_3W_2(1)+a_4W_1(1),$$

$$F_1=\int\limits_0^1 f(\xi)\Big[a_3W_2(1-\xi)+a_4W_1(1-\xi)\Big]d\xi.$$

In the same way, substituting (8) and (9) into the second formula of the system (2), we have

$$B_1D_{0t}^{\alpha-2}u(t)\big|_{t=0}+B_2D_{0t}^{\alpha-1}u(t)\big|_{t=0}+F_2=0,$$

where

$$B_1=b_1+b_3W_1(1)+b_4\Big(\lambda W_\alpha(1)+\mu W_\alpha(1-\tau)\Big),\quad B_2=b_2+b_3W_2(1)+b_4W_1(1),$$

$$F_2=\int\limits_0^1 f(\xi)\Big[b_3W_2(1-\xi)+b_4W_1(1-\xi)\Big]d\xi.$$

Thus, we get the system:

$$\begin{array}{l} A_1D_{0t}^{\alpha-2}u(t)\big|_{t=0}+A_2D_{0t}^{\alpha-1}u(t)\big|_{t=0}=-F_1,\\ B_1D_{0t}^{\alpha-2}u(t)\big|_{t=0}+B_2D_{0t}^{\alpha-1}u(t)\big|_{t=0}=-F_2, \end{array} \quad (10)$$

and the solution to that system (10) equals:

$$D_{0t}^{\alpha-2}u(t)\big|_{t\to0}=\frac{-F_1B_2+F_2A_2}{A_1B_2-A_2B_1},\quad D_{0t}^{\alpha-1}u(t)\big|_{t\to0}=\frac{-A_1F_2+B_1F_1}{A_1B_2-A_2B_1}. \quad (11)$$

Using (11) and the Cauchy problem solution (7), we get the equality:

$$u(t)=\int_0^t f(\xi)W_\alpha(t-\xi)d\xi+\frac{-F_1B_2+F_2A_2}{A_1B_2-A_2B_1}W_{\alpha-1}(t)+\frac{-A_1F_2+B_1F_1}{A_1B_2-A_2B_1}W_\alpha(t)=$$

$$=\int_0^t f(\xi)W_\alpha(t-\xi)d\xi+\frac{B_1W_\alpha(t)-B_2W_{\alpha-1}(t)}{A_1B_2-A_2B_1}\int_0^1 f(\xi)[a_3W_2(1-\xi)+a_4W_1(1-\xi)]d\xi-$$

$$-\frac{A_1W_\alpha(t)-A_2W_{\alpha-1}(t)}{A_1B_2-A_2B_1}\int_0^1 f(\xi)[b_3W_2(1-\xi)+b_4W_1(1-\xi)]d\xi,$$

or

$$\int_0^1 f(\xi)\left[H(t-\xi)W_\alpha(t-\xi)+W_\alpha(t)\left(\frac{a_4B_1-b_4A_1}{\triangle}W_1(1-\xi)+\frac{a_3B_1-b_3A_1}{\triangle}W_2(1-\xi)\right)-\right.$$

$$\left.-W_{\alpha-1}(t)\left(\frac{a_4B_2-b_4A_2}{\triangle}W_1(1-\xi)+\frac{a_3B_2-b_3A_2}{\triangle}W_2(1-\xi)\right)\right],$$

where

$$\triangle=A_1B_2-A_2B_1. \tag{12}$$

Green function

Assume $G(t,\xi)$ is given by

$$G(t,\xi)=H(t-\xi)W_\alpha(t-\xi)+W_\alpha(t)\left(\frac{a_4B_1-b_4A_1}{\triangle}W_1(1-\xi)+\frac{a_3B_1-b_3A_1}{\triangle}W_2(1-\xi)\right)-$$

$$-W_{\alpha-1}(t)\left(\frac{a_4B_2-b_4A_2}{\triangle}W_1(1-\xi)+\frac{a_3B_2-b_3A_2}{\triangle}W_2(1-\xi)\right) \tag{13}$$

with λ and μ satisfying the condition (12). Here the function $W_\nu(t)$ is defined via (4).

Function $G(t,\xi)$ (13) satisfies the following properties.

1. The function $G(t,\xi)$ is continuous for all values of t and ξ from the closed interval $[0,1]$.
2. The function $G(t,\xi)$ satisfies the conditions

$$\lim_{\varepsilon\to0}[D_{0t}^{\alpha-2}G_\xi(t,\xi)|_{\xi=t+\varepsilon}-D_{0t}^{\alpha-2}G_\xi(t,\xi)|_{\xi=t-\varepsilon}]=1. \tag{14}$$

3. The function $G(t,\xi)$ is the solution to the equation

$$\partial_{1\xi}^\alpha G(t,\xi)-\lambda G(t,\xi)-\mu H(1-\tau-\xi)G(t,\xi+\tau)=0. \tag{15}$$

Here ∂_{0t}^α is the Caputo derivative [1; 11] defines as

$$\partial_{1t}^\alpha v(t)=D_{1t}^{\alpha-2}v''(t)=\frac{1}{\Gamma(2-\alpha)}\int_1^t\frac{v''(\xi)d\xi}{(t-\xi)^{\alpha-1}}.$$

4. The function $G(t,\xi)$ satisfies the boundary conditions

$$\begin{cases} \frac{d}{d\xi} D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=0} = -c_1 D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=1} + c_3 D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=0}, \\ \frac{d}{d\xi} D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=1} = c_2 D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=1} - c_4 D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=0}. \end{cases} \tag{16}$$

Here the coefficients c_1, c_2, c_3, c_3 were defined via formula (3).

These properties obviously are implied from formula (13), condition (12) and the relations (5), (6).

The function $G(t,\xi)$ that possesses properties 1–4 is called Green function for problem (1), (2).

Existence and uniqueness theorem

Theorem 1. Assume the function $f(t) \in L(0,1) \cap C(0,1)$ and the condition (12) is satisfied. Then there exists a regular solution to problem (1), (2) in the form of

$$u(t) = \int_0^1 f(\xi) G(t,\xi) d\xi \tag{17}$$

and the solution to problem (1), (2) is unique if and only if condition (12) is satisfied.

Proof. We show that it holds true the representation of the solution to problem (1), (2) in the form (17). For this, we multiply both sides of Eq. (1) (given in terms of variable ξ) by $D_{0t}^{\alpha-2} G(t,\xi)$ and integrate it with respect to variable ξ ranging from ε to $1-\varepsilon$ ($\varepsilon \to 0$):

$$\int_\varepsilon^{1-\varepsilon} D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha} u(\xi) d\xi - \lambda \int_\varepsilon^{1-\varepsilon} u(\xi) D_{0t}^{\alpha-2} G(t,\xi) d\xi -$$

$$-\mu \int_\varepsilon^{1-\varepsilon} H(t-\tau) u(\xi-\tau) D_{0t}^{\alpha-2} G(t,\xi) d\xi = \int_\varepsilon^{1-\varepsilon} f(\xi) D_{0t}^{\alpha-2} G(t,\xi) d\xi, \quad 0 < t < 1. \tag{18}$$

Integrate by parts the first term of equality (18):

$$\int_\varepsilon^{1-\varepsilon} D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha} u(\xi) d\xi = D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha-1} u(\xi)\Big|_\varepsilon^{1-\varepsilon} - \int_\varepsilon^{t-\varepsilon} \frac{d}{d\xi} D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha-1} u(\xi) d\xi -$$

$$-\int_{t+\varepsilon}^{1-\varepsilon} D_{0t}^{\alpha-2} G_\xi(t,\xi) D_{0\xi}^{\alpha-1} u(\xi) d\xi = D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha-1} u(\xi)\Big|_\varepsilon^{1-\varepsilon} - \frac{d}{dx} D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha-2} u(\xi)\Big|_\varepsilon^{t-\varepsilon} -$$

$$-\frac{d}{dx} D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha-2} u(\xi)\Big|_{t+\varepsilon}^{1-\varepsilon} + \int_\varepsilon^{1-\varepsilon} D_{0t}^{\alpha-2} G_{\xi\xi}(t,\xi) D_{0\xi}^{\alpha-2} u(\xi) d\xi =$$

$$= D_{0t}^{\alpha-2} u(t) \left[\frac{d}{dx} D_{0t}^{\alpha-2} G(t,\xi)\Big|_{t+\varepsilon} - \frac{d}{dx} D_{0t}^{\alpha-2} G(t,\xi)\Big|_{t-\varepsilon} \right] + \int_\varepsilon^{1-\varepsilon} D_{0t}^{\alpha-2} G_{\xi\xi}(t,\xi) D_{0\xi}^{\alpha-2} u(\xi) d\xi +$$

$$+ D_{0\xi}^{\alpha-2} u(\xi)\Big|_{\xi=0} \left[c_1 D_{0t}^{\alpha-2} G(t,\xi)\Big|_{\xi=1-\varepsilon} - c_3 D_{0t}^{\alpha-2} G(t,\xi)\Big|_{\xi=\varepsilon} + \frac{d}{d\xi} D_{0t}^{\alpha-2} G(t,\xi)\Big|_{\xi=\varepsilon} \right] +$$

$$+D_{0\xi}^{\alpha-2}u(\xi)\Big|_{\xi=1}\left[c_2D_{0t}^{\alpha-2}G(t,\xi)\Big|_{\xi=1-\varepsilon}-c_4D_{0t}^{\alpha-2}G(t,\xi)\Big|_{\xi=\varepsilon}-\frac{d}{d\xi}D_{0t}^{\alpha-2}G(t,\xi)\Big|_{\xi=1-\varepsilon}\right].$$

Using properties of the Green function (14) and (16), we get the identity

$$\int\limits_{\varepsilon}^{1-\varepsilon} D_{0t}^{\alpha-2}G(t,\xi)D_{0\xi}^{\alpha}u(\xi)d\xi = D_{0t}^{\alpha-2}u(t)+\int\limits_{\varepsilon}^{1-\varepsilon} D_{0t}^{\alpha-2}G_{\xi\xi}(t,\xi)D_{0\xi}^{\alpha-2}u(\xi)d\xi. \tag{19}$$

In the third integral of the equality (18) we replace ξ into $\xi-\tau$:

$$\int\limits_0^1 H(\xi-\tau)u(\xi-\tau)G(t,\xi)d\xi = \int\limits_0^1 H(1-\tau-\xi)u(\xi)G(t,\xi+\tau)d\xi. \tag{20}$$

Substituting (19) and (20) into Eq. (18) and using the formula for fractional integration by parts [20; 15]

$$\int\limits_a^b g(s)D_{as}^{\alpha}h(s)ds = \int\limits_a^b h(s)D_{bs}^{\alpha}g(s)ds,$$

we have the identity

$$D_{0t}^{\alpha-2}u(\xi)+D_{0t}^{\alpha-2}\int\limits_0^1 u(\xi)\left[D_{1\xi}^{\alpha-2}G_{\xi\xi}(t,\xi)-\lambda G(t,\xi)-\mu H(1-t-\xi)G(t,\xi+\tau)\right]d\xi =$$

$$= D_{0t}^{\alpha-2}\int\limits_0^1 f(\xi)G(t,\xi)d\xi,$$

and,using the Green function property (15) and finding the derivative of order $D_{0t}^{2-\alpha}$ we get solution (17).

Next, we show that the function (17) is the solution to equation (1). Formula (17) can be written out in the form of bellow:

$$u(t)=\nu_1+\nu_2+\nu_3,$$

where

$$\nu_1=\int\limits_0^t f(\xi)W_{\alpha}(t-\xi)d\xi, \quad \nu_2=\frac{-F_1B_2+F_2A_2}{A_1B_2-A_2B_1}W_{\alpha-1}(t), \quad \nu_3=\frac{-A_1F_2+B_1F_1}{A_1B_2-A_2B_1}W_{\alpha}(t).$$

Denote $D_{0t}^{\alpha}\nu_1$, $D_{0t}^{\alpha}\nu_2$ and $D_{0t}^{\alpha}\nu_3$. We have

$$D_{0t}^{\alpha}\nu_1=\frac{d}{dt}D_{0t}^{\alpha-1}\int\limits_0^t f(\xi)W_{\alpha}(t-\xi)d\xi=\frac{d}{dt}\int\limits_0^t f(\xi)W_1(t-\xi)d\xi=$$

$$=\int\limits_0^t f(\xi)\frac{d}{dt}(\lambda W_{\alpha+1}(t-\xi)+\mu W_{\alpha+1}(t-\xi-\tau))d\xi+f(t)=\int\limits_0^t f(\xi)(\lambda W_{\alpha}(t-\xi)+\mu W_{\alpha}(t-\xi-\tau))d\xi+f(t);$$

$$D_{0t}^{\alpha}\nu_2=\frac{-F_1B_2+F_2A_2}{A_1B_2-A_2B_1}\frac{d^2}{dt^2}D_{0t}^{\alpha-2}W_{\alpha-1}(t)=\frac{-F_1B_2+F_2A_2}{A_1B_2-A_2B_1}\frac{d^2}{dt^2}W_1(t)=$$

$$= \frac{-F_1B_2 + F_2A_2}{A_1B_2 - A_2B_1}\frac{d^2}{dt^2}(\lambda W_{\alpha+1}(t) + \mu W_{\alpha+1}(t-\tau)) = \frac{-F_1B_2 + F_2A_2}{A_1B_2 - A_2B_1}(\lambda W_{\alpha-1}(t) + \mu W_{\alpha-1}(t-\tau))\,;$$

$$D^{\alpha}_{0t}\nu_3 = \frac{-A_1F_2 + B_1F_1}{A_1B_2 - A_2B_1}\frac{d}{dt}D^{\alpha-}_{0t}W_\alpha(t) = \frac{-A_1F_2 + B_1F_1}{A_1B_2 - A_2B_1}\frac{d}{dt}(\lambda W_{\alpha+1}(t) + \mu W_{\alpha+1}(t-\tau)) =$$

$$= \frac{-A_1F_2 + B_1F_1}{A_1B_2 - A_2B_1}(\lambda W_\alpha(t) + \mu W_\alpha(t-\tau)).$$

Next, using formulas (5), (6) we obtain by the previous relation that

$$D^{\alpha}_{0t}u(t) = f(t) + \lambda\int_0^1 f(\xi)G(t,\xi)d\xi + \mu\int_0^1 f(\xi)G(t,\xi-\tau)d\xi,$$

that is that (17) satisfies (1).

Remark. For $\lambda = 0, \mu > 0$ and

$$a_1 > b_1, \quad a_2 < b_2, \quad a_3 = b_3, \quad a_4 = b_4$$

condition (12) is always satisfied.

On the finiteness of the number of real eigenvalues

Definition. The eigenvalues of problem (1), (2) are the values λ, such that problem (1), (2) has a regular solution that is not the identically zero.

The set of real eigenvalues for problem (1), (2) coincides with the set of real zeros for the function

$$\Phi(\lambda) = \Big[a_1 + a_3W_1(1) + a_4\Big(\lambda W_\alpha(1) + \mu W_\alpha(1-\tau)\Big)\Big]\Big[B_2 = b_2 + b_3W_2(1) + b_4W_1(1)\Big] -$$

$$-\Big[a_2 + a_3W_2(1) + a_4W_1(1)\Big]\Big[b_1 + b_3W_1(1) + b_4\Big(\lambda W_\alpha(1) + \mu W_\alpha(1-\tau)\Big)\Big]. \tag{21}$$

Theorem 2. Problem (1), (2) has only a finite number of real eigenvalues.

The function $W_\nu(\lambda)$ can be written out as [3; 45]

$$W_\nu(1,\tau;\lambda,\mu) = \sum_{m=0}^{\infty}\frac{\mu^m}{m!}(1-m\tau)_+^{\alpha m+\nu-1}\,{}_1\Psi_1\left[\begin{array}{c}(m+1,1)\\(\alpha m+\nu,\alpha)\end{array}\middle|\,\lambda(1-m\tau)_+^{\alpha}\right], \tag{22}$$

where

$${}_p\Psi_q\left[\begin{array}{c}(a_l,\alpha_l)_{1,p}\\(b_l,\beta_l)_{1,q}\end{array}\middle|\,z\right] = \sum_{k=0}^{\infty}\frac{\prod_{l=1}^{p}\Gamma(a_l+\alpha_lk)}{\prod_{l=1}^{q}\Gamma(b_l+\beta_lk)}\frac{z^k}{k!}$$

is the generalized Wright function [23].

As $\lambda \to +\infty$ the following asymptotic formula holds true for the generalized Wright function [23], [24]:

$${}_1\Psi_1\left[\begin{array}{c}(m+1,1)\\(\alpha m+\nu,\alpha)\end{array}\middle|\,\lambda(1-m\tau)_+^{\alpha}\right] = \alpha^{-m}\lambda^{\frac{m(1-\alpha)-\nu+1}{\alpha}}(1-m\tau)_+^{m(1-\alpha)-\nu+1}e^{\lambda^{1/\alpha}(1-m\tau)_+}\left[1 + O\left(\frac{1}{\lambda^{\frac{1}{\alpha}}}\right)\right],$$

and the asymptotic formula for the generalized Wright function as $\lambda \to -\infty$ has form [23], [24]

$${}_1\Psi_1\left[\begin{array}{c}(m+1,1)\\(\alpha m+\nu,\alpha)\end{array}\middle|\,\lambda(1-m\tau)_+^{\alpha}\right] = \sum_{l=0}^{n}\frac{(-1)^{m+l+1}(l+m)!(1-m\tau)_+^{-\alpha(m+l+1)}}{|\lambda|^{m+l+1}\Gamma(\nu-\alpha-\alpha l)(m+l+1)!} + O\left(\frac{1}{|\lambda|^m}\right).$$

Let N be the maximum value of m that satisfies the inequality $(1 - m\tau) > 0$. From these formulas we get the asymptotic formulas for function (22) as $\lambda \to +\infty$

$$W_\nu(1,\tau;\lambda,\mu) = \sum_{m=0}^{N} \frac{\mu^m \alpha^{-m}}{m!} \lambda^{\frac{m(1-\alpha)-\nu+1}{\alpha}} (1 - m\tau)_+^m e^{\lambda^{\frac{1}{\alpha}}(1-m\tau)_+} \left[1 + O\left(\frac{1}{\lambda^{\frac{1}{\alpha}}}\right)\right], \qquad (23)$$

and $\lambda \to -\infty$ in the form

$$W_\nu(1,\tau;\lambda,\mu) = \sum_{m=0}^{N} \mu^m \sum_{l=0}^{n} \frac{(-1)^{m+l+1}(m+1)_l (1 - m\tau)_+^{-\alpha(l+1)+\nu-1}}{|\lambda|^{m+l+1}\Gamma(\nu - \alpha - \alpha l)(m + l + 1)!} + O\left(\frac{1}{|\lambda|^m}\right). \qquad (24)$$

From the representation (21) and asymptotic formula (23) we see that letting $\lambda \to \infty$ the function (21) increases without limit.

As $\lambda \to -\infty$, since $\Phi(\lambda)$ is an entire function of the variable λ, it follows from asymptotic formula (24) that the function (21) may have only a finite number of real zeros.

References

1 Нахушев А.М. Дробное исчисление и его применение / А.М. Нахушев. — М.: Физматлит, 2003. — 272 с.

2 Podlubny I. Fractional differential equations / I. Podlubny. — AP, New York, 1999. — 341 p.

3 Kilbas A.A. Theory and applications of factional differential equations / A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. — Elsevier, Amsterdam, 2006. — 523 p.

4 Псху А.В. Уравнения в частных производных дробного порядка / А.В. Псху. — М.: Наука, 2005. — 199 с.

5 Oldham K.B. The fractional calculus / K.B. Oldham, J. Spanier. — N.-Y. L.: AP, 1974. — 234 p.

6 Miller K.S. An introduction to the fractional calculus and fractional differential equations / K.S. Miller, B. Ross. — John Wiley & sons, New York, 1993. — 366 p.

7 Barrett J.H. Differential equation of non-integer order / J.H. Barrett // Canad. J. Math. — 1954. — 6. — No. 4. — P. 529–541.

8 Джрбашян М.М. Дробные производные и задача Коши для дифференциальных уравнений дробного порядка / М.М. Джрбашян, А.Б. Нерсесян // Изв. Акад. наук АрмянССР. — 1968. — 3. — № 1. — С. 3–28.

9 Джрбашян М.М. Краевая задача для дифференциального оператора дробного порядка типа Штурма-Лиувилля / М.М. Джрбашян // Изв. Акад. наук АрмянССР.

10 Pskhu A.V. Initial-value problem for a linear ordinary differential equation of noninteger order / A.V. Pskhu // Sbornik: Mathematics. — 2011. — 202. — No. 4. — P. 571–582.

11 Норкин С.Б. О решениях линейного однородного дифференциального уравнения второго порядка с запаздывающим аргументом / С.Б. Норкин // Успехи мат. наук. — 1959. — 14. — № 1. — С. 199–206.

12 Bellman R.E. Differential-Difference Equations / R.E. Bellman, K.L. Cooke. — Acad. Press, New York–London, 1963. — 462 p.

13 Эльсгольц Л.Э. Введение в теорию дифференциальных уравнений с запаздывающим аргументом / Л.Э. Эльсгольц, С.Б. Норкин. — М.: Наука, 1971. — 296 с.

14 Мышкис А.Д. Линейные дифференциальные уравнения с запаздывающим аргументом / А.Д. Мышкис. — М.: Наука, 1972. — 352 с.

15 Hale J.K. Introduction to Functional Differential Equations / J.K. Hale, S.M.V. Lunel. — Springer, New-York, 1993. — 448 p.

16 Мажгихова М.Г. Задача Коши для обыкновенного дифференциального уравнения с оператором Римана–Лиувилля с запаздывающим аргументом / М.Г. Мажгихова // Изв. Кабард.-Балкар. научн. центра РАН. — 2017. — 1. — № 75. — С. 24–28.

17 Мажгихова М.Г. Краевые задачи для линейного обыкновенного дифференциального уравнения дробного ряда с запаздывающим аргументом / М.Г. Мажгихова // Сиб. электрон. мат. изв. — 2018. — 15. — С. 685–695.

18 Mazhgikhova M.G. Green function method for a fractional–order delay differential equation / M.G. Mazhgikhova // Bulletin of the Karaganda university. Mathematics series. — 2020. — 97. — No. 1. — P. 87–96.

19 Efendiev B.I. Steklov problem for a second-order ordinary differential equation with a continual derivative / B.I. Efendiev // Differential Equations. — 2013. — 49. — No. 4. — P. 450–456.

20 Мажгихова М.Г. Задача Стеклова первого класса для дифференциального уравнения дробного порядка с запаздывающим аргументом / М.Г. Мажгихова // Вестн. КРАУНЦ. Физ.-мат. науки. — 2021. — 12. — № 1. — С. 5–12.

21 Mazhgikhova M.G. Dirichlet problem for a fractional-order ordinary differential equation with retarded argument / M.G. Mazhgikhova // Differential equations. — 2018. — 54. — No. 2. — P. 187–194.

22 Prabhakar T.R. A singular integral equation with a generalized Mittag-Leffler function in the kernel / T.R. Prabhakar // Yokohama Math. J. — 1971. — 19. — P. 7–15.

23 Wright E.M. The asymptotic expansion of the generalized hypergeometric function / E.M. Wright // J. London Math. Soc. — 1935. — 10. — P. 286–293.

24 Wright E.M. The asymptotic expansion of the generalized hypergeometric function / E.M. Wright // Proc. London Math. Soc. — 1940. — 46. — No. 2. — P. 389–408.

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Аргументі кешігетін Риман-Лиувилль бөлшек туындысы бар сызықтық қарапайым дифференциалдық теңдеу үшін Стеклов есебі

Мақалада тұрақты коэффициенттері бар аргументі кешігетін бөлшек ретті сызықты қарапайым дифференциалдық теңдеу үшін бірінші типті Стеклов шарттарымен жергілікті емес шеттік есептер зерттелген. Грин функциясы табылып, оның қасиеттері дәлелденді. Зерттелетін есептің шешімі Грин функциясы тұрғысынан айқын түрде алынды. Есептің бірегей шешілу шарты, сондай-ақ шешілу шарты сөзсіз орындалатын шарттар табылды. Бар болу және жалғыздық теоремасы дәлелденді. Теорема Грин функциясын және оның қасиеттерін, сондай-ақ теңдеудің іргелі шешімін және оның қасиеттерін көрсету арқылы дәлелденген. Меншікті мәндер сұрағы зерттелді. Зерттелетін есеп нақты меншікті мәндердің шектеулі санына ғана ие болуы мүмкін екендігі теоремамен дәлелденді. Теорема шешімнің жалпыланған Райт функциясы тұрғысынан белгіленуді қолданып, сондай-ақ $\lambda \to \infty$ және $\lambda \to -\infty$ үшін жалпыланған Райт функциясының асимптотикалық қасиеттері арқылы дәлелденді.

Кілт сөздер: бөлшек ретті сызықты дифференциалдық теңдеу, аргументі кешігетін дифференциалдық теңдеу, Стеклов шеттік есебі, Грин функциясы, Миттаг-Леффлераның жалпыланған функциясы, Райттың жалпыланған функциясы.

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Задача Стеклова для линейного обыкновенного дифференциального уравнения с дробной производной Римана–Лиувилля с запаздывающим аргументом

В статье исследована нелокальная краевая задача с условиями Стеклова первого типа для линейного обыкновенного дифференциального уравнения дробного порядка с запаздывающим аргументом с постоянными коэффициентами. Найдена функция Грина и доказаны ее свойства. Решение исследуемой задачи получено в явном виде в терминах функции Грина. Найдено условие однозначной разрешимости задачи, а также условия, при которых условие разрешимости заведомо выполняется. Доказана теорема существования и единственности, с использованием представления функции Грина, ее свойств, а также фундаментального решения уравнения и ее свойств. Исследован вопрос о собственных значениях. Доказана теорема о том, что исследуемая задача может иметь только конечное число действительных собственных значений. Теорема доказана с применением записи решения в терминах обобщенной функции Райта, а также асимптотических свойств обобщенной функции Райта при $\lambda \to \infty$ и $\lambda \to -\infty$.

Ключевые слова: дифференциальное уравнение дробного порядка, дифференциальное уравнение с запаздывающим аргументом, краевая задача Стеклова, функция Грина, обобщенная функция Миттаг-Леффлера, обобщенная функция Райта.

References

1 Nakhushev, A.M. (2003). *Drobnoe ischislenie i ego primenenie [Fractional calculus and its application]*. Moscow: Fizmatlit [in Russian].

2 Podlubny, I. (1999). *Fractional differential equations*. New York: AP.

3 Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and applications of fractional differential equations*. Amsterdam: Elsevier.

4 Pskhu, A.V. (2005). *Uravneniia v chastnykh proizvodnykh drobnogo poriadka [Partial differential equations of fractional order]*. Moscow: Nauka [in Russian].

5 Oldham, K.B., & Spanier, J. (1974). *The fractional calculus*. N.-Y. L.: Acad. press.

6 Miller, K.S., & Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations*. New York: John Wiley & sons.

7 Barrett, J.H. (1954). Differential equation of non-integer order. *Canad. J. Math., 6*(4), 529–541.

8 Dzhrbashyan, M.M., & Nersesyan, A.B. (1968). Drobnye proizvodnye i zadacha Koshi dlia differentsialnykh uravnenii drobnogo poriadka [Fractional derivatives and the Cauchy problem for differential equations of fractional order]. *Izvestiia Akademii nauk Armianskoi SSR – Reports of the Academy of Sciences of the Armenian SSR, 3*(1), 3–28 [in Russian].

9 Dzhrbashyan, M.M. (1970). Kraevaia zadacha dlia differentsialnogo operatora drobnogo poriadka tipa Shturma-Liuvillia [Boundary problem for fractional differential operator of Sturm–Liouville type]. *Izvestiia Akademii nauk Armianskoi SSR – Reports of the Academy of Sciences of the Armenian SSR, 5*(2), 71–96 [in Russian].

10 Pskhu, A.V. (2011). Initial-value problem for a linear ordinary differential equation of noninteger order. *Sbornik: Mathematics, 202*(4), 571–582.

11 Norkin, S.B. (1959). O resheniiakh lineinogo odnorodnogo differentsialnogo uravneniia vtorogo poriadka s zapazdyvaiushchim argumentom [About the solutions to a linear homogeneous second order differential equation with delay]. *Uspekhi matematicheskikh nauk – Advances in Mathematical Sciences, 14*(1), 199–206 [in Russian].

12 Bellman, R.E., & Cooke, K.L. (1963). *Differential-Difference Equations.* New York–London: Acad. Press.

13 Elsgolts, L.E., & Norkin, S.B. (1971). *Vvedenie v teoriiu differentsialnykh uravnenii s zapazdyvaiushchim argumentom [Introduction to the theory of differential equations with delay].* Moscow: Nauka [in Russian].

14 Myshkis, A.D. (1972). *Lineinye differentsialnye uravneniia s zapazdyvaiushchim argumentom [Linear differential equations with delay].* Moscow: Nauka [in Russian].

15 Hale, J.K., & Lunel, S.M.V. (1993). *Introduction to Functional Differential Equations.* New-York: Springer.

16 Mazhgikhova, M.G. (2017). Zadacha Koshi dlia obyknovennogo differentsialnogo uravneniia s operatorom Rimana–Liuvillia s zapazdyvaiushchim argumentom [Cauchy problem for an ordinary delay differential equation with a Riemann–Liouville operator]. *Izvestiia Kabardino-Balkarskogo nauchnogo tsentra RAN – Proceedings of the Kabardino-Balkarian Scientific Center of the RAS, 75*(1), 24–28 [in Russian].

17 Mazhgikhova, M.G. (2018). Kraevye zadachi dlia lineinogo obyknovennogo differentsialnogo uravneniia drobnogo riada s zapazdyvaiushchim argumentom [Boundary value problems for a linear ordinary differential equation of fractional order with delay]. *Sibirskie elektronnye matematicheskie izvestiia – Siberian Electronic Mathematical Reports, 15*, 685–695 [in Russian].

18 Mazhgikhova, M.G. (2020). Green function method for a fractional–order delay differential equation. *Bulletin of the Karaganda University. Mathematics series, 97*(1), 87–96.

19 Efendiev, B.I. (2013). Steklov problem for a second-order ordinary differential equation with a continual derivative. *Differential Equations, 49*(4), 450–456.

20 Mazhgikhova, M.G. (2021). Zadacha Steklova pervogo klassa dlia differentsialnogo uravneniia drobnogo poriadka s zapazdyvaiushchim argumentom [Steklov problem of the first class for a fractional order delay differential equation]. *Bulletin KRASEC. Physical and Mathematical Sciences, 1*(12), 5–12 [in Russian].

21 Mazhgikhova, M.G. (2018). Dirichlet problem for a fractional-order ordinary differential equation with retarded argument. *Differential equations, 54*(2), 187–194.

22 Prabhakar, T.R. (1971). A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math. J., 19*, 7–15.

23 Wright, E.M. (1935). The asymptotic expansion of the generalized hypergeometric function. *J. London Math. Soc. 10*, 286–293.

24 Wright, E.M. (1940). The asymptotic expansion of the generalized hypergeometric function. *Proc. London Math. Soc., 46*(2), 389–408.

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Existentially positive Mustafin theories of S-acts over a group

The paper is connected with the study of Jonsson spectrum notion of the fixed class of models of S-acts signature, assuming a group as a monoid of S-acts. The Jonsson spectrum notion is effective when describing theoretical-model properties of algebras classes whose theories admit joint embedding and amalgam properties. It is usually sufficient to consider universal-existential sentences true on models of this class. Up to the present paper, the Jonsson spectrum has tended to deal only with Jonsson theories. The authors of this study define the positive Jonsson spectrum notion, the elements of which can be, non-Jonsson theories. This happens because in the definition of the existentially positive Mustafin theories considered in a given paper involve not only isomorphic embeddings, but also immersions. In this connection, immersions are considered in the definition of amalgam and joint embedding properties. The resulting theories do not necessarily have to be Jonsson. We can observe that the above approach to the Jonsson spectrum study proves to be justified because even in the case of a non-Jonsson theory there exists regular method for finding such Jonsson theory that satisfies previously known notions and results, but that will also be directly related to the existentially positive Mustafin theory in question.

Keywords: Jonsson theory, perfect Jonsson theory, positive model theory, Jonsson spectrum, positive Jonsson theory, immersion, S-acts, Jonsson S-acts theory, $\exists PM$-theory, cosemanticity.

Introduction

This study is a continuation of previous works by the first two authors of the given paper, related to the study of the theoretical-model properties of positive Jonsson theories [1–5] and Jonsson spectrum of models classes of fixed signature [6–8]. Note that the Jonsson theories form a subclass of inductive theories and, by virtue of their definition, are not, complete. However, they distinguish a rather wide class of classical algebras, such as groups, abelian groups, fixed characteristic fields, Boolean algebras, S-acts, etc. More information about Jonsson theories can be found in [9–17]. The famous American mathematician J. Keisler in his article [18] has conventionally allocated two directions of Model Theory, «western» and «eastern», the names of which are connected with the geographical place of residence of two different directions founders of the model theory A. Robinson and A. Tarsky. It can be noted that the «Western» model theory predominantly studies complete theories and the «Eastern» Jonsson theories and each direction has its own special concepts and methods. In Jonsson writings [19, 20], classes of models of an arbitrary signature satisfying certain well-known theoretical-model and algebraic properties, in the study of which the notion of Jonsson theory has emerged originally, have been defined [18; 80]. It is clear that Jonsson theories define a class of incomplete theories and the interest in studying such theories is also fuelled by the difference between the definitions of the «Western» and «Eastern» model theories concerning the notions of model's universality and homogeneity. In consequence of this difference, which was first noticed by E.A. Palyutin [21], T.G. Mustafin has identified perfect Jonsson theories that eliminate this difference. Subsequently, T.G. Mustafin defined and studied the generalised Jonsson theories [22] and using the technique defined in this direction. In paper [22], he described generalised Jonsson theories of Boolean algebras. In a further study of Jonsson theories, several new classes of positive Jonsson theories were defined [23–25]. Interest in positivity theory arose after the appearance of the works [26–28]. In these works, it was shown that the whole classical first-order model

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theory is a special case of the positive model theory defined under these works. Subsequently, in this framework of positivity [28], there were identified positive Jonsson theories [3].

In the present paper, we do not go into positivity in the sense of work [3], we remain in the first-order model theory framework, but generalise the concepts of those classes of theories that have been considered in [22–25]. Let us focus on a brief description of some key concepts from the works [26, 27], which we need in this paper. Namely, the notions of minimal fragment and those morphisms that coincide with and are used in the positive Jonsson theories study in the works [23–25].

The main result of the paper is related to the study of properties of the positive Jonsson spectrum of a S-acts theory models' class over a group. Interest in the study of the theoretical-model aspects of S-acts theory has arisen relatively recently and is related to the works of W. Gould [29] and T.G. Mustafin [30, 31]. In work [31], T.G. Mustafin proved the fact that any complete theory is similar in some sense to some S-acts theory. Jonsson theories are also closely related to S-acts theory. Thus, in paper [32], there has been derived a connection between an existentially complete perfect Jonsson theory and some Jonsson S-acts theory. In paper [33], a description of Jonsson S-acts theories over a group was obtained. This paper obtains results generalizing the results from [33] as part of a positive spectrum study of $\exists PM$-theories of S-acts over a group.

1 Necessary concepts and results of positive model theory

Let us recall the basic definitions of the positive logic concepts and the results obtained in [26, 27].

A positive fragment (in L) is a subset $\Delta \subseteq L$ containing all atomic formulas and closed with respect to variable substitution, positive Boolean combinations and subformulas. For a given Δ the following sets of formulas are defined:

$$\Sigma = \Sigma(\Delta) = \{\exists y \varphi(x, y) : \varphi \in \Delta\},$$

$$\Pi = \Pi(\Delta) = \{\forall y \varphi(x, y) : \varphi \in \Delta\} = \{\neg\psi : \psi \in \Sigma(\Delta)\}.$$

Definition 1. ([27]) Let M and N be a structures of the language, $A \subseteq M$ and $f : A \to N$ be a map (that is, $f : M \to N$ is a partial map with $dom(f) = A$). Then f is a partial Δ-homomorphism if for every $a \in A$ and every formula $\varphi(x) \in \Delta$ from $M \models \varphi(a)$ follows that $N \models \varphi(f(a))$.

If $dom(f) = M$, then $f : M \to N$ is a Δ-homomorphism; if $M = N$, then f is a (partial) endomorphism.

Definition 2. ([26]) A Π-theory is a set of Π-sentences, closed with respect to deducibility.

Definition 3. ([27]) Let κ be a relatively large cardinal (at least $\kappa > |\Delta|$), and U the structure of the language. Then U is κ-universal domain if it satisfies the following properties:

1) κ-homogeneity: Let $f : U \to U$ be partial endomorphism U, and suppose that $|dom(f)| < \kappa$. Then f extends to automorphism U.

2) κ-compactness: Let $\Gamma \subset \Delta$ such that $|\Gamma| < \kappa$ and suppose that every finite subset of the set Γ is realizable in U. Then Γ is realizable in U.

Definition 4. ([26]) A model $M \models T$ is existentially closed if every Δ-homomorphism $f : M \to N$ such that $N \models T$, is a Σ-embedding.

Definition 5. ([27]) Let U be a universal domain and $T = Th_{\Pi}(U)$. Then we say that U is a universal domain for T.

Definition 6. ([27]) Π-theory T is complete if it is equal to $Th_{\Pi}(M)$ for some structure M of the language of the theory T.

If T is not complete, then the completion of the theory T is a minimal (with respect to inclusion) complete Π-theory containing T. In this case, a universal domain of the theory T is any universal domain of its extensions, i.e., a universal domain whose Π-theory is a complement of the theory T.

Lemma 1. ([27]) Let T be Π-theory. Then for every model $M \models T$ there exists an existentially closed model N and morphism $M \to N$.

Theorem 1. ([27])

1) The completion of the Π-theory are exactly the Π-theories of its various existentially closed models.

2) A Π-theory is positive Robinson if and only if all its completions are positive Robinson.

3) A complete Π-theory is positive Robinson if and only if it has a universal domain.

Theorem 2. ([27]) For Π-theory T the following conditions are equivalent:

1. T is positive Robinson theory.
2. The class of existentially closed models of theory T is axiomatic.

2 Existentially positive Mustafin theories and their properties

Let us define the notion of existentially positive Mustafin theory ($\exists PM$-theory). The main difference of this concept from the classical notion of the theory is that only positive sentences are involved in the axioms defining the theory. Thus, this class of theories is persistent with respect to homomorphisms. If at some fixed Δ, the considered $\exists PM$-theory is Jonsson in the classical sense, then we apply to it all notations and results known earlier, e.g., as in [9].

Let L be a first-order language, At be the set of atomic formulas of L, $B^+(At)$ be the closed set of relatively positive Boolean combinations (conjunctions and disjunctions) of all atomic formulas, their subformulas and substitution of variables. $Q(B^+(At))$ is the set of formulas in prenex normal form obtained by applying quantifiers ($\forall$ and $\exists$) to $B^+(At)$. We call a formula positive if it belongs to the set $Q(B^+(At)) = L^+$. A theory is called positively axiomatizable if its axioms are positive. $B(L^+)$ is an arbitrary Boolean combination of formulas from L^+. It is easy to see that $\Pi(\Delta) \subseteq B(L^+)$ when $\Delta = B^+(At)$, where $\Pi(\Delta)$ is such as described earlier.

Following [26, 27] define Δ-morphisms between structures.

Let M and N be structures of the language, $\Delta \subseteq B(L^+)$. A map $h : M \to N$ is called Δ-homomorphism (symbolically $h : M \to_\Delta N$) if for any $\varphi(\overline{x}) \in \Delta$, $\forall \overline{a} \in M$ from the fact that $M \models \varphi(\overline{a})$, it follows that $N \models \varphi(h(\overline{a}))$. The model M is called the beginning in N and we say that M continues in N, with $h(M)$ called the continuation of M. If the map h is injective, then we say that the map h immerses M into N (symbolically $h : M \leftrightarrow_\Delta N$).

Hereafter we will use the term Δ-extension and Δ-immersion. Within this definition (Δ-homomorphism), it is easy to see that isomorphic embedding and elementary embedding are Δ-imbeddings when $\Delta = B(At)$ and $\Delta = L$, correspondingly.

Definition 7. If C is a class of L-structures, then we note that an element M of C is Δ-positively existentially closed in C if every Δ-homomorphism from M to any element of C is Δ-immersion. We denote the class of all Δ-positively existentially closed models by $(E_C^\Delta)^+$; if $C = ModT$ for some theory T, then by E_T, $(E_T^\Delta)^+$ we mean respectively the class of existentially closed and Δ-positively existentially closed models of that theory. If $\Delta = L$ we obtain a class of positively existentially closed models of this theory and denote it by E_T^+.

Hereinafter throughout the paper $\Delta = B^+(At)$ and in the case where the considered theory is not Jonsson due to the considered positivity (since, n-immersion is not the same as n-embedding), we will use the universal domain from [26] instead of the semantic model considered theory. $\Delta = B^+(At)$, consistent with the above definitions, satisfies the minimal fragment from [26] and is consistent with the definition of $\exists PM$-theory.

Let $0 \leq n \leq \omega$. Π_n^+-formula be a formula of language L^+ whose prenex normal form has n variable quantifiers and begins with $\forall$-quantifier. Similarly, Σ_n^+-formula is a formula of L^+ whose prenex normal form has n variable quantifiers and begins with quantifier $\exists$.

Definition 8. Model A of theory T will be called positively existentially closed with respect to Σ_n-formulas if $\forall\varphi(x) \in \Sigma_n^+$, $\forall a \in A$, for any model $B \supset A$, from the fact that $B \models \varphi(a)$ follows that $A \models \varphi(a)$.

The set of all positive existentially closed with respect to Σ_n-formulas of models of the theory T we will denote as ${}_nE_T^+$.

Definition 9. We consider that theory T admits $\exists_n JEP$, if for any two $A, B \in {}_nE_T^+$ there exists $C \in {}_nE_T^+$ and Δ-homomorphisms $h_1 : A \to_\Delta C$, $h_2 : B \to_\Delta C$.

Definition 10. We say that theory T admits $\exists_n AP$, if for any $A, B, C \in {}_nE_T^+$ such that $h_1 : A \to_\Delta C$, $g_1 : A \to_\Delta B$, where h_1, g_1 are Δ-homomorphisms, there exists $D \in {}_nE_T^+$ and $h_2 : C \to_\Delta D$, $g_2 : B \to_\Delta D$, where h_2, g_2 are Δ-homomorphisms, such that $h_2 \circ h_1 = g_2 \circ g_1$.

If we consider only Δ-immersions as Δ-homomorphisms, then we get the definition of the so-called $\exists PM$-theory.

Definition 11. Let $0 \leq n \leq \omega$. The theory T is called an existentially positive Mustafin ($\exists PM$-theory) if

1) the theory T has infinite models,

2) theory T is Π_{n+2}^+-axiomatizable,

3) theory T admits $\exists_n JEP$,

4) theory T admits $\exists_n AP$.

Definition 12. The $\exists PM$-theory at $n = 0$ will be called the $\exists PJ$-theory.

Hereafter, all definitions of concepts relating to Jonsson theories (in the ordinary sense) are considered to be known and can be extracted, for example, from [9].

In the study of Jonsson theories the main tool of their investigation is the semantic method, which consists in the following: The elementary properties of the centre of Jonsson theory are «translated» onto the theory itself. In this case, the elementary theory of the semantic model of Jonsson theory is similar to the positive Robinson theory, and is invariant to this Jonsson theory because all semantic models of the same Jonsson theory are elementary equivalent to each other. In this connection, if $\exists PJ$-theory is not Jonsson in the classical sense, then by its semantic model we will mean any of its universal domain U (as in [26]) and by the centre T^* we will mean the following set of sentences $T^0 = Th_{\forall\exists}(U)$.

Note the following fact from the work [34].

Fact 1. ([34]) Inductive theory T is Jonsson if and only if there is a semantic model of theory T.

Definition 13. If $\exists PJ$-theory T is Jonsson, then its semantic model is T-$\exists PJ$-universal T-$\exists PJ$-homogeneous model of theory T of cardinality κ, where κ is a fixed unreachable cardinal.

Definition 14. $\exists PJ$-Jonsson theory T is called perfect if its semantic model C is a saturated model of the theory $Th(C)$.

Let us recall the following fact, which describes the perfect Jonsson theories:

Theorem 3. ([9]) Let T be a perfect Jonsson theory. Then the following conditions are equivalent:

1) T^* is model companion T;

2) $Mod(T^*) = E_T = E_{T^*}$;

3) $T^* = T^f = T^0$,

where $T^* = Th(C)$ is the center of theory T (C is semantic model of theory T), T^0 is Kaiser hull (maximal $\forall\exists$-theory mutually model-consistent with T), $T^f = Th(F_T)$, where F_T is class of generic models of the theory T (in terms of Robinson finite forcing).

The positive Robinson theory in the sense of [26, 27] is a generalization of the Kaiser hull concept T^0 for the Jonsson theory T. It follows from the Theorem 3 that when $\Delta = B(At)$ and $\exists PJ$-theory is perfect, the notion of semantic model and universal domain coincide.

Definition 15. Let A be some infinite model of signature σ. A is called $\exists PJ$-model if the set of sentences $Th_{\forall\exists^+}(A)$ is $\exists PJ$-theory.

In all the following, we will denote the $Th_{\forall\exists^+}(A)$ theory by $\forall\exists^+(A)$.

The following result generalizes Proposition 1 of [35].

Lemma 2. Let T be $\exists PJ$-theory complete for existential sentences. Then any infinite model of theory T is a $\exists PJ$-model.

Definition 16. Models A and B will be called $\exists PJ$-equivalent and denoted by $A \equiv_{\exists PJ} B$ if for any $\exists PJ$-theory T $A \models T \Leftrightarrow B \models T$.

The following result generalises Theorem 1 of [35].

Lemma 3. Let A and B be models of signature σ. Then the following conditions are equivalent:

1) $A \equiv_{\exists PJ} B$,

2) $\forall\exists^+(A) = \forall\exists^+(B)$.

Definition 17. Two $\exists PJ$-theories T_1 and T_2 are called $\exists PJ$-cosemantic ($T_1 \bowtie_{\exists PJ} T_2$) if they have the same semantic model, in case if T_1 and T_2 are Jonsson theories; and they have the same universal domain, in case they are not Jonsson.

Definition 18. ([9]) Models A and B of the signature σ are called $\exists PJ$-cosemantic ($A \bowtie_{\exists PJ} B$), if for any $\exists PJ$-theory T_1 such that $A \models T_1$, there is a $\exists PJ$-theory T_2, $\exists PJ$-cosemantic with T_1, such that $B \models T_2$. And vice versa.

Lemma 4. For any models A and B, the following implication is true:

$$A \equiv B \Rightarrow A \equiv_{\exists PJ} B \Rightarrow A \bowtie_{\exists PJ} B.$$

Similarly, the notion of $\exists PM$-cosemanticity between $\exists PM$-theories and respectively their models is defined.

The following convention is paramount. We will talk about the semantic aspect of $\exists PJ$-theory. If $\exists PJ$-theory T is Jonsson, then we work with E_T as a class of models of some Jonsson theory. If $\exists PJ$-theory T is not Jonsson, then we consider as E_T the class of its positively existentially closed models E_T^+. Such an approach for the class E_T, a class of existentially closed models of an arbitrary universal theory T, has been considered in [36].

Since two cases are possible with respect to Jonsson theories: perfect and imperfect, we will stick to the following. According to [9], if a Jonsson theory T is perfect, then the class of its existentially closed models E_T is elementary and coincides with E_{T^*}, where T^* is its center. If the theory T is imperfect, we do as in [36], i.e., instead of E_T work with the class E_T^+.

When an arbitrary $\exists PJ$-theory T is considered, the class E_T^+ is considered an extension of E_T (both classes always exist), and depending on the perfection or imperfection of the theory T, the theoretical-model properties of the class E_T^+ are of special interest.

For any theory T we will denote by $T_{\forall+}$ the theory which axioms are positive universal corollaries of the theory T.

Lemma 5. Let T_1 and T_2 be $\exists PJ$-theories, with C_1 being the semantic model of T_1 and C_2 the semantic model of T_2. If $(T_1)_{\forall+} = (T_2)_{\forall+}$, then $T_1 \bowtie_{\exists PJ} T_2$.

Theorem 4. Let T_1 and T_2 be $\exists PJ$-theories, with C_1 being the semantic model of T_1 and C_2 being the semantic model of T_2. Then the following conditions are equivalent:

1) $C_1 \bowtie_{\exists PJ} C_2$,

2) $C_1 \equiv_{\exists PJ} C_2$,

3) $C_1 = C_2$.

3 Positive Jonsson spectrum of $\exists PM$-theories of a fixed class of S-acts theory models over a group. Main results

The main result of the paper will be the characterization of Jonsson spectra $\exists PM$-theories of S-acts over a group with respect to cosemanticity by means of some invariants which have been defined in paper [33].

Let us give the basic definitions and statements from [33] necessary to formulate and prove the results of the paper.

Let us recall the definition of a S-act.

Definition 19. ([33]) Let A be non-empty set, $\langle S; \cdot, e\rangle$ is monoid. Algebraic system $\langle A; \langle f_\alpha : \alpha \in S\rangle\rangle$ with unary operations f_α, $\alpha \in S$, is called a S-act over S, if the following conditions hold:

$f_e(a) = a$ for all $a \in A$;

$f_{\alpha\beta}(a) = f_\alpha(f_\beta(a))$ for all $a \in A$ and all $\alpha, \beta \in S$.

Let $a \in A$, then $S_a = \{f_\alpha(a) : \alpha \in S\}$; if $\bar{a}$ is tuple of elements from A, then $S_{\bar{a}} = \bigcup\limits_{a_i \in \bar{a}} S_{a_i}$. The set $C_a = \{b \in A : b \in S_a \text{ or } a \in S_b\}$ is called a component.

Proposition 1. ([33]) If T is a S-act theory and for any $f : S_{\overline{a}} \backsimeq S_{\overline{b}}$ there exists a $g \supset f$ such that $g : C_{\overline{a}} \backsimeq C_{\overline{b}}$, then T admits the elimination of the quantifiers.

Hereafter, we consider S-acts over the group G and correspondingly the theory of S-acts over the group.

If A is a S-act over the group G, $a \in A$, then

$$id(a) = \{g \in G : f_g(a) = a\}; \quad \mathfrak{p}(G) = \{H : H \preceq G\}.$$

If $H \preceq G$, then $\mathfrak{F}(H) = |\{gH : g \in G, \{\varphi \in G : \varphi gH = gH\} = H\}|$.

Definition 20. ([33]) 1) If Γ is a family or type of sentence, then $T_\Gamma = \{\psi : \{\varphi \in \Gamma : T \vdash \varphi\} \vdash \psi\}$;

2) $\nabla = \Pi_1 \cup \Sigma_1$, i.e., ∇ is the family of all universal or existential formulas.

Definition 21. ([33]) If $T = T_\nabla$, then the theory T will be called a primitive.

Let us write a known fact about primitives.

Fact 2. ([33]) For a complete theory T the following conditions are equivalent:

1) T is a primitive;

2) if $\mathfrak{A}, \mathfrak{B} \models T$ and $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$, then $\mathfrak{C} \models T$

Definition 22. ([33]) An expression of the form $g \in X$ will be called an atomic figure, where $g \in G$, X is a fixed symbol. A figure is any formal Boolean combination of atomic figures. Denote by Φ the set of all figures. For each figure $\varphi(X)$ we define by induction $U(\varphi) \subseteq \mathfrak{p}(G)$ and formula $\theta(\varphi, a)$ S-act language for any element a of any S-act

1) if $\varphi(X) = g \in X$, then $U(\varphi) = \{H \preceq G : g \in H\},\ \theta(\varphi, a) = (f_g(a) = a)$;

2) if $\varphi(X) = \neg\psi(X)$, then $U(\varphi) = \mathfrak{p}(G) - U(\psi),\ \theta(\varphi, a) = \neg\theta(\psi, a)$;

3) if $\varphi(X) = \psi_1(X)\&\psi_2(X)$, then $U(\varphi) = U(\psi_1) \cap U(\psi_2)$, $\theta(\varphi, a) = \theta(\psi_1, a)\&\theta(\psi_2, a)$.

Let us use the following notations from [33].

Let [ ] be a closure operator induced by a topology over $\mathfrak{p}(G)$ which base of open neighborhood is $\{U(\varphi) : \varphi \in \Phi\}$. If $\mathfrak{h} \subseteq \mathfrak{p}(G)$, then

$$< \mathfrak{h} >= \{gHg^{-1} : g \in G, H \in \mathfrak{h}\}.$$

Let ( ) denote the Poizat operator, i.e., the smallest closure operator on $\mathfrak{p}(G)$ with property $(\mathfrak{h}) \supseteq [\mathfrak{h}] \cup < \mathfrak{h} >$.

$Q = \{H \preceq G : \exists\varphi \in \Phi(U(\varphi) = [H]) \text{ and } \mathfrak{F}(H) < \infty\}$.

Definition 23. ([33]) A pair $< \mathfrak{h}, \varepsilon >$ is called a characteristic if $\mathfrak{h} \subseteq \mathfrak{p}(G), \mathfrak{h} = (\mathfrak{h}), \varepsilon : Q \to [\infty] \cup \omega$ and $\varepsilon(H) = 0 \Leftrightarrow H \notin \mathfrak{h}$.

Definition 24. ([33]) If $n < \omega$, T is a S-act theory, then

$$T^{(n)}(G) = \{< H_1, ..., H_n >\in G^n : \exists\mathfrak{A} \models T, < a_1, ..., a_n >\in \mathfrak{A}^n(\&_{m=1}^n H_m = id(a_m))\}.$$

Definition 25. ([33]) If T is S-act theory, then $\varepsilon_T : Q \to [\infty] \cup \omega$ such that

$$\varepsilon_T(H) = \begin{cases} k, \text{ if } k = max\{|\{G_a : a \in \mathfrak{A}, id(a) = H\}| : \mathfrak{A} \models T\} < \omega; \\ \\ \infty, \text{ if no such maximum exists.} \end{cases}$$

Let $ch(T) =< T^1(G), \varepsilon_T >$.

Proposition 2. ([33]) $ch(T)$ is a characteristic.

Theorem 5. ([33]) Let S-acts theory T have an infinite model. Then

(1) T is inductive;

(2) if T has the property of joint embedding, then it also has the property of amalgamation;

(3) if T is complete, then it admits the elimination of quantifiers and is primitive.

Theorem 6. ([33]) 1) Every α-Jonsson theory of S-acts is perfect and is Jonsson, $0 \leq \alpha \leq \omega$.

2) The S-acts theory T is a Jonsson $\Leftrightarrow \forall 1 \leq n \leq \omega$ $(T^{(n)}(G) = (T^{(1)}(G))^n)$.

Similarly to Theorem 6, let us formulate and prove the following result.

Theorem 7. For every $\exists PM$-theory T of S-acts over a group two cases are possible:

1. a) T is a Jonsson theory, then T is perfect;

b) $\exists PJ$-theory T of S-acts is a Jonsson $\Leftrightarrow \forall 1 \leq n \leq \omega$ $(T^{(n)}(G) = (T^{(1)}(G))^n)$.

2. T is not a Jonsson theory. Then there exists some $\exists PM$-theory T' such that T' is a Jonsson theory and is a Kaiser hull for theory T.

Let us first prove the lemma.

Lemma 6. Let T be $\exists PM$-theory of S-acts over a group and all completions of T admit the elimination of quantifiers. Then

(1) T is perfect;

(2) T is $\exists PJ$-theory.

Proof. (1) Let C be the semantic model of theory T, $T^* = Th(C)$ and C^* is saturated model of theory T^*. $C^* \subseteq_{\Sigma_n^+} C$, $C^* \in E_T^+$ and $D(C^*) = D(C)$. From homogeneity and equality of diagrams follows that $C \cong C^*$, i.e., T is perfect.

(2) Let C be the semantic model for T (saturated for T^*). Obviously C is $\exists PJ$-universal, we have to show that C is $\exists PJ$-homogeneous. Let $A, B \in E_T^+$, with $A \cong B$ by f. Suppose the contrary, that is, the model C is not $\exists PJ$-homogeneous and there exist such existentially closed submodels A' and B' of the semantic model C such that $A \subseteq A'$ and $B \subseteq B'$. This means that there exists an existential formula $\varphi(x)$ such that $A' \models \varphi(x)$ but $B' \not\models \varphi(x)$. It follows that $A \models \varphi(x)$ and $B \not\models \varphi(x)$ due to existential closure of A and B, which contradicts isomorphism f. By virtue of the fact that T^* admits the quantifier elimination then $(C, a)_{a \in A} \equiv (C, f(a))_{a \in A}$, which means that f is an automorphism.

Proof of Theorem 7.

1. a) It follows from Lemma 6.

1. b) It is easy to show that from the condition $\forall n < \omega, T^{(n)}(G) = (T^{(1)}(G))^n$ follows the joint embedding property and vice versa.

2. Let T be $\exists PM$-theory not Jonsson, then since $\Delta = B^+(At)$, we can use the universal domain U for the minimal fragment $\Delta = B^+(At)$ from [26]. Consider all $\forall\exists$-sequences true in U, that is, consider the theory $Th_{\forall\exists}(U) = \Delta$. There are 2 possible cases: $U \in E_\Delta^+$ and $U \notin E_\Delta^+$.

If $U \in E_\Delta^+$, let us consider the theory $Th_{\forall\exists}(U) = \Delta$. Let us show that this theory is Jonsson. To do this, we will use Fact 1. The semantic model of Δ will be the family of maximal components of the theory of all S-acts over the group. It is easy to see that by virtue of Theorem 6, this model is saturated in its cardinality, hence Δ is a perfect Jonsson $\exists PM$-theory and is a Kaiser hull for theory T.

If $U \notin E_\Delta^+$, then, since Δ is an inductive theory, there exists a model $D \in E_\Delta^+$ such that U is isomorphically embedded in D. Consider the theory $\Delta' = Th_{\forall\exists}(D)$. Similarly, it is easy to prove that Δ' is a perfect Jonsson $\exists PM$-theory and that Δ' is a Kaiser hull for theory T.

We will need the following definition and theorem from paper [33].

Definition 26. ([33]) If $<\mathfrak{h},\varepsilon>$ is a characteristic, then

$T_1(\mathfrak{h},\varepsilon)=\{\forall y\neg\theta(\varphi,y):(\varphi\in\Phi,U(\varphi)\cap\mathfrak{h}=\varnothing\}\cup\{\forall y_1,...,y_{\varepsilon(H)\mathfrak{F}(H)+i}(\&_i\theta(\varphi,y_i)\rightarrow\bigvee_{i\neq j}(y_i=y_j)):$ $H\in Q\cap\mathfrak{h},\ \varphi\in\Phi,\ \varepsilon(H)<\infty,\ U(\varphi)=[H]\}$,

$T_2(\mathfrak{h},\varepsilon)=T_1(\mathfrak{h},\varepsilon)\cup\{\exists y_1,...,y_{\varepsilon(H)\mathfrak{F}(H)}(\&_i\theta(\varphi,y_i)\&\&_{i\neq j}(y_i\neq y_j)):H\in Q\cap\mathfrak{h},\ \varepsilon(H)<\infty,\ U(\varphi)=$ $[H]\}\cup\{\exists y_1,...,y_n(\&_i\theta(\varphi,y_i)):U(\varphi)\cap(\mathfrak{h}-Q)\neq\varnothing\vee\exists H\in U(\varphi)\cap Q(\varepsilon(H)=\infty),n<\omega\}$.

Theorem 8. ([33]) 1) $ch(T_1(\mathfrak{h},\varepsilon))=ch(T_2(\mathfrak{h},\varepsilon))=<\mathfrak{h},\varepsilon>$ for any characteristic $<\mathfrak{h},\varepsilon>$;

2) Jonsson S-acts theories T_1 and T_2 are cosemantic $\Leftrightarrow ch(T_1)=ch(T_2)$;

3) T is Jonsson S-acts theory and $ch(T)=<\mathfrak{h},\varepsilon>$ if and only if $T_1(\mathfrak{h},\varepsilon)\subseteq T\subseteq T_2(\mathfrak{h},\varepsilon)$.

Similar to Theorem 8, we have a result for the case of $\exists PM$-theory.

Theorem 9. Let T_1 and T_2 be $\exists PM$-theory of S-acts over group for fixed $0\leq n\leq\omega$. Then:

(1) $\mathrm{ch}(T_1(\mathfrak{h},\varepsilon))=ch(T_2(\mathfrak{h},\varepsilon))=<\mathfrak{h},\varepsilon>$ for any characteristic $<\mathfrak{h},\omega>$;

(2) $T_1\bowtie_{\exists PM}T_2\Leftrightarrow ch(T_1)=ch(T_2)$;

(3) There is $\exists PM$-theory T of S-acts over group such that $ch(T_1)=<\mathfrak{h},\varepsilon>$ iff $T_1(\mathfrak{h},\varepsilon)\subseteq T\subseteq$ $T_2(\mathfrak{h},\varepsilon)$

The proof is the same as for Theorem 8.

The result of Theorem 9 has a natural continuation in the context of the theoretical-model properties study of the positive spectrum of a fixed class of S-acts over the group.

Let K be a class of structures of fixed signature σ. Consider positive spectrum of $\exists PM$-theories of class K:

$$PSp(K)=\{T\mid T is\ \exists PM\text{-theory in language } K\subseteq\mathrm{Mod}(T)\text{ for a fixed } 0\leq n\leq\omega\}.$$

Note that the cosemanticity relation on a set of theories is an equivalence relation. Therefore, we can consider the factor set $PSp(K)/_{\bowtie_{\exists PM}}$ of the positive spectrum class K with respect to the relation $\bowtie_{\exists PM}$.

The result is as follows:

Theorem 10. Let K_Π be a class of all S-acts over group, $[T_1],[T_2]\in PSp(K_\Pi)/_{\bowtie_{\exists PM}}$. Then

1) if $[T_1]$ and $[T_2]$ are classes of Jonsson $\exists PM$-theories then $C_{[T_1]}\bowtie_{\exists PM}C_{[T_2]}\Leftrightarrow ch([T_1]^*)=$ $ch([T_2]^*)$;

2) if $[T_1]$ and $[T_2]$ are classes of not Jonsson $\exists PM$-theories, then there are such classes of Jonsson $\exists PM$-theories $[\Delta_1],[\Delta_2]\in PSp(K_\Pi)/_{\bowtie_{\exists PM}}$, that Δ_i is the Kaiser hull for T_i, where $i=1,2$ $C_{[\Delta_1]}\bowtie_{\exists PM}$ $C_{[\Delta_2]}\Leftrightarrow ch([\Delta_1]^*)=ch([\Delta_2]^*)$;

3) if $[T_1]$ is a class of Jonsson $\exists PM$-theories, and $[T_2]$ is a class of not Jonsson $\exists PM$-theories, then there is such Jonsson $\exists PM$-theory Δ, that $C_{[T_1]}\bowtie_{\exists PM}C_{[\Delta]}\Leftrightarrow ch([T_1]^*)=ch([\Delta]^*)$.

Proof.

1) $\Rightarrow$: Let $[T_1],[T_2]\in PSp(K_\Pi)/_{\bowtie_{\exists PM}}$ be classes of Jonsson $\exists PM$-theories and $C_{[T_1]}\bowtie_{\exists PM}C_{[T_2]}$. Since $[T_1]$ and $[T_2]$ are classes of Jonsson S-acts theories over a group, then $[T_1]$ and $[T_2]$ are classes of perfect Jonsson theories, hence, by Theorem 2.12 from [9], $[T_1]^*$ and $[T_2]^*$ are Jonsson S-acts theories over a group. Then according to 2) of Theorem 8 $ch([T_1]^*)=ch([T_2]^*)$ since $[T_1]^*$ and $[T_2]^*$ are complete theories.

$\Leftarrow$: Let $[T_1]$ and $[T_2]$ be classes of Jonsson $\exists PM$-theories of S-acts over a group and $ch([T_1]^*)=$ $ch([T_2]^*)$. Then $[T_1]$ and $[T_2]$ are classes of perfect Jonsson theories, then $[T_1]^*$ and $[T_2]^*$ are complete Jonsson $\exists PM$-theories of S-acts over a group. Since $ch([T_1]^*)=ch([T_2]^*)$, it follows from 2) of Theorem 9 that $[T_1]^*\bowtie_{\exists PM}[T_2]^*$. From the definition of cosemanticity, it follows that $C_{[T_1]^*}=C_{[T_2]^*}$. However, since $[T_1]^*$ and $[T_2]^*$ are complete Jonsson $\exists PM$-theories, then $[T_1]^*\in[T_1]$ and $[T_2]^*\in[T_2]$, i.e., $C_{[T_1]}=C_{[T_2]}$, from which it follows that $C_{[T_1]}\bowtie_{\exists PM}C_{[T_2]}$.

2) Let $[T_1],[T_2]\in PSp(K_\Pi)/_{\bowtie_{\exists PM}}$ be classes of not Jonsson $\exists PM$-theories, $C_{[T_1]}=U_1,C_{[T_2]}=U_2$ and $[T_1]^*=Th_{\forall\exists}(U_1),[T_2]^*=Th_{\forall\exists}(U_2)$. Since $[T_1]^*$ and $[T_2]^*$ are inductive theories, there are positive

existentially closed models D_1 and D_2 of these theories such that U_1 is isomorphically embedded in D_1 and U_2 is isomorphically embedded in D_2. Consider the theories $\Delta_1 = Th_{\forall\exists}(D_1)$ and $\Delta_2 = Th_{\forall\exists}(D_2)$. They are Jonsson perfect $\exists PM$-theories. The existence of theories Δ_1 and Δ_2 follows from Theorem 7 and they are Kaiser hulls for T_1 and T_2 respectively. Then it follows from 1) of this theorem that $C_{[\Delta_1]} \bowtie_{\exists PM} C_{[\Delta_2]} \Leftrightarrow ch([\Delta_1]^*) = ch([\Delta_2]^*)$.

3) Let $[T_1]$ be the class of Jonsson $\exists PM$-theories and $[T_2]$ be the class of not Jonsson $\exists PM$-theories. Then, similarly to 2), using Theorem 7, we can find such a Jonsson $\exists PM$-theory Δ, which is a Kaiser hull for theory T_2 and according to 1) hold $C_{[T_1]} \bowtie_{\exists PM} C_{[\Delta]} \Leftrightarrow ch([T_1]^*) = ch([\Delta]^*)$.

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References

1 Yeshkeyev A.R. The structure of lattices of positive existential formulae of (Δ-PJ)-theories / A.R. Yeshkeyev // ScienceAsia. — 2013. — 39(SUPPL.1). — P. 19–24.

2 Yeshkeyev A.R. The Properties of Positive Jonsson's Theories and Their Models / A.R. Yeshkeev // International Journal of Mathematics and Computation. — 2014. — 22. — No. 1. — P. 161–171.

3 Poizat B. Positive Jonsson Theories / B. Poizat, A. Yeshkeyev // Logica Universalis. — 2018. — 12. — 1-2. — P. 101–127.

4 Yeshkeyev A.R. Chains of existentially closed models of positive (n(1), n(2))-Jonsson theories / A.R. Yeshkeyev, M.T. Omarova // Bulletin of the Karaganda University-Mathematics. — 2019. — 96. — No. 4. — P. 69–74.

5 Yeshkeyev A.R. The hybrids of the $\Delta - PJ$ theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University-Mathematics. — 2020. — 98. — No. 2. — P. 174–180.

6 Yeshkeyev A.R. JSp-cosemanticness and JSB property of Abelian groups / A.R. Yeshkeyev, O.I. Ulbrikht // Siberian Electronic Mathematical Reports. — 2016. — 13. — P. 861–874.

7 Yeshkeyev A.R. JSp-cosemanticness of R-modules / A.R. Yeshkeyev, O.I. Ulbrikht // Siberian Electronic Mathematical Reports. — 2019. — 16. — P. 1233–1244.

8 Yeshkeyev A.R. Model-theoretical questions of the Jonsson spectrum / A.R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2020. — 98. — No. 2. — P. 165–173.

9 Ешкеев А.Р. Йонсоновские теории / А.Р. Ешкеев. — Караганда: Изд-во КарГУ, 2009. — 249 с.

10 Yeshkeyev A.R. Δ-cl-atomic and prime sets / A.R. Yeshkeyev, A.K. Issayeva // Bulletin of the Karaganda University-Mathematics. — 2019. — 93. — No. 1. — P. 88–94.

11 Yeshkeyev A.R. The atomic definable subsets of semantic model / A.R. Yeshkeyev, A.K. Issayeva, N.M. Mussina // Bulletin of the Karaganda University-Mathematics. — 2019. — 94. — No. 2. — P. 84–91.

12 Yeshkeyev A.R. Properties of hybrids of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University-Mathematics. — 2018. — 92. — No. 4. — P. 99–104.

13 Yeshkeyev A.R. Companions of the fragments in the Jonsson enrichment / A.R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2017. — 85. — No. 1. — P. 41–45.

14 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University-Mathematics. — 2015. — 79. — No. 3. — P. 33–39.

15 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2015. — 80. — No. 4. — P. 47–51.

16 Yeshkeyev A.R. The Properties of Similarity for Jonsson's Theories and Their Models / A.R. Yeshkeyev // Bulletin of the Karaganda University-Mathematics. — 2015. — 80. — No. 4. — P. 52–59.

17 Yeshkeyev A.R. The J-minimal sets in the hereditary theories / A.R. Yeshkeyev, M.T. Omarova, G.E. Zhumabekova // Bulletin of the Karaganda University-Mathematics. — 2019. — 94. — No. 2. — P. 92–98.

18 Кейслер Х.Дж. Основы теории моделей. Справочная книга по математической логике / Х.Дж. Кейслер. — Теория моделей. — М.: Наука, 1982. — С. 55–108.

19 Jonsson B. Universal relational systems / B. Jonsson // Mathematica Scandinavica. — 1956. — 4. — P. 193–208.

20 Jonsson B. Homogeneous universal relational systems / B. Jonsson // Mathematica Scandinavica. — 1960. — 8. — P. 137–142.

21 Палютин Е.А. О N^*-однородных моделях / Е.А. Палютин // Сиб. мат. журн. — 1971. — 12. — № 4. — С. 920, 921.

22 Мустафин Т.Г. Обобщенные условия Йонсона и описание обобщенно-йонсоновских теорий булевых алгебр / Т.Г. Мустафин // Математические труды Ин-та мат. — 1998. — 1. — № 2. — С. 135–197.

23 Ешкеев А.Р. Категоричные позитивные йонсоновские теории / А.Р. Ешкеев // Вестн. Караганд. ун-та. Сер. Математика. — 2006. — 44. — № 4. — С. 10–16.

24 Ешкеев А.Р. О классе Δ-позитивно экзистенциально замкнутых моделей Δ-PJ и Δ-PR-теорий / А.Р. Ешкеев // Вестн. Караганд. ун-та. Сер. Математика. — 2007. — 48. — № 4. — С. 49–56.

25 Ешкеев А.Р. Счетная категоричность Δ-PM-теорий / А.Р. Ешкеев // Тез. докл. 12-й Межвуз. конф. по математике, механике и информатике. — Алматы: Қазақ университетi, 2008. — С. 67.

26 Ben-Yaacov I. Positive model theory and compact abstract theories / I. Ben-Yaacov // Journal of Mathematical Logic. — 2003. — 3. — No. 1. — P. 85–118.

27 Ben-Yaacov I. Compactness and independence in non first order frameworks / I. Ben-Yaacov // Bulletin of Symbolic Logic. — 2005. — 11. — No. 1. — P. 28–50.

28 Ben-Yaacov I. Fondements de la logique positive / I. Ben-Yaacov, B. Poizat // Journal of Symbolic Logic. — 2007. — 72. — No. 4. — P. 1141–1162.

29 Gould V. Axiomatisability problems for S-systems / V. Gould // J. London Math. Soc. — 1987. — 35. — No. 2. — P. 193–201.

30 Мустафин Т.Г. О стабильностной теории полигонов / Т.Г. Мустафин // Теория моделей и ее применение. — Новосибирск: Наука, Сиб. отд-е, 1988. — 8. — С. 92–107.

31 Mustafin T.G. On similarities of complete theories / T.G. Mustafin // Logic Colloquium '90. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic. — Helsinki. — 1990. — P. 259–265.

32 Yeshkeyev A.R. Connection of Jonsson theory with some Jonsson polygons theories / A.R. Yeshkeyev, G.A. Urken // Bulletin of the Karaganda University-Mathematics. — 2019. — 95. — No. 3. — P. 74–78.

33 Мустафин Т.Г. Описание йонсоновских теорий полигонов над группой / Т.Г. Мустафин, Е.С. Нурхайдаров // Исследования в теории алгебраических систем: сб. науч. тр. (межвуз.). — Караганда: Изд-во КарГУ, 1995 — С. 67–73.

34 Mustafin Y.T. Quelques proprietes des theories de Jonsson / Y.T. Mustafin // The Journal of Symbolic Logic. — 2002. — 67. — No. 2. — P. 528–536.

35 Mustafin E. Jonsson equivalent and cosemantical models / E. Mustafin, E. Nurkhaidarov // Quatrieme Colloque Franco-Touranien de Theorie des Modeles, Resumes des Conferences. — Marseille. — 1997. — P. 13–15.

36 Pillay A. Forking in the category of existentially closed structures / A. Pillay // Connection between Model Theory and Algebraic and Analytic Geometry, Quaderni di Matematica: University of Naples. — 2000. — 6. — P. 1–18.

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Группалар маңайындағы полигондардың экцистенциалды позитивті мұстафиндік теориясы

Мақала полигондар сигнатурасының бекітілген модельдер класының йонсондық спектрінің ұғымын зерттеумен байланысты. Сонымен бірге полигонның моноиды ретінде группа қарастырылған. Йонсондық спектр ұғымы алгебралар класын модельді-теоретикалық қасиеттерін сипаттау үшін эффективті болып табылады. Теориялар үйлесімді енгізілуге және амальгама қасиетіне ие. Бұл жағдайда, әдетте, осы модельдер класы бойынша ақиқат болатын әмбебап-экзистенциалды ұсыныстарды қарастыру жеткілікті. Осы уақытқа дейін йонсондық спектр, әдетте, тек йонсондық теорияларымен жұмыс істеді. Авторлар мақалада позитивті йонсондық спектрі түсінігін анықтайды, оның элементтері, жалпы алғанда, йонсондық емес теориялар болуы мүмкін. Бұл мақалада қарастырылатын теорияларды анықтауда изоморфтық енгізулер ғана емес, сонымен қатар батулар (яғни, экзистенциалды позитивті мұстафиндік теория) қатыстылығымен түсіндіріледі. Осыған байланысты амальгама қасиеттерін және бірлескен үйлесімді қасиеттерін анықтауда батулар қарастырылады. Нәтижесінде, теорияның осындай өзгерістеріне байланысты алынған теориялар йонсондық болуы міндетті емес. Осы мақаланың негізгі нәтижелерін талдай отырып, йонсондық емес спектрді зерттеудің жоғарыда аталған тәсілі, ең болмағанда, йонсондық емес теория жағдайында да, бұрын белгілі ұғымдар мен нәтижелерді қанағаттандыратын, бірақ сонымен бірге қарастырылатын экзистенциалды позитивті мұстафиндік теориясымен тікелей байланысты болатын йонсондық теорияны табудың тұрақты әдісі бар екені байқалады.

Кілт сөздер: йонсондық теория, кемел йонсондық теория, позитивті модельдер теориясы, йонсондық спектр, позитивті йонсондық теория, бату, полигон, полигондардың йонсондық теориясы, $\exists PM$-теория, косеманттылық.

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Экзистенциально позитивные мустафинские теории полигонов над группой

Статья связана с изучением понятия йонсоновского спектра фиксированного класса моделей сигнатуры полигонов, причём в качестве моноида полигона рассматривается группа. Понятие йонсоновского спектра является эффективно работающим при описании теоретико-модельных свойств классов алгебр, теории которых допускают свойства совместного вложения и амальгамы. При этом, как правило, достаточно рассматривать универсально-экзистенциальные предложения, истинные на моделях этого класса. До настоящей работы йонсоновский спектр, как правило, оперировал только йонсоновскими теориями. Авторами статьи определено понятие позитивного йонсоновского спектра, элементами которого могут быть, вообще говоря, не йонсоновские теории. Это происходит из-за того, что в определении рассматриваемых теорий в данной статье (а именно, экзистенциально позитивных мустафинских теорий) участвуют не только изоморфные вложения, но и погружения. В связи с этим в определении свойства амальгамы и свойства совместного вложения рассмотрены погружения. Как следствие, полученные в силу таких изменений теории не обязательно должны быть йонсоновскими. Анализируя основные полученные результаты данной статьи, мы можем заметить, что указанный выше подход к изучению йонсоновского спектра оказывается оправданным, хотя бы в силу того, что даже в случае не йонсоновской теории существует регулярный метод нахождения такой йонсоновской теории, которая удовлетворяет ранее известным понятиям и результатам, но которая также будет непосредственно связана с рассматриваемой экзистенциально позитивной мустафинской теорией.

Ключевые слова: йонсоновская теория, совершенная йонсоновская терия, позитивная теория моделей, йонсоновский спектр, позитивная йонсоновская теория, погружение, полигон, йонсоновская теория полигонов, $\exists PM$-теория, косемантичность.

References

1 Yeshkeyev, A.R. (2013). The structure of lattices of positive existential formulae of (Δ-PJ)-theories. *ScienceAsia, 39*(1), 19–24.

2 Yeshkeyev, A.R. (2014). The Properties of Positive Jonsson's Theories and Their Models. *International Journal of Mathematics and Computation, 22*(1), 161–171.

3 Poizat, B., & Yeshkeyev, A. (2018). Positive Jonsson theories. *Logica Universalis, 12*(1-2), 101-–127.

4 Yeshkeyev, A.R., & Omarova, M.T. (2019). Chains of existentially closed models of positive (N1, N2)-Jonsson theories. *Bulletin of the Karaganda University-Mathematics, 96*(4), 69—74.

5 Yeshkeyev, A.R., & Mussina, N.M. (2020). The hybrids of the $\Delta - PJ$-theories. *Bulletin of the Karaganda University-Mathematics, 98*(2), 174—180.

6 Yeshkeyev, A.R., & Ulbrikht, O.I. (2016). JSp-cosemanticness and JSB property of Abelian groups. *Siberian Electronic Mathematical Reports, 13*, 861–874.

7 Yeshkeyev, A.R., & Ulbrikht, O.I. (2019). JSp-cosemanticness of R-modules. *Siberian Electronic Mathematical Reports, 16*, 1233-–1244.

8 Yeshkeyev, A.R. (2020). Model-theoretical questions of the jonsson spectrum. *Bulletin of the Karaganda University-Mathematics, 98*(2), 165—173.

9 Yeshkeyev, A.R. (2009). *Ionsonovskie teorii [Jonsson theories].* Karaganda: Izdatelstvo Karagandinskogo gosudarstvennogo universiteta [in Russian].

10 Yeshkeyev, A.R., & Issayeva, A.K. (2019). Δ-cl-atomic and prime sets. *Bulletin of the Karaganda University-Mathematics, 93*(1), 88-–94.

11 Yeshkeyev, A.R., Issaeva, A.K., & Mussina, N.M. (2019). The atomic definable subsets of semantic model. *Bulletin of the Karaganda University-Mathematics, 94*(2), 84—91.

12 Yeshkeyev, A.R., & Mussina, N.M. (2018). Properties of hybrids of jonsson theories. *Bulletin of the Karaganda University-Mathematics, 92*(4), 99—104.

13 Yeshkeyev, A.R. (2017). Companions of the fragments in the Jonsson Enrichment. *Bulletin of the Karaganda University-Mathematics, 85*(1), 41–45.

14 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University-Mathematics, 79*(3), 33–39.

15 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University-Mathematics, 80*(4), 47–51.

16 Yeshkeyev, A.R. (2015). The Properties of Similarity for Jonsson's Theories and Their Models. *Bulletin of the Karaganda University-Mathematics, 80*(4), 52–59.

17 Yeshkeyev, A.R., Omarova, M.T., & Zhumabekova, G.E. (2019). The J-minimal sets in the hereditary theories. *Bulletin of the Karaganda University-Mathematics, 94*(2), 92—98.

18 Keisler, H.J. (1982). *Osnovy teorii modelei. Spravochnaia kniga po matematicheskoi logike [Fundamentals of Model Theory. Reference book on mathematical logic]*. Moscow: Nauka [in Russian].

19 Jonsson, B. (1956). Universal Relational Systems. *Mathematica Scandinavica, 4*, 193–208.

20 Jonsson, B. (1960). Homogeneous Universal Relational Systems. *Mathematica Scandinavica, 8*, 137–142.

21 Palyutin, E.A. (1972). O N^*-odnorodnykh modeliakh [On N^*-homogeneous models]. *Siberskii Mathematicheskii Jurnal — Siberian Mathematical Journal, 12*(4), 665–666 [in Russian].

22 Mustafin, T.G. (1998). Obobshchennye usloviia Ionsona i opisanie obobshchenno-ionsonovskikh teorii bulevykh algebr [Generalised Jonsson conditions and description of generalised Jonsson theories of Boolean algebras]. *Matematicheskie trudy Instituta matematiki – Proceedings of Math. Inst., 1*(2), 135–197 [in Russian].

23 Yeshkeyev, A.R. (2006). Kategorichnye pozitivnye ionsonovskie teorii [Categorical positive Jonsson theories]. *Vestnik Karagandinskogo universiteta. Seriia Matematika – Bulletin of the Karaganda University-Mathematics, 44*(4), 10–16 [in Russian].

24 Yeshkeyev, A.R. (2007). O klasse Δ-pozitivno ekzistentsialno zamknutykh modelei Δ-PJ i Δ-PR-teorii [On a class of Δ-positively existentially closed models of Δ-PJ and Δ-PR-theories]. *Vestnik Karagandinskogo universiteta. Seriia Matematika – Bulletin of the Karaganda University-Mathematics, 48*(4), 49–56 [in Russian].

25 Yeshkeyev, A.R. (2008). Schetnaia kategorichnost Δ-PM-teorii [Countable categoricality of Δ-PM-theories]. *Tezisy dokladov 12-oi Mezhvuzovskoi konferentsii po matematike, mekhanike i informatike – Abstracts from the 12th Interuniversity Conference on Mathematics, Mechanics and Computer Science, Almaty*, 67 [in Russian].

26 Ben-Yaacov, I. (2003). Positive model theory and compact abstract theories. *Journal of Mathematical Logic, 3*(1), 85–118.

27 Ben-Yaacov, I. (2005). Compactness and independence in non first order frameworks. *Bulletin of Symbolic Logic, 11*(1), 28–50.

28 Ben-Yaacov, I., & Poizat, B. (2007). Fondements de la logique positive. *Journal of Symbolic Logic, 72*(4), 1141-–1162.

29 Gould, V. (1987). Axiomatisability problems for s-systems. *Journal of the London Mathematical Society, s2-35*(2), 193—201.

30 Mustafin, T.G. (1988). O stabilnostnoi teorii poligonov [On the stability theory of S-acts]. *Teoriia modelei i ee primenenie – Model Theory and its application, 8*, 92–107 [in Russian].

31 Mustafin, T.G. (1990). On similarities of complete theories. *Logic Colloquium '90. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic, Helsinki*, 259–265.

32 Yeshkeyev, A.R., & Urken, G. A. (2019). Connection of Jonsson theory with some Jonsson polygons theories. *Bulletin of the Karaganda University-Mathematics, 95*(3), 74—78.

33 Mustafin, T.G. (1995). Opisanie ionsonovskikh teorii poligonov nad gruppoi [Description of Jonsson S-acts theories over a group]. *Issledovaniia v teorii algebraicheskikh sistem – Research in the Algebraic System Theory*, 67–73 [in Russian].

34 Mustafin, Y. (2002). Quelques proprietes des theories de Jonsson. *Journal of Symbolic Logic, 67*(2), 528—536.

35 Mustafin, E., & Nurkhaidarov, E. (1997). Jonsson equivalent and cosemantical models. *Quatrieme Colloque Franco-Touranien de Theorie des Modeles, Resumes des Conferences*, 13–15.

36 Pillay, A. (2000). Forking in the category of existentially closed structures. Connection between Model Theory and Algebraic and Analytic Geometry (A. Macintyre, ed). *Quaderni di Matematica, 6*, University of Naples, 1–18.

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On a mixed problem for Hilfer type differential equation of higher order

The study considers the solvability of a mixed problem for a Hilfer type partial differential equation of the even order with initial value conditions and small positive parameters in mixed derivatives in three-dimensional domain. It studies the solution to this fractional differential equation of higher order in the class of regular functions. The case, when the order of fractional operator is $1 < \alpha < 2$, is examined. During this study the authors use the Fourier series method and obtain a countable system of ordinary differential equations. The initial value problem is integrated as an ordinary differential equation and the integrated constants find by the aid of given initial value conditions. Using the Cauchy–Schwarz inequality and the Bessel inequality, it is proved the absolute and uniform convergence of the obtained Fourier series. The stability of the solution to the mixed problem on the given functions is studied.

Keywords: fractional order, Hilfer operator, mixed problem, Fourier series, initial value conditions, unique solvability.

Introduction

The theory of the mixed problems is one of the most important directions of the modern theory of differential equations. A large number of works are devoted to the study of the mixed problems for differential and integro-differential equations (see, for example, [1–12]). Many problems of gas dynamics, theory of elasticity, plates, and shells are described by higher-order partial differential equations.

Fractional calculus plays an important role for the mathematical modeling in many applied problems. In [13], it is considered problems of continuum and statistical mechanics. The work [14] studies the mathematical problems of the Ebola epidemic model. The studies [15] and [16] investigate the fractional model for the dynamics of tuberculosis infection and novel coronavirus (nCoV-2019), respectively. The construction of various models of theoretical physics by the aid of fractional calculus is described in [17, Vol. 4, 5], [18], [19]. Some applications of fractional calculus in solving applied problems are given in [17, Vol. 6–8], [20]. In [21], the solvability of an initial value problem for Hilfer type fractional differential equation with nonlinear maxima is studied. In [22], by analytical method, the unique solvability of boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator is studied. In [23], the solvability of nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator is examined. In [24], it is considered an inverse problem for a mixed type integro-differential equation with fractional order Gerasimov-Caputo operators. The research works [25–34] obtained the results on the direction of applications of fractional derivatives to solve partial differential equations.

Let $\left(t_0; T\right) \subset \mathbb{R}^+ \equiv [0; \infty)$ be an interval on the set of positive real numbers, where $0 \leq t_0 < T < \infty$. The Riemann–Liouville $0 < \alpha$-order fractional integral of a function $\eta(t)$ is defined as follows:

$$I_{t_0+}^{\alpha}\eta(t) = \frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}\eta(s)ds, \quad \alpha > 0, \quad t \in \left(t_0; T\right),$$

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where $\Gamma(\alpha)$ is the Gamma function.

Let $n-1<\alpha\le n,\ \ n\in\mathbb{N}$. The Riemann–Liouville α-order fractional derivative of a function $\eta(t)$ is defined as follows:

$$D_{t_0+}^{\alpha}\eta(t)=\frac{d^n}{dt^n}I_{t_0+}^{n-\alpha}\eta(t),\quad t\in\big(t_0;\,T\big).$$

The Hilfer fractional derivatives of α-order ($n-1<\alpha\le n,\ \ n\in\mathbb{N}$) and β-type ($0\le\beta\le 1$), are defined by the following composition of three operators:

$$D_{t_0+}^{\alpha,\,\beta}\eta(t)=I_{t_0+}^{\beta(n-\alpha)}\frac{d^n}{dt^n}I_{t_0+}^{(1-\beta)(n-\alpha)}\eta(t),\quad t\in\big(t_0;\,T\big).$$

Let $\gamma=\alpha+\beta n-\alpha\beta$. It is easy to see that $\alpha\le\gamma\le n$. Then it is convenient to use another designation for the operator $D^{\alpha,\,\gamma}\eta(t)=D_{t_0+}^{\alpha,\,\beta}\eta(t)$. Hilfer operator is generalization of the Riemann–Liouville operator and was introduced by R. Hilfer based on fractional time evolutions that arise during the transition from the microscopic scale to the macroscopic time scale (see [17]).

In this paper, for the case $1<\alpha<2$ we study the regular solvability of mixed value problem for a Hilfer type partial differential equation of higher even order with positive small parameters. The stability of the solution from the given functions is proved.

In three-dimensional domain $\Omega=\{(t,\,x,\,y)\,|\,t_0<t<T,\ 0<x,\,y<l\}$ a higher order partial differential equation of the following form is considered

$$D_{\varepsilon_1,\,\varepsilon_2}^{\alpha,\,\gamma}\,[U]=a\,(t)\,b\,(x,\,y) \tag{1}$$

with initial value conditions

$$\lim_{t\to+t_0}J_{t_0+}^{2-\gamma}U(t,\,x,\,y)=\varphi_1(x,\,y),\quad \lim_{t\to+t_0}\frac{d}{dt}J_{t_0+}^{2-\gamma}U(t,\,x,\,y)=\varphi_2(x,\,y), \tag{2}$$

where T and l are given positive real numbers, $0\le t_0<T$,

$$D_{\varepsilon_1,\,\varepsilon_2}^{\alpha,\,\gamma}\,[U]=\Bigg[D^{\alpha,\,\gamma}-D^{\alpha,\,\gamma}\left(\varepsilon_1\left(\frac{\partial^{2k}}{\partial\,x^{2k}}+\frac{\partial^{2k}}{\partial\,y^{2k}}\right)-\varepsilon_2\left(\frac{\partial^{4k}}{\partial\,x^{4k}}+\frac{\partial^{4k}}{\partial\,y^{4k}}\right)\right)-$$

$$-\,\omega\left(\left(\frac{\partial^{2k}}{\partial\,x^{2k}}+\frac{\partial^{2k}}{\partial\,y^{2k}}\right)-\left(\frac{\partial^{4k}}{\partial\,x^{4k}}+\frac{\partial^{4k}}{\partial\,y^{4k}}\right)\right)\Bigg]\,U\,(t,\,x,\,y),$$

ω is positive parameter, ε_1 and ε_2 are positive small parameters, $1<\alpha<\gamma\le 2,\ k$ is given positive integer, $a\,(t)\in C\left(\Omega_T\right),\ \Omega_T\equiv\big[t_0;\,T\big],\ \Omega_l\equiv[0;\,l],\ b\,(x,\,y)\in C\left(\Omega_l^2\right)$ is known function, $\varphi_1(x,\,y)$ and $\varphi_2(x,\,y)$ are given continuous functions, $\Omega_l^2\equiv\Omega_l\times\Omega_l$. We assume that for given functions the following boundary conditions hold

$$\varphi_i(0,\,y)=\varphi_i(l,\,y)=\varphi_i(x,\,0)=\varphi_i\,(x,\,l)=0,\ \ i=1,\,2,$$

$$b\,(0,\,y)=b\,(l,\,y)=b\,(x,\,0)=b\,(x,\,l)=0.$$

Problem Statement. We find the function $U\,(t,\,x,\,y)$, which satisfies differential equation (1), initial value conditions (2), zero boundary value conditions for $t_0\le t\le T$

$$U\,(t,\,0,\,y)=U\,(t,\,l,\,y)=U\,(t,\,x,\,0)=U\,(t,\,x,\,l)=$$

$$=\frac{\partial^2}{\partial\,x^2}U\,(t,\,0,\,y)=\frac{\partial^2}{\partial\,x^2}U\,(t,\,l,\,y)=\frac{\partial^2}{\partial\,x^2}U\,(t,\,x,\,0)=\frac{\partial^2}{\partial\,x^2}U\,(t,\,x,\,l)=$$

$$=\frac{\partial^2}{\partial y^2}U\,(t,\,0,\,y)=\frac{\partial^2}{\partial y^2}U\,(t,\,l,\,y)=\frac{\partial^2}{\partial y^2}U\,(t,\,x,\,0)=\frac{\partial^2}{\partial y^2}U\,(t,\,x,\,l)=\ldots=$$

$$= \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t,0,y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t,l,y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t,x,0) = \frac{\partial^{4k-2}}{\partial x^{4k-2}} U(t,x,l) =$$

$$= \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,0,y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,l,y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,x,0) = \frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,x,l) = 0, \qquad (3)$$

class of functions

$$\left[\begin{array}{l} (t-t_0)^{2-\gamma} U(t,x,y) \in C(\overline{\Omega}), \\ D^{\alpha,\gamma} U(t,x,y) \in C^{4k,4k}_{x,y}(\Omega) \cap C^{4k+0}_{x,y}(\Omega) \cap C^{0+4k}_{x,y}(\Omega), \end{array} \right. \qquad (4)$$

where $C^{4k+0}_{x,y}(\Omega)$ is the class of continuous functions $\frac{\partial^{4k} U(t,x,y)}{\partial x^{4k}}$ on Ω, while $C^{0+4k}_{x,y}(\Omega)$ is the class of continuous functions $\frac{\partial^{4k} U(t,x,y)}{\partial y^{4k}}$ on Ω, $\frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,x,l)$ we understand as $\frac{\partial^{4k-2}}{\partial y^{4k-2}} U(t,x,y)\Big|_{y=l}$, $\overline{\Omega} = \{(t,x,y) \,|\, t_0 \le t \le T,\ 0 \le x, y \le l\}$.

1 Transform of fractional differential equation

Lemma. The solution to the ordinary fractional differential equation

$$D^{\alpha,\gamma} v(t) + \omega\, v(t) = f(t, v(t)) \qquad (5)$$

with initial value condition

$$\lim_{t \to +t_0} J^{2-\gamma}_{t_0+} v(t) = v_0, \quad \lim_{t \to +t_0} \frac{d}{dt} J^{2-\gamma}_{t_0+} v(t) = v_1, \qquad (6)$$

is represented as follows

$$v(t) = v_0 (t-t_0)^{\gamma-2} E_{\alpha,\gamma-1}(\omega\,(t-t_0)^{\alpha}) + v_1 (t-t_0)^{\gamma-1} E_{\alpha,\gamma}(\omega\,(t-t_0)^{\alpha}) +$$

$$+ \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega (t-s)^{\alpha})\, f(s, v(s))\, ds, \qquad (7)$$

where $E_{\alpha,\gamma}(z)$ is the Mittag–Leffler function and has the form [17, vol. 1, 269–295]

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}, \quad z, \alpha, \gamma \in (0; \infty),$$

$f(t,v) \in C(\Omega_1)$, $0 < \omega$ is real parameter, $v_0, v_1 = \text{const}$, $\Omega_1 \equiv [t_0; T] \times X$, $0 \le t_0$, $X \subset \mathbb{R} \equiv (-\infty; \infty)$, X is closed set,

$$D^{\alpha,\gamma} = J^{\gamma-\alpha}_{t_0+} \frac{d^2}{dt^2} J^{2-\gamma}_{t_0+},\ 1 < \alpha < \gamma \le 2,\ \gamma = \alpha + 2\beta - \alpha\beta.$$

Proof. We rewrite the differential equation (5) in the form

$$J^{\gamma-\alpha}_{t_0+} D^{\gamma}_{t_0+} v(t) = -\omega\, v(t) + f(t,v).$$

Applying the operator $J^{\alpha}_{t_0+}$ to both sides of this equation (5), taking into account the linearity of this operator and the formula [35]

$$J^{\delta}_{t_0+} D^{\delta}_{t_0+} v(t) = v(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^{\delta+k-n}}{\Gamma(\delta+k+1-n)} \lim_{t \to t_0+} \frac{d^k}{dt^k} J^{n-\delta}_{t \to t_0+} v(t), \quad \delta \in (n-1; n],$$

we obtain

$$v\left(t\right)=-\omega J_{t_0+}^{\alpha}v\left(t\right)\frac{v_0}{\Gamma(\gamma-1)}\left(t-t_0\right)^{\gamma-2}+\frac{v_1}{\Gamma(\gamma)}\left(t-t_0\right)^{\gamma-1}+J_{t_0+}^{\alpha}f(t,\,v\left(t\right)). \tag{8}$$

Using the lemma from [26], we represent the solution to equation (8) in the form

$$v\left(t\right)=\frac{v_0}{\Gamma(\gamma-1)}\left(t-t_0\right)^{\gamma-2}+\frac{v_1}{\Gamma(\gamma)}\left(t-t_0\right)^{\gamma-1}+J_{t_0+}^{\alpha}f(t,\,v\left(t\right))-$$

$$-\omega\int\limits_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\,\left(t-s\right)^{\alpha}\right)\times$$

$$\times\left[\frac{v_0}{\Gamma(\gamma-1)}\left(s-t_0\right)^{\gamma-2}+\frac{v_1}{\Gamma(\gamma)}\left(s-t_0\right)^{\gamma-1}+J_{t_0+}^{\alpha}f(s,\,v\left(s\right))\right]ds. \tag{9}$$

We rewrite the presentation (9) as the sum of two expressions:

$$I_1(t)=\frac{v_0}{\Gamma(\gamma-1)}\left[(t-t_0)^{\gamma-2}-\omega\int\limits_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\,\left(s-t_0\right)^{\alpha}\right)\left(s-t_0\right)^{\gamma-2}d\,s\right]+$$

$$+\frac{v_1}{\Gamma(\gamma)}\left[(t-t_0)^{\gamma-1}-\omega\int\limits_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\,\left(t-s\right)^{\alpha}\right)\left(s-t_0\right)^{\gamma-1}ds\right], \tag{10}$$

$$I_2\left(t\right)=J_{t_0+}^{\alpha}f\left(t,\,v\left(t\right)\right)-\omega\int\limits_{t_0}^{t}\left(t-s\right)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\cdot\left(t-s\right)^{\alpha}\right)J_{t_0+}^{\alpha}f\left(s,\,v\left(s\right)\right)d\,s. \tag{11}$$

We apply the following presentations [17, vol. 1, 269–295]

$$E_{\alpha,\,\mu}(z)=\frac{1}{\Gamma(\mu)}+z\cdot E_{\alpha,\,\mu+\alpha}(t),\quad \alpha>0,\quad \mu>0, \tag{12}$$

$$\frac{1}{\Gamma(\nu)}\int\limits_{0}^{z}(z-t)^{\nu-1}E_{\alpha,\,\beta}\left(\lambda\,t^{\alpha}\right)t^{\beta-1}d\,t=z^{\beta+\nu-1}\cdot E_{\alpha,\,\beta+\nu}\left(\lambda\,z^{\alpha}\right),\quad \nu>0,\ \beta>0. \tag{13}$$

Then for the integral (10) we obtain the presentation

$$I_1(t)=v_0\left(t-t_0\right)^{\gamma-2}E_{\alpha,\,\gamma-1}\left(-\omega\,\left(t-t_0\right)^{\alpha}\right)+v_1\,\left(t-t_0\right)^{\gamma-1}E_{\alpha,\,\gamma}\left(-\omega\,\left(t-t_0\right)^{\alpha}\right). \tag{14}$$

The integral in (11) is easily transformed to the form

$$\int\limits_{t_0}^{t}\left(t-\xi\right)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\cdot\left(t-\xi\right)^{\alpha}\right)\,J_{t_0+}^{\alpha}f\left(\xi,\,v\left(\xi\right)\right)d\,\xi=$$

$$=\frac{1}{\Gamma\left(\alpha\right)}\int\limits_{t_0}^{t}\left(t-\xi\right)^{\alpha-1}E_{\alpha,\,\alpha}\left(-\omega\cdot\left(t-\xi\right)^{\alpha}\right)\,d\,\xi\int\limits_{t_0}^{\xi}\left(\xi-s\right)^{\alpha-1}f\left(s,\,v\left(s\right)\right)d\,s=$$

$$= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} f(s, v(s))\, ds \int_{s}^{t} (t-\xi)^{\alpha-1} (\xi - s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega \cdot (t-\xi)^{\alpha}\right) d\xi. \tag{15}$$

Taking into account the (13) the second integral in the last equality of (15) can be written as

$$\int_{s}^{t} (t-\xi)^{\alpha-1} (\xi - s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega \cdot (t-\xi)^{\alpha}\right) d\xi = \Gamma(\alpha)\,(t-\xi)^{2\alpha-1} E_{\alpha,2\alpha}\left(-\omega \cdot (t-\xi)^{\alpha}\right).$$

Then, taking into account (12), we represent (11) in the following form

$$I_2(t) = \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega \cdot (t-s)^{\alpha}\right) f(s, v(s))\, ds. \tag{16}$$

Substituting (14) and (16) into the sum $v(t) = I_1(t) + I_2(t)$, we obtain (7). The Lemma is proved.

2 Expansion of the solution into Fourier series

Nontrivial solutions to the problem are sought as a Fourier series

$$U(t, x, y) = \sum_{n,m=1}^{\infty} u_{n,m}(t)\, \vartheta_{n,m}(x, y), \tag{17}$$

where

$$u_{n,m}(t) = \int_0^l \int_0^l U(t, x, y)\, \vartheta_{n,m}(x, y)\, dx\, dy, \tag{18}$$

$$\vartheta_{n,m}(x, y) = \frac{2}{l} \sin \frac{\pi n}{l} x \sin \frac{\pi m}{l} y, \;\; n, m = 1, 2, \ldots$$

We also suppose that the following function is expanded to Fourier series

$$b(x, y) = \sum_{n,m=1}^{\infty} b_{n,m}\, \vartheta_{n,m}(x, y), \tag{19}$$

where

$$b_{n,m} = \int_0^l \int_0^l b(x, y)\, \vartheta_{n,m}(x, y)\, dx\, dy. \tag{20}$$

Substituting Fourier series (17) and (19) into partial differential equation (1), we obtain the countable system of ordinary fractional differential equations of order: $1 < \alpha, \gamma < 2$

$$D^{\alpha,\gamma} u_{n,m}(t) + \lambda_{n,m}^{2k}(\varepsilon_1, \varepsilon_2)\, \omega\, u_{n,m}(t) = \frac{a(t)\, b_{n,m}}{1 + \mu_{n,m}^{2k}\left(\varepsilon_1 + \varepsilon_2 \mu_{n,m}^{2k}\right)}, \tag{21}$$

where

$$\lambda_{n,m}^{2k}(\varepsilon_1, \varepsilon_2) = \frac{\mu_{n,m}^{2k}\left(1 + \mu_{n,m}^{2k}\right)}{1 + \mu_{n,m}^{2k}\left(\varepsilon_1 + \varepsilon_2 \mu_{n,m}^{2k}\right)}, \;\; \mu_{n,m}^{k} = \left(\frac{\pi}{l}\right)^k \sqrt{n^{2k} + m^{2k}}.$$

According to the Lemma, the general solution to countable system of differential equations (21) has the form

$$u_{n,m}(t) = C_{1\,n,m}\,(t-t_0)^{\gamma-2}\,E_{\alpha,\gamma-1}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+$$

$$+C_{2\,n,m}\,(t-t_0)^{\gamma-1}\,E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+b_{n,m}h_{n,m}(t), \tag{22}$$

where

$$h_{n,m}(t)=\frac{1}{1+\mu_{n,m}^{2k}\left(\varepsilon_1+\varepsilon_2\mu_{n,m}^{2k}\right)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\,\omega\,(t-s)^{\alpha}\right)\,a\,(s)\,d\,s,$$

$C_{1\,n,m}$ and $C_{2\,n,m}$ are arbitrary constants.

By Fourier coefficients (18), the initial conditions (2) we rewrite in the forms

$$\lim_{t\to+t_0}J_{t_0+}^{2-\gamma}u_{n,m}(t)=\int\limits_0^l\int\limits_0^l\lim_{t\to+t_0}J_{t_0+}^{2-\gamma}U\;(t,\,x,\,y)\;\vartheta_{n,m}(x,\,y)\,d\,x\,d\,y=$$

$$=\int\limits_0^l\int\limits_0^l\varphi_1(x,\,y)\,\vartheta_{n,m}(x,\,y)\,d\,x\,d\,y=\varphi_{1\,n,m}, \tag{23}$$

$$\lim_{t\to+t_0}\frac{d}{dt}J_{t_0+}^{2-\gamma}u_{n,m}(t)=\int\limits_0^l\int\limits_0^l\lim_{t\to+t_0}\frac{d}{dt}J_{t_0+}^{2-\gamma}U\,(t,\,x,\,y)\;\vartheta_{n,m}(x,\,y)\,d\,x\,d\,y=$$

$$=\int\limits_0^l\int\limits_0^l\varphi_2(x,\,y)\,\vartheta_{n,m}(x,\,y)\,d\,x\,d\,y=\varphi_{2\,n,m}\,. \tag{24}$$

To find the unknown coefficients $C_{1\,n,m}$ and $C_{2\,n,m}$ in (22), we use conditions (23) and (24). Then from (22) we have

$$u_{n,m}(t)=\varphi_{1\,n,m}\,(t-t_0)^{\gamma-2}\,E_{\alpha,\gamma-1}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+$$

$$+\varphi_{2\,n,m}\,(t-t_0)^{\gamma-1}\,E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+$$

$$+\frac{b_{n,m}}{1+\mu_{n,m}^{2k}\left(\varepsilon_1+\varepsilon_2\mu_{n,m}^{2k}\right)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\,\omega\,(t-s)^{\alpha}\right)\,a\,(s)\,d\,s. \tag{25}$$

Substituting the presentation of the Fourier coefficients (25) of main unknown function into Fourier series (6), we obtain

$$U\,(t,\,x,\,y)=\sum_{n,\,m=1}^{\infty}\vartheta_{n,m}(x,\,y)\left[\varphi_{1\,n,m}\,(t-t_0)^{\gamma-2}\,E_{\alpha,\gamma-1}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+\right.$$

$$+\varphi_{2\,n,m}\,(t-t_0)^{\gamma-1}\,E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\;\omega\;(t-t_0)^{\alpha}\right)+$$

$$\left.+\frac{b_{n,m}}{1+\mu_{n,m}^{2k}\left(\varepsilon_1+\varepsilon_2\mu_{n,m}^{2k}\right)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_{n,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\,\omega\,(t-s)^{\alpha}\right)\,a\,(s)\,d\,s\right]. \tag{26}$$

This Fourier series (26) is a formal solution to the initial value problem (1)–(4).

3 Convergence of the Fourier series (26)

We prove absolute and uniform convergence of the Fourier series (26). We need to use the concepts of the following Banach spaces. Hilbert coordinate space ℓ_2 of number sequences $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ with norm

$$\|\varphi\|_{\ell_2} = \sqrt{\sum_{n,m=1}^{\infty} |\varphi_{n,m}|^2} < \infty.$$

The space $L_2(\Omega_l^2)$ of square-summable functions on the domain $\Omega_l^2 = \Omega_l \times \Omega_l$ with norm

$$\|\vartheta(x, y)\|_{L_2(\Omega_l^2)} = \sqrt{\int_0^l \int_0^l |\vartheta(x, y)|^2 \, dx\, dy} < \infty.$$

Conditions of smoothness. Let for functions

$$\varphi_i(x, y)\ (i = 1, 2),\ b(x, y) \in C^{4k}(\Omega_l^2)$$

there exist piecewise continuous $4k+1$ order derivatives. Then by integrating in parts the functions (20), (23) and (24) $4k+1$ times over every variable x, y, we obtain the following relations

$$|\varphi_{i\,n,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|\varphi_{i\,n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \ |b_{n,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|b_{n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \tag{27}$$

$$\left\|\varphi_{i\,n,m}^{(8k+2)}\right\|_{\ell_2} \leq \frac{2}{l} \left\| \frac{\partial^{8k+2}\varphi_i(x, y)}{\partial x^{4k+1}\, \partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)}, \tag{28}$$

$$\left\|b_{n,m}^{(8k+2)}\right\|_{\ell_2} \leq \frac{2}{l} \left\| \frac{\partial^{8k+2} b(x, y)}{\partial x^{4k+1}\partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)}, \tag{29}$$

where

$$\varphi_{i\,n,m}^{(8k+2)} = \int_0^l \int_0^l \frac{\partial^{8k+2}\varphi_i(x, y)}{\partial x^{4k+1}\, \partial y^{4k+1}}\, \vartheta_{n,m}(x, y)\, dx\, dy, \quad i = 1, 2,$$

$$b_{n,m}^{(8k+2)} = \int_0^l \int_0^l \frac{\partial^{8k+2} b(x, y)}{\partial x^{4k+1}\, \partial y^{4k+1}}\, \vartheta_{n,m}(x, y)\, dx\, dy.$$

To obtain estimates for solution, we use the properties of the Mittag–Leffler function [36]. Let $\alpha \in (0; 2)$ and $\gamma \in \mathbb{R}$. If $\arg z = \pi$, then there takes place the following estimate

$$|E_{\alpha,\gamma}(z)| \leq \frac{M_1}{1+|z|},$$

where $0 < M_1 = \text{const}$ does not depend from z.

Therefore, it is easy to see that there exists constant M_2 such that

$$\max_{t_0 \leq t \leq T} \left| E_{\alpha,\gamma-1}\left(-\lambda_{n,m}^{2k}(\varepsilon_1, \varepsilon_2)\ \omega\ (t-t_0)^{\alpha}\right) \right| \leq M_2 < \infty, \tag{30}$$

$$\max_{t_0 \leq t \leq T} \left| E_{\alpha,\gamma}\left(-\lambda_{n,m}^{2k}(\varepsilon_1, \varepsilon_2)\ \omega\ (t-t_0)^{\alpha}\right) \right| \leq M_2 < \infty, \tag{31}$$

$$\max_{t_0\le t\le T}\left|\int\limits_{t_0}^{t}(t-s)^{\alpha-1}\,(t-t_0)^{2-\gamma}\,E_{\alpha,\,\alpha}\left(-\lambda_{n,\,m}^{2k}\,(\varepsilon_1,\,\varepsilon_2)\,\omega\,(t-s)^{\alpha}\right)\,a\,(s)\,d\,s\right|\le M_2<\infty. \tag{32}$$

Theorem 1. Suppose that the conditions of smoothness and estimates (27)–(29) are fulfilled. Then Fourier series (26) convergence is absolute and uniform.

Proof. We apply the formulas (27)–(29) and estimates (30)–(32) to estimate the series (26). Using the Cauchy–Schwartz inequality for series (26), we get the estimate

$$\left|(t-t_0)^{2-\gamma}\,U\,(t,\,x,\,y)\right|\le M_2\sum_{n,\,m=1}^{\infty}|\,\vartheta_{\,n,\,m}(x,\,y)\,|\cdot[|\,\varphi_{\,1\,n,\,m}\,|+|\,\varphi_{\,2\,n,\,m}\,|+|\,b_{\,n,\,m}\,|]\le$$

$$\le\frac{2}{l}\,M_2\left[\sum_{n,\,m=1}^{\infty}|\,\varphi_{\,1\,n,\,m}\,|+\sum_{n,\,m=1}^{\infty}|\,\varphi_{\,2\,n,\,m}\,|+\sum_{n,\,m=1}^{\infty}|\,b_{\,n,\,m}|\right]\le$$

$$\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2}M_2\left[\sum_{n,\,m=1}^{\infty}\frac{\left|\varphi_{\,1\,n,\,m}^{(8k+2)}\right|}{n^{\,4k+1}m^{\,4k+1}}+\sum_{n,\,m=1}^{\infty}\frac{\left|\varphi_{\,2\,n,\,m}^{(8k+2)}\right|}{n^{\,4k+1}m^{\,4k+1}}+\sum_{n,\,m=1}^{\infty}\frac{\left|b_{n,\,m}^{(8k+2)}\right|}{n^{\,4k+1}m^{\,4k+1}}\right]\le$$

$$\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2}M_{\,2}M_{\,3}\left[\left\|\varphi_{\,1\,n,\,m}^{\,(8k+2)}\right\|_{\ell_{\,2}}+\left\|\varphi_{\,2\,n,\,m}^{\,(8k+2)}\right\|_{\ell_{\,2}}+\left\|b_{n,\,m}^{(8k+2)}\right\|_{\ell_{\,2}}\right]\le$$

$$\le\gamma_1\left[\sum_{i=1}^{2}\left\|\frac{\partial^{\,8k+2}\varphi_i(x,\,y)}{\partial\,x^{4k+1}\partial\,y^{4k+1}}\right\|_{L_{\,2}\left(\Omega_l^2\right)}+\left\|\frac{\partial^{\,8k+2}b\,(x,\,y)}{\partial\,x^{4k+1}\partial\,y^{4k+1}}\right\|_{L_{\,2}\left(\Omega_l^2\right)}\right]<\infty, \tag{33}$$

where

$$\gamma_1=M_{\,2}\,M_{\,3}\left(\frac{2}{l}\right)^{2}\left(\frac{l}{\pi}\right)^{8k+2},\quad M_{\,3}=\sqrt{\sum_{n,\,m=1}^{\infty}\frac{1}{n^{\,8k+2}\,m^{\,8k+2}}}<\infty.$$

From the estimate (33) the absolute and uniform convergence of Fourier series (26) implies. The Theorem 1 is proved.

4 Uniqueness of the solution

To establish the uniqueness of the function $U\,(t,\,x,\,y)$ we suppose that there are two functions U_1 and U_2 that satisfy the given conditions (1)–(4). Then their difference $U=U_1-U_2$ is a solution to differential equation (1), satisfying conditions (2)–(4) with zero functions $\varphi_1(x,\,y)=\varphi_2(x,\,y)=0$. By virtue of relations (23) and (24) we have that $\varphi_{\,1\,n,\,m}=\varphi_{\,2\,n,\,m}=0$. Hence, we that obtain from formulas (18) and (26) in the domain Ω follows the zero identity

$$\int\limits_0^l\int\limits_0^l(t-t_0)^{2-\gamma}\,U\,(t,\,x,\,y)\,\vartheta_{\,n,\,m}(x,\,y)\,d\,x\,d\,y\equiv 0.$$

Hence, by virtue of the completeness of the systems of eigenfunctions $\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi\,n}{l}x\right\}$, $\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi\,m}{l}y\right\}$ in $L_{\,2}\left(\Omega_l^{\,2}\right)$ we deduce that $U\,(t,\,x,\,y)\equiv 0$ for all $x\,\in\Omega_l^{\,2}\equiv[0,\,l]^{\,2}$ and $t\in\Omega_T\equiv[0;\,T]$.

Since $(t-t_0)^{\,2-\gamma}\,U\,(t,\,x,\,y)\in C\,\left(\overline{\Omega}\right)$, then $(t-t_0)^{\,2-\gamma}\,U\,(t,\,x,\,y)\equiv 0$ in the domain $\overline{\Omega}$. Therefore, the solution to the initial value problem (1)–(4) is unique in the domain $\overline{\Omega}$.

5 Term-by-term differentiation possibility

Theorem 2. Let the conditions of the Theorem 1 be fulfilled. Then term-by-term differentiation of the series (26) is possible.

Proof. The function (26) we differentiate the required number of times

$$\frac{\partial^{4k}}{\partial x^{4k}}(t-t_0)^{2-\gamma}U(t,x,y)=\sum_{n,m=1}^{\infty}\left(\frac{\pi n}{l}\right)^{4k}\vartheta_{n,m}(x,y)\,(t-t_0)^{2-\gamma}u_{n,m}(t), \tag{34}$$

$$\frac{\partial^{4k}}{\partial y^{4k}}(t-t_0)^{2-\gamma}U(t,x,y)=\sum_{n,m=1}^{\infty}\left(\frac{\pi m}{l}\right)^{4k}\vartheta_{n,m}(x,y)\,(t-t_0)^{2-\gamma}u_{n,m}(t), \tag{35}$$

where $u_{n,m}(t)$ is defined from the presentation (25).

The expansion of the following functions into Fourier series are defined in a similar way

$$(t-t_0)^{2-\gamma}D^{\alpha,\gamma}U(t,x,y),\ \ \frac{\partial^{4k}}{\partial x^{4k}}(t-t_0)^{2-\gamma}D^{\alpha,\gamma}U(t,x,y),\ \ \frac{\partial^{4k}}{\partial y^{4k}}(t-t_0)^{2-\gamma}D^{\alpha,\gamma}U(t,x,y).$$

We show the convergence of series (34) and (35). Analogously to the case of estimate (33), applying the Cauchy–Schwarz inequality, we obtain

$$\left|\frac{\partial^{4k}}{\partial x^{4k}}(t-t_0)^{2-\gamma}U(t,x,y)\right|\le\sum_{n,m=1}^{\infty}\left(\frac{\pi n}{l}\right)^{4k}\left|(t-t_0)^{2-\gamma}u_{n,m}(t)\right|\cdot|\vartheta_{n,m}(x,y)|\le$$

$$\le\frac{2}{l}\left(\frac{\pi}{l}\right)^{4k}M_2\left[\sum_{n,m=1}^{\infty}n^{4k}|\varphi_{1n,m}|+\sum_{n,m=1}^{\infty}n^{4k}|\varphi_{2n,m}|+\sum_{n,m=1}^{\infty}n^{4k}|b_{n,m}|\right]\le$$

$$\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}M_2\left[\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{1n,m}^{(8k+2)}\right|}{n\,m^{4k+1}}+\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{2n,m}^{(8k+2)}\right|}{n\,m^{4k+1}}+\sum_{n,m=1}^{\infty}\frac{\left|b_{n,m}^{(8k+2)}\right|}{n\,m^{4k+1}}\right]\le$$

$$\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}M_2M_4\left[\left\|\varphi_{1n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|\varphi_{2n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|b_{n,m}^{(8k+2)}\right\|_{\ell_2}\right]\le$$

$$\le\gamma_2\left[\sum_{i=1}^{2}\left\|\frac{\partial^{8k+2}\varphi_i(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}b(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right]<\infty, \tag{36}$$

where $\gamma_2=\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{4k+2}M_2M_4$, $M_4=\sqrt{\sum\limits_{n,m=1}^{\infty}\frac{1}{n\,m^{8k+2}}}<\infty$;

$$\left|\frac{\partial^{4k}}{\partial y^{4k}}(t-t_0)^{2-\gamma}U(t,x,y)\right|\le\sum_{n,m=1}^{\infty}\left(\frac{\pi m}{l}\right)^{4k}\left|(t-t_0)^{2-\gamma}u_{n,m}(t)\right|\cdot|\vartheta_{n,m}(x,y)|\le$$

$$\le\frac{2}{l}\left(\frac{\pi}{l}\right)^{4k}M_2\left[\sum_{n,m=1}^{\infty}m^{4k}|\varphi_{1n,m}|+\sum_{n,m=1}^{\infty}m^{4k}|\varphi_{2n,m}|+\sum_{n,m=1}^{\infty}m^{4k}|b_{n,m}|\right]\le$$

$$\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}M_2\left[\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{1n,m}^{(8k+2)}\right|}{n^{4k+1}m}+\sum_{n,m=1}^{\infty}\frac{\left|\varphi_{2n,m}^{(8k+2)}\right|}{n^{4k+1}m}+\sum_{n,m=1}^{\infty}\frac{\left|b_{n,m}^{(8k+2)}\right|}{n^{4k+1}m}\right]\le$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2} M_2 M_5 \left[\left\| \varphi_{1\,n,\,m}^{(8k+2)} \right\|_{\ell_2} + \left\| \varphi_{2\,n,\,m}^{(8k+2)} \right\|_{\ell_2} + \left\| b_{n,\,m}^{(8k+2)} \right\|_{\ell_2}\right] \leq$$

$$\leq \gamma_3 \left[\sum_{i=1}^{2} \left\| \frac{\partial^{8k+2}\varphi_i(x,\,y)}{\partial\, x^{4k+1}\partial\, y^{4k+1}} \right\|_{L_2(\Omega_l^2)} + \left\| \frac{\partial^{8k+2} b\,(x,\,y)}{\partial\, x^{4k+1}\partial\, y^{4k+1}} \right\|_{L_2(\Omega_l^2)}\right] < \infty, \tag{37}$$

where $\gamma_3 = \left(\frac{2}{l}\right)^2 \left(\frac{l}{\pi}\right)^{4k+2} M_2 M_5,\;\; M_5 = \sqrt{\sum\limits_{n,\,m=1}^{\infty} \frac{1}{n^{8k+2}m}} < \infty.$

The convergence of Fourier series for functions

$$(t-t_0)^{2-\gamma}\, D^{\alpha,\gamma} U\,(t,\,x,\,y),\quad \frac{\partial^{4k}}{\partial\, x^{4k}}\,(t-t_0)^{2-\gamma}\, D^{\alpha,\gamma} U\,(t,\,x,\,y),\quad \frac{\partial^{4k}}{\partial\, y^{4k}}\,(t-t_0)^{2-\gamma}\, D^{\alpha,\gamma} U\,(t,\,x,\,y)$$

is easy to prove, and the necessary estimates are obtained similarly to the cases of estimates (33), (36), and (37). Therefore, the function $U\,(t,\,x,\,y)$ belongs to the class of functions (4). Theorem 2 is proved.

6 Stability of the solution $U(t,x,y)$ with respect to given functions

Theorem 3. Suppose that all the conditions of Theorem 2 are fulfilled. Then, the function $U\,(t,\,x,\,y)$ as a solution to the problem (1)–(4) is stable with respect to given functions $\varphi_1(x,\,y),\;\varphi_2(x,\,y)$.

Proof. We show that the solution to the differential equation (1) $U\,(t,\,x,\,y)$ is stable with respect to a given functions $\varphi_1(x,\,y),\;\varphi_2(x,\,y)$. Let $U_1(t,\,x,\,y)$ and $U_2(t,\,x,\,y)$ be two different solutions to the initial value problem (1)–(4), corresponding to two different values of the functions $\varphi_{11}(x,\,y)$, $\varphi_{12}(x,\,y)$ and $\varphi_{21}(x,\,y),\;\varphi_{22}(x,\,y)$, respectively.

We put that $|\,\varphi_{11\,n,m} - \varphi_{12\,n,m}\,| + |\,\varphi_{21\,n,m} - \varphi_{22\,n,m}\,| < \delta_{n,m}$, where $0 < \delta_{n,m}$ is sufficiently small positive quantity and the series $\sum\limits_{n,\,m=1}^{\infty} |\,\delta_{n,m}\,|$ is convergent. Then, considering this, by virtue of the conditions of the theorem, from the Fourier series (26) it is easy to obtain that

$$\left|\, t^{2-\gamma}\,[U_1(t,\,x,\,y) - U_2(t,\,x,\,y)]\,\right| \leq$$

$$\leq \frac{2}{l}\sigma_3 \sum_{n,\,m=1}^{\infty} [|\,\varphi_{11\,n,m} - \varphi_{12\,n,m}\,| + |\,\varphi_{21\,n,m} - \varphi_{22\,n,m}\,|] < \frac{2}{l}\sigma_3 \sum_{n,\,m=1}^{\infty} |\,\delta_{n,m}\,| < \infty.$$

If we put $\varepsilon = \frac{2}{l}\sigma_3 \sum\limits_{n,\,m=1}^{\infty} |\,\delta_{n,m}\,| < \infty$, then from last estimate we finally obtain assertions about the stability of the solution to the differential equation (1) with respect to a given functions $\varphi_1(x,\,y),\;\varphi_2(x,\,y)$ in (2). The theorem 3 is proved.

Similarly, it is proved that there holds the following theorem.

Theorem 4. Suppose that all the conditions of Theorem 2 are fulfilled. Then, the function $U\,(t,\,x,\,y)$ as a solution to the problem (1)–(4) is stable with respect to given function $b\,(x,\,y)$ in the right-hand side of the differential equation (1).

Conclusions

In three-dimensional domain, the solvability of a mixed problem for a Hilfer type partial differential equation (1) of the higher even order with initial value conditions (2) and small positive parameters in mixed derivatives is considered. Suppose that the conditions of smoothness are fulfilled. Then the solution to this fractional differential equation of higher order for $1 < \alpha < \gamma \leq 2$ is studied in the class of regular functions. The Fourier series method is used and a countable system of ordinary differential equations is obtained (21). The initial value problem is integrated as an ordinary differential equation. We obtained the presentation for unknown function $U(t, x, y)$. Using the Cauchy–Schwarz inequality and the Bessel inequality, we proved the absolute and uniform convergence of the obtained Fourier series (26) for function $U(t, x, y)$ and its derivatives. It is proved that solution to the problem (1)–(4) $U(t, x, y)$ is stable with respect to given functions $b(x, y)$ and $\varphi_i(x, y)$, $i = 1, 2$.

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References

1 Асанова А.Т. Об однозначной разрешимости начально-краевой задачи для системы дифференциальных уравнений в частных производных третьего порядка / А.Т. Асанова // Дифференц. уравнения. — 2021. — 57. — № 1. — С. 125–129.

2 Асанова А.Т. О решении начально-краевой задачи для системы дифференциальных уравнений в частных производных третьего порядка / А.Т. Асанова // Изв. вузов. Математика. — 2019. — № 4. — С. 15–26.

3 Assanova A.T. An initial-boundary value problem for a higher-order partial differential equation / A.T. Assanova, A.D. Abildayeva, A.P. Sabalakhova // News of the National Academy of Sciences of the Republic of Kazakhstan. Series Physico-Mathematical. — 2020. — 302. — No. 2. — P. 133–141.

4 Assanova A.T. Initial-boundary-value problem for an integro-differential equation of the third order / A.T. Assanova, G.K. Vasilina, A.E. Imanchiev // Journal of Mathematical Sciences. — 2021. — 253. — No. 2. — P. 181–203.

5 Assanova A.T. A nonlocal problem for loaded partial differential equations of fourth order / A.T. Assanova, A.E. Imanchiyev, Zh.M. Kadirbayeva // Bulletin of the Karaganda University. Mathematics series. — 2020. — 97. — No. 1. — P. 6–16.

6 Барановская С.Н. Задача Коши и вторая смешанная задача для параболических уравнений с потенциалом Дирака, сосредоточенным в конечном числе заданных точек / С.Н. Барановская, Н.И. Юрчук // Дифференц. уравнения. — 2019. — 55. — № 3. — С. 356–360.

7 Ильин В.А. Единственность обобщенных решений смешанных задач для волнового уравнения с нелокальными граничными условиями // В.А. Ильин // Дифференц. уравнения. — 2008. — 44. — № 5. — С. 672–680.

8 Ильин В.А. Об эквивалентности двух определений обобщенного из класса L_p решения смешанной задачи для волнового уравнения / В.А. Ильин, А.А. Кулешов // Тр. МИАН. — 2014. — 284. — С. 163–168.

9 Ильин В.А. Смешанные задачи, описывающие придольные колебания стержня, состоящего из двух участков, имеющих разные плотности и разные упругости, но одинаковые импедансы / В.А. Ильин, П.В. Луференко // Докл. РАН. — 2009. — 428. — № 1. — С. 12–15.

10 Дженалиев М.Т. К обобщенной разрешимости нагруженного волнового уравнения / М.Т. Дженалиев // Дифференц. уравнения. — 1994. — 30. — № 4. — С. 723, 724.

11 Юлдашев Т.К. Нелокальная смешанная задача для интегро-дифференциального уравнения типа Буссинеска с вырожденным ядром / Т.К. Юлдашев // Укр. мат. журн. — 2016. — 68. — № 8. — С. 1115–1131.

12 Юлдашев Т.К. Смешанная задача для псевдопараболического интегро-дифференциального уравнения с вырожденным ядром / Т.К. Юлдашев // Дифференц. уравнения. — 2017. — 53. — № 1. — С. 101–110.

13 Mainardi F. Fractional calculus: some basic problems in continuum and statistical mechanics / F. Mainardi // In: Carpinteri, A., Mainardi, F. (eds.) Fractals and Fractional Calculus in Continuum Mechanics. Springer, Wien, 1997.

14 Area I. On a fractional order Ebola epidemic model / I. Area, H. Batarfi, J. Losada, J.J. Nieto, W. Shammakh, A. Torres // Advances in Difference Equations. — 2015. — 2015. — № 1. — ID 278. — P. 1–12.

15 Hussain A. Existence of solution and stability for the fractional order novel coronavirus (nCoV-2019) model / A. Hussain, D. Baleanu, M. Adeel // Advances in Difference Equations. — 2020. — 384.

16 Ullah S. A fractional model for the dynamics of tuberculosis infection using Caputo–Fabrizio derivative / S. Ullah, M.A. Khan, M. Farooq, Z. Hammouch, D. Baleanu // Discrete and Continuous Dynamical Systems. Ser. S. — 2020. — 13. — № 3. — P. 975–993.

17 Handbook of Fractional Calculus with Applications. Vols. 1-8. Tenreiro Machado J.A. (ed.). Berlin, Boston: Walter de Gruyter GmbH, 2019.

18 Kumar D. Editorial: fractional calculus and its applications in physics / D. Kumar, D. Baleanu // Frontiers Physics. — 2019. — 7. — № 6.

19 Sun H. A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications / H. Sun, A. Chang, Y. Zhang, W. Chen // Fractional Calculus and Applied Analysis. — 2019. — 22. — № 1. — P. 27–59.

20 Patnaik S. Applications of variable-order fractional operators: a review / S. Patnaik, J.P. Hollkamp, F. Semperlotti // Proceedings of the Royal Society, A. — 2020. — 476. — No. 2234. — P. 1–32.

21 Yuldashev T.K. On solvability of an initial value problem for Hilfer type fractional differential equation with nonlinear Maxima / T.K. Yuldashev, B.J. Kadirkulov // Daghestan Electronic Mathematical Reports. — 2020. — 14. — P. 48–65.

22 Yuldashev T.K. Boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator / T.K. Yuldashev, B.J. Kadirkulov // Axioms. — 2020. — 9. — No. 2. — ID 68. — P. 1–19.

23 Yuldashev T.K. Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator / T.K. Yuldashev, B.J. Kadirkulov // Ural Mathematical Journal. — 2020. — 6. — No. 1. — P. 153–167.

24 Yuldashev T.K. Inverse problem for a mixed type integro-differential equation with fractional order Caputo operators and spectral parameters / T.K. Yuldashev, E.T. Karimov // Axioms. — 2020. — 9. — 4. — ID 121. — P. 1–19.

25 Abdullaev O.Kh. Non-local problems with integral gluing condition for loaded mixed type equations involving the Caputo fractional derivative / O.Kh. Abdullaev, K.B. Sadarangani // Electronic Journal of Differential Equations. — 2016. — 2016. — No. 164. — 1–10 p.

26 Бердышев А.С. Об одной нелокальной задаче для параболического уравнения четвертого порядка с дробным оператором Джрбашяна-Нерсесяна / А.С. Бердышев, Б.Ж. Кадиркулов // Дифференц. уравнения. — 2016. — 52. — No. 1. — С. 123–128.

27 Islomov B.I. On a boundary value problem for a parabolic-hyperbolic equation with fractional order Caputo operator in rectangular domain / B.I. Islomov, U.Sh. Ubaydullayev // Lobachevskii Journal of Mathematics. — 2020. — 41. — No. 9. — P. 1801–1810.

28 Karimov E.T. Frankl-type problem for a mixed type equation with the Caputo fractional derivative / E.T. Karimov // Lobachevskii Journal of Mathematics. — 2020. — 41. — No. 9. — P. 1829–1836.

29 Karimov E. Nonlocal initial problem for second order time-fractional and space-singular equation / E. Karimov, M. Mamchuev, M. Ruzhansky // Hokkaido Mathematical Journal. 2020. — 49. — No. 2. — P. 349–361.

30 Malik S.A. An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions / S.A. Malik, S. Aziz // Computers and Mathematics with Applications. — 2017. — 73. — No. 12. — P. 2548–2560.

31 Serikbaev D. A source inverse problem for the pseudo-parabolic equation with the fractional Sturm–Liouville operator / D. Serikbaev, N. Tokmagambetov // Bulletin of the Karaganda university. Mathematics series. — 2020. — No. 4 (99). — P. 143–151.

32 Sadarangani K.B. A non-local problem with discontinuous matching condition for loaded mixed type equation involving the Caputo fractional derivative / K.B. Sadarangani, O.Kh. Abdullaev // Advances in Difference Equations. — 2016. — No. 241. — P. 1–10.

33 Sadarangani K.B. About a Problem for Loaded Parabolic-Hyperbolic Type Equation with Fractional Derivatives / K.B. Sadarangani, O.Kh. Abdullaev // International Journal of Differerential Equations. — 2016. — ID 9815796. — P. 1–6.

34 Yuldashev T.K. Inverse boundary value problem for a fractional differential equations of mixed type with integral redefinition conditions / T.K. Yuldashev, B.J. Kadirkulov // Lobachevskii Journal of Mathematics. — 2021. — 42. — No. 3. — P. 649–662.

35 Kim M.Ha. Operational method for solving multi-term fractional differential equations with the generalized fractional derivatives / M.Ha. Kim, Ri.G. Chol, O.H. Chol // Fractional Calculus and Applied Analyses. — 2014. — 17. — No. 1. — P. 79–95.

36 Джрбашян М.М. Интегральные преобразования и представления функций в комплексной области. — М.: Наука, 1966. — 671 с.

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Жоғары ретті Гильфер типті дифференциалдық теңдеу үшін аралас есеп туралы

Үшөлшемді облыста аралас туындыларда бастапқы шарттары және шағын оң параметрлері бар жұп ретті Гильфер типті дербес туындылы теңдеу үшін аралас есептің шешілу мүмкіндігі қарастырылған. Бұл жоғары ретті бөлшек дифференциалдық теңдеудің шешімі тұрақты функциялар класында зерттелген. Бөлшек операторының реті $1 < \alpha < 2$ тең болатын жағдай зерттелді. Фурье қатарларының әдісі қолданылып, қарапайым дифференциалдық теңдеулердің есептелетін жүйесі алынды. Бастапқы есеп қарапайым дифференциалдық теңдеу ретінде интегралданады, ал интегралдық тұрақтылар берілген бастапқы шарттарды пайдалана отырып табылады. Коши-Шварц теңсіздігі мен Бессель теңсіздігін пайдаланып, алынған Фурье қатарының абсолютті және біркелкі жинақтылығы дәлелденді. Берілген функцияларға қатысты есептің шешімінің тұрақтылығы да зерттелді.

Кілт сөздер: бөлшек реті, Гильфер операторы, аралас есеп, Фурье қатары, бастапқы шарттар, бірегей шешім.

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О смешанной задаче для дифференциального уравнения типа Хильфера высшего порядка

В трехмерной области рассмотрена разрешимость смешанной задачи для дифференциального уравнения в частных производных типа Хильфера четного порядка с начальными условиями и малыми положительными параметрами в смешанных производных. Решение этого дробного дифференциального уравнения высшего порядка изучено в классе регулярных функций. Исследован случай, когда порядок дробного оператора равен $1 < \alpha < 2$. Применен метод рядов Фурье, и получена счетная система обыкновенных дифференциальных уравнений. Начальная задача интегрируется как обыкновенное дифференциальное уравнение, и интегральные константы находятся с помощью заданных начальных условий. С помощью неравенства Коши–Шварца и неравенства Бесселя доказана абсолютная и равномерная сходимость полученного ряда Фурье. Изучена также устойчивость решения задачи по заданным функциям.

Ключевые слова: дробный порядок, оператор Хильфера, смешанная задача, ряды Фурье, начальные условия, однозначная разрешимость.

References

1 Assanova, A.T. (2021). Ob odnoznachnoi razreshimosti nachalno-kraevoi zadachi dlia sistemy differentsialnykh uravnenii v chastnykh proizvodnykh tretego poriadka [Unique solvability of an initial-boundary value problem for a system of third-order partial differential equations]. *Differentsialnye uravneniia – Differential Equations, 57*, 1, 125–129 [in Russian].

2 Assanova, A.T. (2019). O reshenii nachalno-kraevoi zadachi dlia sistemy differentsialnykh uravnenii v chastnykh proizvodnykh tretego poriadka [Solution to initial-boundary value problem for a system of partial differential equations of the third order]. *Izvestiia vuzov. Matematika – Russian Mathematics (Izv. VUZ)*, 4, 15–26 [in Russian].

3 Assanova, A.T., & Abildayeva, A.D. (2020). Sabalakhova A.P. An initial-boundary value problem for a higher-order partial differential equation. *News of the National Academy of Sciences of the Republic of Kazakhstan. Series Physico-Mathematical, 302*, 2, 133–141.

4 Assanova, A.T., Vasilina, G.K., & Imanchiev, A.E. (2021). Initial-boundary-value problem for an integro-differential equation of the third order. *Journal of Mathematical Sciences, 253*, 2, 181–203.

5 Assanova, A.T., Imanchiyev, A.E., & Kadirbayeva, Zh.M. (2020). A nonlocal problem for loaded partial differential equations of fourth order. *Bulletin of the Karaganda University. Mathematics series, 97*, 1, 6–16.

6 Baranovskaya, S.N., & Yurchuk, N.I. (2019). Zadacha Koshi i vtoraia smeshannaia zadacha dlia parabolicheskikh uravnenii s potentsialom Diraka, sosredotochennym v konechnom chisle zadannykh tochek [Cauchy problem and the second mixed problem for parabolic equations with Dirac potential concentrated in a finite number of given points]. *Differentsialnye uravneniia – Differential Equations, 55*, 3, 356–360 [in Russian].

7 Il'in, V.A. (2008). Edinstvennost obobshchennykh reshenii smeshannykh zadach dlia volnovogo uravneniia s nelokalnymi granichnymi usloviiami [Uniqueness of generalized solutions to mixed problems for the wave equation with nonlocal boundary conditions]. *Differentsialnye uravneniia – Differential Equations, 44*, 5, 672–680 [in Russian].

8 Ilin, V.A., & Kuleshov, A.A. (2014). Ob ekvivalentnosti dvukh opredelenii obobshchennogo iz klassa L_p resheniia smeshannoi zadachi dlia volnovogo uravneniia [Equivalence of two definitions of a generalized L_p solution to the initial–boundary value problem for the wave equation]. *Trudy MIAN – Proceedings of Steklov Mathematical Institute, 284,* 163–168.

9 Ilin, V.A., & Luferenko, P.V. (2009). Smeshannye zadachi, opisyvaiushchie pridolnye kolebaniia sterzhnia, sostoiashchego iz dvukh uchastkov, imeiushchikh raznye plotnosti i raznye uprugosti, no odinakovye impedansy [Mixed problems describing longitudinal oscillations of a rod consisting of two segments with different densities and different elasticities but equal impedances]. *Doklady Rossiiskoi akademii nauk – Reports of Academy of Sciences, 428*, 1, 12–15 [in Russian].

10 Jenaliev, M.T. (1994). K obobshchennoi razreshimosti nagruzhennogo volnovogo uravneniia [To the generalized solvability of a loaded wave equation]. *Differentsialnye uravneniia – Differential Equations, 30*, 4, 723–724.

11 Yuldashev, T.K. (2016). Nelokalnaia smeshannaia zadacha dlia integro-differentsialnogo uravneniia tipa Bussineska s vyrozhdennym yadrom [Nonlocal mixed-value problem for a Boussinesq-type integro-differential equation with degenerate kernel]. *Ukrainskii matematicheskii zhurnal — Ukrainian Mathematical Journal, 68*, 8, 1115–1131 [in Russian].

12 Yuldashev, T.K. (2017). Smeshannaia zadacha dlia psevdoparabolicheskogo integro-differentsialnogo uravneniia s vyrozhdennym yadrom [Mixed problem for pseudoparabolic integro-differential equation with degenerate kernel]. *Differentsialnye uravneniia – Differential Equations, 53*, 1, 101–110 [in Russian].

13 Mainardi, F. (1997). Fractional calculus: some basic problems in continuum and statistical mechanics. In: Carpinteri, A., Mainardi, F. (Eds.). Fractals and Fractional Calculus in Continuum Mechanics. Springer, Wien, 1997.

14 Area, I., Batarfi, H., Losada, J., Nieto, J.J., Shammakh, W., & Torres, A. (2015). On a fractional order Ebola epidemic model. *Advances in Difference Equations, 2015*, 1, ID 278, 1–12.

15 Hussain, A., Baleanu, D., & Adeel, M. (2020). Existence of solution and stability for the fractional order novel coronavirus (nCoV-2019) model. *Advances in Difference Equations, 2020*, 384.

16 Ullah, S., Khan, M.A., Farooq, M., Hammouch, Z., & Baleanu, D. (2020). A fractional model for the dynamics of tuberculosis infection using Caputo–Fabrizio derivative. *Discrete and Continuous Dynamical Systems, Ser. S, 13*, 3, 975–993.

17 Tenreiro Machado, J.A. (Ed.). (2019). Handbook of Fractional Calculus with Applications. Vols. 1-8. Berlin, Boston: Walter de Gruyter GmbH, 2019.

18 Kumar, D., & Baleanu, D. (2019). Editorial: fractional calculus and its applications in physics. *Frontiers Physics, 7*, 6.

19 Sun, H., Chang, A., Zhang, Y., & Chen, W. (2019). A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications. *Fractional Calculus and Applied Analysis, 22*, 1, 27–59.

20 Patnaik, S., Hollkamp, J.P., & Semperlotti, F. (2020). Applications of variable-order fractional operators: a review. *Proceedings of the Royal Society, A., 476*, 2234, 1–32.

21 Yuldashev, T.K., & Kadirkulov, B.J. (2020). On solvability of an initial value problem for Hilfer type fractional differential equation with nonlinear Maxima. *Daghestan Electronic Mathematical Reports, 14,* 48–65.

22 Yuldashev, T.K., & Kadirkulov, B.J. (2020). Boundary value problem for weak nonlinear partial differential equations of mixed type with fractional Hilfer operator. *Axioms, 9*, 2, ID 68, 1–19.

23 Yuldashev, T.K., & Kadirkulov, B.J. (2020). Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator. *Ural Mathematical Journal, 6*, 1, 153–167.

24 Yuldashev, T.K., & Karimov, E.T. (2020). Inverse problem for a mixed type integro-differential equation with fractional order Caputo operators and spectral parameters. *Axioms, 9*, 4, ID 121, 1–24.

25 Abdullaev, O.Kh., & Sadarangani, K.B. (2016). Nonlocal problems with integral gluing condition for loaded mixed type equations involving the Caputo fractional derivative. *Electronic Journal of Differential Equations, 2016*, 164, 1–10.

26 Berdyshev, A.S., & Kadirkulov, B.J. (2016). Ob odnoi nelokalnoi zadache dlia parabolicheskogo uravneniia chetvertogo poriadka s drobnym operatorom Dzhrbashiana–Nersesiana [On a nonlocal problem for a fourth-order parabolic equation with the fractional Dzhrbashyan–Nersesyan operator]. *Differentsialnye uravneniia – Differential Equations, 52*, 1, 123–128 [in Russian].

27 Islomov, B.I., & Ubaydullayev, U.Sh. (2020). On a boundary value problem for a parabolic-hyperbolic equation with fractional order Caputo operator in rectangular domain. *Lobachevskii Journal of Mathematics, 41*, 9, 1801–1810.

28 Karimov, E.T. (2020). Frankl-type problem for a mixed type equation with the Caputo fractional derivative. *Lobachevskii Journal of Mathematics, 41*, 9, 1829–1836.

29 Karimov, E., Mamchuev, M., & Ruzhansky, M. (2020). Nonlocal initial problem for second order time-fractional and space-singular equation. *Hokkaido Mathematical Journal, 49*, 2, 349–361.

30 Malik, S.A., & Aziz, S. (2017). An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions. *Computers and Mathematics with Applications, 73*, 12, 2548–2560.

31 Serikbaev, D., & Tokmagambetov, N. (2020). A source inverse problem for the pseudo-parabolic equation with the fractional Sturm-Liouville operator. *Bulletin of the Karaganda University. Mathematics series, 4*(99), 143–151.

32 Sadarangani, K.B., & Abdullaev, O.Kh. (2016). A nonlocal problem with discontinuous matching condition for loaded mixed type equation involving the Caputo fractional derivative. *Advances in Difference Equations, 2016*, 241, 1–12.

33 Sadarangani, K.B., & Abdullaev, O.Kh. (2016). About a Problem for Loaded Parabolic-Hyperbolic Type Equation with Fractional Derivatives. *International Journal of Differential Equations, 2016*, ID 9815796, 1–6.

34 Yuldashev, T.K., & Kadirkulov, B.J. (2021). Inverse boundary value problem for a fractional differential equations of mixed type with integral redefinition conditions. *Lobachevskii Journal of Mathematics, 42*, 3, 649–662.

35 Kim, M.Ha., Chol, Ri.G., & Chol, O.H. (2014). Operational method for solving multi-term fractional differential equations with the generalized fractional derivatives. *Fractional Calculus and Applied Analyses, 17*, 1, 79–95.

36 Dzhrbashian, M.M. (1966). Integralnye preobrazovaniia i predstavleniia funktsii v kompleksnoi oblasti [Integral transformations and representations of functions in complex domain]. Moscow: Nauka [in Russian].

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A problem with shift for a mixed-type model equation of the second kind in an unbounded domain

This article studies a problem with shift in the characteristics of different families in an unbounded domain for a mixed-type model equation of the second kind. The elliptic part of this problem is the vertical half-strip; the hyperbolic part is the characteristic triangle bounded by the characteristics of the equation. Using the extremum principle we prove the uniqueness of the solution. With the integral equations method we prove the existence of the solution.

Keywords: mixed-type equation of the second kind, problem with a shift, uniqueness and existence of a solution, extremum principle, method of integral equations.

1 Statement of the problem

Consider the following equation

$$u_{xx} + \operatorname{sign} y|y|^m u_{yy} = 0, \quad 0 < m < 1 \tag{1}$$

in unbounded mixed domain $\Omega = \Omega_1 \cup J \cup, \Omega_2$, where $\Omega_1 = \{(x,y) : 0 < x < 1,\ 0 < y < +\infty\}$ $J = \{(x,y) : 0 < x < 1,\ y = 0\}$ and Ω_2 is the domain of half-plane $y < 0$, bounded by the characteristics of equation (1)

$$AC:\ x - [2/(2-m)]\,(-y)^{(2-m)/2} = 0, \quad BC:\ x + [2/(2-m)]\,(-y)^{(2-m)/2} = 1,$$

going out of points $A(0,0)$ and $B(1,0)$ and intersecting at point $C(\frac{1}{2}, -\left(\frac{2-m}{4}\right)^{\frac{2}{2-m}})$, and by the AB segment of the abscissa axis, we assume the following notation:

$$\beta = m/(2m-4)\,,\ J_1 = \{(x,y) : 0 < y < +\infty,\ x = 0\},\ J_2 = \{(x,y) : 0 < y < +\infty,\ x = 1\},$$

$$\theta_0(x) = \left(\frac{x}{2}, -\left[\frac{2-m}{2}\cdot\frac{x}{2}\right]^{\frac{2}{2-m}}\right), \quad \theta_1(x) = \left(\frac{1+x}{2}, -\left[\frac{2-m}{2}\cdot\frac{1-x}{2}\right]^{\frac{2}{2-m}}\right).$$

Problem S^{∞}. Find function $u(x,y)$ that satisfies the following conditions: $u(x,y) \in C(\Omega \cup \overline{J_1} \cup \overline{J_2} \cup AC \cup BC) \cap C^1(\Omega_1 \cup J) \cap C^1(\Omega_2 \cup J) \cap C^2(\Omega_1 \cup \Omega_2)$, it satisfies equation (1) in domains Ω_1 and Ω_2, and has the following property $u_y(x,+0) = \nu(x) \in C^1(J)$ and at the ends of the interval it can turn to infinity of order -2β for $x = 0$ and of order $\frac{1}{2} - \beta$ for $x = 1$ with the following boundary conditions:

$$u(0,y) = \varphi_1(y)\,,\ u(1,y) = \varphi_2(y)\,,\ 0 \le y < +\infty, \tag{2}$$

$$\lim_{y\to+\infty} u(x,y) = 0,\ \text{uniformly in } x \in [0,1]\,, \tag{3}$$

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$$a\left(x\right) D_{0x}^{1-\beta}u\left[\theta_0\left(x\right)\right]+b(x)D_{x1}^{1-\beta}u\left[\theta_1\left(x\right)\right]=c(x),\ \forall x\in J, \tag{4}$$

$$u\left(x,-0\right)=u\left(x,+0\right), \quad \frac{\partial u\left(x,-0\right)}{\partial y}=-\frac{\partial u\left(x,+0\right)}{\partial y}. \tag{5}$$

Here $a^2\left(x\right)+b^2(x)\neq 0$, $\forall x\in \overline{J}$; $a\left(x\right)x^{-\beta}+b(x)(1-x)^{-\beta}\neq 0$, $\forall x\in \overline{J}$; the functions $\varphi_i\left(y\right)\in C(J_i)$ are such that $\varphi_1\left(0\right)=0$, $\varphi_2\left(0\right)=0$, and the integrals

$$\int\limits_0^\infty s^{-\frac{m}{2(2-m)}}\left|\varphi_i\left[\left(\frac{2-m}{2}s\right)^{\frac{2}{2-m}}\right]\right|ds, \quad \int\limits_0^\infty s^{-\frac{m}{(2-m)}}\left|\varphi_i\left[\left(\frac{2-m}{2}s\right)^{\frac{2}{2-m}}\right]\right|ds \ \ (i=1,2)$$

converge; $-1\leq \frac{b(x)(1-x)^{-\beta}}{a(x)x^{-\beta}+b(x)(1-x)^{-\beta}}\leq 0$; $a(x)=a_1(x)x^p,\ p<\beta,\ a_1\left(x\right),b\left(x\right),c\left(x\right)\in C(\overline{J})\cap C^3(J)$; here $D_{sx}^{\alpha}[f(x)]$ is the operator of fractional integro-differentiation in the sense of Riemann-Liouville [1].

2 Uniqueness of the solution

The solution to the Cauchy problem has the following form [2]:

$$u(x,y)=\int\limits_0^1 T\left\{\left[x-(1-2\beta)(-y)^{\frac{1}{1-2\beta}}\right]t\right\}\left[x+(1-2\beta)(-y)^{\frac{1}{1-2\beta}}-xt+(1-2\beta)(-y)^{\frac{1}{1-2\beta}}t\right]^{-\beta}\times$$

$$\times\left[x-(1-2\beta)(-y)^{\frac{1}{1-2\beta}}\right]^{1-\beta}(1-t)^{-\beta}dt+\frac{1}{2\cos\pi\beta}[2(1-2\beta)]^{1-2\beta}\times$$

$$\times\int\limits_0^1 T\left[x-(1-2\beta)(-y)^{\frac{1}{1-2\beta}}(2t-1)\right](-y)t^{-\beta}\left(1-t\right)^{-\beta}dt-\frac{\Gamma(2-2\beta)}{\Gamma^2(1-\beta)}\times$$

$$\times\int\limits_0^1 \nu_1\left[x-(1-2\beta)(-y)^{\frac{1}{1-2\beta}}(2t-1)\right](-y)t^{-\beta}(1-t)^{-\beta}dt, \tag{6}$$

where

$$\nu_1(x)=u_y(x,\ -0), \quad u(x,0)=\tau(x)=\Gamma(1-2\beta)D_{0x}^{2\beta-1}T(x). \tag{7}$$

Considering the definitions and properties of operators of fractional integro-differentiation in the sense of Riemann-Liouville from (6), we have (8)

$$U\left[\theta_0(x)\right]=\frac{\Gamma\left(1-\beta\right)}{2\cos\left(\pi\beta\right)}D_{0x}^{\beta-1}T(x)x^{-\beta}-\frac{\Gamma(2-2\beta)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}}D_{x1}^{\beta-1}\nu_1(x)(1-x)^{-\beta}. \tag{8}$$

$$U\left[\theta_1(x)\right]=\Gamma(1-\beta)D_{0x}^{\beta-1}T(x)(1-x)^{-\beta}+\frac{\Gamma(1-\beta)}{2\cos\pi\beta}D_{0x}^{\beta-1}T(x)(1-x)^{-\beta}-$$

$$-\frac{\Gamma(2-2\beta)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}}D_{0x}^{\beta-1}\nu_1(x)x^{-\beta}. \tag{9}$$

Now, substituting (8) and (9) into the boundary condition (4) considering (7), we obtain

$$\frac{\Gamma(2-2\beta)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}}\left[a(x)x^{-\beta}+b(x)(1-x)^{-\beta}\right]\nu_1(x)=$$

$$= c(x) + \frac{\Gamma(1-\beta)}{2\cos\pi\beta}\left[a(x)x^{-\beta} + b(x)(1-x)^{-\beta}\right]T(x) + \tag{10}$$

$$+ b(x)\Gamma(1-\beta)D_{x1}^{1-\beta}D_{0x}^{\beta-1}T(x)(1-x)^{-\beta}.$$

Next, consider the superposition of two operators

$$D_{x1}^{1-\beta}D_{0x}^{\beta-1}T(x)(1-x)^{-\beta},$$

where function $T(x)$ is continuous in the interval $(0,1)$ and integrable on the segment $[0,1]$. The following equality holds

$$D_{x1}^{1-\beta}D_{0x}^{\beta-1}T(x)(1-x)^{-\beta} = T(x)(1-x)^{-\beta}\cos\pi\beta + \frac{\sin\pi\beta}{\pi}\int_0^1 T(t)\frac{(1-t)^{1-2\beta}}{(1-x)^{1-\beta}(t-x)} \tag{11}$$

(the integral here is understood in the sense of the Cauchy's principal value). From (10), considering the properties mentioned above, we conclude that $\nu_1(x)$ belongs to the class of functions integrable on the segment $[0,1]$ and continuous in the interval $(0,1)$.

Theorem. Problem S^{∞} cannot have more than one solution.

Proof. Let $u(x,y)$ be the solution to homogeneous problem S^{∞}. At that $c(x) \equiv 0$. We can prove that $u(x,y) \equiv 0$ in $\Omega \cup J_1 \cup J_2 \cup \overline{AC} \cup \overline{BC}$.

First, we prove that $u(x,y) \equiv 0$ in $\Omega_1 \cup J_1 \cup J_2 \cup \overline{AB}$. Let us assume the opposite. Then there is domain $\Omega_{1\rho} = \{(x,y):\ 0<x<1,\ 0<y<\rho\}$, in which $u(x,y) \not\equiv 0$. Therefore, $\sup\limits_{\overline{\Omega}_{1\rho}} |u(x,y)| > 0$ and this value is reached at some point $(\xi,\eta) \in \overline{\Omega}_{1\rho}$.

We introduce the notation $\partial\Omega_{1\rho} = AB \cup BD \cup DP \cup PA$, where

$$AB = \{(x,y):\ 0<x<1,\ y=0\},\ BD = \{(x,y):\ x=1,\ 0<y<\rho\},$$

$$DP = \{(x,y):\ 0<x<1,\ y=\rho\},\ PA = \{(x,y):\ x=0,\ 0<y<\rho\}.$$

According to the extremum principle for elliptic equations [3], it follows that $(\xi,\eta) \notin \Omega_{1\rho}$. Due to homogeneous conditions (2) $(\xi,\eta) \notin \overline{BD} \cup \overline{PA}$. Then $(\xi,\eta) \in AB \cup \overline{DP}$. Let $(\xi,\eta) \in AB$, i.e., $\sup\limits_{\overline{\Omega}_{1\rho}} |u(x,y)| = \sup\limits_{\overline{AB}} |u(x,y)| = |u(\xi,0)| > 0,\ 0<\xi<1$. Then if $u(\xi,0) > 0\ (<0)$, i.e., $(\xi,0)$ is a point of positive maximum (negative minimum) of function $u(x,y)$, then according to the sign lemma proved in [4], and due to the Zaremba-Giraud principle [3], it follows that $(\xi,\eta) \notin AB$. Therefore, $(\xi,\eta) \in \overline{DP}$, i.e. $\sup\limits_{\overline{\Omega}_{1\rho}} |u(x,y)| = \sup\limits_{0\le x\le 1} |u(x,\rho)| > 0$. Taking arbitrary number $\rho_1 > \rho$, we obtain by the same method $\sup\limits_{\overline{\Omega}_{1\rho_1}} |u(x,y)| = \sup\limits_{0\le x\le 1} |u(x,\rho_1)| > 0$. Since $\Omega_{1\rho} \subset \Omega_{1\rho_1}$, then $\sup\limits_{\overline{\Omega}_{1\rho_1}} |u(x,y)| \ge$ $\ge \sup\limits_{\overline{\Omega}_{1\rho}} |u(x,y)| > 0$, i.e. $\sup\limits_{0\le x\le 1} |u(x,\rho_1)| \ge \sup\limits_{0\le x\le 1} |u(x,\rho)| > 0$. This implies that $\lim\limits_{y\to+\infty} u(x,y) \ne 0$, which contradicts condition (3). Therefore, $u(x,y) \equiv 0$, $(x,y) \in \Omega_1 \cup l_1 \cup l_2 \cup \overline{AB}$. Hence, from (6) and (10), it follows that $u(x,y) \equiv 0$ in $\overline{\Omega}_2$. Therefore, $u(x,y) \equiv 0$, $(x,y) \in \Omega \cup l_1 \cup l_2 \cup \overline{AC} \cup \overline{BC}$, whence follows the assertion of the theorem.

3 Existence of the solution

Solving the problem N in the area of ellipticity of equation (1) according to the S.V. Falkovich method [5], we obtain the solution in the following form:

$$u(x,y)=k\sqrt{y}\int\limits_0^1\nu\left(t\right)\sum_{n=1}^{\infty}\frac{\sin n\pi x\sin n\pi t}{n^{\alpha}}K_{\alpha}\left(2n\pi\alpha y^{\frac{1}{2\alpha}}\right)dt+$$

$$+\sqrt{y}\int\limits_0^{\infty}\left[\frac{2\alpha}{s}\right]^{\alpha}\varphi_1\left[\left(\frac{s}{2\alpha}\right)^{2\alpha}\right]sds\int\limits_0^{\infty}\lambda\frac{sh\left(1-x\right)\lambda}{sh\lambda}J_{-\alpha}\left(2\lambda\alpha y^{\frac{1}{2\alpha}}\right)J_{-\alpha}\left(\lambda s\right)d\lambda+$$

$$+\sqrt{y}\int\limits_0^{\infty}\left[\frac{2\alpha}{s}\right]^{\alpha}\varphi_2\left[\left(\frac{s}{2\alpha}\right)^{2\alpha}\right]sds\int\limits_0^{\infty}\lambda\frac{shx\lambda}{sh\lambda}J_{-\alpha}\left(2\lambda\alpha y^{\frac{1}{2\alpha}}\right)J_{-\alpha}\left(\lambda s\right)d\lambda, \qquad (12)$$

where $\alpha=\frac{1}{2-m}$, $k=-\frac{4\sin\pi\alpha\Gamma(1+\alpha)}{\pi(\pi\alpha)^{\alpha}}$, $\Gamma\left(z\right)$ is the gamma function [1], $K_{\alpha}\left(z\right)$ and $J_{\alpha}\left(z\right)$ are the Macdonald and Bessel functions, respectively [6]. Passing to the limit as $y\to 0$ in formula (12), we obtain the main functional relation between $\tau\left(x\right)$ and $\nu\left(x\right)$ brought from the area of ellipticity of equation (1):

$$\tau\left(x\right)=-\frac{2\Gamma\left(1+\alpha\right)}{\Gamma\left(1-\alpha\right)}\left(\frac{1}{\pi\alpha}\right)^{2\alpha}\int\limits_0^1\nu\left(t\right)\sum_{n=1}^{\infty}\frac{\sin n\pi x\sin n\pi t}{n^{2\alpha}}dt+F_1\left(x\right). \qquad (13)$$

From the hyperbolic area we have relation (10) between $\nu\left(x\right)$ and $T\left(x\right)$ which, considering (11), has the following form:

$$\frac{\Gamma\left(2-2\beta\right)}{\Gamma\left(1-\beta\right)\left[2\left(1-2\beta\right)\right]^{1-2\beta}}\nu_1\left(x\right)=-\frac{c\left(x\right)}{a\left(x\right)x^{-\beta}+b\left(x\right)\left(1-x\right)^{-\beta}}+\frac{\Gamma\left(1-\beta\right)}{2\cos\pi\beta}T\left(x\right)+ \qquad (14)$$

$$+\frac{\Gamma\left(1-\beta\right)b\left(x\right)}{a\left(x\right)x^{-\beta}+b\left(x\right)\left(1-x\right)^{-\beta}}\left[T\left(x\right)\left(1-x\right)^{-\beta}\cos\pi\beta+\frac{\sin\pi\beta}{\pi(1-x)^{1-\beta}}\int\limits_0^1\frac{T\left(t\right)dt}{\left(1-t\right)^{2\beta-1}\left(t-x\right)}\right].$$

Taking into account the gluing conditions (5), we eliminate $T\left(x\right)$ from (13) and (14). After some transformations, we obtain a singular integral equation:

$$A\left(x\right)\rho(x)+\frac{B\left(x\right)}{\pi i}\int\limits_0^1\rho\left(t\right)\left(\frac{1}{t-x}+\frac{1-2t}{t+x-2xt}\right)dt+$$

$$+\cos\pi\beta\mu\left(x\right)\int\limits_0^1\rho\left(t\right)K_1\left(x,t\right)dt=F\left(x\right), \qquad (15)$$

where $A\left(x\right)=1-\sin\pi\beta$, $B\left(x\right)=-i\cos\pi\beta\left[1+2\mu\left(x\right)\right]$, $\rho\left(x\right)=\nu\left(x\right)x^{-2\beta}$,

$$\mu\left(x\right)=\frac{b\left(x\right)\left(1-x\right)^{-\beta}}{a\left(x\right)x^{-\beta}+b\left(x\right)\left(1-x\right)^{-\beta}},$$

$$F\left(x\right)=\frac{\Gamma\left(1-\beta\right)\left[2\left(1-2\beta\right)\right]^{1-2\beta}x^{-2\beta}}{\Gamma\left(2-2\beta\right)}\left[\frac{\Gamma\left(1+2\beta\right)}{\Gamma\left(1-2\beta\right)}D_{0x}^{1-2\beta}F_1\left(x\right)-\frac{\Gamma\left(1+2\beta\right)c\left(x\right)}{a\left(x\right)x^{-\beta}+b\left(x\right)\left(1-x\right)^{-\beta}}\right],$$

$$K_1\left(x,t\right)=\left(\frac{t}{x}\right)^{1-2\beta}\left[\frac{1}{t+x-2xt}-\frac{1}{t+x}+\sum_{n=1}^{\infty}\left[\left(\frac{2n+t}{t}\right)^{1-2\beta}\frac{1}{2n-x+t}+\right.\right.$$

$$+\left(\frac{2n-t}{t}\right)^{1-2\beta}\frac{1}{2n-t+x}-\left(\frac{2n-t}{t}\right)^{1-2\beta}\frac{1}{2n-x-t}-\left(\frac{2n+t}{t}\right)\frac{1}{2n+x+t}\Bigg]$$

is the weakly-singular kernel. Since $A^2(x)-B^2(x)\neq 0$, therefore, the singular integral equation (15) is of the normal type. Now, changing the variables

$$z=\frac{t^2}{1-2t+t^2}\quad\text{and}\quad w=\frac{x^2}{1-2x+x^2}.$$

equation (15) is reduced to a singular integral equation with the Cauchy kernel. Then, applying the Carleman-Vekua regularization method [7, 8], we obtain an equivalent Fredholm equation of the second kind, the unconditional solvability of which follows from the uniqueness of the problem solution.

References

1 Смирнов М.М. Уравнения смешанного типа / М.М. Смирнов. — М.: Высш. шк., 1985. — 304 с.

2 Кароль А.И. Дзета-функция вырождающихся эллиптических операторов: автореф. дис. ... канд. физ.-мат. наук / А.И. Кароль. — Л.: ЛГУ, 1981. — 13 с.

3 Бицадзе А.В. Краевые задачи для эллиптических уравнений второго порядка / А.В Бицадзе. — М.: Наука, 1966. — 203 с.

4 Ивашкина Г.А. Краевая задача со смещением для уравнения второго рода / Г.А. Ивашкина, Л.М. Невоструев // Диффренц. уравнения. — 1978. — 14. — № 2. — С. 281–290.

5 Фалькович С.В. Об одном случае решения задачи Трикоми / С.В. Фалькович // Трансзвуковые течения газа: межвуз. сб. — Вып. 1. — Саратов, 1964. — С. 3–8.

6 Кузнецов Д.С. Специальные функции / Д.С. Кузнецов. — М.: Высш. шк., 1962. — 424 с.

7 Мусхелишвили Н.И. Сингулярные интегральные уравнения / Н.И. Мусхелишвили. — М.: Физматлит, 1962. — 600 с.

8 Гахов Ф.Д. Краевые задачи / Ф.Д. Гахов. — М.: ГИФМЛ, 1963. — 639 с.

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Шексіз облыста екінші текті аралас типті модельдік теңдеу үшін ығысу есебі

Мақалада екінші текті аралас типті модельдік теңдеу үшін шектелмеген облыстағы әртүрлі сипаттамаларының ығысу есебі зерттелген. Облыстық эллипстік бөлігі – тік жарты жолақ, ал гиперболалық бөлігі – теңдеу сипаттамаларымен шектелген сипаттамалық үшбұрыш. Шешімнің бірегейлігі экстремум принципі арқылы, ал шешімнің бар екендігі интегралдық теңдеулер әдісімен дәлелденген.

Кілт сөздер: екінші текті аралас типті теңдеу, ығысуы бар есеп, шешімнің жалғыздығы және бар болуы, экстремум принципі, интегралдық теңдеулер әдісі.

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Задача со смещением для модельного уравнения смешанного типа второго рода в неограниченной области

В статье в неограниченной области для модельного уравнения смешанного типа второго рода исследована задача со смещением на характеристиках различных семейств. Эллиптическая часть области представляет собой вертикальную полуполосу, а гиперболическая часть — характеристический треугольник, ограниченный характеристиками уравнения. Единственность решения доказана с помощью принципа экстремума, а существование решения — методом интегральных уравнений.

Ключевые слова: уравнение смешанного типа второго рода, задача со смещением, единственность и существование решения, принцип экстремума, метод интегральных уравнений.

References

1 Smirnov, M.M. (1985). *Uravneniia smeshannogo tipa [Mixed-type equations]*. Moscow: Vysshaia shkola [in Russian].

2 Karol, A.I. (1981). Dzeta-funktsiia vyrozhdaiushchikhsia ellipticheskikh operatorov [Zeta function of degenerate elliptic operators]. *Extended abstract of candidate's thesis.* Leningrad: Leningradkii gosudarstvennyi universitet [in Russian].

3 Bitsadze, A.V. (1966). *Kraevye zadachi dlia ellipticheskikh uravnenii vtorogo poriadka [Boundary value problems for second order elliptic equations]*. Moscow: Nauka [in Russian].

4 Ivashkina, G.A., & Nevostruev, L.M. (1978). Kraevaia zadacha so smeshcheniem dlia uravneniia vtorogo roda [Boundary value problem with a shift for an equation of the second kind]. *Differentsialnye uravneniia — Differential Equations, 14*, 2, 281–290 [in Russian].

5 Falkovich, S.V. (1964). Ob odnom sluchae resheniia zadachi Trikomi [On a case of solving the Tricomi problem] *Okolozvukovye potoki gaza — Transonic gas flows.* Saratov: Saratov University, 3–6 [in Russian].

6 Kuznetsov, D.S. (1962). *Spetsialnye funktsii [Special functions]*. Moscow: Vysshaia shkola [in Russian].

7 Muskhelishvili, N.I. (1962). *Singuliarnye integralnye uravneniia [Singular integral equations]*. Moscow: GIFML [in Russian].

8 Gakhov, F.D. (1963). *Kraevye zadachi [Boundary value problems]*. Moscow: Fizmatgiz [in Russian].

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