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E-mail: vestnikku@gmail.com. Web-site: mathematics-vestnik.ksu.kz

Editors
Zh.T. Nurmukhanova, S.S. Balkeyeva, Z.E. Ramazanova
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A.A. Aniyarov[1], S.A. Jumabayev[2], D.B. Nurakhmetov[1,*], R.K. Kussainov[1,3]

[1] *Institute of Mathematics and Mathematical Modelling, Almaty, Kazakhstan;*
[2] *Institute of Management, Academy of Public Administration under the President of the Republic of Kazakhstan, Nur-Sultan, Kazakhstan;*
[3] *Shakarim University of Semey, Semey, Kazakhstan;*
(E-mail: aniyarov@math.kz, Serik.Jumabayev@apa.kz, nurakhmetov@math.kz, rinat.k.kus@mail.ru)

A hybrid algorithm for solving inverse boundary problems with respect to intermediate masses on a beam

The inverse problem of determining the weight of three intermediate masses on a uniform beam from the known three natural frequencies has been solved. The performed numerical analysis allows restoring the value of only the second mass in a unique way. The inverse problem of determining the weight of three intermediate masses has been solved uniquely except in the case when the first and the third masses are located geometrically symmetric relative to the middle of the beam. The hybrid algorithm for the unique solving inverse problem of determining the weight of three intermediate masses has been developed. The first three natural frequencies of the beam are calculated numerically by using the Maple computer package. Analytical relations between the masses are found.

Keywords: natural frequencies, beam equation, characteristic determinant, inverse problem, intermediate elements.

Introduction

Oscillatory systems with attached masses or attached masses and elastic couplings have been studied since the 18th century and enormous number of works were devoted to them [1–9]. In these works, mainly, problems of eigenvalues of the beam were investigated. In the listed works above, firstly, the influence on the spectrum of the geometry of the region on which the additional element is concentrated was illustrated. Secondly, the difference in the behavior of natural frequencies at large and small loads was demonstrated.

In recent years, methods of analysis of direct and inverse problems for differential operators with concentrated masses and elastic connections were actively developing [10–17]. These methods are paramount as they make it possible to develop technologies to ensure the safety of people. In contrast to works [18–22], in this paper, we develop a hybrid algorithm for solving the unique solution of the inverse problem of determining the weight of intermediate masses at points non-end of the beam from the three known natural frequencies. The novelty of this work is the geometrical symmetry of the location of the first and third masses relative to the middle of the beam for ambiguous definition of concentrated masses, which has been found on the basis of numerical calculations (see subsection 3.2).

[*] Corresponding author.
E-mail: nurakhmetov@math.kz

The problems of diagnosing the value of one concentrated mass were investigated numerically and experimentally in [18, 19]. Since changes in the values of concentrated masses on a beam can characterize the degree of disc frazzle, it is relevant to study mechanical systems with n elements ($1 < n < \infty$) ([20], $n = 2$). The masses were found uniquely up to rearrangement of their places in [20]. The sufficient conditions for the existence of a unique solution to the problem of identifying the concentrated mass and spring stiffness at the points non-end of the beam from the known first two natural frequencies were found in [21]. The inverse problem of determining the stiffness coefficients of intermediate springs on the beam from the two known natural frequencies was solved in [22]. The conditions on the disposition of intermediate springs were found in [22] where the spring stiffness coefficients were accurately determined up to their transposition.

In this paper, a beam with three intermediate masses m_1, m_2, and m_3 (kg) is considered. Units of measurement and abbreviations for all physical parameters considered in the article are standard.

The main goal of this work is to reveal the conditions for the geometric disposition of the concentrated masses for the nonunique solution of the inverse problem of restoring the concentrated masses with the known first three eigenfrequencies in advance.

The authors of this paper propose a hybrid algorithm that allows to calculate all three weights for concentrated masses with geometric symmetry of the location of the first and third masses relative to the middle of the beam. Note that the numerical method for solving the inverse problem allows to determine only the value of the second mass.

To solve the inverse problem, methods of the spectral theory of differential operators are applied. Justification of the proposed hybrid algorithm is carried out by using numerical calculations and analytical relationships (see subsection 3.3). The results of this study will contribute to the development of methods for solving inverse problems with multipoint internal elements.

2 Formulation of the main problem

Let the first mass be located at a distance a from the left end of the beam, the second mass at a distance b, respectively, and the third mass at a distance c, respectively (Fig. 1). As a result, the beam is divided into 4 sections: $-\frac{l}{2} < x < a - \frac{l}{2}$, $a - \frac{l}{2} < x < b - \frac{l}{2}$, $b - \frac{l}{2} < x < c - \frac{l}{2}$, $c - \frac{l}{2} < x < \frac{l}{2}$.

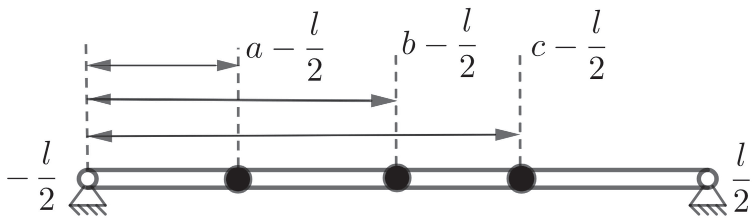

Figure 1. A beam with intermediate masses

The equation of free transverse vibrations of a beam of length l at $-\frac{l}{2} < x < \frac{l}{2}, t > 0$ is written as

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + EJ \frac{\partial^4 w(x,t)}{\partial x^4} = 0,$$

where $w(x,t)$ is transverse displacement, m; ρ is material density, kg/m^3 ; A is the cross-sectional area, m^2; E is the elastic modulus of material, N/m^2 ; J is the moment of inertia of the beam cross-section, m^4.

We use a method that is also applicable to beams with different types of fixation. For definiteness, we consider only a hinged-hinged beam. The problem of transverse vibrations of a beam of length l by replacement $w(x,t) = y(x)\sin(\omega t)$ is reduced to the following spectral problem:

$$EJy^{IV}(x) = \omega^2 \rho A y(x), \qquad x \neq a - \frac{l}{2},\ x \neq b - \frac{l}{2},\ x \neq c - \frac{l}{2}, \tag{1}$$

$$\left[EJy'''(x)\right]_{x=P_i} = -m_i \omega^2 y\left(P_i\right), \qquad i = 1, 2, 3. \tag{2}$$

$$\left[y(x)\right]_{x=P_i} = 0, \quad \left[Ey'(x)\right]_{x=P_i} = 0, \quad \left[EJy''(x)\right]_{x=P_i} = 0, \tag{3}$$

$$y(x)|_{x=-\frac{l}{2}} = 0, \quad EJy''(x)|_{x=-\frac{l}{2}} = 0, \tag{4}$$

$$y(x)|_{x=\frac{l}{2}} = 0, \quad EJy''(x)|_{x=\frac{l}{2}} = 0, \tag{5}$$

where $P_1 = a - \frac{l}{2}$, $P_2 = b - \frac{l}{2}$, $P_3 = c - \frac{l}{2}$ and

$$[f(x)]_{x=c} = \lim_{\varepsilon \to +0} [f(c-\varepsilon) - f(c+\varepsilon)] .$$

It means the jump of the function at the point $x = c$. Denote $p^4 = \frac{\omega^2 \rho A}{EJ}$, where ω is frequency parameter, Hz.

3 Material and methods

This section describes the main methods for solving the inverse problem of determining the concentrated masses from three known natural frequencies of the hinged-hinged Euler-Bernoulli beam. For this, it is necessary to write down an explicit form of the characteristic determinant of problem (1)–(5), which is important for calculating the first three natural frequencies. Then, we obtain a system of three nonlinear equations with three unknowns by using the known first three natural frequencies and the explicit form of the characteristic determinant. To find the physical parameters of the concentrated masses, the Maple computer package is used [23]. Some explicit relationships between the masses are found with the help of recurrent transformations, which are confirmed by numerical calculations.

3.1 The problem of transverse vibrations of a beam with intermediate masses

To calculate the natural frequencies of problem (1)–(5), an explicit form of the characteristic determinant is required. Let us formulate the main lemma.

Lemma 1. The values of the natural frequencies of problem (1)–(5) are determined from the equation

$$\begin{gathered}\Delta(a,b,c,l,p,m_1,m_2,m_3) = \alpha(a,b,c,l,p)\, m_1 m_2 m_3 + \beta_1(a,b,l,p)\, m_1 m_2 + \\ +\beta_2(a,c,l,p)\, m_1 m_3 + \beta_3(a,b,l,p)\, m_2 m_3 + \gamma_1(a,l,p)\, m_1 + \\ +\gamma_2(b,l,p)\, m_2 + \gamma_3(c,l,p)\, m_3 + \Delta_0(l,p) = 0,\end{gathered} \tag{6}$$

where

$$\begin{gathered}\alpha(a,b,c,l,p) = -\frac{p^3}{\rho^3 A^3}\left(2\left(\sin(p(a+b-l)) + \sin(p(a-b+l))\right)\left(\cosh(p(a-b+l)) - \cosh(p(a+b-l))\right) + \right. \\ +2\left(\sin(p(b+c-l)) - \sin(p(b-c+l))\right)\left(\cosh(p(b+c-l)) - \cosh(p(b-c+l))\right) + \\ +2\left(\cos(p(a+b-l)) - \cos(p(a-b+l))\right)\left(\sinh(p(a-b+l)) + \sinh(p(a+b-l))\right) + \\ +2\left(\cos(p(b+c-l)) - \cos(p(b-c+l))\right)\left(\sinh(p(b-c+l)) - \sinh(p(b+c-l))\right) + \\ +2\left(\cos(p(b+c-l)) - \cos(p(b-c+l))\right)\left(\sinh(p(2a-b+c-l)) - \sinh(p(2a-b-c+l))\right) + \\ +2\left(\cos(p(a-b+l)) - \cos(p(a+b-l))\right)\left(\sinh(p(a+b-2c+l)) + \sinh(p(a-b+2c-l))\right) + \\ +2\left(\cosh(p(a+b-l)) - \cosh(p(a-b+l))\right)\left(\sin(p(a+b-2c+l)) + \sin(p(a-b+2c-l))\right) + \\ +2\left(\cosh(p(b-c+l)) - \cosh(p(b+c-l))\right)\left(\sin(p(2a-b+c-l)) - \sin(p(2a-b-c+l))\right) + \\ \\ + \sin(p(2a-l))\left(\cosh(p(2b-l)) + \cosh(p(2c-l))\right) + \sin(p(2b-l))\left(\cosh(p(2a-l)) - \cosh(p(2c-l))\right) - \\ - \sin(p(2c-l))\left(\cosh(p(2a-l)) + \cosh(p(2b-l))\right) + \cos(p(2a-l))\left(\sinh(p(2c-l)) - \sinh(p(2b-l))\right) + \\ + \cos(p(2b-l))\left(\sinh(p(2c-l)) - \sinh(p(2a-l))\right) + \cos(p(2c-l))\left(\sinh(p(2b-l)) - \sinh(p(2a-l))\right) + \\ + \sin(p(2a-2b+l))\left(\cosh(pl) - \cosh(p(2c-l))\right) + \sin(p(2a-2c+l))\left(\cosh(pl) - \cosh(p(2b-l))\right) + \\ + \sin(p(2b-2c+l))\left(\cosh(pl) - \cosh(p(2a-l))\right) + \sinh(p(2a-2b+l))\left(\cos(p(2c-l)) - \cos(pl)\right) + \\ + \sinh(p(2a-2c+l))\left(\cos(p(2b-l)) - \cos(pl)\right) + \sinh(p(2b-2c+l))\left(\cos(p(2a-l)) - \cos(pl)\right) + \\ +2\left(\sinh(p(2a-l)) - \sinh(p(2c-l))\right)\cos(pl) + 2\left(\sin(p(2c-l)) - \sin(p(2a-l))\right)\cosh(pl) + \\ +2\left(\cosh(p(2b-l)) - \cosh(pl)\right)\sin(pl) + 2\left(\cos(pl) - \cos(p(2b-l))\right)\sinh(pl) - \\ - \left(\cosh(p(2a-2b+l)) + \cosh(p(2b-2c+l)) - \cosh(p(2a-2c+l))\right)\sin(pl) + \\ + \left(\cos(p(2a-2b+l)) + \cos(p(2b-2c+l)) - \cos(p(2a-2c+l))\right)\sinh(pl) + \\ \left. + \cosh(p(2a-2b+2c-l))\sin(pl) - \cos(p(2a-2b+2c-l))\sinh(pl)\right),\end{gathered}$$

$$\beta_1(a,b,l,p) = -\frac{1}{\rho^2A^2}\left(4p^2\left(\sin(pl)\sinh(p(2a-2b+l)) - \sinh(pl)\sin(p(2a-2b+l)\right) + \right.$$
$$+(\cos(pl)-\cos(p(2b-l)))\cosh(p(2a-l)) + (\cos(pl)-\cos(p(2a-l)))\cosh(p(2b-l)) +$$
$$+(\sinh(p(2b-l)) - \sinh(p(2a-l)))\sin(pl) + (\sin(p(2a-l)) - \sin(p(2b-l)))\sinh(pl) +$$
$$+2(\cos(p(a-b+l)) - \cos(p(a+b-l)))\cosh(p(a-b+l)) +$$
$$+2(\cos(p(a+b-l)) - \cos(p(a-b+l)))\cosh(p(a+b-l)) +$$
$$\left. + \ (\cos(p(2a-l)) + \cos(p(2b-l)))\cosh(pl) - 2\cosh(pl)\cos(pl))\right),$$

$$\beta_2(a,c,l,p) = -\frac{1}{\rho^2A^2}\left(4p^2\left(\sinh(pl)\sin(p(2a-2c+l)) - \sin(pl)\sinh(p(2a-2c+l)\right) + \right.$$
$$+(\cosh(p(2a-l)) - \cosh(pl))\cos(p(2c-l)) + (\cosh(p(2c-l)) - \cosh(pl))\cos(p(2a-l)) +$$
$$+(\sinh(p(2a-l)) - \sinh(p(2c-l)))\sin(pl) + (\sin(p(2c-l)) - \sin(p(2a-l)))\sinh(pl)$$
$$+2\left(\cosh(p(a+c-l)) - \cosh(p(a-c+l))\right)\cos(p(a-c+l)) +$$
$$+2\left(\cosh(p(a-c+l)) - \cosh(p(a+c-l))\right)\cos(p(a+c-l)) -$$
$$\left. - \ (\cosh(p(2a-l)) + \cosh(p(2c-l)))\cos(pl) + 2\cosh(pl)\cos(pl))\right),$$

$$\beta_3(b,c,l,p) = -\frac{1}{\rho^2A^2}\left(4p^2\left(\sin(pl)\sinh(p(2b-2c+l)) - \sinh(pl)\sin(p(2b-2c+l)\right) + \right.$$
$$+(\cos(pl)-\cos(p(2c-l)))\cosh(p(2b-l)) + (\cos(pl)-\cos(p(2b-l)))\cosh(p(2c-l)) +$$
$$+(\sinh(p(2c-l)) - \sinh(p(2b-l)))\sin(pl) + (\sin(p(2b-l)) - \sin(p(2c-l)))\sinh(pl) +$$
$$+2(\cos(p(b-c+l)) - \cos(p(b+c-l)))\cosh(p(b-c+l)) +$$
$$+2(\cos(p(b+c-l)) - \cos(p(b-c+l)))\cosh(p(b+c-l)) -$$
$$\left. + \ (\cos(p(2b-l)) + \cos(p(2c-l)))\cosh(pl) - 2\cosh(pl)\cos(pl))\right),$$

$$\gamma_1(a,l,p) = \frac{16p}{\rho A}\left((\cosh(pl) - \cosh(p(2a-l)))\sin(pl) + (\cos(pl) - \cos(p(2a-l)))\sinh(pl)\right),$$
$$\gamma_2(b,l,p) = -\frac{16p}{\rho A}\left((\cosh(pl) - \cosh(p(2b-l))\sin(pl) + (\cos(pl) - \cos(p(2b-l)))\sinh(pl)\right),$$
$$\gamma_3(c,l,p) = -\frac{16p}{\rho A}\left((\cosh(pl) - \cosh(p(2c-l)))\sin(pl) + (\cos(pl) - \cos(p(2c-l)))\sinh(pl)\right),$$
$$\Delta_0(l,p) = -64\sin(pl)\sinh(pl).$$

Here $\Delta_0(l,p)$ is the characteristic determinant without masses. To find ω the vibration frequencies from relation (6), the values p are first determined and then we find $\omega = \sqrt{\frac{EJ}{\rho A}}p^2$.

The proof of this lemma is carried out similarly to the method of in [22], and checking it is easy. To limit the volume of the paper, we present only a scheme for the proof of Lemma 1.

The scheme of the proof:

1. Let us write the fundamental systems solving equation (1) in four intervals.
2. We construct the solution of equation (1) on four intervals, which contain 16 constants.
3. Further, it is required that the solution of equation (1) satisfies the internal conditions (2), (3), and the boundary conditions (4), (5). Thus, we obtain a system of homogeneous nonlinear equations.
4. A priory for the existence of natural frequencies, it is necessary that the determinant of the resulting system of nonlinear equations is equal to zero.

3.2 Numerical calculations

In this subsection, a series of numerical calculations are carried out to reconstruct the quantity of the concentrated masses from the known first three natural frequencies. The experimental model consists of a steel beam with a radius of 0.01 m, a length of 6 m and simply supported at the ends. Then $EJ = 1649.34$ (Nm^2), $\rho = 7800$ (kg m^{-3}), $A = 3.14 \cdot 10^{-4}$ (m^2).

Table 1

Determination of the quantity of the masses from the known first three natural frequencies

ω_1	ω_2	ω_3	a	b	c	m_1	m_2	m_3
6.796	27.355	63.587	2	3	4	0.499	0.1	0.3
						0.3	0.1	0.499
6.926	27.35	60.08	1	3	5	0.5	0.2	0.299
						0.299	0.2	0.5
6.9	27.737	61.477	1.3	3	4.7	0.149	0.299	0.25
						0.25	0.299	0.149

Table 2

Determination of the quantity of the masses from the known first three natural frequencies

ω_1	ω_2	ω_3	a	b	c	m_1	m_2	m_3
6.799	27.308	63.799	2	2.5	4	0.3	0.1	0.499
						0.433	0.099	0.366
6.801	27.282	63.925	2	3.7	4	0.3	0.099	0.5
						0.585	0.1	0.215
6.982	27.228	60.863	1	4	5	0.299	0.1	0.5
						0.344	0.1	0.454

Table 3

Determination of the quantity of the masses from the known first three natural frequencies

ω_1	ω_2	ω_3	a	b	c	m_1	m_2	m_3
6.785	27.459	63.479	2.2	3	4	0.3	0.099	0.499
						-0.162	0.139	0.953
6.827	27.178	63.170	2	3	4.3	0.299	0.099	0.5

The results of numerical calculations illustrate that the geometrical arrangement of the first and third masses plays an important role in the ambiguous reconstruction of these quantities. It follows from Tables 1, 2 that the symmetrical arrangement of the first and third masses relative to the middle of the beam leads to ambiguous restoration of the values of these masses. Note that the location of the second mass between the first and third has no significant effect. It can be seen from Table 3 that the violation of symmetry with respect to the middle of the beam when the masses are located makes it possible to restore the values of all three masses uniquely. We calculate the natural frequencies of problem (1)–(5) with an accuracy of $\varepsilon = 10^{-6}$. Here ε means that for fixed values of a, b, c, l, m_1, m_2, m_3 condition $|\Delta(a, b, c, l, m_1, m_2, m_3)| < \varepsilon$, $i = 1, 2, 3$ is satisfied.

3.3 A relationship between the concentrated masses

It follows from tables 1-3 that the second mass is determined uniquely regardless of the geometric location. Therefore, in this section, the analytical relationships of the first and third masses between the second mass are shown.

Consider the inverse problem for determining the values of the mass. Assume that we know all the physical parameters, the location of the intermediate masses, as well as the first three natural frequencies of the transverse vibrations of the beam. It is required to determine the value of the first and third mass. Here, we assume that the second mass is uniquely determined numerically; therefore, the parameter is assumed to be known.

Lemma 2. The parameters m_1 and m_3 are determined by the following formulas, respectively:

If $b_{m_1} < 0$, then

$$m_1 = -\frac{2c_{m_1}}{b_{m_1} - \sqrt{b_{m_1}^2 - 4a_{m_1}c_{m_1}}}, \qquad m_3 = -\frac{a_2 m_1 m_2 + a_5 m_1 + a_6 m_2 + a_8}{a_1 m_1 m_2 + a_3 m_1 + a_4 m_1 + a_7}, \tag{7}$$

If $b_{m_1} \geq 0$, then

$$m_1 = -\frac{2c_{m_1}}{b_{m_1} + \sqrt{b_{m_1}^2 - 4a_{m_1}c_{m_1}}}, \qquad m_3 = -\frac{a_2 m_1 m_2 + a_5 m_1 + a_6 m_2 + a_8}{a_1 m_1 m_2 + a_3 m_1 + a_4 m_1 + a_7}, \tag{8}$$

where

$$a_{m_1} = (a_1b_2 - a_2b_1)m_2^2 + (a_1b_5 - a_2b_3 + a_3b_2 - a_5b_1)m_2 - b_3a_5 + b_5a_3,$$
$$b_{m_1} = (a_1b_6 - a_2b_4 + a_4b_2 - a_6b_1)m_2^2 - a_8b_3 + b_5a_7 - b_7a_5 + b_8a_3+$$
$$+(a_1b_8 - a_2b_7 + a_3b_6 + a_4b_5 - a_5b_4 - a_6b_3 + a_7b_2 - a_8b_1)m_2,$$
$$c_{m_1} = (a_4b_6 - a_6b_4)m_2^2 + (a_4b_8 - a_6b_7 + a_7b_6 - a_8b_4)m_2 - a_8b_7 + a_7b_8.$$

Proof of Lemma 2. Let p_1, p_2 and p_3 be zeros of $\Delta(p) := \Delta(a, b, c, l, p, m_1, m_2, m_3)$. Then, three equalities hold

$$\begin{cases} a_1m_1m_2m_3 + a_2m_1m_2 + a_3m_1m_3 + a_4m_2m_3 + a_5m_1 + a_6m_2 + a_7m_3 + a_8 = 0, \\ b_1m_1m_2m_3 + b_2m_1m_2 + b_3m_1m_3 + b_4m_2m_3 + b_5m_1 + b_6m_2 + b_7m_3 + b_8 = 0, \\ c_1m_1m_2m_3 + c_2m_1m_2 + c_3m_1m_3 + c_4m_2m_3 + c_5m_1 + c_6m_2 + c_7m_3 + c_8 = 0, \end{cases} \tag{9}$$

Here

$$a_1 = \alpha(a, b, c, l, p_i), \quad a_2 = \beta_1(a, b, l, p_i), \quad a_3 = \beta_2(a, c, l, p_i), \quad a_4 = \beta_3(b, c, l, p_i),$$
$$a_5 = \gamma_1(a, l, p_i), \quad a_6 = \gamma_2(b, l, p_i), \quad a_7 = \gamma_3(c, l, p_i), \quad a_8 = \Delta_0(l, p_i), \quad i = 1.$$

Similar designations are valid for $b_k, c_k,\ k = \overline{1,8}$ for $i = 2, 3$, respectively. We transform the system of nonlinear equations (9) into the following form

$$\begin{cases} (a_1m_1m_2 + a_3m_1 + a_4m_2 + a_7)m_3 + a_2m_1m_2 + a_5m_1 + a_6m_2 + a_8 = 0, \\ (b_1m_1m_2 + b_3m_1 + b_4m_2 + b_7)m_3 + b_2m_1m_2 + b_5m_1 + b_6m_2 + b_8 = 0. \\ (c_1m_1m_2 + c_3m_1 + c_4m_2 + c_7)m_3 + c_2m_1m_2 + c_5m_1 + c_6m_2 + c_8 = 0, \end{cases} \tag{10}$$

Using a linear combination from system (10), we obtain a quadratic equation with respect to m_1:

$$\begin{gathered} ((a_1b_2 - a_2b_1)m_2^2 + (a_1b_5 - a_2b_3 + a_3b_2 - a_5b_1)m_2 - b_3a_5 + b_5a_3)m_1^2+ \\ +((a_1b_6 - a_2b_4 + a_4b_2 - a_6b_1)m_2^2 - a_8b_3 + b_5a_7 - b_7a_5 + b_8a_3+ \\ +(a_1b_8 - a_2b_7 + a_3b_6 + a_4b_5 - a_5b_4 - a_6b_3 + a_7b_2 - a_8b_1)m_2)m_1+ \\ +(a_4b_6 - a_6b_4)m_2^2 + (a_4b_8 - a_6b_7 + a_7b_6 - a_8b_4)m_2 - a_8b_7 + a_7b_8 = 0 \end{gathered} \tag{11}$$

Let $b_{m_1} < 0$. Then the corresponding solution to a quadratic equation of the form $ax^2 + bx + c = 0$ is defined as

$$x = \frac{-2c}{b - \sqrt{b^2 - 4ac}} \tag{12}$$

which is used in Muller's method. Using formula (12) for the quadratic equation (11), we obtain the first formula from (7). After finding m_1, the second formula of (7) follows from (10). For the case $b_{m_1} \geq 0$, the proof of (8) is similar. Lemma 2 is proved.

3.4 The hybrid algorithm for solving the uniqueness of the inverse problem

Subsections 3.2 and 3.3 allow to formulate the hybrid algorithm for solving the uniqueness of the inverse problem of three concentrated masses from the known first three natural frequencies.

Hybrid algorithm:

1. All physical parameters of the beam are fixed, except concentrated masses.
2. The parameters of the first three natural frequencies are entered.
3. The value of m_2 is numerically found.
4. Using formula (7), we find m_1 and m_3.

To test the proposed hybrid algorithm, consider an example.

1. The experimental model consists of a steel beam with the radius of 0.01 m, the length of 6 m and the hinged fixation at the end. Then $EJ = 1649.34$ (Nm^2), $\rho = 7800$ (kg m^{-3}), $A = 3.14 \cdot 10^{-4}$ (m^2). The masses $m_1,\ m_2$, and m_3 are located from the left end of the beam at distances of 2 m, 3 m, and 4 m, respectively.
2. $\omega_1 = 6,796, \qquad \omega_2 = 27,355, \qquad \omega_3 = 63,587.$
3. $m_2 = 0,1.$
4. $m_1 = 0,3, \qquad m_3 = 0,499.$

The considered example confirms the validity of the proposed hybrid algorithm. Note that for the proposed algorithm the geometrical disposition of the concentrated masses does not matter. In the future, the practical interest will be to investigate the inverse problem for a beam with a variable foundation coefficient when the beam comprises some concentrated elements. The beams with the variable foundation coefficient without concentrated masses for various fixations were investigated in [24].

Conclusion

It can be concluded that for solving the inverse problem regarding intermediate masses on the beam, the geometric symmetry of the location of the first and third masses relative to the middle of the beam is essential based on the performed numerical calculations (see Table 1, 2). The numerical analysis allows restoring uniquely the value of the second mass, regardless of its location (see Table 3). The last formulated fact allowed us to find the analytical relationships of the first and third masses between the second. We have developed the hybrid algorithm for solving inverse problems for determining the weight of intermediate masses on a uniform beam from the known first three natural frequencies based on the revealed patterns. Our results can be useful for the development of methods of inverse problems in beam systems with attached elements.

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А.А. Аniяров[1], С.А. Джумабаев[2], Д.Б. Нұрахметов[1], Р.К. Құсаинов[1,3]

[1] *Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[2] *Қазақстан Республикасы Президентінің жанындағы Мемлекеттік басқару академиясы, Нұр-Сұлтан, Қазақстан;*
[3] *Семей қаласының Шәкәрім атындағы университеті, Семей, Қазақстан*

Бөренедегі аралық массаларға қатысты шекаралық кері есептерді шешуге гибридті алгоритм туралы

Белгілі үш жиілікпен біртекті бөренедегі үш аралық массаның салмақтарын анықтайтын кері есеп шешілді. Жүргізілген сандық есептеулер тек қана екінші массаның салмағын анықтауға мүмкіндік береді. Бөренедегі үш аралық массаның салмақтарын анықтайтын кері есеп бірінші мен екінші масса бөрененің ортасына қатысты геометриялық симметриялы орналасқан жағдайдан басқа жағдайларда бірмәнді шешіледі. Барлық үш аралық массаның салмақтарын анықтайтын кері есептің бірмәнді шешімі бар болуы үшін гибридті алгоритм жасалды. Алғашқы үш меншікті жиілік Maple компьютерлік пакеті арқылы есептелді. Массалар арасында аналитикалық қатынас табылды.

Кілт сөздер: меншікті жиіліктер, бөрене теңдеуі, сипаттамалық анықтауыш, кері есеп, аралық элементтер.

А.А. Аніяров[1], С.А. Джумабаев[2], Д.Б. Нурахметов[1], Р.К. Кусаинов[1,3]

[1] *Институт математики и математического моделирования, Алматы, Казахстан;*
[2] *Академия государственного управления при Президенте Республики Казахстан, Нур-Султан, Казахстан;*
[3] *Университет имени Шакарима города Семей, Семей, Казахстан*

Об одном гибридном алгоритме решения обратных граничных задач относительно промежуточных масс на балке

Решена обратная задача определения веса трех промежуточных масс на однородной балке по известным трем собственным частотам. Проведенный численный анализ позволяет единственным образом восстанавливать величину только второй массы. Обратная задача определения веса трех промежуточных масс решается однозначно, кроме случая, когда первая и третья массы расположены геометрически симметрично относительно середины балки. Для однозначного решения обратной задачи определения веса трех промежуточных масс разработан гибридный алгоритм. Первые три собственные частоты стержня вычислены численно с помощью компьютерного пакета Maple. Найдено аналитическое соотношение между массами.

Ключевые слова: собственные частоты, уравнение балки, характеристический определитель, обратная задача, промежуточные элементы.

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B.R. Bayraktar[1], A.Kh. Attaev[2,*]

[1] *Uludağ University, Bursa, Turkey;*
[2] *IAMA KBSC RAS, Nalchik, Russia*
(E-mail: bbayraktar@uludag.edu.tr, attaev.anatoly@yandex.ru)

Fractional Integral Inequalities for Some Convex Functions

In this paper, we obtained several new integral inequalities using fractional Riemann-Liouville integrals for convex s-Godunova-Levin functions in the second sense and for quasi-convex functions. The results were gained by applying the double Hermite-Hadamard inequality, the classical Holder inequalities, the power mean, and weighted Holder inequalities. In particular, the application of the results for several special computing facilities was given. Some applications to special means for arbitrary real numbers: arithmetic mean, logarithmic mean, and generalized log-mean, are provided.

Keywords: integral inequalıty, Hermite-Hadamard inequalitiy, convex function, s-Godunova-Levin convex, quasiconvex, fractional integral, Hölder inequality, power mean inequality.

Introduction

Here, we give some well-known definitions in the literature that have attracted the attention of many scientists in the field of convex analysis.

Definition 1. A function ψ defined on the interval $[\upsilon_1, \upsilon_2] \subset \mathbb{R}$ is convex on this interval if the inequality

$$\psi\left(\tau\xi + (1-\tau)\,\zeta\right) \leq \tau\psi\left(\xi\right) + (1-\tau)\,\psi\left(\zeta\right)$$

holds for all $\xi, \zeta \in [\upsilon_1, \upsilon_2]$ and $\tau \in [0,1]$.

Definition 2. ([1]) We say that $\psi : I \subset \mathbb{R} \to \mathbb{R}$ is a Godunova-Levin function or that ψ belongs to the class $Q(I)$, if f is non-negative and

$$\psi(\tau\xi + (1-\tau)\zeta) \leq \frac{\psi(\xi)}{\tau} + \frac{\psi(\zeta)}{1-\tau}, \forall \xi, \zeta \in I, \tau \in (0,1).$$

Definition 3. ([2]) A function $\psi : [\upsilon_1, \upsilon_2] \to \mathbb{R}$ is said to be $s-$Godunova$-$Levin convex, with $s \in [0,1]$, if inequality

$$\psi\left(\tau\xi + (1-\tau)\,\zeta\right) \leq \frac{\psi\left(\xi\right)}{\tau^s} + \frac{\psi\left(\zeta\right)}{\left(1-\tau\right)^s}$$

holds for all $\tau \in (0,1)$ and $\xi, \zeta \in [\upsilon_1, \upsilon_2]$.

Definition 4. ([2]) A function $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ is P function or that f belongs to the class of $P(I)$; if it is non-negative and for all $\xi, \zeta \in I$ and $\tau \in [0;1]$; satisfies the following inequality:

$$\psi(\tau\xi + (1-\tau)\zeta) \leq \psi(\xi) + \psi(\zeta).$$

It is obvious that $s-$Godunova-Levin type functions for $s = 0$ yield P functions.

Definition 5. ([3]) A function ψ defined on the interval $[\upsilon_1, \upsilon_2]$, if it satisfies the inequality

$$\psi\left(\tau\xi + (1-\tau)\,\zeta\right) \leq \max\left\{\psi\left(\xi\right),\ \psi\left(\zeta\right)\right\}$$

for all $\xi, \zeta \in I \subset \mathbb{R}$ and $\tau \in [0,1]$, then the function is called quasi-convex on this interval or ψ belongs to the class $QC(I)$.

It was established that any convex function is a quasiconvex function, but the converse is not true.

*Corresponding author.
E-mail: attaev.anatoly@yandex.ru

There are many classes of convexity of functions in the literature. These functional classes, which have different properties, often generalize each other. Over the past few decades, researchers have obtained a few interesting results related to mathematical inequalities using fractional integration operators for various classes of convex functions.

Fractional integral operators generalize the notion of definite integration and have a significant effect on generalizing mathematical inequalities, so these operators play a vital role in the development of science and technology. In recent decades, many different fractional integration operators have been formulated, many of which supplement and extend the classical operators. A special place among these operators is occupied by the Riemann – Liouville fractional integration operators.

In recent years, scientists have obtained some mathematical inequalities associated with various operators, for example, see [4–7]. Among these operators, the Riemann-Liouville operators, which have become classical, occupy a special place.

Definition 6. ([8]) Let $\psi \in L[\upsilon_1, \upsilon_2]$. Then the left and right Riemann-Liouville integrals of order $\alpha > 0$ with $\upsilon_1 \geq 0$ are defined by

$$J^{\alpha}_{\upsilon_1^+}\psi\ (\xi) = \frac{1}{\Gamma(\alpha)}\int_{\upsilon_1}^{\xi} (\xi-\tau)^{\alpha-1}\psi(\tau)\,d\tau\ , \qquad \xi > \upsilon_1$$

and

$$J^{\alpha}_{\upsilon_2 -}\psi\ (\xi) = \frac{1}{\Gamma(\alpha)}\int_{\xi}^{\upsilon_2} (\tau-\xi)^{\alpha-1}\psi(\tau)\,d\tau\ , \qquad \xi < \upsilon_2$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{\upsilon_1^+}\psi(\xi) = J^0_{\upsilon_2^-}\psi(\xi) = \psi(\xi)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

In the theory of inequalities, a important role is played by the double Hermite-Hadamard inequality:

Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $\upsilon_1, \upsilon_2 \in I$, with $\upsilon_1 < \upsilon_2$. The following double inequality holds

$$\psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \leq \frac{1}{\upsilon_2-\upsilon_1}\int\limits_{\upsilon_1}^{\upsilon_2}\psi(\xi)d\xi \leq \frac{\psi(\upsilon_1)+\psi(\upsilon_2)}{2}.$$

If ψ is a concave function, then the inequalities are in the opposite direction.

A few studies (for example, [5–16] and references therein) are devoted to obtaining, improving, and generalizing integral inequalities in terms of various fractional integral operators.

It is known that the class of quasiconvex functions includes the class of convex functions defined on finite closed intervals. Some references to quasiconvex functions and their applications can be seen in [7], [17–21] and references therein. Studies devoted to Godunova-Levin type convex functions can be seen, for example, in the works [22–27] and references therein.

In addition to the classical integral Hölder inequality and its version-the power mean, we use the weighted Hölder inequality([28]):

$$\left|\int_I \psi(\tau)\,s(\tau)\,h(\tau)\,d\tau\right| \leq \left(\int_I |\psi(\tau)|^p\,h(\tau)\,d\tau\right)^{\frac{1}{p}}\left(\int_I |s(\tau)|^q\,h(\tau)\,d\tau\right)^{\frac{1}{q}}$$

for $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and $h(\tau) \geq 0$, $\forall\tau \in I$.

Bayraktar in [9], proved the following identity:

Lemma 1. (Lemma 2.1, for $m = 1$) Let $\psi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, where $\upsilon_1, \upsilon_2 \in I$ with $\tau \in [0,1]$. If $\psi'' \in L[\upsilon_1, \upsilon_2]$. Then $\forall\alpha > 1$, with properties of Gamma function we have

$$\frac{2^{\alpha-2}\Gamma(\alpha)}{(\upsilon_2-\upsilon_1)^{\alpha-1}}\left[J^{\alpha-1}_{\frac{\upsilon_1+\upsilon_2}{2}+}\ \psi(\upsilon_2) + J^{\alpha-1}_{\frac{\upsilon_1+\upsilon_2}{2}-}\ \psi(\upsilon_1)\right] - \psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \qquad (1)$$

$$= \frac{(\upsilon_2-\upsilon_1)^2}{\alpha 2^{2-\alpha}}\left[\int_0^{\frac{1}{2}} \tau^{\alpha}\psi''(\tau\upsilon_1+(1-\tau)\upsilon_2)\,d\tau\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}(1-\tau)^{\alpha}\,\psi''(\tau\upsilon_1+(1-\tau)\upsilon_2)\,d\tau\right].$$

In [15], Sarıkaya *et al.* proved the following identity:

Lemma 2. ([15]) Let $\psi : [\upsilon_1, \upsilon_2] \to R$, be a differentiable mapping on (υ_1, υ_2). If $\psi' \in L[\upsilon_1, \upsilon_2]$, then the equality:

$$\frac{\psi(\upsilon_1) + \psi(\upsilon_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\upsilon_2 - \upsilon_1)^{\alpha}} \left[J^{\alpha}_{\upsilon_1^+} \psi(\upsilon_2) + J^{\alpha}_{\upsilon_2^-} \psi(\upsilon_1)\right] = \tag{2}$$

$$= \frac{\upsilon_2 - \upsilon_1}{2} \int_0^1 \left[(1-\tau)^{\alpha} - \tau^{\alpha}\right] \psi'(\tau\upsilon_1 + (1-\tau)\upsilon_2) \, d\tau$$

holds, where $\alpha > 0$.

The main goal of the article is to obtain new integral inequalities in terms of fractional integration operators of the Riemann−Liouville type on the basis of the formulated identities for $s-$Godunova−Levin and quasiconvex functions.

Main results

Theorem 1. Let $\psi : I = [\upsilon_1, \upsilon_2] \to \mathbb{R}$ be a differentiable function on (υ_1, υ_2) such that $\psi'' \in L[\upsilon_1, \upsilon_2]$. If $|\psi''|^q \in QC(I)$ and $q \geq 1$, then the inequality

$$\left| \frac{2^{\alpha-2}\Gamma(\alpha)}{(\upsilon_2 - \upsilon_1)^{\alpha-1}} \left[J^{\alpha-1}_{\frac{\upsilon_1+\upsilon_2}{2}^+} \psi(\upsilon_2) + J^{\alpha-1}_{\frac{\upsilon_1+\upsilon_2}{2}^-} \psi(\upsilon_1)\right] - \psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \right| \leq$$

$$\leq \left(\frac{\upsilon_2 - \upsilon_1}{2}\right)^2 \frac{1}{\alpha(\alpha+1)} \left[\max\left(|\psi''(\upsilon_1)|^q, \ |\psi''(\upsilon_2)|^q\right)\right]^{\frac{1}{q}}$$

holds, where $\alpha > 1$.

Proof. From (1) using the properties of the modulus and the power mean inequality, we get

$$\frac{2^{\alpha-2}\Gamma(\alpha)}{\alpha(\upsilon_2 - \upsilon_1)^{\alpha-1}} \left[J^{\alpha-1}_{\frac{\upsilon_1+\upsilon_2}{2}^+} \psi(\upsilon_2) + J^{\alpha-1}_{\frac{\upsilon_1+\upsilon_2}{2}^-} \psi(\upsilon_1)\right] - \psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right) = U$$

$$|U| \leq \frac{(\upsilon_2 - \upsilon_1)^2}{\alpha 2^{2-\alpha}} \left[\int_0^{\frac{1}{2}} \tau^{\alpha} |\psi''(\tau\upsilon_1 + (1-\tau)\upsilon_2)| \, d\tau + \right.$$

$$\left. + \int_{\frac{1}{2}}^1 (1-\tau)^{\alpha} |\psi''(\tau\upsilon_1 + (1-\tau)\upsilon_2)| \, d\tau \right] \leq$$

$$\leq \frac{(\upsilon_2 - \upsilon_1)^2}{\alpha 2^{2-\alpha}} \left[\left(\int_0^{\frac{1}{2}} \tau^{\alpha} d\tau\right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \tau^{\alpha} |\psi''(\tau\upsilon_1 + (1-\tau)\upsilon_2)|^q \, d\tau\right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(\int_{\frac{1}{2}}^1 (1-\tau)^{\alpha} d\tau\right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 (1-\tau)^{\alpha} |\psi''(\tau\upsilon_1 + (1-\tau)\upsilon_2)|^q \, d\tau\right)^{\frac{1}{q}} \right] =$$

$$= \frac{(\upsilon_2 - \upsilon_1)^2}{\alpha 2^{2-\alpha}} \frac{1}{2^{\alpha}(\alpha+1)} \left[\max\left(|\psi''(\upsilon_1)|^q, \ |\psi''(\upsilon_2)|^q\right)\right]^{\frac{1}{q}} =$$

$$= \left(\frac{\upsilon_2 - \upsilon_1}{2}\right)^2 \frac{1}{\alpha(\alpha+1)} \left[\max\left(|\psi''(\upsilon_1)|^q, \ |\psi''(\upsilon_2)|^q\right)\right]^{\frac{1}{q}}$$

Here, we used the quasi-convex of $|\psi''|^q$ on $[\upsilon_1, \upsilon_2]$ and it can be easily checked that

$$\int_{\frac{1}{2}}^1 (1-\tau)^{\alpha} d\tau = \int_0^{\frac{1}{2}} \tau^{\alpha} d\tau = \frac{1}{2^{\alpha+1}(\alpha+1)}.$$

This completes the proof.

Theorem 2. Let $\psi : I = [\upsilon_1, \upsilon_2] \to \mathbb{R}$ be a differentiable function on (υ_1, υ_2) such that $\psi'' \in L[\upsilon_1, \upsilon_2]$. If $|\psi''|$ is $s-$Godunova-Levin function on $[\upsilon_1, \upsilon_2]$ and $q \geq 1$, then the inequality

$$\left| \frac{2^{\alpha-2}\Gamma(\alpha)}{(\upsilon_2-\upsilon_1)^{\alpha-1}} \left[J^{\alpha-1}_{\frac{\upsilon_1+\upsilon_2}{2}+} \psi(\upsilon_2) + J^{\alpha-1}_{\frac{\upsilon_1+\upsilon_2}{2}-} \psi(\upsilon_1) \right] - \psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \right|$$
$$\leq \frac{(\upsilon_2-\upsilon_1)^2 \left[|\psi''(\upsilon_1)| + |\psi''(\upsilon_2)|\right]}{2^{2-\alpha}} \left[B_{\frac{1}{2}}(\alpha+1, \alpha-s+1) + \frac{1}{2^{\alpha-s+1}(\alpha-s+1)} \right]$$

holds, where $\alpha > 1, s \in [0,1], B_x(a,b) = \int_0^x \tau^{a-1}(1-\tau)^{b-1} d\tau$ is the incomplete Euler Beta function.

Proof. From (1) using the properties of the modulus and the power mean inequality, we get

$$\frac{2^{\alpha-2}\Gamma(\alpha)}{(\upsilon_2-\upsilon_1)^{\alpha-1}} \left[J^{\alpha-1}_{\frac{\upsilon_1+\upsilon_2}{2}+} \psi(\upsilon_2) + J^{\alpha-1}_{\frac{\upsilon_1+\upsilon_2}{2}-} \psi(\upsilon_1) \right] - \psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right) = U$$

$$|U| \leq \frac{(\upsilon_2-\upsilon_1)^2}{2^{2-\alpha}} \left[\int_0^{\frac{1}{2}} \tau^\alpha |\psi''(\tau\upsilon_1 + (1-\tau)\upsilon_2)| d\tau \right.$$
$$\left. + \int_{\frac{1}{2}}^1 (1-\tau)^\alpha |\psi''(\tau\upsilon_1 + (1-\tau)\upsilon_2)| d\tau \right]$$
$$\leq \frac{(\upsilon_2-\upsilon_1)^2}{\alpha 2^{2-\alpha}} \left[\int_0^{\frac{1}{2}} \tau^\alpha |\psi''(\tau\upsilon_1 + (1-\tau)\upsilon_2)| d\tau \right.$$
$$\left. + \int_{\frac{1}{2}}^1 (1-\tau)^\alpha |\psi''(\tau\upsilon_1 + (1-\tau)\upsilon_2)| d\tau \right].$$

Sinse $|\psi''|$ is a s-Godunova-Levin type function on $[\upsilon_1, \upsilon_2]$ for the integrals we can write

$$\int_0^{\frac{1}{2}} \tau^\alpha |\psi''(\tau\upsilon_1 + (1-\tau)\upsilon_2)| d\tau \leq$$
$$\leq \quad |\psi''(\upsilon_1)| \int_0^{\frac{1}{2}} \tau^{\alpha-s} d\tau + |\psi''(\upsilon_2)| \int_0^{\frac{1}{2}} \tau^\alpha (1-\tau)^{\alpha-s} d\tau \leq$$
$$\leq \quad \frac{|\psi''(\upsilon_1)|}{2^{\alpha-s+1}(\alpha-s+1)} + B_{\frac{1}{2}}(\alpha+1, \alpha-s+1)|\psi''(\upsilon_2)|.$$

And for the integral, we get

$$\int_{\frac{1}{2}}^1 (1-\tau)^\alpha |\psi''(\tau\upsilon_1 + (1-\tau)\upsilon_2)|^q d\tau \leq$$
$$\leq \quad |\psi''(\upsilon_1)| \int_{\frac{1}{2}}^1 \tau^{\alpha-s} (1-\tau)^\alpha d\tau + |\psi''(\upsilon_2)| \int_{\frac{1}{2}}^1 (1-\tau)^{\alpha-s} d\tau \leq$$
$$\leq \quad B_{\frac{1}{2}}(\alpha+1, \alpha-s+1)|\psi''(\upsilon_1)| + \frac{|\psi''(\upsilon_2)|}{2^{\alpha-s+1}(\alpha-s+1)}.$$

In this way

$$|U| \quad \leq \quad \frac{(\upsilon_2-\upsilon_1)^2}{2^{2-\alpha}} \left(B_{\frac{1}{2}}(\alpha+1, \alpha-s+1) + \frac{1}{2^{\alpha-s+1}(\alpha-s+1)} \right) \times$$
$$\times (|\psi''(\upsilon_1)| + |\psi''(\upsilon_2)|).$$

This completes the proof.

Corollary 1. If we choose $s = 1$ and $\alpha = 2$ in Theorem, then for Godunova-Levin function, we get the inequality

$$\left| \frac{1}{\upsilon_2-\upsilon_1} \int_{\upsilon_1}^{\upsilon_2} \psi(\tau) d\tau - \psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \right| \leq \frac{29(\upsilon_2-\upsilon_1)^2}{192} \left[|\psi''(\upsilon_1)| + |\psi''(\upsilon_2)|\right]. \tag{3}$$

Corollary 2. If we choose $s = 0$ and $\alpha = 2$ in Theorem , then for P function, we get

$$\left| \frac{1}{\upsilon_2 - \upsilon_1} \int_{\upsilon_1}^{\upsilon_2} \psi(\tau)\, d\tau - \psi\left(\frac{\upsilon_1 + \upsilon_2}{2} \right) \right| \leq \frac{7\left(\upsilon_2 - \upsilon_1\right)^2}{120} \left[|\psi''(\upsilon_1)| + |\psi''(\upsilon_2)| \right]. \tag{4}$$

Remark 1. Estimates (3) and (4) show that if the function $|\psi''|$ is $s-$Godunova-Levin convex or is P function, then the upper bound of the midpoint inequality is worse than the estimates available in the literature (for example, in [9] Corollary 2.1 and in [29] Proposition 1).

Lemma 3. $\psi : I = [\upsilon_1, \upsilon_2] \to \mathbb{R}$ be a differentiable function on (υ_1, υ_2). If $\psi' \in L[\upsilon_1, \upsilon_2]$, then $\forall \xi, \zeta \in [\upsilon_1, \upsilon_2]$, with $\xi < \zeta$ and $\alpha > 0$ we have:

$$\frac{1}{\zeta - \xi} \psi(\zeta) - \frac{\Gamma(\alpha + 1)}{(\zeta - \xi)^{\alpha+1}} J_{\zeta-}^{\alpha} \psi(\xi) = \int_0^1 (1 - \tau)^{\alpha} \psi'(\tau\xi + (1 - \tau)\zeta)\, d\tau.$$

Proof. By integrating parts

$$\begin{aligned} &\int_0^1 (1 - \tau)^{\alpha} \psi'(\tau\xi + (1 - \tau)\zeta)\, d\tau \\ &= \frac{1}{\zeta - \xi} \psi(\zeta) - \frac{\alpha}{\zeta - \xi} \int_0^1 (1 - \tau)^{\alpha - 1} \psi(\tau\xi + (1 - \tau)\zeta)\, d\tau \end{aligned}$$

and, applying the change of variable $u = \tau\xi + (1 - \tau)\zeta$ to the resulting integral, we obtain

$$\begin{aligned} &\frac{1}{\zeta - \xi} \psi(\zeta) - \frac{\alpha}{(\zeta - \xi)^{\alpha+1}} \int_{\xi}^{\zeta} (u - \xi)^{\alpha - 1} \psi(u)\, du \\ &= \frac{1}{\zeta - \xi} \psi(\zeta) - \frac{\alpha}{(\zeta - \xi)^{\alpha+1}} \Gamma(\alpha) J_{\zeta-}^{\alpha} \psi(\xi). \end{aligned}$$

The proof is finished.

Remark 2. If we choose $\xi = \upsilon_1$ and $\zeta = \upsilon_2$ in Lemma 3, we obtain

$$\frac{1}{\upsilon_2 - \upsilon_1} \psi(\upsilon_2) - \frac{\alpha}{(\upsilon_2 - \upsilon_1)^{\alpha+1}} \Gamma(\alpha) J_{\upsilon_2-}^{\alpha} \psi(\upsilon_1) = \int_0^1 (1 - \tau)^{\alpha} \psi'(\tau\upsilon_1 + (1 - \tau)\upsilon_2)\, d\tau.$$

Theorem 3. Let $I = [\upsilon_1, \upsilon_2] \to \mathbb{R}$, be a differentiable function on I° such that $\psi' \in L[\upsilon_1, \upsilon_2]$, with $\xi, \zeta \in [\upsilon_1, \upsilon_2]$. If $\psi' \in QC([\xi, \zeta])$. Then, for all $\alpha > 0$ we have

$$\frac{1}{\zeta - \xi} \psi(\zeta) - \frac{\Gamma(\alpha + 1)}{(\zeta - \xi)^{\alpha+1}} J_{\zeta-}^{\alpha} \psi(\xi) \leq \frac{1}{(\alpha + 1)} \max\{\psi'(\xi), \psi'(\zeta)\}.$$

Proof. Since $\psi'(\tau\xi + (1 - \tau)\zeta) \leq \max\{\psi'(\xi), \psi'(\zeta)\}$ for $\tau \in [0, 1]$ and from Lemma 3, we obtain

$$\begin{aligned} \frac{1}{\zeta - \xi} \psi(\zeta) - \frac{\alpha}{(\zeta - \xi)^{\alpha+1}} \Gamma(\alpha) J_{\zeta-}^{\alpha} \psi(\xi) &= \int_0^1 (1 - \tau)^{\alpha} \psi'(\tau\xi + (1 - \tau)\zeta)\, d\tau \\ &\leq \max\{\psi'(\xi), \psi'(\zeta)\} \int_0^1 (1 - \tau)^{\alpha}\, d\tau \\ &= \frac{1}{(\alpha + 1)} \max\{\psi'(\xi), \psi'(\zeta)\} \end{aligned}$$

this completes the proof of theorem.

Corollary 3. Under the conditions of Theorem 3, if we choose $\xi = \upsilon_1$ and $\zeta = \upsilon_2$ and the function ψ is increasing, then we get:

$$\begin{aligned} &\frac{1}{\upsilon_2 - \upsilon_1} \psi(\upsilon_2) - \frac{\alpha}{(\upsilon_2 - \upsilon_1)^{\alpha+1}} \Gamma(\alpha) J_{\upsilon_2-}^{\alpha} \psi(\upsilon_1) \\ &\leq \frac{1}{(\alpha + 1)} \max\{\psi'(\upsilon_1), \psi'(\upsilon_2)\} \leq \|\psi\|_{\infty} \frac{1}{\alpha + 1}. \end{aligned} \tag{5}$$

Corollary 4. In inequality (5), if we choose $\alpha = 1$, we have

$$\psi(\upsilon_2) - \frac{1}{\upsilon_2 - \upsilon_1}\int_{\upsilon_1}^{\upsilon_2} \psi(\xi)d\xi \leq \frac{(\upsilon_2 - \upsilon_1)}{2}\max\{\psi'(\upsilon_1), \psi'(\upsilon_2)\}.$$

Theorem 4. Let $\psi : I = [\upsilon_1, \upsilon_2] \to \mathbb{R}$, be a differentiable function on (υ_1, υ_2) such that $\psi' \in L[\upsilon_1, \upsilon_2]$, $\upsilon_1 \leq \xi < \zeta \leq \upsilon_2$. If $|\psi'|^q \in QC([\xi, \zeta])$, $q > 1$, $p = \frac{q}{q-1}$, then $\forall \alpha > 0$, we have

$$\left|\frac{1}{\zeta - \xi}\psi(\zeta) - \frac{\alpha\Gamma(\alpha)}{(\zeta - \xi)^{\alpha+1}} J_{\zeta-}^{\alpha}\psi(\xi)\right| \leq \frac{\max\left\{|\psi'(\xi)|^q, |\psi'(\zeta)|^q\right\}^{\frac{1}{q}}}{\alpha + 1}. \tag{6}$$

Proof. From Lemma 3, using the properties of the modulus and Hölder's inequality, taking into account that $|\psi'|^q$ is a quasiconvex function, we obtain:

$$\begin{aligned}
&\left|\frac{1}{\zeta - \xi}\psi(\zeta) - \frac{\alpha}{(\zeta - \xi)^{\alpha+1}}\Gamma(\alpha) J_{\zeta-}^{\alpha}\psi(\xi)\right| = \left|\int_0^1 (1-\tau)^{\alpha}\psi'(\tau\xi + (1-\tau)\zeta)\, d\tau\right| \\
&\leq \int_0^1 |(1-\tau)^{\alpha}|\, |\psi'(\tau\xi + (1-\tau)\zeta)|\, d\tau \\
&= \int_0^1 (1-\tau)^{\alpha\left(1-\frac{1}{q}\right)} (1-\tau)^{\alpha\frac{1}{q}} |\psi'(\tau\xi + (1-\tau)\zeta)|\, d\tau \\
&\leq \left(\int_0^1 (1-\tau)^{\alpha} d\tau\right)^{\frac{1}{p}} \left(\int_0^1 (1-\tau)^{\alpha} |\psi'(\tau\xi + (1-\tau)\zeta)|^q d\tau\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{\alpha + 1}\right)\max\left\{|\psi'(\xi)|^q, |\psi'(\zeta)|^q\right\}^{\frac{1}{q}},
\end{aligned}$$

this completes proof.

Corollary 5. If we choose $\xi = \upsilon_1$, $\zeta = \upsilon_2$ and $\alpha = 1$ in inequality (6), then

$$\left|\psi(\upsilon_2) - \frac{1}{\upsilon_2 - \upsilon_1}\int_{\upsilon_1}^{\upsilon_2}\psi(\xi)d\xi\right| \leq \frac{\upsilon_2 - \upsilon_1}{2}\max\left\{|\psi'(\upsilon_1)|^q, |\psi'(\upsilon_2)|^q\right\}^{\frac{1}{q}}.$$

Lemma 4. Let $\psi : I = [\upsilon_1, \upsilon_2] \to \mathbb{R}$ be a differentiable function on (υ_1, υ_2). If $\psi' \in L[\upsilon_1, \upsilon_2]$, then for all $\upsilon_1 \leq \xi < \zeta \leq \upsilon_2$ and $\alpha > 0$, we have.

$$\begin{aligned}
&\frac{1}{\xi - \zeta}[\psi(\xi) - \psi(\zeta)] + \frac{\alpha\Gamma(\alpha)}{(\zeta - \xi)}\left[J_{\xi+}^{\alpha}\psi(\zeta) - J_{\zeta-}^{\alpha}\psi(\xi)\right] \qquad (7)\\
&= \int_0^1 \tau^{\alpha}\psi'(\tau\xi + (1-\tau)\zeta)\, d\tau + \int_0^1 (1-\tau)^{\alpha}\psi'(\tau\xi + (1-\tau)\zeta)\, d\tau
\end{aligned}$$

Proof. Applying the method of integration by parts for each integral, we get:

$$\begin{aligned}
&\int_0^1 \tau^{\alpha}\psi'(\tau\xi + (1-\tau)\zeta)\, d\tau \\
&= \frac{1}{\xi - \zeta}\psi(\xi) + \frac{\alpha}{(\zeta - \xi)^{\alpha+1}}\int_{\xi}^{\zeta}(\zeta - u)^{\alpha-1}\psi(u)\, du, \qquad u = \tau\xi + (1-\tau)\zeta \\
&= \frac{1}{\xi - \zeta}\psi(\xi) + \frac{\alpha\Gamma(\alpha)}{(\zeta - \xi)^{\alpha+1}} J_{\xi+}^{\alpha}\psi(\zeta)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 (1-\tau)^{\alpha}\psi'(\tau\xi + (1-\tau)\zeta)\, d\tau \\
&= -\frac{1}{\xi - \zeta}\psi(\zeta) - \frac{\alpha}{(\zeta - \xi)^{\alpha+1}}\int_{\xi}^{\zeta}(u - \xi)^{\alpha-1}\psi(u)\, du, \quad u = \tau\xi + (1-\tau)\zeta \\
&= -\frac{1}{\xi - \zeta}\psi(\zeta) - \frac{\alpha\Gamma(\alpha)}{(\zeta - \xi)^{\alpha+1}} J_{\zeta-}^{\alpha}\psi(\xi).
\end{aligned}$$

Adding the last two equalities, we obtain (7).

Theorem 5. Let $\psi : I = [\upsilon_1, \upsilon_2] \to \mathbb{R}$ be a differentiable function on (υ_1, υ_2) . If $\psi' \in L[\upsilon_1, \upsilon_2]$ and $|\psi'|$ is $s-$Godunova-Levin type function. Then, for all $\upsilon_1 \leq \xi < \zeta \leq \upsilon_2$ and $\alpha > 0,\ s \in [0, 1)$ we have

$$\left|\frac{1}{\xi - \zeta}\left[\psi(\xi) - \psi(\zeta)\right] + \frac{\alpha\Gamma(\alpha)}{(\zeta - \xi)}\left[J^{\alpha}_{\xi+}\psi(\zeta) - J^{\alpha}_{\zeta-}\psi(\xi)\right]\right| \tag{8}$$

$$\leq \left[B(\alpha + 1, 1 - s) + \frac{1}{\alpha - s + 1}\right](|\psi(\xi)| + |\psi(\zeta)|),$$

where $B(\xi, \zeta) = \int_0^1 \tau^{\xi-1}(1 - \tau)^{\zeta-1}\, d\tau,\ \xi > 1,\ \zeta > 0$ is Euler Beta function.

Proof. From Lemma 4 and with properties of modulus

$$\left|\int_0^1 \tau^{\alpha}\psi'(\tau\xi + (1 - \tau)\zeta)\, d\tau + \int_0^1 (1 - \tau)^{\alpha}\psi'(\tau\xi + (1 - \tau)\zeta)\, d\tau\right|$$
$$\leq \int_0^1 \tau^{\alpha}|\psi'(\tau\xi + (1 - \tau)\zeta)|\, d\tau + \int_0^1 (1 - \tau)^{\alpha}|\psi'(\tau\xi + (1 - \tau)\zeta)|\, d\tau.$$

Since $|\psi'|$ is $s-$Godunova-Levin type function, we get

$$\int_0^1 \tau^{\alpha}|\psi'(\tau\xi + (1 - \tau)\zeta)|\, d\tau \leq |\psi'(\xi)|\int_0^1 \tau^{\alpha-s}d\tau + |\psi'(\zeta)|\int_0^1 \tau^{\alpha}(1 - \tau)^{-s}\, d\tau$$
$$= \left[\frac{1}{\alpha - s + 1}|\psi'(\xi)| + |\psi'(\zeta)|\, B(\alpha + 1, 1 - s)\right]$$

and

$$\int_0^1 (1 - \tau)^{\alpha}|\psi'(\tau\xi + (1 - \tau)\zeta)|\, d\tau = |\psi'(\xi)|\, B(1 - s, \alpha + 1) + |\psi'(\zeta)|\frac{1}{\alpha - s + 1}.$$

Finally, since $B(\xi, \zeta) = B(\zeta, \xi)$, we have (8). This completes the proof.

Theorem 6. Let $\psi : I = [\upsilon_1, \upsilon_2] \to \mathbb{R}$, be a differentiable function on (υ_1, υ_2) such that $\psi, g \in L[\upsilon_1, \upsilon_2]$ and $0 \leq \upsilon_1 < \upsilon_2$. If $|\psi|^p, |g|^q \in QC(I)$ and increasing on $[\upsilon_1, \upsilon_2]$, $q > 1$. Then for all for $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ the inequality

$$\frac{1}{\upsilon_2 - \upsilon_1}\left|\int_{\upsilon_1}^{\upsilon_2} \psi(\xi)\, g(\xi)\, h(\xi)\, d\xi\right| \leq \frac{\|\psi\|_{\infty}\|g\|_{\infty}}{2}$$
$$\times\left[\frac{\Gamma(\alpha + 1)}{(\upsilon_2 - \upsilon_1)^{\alpha+1}}\left[J^{\alpha}_{\upsilon_1^+}\psi(\upsilon_2) + J^{\alpha}_{\upsilon_2^-}\psi(\upsilon_1)\right] - \frac{1}{\upsilon_2 - \upsilon_1}\int_{\upsilon_1}^{\upsilon_2} \psi(\xi)\, d\xi\right]$$

holds, where

$$h(\tau\upsilon_1 + (1 - \tau)\upsilon_2) = \left[(1 - \tau)^{\alpha} + (\tau^{\alpha} - 1)\right]\psi(\tau\upsilon_1 + (1 - \tau)\upsilon_2) \geq 0$$

$\forall\ \tau \in [0, 1]$ and $\alpha \in [0, 1]$.

Proof. The following equality is obvious:

$$\frac{1}{\upsilon_2 - \upsilon_1}\int_{\upsilon_1}^{\upsilon_2} \psi(\xi)\, g(\xi)\, h(\xi)\, d\xi$$
$$= \int_0^1 \psi(\tau\upsilon_1 + (1 - \tau)\upsilon_2)\, g(\tau\upsilon_1 + (1 - \tau)\upsilon_2)\, h(\tau\upsilon_1 + (1 - \tau)\upsilon_2)\, d\tau$$

Now, we use the weighted Hölder inequality:

$$\frac{1}{\upsilon_2-\upsilon_1}\left|\int_{\upsilon_1}^{\upsilon_2}\psi(\xi)\,g(\xi)\,h(\xi)\,d\xi\right| \le \left(\int_0^1 |\psi(\tau\upsilon_1+(1-\tau)\upsilon_2)|^p\,h(\tau\upsilon_1+(1-\tau)\upsilon_2)\,d\tau\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |g(\tau\upsilon_1+(1-\tau)\upsilon_2)|^q\,h(\tau\upsilon_1+(1-\tau)\upsilon_2)\,d\tau\right)^{\frac{1}{q}}$$

$$\le \left[\max\left\{|\psi(\upsilon_1)|^p,|\psi(\upsilon_2)|^p\right\}\right]^{\frac{1}{p}}\left[\max\left\{|g(\upsilon_1)|^q,|g(\upsilon_2)|^q\right\}\right]^{\frac{1}{q}}$$

$$\times\left(\int_0^1 h(\tau\upsilon_1+(1-\tau)\upsilon_2)\,d\tau\right)^{\frac{1}{p}}\left(\int_0^1 h(\tau\upsilon_1+(1-\tau)\upsilon_2)\,d\tau\right)^{\frac{1}{q}}$$

$$= \left[\max\left\{|\psi(\upsilon_1)|^p,|\psi(\upsilon_2)|^p\right\}\right]^{\frac{1}{p}}\left[\max\left\{|g(\upsilon_1)|^q,|g(\upsilon_2)|^q\right\}\right]^{\frac{1}{q}}$$

$$\times\left(\int_0^1 h(\tau\upsilon_1+(1-\tau)\upsilon_2)\,d\tau\right)^{\frac{1}{p}+\frac{1}{q}}$$

$$= \left[\max\left\{|\psi(\upsilon_1)|^p,|\psi(\upsilon_2)|^p\right\}\right]^{\frac{1}{p}}\left[\max\left\{|g(\upsilon_1)|^q,|g(\upsilon_2)|^q\right\}\right]^{\frac{1}{q}}$$

$$\times\left(\int_0^1 h(\tau\upsilon_1+(1-\tau)\upsilon_2)\,d\tau\right)$$

$$= \|\psi\|_\infty\|g\|_\infty\left(\int_0^1 h(\tau\upsilon_1+(1-\tau)\upsilon_2)\,d\tau\right)$$

$$= \|\psi\|_\infty\|g\|_\infty\left(\int_0^1 \left[(1-\tau)^\alpha+(\tau^\alpha-1)\right]\psi(\tau\upsilon_1+(1-\tau)\upsilon_2)\,d\tau\right)$$

$$= \frac{\|\psi\|_\infty\|g\|_\infty}{2}\left[\frac{\Gamma(\alpha+1)}{(\upsilon_2-\upsilon_1)^{\alpha+1}}\left[J^\alpha_{\upsilon_2^-}\psi(\upsilon_1)+J^\alpha_{\upsilon_1^+}\psi(\upsilon_2)\right]-\frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2}\psi(\xi)\,d\xi\right],$$

this completes the proof.

Theorem 7. Let $\psi: I=[\upsilon_1,\upsilon_2]\to\mathbb{R}$, be a differentiable function on (υ_1,υ_2). If $|\psi'|^q\in QC(I)$ and $|\psi'|^q$ is increasing. Then for all $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\in[0,1]$ the inequality

$$\Gamma(\alpha+1)\left(J^\alpha_{\upsilon_1^+}\psi(\upsilon_2),J^\alpha_{\upsilon_2^-}\psi(\upsilon_1)\right)_{\alpha\in[0,1]} \tag{9}$$

$$\le\frac{\upsilon_2-\upsilon_1}{2}\left(\frac{1}{\alpha p^2-\alpha p+\alpha+1}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha-\alpha p+1}\right)^{\frac{1}{q}}\|\psi'\|_\infty$$

holds, where

$$\Gamma(\alpha+1)\left(J^\alpha_{\upsilon_1^+}\psi(\upsilon_2),J^\alpha_{\upsilon_2^-}\psi(\upsilon_1)\right)_{\alpha\in[0,1]}$$

$$=\left|\frac{\psi(\upsilon_1)+\psi(\upsilon_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\upsilon_2-\upsilon_1)^\alpha}\left[J^\alpha_{\upsilon_1^+}\psi(\upsilon_2)+J^\alpha_{\upsilon_2^-}\psi(\upsilon_1)\right]\right|.$$

Proof. From (2), taking into account the properties of the module, we get:

$$\Gamma(\alpha+1)\left(J^\alpha_{\upsilon_1^+}\psi(\upsilon_2),J^\alpha_{\upsilon_2^-}\psi(\upsilon_1)\right)_{\alpha\in[0,1]} \tag{10}$$

$$\leq \frac{\upsilon_2-\upsilon_1}{2}\left[\int_0^1 \left|(1-\tau)^{\alpha}-\tau^{\alpha}\right|\left|\psi'\left(\tau\upsilon_1+(1-\tau)\upsilon_2\right)\right|d\tau\right].$$

We know that for $\alpha \in [0,1]$ and $\forall \tau_1, \tau_2 \in [0,1]$,

$$\left|\tau_1^{\alpha}-\tau_2^{\alpha}\right| \leq \left|\tau_1-\tau_2\right|^{\alpha} \ ,$$

that is

$$\int_0^1 \left|(1-\tau^{\alpha})-\tau^{\alpha}\right|d\tau \leq \int_0^1 \left|1-2\tau\right|^{\alpha}d\tau.$$

Taking into account last inequality and using the power mean inequality for (10), we obtain:

$$\begin{aligned}
&\Gamma(\alpha+1)\left(J^{\alpha}_{\upsilon_1^+}\psi(\upsilon_2), J^{\alpha}_{\upsilon_2^-}\psi(\upsilon_1)\right)_{\alpha\in[0,1]}\\
&\leq \frac{\upsilon_2-\upsilon_1}{2}\left[\int_0^1 |1-2\tau|^{\alpha}\left|\psi'\left(\tau\upsilon_1+(1-\tau)\upsilon_2\right)\right|d\tau\right]\\
&= \frac{\upsilon_2-\upsilon_1}{2}\left[\int_0^1 |1-2\tau|^{\alpha p}\left|\psi'\left(\tau\upsilon_1+(1-\tau)\upsilon_2\right)\right||1-2\tau|^{\alpha(1-p)}d\tau\right]\\
&\leq \frac{\upsilon_2-\upsilon_1}{2}\left(\int_0^1 |1-2\tau|^{\alpha p^2}|1-2\tau|^{\alpha(1-p)}d\tau\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1 \left|\psi'\left(\tau\upsilon_1+(1-\tau)\upsilon_2\right)\right|^q|1-2\tau|^{\alpha(1-p)}d\tau\right)^{\frac{1}{q}}\\
&= \frac{\upsilon_2-\upsilon_1}{2}\left(\int_0^1 |1-2\tau|^{\alpha p^2+\alpha(1-p)}d\tau\right)^{\frac{1}{p}}\left(\int_0^1 |1-2\tau|^{\alpha(1-p)}d\tau\right)^{\frac{1}{q}}\\
&\times\left(\max\left\{\left|\psi'(\upsilon_1)\right|^q, \left|\psi'(\upsilon_2)\right|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^1 |1-2\tau|^{\alpha\left(p^2-p-1\right)}d\tau &= \int_0^{\frac{1}{2}}(1-2\tau)^{\alpha\left(p^2-p-1\right)}d\tau+\int_{\frac{1}{2}}^1(2\tau-1)^{\alpha\left(p^2-p-1\right)}d\tau\\
&= \frac{1}{\alpha p^2-\alpha p+\alpha+1},
\end{aligned}$$

$$\begin{aligned}
\int_0^1 |1-2\tau|^{\alpha(1-p)}d\tau &= \int_0^{\frac{1}{2}}(1-2\tau)^{\alpha(1-p)}d\tau+\int_{\frac{1}{2}}^1(2\tau-1)^{\alpha(1-p)}d\tau\\
&= \frac{1}{\alpha-\alpha p+1}.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&\Gamma(\alpha+1)\left(J^{\alpha}_{\upsilon_1^+}\psi(\upsilon_2), J^{\alpha}_{\upsilon_2^-}\psi(\upsilon_1)\right)_{\alpha\in[0,1]}\\
&\leq \frac{\upsilon_2-\upsilon_1}{2}\left(\frac{1}{\alpha p^2-\alpha p+\alpha+1}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha-\alpha p+1}\right)^{\frac{1}{q}}\\
&\times\left(\max\left\{\left|\psi'(\upsilon_1)\right|^q, \left|\psi'(\upsilon_2)\right|^q\right\}\right)^{\frac{1}{q}}\\
&\leq \frac{\upsilon_2-\upsilon_1}{2}\left(\frac{1}{\alpha p^2-\alpha p+\alpha+1}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha-\alpha p+1}\right)^{\frac{1}{q}}\left\|\psi'\right\|_{\infty},
\end{aligned}$$

this completes the required proof.

Remark 3. For $p \in (1, \infty)$, we have

$$\lim_{p \to \infty} \left(\frac{1}{1 + \alpha - \alpha p} \right)^{1 - \frac{1}{p}} = 1, \quad \lim_{p \to 1^+} \left(\frac{1}{1 + \alpha - \alpha p} \right)^{\frac{1}{q}} = 1.$$

and

$$\lim_{p \to \infty} \left(\frac{1}{\alpha p^2 - \alpha p + \alpha + 1} \right)^{\frac{1}{p}} = 1, \lim_{p \to 1^+} \left(\frac{1}{\alpha p^2 - \alpha p + \alpha + 1} \right)^{\frac{1}{p}} = \frac{1}{\alpha + 1} < 1 ,$$

$$\frac{1}{\alpha + 1} < \left(\frac{1}{\alpha p^2 - \alpha p + \alpha + 1} \right)^{\frac{1}{p}} < 1,$$

Thus, we can rewrite inequality (9) as follows

$$\left| \frac{\psi(\upsilon_1) + \psi(\upsilon_2)}{2} - \frac{\Gamma(\alpha + 1)}{2(\upsilon_2 - \upsilon_1)^\alpha} \left[J^\alpha_{\upsilon_1^+} \psi(\upsilon_2) + J^\alpha_{\upsilon_2^-} \psi(\upsilon_1) \right] \right| \leq \frac{\upsilon_2 - \upsilon_1}{2} \|\psi'\|_\infty .$$

Applications to special means

In this section, we consider some special means for arbitrary real numbers:

1 *Arithmetic mean* :

$$A(\upsilon_1, \upsilon_2) = \frac{\upsilon_1 + \upsilon_2}{2}, \quad \upsilon_1, \upsilon_2 \in \mathbb{R}^+,$$

2 *Logarithmic mean*:

$$L(\upsilon_1, \upsilon_2) = \frac{\upsilon_1 - \upsilon_2}{\ln|\upsilon_1| - \ln|\upsilon_2|}, \quad \upsilon_1 \neq \upsilon_2, \ \upsilon_1, \upsilon_2 \neq 0, \ \upsilon_1, \upsilon_2 \in \mathbb{R}^+,$$

3 *Generalized log – mean*:

$$L_n(\upsilon_1, \upsilon_2) = \left[\frac{\upsilon_2^{n+1} - \upsilon_1^{n+1}}{(n+1)(\upsilon_2 - \upsilon_1)} \right]^{\frac{1}{n}}, \quad n \in \mathbb{Z} \backslash \{-1, 0\}, \upsilon_1, \upsilon_2 \in \mathbb{R}^+.$$

Now, using the some results , we give some applications to special means of real numbers.

Proposition 1. Let $\upsilon_1, \upsilon_2 \in \mathbb{R}^+$, $\upsilon_1 < \upsilon_2$ and $n \in \mathbb{Z}$. Then, we have

$$\upsilon_2^n - L_n^n(\upsilon_1, \upsilon_2) \leq 0.5n(\upsilon_2 - \upsilon_1) \max\left\{ \upsilon_1^{n-1}, {\upsilon_2}^{n-1} \right\}.$$

Proof. The proof follows from the Corollary 4 applied to the map $f(x) = x^n$, $x \in \mathbb{R}$.
Proposition 2. Let $\upsilon_1, \upsilon_2 \in \mathbb{R}^+$, $\upsilon_1 < \upsilon_2$ and $n \in \mathbb{Z}$. Then, for all $q \geq 1$, we have

$$|\upsilon_2^n - L_n^n(\upsilon_1, \upsilon_2)| \leq 0.5n(\upsilon_2 - \upsilon_1) \left(\max\left\{ \left(|\upsilon_1|^{n-1} \right)^q, \left(|\upsilon_2|^{n-1} \right)^q \right\} \right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Corollary 5 applied to the quasi-convex mapping $\psi(\xi) = \xi^n$, $\xi \in \mathbb{R}$.

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Б.Р. Байрактаров[1], А.Х. Аттаев[2]

[1] *Улудаг университетi, Бурса, Түркия;*
[2] *РГА КБГО ҚМАИ, Нальчик, Ресей*

Кейбiр дөңес функциялар үшiн бөлшектi-интегралды теңсiздiктер

Мақалада екiншi мағынадағы дөңес s – Годунов-Левин функциялары үшiн және квази–дөңес функциялар үшiн Риман-Лиувиллдiң бөлшектi интегралдары арқылы бiрнеше жаңа интегралдық теңсiздiктер алынған. Нәтижелер классикалық Гельдер теңсiздiгiн, Эрмит-Адамардың екi еселi теңсiздiгiн, орташа дәрежелiк және өлшенген Гельдер теңсiздiгiн арқылы алынды. Соның iшiнде кейбiр арнайы есептеу құралдарына арналған нәтижелердiң қолданулары берiлген. Мақалада кезкелген нақты сандар үшiн арнайы жағдайлардың қолданылулары берiлген: арифметикалық, логарифмдiк, жалпыланған логарифмдiк жағдай.

Кiлт сөздер: интегралдық теңсiздiк, Эрмит-Адамар теңсiздiгi, дөңес функция, квази-дөңес функция, бөлшектi интеграл, Гельдер теңсiздiгi, орташа дәрежелiк теңсiздiк.

Б.Р. Байрактаров[1], А.Х. Аттаев[2]

[1] *Университет Улудаг, Бурса, Турция;*
[2] *ИПМА КБНЦ РАН, Нальчик, Россия*

Дробно-интегральные неравенства для некоторых выпуклых функций

В статье получены несколько новых интегральных неравенств с помощью дробных интегралов Римана–Лиувилля для выпуклых s-Годунова–Левина функций во втором смысле и для квазивыпуклых функций. Результаты получены с использованием двойного неравенства Эрмита–Адамара, классических неравенств Гёльдера, среднего степенного и взвешенного неравенства Гёльдера. В том числе дано приложение результатов для некоторых специальных вычислительных средств. Авторами приведены приложения некоторых специальных случаев для произвольных действительных чисел: арифметический случай, логарифмический случай, случай обобщённого логарифма.

Ключевые слова: интегральное неравенство, неравенство Эрмита–Адамара, выпуклая функция, квазивыпуклая функция, дробный интеграл, неравенство Гёльдера, среднее степенное неравенство.

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K.A. Bekmaganbetov[1], K.Ye. Kervenev[2], Ye. Toleugazy[1,*]

[1] *M.V. Lomonosov Moscow State University, Kazakhstan Branch, Nur-Sultan, Kazakhstan;*
[2] *Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan*
(E-mail: bekmaganbetov-ka@yandex.kz, kervenev@bk.ru, toleugazy-y@yandex.ru)

The embedding theorems for anisotropic Nikol'skii-Besov spaces with generalized mixed smoothness

The theory of embedding of spaces of differentiable functions studies the important relations of differential (smoothness) properties of functions in various metrics and has a wide application in the theory of boundary value problems of mathematical physics, approximation theory, and other fields of mathematics. In this article, we prove the embedding theorems for anisotropic spaces Nikol'skii-Besov with a generalized mixed smoothness and mixed metric, and anisotropic Lorentz spaces. The proofs of the obtained results are based on the inequality of different metrics for trigonometric polynomials in Lebesgue spaces with mixed metrics and interpolation properties of the corresponding spaces.

Keywords: anisotropic Lorentz spaces, anisotropic Nikol'skii-Besov spaces, generalized mixed smoothness, mixed metric, embedding theorems.

Introduction

One of the first results related to the theory of embedding spaces of differentiable functions belongs to S.L. Sobolev [1]. This theory studies important relations of differential (smoothness) properties of functions in various metrics. Further, the development of this theory is associated with new classes of function spaces defined and studied in the works of S.M. Nikol'skii [2, 3], O.V. Besov [4, 5], P.I. Lizorkin [6], H. Triebel [7, 8], and many others. The development of this research was determined both by its internal problems and by its applications in the theory of boundary value problems of mathematical physics and approximation theory (see, for example, [9–14]).

In this paper, embedding theorems for spaces with generalized mixed smoothness and with mixed metrics and anisotropic Lorentz spaces are obtained. The proofs of the achieved results are based on the inequalities of different metrics for trigonometric polynomials and interpolation theorems from the works of E.D. Nursultanov [15] and the authors [16].

1 Definitions and auxiliary results

Let $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, $\mathbb{T}^{\mathbf{d}} = \{\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) : \mathbf{x}_i \in \mathbb{T}^{d_i} = [0, 2\pi)^{d_i}, i = 1, \ldots, n\}$ and $f(\mathbf{x}) = f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be measurable function on $\mathbb{T}^{\mathbf{d}}$.

Let $\mathbf{1} \le \mathbf{p} = (p_1, \ldots, p_n) \le \infty$. We say that the function f belongs to the Lebesgue space with mixed metric $L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})$ if

$$\|f\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} = \left(\int_{\mathbb{T}^{d_n}} \left(\ldots \left(\int_{\mathbb{T}^{d_1}} |f(\mathbf{x}_1, \ldots, \mathbf{x}_n|^{p_1} d\mathbf{x}_1 \right)^{p_2/p_1} \ldots \right)^{p_n/p_{n-1}} d\mathbf{x}_n \right)^{1/p_n} < \infty.$$

In a case when $p_i = \infty$ the expression $\left(\int_{\mathbb{T}^{d_i}} |f(\mathbf{x}_i)|^{p_i} \, d\mathbf{x}_i \right)^{1/p_i}$ means that $\operatorname*{ess\,sup}_{\mathbf{x}_i \in \mathbb{T}^{d_i}} |f(\mathbf{x}_i)|$.

*Corresponding author.
E-mail: toleugazy-y@yandex.ru

Let us denote by

$$\Delta_{\mathbf{s}}(f,\mathbf{x}) = \sum_{\mathbf{k}\in\rho(\mathbf{s})} a_{\mathbf{k}} e^{i\langle \mathbf{k},\mathbf{x}\rangle_{\mathbf{d}}}$$

the trigonometric series of $f \sim \sum_{\mathbf{k}\in\mathbb{Z}^{\mathbf{d}}} a_{\mathbf{k}} e^{i\langle \mathbf{k},\mathbf{x}\rangle_{\mathbf{d}}}$, where $\langle \mathbf{k},\mathbf{x}\rangle_{\mathbf{d}} = \sum_{i=1}^{n}\sum_{j=1}^{d_i} k_j^i x_j^i$ is the (modified) inner product, $\rho(\mathbf{s}) = \{\mathbf{k} = (\mathbf{k}_1,\ldots,\mathbf{k}_n) \in \mathbb{Z}^{\mathbf{d}} : \left[2^{s_i-1}\right] \le \max\limits_{j=1,\ldots,d_i} |k_j^i| < 2^{s_i}, i = 1,\ldots,n\}$ and $[a]$ is the integer part of the number a.

Let $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^n$, $\mathbf{1} \le \mathbf{q} = (\mathbf{q_1},\ldots,\mathbf{q_n}) \le \infty$ and $\mathbf{1} < \mathbf{p} = (p_1,\ldots,p_n) < \infty$.

The anisotropic Nikol'skii-Besov space with generalized mixed smoothness and mixed metric $B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ is a set of the series $f \sim \sum_{\mathbf{k}\in\mathbb{Z}^{\mathbf{d}}} a_{\mathbf{k}} e^{i\langle \mathbf{k},\mathbf{x}\rangle_{\mathbf{d}}}$ such that

$$\|f\|_{B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = \left\|\left\{2^{(\alpha,\mathbf{s})}\|\Delta_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})}\right\}\right\|_{l_{\mathbf{q}}} < \infty,$$

where $(\alpha,\mathbf{s}) = \sum_{i=1}^{n}\alpha_i s_i$ is the inner product and $\|\cdot\|_{l_{\mathbf{q}}}$ is the norm of a discrete Lebesgue space with mixed metric $l_{\mathbf{q}}$.

Here $B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ is a version of spaces, which was introduced and studied in [15].

Remark 1. The anisotropic Nikol'skii-Besov space with generalized mixed smoothness $B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ mentioned above is a hybrid structure of Nikol'skii-Besov space (concerning variables included in one multi-variable) [2, 4] and spaces with dominant mixed derivative (concerning variables included in different multi-variables) [17, 18]. In the isotropic case, when p and q are scalars, analogs of these spaces were studied by D.B. Bazarkhanov [19].

Below we define an anisotropic interpolation method (see [20]) and interpolation theorems for Lebesgue spaces with mixed metric and anisotropic Nikol'skii-Besov spaces with generalized mixed smoothness.

Let $E = \{\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n) : \varepsilon_i = 0 \text{ or } \varepsilon_i = 1, i = 1,\ldots,n\}$ be vertices of the n-dimensional unit cube in $\mathbb{R}^n$, $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon\in E}$ be compatible family of Banach spaces (this means that they are all embedded in a linear Hausdorff space). Let us define functional for $a \in \sum_{\varepsilon\in E} A_\varepsilon$

$$K(\mathbf{t},a;\mathbf{A}) = \inf_{a=\sum_{\varepsilon\in E} a_\varepsilon} \sum_{\varepsilon\in E} \mathbf{t}^\varepsilon \|a_\varepsilon\|_{A_\varepsilon},$$

here $t^\varepsilon = t_1^{\varepsilon_1}\cdot\ldots\cdot t_n^{\varepsilon_n}$.

Moreover, for $\mathbf{0} < \theta = (\theta_1,\ldots,\theta_n) < \mathbf{1}$, $\mathbf{0} < \mathbf{r} = (r_1,\ldots,r_n) \le \infty$, we denote by $\mathbf{A}_{\theta\mathbf{r}} = (A_\varepsilon;\varepsilon\in E)_{\theta\mathbf{r}}$ the linear subset of the set $\sum_{\varepsilon\in E} A_\varepsilon$, such that

$$\|a\|_{\mathbf{A}_{\theta\mathbf{r}}} =$$

$$= \left(\int_0^\infty \left(t_n^{-\theta_n}\ldots\left(\int_0^\infty \left(t_1^{-\theta_1}K(\mathbf{t},a;\mathbf{A})\right)^{r_1}\frac{dt_1}{t_1}\right)^{r_2/r_1}\ldots\right)^{r_n/r_{n-1}}\frac{dt_n}{t_n}\right)^{1/r_n} < \infty.$$

Lemma 1 ([20]). Let $\mathbf{0} < \theta < \mathbf{1}$, $\mathbf{0} < \mathbf{r} \le \infty$, $\mathbf{A} = \{A_\varepsilon\}_{\varepsilon\in E}, \mathbf{B} = \{B_\varepsilon\}_{\varepsilon\in E}$ be two compatible families of Banach spaces. If there exist two vectors $\mathbf{M_0} = (M_1^0,\ldots,M_n^0), \mathbf{M_1} = (M_1^1,\ldots,M_n^1)$ with positive components such that for the linear operator T the following mapping holds

$$T : \mathbf{A}_\varepsilon \to \mathbf{B}_\varepsilon$$

with the norm estimation $C_\varepsilon \prod_{i=1}^{n} M_i^{\varepsilon_i}$ for every $\varepsilon \in E$, then

$$T : \mathbf{A}_{\theta\mathbf{r}} \to \mathbf{B}_{\theta\mathbf{r}},$$

with the norm $\|T\|_{\mathbf{A}_{\theta\mathbf{r}}\to\mathbf{B}_{\theta\mathbf{r}}} \le \max\limits_{\varepsilon\in E} C_\varepsilon \prod_{i=1}^{n}\left(M_i^0\right)^{1-\theta_i}\left(M_i^1\right)^{\theta_i}$.

Let the multi-indices $\mathbf{p} = (p_1,\ldots,p_n)$, $\mathbf{r} = (r_1,\ldots,r_n)$ be such that if $1 \le p_i < \infty$, then $1 \le r_i \le \infty$, and if $p_i = \infty$, then $r_i = \infty$ $(i = 1,\ldots,n)$.

The anisotropic Lorentz space $L_{\mathbf{pr}}(\mathbb{T}^{\mathbf{d}})$ (see [15]) is a set of functions such that

$$\|f\|_{L_{\mathbf{pr}}(\mathbb{T}^{\mathbf{d}})} =$$

$$= \left(\int_0^{(2\pi)^{d_n}} \left(t_n^{1/p_n} \dots \left(\int_0^{(2\pi)^{d_1}} \left(t_1^{1/p_1} f^{*_1,\dots,*_n}(t_1,\dots,t_n) \right)^{r_1} \frac{dt_1}{t_1} \right)^{r_2/r_1} \dots \right)^{r_n/r_{n-1}} \frac{dt_n}{t_n} \right)^{1/r_n} < \infty,$$

where $f^*(\mathbf{t}) = f^{*_1,\dots,*_n}(t_1,\dots,t_n)$ is repeated non-increasing rearrangement of a function $f(\mathbf{x}) = f(\mathbf{x}_1,\dots,\mathbf{x}_n)$. In what follows by $f^*(\mathbf{t}) = f^{*_1,\dots,*_n}(t_1,\dots,t_n)$ we mean the non-increasing rearrangement first with respect to $\mathbf{x}_1$ with fixed $\mathbf{x}_2,\dots,\mathbf{x}_n$ and then with respect to $\mathbf{x}_2$ with fixed other (multi)variables and so on.

Let us denote $\mathbf{b}_\varepsilon = (b_1^{\varepsilon_1},\dots,b_n^{\varepsilon_n})$ for multi-indices $\mathbf{b}_0 = (b_1^0, \dots, b_n^0), \mathbf{b}_1 = (b_1^1, \dots, b_n^1)$ and $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in E$.

Lemma 2 ([15]). Let $\mathbf{1} \le \mathbf{p_0} = (p_1^0,\dots,p_n^0) < \mathbf{p_1} = (p_1^1,\dots,p_n^1) \le \infty$. Then for $\mathbf{0} < \theta = (\theta_1,\dots,\theta_n) < \mathbf{1}$ and $\mathbf{1} \le \mathbf{r} = (r_1,\dots,r_n) \le \infty$

$$\left(L_{\mathbf{p}_\varepsilon}(\mathbb{T}^{\mathbf{d}}); \varepsilon \in E\right)_{\theta\mathbf{r}} = L_{\mathbf{pr}}(\mathbb{T}^{\mathbf{d}}),$$

where $\mathbf{1}/\mathbf{p} = (\mathbf{1} - \theta)/\mathbf{p_0} + \theta/\mathbf{p_1}$.

Lemma 3 ([16]). Let $\mathbf{1} < \mathbf{p} = (p_1,\dots,p_n) < \infty$, $-\infty < \alpha_{\mathbf{0}} = (\alpha_1^0,\dots,\alpha_n^0) < \alpha_{\mathbf{1}} = (\alpha_1^1,\dots,\alpha_n^1) < \infty$, $\mathbf{1} \le \mathbf{q_0} = (q_1^0,\dots,q_n^0), \mathbf{q_1} = (q_1^1,\dots,q_n^1) \le \infty$, $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in E$. Then for $\mathbf{0} < \theta = (\theta_1,\dots,\theta_n) < \mathbf{1}$ and $\mathbf{1} \le \mathbf{q} = (q_1,\dots,q_n) \le \infty$

$$\left(B_{\mathbf{p}}^{\alpha_\varepsilon \mathbf{q}_\varepsilon}(\mathbb{T}^{\mathbf{d}}); \varepsilon \in E\right)_{\theta\mathbf{q}} = B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}}),$$

here $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

2 Embedding theorems for anisotropic Nikol'skii-Besov spaces with generalized mixed smoothness

In this section, the limit embedding theorems for different metrics for anisotropic Nikol'skii-Besov spaces with generalized mixed smoothness and anisotropic Lorentz spaces are proved.

Lemma 4 (Inequality of different metrics [2]). Let $T_{\mathbf{s}}(\mathbf{x})$ be trigonometric polynomial with the order no more than $\mathbf{s} = (s_1^1,\dots,s_{d_1}^1;\dots;s_1^n,\dots,s_{d_n}^n)$ by multi-variable $\mathbf{x} = (x_1^1,\dots,x_{d_1}^1;\dots;x_1^n,\dots,x_{d_n}^n)$. Then for $\mathbf{1} < \mathbf{p_1} = (p_1^1,\dots,p_n^1) \le \mathbf{p_2} = (p_1^2,\dots,p_n^2) < \infty$

$$\|T_{\mathbf{s}}\|_{L_{\mathbf{p_2}}(\mathbb{T}^{\mathbf{d}})} \le C \prod_{\{i:\, p_i^1 < p_i^2\}} \prod_{j=1}^{d_i} \left(s_j^i\right)^{1/p_i^1 - 1/p_i^2} \|T_{\mathbf{s}}\|_{L_{\mathbf{p_1}}(\mathbb{T}^{\mathbf{d}})},$$

where C is the positive constant which does not depend on $\mathbf{s}$.

Theorem 1. Let $-\infty < \alpha_{\mathbf{0}} = (\alpha_1^0,\dots,\alpha_n^0) \le \alpha_{\mathbf{1}} = (\alpha_1^1,\dots,\alpha_n^1) < \infty$, $\mathbf{1} \le \mathbf{q} = (q_1,\dots,q_n) \le \infty$ and $\mathbf{1} < \mathbf{p_0} = (p_1^0,\dots,p_n^0), \mathbf{p_1} = (p_1^1,\dots,p_n^1) < \infty$. Then the embedding

$$B_{\mathbf{p_1}}^{\alpha_{\mathbf{1}}\mathbf{q}}(\mathbb{T}^{\mathbf{d}}) \hookrightarrow B_{\mathbf{p_0}}^{\alpha_{\mathbf{0}}\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$$

holds for $\alpha_{\mathbf{0}} - \mathbf{d}/\mathbf{p_0} = \alpha_{\mathbf{1}} - \mathbf{d}/\mathbf{p_1}$.

Proof. Let $f \in B_{\mathbf{p_1}}^{\alpha_{\mathbf{1}}\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$, then, according to the inequality of different metrics (Lemma 4), we obtain

$$\|f\|_{B_{\mathbf{p_0}}^{\alpha_{\mathbf{0}}\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = \left\| \left\{ 2^{\langle \alpha_{\mathbf{0}}, \mathbf{s} \rangle} \|\Delta_{\mathbf{s}}(f)\|_{L_{\mathbf{p_0}}(\mathbb{T}^{\mathbf{d}})} \right\} \right\|_{l_{\mathbf{q}}} \le$$

$$\le C_1 \left\| \left\{ 2^{\langle \alpha_{\mathbf{0}} + \mathbf{d}(\mathbf{1}/\mathbf{p_1} - \mathbf{1}/\mathbf{p_0}), \mathbf{s} \rangle} \|\Delta_{\mathbf{s}}(f)\|_{L_{\mathbf{p_1}}(\mathbb{T}^{\mathbf{d}})} \right\} \right\|_{l_{\mathbf{q}}} =$$

$$= C_1 \left\| \left\{ 2^{\langle \alpha_{\mathbf{1}}, \mathbf{s} \rangle} \|\Delta_{\mathbf{s}}(f)\|_{L_{\mathbf{p_1}}(\mathbb{T}^{\mathbf{d}})} \right\} \right\|_{l_{\mathbf{q}}} = C_1 \|f\|_{B_{\mathbf{p_1}}^{\alpha_{\mathbf{1}}\mathbf{q}}(\mathbb{T}^{\mathbf{d}})}.$$

This completes the proof.

Theorem 2. Let $\mathbf{1} < \mathbf{p} = (p_1,\dots,p_n) < \mathbf{q} = (q_1,\dots,q_n) < \infty$ and $\mathbf{1} \le \tau = (\tau_1,\dots,\tau_n) \le \infty$. Then the embedding

$$B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^{\mathbf{d}}) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{T}^{\mathbf{d}})$$

holds for $\alpha = (\mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q})\mathbf{d}$.

Proof. According to the Minkowski inequality and the inequality of different metrics (Lemma 4), we receive

$$\|f\|_{L_{\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = \left\| \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \Delta_{\mathbf{k}}(f) \right\|_{L_{\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} \leq$$

$$\leq \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \|\Delta_{\mathbf{k}}(f)\|_{L_{\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} \leq \sum_{\mathbf{k}=\mathbf{0}}^{\infty} 2^{((\mathbf{1/p-1/q})\mathbf{d},\mathbf{k})} \|\Delta_{\mathbf{k}}(f)\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} = \|f\|_{B_{\mathbf{p}}^{\alpha\mathbf{1}}(\mathbb{T}^{\mathbf{d}})},$$

where $\alpha = (\mathbf{1/p} - \mathbf{1/q})\,\mathbf{d}$.

Therefore, for $\alpha = (\mathbf{1/p} - \mathbf{1/q})\,\mathbf{d}$ we get

$$B_{\mathbf{p}}^{\alpha\mathbf{1}}(\mathbb{T}^{\mathbf{d}}) \hookrightarrow L_{\mathbf{q}}(\mathbb{T}^{\mathbf{d}}).$$

Let us fix $\mathbf{p} = (p_1, \ldots, p_n)$ and let us choose $\alpha_i = (\alpha_1^i, \ldots, \alpha_n^i)$ and $\mathbf{q}_i = (q_1^i, \ldots, q_n^i)$ such that $\alpha_j^i = \left(1/p_j - 1/q_j^i\right) d_j$, $i = 0, 1$, $j = 1, \ldots, n$. Then for every $\varepsilon \in E$ we have

$$B_{\mathbf{p}}^{\alpha_\varepsilon\mathbf{1}}(\mathbb{T}^{\mathbf{d}}) \hookrightarrow L_{\mathbf{q}_\varepsilon}(\mathbb{T}^{\mathbf{d}}).$$

According to Lemma 2 and Lemma 3, we obtain

$$\left(B_{\mathbf{p}}^{\alpha_\varepsilon\mathbf{1}}(\mathbb{T}^{\mathbf{d}}); \varepsilon \in E\right)_{\theta\tau} \hookrightarrow \left(L_{\mathbf{q}_\varepsilon\mathbf{r}}(\mathbb{T}^{\mathbf{d}}); \varepsilon \in E\right)_{\theta\tau}$$

or

$$B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^{\mathbf{d}}) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{T}^{\mathbf{d}}),$$

where $\alpha = (\mathbf{1} - \theta)\alpha_0 + \theta\alpha_1$, $\mathbf{1/q} = (\mathbf{1} - \theta)/\mathbf{q}_0 + \theta/\mathbf{q}_1$.

Let us check the relationship between the parameters α, $\mathbf{p}$ and $\mathbf{q}$

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1 = (1-\theta)\left(\mathbf{1/p} - \mathbf{1/q}_0\right)\mathbf{d} + \theta\left(\mathbf{1/p} - \mathbf{1/q}_1\right)\mathbf{d} =$$

$$= ((1-\theta)/\mathbf{p} + \theta/\mathbf{p})\,\mathbf{d} - ((1-\theta)\mathbf{q}_0 + \theta/\mathbf{q}_1)\,\mathbf{d} = (\mathbf{1/p} - \mathbf{1/q})\,\mathbf{d}.$$

The proof is complete.

Theorem 3. Let $\mathbf{1} < \mathbf{q} = (q_1, \ldots, q_n) < \mathbf{p} = (p_1, \ldots, p_n) < \infty$ and $\mathbf{1} \leq \tau = (\tau_1, \ldots, \tau_n) \leq \infty$. Then the embedding

$$L_{\mathbf{q}\tau}(\mathbb{T}^{\mathbf{d}}) \hookrightarrow B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^{\mathbf{d}})$$

holds for $\alpha = (\mathbf{1/p} - \mathbf{1/q})\mathbf{d}$.

Proof. According to the inequality of different metrics (Lemma 4) and M. Riesz's theorem on the boundedness of parallelepiped partial sums, we obtain

$$\|f\|_{B_{\mathbf{p}}^{\alpha\infty}(\mathbb{T}^{\mathbf{d}})} = \sup_{\mathbf{k}\geq\mathbf{0}} 2^{(\alpha,\mathbf{k})} \|\Delta_{\mathbf{k}}(f)\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \leq$$

$$\leq C_1 \sup_{\mathbf{k}\geq\mathbf{0}} 2^{(\alpha+(\mathbf{1/q-1/p})\mathbf{d},\mathbf{k})} \|\Delta_{\mathbf{k}}(f)\|_{L_{\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = C_1 \sup_{\mathbf{k}\geq\mathbf{0}} \|\Delta_{\mathbf{k}}(f)\|_{L_{\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} \leq C_2 \|f\|_{L_{\mathbf{q}}(\mathbb{T}^{\mathbf{d}})},$$

since $\alpha = (\mathbf{1/p} - \mathbf{1/q})\,\mathbf{d}$.

Thus, for $\alpha = (\mathbf{1/p} - \mathbf{1/q})\,\mathbf{d}$ we have

$$L_{\mathbf{q}}(\mathbb{T}^{\mathbf{d}}) \hookrightarrow B_{\mathbf{p}}^{\alpha\infty}(\mathbb{T}^{\mathbf{d}}).$$

Let us fix $\mathbf{p} = (p_1, \ldots, p_n)$ and let us choose parameters $\alpha_i = (\alpha_1^i, \ldots, \alpha_n^i)$ and $\mathbf{q}_i = (q_1^i, \ldots, q_n^i)$ such that $\alpha_j^i = \left(1/p_j - 1/q_j^i\right) d_j$, $i = 0, 1$, $j = 1, \ldots, n$. Then for every $\varepsilon \in E$ we receive

$$L_{\mathbf{q}_\varepsilon}(\mathbb{T}^{\mathbf{d}}) \hookrightarrow B_{\mathbf{p}}^{\alpha_\varepsilon\infty}(\mathbb{T}^{\mathbf{d}}).$$

According to Lemma 2 and Lemma 3 we obtain

$$\left(L_{\mathbf{q}_\varepsilon}(\mathbb{T}^{\mathbf{d}}); \varepsilon \in E\right)_{\theta\tau} \hookrightarrow \left(B_{\mathbf{p}}^{\alpha_\varepsilon\infty}(\mathbb{T}^{\mathbf{d}}); \varepsilon \in E\right)_{\theta\tau}$$

or

$$L_{\mathbf{q}\tau}(\mathbb{T}^{\mathbf{d}}) \hookrightarrow B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^{\mathbf{d}}),$$

where $\alpha = (\mathbf{1}-\theta)\alpha_0 + \theta\alpha_1$, $\mathbf{1}/\mathbf{q} = (\mathbf{1}-\theta)/\mathbf{q}_0 + \theta/\mathbf{q}_1$.

Let us check the relationship between the parameters α, $\mathbf{p}$ and $\mathbf{q}$

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1 = (1-\theta)\left(\mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}_0\right)\mathbf{d} + \theta\left(\mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}_1\right)\mathbf{d} =$$

$$= \left((\mathbf{1}-\theta)/\mathbf{p} + \theta/\mathbf{p}\right)\mathbf{d} - \left((\mathbf{1}-\theta)/\mathbf{q}_0 + \theta/\mathbf{q}_1\right)\mathbf{d} = \left(\mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}\right)\mathbf{d}.$$

The proof is complete.

Remark 2. 1) It is possible to show that the conditions of Theorems 1 – 3 are unimprovable. The proof of these facts can be carried out by analogy with the corresponding proof from the paper [21].

2) In a case when $\mathbf{d} = (1, \ldots, 1)$ the results of Theorems 1 – 3 were announced in the paper by E.D. Nursultanov [15].

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Қ.А. Бекмағанбетов[1], Қ.Е. Кервенев[2], Е. Төлеуғазы[1]

[1] *М.В. Ломоносов атындағы Мәскеу мемлекеттік университеті, Қазақстан филиалы, Нұр-Сұлтан, Қазақстан;*
[2] *Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан*

Жалпыланған аралас тегістілігі бар Никольский-Бесовтың анизотропты кеңістіктері үшін енгізу теоремалары

Дифференциалданатын функциялар кеңістіктерінің енгізу теориясы әртүрлі метрикалардағы функциялардың дифференциалдық (тегістіліктік) қасиеттерінің маңызды байланыстары мен қатынастарын зерттейді. Математикалық физиканың шектік есептер теориясында, жуықтау теориясында және математиканың басқа да салаларында кеңінен колданысқа ие. Мақалада жалпыланған аралас тегістілігі және аралас метрикасы бар Никольский-Бесовтың анизотропты кеңістіктері үшін және Лоренцтің анизотропты кеңістіктері үшін енгізу теоремалары дәлелдеген. Алынған нәтижелердің дәлелдеулері аралас метрикасы бар Лебег кеңістіктеріндегі тригонометриялық полиномдар үшін әртүрлі метрикалар теңсіздіктерін және сәйкес кеңістіктердің интерполяциялық қасиеттерін қолдануға негізделген.

Кілт сөздер: Лоренцтің анизотропты кеңістіктері, Никольский-Бесовтың анизотропты кеңістіктері, жалпыланған аралас тегістілік, аралас метрика, енгізу теоремалары.

К.А. Бекмаганбетов[1], К.Е. Кервенев[2], Е. Толеугазы[1]

[1] *Московский государственный университет имени М.В. Ломономова, Казахстанский филиал, Нур-Султан, Казахстан;*
[2] *Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан*

Теоремы вложения для анизотропных пространств Никольского–Бесова с обобщенной смешанной гладкостью

Теория вложения пространств дифференцируемых функций изучает важные связи и соотношения дифференциальных (гладкостных) свойств функций в различных метриках и имеет широкое применение в теории краевых задач математической физики, теории приближений и других разделах математики. В статье авторами доказаны теоремы вложения для анизотропных пространств Никольского–Бесова с обобщенной смешанной гладкостью, со смешанной метрикой и для анизотропных пространств Лоренца. Доказательства полученных результатов основаны на использовании неравенства разных метрик для тригонометрических полиномов в пространствах Лебега со смешанной метрикой и интерполяционных свойствах соответствующих пространств.

Ключевые слова: анизотропные пространства Лоренца, анизотропные пространства Никольского–Бесова, обобщенная смешанная гладкость, смешанная метрика, теоремы вложения.

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A.S. Berdyshev[1,2], A.R. Ryskan[1*]

[1]*Abai Kazakh National Pedagogical University, Almaty, Kazakhstan;*
[2]*Institute of Information and Computational Technologies, Almaty, Kazakhstan*
(E-mail: berdyshev@mail.ru, ryskan.a727@gmail.com)

Boundary value problem for the four-dimensional Gellerstedt equation

In this work, the solvability of the problem with Neumann and Dirichlet boundary conditions for the Gellerstedt equation in four variables was investigated. The energy integral method was used to prove the uniqueness of the solution to the problem. In addition to it, formulas for differentiation, autotransformation, and decomposition of hypergeometric functions were applied. The solution was obtained explicitly and expressed by Lauricella's hypergeometric function.

Keywords: Gellerstedt equation, boundary value problem with mixed conditions, fundamental solution,

Introduction

The study of boundary value problems for degenerate equations is one of the important directions of modern theory of partial differential equations. The solution of many boundary value problems for partial differential equations has an applied nature [1–2]. The boundary value problems for degenerate elliptic equations were well studied in works [3–6].

In the formulation of problems and questions of the solvability of local and nonlocal boundary value problems for degenerate elliptic equations, fundamental solutions of these equations are essentially used [7]. The explicit form of the fundamental solutions makes it possible to correctly formulate the problem statement and study in detail the various properties of the considered equation solutions.

Fundamental solutions of degenerate elliptic equations are expressed in terms of the Lauricella's multidimensional hypergeometric functions and the Gauss hypergeometric function of one variable. Many problems of natural science, such as problems of dynamics and heat conduction, the theory of electromagnetic oscillations, aerodynamics, quantum mechanics, quantum chemistry, potential theory, etc. lead to the study of various properties of multidimensional hypergeometric functions [8–14].

For two-dimensional and three-dimensional elliptic equations with singular coefficients, fundamental solutions were constructed, which were applied in the study of the various problems solvability in many works [15–19].

In [20] fundamental solutions for the generalized Gellerstedt equation of four variables were constructed

$$y^m z^k t^l u_{xx} + x^n z^k t^l u_{yy} + x^n y^m t^l u_{zz} + x^n y^m z^k u_{tt} = 0, \;\; m,n,k,l > 0, \;\; m,n,k,l \equiv const.$$

Since the generalized Gellerstedt equation has four hypersurfaces of degeneration of the equation type, accordingly sixteen fundamental solutions were obtained. It was proved that the fundamental solutions have a singularity of the order $\frac{1}{r^2}$, at $r \to 0$, where $r = \sqrt{x^2 + y^2 + z^2 + t^2}$.

These fundamental solutions are expressed in terms of Lauricella's hypergeometric functions, each of the fundamental solutions is applied in solving the corresponding boundary value problems [21–23].

1. Preliminary information

By definition, the Gauss hypergeometric function has the form

*Corresponding author.
E-mail: ryskan.a727@gmail.com

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n,\quad c\neq 0,\ -1,\ -2,\ \dots,$$

where

$$(a)_n=a(a+1)(a+2)\dots(a+n-1)\ =\frac{\Gamma(a+n)}{\Gamma(a)}\quad (n=0,1,2,3,\dots)$$

is Pochhammer symbol. Here $\Gamma(a)$ is Euler's gamma function, for it the formula of the doubled argument is valid [24; 19, (15)]

$$\Gamma(2z)=2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{1}$$

We present the main properties of the Euler's gamma function, the Gauss hypergeometric function, and Lauricella's hypergeometric function of many variables, which will be used in what follows.

The Gauss hypergeometric function has the following property [25; 3, (5)]:

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad \operatorname{Re}(c-a-b)>0. \tag{2}$$

The Gauss hypergeometric function satisfies the Bolz autotransformation formula [26; 64, (22)]:

$$F(a,b;c;x)=(1-x)^{-b}F\left(c-a,b;c;\frac{x}{x-1}\right). \tag{3}$$

Lauricella's hypergeometric function of n variables [25; 114]

$$\begin{gathered}F_D^{(n)}(a;b_1,b_2,b_3,...,b_n;c;x_1,x_2,...,x_n)=\sum_{m_1,...,m_n=0}^{\infty}\frac{(a)_{m_1+...+m_n}(b_1)_{m_1}\dots(b_n)_{m_n}}{(c)_{m_1+...+m_n}m_1!\dots m_n!}x_1^{m_1}...x_n^{m_n},\\ (|x_1|<1,\ |x_2|<1,\ \dots\ ,|x_n|<1),\end{gathered} \tag{4}$$

the form [25; 117]

$$\begin{gathered}F_D^{(n)}(a;b_1,b_2,b_3,...,b_n;c;1,1,...,1)=\frac{\Gamma(c)\Gamma(c-a-b_1-b_2-...-b_n)}{\Gamma(c-a)\Gamma(c-b_1-b_2-...-b_n)},\ \ n=1,2,\dots\\ \operatorname{Re}(c-a-b_1-b_2-...-b_n)>0,\ \ c\neq 0,-1,-2,\dots.\end{gathered} \tag{5}$$

in the case when all variables in (4) take the value 1.

Lauricella's hypergeometric function in the case of four variables has the form [25; 114, (1)]:

$$\begin{gathered}F_A^{(4)}(a;b_1,b_2,b_3,b_4;c_1,c_2,c_3,c_4;x,y,z,t)=\sum_{m,n,p,q}^{\infty}\frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q m!n!p!q!}x^my^nz^pt^q,\\ (|x|+|y|+|z|+|t|<1).\end{gathered} \tag{6}$$

The validity of the decomposition formula for a hypergeometric function of three variables, was proved in [27]:

$$\begin{gathered}F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(a)_{n_1+n_2+n_3}(b_1)_{n_1+n_2}(b_2)_{n_1+n_3}(b_3)_{n_2+n_3}}{(c_1)_{n_1+n_2}(c_2)_{n_1+n_3}(c_3)_{n_2+n_3}n_1!n_2!n_3!}\times\\ \times x^{n_1+n_2}y^{n_1+n_3}z^{n_2+n_3}F(a+n_1+n_2,b_1+n_1+n_2;c_1+n_1+n_2;x)\\ \times F(a+n_1+n_2+n_3,b_2+n_1+n_3;c_2+n_1+n_3;y)\\ \times F(a+n_1+n_2+n_3,b_3+n_2+n_3;c_3+n_2+n_3;z).\end{gathered} \tag{7}$$

We also use the formula for the differentiation of hypergeometric functions of three variables [25]

$$\begin{aligned}&\frac{\partial^{i+j+k}}{\partial x^i\partial y^j\partial z^k}F_A^{(3)}\left(\alpha;\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;x,y,z\right)=\frac{(\alpha)_{i+j+k}(\beta_1)_i(\beta_2)_j(\beta_3)_k}{(\gamma_1)_i(\gamma_2)_j(\gamma_3)_k}\times\\&\times F_A^{(3)}\left(\alpha+i+j+k;\beta_1+i,\beta_2+j,\beta_3+k;\gamma_1+i,\gamma_2+j,\gamma_3+k;x,y,z\right),\\&i,j,k\in\mathbb{N}_0=\{0,1,2,...\}.\end{aligned}\tag{8}$$

For Lauricella's function $F_A^{(n)}$ the following adjacent relations are valid

$$\begin{aligned}&\frac{b_1}{c_1}x_1F_A\left(a+1;b_1+1,b_2,\ldots,b_n;c_1+1,c_2,\ldots,c_n;x_1,\ldots,x_n\right)\\&+\frac{b_2}{c_2}x_2F_A\left(a+1;b_1,b_2+1,\ldots,b_n;c_1,c_2+1,\ldots,c_n;x_1,\ldots,x_n\right)\\&+\ldots+\frac{b_n}{c_n}x_nF_A\left(a+1;b_1,b_2,\ldots,b_n+1;c_1,c_2,\ldots,c_n+1;x_1,\ldots,x_n\right)=\\&=F_A\left(a+1;b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right)-F_A\left(a;b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n\right).\end{aligned}\tag{9}$$

To calculate the value of a multiple integral, we use the formula [28; 637, (3)]

$$\begin{aligned}&\int\limits_0^\infty\int\limits_0^\infty\cdots\int\limits_0^\infty\frac{x_1^{p_1-1}x_2^{p_2-1}\ldots x_n^{p_n-1}}{\left[1+(r_1x_1)^{q_1}+(r_2x_2)^{q_2}+\ldots+(r_nx_n)^{q_n}\right]^s}dx_1dx_2\ldots dx_n=\\&=\frac{\Gamma\left(\frac{p_1}{q_1}\right)\Gamma\left(\frac{p_2}{q_2}\right)\ldots\Gamma\left(\frac{p_n}{q_n}\right)}{q_1q_2\ldots q_n{r_1}^{p_1q_1}{r_2}^{p_2q_2}\ldots{r_n}^{p_nq_n}}\frac{\Gamma\left(s-\frac{p_1}{q_1}-\frac{p_2}{q_2}-\ldots-\frac{p_n}{q_n}\right)}{\Gamma\left(s\right)},\ \left(p_i>0,q_i>0,r_i>0,s>0\right),\end{aligned}\tag{10}$$

and for integrals expressed in terms of the beta function, the formulas [24; 25, (16), (19)]

$$\int\limits_0^\infty\left(1+bt^z\right)^{-y}t^xdt=z^{-1}b^{-\frac{x+1}{z}}\beta\left(\frac{x+1}{z},y-\frac{x+1}{z}\right),\ \left(z>0,\ b>0,\ 0<\mathrm{Re}\,\frac{x+1}{z}<\mathrm{Re}\,y\right),\tag{11}$$

$$\int\limits_0^{\frac{\pi}{2}}\left(\sin t\right)^{2x-1}\left(\cos t\right)^{2y-1}dt=\frac{1}{2}\beta\left(x,y\right),\ \left(\mathrm{Re}\,x>0,\mathrm{Re}\,y>0\right).\tag{12}$$

2. Statement of the problem

Considering the generalized Gellerstedt equation:

$$H\left(u\right)=y^mz^kt^lu_{xx}+x^nz^kt^lu_{yy}+x^ny^mt^lu_{zz}+x^ny^mz^ku_{tt}=0,m,n,k,l>0,\ \ m,n,k,l\equiv const,\tag{13}$$

we introduce the following notations:

$$D=\left\{(x,y,z,t):\ x>0,y>0,\ z>0,t>0\right\},$$

$$S_1=\left\{(0,y,z,t):\ x=0,y>0,\ z>0,t>0\right\},$$

$$S_2=\left\{(x,0,z,t):\ x>0,y=0,\ z>0,t>0\right\},$$

$$S_3=\left\{(x,y,0,t):\ x>0,y>0,\ z=0,t>0\right\},$$

$$S_4=\left\{(x,y,z,0):\ x>0,y>0,\ z>0,t=0\right\},$$

$$R^2=\frac{4}{(n+2)^2}x^{n+2}+\frac{4}{(m+2)^2}y^{m+2}+\frac{4}{(k+2)^2}z^{k+2}+\frac{4}{(l+2)^2}t^{l+2}.$$

Problem ND_2. Find a regular solution $u\left(x,y,z,t\right)$ of the equation (13) from the class $C\left(\overline{D}\right)\cap C^1\left(D\cup\overline{S_3}\cup\overline{S_4}\right)\cap C^2\left(D\right)$ satisfying the condition:

$$u(x,y,z,t)|_{x=0} = \tau_1(y,z,t), \quad (y,z,t) \in \overline{S_1}, \tag{14}$$

$$u(x,y,z,t)|_{y=0} = \tau_2(x,z,t), \quad (x,z,t) \in \overline{S_2}, \tag{15}$$

$$\left.\frac{\partial}{\partial z} u(x,y,z,t)\right|_{z=0} = \nu_3(x,y,t), \quad (x,y,t) \in S_3, \tag{16}$$

$$\left.\frac{\partial}{\partial t} u(x,y,z,t)\right|_{t=0} = \nu_4(x,y,z), \quad (x,y,z) \in S_4, \tag{17}$$

$$\lim_{R\to\infty} u(x,y,z,t) = 0, \tag{18}$$

where $\tau_1(y,z,t), \tau_2(x,z,t), \nu_3(x,y,t), \nu_4(x,y,z) \in \mathbb{C}$ are given continuous functions, moreover the function $\nu_3(x,y,t), \nu_4(x,y,z)$ at the origin of coordinates can go to integrable order infinity. Also, for the large enough values R, the following inequalities hold:

$$|\tau_1(y,z,t)| \le \frac{c_1}{\left[1 + \frac{4}{(n+2)^2} y^{m+2} + \frac{4}{(k+2)^2} z^{k+2} + \frac{4}{(l+2)^2} t^{l+2}\right]^{\varepsilon_1}}, \tag{19}$$

$$|\tau_2(x,z,t)| \le \frac{c_2}{\left[1 + \frac{4}{(n+2)^2} x^{n+2} + \frac{4}{(k+2)^2} z^{k+2} + \frac{4}{(l+2)^2} t^{l+2}\right]^{\varepsilon_2}}, \tag{20}$$

$$|\nu_3(x,y,t)| \le \frac{c_3}{\left[1 + \frac{4}{(n+2)^2} x^{n+2} + \frac{4}{(m+2)^2} y^{m+2} + \frac{4}{(l+2)^2} t^{l+2}\right]^{\frac{1-2\gamma+\varepsilon_3}{2}}}, \tag{21}$$

$$|\nu_4(x,y,z)| \le \frac{c_4}{\left[1 + \frac{4}{(n+2)^2} x^{n+2} + \frac{4}{(m+2)^2} y^{m+2} + \frac{4}{(k+2)^2} z^{k+2}\right]^{\frac{1-2\delta+\varepsilon_4}{2}}}, \tag{22}$$

here $c_1, c_2, c_3, c_4 > 0$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are small enough positive numbers.

Theorem 1. The boundary value problem ND_2 has at most one solution.

Proof. Let $u(x,y,z,t)$ be the solution of a homogeneous problem ND_2, i.e. $u(x,y,z,t)$ is the solution of the equation (13) satisfying the conditions (14)–(18).

By D_R we denote the bounded domain with the boundary $\partial D_R = S_{1R} \cup S_{2R} \cup S_{3R} \cup S_{4R} \cup \sigma_R$, where $S_{1R} = S_1 \cap \{x = 0, 0 < y < R, 0 < z < R, 0 < t < R\}$, $S_{2R} = S_2 \cap \{0 < x < R, y = 0, 0 < z < R, 0 < t < R\}$, $S_{3R} = S_3 \cap \{0 < x < R, 0 < y < R, z = 0, 0 < t < R\}$, $S_{4R} = S_4 \cap \{0 < x < R, 0 < y < R, 0 < z < R, t = 0\}$, $\sigma_R = \left\{(x, y, z, t) : \frac{4}{(n+2)^2} x^{n+2} + \frac{4}{(m+2)^2} y^{m+2} + \frac{4}{(k+2)^2} z^{k+2} + \frac{4}{(l+2)^2} t^{l+2} = R^2, x \ge 0, y \ge 0, z \ge 0, t \ge 0\right\}$.

Choosing large enough R, we integrate equation (13) over the domain D_R, previously multiplied it by a function $u(x,y,z,t)$, we obtain

$$\iiiint\limits_{D_R} \left[y^m z^k t^l u u_{xx} + x^n z^k t^l u u_{yy} + x^n y^m t^l u u_{zz} + x^n y^m z^k u u_{tt}\right] dxdydzdt = 0. \tag{23}$$

Taking into account (23) we obtain the following equalities:

$$y^m z^k t^l u u_{xx} = \frac{\partial}{\partial x}\left(y^m z^k t^l u u_x\right) - y^m z^k t^l u_x^2, \quad x^n z^k t^l u u_{yy} = \frac{\partial}{\partial y}\left(x^n z^k t^l u u_y\right) - x^n z^k t^l u_y^2,$$

$$x^n y^m t^l u u_{zz} = \frac{\partial}{\partial z}\left(x^n y^m t^l u u_z\right) - x^n y^m t^l u_z^2, \quad x^n y^m z^k u u_{tt} = \frac{\partial}{\partial t}\left(x^n y^m z^k u u_t\right) - x^n y^m z^k u_t^2,$$

after applying the Gauss-Ostrogradsky formula, we have

$$\iiiint\limits_{D_R} \left[y^m z^k t^l u_x^2 + x^n z^k t^l u_y^2 + x^n y^m t^l u_z^2 + x^n y^m z^k u_t^2\right] dxdydzdt =$$

$$= \iiint\limits_{S_{1R}} y^m z^k t^l \tau_1 u_x dydzdt + \iiint\limits_{S_{2R}} x^n z^k t^l \tau_2 u_y dxdzdt + \iiint\limits_{S_{3R}} x^n y^m t^l u \nu_3 dxdydt$$

$$+ \iiint\limits_{S_{4R}} x^n y^m z^k u \nu_4 dxdydz + \iiint\limits_{\sigma_R} x^n y^m z^k t^l u \frac{\partial u}{\partial n} dS, \tag{24}$$

where,

$$\frac{\partial u}{\partial n} = u_x \cos(n,x) + u_y \cos(n,y) + u_z \cos(n,z) + u_t \cos(n,t),$$

$\cos(n,x)\,dS = dydzdt$, $\cos(n,y)\,dS = dxdzdt$, $\cos(n,z)\,dS = dxdydt$, $\cos(n,t)\,dS = dxdydz$, n is outer normal to ∂D_R.

Since for the function u $\tau_1 = \tau_2 = \nu_3 = \nu_4 = 0$, then from (24) we have

$$\iiiint\limits_{D_R} \left[y^m z^k t^l u_x^2 + x^n z^k t^l u_y^2 + x^n y^m t^l u_z^2 + x^n y^m z^k u_t^2\right] dxdydzdt = \iiint\limits_{\sigma_R} x^n y^m z^k t^l u \frac{\partial u}{\partial n} dS. \tag{25}$$

By virtue of condition (18) for $R \to \infty$ $\lim\limits_{R\to\infty} \iiint\limits_{\sigma_R} x^n y^m z^k t^l u \frac{\partial u}{\partial n} dS = 0$, then from (25) we have

$$\iiiint\limits_{D} \left[y^m z^k t^l u_x^2 + x^n z^k t^l u_y^2 + x^n y^m t^l u_z^2 + x^n y^m z^k u_t^2\right] dxdydzdt \equiv 0. \tag{26}$$

From (26), we get $u_x = u_y = u_z = u_t = 0$, which means $u = const$, and from the conditions $u|_{x=0} = u|_{y=0} = u_z|_{z=0} = u_t|_{t=0} = 0$ follows that $u \equiv 0$. So, we have proved the uniqueness of the problem ND_2 .

3. Existence of a problem solution

The solution to the ND_2 problem has the form

$$\begin{aligned} u(x_0,y_0,z_0,t_0) = &\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty y^m z^k t^l \tau_1(y,z,t)\, \frac{\partial}{\partial x} g_6(x,y,z,t;x_0,y_0,z_0,t_0)\Big|_{x=0} dydzdt+ \\ &+ \int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty x^n z^k t^l \tau_2(x,z,t)\, \frac{\partial}{\partial y} g_6(x,y,z,t;x_0,y_0,z_0,t_0)|_{y=0} dxdzdt- \\ &- \int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty x^n y^m t^l \nu_3(x,y,t)\, g_6(x,y,0,t;x_0,y_0,z_0,t_0)\, dxdydt- \\ &- \int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty x^n y^m z^k \nu_4(x,y,z)\, g_6(x,y,z,0;x_0,y_0,z_0,t_0)\, dxdydz, \end{aligned} \tag{27}$$

where

$$g_6(x,y,z,t;x_0,y_0,z_0,t_0) = k_6 \left(\frac{4}{n+2}\right)^{\frac{4}{n+2}} \left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} \left(r^2\right)^{\alpha+\beta-\gamma-\delta-3} xyx_0y_0 \times$$
$$\times F_A^{(4)}(3-\alpha-\beta+\gamma+\delta; 1-\alpha, 1-\beta, \gamma, \delta; 2-2\alpha, 2-2\beta, 2\gamma, 2\delta; \xi, \eta, \zeta, \varsigma)$$

is fundamental solution to the equation (13). Here function $F_A^{(4)}$ is Lauricella's function (6),

$$\begin{aligned} k_6 = \frac{1}{4\pi^2} \left(\frac{4}{n+2}\right)^{2\alpha} \left(\frac{4}{m+2}\right)^{2\beta} \left(\frac{4}{k+2}\right)^{2\gamma} \left(\frac{4}{l+2}\right)^{2\delta} \times \\ \times \frac{\Gamma(3-\alpha-\beta+\gamma+\delta)\,\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\gamma)\,\Gamma(\delta)}{\Gamma(2-2\alpha)\,\Gamma(2-2\beta)\,\Gamma(2\gamma)\,\Gamma(2\delta)}, \end{aligned} \tag{28}$$

$$\xi = \frac{r^2 - r_1^2}{r^2},\;\; \eta = \frac{r^2 - r_2^2}{r^2},\;\; \zeta = \frac{r^2 - r_3^2}{r^2},\;\; \varsigma = \frac{r^2 - r_4^2}{r^2},$$

$$\left.\begin{matrix} r^2 \\ r_1^2 \\ r_2^2 \\ r_3^2 \\ r_4^2 \end{matrix}\right\} = \left(\frac{2}{n+2} x^{\frac{n+2}{2}} \begin{matrix} - \\ + \\ - \\ - \\ - \end{matrix} \frac{2}{n+2} x_0^{\frac{n+2}{2}} \right)^2 + \left(\frac{2}{m+2} y^{\frac{m+2}{2}} \begin{matrix} - \\ - \\ + \\ - \\ - \end{matrix} \frac{2}{m+2} y_0^{\frac{m+2}{2}} \right)^2$$

$$+ \left(\frac{2}{k+2} z^{\frac{k+2}{2}} \begin{matrix} - \\ - \\ - \\ + \\ - \end{matrix} \frac{2}{k+2} z_0^{\frac{k+2}{2}} \right)^2 + \left(\frac{2}{l+2} t^{\frac{l+2}{2}} \begin{matrix} - \\ - \\ - \\ - \\ + \end{matrix} \frac{2}{l+2} t_0^{\frac{l+2}{2}} \right)^2,$$

$$\alpha = \frac{n}{2\left(n+2\right)},\ \ \beta = \frac{m}{2\left(m+2\right)},\ \ \gamma = \frac{k}{2\left(k+2\right)},\ \ \delta = \frac{l}{2\left(l+2\right)}.$$

Since the function q_6 is a fundamental solution to equation (13), it is obvious that the solution to problem (27) satisfies equation (13).

Let us prove that function (27) satisfies conditions (14) - (17) of problem ND_2. We apply differentiation formulas (8) and decomposition of hypergeometric functions (9) to (27) and represent (27) as the sum:

$$u\left(x_0, y_0, z_0, t_0\right) = I_1\left(x_0, y_0, z_0, t_0\right) + I_2\left(x_0, y_0, z_0, t_0\right) + I_3\left(x_0, y_0, z_0, t_0\right) + I_4\left(x_0, y_0, z_0, t_0\right), \tag{29}$$

where

$$I_1\left(x_0, y_0, z_0, t_0\right) = k_6 \left(\frac{4}{n+2}\right)^{\frac{4}{n+2}} \left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} x_0 y_0 \int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_0^{\infty} y^{m+1} z^k t^l \tau_1\left(y, z, t\right) \times$$
$$\times \left(r^2\right)^{\alpha+\beta-\gamma-\delta-3} F_A^{(3)}\left(3-\alpha-\beta+\gamma+\delta; 1-\beta, \gamma, \delta; 2-2\beta, 2\gamma, 2\delta; \eta, \zeta, \varsigma\right)\Big|_{x=0} dydzdt, \tag{30}$$

$$I_2\left(x_0, y_0, z_0, t_0\right) = k_6 \left(\frac{4}{n+2}\right)^{\frac{4}{n+2}} \left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} x_0 y_0 \int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_0^{\infty} x^{n+1} z^k t^l \tau_2\left(x, z, t\right) \times$$
$$\times \left(r^2\right)^{\alpha+\beta-\gamma-\delta-3} F_A^{(3)}\left(3-\alpha-\beta+\gamma+\delta; 1-\alpha, \gamma, \delta; 2-2\alpha, 2\gamma, 2\delta; \xi, \zeta, \varsigma\right)\Big|_{y=0} dxdzdt, \tag{31}$$

$$I_3\left(x_0, y_0, z_0, t_0\right) = -k_6 \left(\frac{4}{n+2}\right)^{\frac{4}{n+2}} \left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} x_0 y_0 \int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_0^{\infty} x^{n+1} y^{m+1} t^l \nu_3\left(x, y, t\right) \times$$
$$\times \left(r^2\right)^{\alpha+\beta-\gamma-\delta-3} F_A^{(3)}\left(3-\alpha-\beta+\gamma+\delta; 1-\alpha, 1-\beta, \delta; 2-2\alpha, 2-2\beta, 2\delta; \xi, \eta, \varsigma\right)\Big|_{z=0} dxdydt, \tag{32}$$

$$I_4\left(x_0, y_0, z_0, t_0\right) = -k_6 \left(\frac{4}{n+2}\right)^{\frac{4}{n+2}} \left(\frac{4}{m+2}\right)^{\frac{4}{m+2}} x_0 y_0 \int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_0^{\infty} x^{n+1} y^{m+1} z^k \nu_4\left(x, y, z\right) \times$$
$$\left(r^2\right)^{\alpha+\beta-\gamma-\delta-3} F_A^{(3)}\left(3-\alpha-\beta+\gamma+\delta; 1-\alpha, 1-\beta, \gamma; 2-2\alpha, 2-2\beta, 2\gamma; \xi, \eta, \zeta\right)\Big|_{t=0} dxdydz. \tag{33}$$

Let us check condition (14). Consider the first term of the solution, written in the form (29), function (30). We decompose the function $F_A^{(3)}$ in (30) by formula (7), then after performing some transformations in (30) and applying the Bolz autotransformation formula (3), we obtain

$$F_A\left(3-\alpha-\beta+\gamma+\delta; 1-\beta, \gamma, \delta; 2-2\beta, 2\gamma, 2\delta; \eta, \zeta, \varsigma\right) =$$
$$= \left(r^2\right)^{1-\beta+\gamma+\delta} \left(r_2^2\right)^{\beta-1} \left(r_3^2\right)^{-\gamma} \left(r_4^2\right)^{-\delta} P_1\left(0, y, z, t; x_0, y_0, z_0, t_0\right), \tag{34}$$

where

$$P_1(0,y,z,t;x_0,y_0,z_0,t_0)=\sum_{l_1,l_2,l_3=0}^{\infty}\frac{(3-\alpha-\beta+\gamma+\delta)_{l_1+l_2+l_3}(1-\beta)_{l_1+l_2}(\gamma)_{l_1+l_3}(\delta)_{l_2+l_3}}{(2-2\beta)_{l_1+l_2}(2\gamma)_{l_1+l_3}(2\delta)_{l_2+l_3}l_1!l_2!l_3!}$$
$$\times\left(\frac{r_2^2-r^2}{r_2^2}\right)^{l_1+l_2}\left(\frac{r_3^2-r^2}{r_3^2}\right)^{l_1+l_3}\left(\frac{r_4^2-r^2}{r_4^2}\right)^{l_2+l_3}$$
$$\times F\left(\alpha-\beta-\gamma-\delta-1,1-\beta+l_1+l_2;2-2\beta+l_1+l_2;\frac{r_2^2-r^2}{r_2^2}\right)$$
$$\times F\left(\alpha+\beta+\gamma-\delta-3-l_2,\gamma+l_1+l_3;2\gamma+l_1+l_3;\frac{r_3^2-r^2}{r_3^2}\right)$$
$$\times F\left(\alpha+\beta-\gamma+\delta-3-l_1,\delta+l_2+l_3;2\delta+l_2+l_3;\frac{r_4^2-r^2}{r_4^2}\right). \qquad (35)$$

Thus, substituting (34) into (30), we have

$$I_1(x_0,y_0,z_0,t_0)=k_6\left(\frac{4}{n+2}\right)^{\frac{4}{n+2}}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}}x_0y_0\times$$
$$\times\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}y^{m+1}z^kt^l\tau_1(y,z,t)\left.\frac{P_1(0,y,z,t;x_0,y_0,z_0,t_0)}{(r^2)^{2-\alpha}(r_2^2)^{1-\beta}(r_3^2)^{\gamma}(r_4^2)^{\delta}}\right|_{x=0}dydzdt. \qquad (36)$$

In (36), we make the change of variables

$$\frac{2}{m+2}y^{\frac{m+2}{2}}=\frac{2}{m+2}y_0^{\frac{m+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_1,\ \ \frac{2}{k+2}z^{\frac{k+2}{2}}=\frac{2}{k+2}z_0^{\frac{k+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_2,$$
$$\frac{2}{l+2}t^{\frac{l+2}{2}}=\frac{2}{l+2}t_0^{\frac{l+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_3. \qquad (37)$$

Then, we obtain the following equality

$$I_1(x_0,y_0,z_0,t_0)=k_6\left(\frac{4}{n+2}\right)^{\frac{4}{n+2}}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}}x_0y_0\left(\frac{2}{n+2}x_0^{\frac{n+2}{2}}\right)^3$$
$$\int_{-a}^{\infty}\int_{-b}^{\infty}\int_{-c}^{\infty}\left[\left[\frac{m+2}{2}\left(\frac{2}{m+2}y_0^{\frac{m+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_1\right)\right]^{\frac{2}{m+2}}\right]^{m+1}\left[\left[\frac{k+2}{2}\left(\frac{2}{k+2}z_0^{\frac{k+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_2\right)\right]^{\frac{2}{k+2}}\right]^{k}$$
$$\times\left[\left[\frac{l+2}{2}\left(\frac{2}{l+2}t_0^{\frac{l+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_3\right)\right]^{\frac{2}{l+2}}\right]^{l}\left.\frac{P_1(0,y,z,t;x_0,y_0,z_0,t_0)}{(1+s_1^2+s_2^2+s_3^2)^{2-\alpha}(r_2^2)^{1-\beta}(r_3^2)^{\gamma}(r_4^2)^{\delta}}\right|_{x=0}$$
$$\times\tau_1\left(\left[\frac{m+2}{2}\left(\frac{2}{m+2}y_0^{\frac{m+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_1\right)\right]^{\frac{2}{m+2}},\left[\frac{k+2}{2}\left(\frac{2}{k+2}z_0^{\frac{k+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_2\right)\right]^{\frac{2}{k+2}},\right.$$
$$\left.\left[\frac{l+2}{2}\left(\frac{2}{l+2}t_0^{\frac{l+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_3\right)\right]^{\frac{2}{l+2}}\right)\left[\frac{m+2}{2}\left(\frac{2}{m+2}y_0^{\frac{m+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_1\right)\right]^{-\frac{m}{m+2}}\times$$
$$\times\left[\frac{k+2}{2}\left(\frac{2}{k+2}z_0^{\frac{k+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_2\right)\right]^{-\frac{k}{k+2}}\left[\frac{l+2}{2}\left(\frac{2}{l+2}t_0^{\frac{l+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_3\right)\right]^{-\frac{l}{l+2}}ds_1ds_2ds_3, \qquad (38)$$

where

$$a=\frac{\frac{2}{m+2}y_0^{\frac{m+2}{2}}}{\frac{2}{n+2}x_0^{\frac{n+2}{2}}},b=\frac{\frac{2}{k+2}z_0^{\frac{k+2}{2}}}{\frac{2}{n+2}x_0^{\frac{n+2}{2}}},\ c=\frac{\frac{2}{l+2}t_0^{\frac{l+2}{2}}}{\frac{2}{n+2}x_0^{\frac{n+2}{2}}}.$$

At $x_0 \to 0$ from (35) we have

$$\lim_{x_0\to 0} P_1(0,y,z,t;x_0,y_0,z_0,t_0) = \sum_{l_1,l_2,l_3=0}^{\infty} \frac{(3-\alpha-\beta+\gamma+\delta)_{l_1+l_2+l_3}(1-\beta)_{l_1+l_2}(\gamma)_{l_1+l_3}(\delta)_{l_2+l_3}}{(2-2\beta)_{l_1+l_2}(2\gamma)_{l_1+l_3}(2\delta)_{l_2+l_3}l_1!l_2!l_3!}$$
$$\times F(\alpha-\beta-\gamma-\delta-1, 1-\beta+l_1+l_2; 2-2\beta+l_1+l_2; 1)$$
$$\times F(\alpha+\beta+\gamma-\delta-3-l_2, \gamma+l_1+l_3; 2\gamma+l_1+l_3; 1)$$
$$\times F(\alpha+\beta-\gamma+\delta-3-l_1, \delta+l_2+l_3; 2\delta+l_2+l_3; 1) \qquad (39)$$

Applying formulas (2) and (5) to (39), we determine

$$\lim_{x_0\to 0} P_1(0,y,z,t;x_0,y_0,z_0,t_0) = \frac{\Gamma(2-2\beta)\Gamma(2\gamma)\Gamma(2\delta)\Gamma(2-\alpha)}{\Gamma(3-\alpha-\beta+\gamma+\delta)\Gamma(1-\beta)\Gamma(\gamma)\Gamma(\delta)} \qquad (40)$$

By virtue of (40), from (38) at $x_0 \to 0$, we obtain

$$\lim_{x_0\to 0} I_1(x_0,y_0,z_0,t_0) = k_6\left(\frac{8}{n+2}\right)^{\frac{2}{n+2}}\left(\frac{4}{m+2}\right)^{-2\beta}\left(\frac{4}{k+2}\right)^{-2\gamma}\left(\frac{4}{l+2}\right)^{-2\delta}\times$$
$$\times\frac{\Gamma(2-2\beta)\Gamma(2\gamma)\Gamma(2\delta)\Gamma(2-\alpha)}{\Gamma(3-\alpha-\beta+\gamma+\delta)\Gamma(1-\beta)\Gamma(\gamma)\Gamma(\delta)}\tau_1(y_0,z_0,t_0)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{ds_1ds_2ds_3}{(1+s_1^2+s_2^2+s_3^2)^{2-\alpha}}. \qquad (41)$$

To calculate the triple integral from (41), using formula (10), we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{ds_1ds_2ds_3}{(1+s_1^2+s_2^2+s_3^2)^{2-\alpha}} = 8\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\frac{ds_1ds_2ds_3}{(1+s_1^2+s_2^2+s_3^2)^{2-\alpha}} = \frac{\pi\sqrt{\pi}\Gamma\left(\frac{1}{2}-\alpha\right)}{\Gamma(2-\alpha)}. \qquad (42)$$

Applying formula (1) in (42), as a result, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{ds_1ds_2ds_3}{(1+s_1^2+s_2^2+s_3^2)^{2-\alpha}} = \frac{\pi^2\Gamma(2-2\alpha)}{2^{-2\alpha}\Gamma(2-\alpha)(1-2\alpha)\Gamma(1-\alpha)}. \qquad (43)$$

Substituting (43) into (41), we finally have

$$\lim_{x_0\to 0} I_1(x_0,y_0,z_0,t_0) = 4\pi^2 k_6\left(\frac{4}{n+2}\right)^{-2\alpha}\left(\frac{4}{m+2}\right)^{-2\beta}\left(\frac{4}{k+2}\right)^{-2\gamma}\left(\frac{4}{l+2}\right)^{-2\delta}\times$$
$$\times\frac{\Gamma(2-2\alpha)\Gamma(2-2\beta)\Gamma(2\gamma)\Gamma(2\delta)}{\Gamma(3-\alpha-\beta+\gamma+\delta)\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\gamma)\Gamma(\delta)}\tau_1(y_0,z_0,t_0) \qquad (44)$$

Taking into account (28), from (44), we obtain

$$\lim_{x_0\to 0} I_1(x_0,y_0,z_0,t_0) = \tau_1(y_0,z_0,t_0).$$

It is easy to show that

$$\lim_{x_0\to 0} I_2(x_0,y_0,z_0,t_0) = 0,\ \lim_{x_0\to 0} I_3(x_0,y_0,z_0,t_0) = 0,\ \lim_{x_0\to 0} I_4(x_0,y_0,z_0,t_0) = 0.$$

Accordingly, $\lim_{x_0\to 0} u(x_0,y_0,z_0,t_0) = \tau_1(y_0,z_0,t_0)$, hence, function (29) satisfies condition (14) of the problem ND_2. Similarly, can be convinced that function (29) also satisfies conditions (15), (16), and (17) of the problem ND_2.

Let us show that if the given functions satisfy inequalities (19) – (22) for large enough values of the argument, then the solution (29) of the Problem ND_2 also satisfies condition (18). Indeed, let inequalities (19) – (22) are hold, in expressions (30) – (33) we make the following change of variables

$$\xi_1 = \frac{1}{R_0}\frac{2}{n+2}x^{\frac{n+2}{2}},\ \eta_1 = \frac{1}{R_0}\frac{2}{m+2}y^{\frac{m+2}{2}},\ \zeta_1 = \frac{1}{R_0}\frac{2}{k+2}z^{\frac{k+2}{2}},\ \varsigma_1 = \frac{1}{R_0}\frac{2}{l+2}t^{\frac{l+2}{2}},$$
$$\sigma_1 = \frac{1}{R_0}\frac{2}{n+2}x_0^{\frac{n+2}{2}},\ \sigma_2 = \frac{1}{R_0}\frac{2}{m+2}y_0^{\frac{m+2}{2}},\ \sigma_3 = \frac{1}{R_0}\frac{2}{k+2}z_0^{\frac{k+2}{2}},\ \sigma_4 = \frac{1}{R_0}\frac{2}{l+2}t_0^{\frac{l+2}{2}},$$

where

$$R_0{}^2 = \frac{4}{(n+2)^2} x_0{}^{n+2} + \frac{4}{(m+2)^2} y_0{}^{m+2} + \frac{4}{(k+2)^2} z_0{}^{k+2} + \frac{4}{(l+2)^2} t_0{}^{l+2}.$$

Then at $R_0 \to \infty$ from (30)–(33) we obtain the following inequalities:

$$\lim_{R_0\to\infty} |I_1(x_0,y_0,z_0,t_0)| \leq \frac{k_6 c_1}{R_0^{2\varepsilon_1}} 4^{\frac{2}{n+2}+\frac{2}{m+2}} (\sigma_1)^{\frac{2}{n+2}} (\sigma_2)^{\frac{2}{m+2}} \left(\frac{2}{n+2}\right)^{1-2\alpha} \left(\frac{2}{m+2}\right)^{-2\beta} \times$$
$$\left(\frac{2}{k+2}\right)^{-2\gamma} \left(\frac{2}{l+2}\right)^{-2\delta} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\eta_1 \zeta_1{}^{2\gamma} \varsigma_1{}^{2\delta}}{\left(1+\eta_1{}^2+\zeta_1{}^2+\varsigma_1{}^2\right)^{3-\alpha-\beta+\gamma+\delta} \left(\eta_1{}^2+\zeta_1{}^2+\varsigma_1{}^2\right)^{\varepsilon_1}} d\eta_1 d\zeta_1 d\varsigma_1, \tag{45}$$

$$\lim_{R_0\to\infty} |I_2(x_0,y_0,z_0,t_0)| \leq \frac{k_6 c_2}{R_0^{2\varepsilon_2}} 4^{\frac{2}{n+2}+\frac{2}{m+2}} (\sigma_1)^{\frac{2}{n+2}} (\sigma_2)^{\frac{2}{m+2}} \left(\frac{2}{n+2}\right)^{-2\alpha} \left(\frac{2}{m+2}\right)^{1-2\beta} \times$$
$$\left(\frac{2}{k+2}\right)^{-2\gamma} \left(\frac{2}{l+2}\right)^{-2\delta} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\xi_1 \zeta_1{}^{2\gamma} \varsigma_1{}^{2\delta}}{\left(1+\xi_1{}^2+\zeta_1{}^2+\varsigma_1{}^2\right)^{3-\alpha-\beta+\gamma+\delta} \left(\xi_1{}^2+\zeta_1{}^2+\varsigma_1{}^2\right)^{\varepsilon_2}} d\xi_1 d\zeta_1 d\varsigma_1, \tag{46}$$

$$\lim_{R_0\to\infty} |I_3(x_0,y_0,z_0,t_0)| \leq \frac{k_6 c_3}{R_0^{\varepsilon_3}} 4^{\frac{2}{n+2}+\frac{2}{m+2}} (\sigma_1)^{\frac{2}{n+2}} (\sigma_2)^{\frac{2}{m+2}} \left(\frac{2}{n+2}\right)^{-2\alpha} \left(\frac{2}{m+2}\right)^{-2\beta} \times$$
$$\left(\frac{2}{l+2}\right)^{-2\delta} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\xi_1 \eta_1 \varsigma_1{}^{2\delta}}{\left(1+\xi_1{}^2+\eta_1{}^2+\varsigma_1{}^2\right)^{3-\alpha-\beta+\gamma+\delta} \left(\xi_1{}^2+\eta_1{}^2+\varsigma_1{}^2\right)^{\frac{1-2\gamma+\varepsilon_3}{2}}} d\xi_1 d\eta_1 d\varsigma_1, \tag{47}$$

$$\lim_{R_0\to\infty} |I_4(x_0,y_0,z_0,t_0)| \leq \frac{k_6 c_4}{R_0^{\varepsilon_4}} 4^{\frac{2}{n+2}+\frac{2}{m+2}} (\sigma_1)^{\frac{2}{n+2}} (\sigma_2)^{\frac{2}{m+2}} \left(\frac{2}{n+2}\right)^{-2\alpha} \left(\frac{2}{m+2}\right)^{-2\beta} \times$$
$$\left(\frac{2}{k+2}\right)^{-2\gamma} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\xi_1 \eta_1 \zeta_1{}^{2\gamma}}{\left(1+\xi_1{}^2+\eta_1{}^2+\zeta_1{}^2\right)^{3-\alpha-\beta+\gamma+\delta} \left(\xi_1{}^2+\eta_1{}^2+\zeta_1{}^2\right)^{\frac{1-2\delta+\varepsilon_4}{2}}} d\xi_1 d\eta_1 d\zeta_1. \tag{48}$$

Let us show that the triple integrals in inequalities (45) – (48) are bounded.

Considering the integrals from inequality (45) – (46), these integrals satisfy the identity.

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{x y^{2b} z^{2c} dx dy dz}{(1+x^2+y^2+z^2)^{3-a-b+c+d} (x^2+y^2+z^2)^{\varepsilon}} =$$
$$= \frac{1}{8} \frac{\Gamma\left(\frac{1}{2}+c\right)\Gamma\left(\frac{1}{2}+d\right)\Gamma(2+c+d-\varepsilon)\Gamma(1-a-b+\varepsilon)}{\Gamma(2+c+d)\Gamma(3-a-b+c+d)}, \quad a+b-1<\varepsilon<2+c+d. \tag{49}$$

Indeed, in integral (49), passing into spherical coordinates, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{x y^{2b} z^{2c} dx dy dz}{(1+x^2+y^2+z^2)^{3-a-b+c+d} (x^2+y^2+z^2)^{\varepsilon}} =$$
$$= \int_0^{\frac{\pi}{2}} \sin^{2c}\varphi \cos\varphi d\varphi \int_0^{\frac{\pi}{2}} \sin^{2+2c}\theta \cos^{2d}\theta d\theta \int_0^\infty r^{3+2c+2d-2\varepsilon}\left(1+r^2\right)^{a+b-c-d-3} dr. \tag{50}$$

Using the values of integrals (11) and (12) in expression (50), we obtain the identity (49)

$$\int_0^{\frac{\pi}{2}} \sin^{2c}\varphi \cos\varphi d\varphi \int_0^{\frac{\pi}{2}} \sin^{2+2c}\theta \cos^{2d}\theta d\theta \int_0^\infty r^{3+2c+2d-2\varepsilon}\left(1+r^2\right)^{a+b-c-d-3} dr =$$
$$= \frac{1}{8} \frac{\Gamma\left(\frac{1}{2}+c\right)\Gamma\left(\frac{1}{2}+d\right)\Gamma(2+c+d-\varepsilon)\Gamma(1-a-b+\varepsilon)}{\Gamma(2+c+d)\Gamma(3-a-b+c+d)}, \quad a+b-1<\varepsilon<2+c+d.$$

Thus, inequalities (45) - (46) by virtue of the value of integral (49) the inequalities follow

$$\lim_{R_0\to\infty}|I_1(x_0,y_0,z_0,t_0)|\le\frac{k_6\overline{c_1}}{{R_0}^{2\varepsilon_1}},\ \lim_{R_0\to\infty}|I_2(x_0,y_0,z_0,t_0)|\le\frac{k_6\overline{c_2}}{{R_0}^{2\varepsilon_2}},\tag{51}$$

where $\overline{c_1}$, $\overline{c_2}$ are constants.

Let us show that the integrals in (47) – (48) are bounded. For inequalities (47) – (48), the identity is true

$$\begin{gathered}\int_0^\infty\int_0^\infty\int_0^\infty\frac{xyt^{2d}}{(1+x^2+y^2+t^2)^{3-a-b+c+d}(x^2+y^2+t^2)^{\frac{1-2c+\varepsilon}{2}}}dxdydt=\\=\frac{1}{8}\frac{\Gamma\left(\frac{1}{2}+d\right)}{\Gamma\left(\frac{5}{2}+d\right)}\frac{\Gamma\left(2+c+d-\frac{\varepsilon}{2}\right)\Gamma\left(1-a-b+\frac{\varepsilon}{2}\right)}{\Gamma(3-a-b+c+d)},\quad 2a+2b-2<\varepsilon<4+2c+2d.\end{gathered}\tag{52}$$

Passing into spherical coordinates in (52), we obtain

$$\begin{gathered}\int_0^\infty\int_0^\infty\int_0^\infty\frac{xyt^{2d}}{(1+x^2+y^2+t^2)^{3-a-b+c+d}(x^2+y^2+t^2)^{\frac{1-2c+\varepsilon}{2}}}dxdydt=\\=\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi d\varphi\int_0^{\frac{\pi}{2}}\sin^3\theta r\cos^{2d}\theta d\theta\int_0^\infty\left(1+r^2\right)^{a+b-c-d-3}r^{3+2c+2d-\varepsilon}dr.\end{gathered}\tag{53}$$

Using formulas (11) and (12) to the right-hand side of (53), we define

$$\begin{gathered}\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi d\varphi\int_0^{\frac{\pi}{2}}\sin^3\theta r\cos^{2d}\theta d\theta\int_0^\infty\left(1+r^2\right)^{a+b-c-d-3}r^{3+2c+2d-\varepsilon}d=\\=\frac{1}{8}\frac{\Gamma\left(\frac{1}{2}+d\right)}{\Gamma\left(\frac{5}{2}+d\right)}\frac{\Gamma\left(2+c+d-\frac{\varepsilon}{2}\right)\Gamma\left(1-a-b+\frac{\varepsilon}{2}\right)}{\Gamma(3-a-b+c+d)},\ 2a+2b-2<\varepsilon<4+2c+2d.\end{gathered}$$

Thus, we have shown that the integrals in inequalities (47) – (48) are bounded; the integrals satisfy the inequalities

$$\lim_{R_0\to\infty}|I_3(x_0,y_0,z_0,t_0)|\le\frac{k_6\overline{c_3}}{{R_0}^{\varepsilon_3}},\lim_{R_0\to\infty}|I_4(x_0,y_0,z_0,t_0)|\le\frac{k_6\overline{c_4}}{{R_0}^{\varepsilon_4}},\tag{54}$$

where $\overline{c_3}$, $\overline{c_4}$ are constants. Inequalities (51) and (54) show that solution (27) at $R_0\to\infty$ tends to zero. Thereby, condition (18) of Problem ND_2 is satisfied. In this connection, solution (27) of Problem ND_2 satisfies all conditions of Problem ND_2.

Conclusions

We have proved the following theorem.

Theorem 2. Let conditions (19) – (22) be satisfied, then a regular solution to problem ND_2 (13), (14) – (18) exists and is expressed by formula (27).

In four-dimensional space in an infinite domain for the degenerate elliptic Gellerstedt equation, the problem ND_2 with two Neumann boundary conditions and with two Dirichlet conditions is solved. The solution is written explicitly. The uniqueness and existence of a solution to the equation are proved.

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А.С. Бердышев[1,2], А.Р. Рысқан[1,2]

[1] *Абай атындағы Қазақ ұлттық педагогикалық университеті, Алматы, Қазақстан;*
[2] *Ақпараттық және есептеуіш технологиялар институты, Алматы, Қазақстан*

Төрт өлшемді Геллерстедт теңдеуі үшін шеттік есеп

Мақалада төрт айнымалы Геллерстедт теңдеу үшін Нейман және Дирихле шарттары бар шеттік есептің шешілуі зерттелген. Есеп шешімінің жалғыздығын дәлелдеу үшін энергия интегралы әдісі қолданылған. Сонымен қатар, шешімінің бар болуына гипергеометриялық функцияларды дифференциациялау, автотрансформациялау және жіктеу формулалары пайдаланылған. Шешім айқын түрде алынған және Лауричелла гипергеометриялық функцияларымен өрнектелген.

Кілт сөздер: Геллерстедт теңдеуі, аралас шарттары бар шеттік есеп, фундаментальді шешім, Лауричелла гипергеометриялық функциясы.

А.С. Бердышев[1,2], А.Р. Рыскан[1,2]

[1] *Казахский национальный педагогический университет имени Абая, Алматы, Казахстан;*
[2] *Институт информационных и вычислительных технологий МОН РК, Алматы, Казахстан*

Краевая задача для четырехмерного уравнения Геллерстедта

В статье исследована разрешимость задачи с краевыми условиями Неймана и Дирихле для уравнения Геллерстедта от четырех переменных. В ходе доказательства единственности решения задачи применен метод интеграла энергии, кроме того, существовании решения в задачи использованы формулы дифференцирования, автотрансформации, разложения гипергеометрических функций. Решение получено в явном виде и выражено гипергеометрическими функциями Лауричеллы.

Ключевые слова: уравнение Геллерстедта, краевая задача со смешанными условиями, фундаментальное решение, гипергеометрическая функция Лауричеллы.

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S. Bitimkhan*

Karagandy University of the name of academician E.A. Buketov, Kazakhstan
(E-mail: bitimsamat10@gmail.com)

Generalization of the Hardy-Littlewood theorem on Fourier series

In the theory of one-dimensional trigonometric series, the Hardy-Littlewood theorem on Fourier series with monotone Fourier coefficients is of great importance. Multidimensional versions of this theorem have been extensively studied for the Lebesgue space. Significant differences of the multidimensional variants in comparison with the one-dimensional case are revealed and the strengthening of this theorem is obtained. The Hardy-Littlewood theorem is also generalized for various function spaces and various types of monotonicity of the series coefficients. Some of these generalizations can be seen in works of M.F. Timan, M.I. Dyachenko, E.D. Nursultanov, S. Tikhonov. In this paper, a generalization of the Hardy-Littlewood theorem for double Fourier series of a function in the space $L_q\varphi(L_q)(0,2\pi]^2$ is obtained.

Keywords: trigonometric series, Fourier series, Lebesgue space, Hardy-Littlewood theorem, Fourier coefficients.

Let $L_q(0,2\pi]^2$, $1 \leq q < +\infty$ be the space of all 2π- periodic for each variable, measurable by Lebesgue functions $f(x,y)$, for which

$$\|f\|_q = \left(\int_0^{2\pi}\int_0^{2\pi} |f(x,y)|^q \, dxdy\right)^{\frac{1}{q}} < +\infty.$$

In this article, we study the condition of belonging to the space $L_q\varphi(L_q)(0,2\pi]^2$ for a function of two variables. Let us recall the definition of the space $L_q\varphi(L_q)(0,2\pi]^2$. Let the function $\varphi(t)$ satisfy the following conditions [1]:

a) $\varphi(t)$ is even, non-negative, non-decreasing on $[0,+\infty)$;

b) $\varphi(t^2) \leq C\cdot\varphi(t)$, $t \in [0,+\infty)$, $C \geq 1$;

c) $\frac{\varphi(t)}{t^\varepsilon} \downarrow$ on $(0,+\infty)$ with some $\varepsilon > 0$.

Measurable 2π- periodic function for each variable $f(x,y) \in L_q\varphi(L_q)(0,2\pi]^2$ if

$$\int\limits_0^{2\pi}\int\limits_0^{2\pi} |f(x,y)|^q \cdot \varphi\left(|f(x,y)|^q\right) dxdy < +\infty.$$

In particular, when $\varphi(t) \equiv 1$ the space $L_q\varphi(L_q)(0,2\pi]^2$ coincides with the Lebesgue space $L_q(0,2\pi]^2$.

We give the following well-known theorem of Hardy-Littlewood.

Theorem (Hardy-Littlewood). Let $a_n \downarrow 0$, $n \to +\infty$. For the trigonometric series

$$\sum_{n=1}^{+\infty} a_n \cos nx$$

to be the Fourier series of some functions $f(x) \in L_q$, $1 < q < +\infty$, it is necessary and sufficient that

$$\sum_{n=1}^{+\infty} n^{q-2}\cdot a_n^q < +\infty.$$

The Hardy-Littlewood theorem is generalized for various function spaces and various types of monotonicity of the series coefficients. Some of these generalizations can be seen, for example, in works [1–9]. Our main goal is to prove the Hardy-Littlewood theorem for the double Fourier series of a function $f(x,y) \in L_q\varphi(L_q)(0,2\pi]^2$.

*Corresponding author.
E-mail: bitimsamat10@gmail.com

To obtain the main result, we need the following Lemma.

Lemma A. Let the function $\Phi(u)$ be even, non-negative, non-decreasing on $[0,+\infty)$. If $\Phi(u^2) \leq C \cdot \Phi(u)$ at some constant $C \geq 1$ and $u \in [0,+\infty)$, then for any nonnegative function $\psi(x,y)$ measurable on $[0,1)^2$ and satisfying the condition

$$B \equiv \int_0^1 \int_0^1 \psi(x,y) \cdot \Phi(\psi(x,y)) dxdy < +\infty, \tag{1}$$

the following inequality holds

$$A \equiv \int_0^1 \int_0^1 \psi(x,y) \cdot \Phi\left(\frac{1}{xy}\right) dxdy < +\infty. \tag{2}$$

Proof. In [10], it was proved that if $\alpha \geq 0$ and $\beta \geq 0$, then

$$\alpha\Phi(\beta) \leq C \cdot \alpha \cdot \Phi(\alpha) + \sqrt{\beta} \cdot \Phi(\beta). \tag{3}$$

Let $x > 0, \quad y > 0$. Let in (3) $\alpha = \psi(x,y)$ and $\beta = \frac{1}{xy}$. By integrating we get

$$A = \int_0^1 \int_0^1 \psi(x,y) \cdot \Phi\left(\frac{1}{xy}\right) dxdy \leq C \cdot \int_0^1 \int_0^1 \psi(x,y) \cdot \Phi(\psi(x,y)) dxdy +$$

$$+ \int_0^1 \int_0^1 \frac{1}{\sqrt{xy}} \cdot \Phi\left(\frac{1}{xy}\right) dxdy. \tag{4}$$

However, a non-decreasing function $\Phi(u) \geq 0$ satisfies the condition $\Phi(u^2) \leq C \cdot \Phi(u)$ and therefore

$$\Phi(uv) \leq \Phi(4) \cdot \left[(\log_2(2+u))^{\log_2 C} + (\log_2(2+v))^{\log_2 C}\right], \tag{5}$$

at $0 \leq u < +\infty, \quad 0 \leq v < +\infty$.

In fact, at $0 \leq uv \leq 4$ inequality (5) is obvious. If $uv \geq 4$, then put $k = [\log_2 \log_2(uv)]$, where $[a]$ means the integer part of a, and then

$$\Phi(uv) \leq C \cdot \Phi((uv)^{\frac{1}{2}}) \leq C^2 \cdot \Phi((uv)^{\frac{1}{2^2}}) \leq \ldots \leq C^k \cdot \Phi((uv)^{\frac{1}{2^k}}) \leq$$

$$\leq C^{\log_2 \log_2 (uv)} \cdot \Phi\left((uv)^{\frac{2}{\log_2(uv)}}\right) = k(\log_2(uv))^{\log_2 C} \cdot \Phi(4) \leq$$

$$\leq \Phi(4) \cdot \left\{(\log_2(u+2))^{\log_2 C} + (\log_2(v+2))^{\log_2 C}\right\},$$

that is, inequality (5) is true when $uv > 4$. From condition (1), inequalities (4) and (5) follows, that (2). The Lemma A is proved.

Theorem. Let the function $\varphi(t)$ satisfies conditions a) - c) and $f(x,y) \in L_1(0,2\pi]^2$, is an even function with Fourier series

$$\sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} a_{n_1,n_2} \cdot \cos n_1 x \cdot \cos n_2 y, \tag{6}$$

where

$$a_{k,l} \geq 0, \; a_{k,l} - a_{k+1,l} - a_{k,l+1} + a_{k+1,l+1} \geq 0, \; a_{k,l} \geq a_{k+1,l}, \; a_{k,l} \geq a_{k,l+1}, \; k,l \in \mathbb{N}.$$

Then in order for some $q \in (1,+\infty)$ function $f \in L_q\varphi(L_q)(0,2\pi]^2$, it is necessary and sufficient that

$$\sum_{n_1=2}^{+\infty} \sum_{n_2=2}^{+\infty} (n_1 n_2)^{q-2} \varphi(n_1 n_2) \cdot a_{n_1,n_2}^q < +\infty. \tag{7}$$

Proof. Sufficiency. Since with every $\varepsilon \in (0,1)$ the inequality is fulfilled

$$\sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} (n_1 n_2)^{q-2} \varphi(n_1 n_2) \cdot a_{n_1,n_2}^q =$$

$$= \sum_{n_1=1}^{+\infty} n_1^{q-2} \sum_{n_2=1}^{+\infty} \left\{ a_{n_1,n_2} \cdot n_2^{-\frac{\varepsilon}{q}} \cdot \varphi^{\frac{1}{q}}(n_1 n_2) n_2 \right\}^q \cdot n_2^{-(2-\varepsilon)},$$

then using the inequality (see. [11; 308]):

$$\sum_{n=1}^{+\infty} n^{-c} \left(\sum_{\nu=1}^{n} a_\nu \right)^l \leq M(c,l) \cdot \sum_{n=1}^{+\infty} n^{-c} (n \cdot a_n)^l, \quad (c > 1, \quad l > 1, \quad a_\nu \geq 0)$$

and using properties of the function $\varphi(t)$ we get:

$$\sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} (n_1 n_2)^{q-2} \varphi(n_1 n_2) \cdot a_{n_1,n_2}^q \geq$$

$$\geq C \sum_{n_1=1}^{+\infty} n_1^{q-2} \cdot \sum_{n_2=1}^{+\infty} n_2^{-(2-\varepsilon)} \left\{ \sum_{\nu_2=1}^{n_2} a_{n_1,\nu_2} \cdot \varphi^{\frac{1}{q}}(n_1 \nu_2) \nu_2^{-\frac{\varepsilon}{q}} \right\}^q =$$

$$= C \cdot \sum_{n_1=1}^{+\infty} n_1^{q-2+\varepsilon} \cdot \sum_{n_2=1}^{+\infty} n_2^{-(2-\varepsilon)} \left\{ \sum_{\nu_2=1}^{n_2} a_{n_1,\nu_2} \cdot \left(\frac{\varphi(n_1 \nu_2)}{n_1^\varepsilon \nu_2^\varepsilon} \right)^{\frac{1}{q}} \right\}^q \geq$$

$$\geq C \cdot \sum_{n_1=1}^{+\infty} n_1^{q-2+\varepsilon} \cdot \sum_{n_2=1}^{+\infty} n_2^{-(2-\varepsilon)} \frac{\varphi(n_1 n_2)}{n_1^\varepsilon n_2^\varepsilon} \left\{ \sum_{\nu_2=1}^{n_2} a_{n_1,\nu_2} \right\}^q =$$

$$= C \cdot \sum_{n_2=1}^{+\infty} n_2^{-2} \cdot \sum_{n_1=1}^{+\infty} n_1^{q-2} \cdot \varphi(n_1 n_2) \left\{ \sum_{\nu_2=1}^{n_2} a_{n_1,\nu_2} \right\}^q =$$

$$= C \cdot \sum_{n_2=1}^{+\infty} n_2^{-2} \cdot \sum_{n_1=1}^{+\infty} \left\{ \left(\sum_{\nu_2=1}^{n_2} a_{n_1,\nu_2} \right) n_1 \cdot \varphi^{\frac{1}{q}}(n_1 n_2) \cdot n_1^{-\frac{\varepsilon}{p}} \right\}^q \cdot n_1^{-(2-\varepsilon)} \geq$$

$$\geq \sum_{n_2=1}^{+\infty} n_2^{-2} \sum_{n_1=1}^{+\infty} n_1^{-(2-\varepsilon)} \cdot \left\{ \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2} \cdot \varphi^{\frac{1}{q}}(\nu_1 n_2) \cdot \nu_1^{-\frac{\varepsilon}{q}} \right\}^q \geq$$

$$\geq C \cdot \sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} (n_1 n_2)^{-2} \cdot \varphi(n_1 n_2) \left\{ \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2} \right\}^q \equiv S(q,\varphi).$$

Since

$$J(q,\varphi) = \int_0^\pi \int_0^\pi |f(x,y)|^q \cdot \varphi\left[|f(x,y)|^q\right] dxdy =$$

$$= \sum_{n_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} \int_{\frac{\pi}{n_1+1}}^{\frac{\pi}{n_1}} \int_{\frac{\pi}{n_2+1}}^{\frac{\pi}{n_2}} |f(x,y)|^q \cdot \varphi\left[|f(x,y)|^q\right] dxdy,$$

and for $\frac{\pi}{n_1+1} \leq x \leq \frac{\pi}{n_1}, \quad \frac{\pi}{n_2+1} \leq y \leq \frac{\pi}{n_2}$

$$|f(x,y)| \leq \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2} + \left| \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=n_2+1}^{+\infty} a_{\nu_1,\nu_2} \cdot \cos \nu_1 x \cdot \cos \nu_2 y \right| +$$

$$+ \left| \sum_{\nu_1=n_1+1}^{+\infty} \sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2} \cdot \cos \nu_1 x \cdot \cos \nu_2 y \right| + \left| \sum_{\nu_1=n_1+1}^{+\infty} \sum_{\nu_2=n_2+1}^{+\infty} a_{\nu_1,\nu_2} \cdot \cos \nu_1 x \cdot \cos \nu_2 y \right| \leq$$

$$\leq 9 \cdot \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} a_{\nu_1,\nu_2},$$

then

$$J(q,\varphi)\le 9^q\pi^2\cdot\sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty}(n_1n_2)^{-2}\cdot\left\{\cdot\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}a_{\nu_1,\nu_2}\right\}^q\varphi\left(\left\{9\cdot\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}a_{\nu_1,\nu_2}\right\}^q\right)\le$$

$$\le 9^q\pi^2\cdot\sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty}(n_1n_2)^{-2}\cdot\left\{\cdot\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}a_{\nu_1,\nu_2}\right\}^q\varphi\left(9^q\cdot a_{1,1}^q\cdot(n_1n_2)\right).$$

From the condition (7) it follows the finiteness of value $S(p,\varphi)$, and by virtue of the properties of the function $\varphi(t)$ it follows the finiteness of value $J(p,\varphi)$, that is (6).

Now, let us prove the necessity. First, we prove that if condition (6) is satisfied, then there exists a number q_0 such that $1<q_0<q$ and $f(x,y)\in L_{q_0}[0,\pi]^2$.

Indeed, applying Holder's inequality at $\theta=\frac{q}{q_0}$, we obtain:

$$\int_0^\pi\int_0^\pi|f(x,y)|^{q_0}dxdy=\int_0^\pi\int_0^\pi|f(x,y)|^{q_0}\cdot\varphi^{\frac{q_0}{q}}\left(\frac{1}{xy}\right)\cdot\varphi^{-\frac{q_0}{q}}\left(\frac{1}{xy}\right)dxdy\le$$

$$\le\left\{\int_0^\pi\int_0^\pi|f(x,y)|^q\cdot\varphi\left(\frac{1}{xy}\right)dxdy\right\}^{\frac{q_0}{q}}\cdot\left\{\int_0^\pi\int_0^\pi\varphi^{-\frac{q_0}{q}\cdot\theta'}\left(\frac{1}{xy}\right)dxdy\right\}^{\frac{1}{\theta'}},$$

where $\frac{1}{\theta}+\frac{1}{\theta'}=1$. Hence, by virtue of Lemma A and the monotony of the function $\varphi(t)$ we obtain $f(x,y)\in L_{p_0}$.

Now, we show that the following series converges

$$\sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty}\frac{a_{n_1,n_2}}{n_1n_2}.$$

Indeed, applying the Hausdorff-Young's theorem, we get:

$$\sum_{n_1=1}^{+\infty}\sum_{n_2=1}^{+\infty}\frac{a_{n_1,n_2}}{n_1n_2}\le$$

$$\le\left(\sum_{n_1,n_2=1}^{+\infty}(a_{n_1,n_2})^{\max(p_0,2)}\right)^{\frac{1}{\max(p_0,2)}}\cdot\left(\sum_{n_1,n_2=1}^{+\infty}(n_1n_2)^{-\min(q_0,2)}\right)^{\frac{1}{\min(q_0,2)}}\le+\infty,$$

where $\frac{1}{q_0}+\frac{1}{p_0}=1$.

Then by Lemma 1 of work [2]

$$a_{n_1,n_2}\le\int_0^{\frac{\pi}{n_1}}\int_0^{\frac{\pi}{n_2}}|f(t,\tau)|dtd\tau.$$

Now, let us estimate the sum:

$$C(f,\varphi,q)=\sum_{n_1=2}^{+\infty}\sum_{n_2=2}^{+\infty}(n_1n_2)^{q-2}\varphi(n_1n_2)\cdot a_{n_1,n_2}^q\le$$

$$\le\sum_{n_1=2}^{+\infty}\sum_{n_2=2}^{+\infty}(n_1n_2)^{q-2}\varphi(n_1n_2)\cdot\left\{\int_0^{\frac{\pi}{n_1}}\int_0^{\frac{\pi}{n_2}}|f(t,\tau)|dtd\tau\right\}^q\le$$

$$\le C\cdot\sum_{n_1=2}^{+\infty}\sum_{n_2=2}^{+\infty}\int_{\frac{\pi}{n_1}}^{\frac{\pi}{n_1-1}}\int_{\frac{\pi}{n_2}}^{\frac{\pi}{n_2-1}}\left\{[\varphi(n_1n_2)]^{\frac{1}{q}}\cdot n_1n_2\cdot\int_0^{\frac{\pi}{n_1}}\int_0^{\frac{\pi}{n_2}}|f(t,\tau)|dtd\tau\right\}^q dxdy\le$$

$$\leq C\cdot\sum_{n_1=2}^{+\infty}\sum_{n_2=2}^{+\infty}\int\limits_{\frac{\pi}{n_1}}^{\frac{\pi}{n_1-1}}\int\limits_{\frac{\pi}{n_2}}^{\frac{\pi}{n_2-1}}\left\{\frac{1}{xy}\cdot\int\limits_0^x\int\limits_0^y|f(t,\tau)|\cdot\left[\varphi\left(\frac{1}{t\tau}\right)\right]^{\frac{1}{q}}dtd\tau\right\}^q dxdy=$$

$$=C\cdot\int\limits_0^{\pi}\int\limits_0^{\pi}\left\{\frac{1}{xy}\cdot\int\limits_0^x\int\limits_0^y|f(t,\tau)|\cdot\left[\varphi\left(\frac{1}{t\tau}\right)\right]^{\frac{1}{q}}dtd\tau\right\}^q dxdy.$$

Hence, by virtue of Lemma 2 of [2] and Lemma A we obtain:

$$C(f,\varphi,q)\leq C\cdot\int\limits_0^{\pi}\int\limits_0^{\pi}|f(t,\tau)|^q\cdot\varphi\left(\frac{1}{t\tau}\right)dtd\tau<+\infty.$$

The theorem is proved.

Remark. This theorem for a function of one variable is proved by M. F. Timan [1].

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С. Бітімхан

Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан

Фурье қатары туралы Харди-Литтлвуд теоремасының жалпыламасы

Бір өлшемді тригонометриялық қатарлар теориясында Фурье коэффициенті бірсарынды болатын Фурье қатары туралы Харди-Литтлвуд теоремасы маңызды орын иеленеді. Бұл теореманың көп өлшемді нұсқалары Лебег кеңістігі үшін кеңінен зерттелген. Көп өлшемді нұсқаларының бір өлшемді жағдайдан айтарлықтай айырмашылықтары бар екені анықталған және бұл теореманың күшейтілулері алынған болатын. Харди-Литтлвуд теоремасы сонымен қатар әртүрлі функциялық кеңістіктер үшін және қатар коэффициентінің біркелкілігі түрлері үшін жалпыланған болатын. Осы жалпылауларды кейбірін М.Ф. Тиман, М.И. Дьяченко, Е.Д. Нурсултанов, С. Тихонов жұмыстарынан көруге болады. Осы жұмыста Харди-Литтлвуд теоремасының $L_q\varphi(L_q)(0,2\pi]^2$ кеңістігінен алынған функцияның екі еселі Фурье қатары үшін жалпыламасы алынған.

Кілт сөздер: тригонометриялық қатар, Фурье қатары, Лебег кеңістігі, Харди-Литтлвуд теоремасы, Фурье коэффициенттері.

С. Битимхан

Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан

Обобщение теоремы Харди-Литтлвуда о рядах Фурье

В теории одномерных тригонометрических рядов важное значение имеет теорема Харди-Литтлвуда о рядах Фурье с монотонными коэффициентами Фурье. Многомерные варианты этой теоремы широко исследованы для пространства Лебега. Выявлены существенные отличия многомерных вариантов по сравнению с одномерным случаем и получено усиление этой теоремы. Теорема Харди-Литтлвуда также обобщена для различных функциональных пространств и видов монотонности коэффициентов ряда. Некоторые из этих обобщений встречается в работах М.Ф. Тимана, М.И. Дьяченко, Е.Д. Нурсултанова и С. Тихонова. В настоящей работе получено обобщение теоремы Харди-Литтлвуда для двойных рядов Фурье функции из пространства $L_q\varphi(L_q)(0,2\pi]^2$.

Ключевые слова: тригонометрический ряд, ряд Фурье, пространство Лебега, теорема Харди-Литтлвуда, коэффициенты Фурье.

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M.A. Bobodzhanova[1], B.T. Kalimbetov[2,*], G.M. Bekmakhanbet[2]

[1]*National Research University, Moscow Power Engineering Institute, Moscow, Russia;*
[2]*Kh.A. Yasawi International Kazakh-Turkish University, Turkestan, Kazakhstan*
(E-mail: bobojanovama@mpei.ru, burkhan.kalimbetov@ayu.edu.kz, bekmakhanbet.97@mail.ru)

Asymptotics solutions of a singularly perturbed integro-differential fractional order derivative equation with rapidly oscillating coefficients

In this paper, the regularization method of S.A.Lomov is generalized to the singularly perturbed integro-differential fractional-order derivative equation with rapidly oscillating coefficients. The main goal of the work is to reveal the influence of the oscillating components on the structure of the asymptotics of the solution to this problem. The case of the absence of resonance is considered, i.e. the case when an integer linear combination of a rapidly oscillating inhomogeneity does not coincide with a point in the spectrum of the limiting operator at all points of the considered time interval. The case of coincidence of the frequency of a rapidly oscillating inhomogeneity with a point in the spectrum of the limiting operator is called the resonance case. This case is supposed to be studied in our subsequent works. More complex cases of resonance (for example, point resonance) require more careful analysis and are not considered in this work.

Keywords: singularly perturbed, fractional order derivation, integro-differential equation, iterative problems, solvability of iterative problems.

Introduction

An initial problem is considered for a singularly perturbed integro-differential equation:

$$L_\varepsilon z(t,\varepsilon) \equiv \varepsilon z^{(\alpha)} - a(t)z - \int\limits_{t_0}^{t} K(t,s)z(s,\varepsilon)ds = h_1(t) + h_2(t)\sin\frac{\beta(t)}{\varepsilon},$$

$$z(t_0,\varepsilon) = y^0, \quad t \in [t_0,T], \quad t_0 > 0 \tag{1}$$

for a scalar unknown function $z(t,\varepsilon)$, in which $a(t),\ h_1(t),\ h_2(t),\ \beta'(t) > 0,\ (\forall t \in [t_0,T])$ are known functions, $0 < \alpha < 1$, z^0 constant number, $\varepsilon > 0$ is a small parameter. The problem is posed of constructing a regularized [1–2] asymptotic solution to problem (1). Previously, systems for ordinary differential equations [3–6] and integro-differential equations with rapidly oscillating coefficients [7–11] were considered.

By definition of the fractional derivative [12], the fractional derivative $z^{(\alpha)}$ in terms of integer derivatives is denoted in the following form $t^{(1-\alpha)}\frac{dz}{dt}$. Accordingly, we rewrite the original fractional order equation (1) in the following form:

$$L_\varepsilon z(t,\varepsilon) \equiv \varepsilon t^{(1-\alpha)}\frac{dz}{dt} - a(t)z - \int\limits_{t_0}^{t} K(t,s)z(s,\varepsilon)ds = h_1(t) + h_2(t)\sin\frac{\beta(t)}{\varepsilon},\ z(t_0,\varepsilon) = z^0,\ t \in [t_0,T]. \tag{2}$$

In problem (2), the frequency of the rapidly oscillating sine is $\beta'(t)$. In what follows, the function $\lambda_1(t) = a(t)$ is called the spectrum of problem (2), and functions $\lambda_2(t) = -i\beta'(t),\ \lambda_3(t) = +i\beta'(t)$ spectrum of a rapidly oscillating sine.

Problem (1) will be considered under the following conditions:

1) $a(t), \beta(t), h_1(t), h_2(t) \in C[t_0,T],\ K(t,s) \in C^\infty(t_0 \le s \le t \le T)$;

2) $a(t) < 0\ \ \forall t \in [t_0,T]$.

We will develop an algorithm for constructing a regularized asymptotic solution [6] of problem (1).

[*]Corresponding author.
E-mail: burkhan.kalimbetov@ayu.edu.kz

Regularization of the problem (2)

Denote by $\sigma_j = \sigma_j(\varepsilon)$ independent of magnitude $\sigma_1 = e^{-\frac{i}{\varepsilon}\beta(t_0)}$, $\sigma_2 = e^{+\frac{i}{\varepsilon}\beta(t_0)}$, and introduce the regularized variables:

$$\tau_1 = \frac{1}{\varepsilon}\int_{t_0}^{t} \theta^{(\alpha-1)}\lambda_1(\theta)d\theta \equiv \frac{\psi_1(t)}{\varepsilon}, \quad \tau_j = \frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_j(\theta)d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, j = 2,3, \tag{3}$$

and instead of problem (2), consider the problem

$$L_\varepsilon \tilde{z}(t,\tau,\sigma,\varepsilon) \equiv \varepsilon t^{(1-\alpha)}\frac{\partial \tilde{z}}{\partial t} + \lambda_1(t)\frac{\partial \tilde{z}}{\partial \tau_1} + t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial \tilde{z}}{\partial \tau_j} - \lambda_1(t)\tilde{z} - \int\limits_{t_0}^{t} K(t,s)\tilde{z}(s,\frac{\psi(s)}{\varepsilon},\sigma,\varepsilon)ds =$$

$$= h_1(t) - \frac{1}{2i}h_2(t)\left(e^{\tau_2}\sigma_1 - e^{\tau_3}\sigma_2\right), \; \tilde{z}(t,\tau,\sigma,\varepsilon)|_{t=t_0,\tau=0} = z^0, \; t \in [t_0,T]. \tag{4}$$

for the function $\tilde{z} = \tilde{z}(t,\tau,\sigma,\varepsilon)$, where is indicated (according (3)): $\tau = (\tau_1,\tau_2,\tau_3)$, $\psi = (\psi_1,\psi_2,\psi_3)$. It is clear that if $\tilde{z} = \tilde{z}(t,\tau,\sigma,\varepsilon)$ is a solution of the problem (4), then the function is $\tilde{z} = \tilde{z}\left(t, \frac{\psi(t)}{\varepsilon}, \sigma, \varepsilon\right)$ an exact solution to problem (3), therefore, problem (4) is extended with respect to problem (2). However, it cannot be considered fully regularized, since it does not regularize the integral

$$J\tilde{z} \equiv J\left(\tilde{z}(t,\tau,\sigma,\varepsilon)|_{t=s,\tau=\psi(s)/\varepsilon}\right) = \int\limits_{t_0}^{t} K(t,s)\tilde{z}(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds.$$

For its regularization, we introduce the class M_ε asymptotically invariant with respect to the operator $J\tilde{z}$ (see [1; 62]). Consider first the space U of vector functions $z(t,\tau,\sigma)$, representable by the sums

$$z(t,\tau,\sigma) = z_0(t,\sigma) + \sum_{i=1}^{3} z_i(t,\sigma)e^{\tau_i}, \quad z_i(t,\sigma) \in C^\infty\left([t_0,T],\mathbb{C}\right), i = \overline{0,3}. \tag{5}$$

In addition, the elements of space U depend on bounded in $\varepsilon > 0$ terms of constants $\sigma_1 = \sigma_1(\varepsilon)$ and $\sigma_2 = \sigma_2(\varepsilon)$ which do not affect the development of the algorithm described below, therefore, in the record of element (5) of this space U, we omit the dependence on $\sigma = (\sigma_1,\sigma_2)$ for brevity. We show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant with respect to the operator J.

For the space U, we take the space of functions $z(t,\tau,\sigma)$, represented by sums

$$J\tilde{z}(t,\tau,\varepsilon) \equiv \int\limits_{t_0}^{t} K(t,s)z_0(s)ds + \int\limits_{t_0}^{t} K(t,s)z_1(s)e^{\frac{1}{\varepsilon}\int\limits_{t_0}^{s}\theta^{(\alpha-1)}\lambda_1(\theta)d\theta}ds +$$

$$+\sum_{i=2}^{3}\int\limits_{t_0}^{t} K(t,s)z_i(s)e^{\frac{1}{\varepsilon}\int\limits_{t_0}^{s}\lambda_i(\theta)d\theta}ds.$$

Integrating by parts, we write the image of the operator J on the element (5) of the space U as a series

$$J\tilde{z}(t,\tau,\varepsilon) = \int\limits_{t_0}^{t} K(t,s)z_0(s)ds +$$

$$+\sum_{i=1}^{3}\sum_{\nu=0}^{\infty}(-1)^\nu\varepsilon^{\nu+1}\left[\left(I_i^\nu\left(K(t,s)z_i(s)\right)\right)_{s=t}e^{\tau_i} - \left(I_i^\nu\left(K(t,s)z_i(s(s)\right)\right)_{s=t_0}\right],$$

where are indicated:

$$I_1^0 = \frac{1}{s^{(\alpha-1)}\lambda_1(s)}\cdot, I_1^\nu = \frac{1}{s^{(\alpha-1)}\lambda_i(s)}\frac{\partial}{\partial s}I_1^{\nu-1},$$

$$I_i^0=\frac{1}{\lambda_i(s)}\cdot, I_i^{\nu}=\frac{1}{\lambda_i(s)}\frac{\partial}{\partial s}I_i^{\nu-1},\ \ i=2,3.$$

It is easy to show (see, for example, [13; 291–294] that this series converges asymptotically for $\varepsilon\to+0$ (uniformly in $t\in[t_0,T]$). This means that the class M_ε is asymptotically invariant (for $\varepsilon\to+0$) with respect to the operator J.

We introduce operators $R_\nu: U\to U$, acting on each element $z(t,\tau)\in U$ of the form (5) according to the law:

$$R_0z(t,\tau)=\int\limits_{t_0}^{t}K(t,s)z_0(s)ds, \tag{6_0}$$

$$R_1z(t,\tau)=\sum_{i=1}^{3}\Big[\left(I_i^0\left(K(t,s)z_i(s)\right)\right)_{s=t}e^{\tau_i}-\left(I_i^0\left(K(t,s)z_i(s)\right)\right)_{s=t_0}\Big], \tag{6_1}$$

$$R_{\nu+1}z\left(t,\tau\right)=\sum_{i=1}^{3}\sum_{\nu=0}^{\infty}\left(-1\right)^{\nu}\varepsilon^{\nu+1}\left[\left(I_i^{\nu}\left(K(t,s)z_i(s)\right)\right)_{s=t}e^{\tau_i}-\left(I_i^{\nu}\left(K(t,s)z_i(s)\right)\right)_{s=t_0}\right],\nu\geq1. \tag{$6_{\nu+1}$}$$

Now, let $\tilde z(t,\tau,\varepsilon)$ be an arbitrary continuous function on $(t,\tau)\in G=[t_0,T]\times\{\tau: Re\tau_1<0,\ Re\tau_j\leq0,$ $j=2,3\}$, with asymptotic expansion

$$\tilde z(t,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k z_k(t,\tau), y_k(t,\tau)\in U \tag{7}$$

converging as $\varepsilon\to+0$ (uniformly in $(t,\tau)\in G$). Then, the image $J\tilde z\left(t,\tau,\varepsilon\right)$ of this function is decomposed into an asymptotic series

$$J\tilde z(t,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k Jz_k(t,\tau)=\sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}z_s(t,\tau)|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing an extension of an operator J on series of the form (7):

$$\tilde J\tilde z\equiv\tilde J\left(\sum_{k=0}^{\infty}\varepsilon^k z_k(t,\tau)\right)=\sum_{r=0}^{\infty}\varepsilon^r\left(\sum_{k=0}^{r}R_{r-k}z_k(t,\tau)\right).$$

Although the operator $\tilde J$ is formally defined, its utility is obvious, since in practice it is usual to construct the N-th approximation of the asymptotic solution of the problem (3), in which impose only N-th partial sums of the series (6), which have not a formal, but a true meaning. Now, one can write a problem that is completely regularized with respect to the original problem (3):

$$L_\varepsilon\tilde z(t,\tau,\sigma,\varepsilon)\equiv\varepsilon t^{(1-\alpha)}\frac{\partial\tilde z}{\partial t}+\lambda_1(t)\frac{\partial\tilde z}{\partial\tau_1}+t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial\tilde z}{\partial\tau_j}-\lambda_1(t)\tilde z-\tilde J\tilde z=$$

$$=h_1(t)-\frac{1}{2i}h_2(t)\left(e^{\tau_2}-e^{\tau_3}\right),\ \tilde z(t_0,0,\sigma,\varepsilon)=z^0,\ t\in[t_0,T]. \tag{8}$$

Iterative problems and their solvability in the space U

Substituting the series (7) into (8) and equating the coefficients of the same powers of ε, we obtain the following iterative problems:

$$Lz_0(t,\tau,\sigma)\equiv\lambda_1(t)\frac{\partial z_0}{\partial\tau_1}+t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial z_0}{\partial\tau_j}-\lambda_1(t)z_0-R_0z_0=$$

$$=h_1(t)-\frac{1}{2i}h_2(t)\left(e^{\tau_2}-e^{\tau_3}\right),\ \ z_0(t_0,0)=z^0; \tag{9_0}$$

$$Lz_1(t,\tau,\sigma) = -t^{(1-\alpha)}\frac{\partial z_0}{\partial t} + R_1 z_0, \quad z_1(t_0,0) = 0; \tag{9_1}$$

$$Lz_2(t,\tau,\sigma) = -t^{(1-\alpha)}\frac{\partial z_1}{\partial t} + R_1 z_1 + R_2 z_0, \quad z_0(t_0,0) = 0; \tag{9_2}$$

....

$$L\,z_k(t,\tau,\sigma) = -t^{(1-\alpha)}\frac{\partial z_{k-1}}{\partial t} + R_k z_0 + \ldots +$$

$$+ \ldots + R_1 z_{k-1},\ z_k(t_0,0) = 0, \quad k \geq 1. \tag{9_k}$$

Each iterative problem (9_k) has the form

$$L\,z(t,\tau,\sigma) \equiv \lambda_1(t)\frac{\partial z}{\partial \tau_1} + t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial z}{\partial \tau_j} - \lambda_1(t)z - R_0 z = H(t,\tau,\sigma),\ \ z(t_0,0) = z^* \tag{10}$$

where $H(t,\tau,\sigma) = H_0(t,\sigma) + \sum\limits_{i=1}^{3} H_i(t,\sigma)e^{\tau_i}$ is the known function of space U, y_* is the known function of the complex space $\mathbf{C}$, and the operator R_0 has the form (see (6_0))

$$R_0 z \equiv R_0\left(z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j} +\right) \triangleq \int_{t_0}^{1} K(t,s)z_0(s)ds.$$

We introduce scalar (for each $t \in [t_0,T]$) product in space U:

$$< u, w > \equiv < u_0(t) + \sum_{j=1}^{3} u_j(t)e^{\tau_j}, w_0(t) + \sum_{j=1}^{3} w_j(t)e^{\tau_j} > \equiv$$

$$\equiv (u_0(t), w_0(t)) + \sum_{j=1}^{3}\ (u_j(t), w_j(t))$$

where we denote by $(*,*)$ the usual scalar product in the complex space $\mathbf{C}:\ (u,v) = u\cdot\bar{v}$. Let us prove the following statement.

Theorem 1. Let conditions (1), (2) be fulfilled and the right-hand side $H(t,\tau,\sigma) = H_0(t,\sigma) + + \sum\limits_{j=1}^{3} H_j(t,\sigma)e^{\tau_j}$ of equation (10) belongs to the space U. Then the equation (10) is solvable in U, if and only if

$$< H(t,\tau), e^{\tau_1} > \equiv 0, \forall t \in [t_0,T]. \tag{11}$$

Proof. We will determine the solution of equation (10) as an element (5) of the space U :

$$z(t,\tau,\sigma) = z_0(t,\sigma) + \sum_{j=1}^{3} z_j(t,\sigma)e^{\tau_j}. \tag{12}$$

Substituting (12) into equation (10), and equating here the free terms and coefficients separately for identical exponents, we obtain the following equations of equations:

$$\lambda_1(t)z_0(t,\sigma) - \int_{t_0}^{t} K(t,s)z_0(s,\sigma)ds = H_0(t,\sigma), \tag{13}$$

$$0\cdot z_1(t,\sigma) = H_1(t,\sigma), \tag{13_1}$$

$$\left[t^{(1-\alpha)}\lambda_j(t) - \lambda_1(t)\right] z_j(t,\sigma) = H_j(t,\sigma),\ j = \overline{2,3}. \tag{13_j}$$

Since the $\lambda_1(t) \neq 0$, the equation (13) can be written as

$$z_0(t,\sigma) = \int\limits_{t_0}^{t} \left(-\lambda_1^{-1}(t)K(t,s)\right) z_0(s,\sigma)ds - \lambda_1^{-1}(t)H_0(t,\sigma). \qquad (13_0)$$

Due to the smoothness of the kernel $\left(-\lambda_1^{-1}(t)K(t,s)\right)$ and heterogeneity $-\lambda_1^{-1}(t)H_0(t,\sigma)$, this Volterra integral equation has a unique solution $z_0(t,\sigma) \in C^\infty([t_0,T],\mathbf{C})$. The equations (13_2) and (13_3) also have unique solutions

$$z_j(t,\sigma) = [\lambda_j(t) - \lambda_1(t)]^{-1} H_j(t,\sigma) \in C^\infty([t_0,T],\mathbf{C}),\, j = 2,3 \qquad (14)$$

since $\lambda_2(t), \lambda_3(t)$ not equal to $\lambda_1(t)$.

The equation (13_1) is solvable in space $C^\infty([t_0,T],\mathbf{C})$ if and only if $(H_1(t,\tau), e^{\tau_1}) \equiv 0\, \forall t \in [t_0,T]$ hold. It is not difficult to see that these identities coincide with identities (10). Thus, condition (10) is necessary and sufficient for the solvability of equations (9) in the space U. Theorem 1 is proved.

Remark 1. If identity (10) holds, then under conditions (1), (2), equation (9) has the following solution in the space U :

$$z(t,\tau,\sigma) = z_0(t,\sigma) + \alpha_1(t,\sigma)e^{\tau_1} + \sum_{j=2}^{3} \left[t^{(1-\alpha)}\lambda_j(t) - \lambda_1(t)\right]^{-1} H_j(t,\sigma)e^{\tau_j} \qquad (15)$$

where $\alpha_1(t,\sigma) \in C^\infty([t_0,T],\mathbf{C})$ are arbitrary function, $z_0(t,\sigma)$ is the solution of an integral equation (13_0).

The unique solvability of the general iterative problem in the space U. Residual term theorem

Let us proceed to the description of the conditions for the unique solvability of equation (10) in space U. Along with problem (10), we consider the equation

$$Lz(t,\tau) = -t^{(1-\alpha)}\frac{\partial z}{\partial t} + R_1 z + Q(t,\tau), \qquad (16)$$

where $z = z(t,\tau)$ is the solution (16) of the equation (10), $Q(z,\tau) \in U$ is the well-known function of the space U. The right part of this equation:

$$G(t,\tau) \equiv -t^{(1-\alpha)}\frac{\partial z}{\partial t} + R_1 z + Q(t,\tau) =$$

$$= -t^{(1-\alpha)}\frac{\partial}{\partial t}\left(z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j}\right) + R_1\left(z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j}\right) + Q(t,\tau)$$

may not belong to space U, if $z = z(t,\tau) \in U$. Indeed, taking into account the form (14) of the function $z = z(t,\tau) \in U$, we consider in $G(t,\tau)$, for example, the terms

$$Z(t,\tau) \equiv \frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left[z_0(t) + \sum_{j=1}^{3} z_j(t)e^{\tau_j} + \sum_{2\le|m|\le N_H}^{*} z^m(t)e^{(m,\tau)}\right] =$$

$$= \frac{g(t)}{2}z_0(t)\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right) + \sum_{j=1}^{3}\frac{g(t)}{2}z_j(t)\left(e^{\tau_j+\tau_2}\sigma_1 + e^{\tau_j+\tau_3}\sigma_2\right) +$$

$$+\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\sum_{2\le|m|\le N_H}^{*} z^m(t)e^{(m,\tau)}.$$

Here, for instance, terms with exponents

$$e^{\tau_2+\tau_3} = e^{(m,\tau)}|_{m=(0,1,1)},\; e^{\tau_2+(m,\tau)}\;(if\; m_1 = 0, m_2 + 1 = m_3),$$

$$e^{\tau_3+(m,\tau)}\;(if\; m_1 = 0,\; m_3 + 1 = m_2),\; e^{\tau_2+(m,\tau)}\;(if\; m_1 = 0, m_2 = m_3), \qquad (*)$$

$$e^{\tau_3+(m,\tau)}\,(if\ m_1=0,\,m_2=m_3)\,,\,e^{\tau_2+(m,\tau)}\,(if\ m_1=1,\,m_2=m_3)\,,$$
$$e^{\tau_3+(m,\tau)}\,(if\ m_1=1,\,m_2=m_3)$$

do not belong to space U, since multi-indexes

$$(0,n,n)\,\in\Gamma_0,\ (0,n+1,n)\in\Gamma_1,\ (0,n,n+1)\in\Gamma_2\,\forall n\in N$$

are resonant. Then, according to the well-known theory (see, [6; 234]), we embed these terms in the space U according to the following rule (see $(*)$):

$$\widehat{e^{\tau_2+\tau_3}}=e^0=1,\,\widehat{e^{\tau_2+(m,\tau)}}=e^0=1\,(if\ m_1=0,m_2+1=m_3)\,,$$
$$\widehat{e^{\tau_3+(m,\tau)}}=e^0=1\,(if\ m_1=0,\,m_3+1=m_2)\,,$$
$$\widehat{e^{\tau_2+(m,\tau)}}=e^{\tau_2}\,(if\ m_1=0,m_2=m_3)\,,\widehat{e^{\tau_3+(m,\tau)}}=e^{\tau_3}\,(if\ m_1=0,\,m_2=m_3)\,,$$
$$\widehat{e^{\tau_2+(m,\tau)}}\,(if\ m_1=1,\,m_2=m_3)=e^{\tau_1},\,\widehat{e^{\tau_3+(m,\tau)}}=e^{\tau_1}\,(if\ m_1=1,\,m_2=m_3)\,.$$

In other words, terms with resonant exponentials $e^{(m,\tau)}$ replaced by members with exponents $e^0,e^{\tau_1},e^{\tau_2},e^{\tau_3}$ according to the following rule:

$$\widehat{e^{(m,\tau)}}|_{m\in\Gamma_0}=e^0=1,\,\widehat{e^{(m,\tau)}}|_{m\in\Gamma_1}=e^{\tau_1},\,\widehat{e^{(m,\tau)}}|_{m\in\Gamma_2}=e^{\tau_2},\,\widehat{e^{(m,\tau)}}|_{m\in\Gamma_3}=e^{\tau_3}.$$

After embedding, the right-hand side of equation (15) will look like

$$\widehat{G}(t,\tau)=-t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t)+\sum_{j=1}^{3}z_j(t)e^{\tau_j}+\sum_{2\leq|m|\leq N_H}^{*}z^m(t)e^{(m,\tau)}+\sum_{j=0}^{3}\sum_{m^j\in\Gamma_j}z^{m^j}(t)\,e^{\tau_j}\right]+Q(t,\tau)$$

As indicated in [6], the embedding $G(t,\tau)\to\widehat{G}(t,\tau)$ will not affect the accuracy of the construction of asymptotic solutions of problem (2), since $G(t,\tau)$ at $\tau=\frac{\psi(t)}{\varepsilon}$ coincides with $\widehat{G}(t,\tau)$.

Theorem 2. Let conditions (1), (2) be fulfilled and the right-hand side $H(t,\tau)=H_0(t)+\sum\limits_{j=1}^{3}H_j(t)e^{\tau_j}+$ $+\sum\limits_{2\leq|m|\leq N_H}^{*}H^m(t)e^{(m,\tau)}\in U$ of equation (10) satisfy condition (11). Then problem (10) under additional conditions

$$<\widehat{G}(t,\tau),e^{\tau_1}>\equiv 0\ \forall t\in[t_0,T] \tag{17}$$

where $Q(t,\tau)=Q_0(t)+\sum\limits_{k=1}^{3}Q_k(t)e^{\tau_i}+\sum\limits_{2\leq|m|\leq N_z}^{*}Q^m(t)e^{(m,\tau)}$ *is the known function of space* U*, is uniquely solvable in* U*.*

Proof. Since the right-hand side of equation (10) satisfies condition (11), this equation has a solution in space U in the form (14), where $\alpha_1(t)\in C^\infty([t_0,T],\mathbf{C})$ is arbitrary function. Submit (14) to the initial condition $y(t_0,0)=y^*$. We get $\alpha_1(t_0,t)=y_*$, where denoted

$$z_*=z^*+A^{-1}(t_0)H_0(t_0)-\frac{H_2(t_0)}{t_0{}^{(1-\alpha)}\lambda_2(t_0)-\lambda_1(t_0)}-\frac{H_3(t_0)}{t_0{}^{(1-\alpha)}\lambda_3(t_0)-\lambda_1(t_0)}-$$
$$-\sum_{2\leq|m|\leq N_H}^{*}[(m,\lambda(t_0))-A(t_0)]^{-1}H^m(t_0).$$

Now, we subordinate the solution (15) to the orthogonality condition (17). We write $G(t,\tau)$ in more detail the right side of equation (10):

$$G(t,\tau)\equiv-t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\leq|m|\leq N_H}^{*}P^m(t)e^{(m,\tau)}\right]+$$

$$+\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+$$

$$+R_1\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+Q(t,\tau).$$

Embedding this function into space U, we will have

$$\hat{G}(t,\tau)\equiv -t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+$$

$$+\left\{\frac{g(t)}{2}z_0(t)e^{\tau_2}\sigma_1+\frac{g(t)}{2}z_0(t)e^{\tau_3}\sigma_2+\sum_{j=1}^{3}\frac{g(t)}{2}z_j(t)e^{\tau_j+\tau_2}\sigma_1+\sum_{j=1}^{3}\frac{g(x)}{2}z_j(t)e^{\tau_j+\tau_3}\sigma_2+\right.$$

$$\left.+\sum_{2\le|m|\le N_H}^{*}\frac{g(t)}{2}z^m(t)e^{(m,\tau)+\tau_2}\sigma_1+\sum_{2\le|m|\le N_H}^{*}\frac{g(t)}{2}z^m(t)e^{(m,\tau)+\tau_3}\sigma_2\right\}^{\wedge}+$$

$$+R_1\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+Q(t,\tau)=$$

$$=-t^{(1-\alpha)}\frac{\partial}{\partial t}\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+$$

$$+\frac{g(t)}{2}\left\{z_0(t)e^{\tau_2}\sigma_1+z_0(t)e^{\tau_3}\sigma_2+\underline{\alpha_1(t)e^{\tau_1+\tau_2}\sigma_1}+h_{21}(t)e^{2\tau_2}\sigma_1+h_{31}(t)e^{\tau_3+\tau_2}\sigma_1+\right.$$

$$+\underline{\alpha_1(t)e^{\tau_1+\tau_3}\sigma_2}+h_{21}(t)e^{\tau_2+\tau_3}\sigma_2+h_{31}(t)e^{2\tau_3}\sigma_2+$$

$$\left.+\underline{\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)+\tau_2}\sigma_1}+\underline{\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)+\tau_3}\sigma_2}\right\}^{\wedge}+$$

$$+R_1\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]+Q(t,\tau).$$

The embedding operation acts only on resonant exponentials, leaving the coefficients unchanged at these exponents. Given that the expression

$$R_1\left[z_0(t)+\alpha_1(t)e^{\tau_1}+h_{21}(t)e^{\tau_2}+h_{31}(t)e^{\tau_3}+\sum_{2\le|m|\le N_H}^{*}P^m(t)e^{(m,\tau)}\right]$$

linearly depends on $\alpha_1(t)$ (see formula (5_1)), we also conclude that after the embedding operation the function $\hat{G}(t,\tau)$ will linearly depend on the scalar function $\alpha_1(t)$. Given that in condition (16) scalar multiplication by functions e^{τ_1}, containing only the exponent e^{τ_1}, in the expression for $\widehat{G}(t,\tau)$ it is necessary to keep only the term with the exponent e^{τ_1}. Then condition (17) takes the form

$$<-t^{(1-\alpha)}\frac{\partial}{\partial t}\left(\alpha_1(t)e^{\tau_1}\right)+\left(\sum_{|m^1|=2:\,m^1\in\Gamma_1}^{N}w^{m^1}\left(\alpha_1(t),t\right)\right)e^{\tau_1}+Q_1(t)e^{\tau_1},e^{\tau_1}>=0\quad\forall t\in[t_0,T]$$

where $w^{m^1}\left(\alpha_1(t),t\right)$ are some functions linearly dependent on $\alpha_1(t)$. Performing scalar multiplication here, we obtain a linear ordinary differential equation (relative t) for a function $\alpha_1(t)$. Given the initial condition

$\alpha_1(t_0) = y_*$, found above, we find uniquely the function $\alpha_1(t) \in C^\infty[t_0,T]$ and therefore, we will uniquely construct a solution to equation (9) in the space U. The theorem is proved.

As mentioned above, the right-hand sides of iterative problems (8_k) (if solved sequentially) may not belong to space U. Then, according to [6; 234], the right-hand sides of these problems must be embedded into U, according to the above rule. As a result, we obtain the following problems:

$$L\,z_0(t,\tau,\sigma) \equiv \lambda_1(t)\frac{\partial z_0}{\partial \tau_1} + t^{(1-\alpha)}\sum_{j=2}^{3}\lambda_j(t)\frac{\partial z_0}{\partial \tau_j} - \lambda_1(t)z_0 - R_0 z_0 = h(t), \quad z_0(t_0,0) = z^0; \qquad (\bar{9}_1)$$

$$Lz_1(t,\tau) = -t^{(1-\alpha)}\frac{\partial z_0}{\partial t} + \left[\frac{g(t)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2)z_0\right]^{\wedge} + R_1 z_0, \quad z_1(t_0,0) = 0; \qquad (\bar{8}_1)$$

$$Lz_2(t,\tau) = -t^{(1-\alpha)}\frac{\partial z_1}{\partial t} + \left[\frac{g(t)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2)z_1\right]^{\wedge} + R_1 z_1 + R_2 z_0, \quad z_2(t_0,0) = 0; \qquad (\bar{8}_2)$$

....

$$Lz_k(t,\tau) = -t^{(1-\alpha)}\frac{\partial z_{k-1}}{\partial t} + \left[\frac{g(t)}{2}(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2)z_{k-1}\right]^{\wedge} + R_k z_0 + \ldots + R_1 z_{k-1},$$

$$z_k(t_0,0) = 0,\ k \geq 1. \qquad (\bar{8}_k)$$

(images of linear operators $\frac{\partial}{\partial t}$ and R_ν do not need to be embedding in space U, since these operators operate from U to U). Such a change will not affect the construction of the asymptotic solution of the original problem (1) (or the equivalent problem (2)), so on the restriction $\tau = \frac{\psi(t)}{\varepsilon}$ series of problems $(\bar{8}_k)$ will coincide with a series of problems (8_k) (see [6; 234–235].

Applying Theorems 1 and 2 to iterative problems (8_k) (in this case, the right-hand sides $H^{(k)}(t,\tau)$ of these problems are embedded in the space U, i.e. $H^{(k)}(t,\tau)$ we replace with $\hat{H}^{(k)}(t,\tau) \in U$), we find uniquely their solutions in space U and construct series (6). Just as in [13, 14], we prove the following statement.

Theorem 3. Suppose that conditions 1), 2) are satisfied for equation (2). Then, when $\varepsilon \in (0,\varepsilon_0](\varepsilon_0 > 0$ is sufficiently small), equation (2) has a unique solution $z(t,\varepsilon) \in C^1([t_0,T],\mathbf{C})$, in this case, the estimate

$$||z(t,\varepsilon) - z_{\varepsilon N}(t)||_{C[t_0,T]} \leq c_N \varepsilon^{N+1}, \ \ N = 0,1,2,\ldots$$

holds true, where $z_{\varepsilon N}(t)$ is the restriction (for $\tau = \frac{\psi(t)}{\varepsilon}$) of the N – partial sum of series (6) (with coefficients $z_k(t,\tau) \in U$, satisfying the iteration problems (8_k)), and the constant $c_N > 0$ does not depend on $\varepsilon \in (0,\varepsilon_0]$.

Construction of the solution of the first iteration problem

Using Theorem 1, we will try to find a solution to the first iteration problem $(\bar{8}_0)$. Since the right side $h(t)$ of the equation $(\bar{8}_0)$ satisfies condition (10), this equation has (according to (15)) a solution in the space U in the form

$$z_0(t,\tau) = z_0^{(0)}(t) + \alpha_1^{(0)}(t)e^{\tau_1} \qquad (18)$$

where $\alpha_1^{(0)}(t) \in C^\infty([t_0,T],\mathbf{C})$ are arbitrary function, $y_0^{(0)}(t)$ is the solution of the integral equation

$$z_0^{(0)}(t) = \int_{t_0}^{t}\left(-\lambda_1^{-1}(t)K(t,s)\right)z_0^{(0)}(s)ds - \lambda_1^{-1}(t)h(t). \qquad (19)$$

Subordinating (18) to the initial condition $z_0(t_0,0) = z^0$, we have

$$z_0^{(0)}(t_0) + \alpha_1^{(0)}(t_0) = z^0 \ \ \Leftrightarrow \ \ \alpha_1^{(0)}(t_0) = z^0 - z_0^{(0)}(t_0) \Leftrightarrow \alpha_1^{(0)}(t_0) = z^0 + \lambda_1^{-1}(t_0)h(t_0).$$

To fully compute the function $\alpha_1^{(0)}(t)$, we proceed to the next iteration problem $(\bar{8}_1)$. Substituting into it the solution (18) of the equation $(\bar{8}_0)$, we arrive at the following equation:

$$Lz_1(t,\tau) = -t^{(1-\alpha)}\frac{\partial}{\partial t}z_0^{(0)}(t) - t^{(1-\alpha)}\frac{\partial}{\partial t}\left(\alpha_1^{(0)}(t)\right)e^{\tau_1} + \left[\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right)\left(z_0^{(0)}(t) + \alpha_1^{(0)}(t)e^{\tau_1}\right)\right]^{\wedge} +$$

$$+\frac{K(t,t)\alpha_1^{(0)}(t)}{t^{(1-\alpha)}\lambda_1(t)}e^{\tau_1}-\frac{K(t,t_0)\alpha_1^{(0)}(t_0)}{t_0{}^{(1-\alpha)}\lambda_1(t_0)}+\sum_{j=2}^{3}\left[\frac{K(t,t)z_j^{(0)}(t)}{\lambda_j(t)}e^{\tau_j}-\frac{K(t,t_0)z_j^{(0)}(t_0)}{\lambda_j(t_0)}\right]$$

(here, we used the expression (6_1) for $R_1 z(t,\tau)$ and took into account that when $z(t,\tau)=z_0(t,\tau)$ the sum (6_1) contains only terms with e^{τ_1}).

Let us calculate

$$M=\left[\frac{g(t)}{2}\left(e^{\tau_2}\sigma_1+e^{\tau_3}\sigma_2\right)\left(z_0^{(0)}(t)+\alpha_1^{(0)}(t)e^{\tau_1}\right)\right]^{\wedge}=$$

$$=\frac{g(t)}{2}\left\{\sigma_1 z_0^{(0)}(t)e^{\tau_2}+\sigma_2 z_0^{(0)}(t)e^{\tau_3}+\sigma_1\alpha_1^{(0)}(t)e^{\tau_2+\tau_1}+\sigma_2\alpha_1^{(0)}(t)e^{\tau_3+\tau_1}\right\}^{\wedge}.$$

Let us analyze the exponents of the second dimension included here for their resonance:

$$e^{\tau_2+\tau_1}|_{\tau=\psi(t)/\varepsilon}=e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\left(-i\beta'(\theta)+A(\theta)\right)d\theta},\qquad e^{\tau_2+\tau_1}|_{\tau=\psi(t)/\varepsilon}=e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\left(-i\beta'(\theta)+A(\theta)\right)d\theta},$$

$$-i\beta'+A=\left[\begin{array}{l}0,\\ A,\\ -i\beta',\\ +i\beta',\end{array}\right.\Leftrightarrow\emptyset,\qquad -i\beta'+A=\left[\begin{array}{l}0,\\ A,\\ -i\beta',\\ +i\beta',\end{array}\right.\Leftrightarrow\emptyset.$$

Thus, exponents $e^{\tau_2+\tau_1}$ and $e^{\tau_3+\tau_1}$ are not resonant. Then, for solvability, equation (18) it is necessary and sufficient that the condition

$$-t^{(1-\alpha)}\frac{\partial}{\partial t}\left(\alpha_1^{(0)}(t)\right)+\frac{K(t,t)\alpha_1^{(0)}(t)}{t^{(1-\alpha)}\lambda_1(t)}=0$$

is satisfied. Attaching the initial condition

$$\alpha_1^{(0)}(t_0)=z^0+\lambda_1^{-1}(t_0)h(t_0)$$

to this equation, we find

$$\alpha_1^{(0)}(t)=\alpha_1^{(0)}(t_0)e^{\int\limits_{t_0}^{t}\left(\frac{K(\theta,\theta)}{\theta^{2(1-\alpha)}\lambda_1(\theta)}\right)d\theta}$$

and therefore, we uniquely calculate the solution (18) of the problem ($\bar{9}_0$) in the space U. Moreover, the main term of the asymptotic of the solution to problem (2) has the form

$$z_{\varepsilon 0}(t)=z_0^{(0)}(t)+\alpha_1^{(0)}(t_0)e^{\int\limits_{t_0}^{t}\left(\frac{K(\theta,\theta)}{\theta^{2(1-\alpha)}\lambda_1(\theta)}\right)d\theta+\frac{1}{\varepsilon}\int\limits_{t_0}^{t}\lambda_1(\theta)d\theta}\qquad(20)$$

where $\alpha_1^{(0)}(t_0)=z^0+\lambda_1^{-1}(t_0)h(t_0)$, $z_0^{(0)}(t)$ is the solution of the integrated equation (19). From expression (20) for $z_{\varepsilon 0}(t)$ it is clear that $z_{\varepsilon 0}(t)$ is independent of rapidly oscillating terms. However, already in the next approximation, their influence on the asymptotic solution of problem (2) is revealed.

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М.А. Бободжанова[1], Б.Т. Калимбетов[2], Г.М. Бекмаханбет[2]

[1] *Ұлттық зерттеу университеті, Мәскеу энергетикалық институты, Мәскеу, Ресей;*
[2] *Х.А. Ясауи атындағы Халықаралық қазақ-түрік университеті, Түркістан, Қазақстан*

Жылдам осцилляцияланатын коэффициентті бөлшек ретті туындылы сингуляр ауытқыған интегро-дифференциалдық теңдеудің асимптотикасы

Мақалада С.А.Ломовтың регуляризация әдісі жылдам осцилляцияланатын коэффициенттері бар бөлшек-ретті туындылы интегро-дифференциалдық теңдеуі жалпыланған. Жұмыстың басты мақсаты — осцилляцияланатын компоненттердің есептік шешімінің асимптотикасының структурасына әсерін зерттеу болып табылады. Резонанстың болмауы жағдайы қарастырылған, яғни, жылдам тербелетін біртектіліксіздіктің бүтін сызықтық комбинациясы берілген уақыт интервалының барлық нүктелеріндегі шекті операторының спектрінің мәндеріне сәйкес келмейтін жағдай зерттелген. Шекті оператор спектрімен жылдам тербелетін біртектіліксіздіктің жиілігінің сәйкес келу жағдайы резонанстық жағдай деп аталады. Бұл жағдайдың зерттелуі келесі еңбекте жоспарланған. Резонанстың күрделі жағдайлары (мысалы, тепе-теңдік резонансы) мұқият талдауды қажет етеді және бұл жұмыста қарастырылмаған.

Кілт сөздер: сингуляр ауытқу, бөлшек ретті туындылы интегро-дифференциалдық теңдеу, итерациялық есептер, итерацион есептердің шешімділігі.

М.А. Бободжанова[1], Б.Т. Калимбетов[2], Г.М. Бекмаханбет[2]

[1] *Национальный исследовательский университет, Московский энергетический институт, Москва, Россия;*
[2] *Международный казахско-турецкий университет имени Х.А. Ясави, Туркестан, Казахстан*

Асимптотика решений сингулярно-возмущенного интегро-дифференциального уравнения дробного порядка с быстро осциллирующими коэффициентами

В статье метод регуляризации С.А. Ломова обобщен на сингулярно-возмущенное интегро-дифференциальное уравнение дробного производного с быстро осциллирующими коэффициентами. Основная цель работы — выявить влияние осциллирующих составляющих на структуру асимптотики решения этой задачи. Рассмотрен случай отсутствия резонанса, т.е. случай, когда целочисленная линейная комбинация быстро осциллирующей неоднородности не совпадает с точкой спектра предельного оператора на всех точках рассматриваемого отрезка времени. Случай совпадения частоты быстро осциллирующей неоднородности с точкой спектра предельного оператора называется резонансным. Данный случай предполагается изучить в наших последующих работах. Более сложные случаи резонанса (например, точечный резонанс) требуют тщательного подхода и в данной работе не будут рассматриваться.

Ключевые слова: сингулярно-возмущенное, интегро-дифференциальное уравнение производного дробного порядка, итерационные задачи, разрешимость итерационных задач.

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UDC 512

A. Daşdemir*

Kastamonu University, Kastamonu, Turkey
(E-mail: ahmetdasdemir37@gmail.com)

On Hadamard Product of Hypercomplex Numbers

Certain product rules take various forms in the set of hypercomplex numbers. In this paper, we introduced a new multiplication form of the hypercomplex numbers that would be called «the Hadamard product», inspired by the analogous product in the real matrix space, and investigated some algebraic properties of that, including the norm of inequality. In particular, we extended our new definition and its applications to the complex matrix theory.

Keywords: Quaternion product, dot product, Hadamard product, hypercomplex number.

Introduction

In 1843, the Irish mathematician Sir William Rowan Hamilton introduced quaternions as an extension of complex numbers to higher spatial dimensions [1]. The set of real quaternions is often denoted by $\mathbb{H}$ in honor of its discoverer and is defined as follows:

$$\mathbb{H} = \{q = q_0 + \tilde{q} : \tilde{q} = iq_1 + jq_2 + kq_3 \text{ and } q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where q_0 is called the scalar part of q, and $\tilde{q}$ is called its vector part. The scalar and vector parts of a quaternion q are denoted by $Sc(q)$ and $Vec(q)$, respectively. The monographs [2], [3] present well-known systematic investigations on the subject. In addition, the papers [4], [5] include some interesting applications of the quaternions.

One can apply the Cayley-Dickson process (also known as Cayley-Dickson doubling) this process to the complex numbers quaternions (dim 4), octonions (dim 8), sedenions (dim 16), ..., 2^N-ons $\left(dim\ 2^N\right)$ in succession. Each one is a sub-algebra of all the preceding ones. Note that an increase in the dimension of the algebra causes the loss of certain algebraic properties. For example, quaternions do not possess the commutative property that complex numbers possess, and octonion algebra loses the associative property. These losses often lead to unexpected results.

A 2^N-ons hyper-complex number is regarded as a linear combination of a canonical basis set of this algebra in the form

$$\omega = \sum_{i=1}^{2^N} w_i \vec{e}_i = w_1\vec{e}_1 + w_2\vec{e}_2 + \cdots + w_{2^N}\vec{e}_{2^N},$$

where $\vec{e}_i$ refers to the components of the basis set and w_i refers to real numbers. Note that the product of any two values of $\vec{e}_i$ is linearly dependent on $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_{2^N}$. Consider that algebraic properties of the hypercomplex numbers are investigated in [6–9]. For example, Table 1 summarizes the multiplication rules for the basis vectors of the quaternion and octonion algebras. This table is provided by Cawagas in reference [10]. The conjugate of ω is as follows:

$$\omega^* = w_1\vec{e}_1 - \sum_{i=2}^{2^N} w_i\vec{e}_i = w_1\vec{e}_1 - w_2\vec{e}_2 - \cdots - w_{2^N}\vec{e}_{2^N},$$

and its norm is

$$N(\omega) = \sqrt{\sum_{i=1}^{2^N} {w_i}^2}. \tag{1}$$

*Corresponding author.
E-mail: ahmetdasdemir37@gmail.com

Table 1

The multiplication rules for quaternions and octonions

·	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	-0	3	-2	5	-4	-7	6
2	2	-3	-0	1	6	7	-4	-5
3	3	2	-1	-0	7	-6	5	-4
4	4	-5	-6	-7	-0	1	2	3
5	5	4	-7	6	-1	-0	-3	2
6	6	7	4	-5	-2	3	-0	-1
7	7	-6	5	4	-3	-2	1	-0

In this paper, we define a convenient operation between two hypercomplex numbers u and v as the Hadamard product $u \circ v$. It turns out that the Hadamard product of hypercomplex numbers is the analog of the Hadamard product of matrices. In addition, we show certain algebraic properties of such a product. This definition was developed particularly to characterize certain structures while working on specific quaternion sequences.

Before presenting our definition and results, we recall the following multiplication rules on the set of $\mathbb{H}$.

Let p, q, and r be any three quaternions.

- The quaternion multiplication of p and q is defined as follows:

$$pq = u_0 + iu_1 + ju_2 + ku_3,$$

  where $u_0 = p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3$, $u_1 = p_1q_0 + p_0q_1 - p_3q_2 + p_2q_3$, $u_2 = p_2q_0 + p_3q_1 + p_0q_2 - p_1q_3$ and $u_3 = p_3q_0 - p_2q_1 + p_1q_2 + p_0q_3$.
- The dot product of p and q is a real number that is defined by

$$p.q = p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3.$$

- The cross product is defined by

$$\tilde{p} \times \tilde{q} = Vec\left(\tilde{p}\tilde{q}\right).$$

- The mixed product is defined by

$$\left(\tilde{p}, \tilde{q}, \tilde{r}\right) = \tilde{p}.\left(\tilde{q} \times \tilde{r}\right).$$

- The quaternion outer product of p and q is defined by

$$(p, q) = p_0\tilde{q} - q_0\tilde{p} - \tilde{p} \times \tilde{q}.$$

- The even product of p and q is defined by

$$[p, q] = p_0q_0 - \tilde{p}\tilde{q} - p_0\tilde{q} + q_0\tilde{p}.$$

Main Results

Let us denote the set of all 2^N-ons hypercomplex numbers in the following form:

$$\mathbb{O} = \left\{w = w_1\vec{e_1} + \tilde{w} : \tilde{w} = w_l\vec{e_l},\ l = 2, 3, \ldots, 2^N \text{ and } w_l \in \mathbb{R}\right\},$$

where $\vec{e_i}$ refers to the components of the basis set. Throughout this paper, $i, j = 1, \ldots, 2^N$ and $l = 2, 3, \ldots, 2^N$. Note that the multiplication rule

$$\vec{e_1}^2 = 1,\ \vec{e_l}^2 = -1,\ \text{and } \vec{e_i}\vec{e_j} + \vec{e_j}\vec{e_i} = 0 \text{ for } i \neq j \tag{2}$$

is valid. At this instance and further in the document, repeated indices are summed over their ranges unless specified otherwise. We can reduce our results to particular cases depending on the choice of N. Clearly, we successively obtain the well-known complex numbers for $N = 1$, quaternions for $N = 2$, octonions for $N = 3$, sedenions for $N = 4$, ..., 2^N-ons for N by starting from the real numbers for $N = 0$.

Now we give the following definition.

Definition 1. For two hypercomplex numbers $p = p_i\vec{e}_i$ and $q = q_i\vec{e}_i$, the Hadamard product of p and q over the set $\mathbb{O}$ is defined as follows:

$$p \circ q = p_i q_i \vec{e}_i = p_1 q_1 \vec{e}_1 + p_2 q_2 \vec{e}_2 + p_3 q_3 \vec{e}_3 + \cdots + p_{2^N} q_{2^N} \vec{e}_{2^N},$$

where $p_i, q_i \in \mathbb{R}$.

This definition involves an element-wise product of hypercomplex numbers similar to the Hadamard product defined on the set of matrices. As a result, this inspired the name «Hadamard product» of hypercomplex numbers. To avoid confusion, juxtaposition of hypercomplex numbers will imply the usual product of hypercomplex numbers, and we will always employ the notation «$\circ$» for the Hadamard product.

Theorem 1. $(\mathbb{O}, +, \circ)$ is a ring but not either an integral domain or a field.

Proof. We know that $(\mathbb{O}, +)$ is a commutative group. On the other hands, we can write

$$p \circ (q \circ r) = [p_i\vec{e}_i] \circ [q_i r_i \vec{e}_i] = p_i q_i r_i \vec{e}_i = [p_i q_i \vec{e}_i] \circ [r_i \vec{e}_i] = (p \circ q) \circ r$$

and

$$r \circ (p + q) = [r_i \vec{e}_i] \circ [(p_i + q_i)\vec{e}_i] = r_i (p_i + q_i)\vec{e}_i = r_i p_i \vec{e}_i + r_i q_i \vec{e}_i = r \circ p + r \circ q.$$

Similarly, $(r + p) \circ q = r \circ q + p \circ q$ can be demonstrated. Now, we investigate the algebraic properties of $(\mathbb{O}^\star, \circ)$, where $\mathbb{O}^\star = \mathbb{O} - \{0\}$. It is clear that $(\mathbb{O}^\star, \circ)$ is closed, associative, distributive, and commutative. However, there are nonzero elements which are not invertible. For example, the element $q = \vec{e}_1 + \vec{e}_3$ is nonzero and not invertible. Since $(\mathbb{O}^\star, \circ)$ is not a group, $(\mathbb{O}, +, \circ)$ is not a field. We will prove that $(\mathbb{O}, +, \circ)$ is not an integral domain later.

Unlike the usual product, the commutative law on the set $\mathbb{O}$ is valid for the Hadamard product. We denote the identity element under the Hadamard product by $\mathcal{I}$. Clearly, the identity $\mathcal{I}$ is a hypercomplex numbers with all entries equal to 1, that is $\mathcal{I} = \vec{e}_1 + \vec{e}_2 + \ldots + \vec{e}_{2^N}$. In addition, we can conclude from the proof of Theorem 1 that a hypercomplex numbers q has an inverse under the Hadamard product only if $q_i \neq 0$ for all $1 \leq i \leq 2^N$.

Remark 1. According to Theorem 1, $(\mathbb{O}, +, \circ)$ is a commutative and associative ring with an identity.

Throughout the paper, we will use the following notations under the Hadamard product: We denote the Hadamard inverse of q by $q^{\circ(-1)}$ and the iterated Hadamard product $\underbrace{p \circ p \circ \cdots \circ p}_{n \text{ times}}$ by $q^{\circ(n)}$, that is $q^{\circ(n)} = {p_i}^n \vec{e}_i$.

Theorem 2. The Hadamard product is linear.

Proof. To prove this theorem, we must show that the Hadamard product satisfies two conditions such that $r \circ (p + q) = r \circ p + r \circ q$ and $\alpha (p \circ q) = (\alpha p) \circ q = p \circ (\alpha q)$. We have already shown the first condition. It is sufficient to complete the proof that we show in the second condition. For $\alpha \in \mathbb{R}$, we can write

$$\alpha (p \circ q) = \alpha (p_i q_i \vec{e}_i) = (\alpha p_i) q_i \vec{e}_i = (\alpha p) \circ q = p_i (\alpha q_i) \vec{e}_i = p \circ (\alpha q).$$

Thus, the theorem has been proved.

The next theorem presents certain fundamental properties.

Theorem 3. Let p and q be any two hypercomplex numbers. Then

i. $\overline{p \circ q} = \overline{p} \circ q = p \circ \overline{q}$

ii. $\overline{p} \circ p = p \circ \overline{p} = \overline{p^{\circ(2)}}$

iii. $(p \circ q)^{\circ(-1)} = p^{\circ(-1)} \circ q^{\circ(-1)}$, where p and q are invertible in the Hadamard sense.

Proof. We can write

$$\overline{p \circ q} = \overline{(p_i q_i \vec{e}_i)} = p_1 q_1 \vec{e}_1 - p_l q_l \vec{e}_l = p_1 q_1 \vec{e}_1 + (-p_l) q_l \vec{e}_l = p_1 q_1 \vec{e}_1 + p_l (-q_l) \vec{e}_l.$$

The last two results give the the proof of 3.*i*. Similarly, we get

$$p \circ \overline{p} = \overline{p} \circ p = \overline{(p_i \vec{e}_i)} \circ (q_i \vec{e}_i) = {p_1}^2 \vec{e}_1 - {p_l}^2 \vec{e}_l = \overline{p^{\circ(2)}}.$$

Finally,

$$\left(p\circ q\right)^{\circ(-1)}=\left(p_iq_i\vec{e}_i\right)^{\circ(-1)}=\frac{1}{p_iq_i}\vec{e}_i,$$

and the result follows.

Theorem 4. $(\mathbb{O},\circ)$ contains zero divisors and nontrivial idempotent elements but does not have nilpotent element.

Proof. Let p and q be two non-zero hypercomplex numbers whose coefficients satisfy one of the conditions such that $p_i\neq 0$ and $q_i=0$ or $p_i=0$ and $q_i\neq 0$. In this case, since $p_iq_i=0$, viz. $p\circ q=0$, we conclude that $(\mathbb{O},\circ)$ has many zero divisiors for hypercomplex numbers in the form mentioned. Suppose that p is not a unit hypercomplex numbers; further, suppose that the entries of p are either 1 or 0. Then, we can write that $p^{\circ(2)}=p$. Further, $p^{\circ(m)}=0$ only if ${p_i}^m=0$. As a result, $(\mathbb{O},\circ)$ does not have nilpotent elements.

Theorem 5. Let p and q be any two hypercomplex numbers. Then, we have

$$N\left(p\circ q\right)\leq N\left(p\right)N\left(q\right).$$

Proof. Considering Eq. (1), we write

$$\left(N\left(p\right)N\left(q\right)\right)^2=\left(\sum_{i=1}^{2^N}{p_i}^2\right)\left(\sum_{i=1}^{2^N}{q_i}^2\right)=\left(p_iq_i\right)^2+R\left(p_i,q_i\right),$$

where $R\left(p_i,q_i\right)$ denotes the remaining terms. Since $R\left(p_i,q_i\right)\geq 0$, the desired result is obtained.

Remark 2. Theorem 5 indicates that $(\mathbb{O},+,\circ)$ is a normed algebra.

Theorem 6. $(\mathbb{O},+,\circ)$ is isomorphic to one of the following four algebras: the real numbers, the complex numbers, the quaternions, and the Cayley numbers.

Proof. The well-known Hurwitz's theorem states that every normed algebra with an identity is isomorphic to one of the following: the algebra of real numbers, the algebra of complex numbers, the algebra of quaternions, and the algebra of Cayley numbers. From Remarks 1 and 2, we can consider that $(\mathbb{O},+,\circ)$ is a normed algebra with an identity. Thus, the proof is completed.

We can say that Def. 1 has many applications in mathematics. One of the most important is that there are usages in the matrix theory. Let $A=(a_{kl})$ and $B=(b_{kl})$ be two $m\times n$ matrices, i.e., of the same dimension but not necessarily square. Then, the Hadamard product between these two matrices, denoted by $A\star B$, is an $m\times n$ matrix given by

$$A\star B=\left(a_{kl}b_{kl}\right).$$

In our investigation, there are two various situations: Case (1) is the usual product of the hypercomplex numbers in the product element, i.e. $P\circledast Q=[p_{kl}q_{kl}]$; and Case (2) is the Hadamard product of the hypercomplex numbers in the product element, i.e. $P\odot Q=[p_{kl}\circ q_{kl}]$. Note that p_{kl} and q_{kl} are any 2^N-ons hypercomplex numbers here.

A finite-dimensional associative algebra over any field F is algebraically isomorphic to a sub-algebra of the full matrix algebra over the considered field. This means that each component in the finite-dimensional associative algebra has a faithful matrix representation over the field. Based on this motivation, we now define a bijective map

$$\varphi:w=w_1\vec{e}_1+w_2\vec{e}_2+\cdots+w_{2^N}\vec{e}_{2^N}\in\mathbb{O}\to\varphi\left(w\right)=\left[\left[w\right]\vec{e}_j\right]_{2^N\times 2^N}.$$

Here, the bracket is the vector representation of the corresponding quantity. Note that $\varphi\left(w\right)$ is a real skew-symmetric matrix. As an example, for $n=2$, we can write the matrix $\varphi\left(w\right)$ as

$$\varphi\left(w\right)=\begin{bmatrix} w_0 & -w_1 & -w_2 & -w_3\\ w_1 & w_0 & -w_3 & w_2\\ w_2 & w_3 & w_0 & -w_1\\ w_3 & -w_2 & w_1 & w_0\end{bmatrix}.$$

Hence, we can give the following significant result.

Corollary 1. $(\mathbb{O},+,\circ)$ is algebraically isomorphic to the matrix algebra

$$\mathcal{M}=\left\{\left[\left[w\right]\vec{e}_j\right]_{2^N\times 2^N}\middle|\,w\in\mathbb{O}\right\}.$$

Further, $\varphi(w)$ is a faithful real matrix representation of w.

Next, we present the following result.

Lemma 1. Given any $w \in \mathbb{O}$, the following unitary similarity factorization equality is valid:

$$\varphi(w) = \mathtt{M} D_w \mathtt{M}^*, \tag{3}$$

where $\mathtt{M} = \left(\sqrt{2}\right)^{-N} [\vec{e}_i E_N]$, $\mathrm{E}_N = \begin{bmatrix} \vec{e}_1 & -\vec{e}_2 & -\vec{e}_3 & \dots & -\vec{e}_{2^N} \end{bmatrix}^T$, $\mathtt{M}^*$ means the complex conjugate of $\mathtt{M}$, and D_w denotes a diagonal matrix such that while the entries outside main diagonal are zero, others are equal to w.

Proof. According to definition of $\mathtt{M}$, we can write that $\mathtt{M}^* = \mathtt{M}$. Hence, we can rearrange Eq. (3) as follows:

$$\varphi(w)\,\mathtt{M} = \mathtt{M} D_w.$$

Using the definition of matrix multiplication and the rules given in (2), the result follows.

We present our main result in the following.

Theorem 7. For any $u, v \in \mathbb{O}$, we have

$$D_u \circledast D_v = \mathtt{M}^* \varphi(uv)\,\mathtt{M} \tag{4}$$

and

$$D_u \odot D_v = \mathtt{M}^* \varphi(u \circ v)\,\mathtt{M}. \tag{5}$$

Proof. Applying matrix multiplication rule and Eq. (2) into each side of Eqs. (4) and (5), the results follow.

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А. Дашдемир

Кастамону университеті, Кастамону, Түркия

Гиперкомплексті сандар Адамардың көбейтіндісі туралы

Гиперкомплексті сандар жиынында әртүрлі форманы алатын белгілі бір көбейту ережелері бар болады. Мақалада нақты матрицалық кеңістіктегі көбейтуге ұқсас гиперкомплексті сандарды көбейтудің жаңа формасы енгізілген, ол «Адамар көбейтіндісі» деп аталады және оның кейбір алгебралық қасиеттері, соның ішінде норманың теңсіздігі зерттелген. Атап айтқанда, жаңа анықтамалар және оның қосымшалары комплексті матрицалар теориясына дейін кеңейтілген.

Кілт сөздер: кватерниондардың көбейтіндісі, скаляр көбейтінді, Адамардың көбейтіндісі, гиперкомплексті сан.

А. Дашдемир

Университет Кастамону, Кастамону, Турция

О произведении Адамара гиперкомплексных чисел

Существуют определённые правила произведения, которые принимают различные формы на множестве гиперкомплексных чисел. В настоящей статье введена новая форма умножения гиперкомплексных чисел, которая будет называться «произведением Адамара», вдохновленная аналогичным произведением в вещественном матричном пространстве, и исследованы некоторые его алгебраические свойства, включая неравенство для нормы. В частности, автор статьи расширил определение и его приложения на теорию комплексных матриц.

Ключевые слова: произведение кватернионов, скалярное произведение, произведение Адамара, гиперкомплексное число.

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M.T. Jenaliyev[1], M.G. Yergaliyev[1], A.A. Assetov[2,*], A.K. Kalibekova[1]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan*
(E-mail: muvasharkhan@gmail.com, ergaliev.madi.g@gmail.com, bekaaskar@mail.ru, kalibekovaaidana2014@gmail.com)

On One Initial Boundary Value Problem for the Burgers Equation in a Rectangular Domain

We consider some initial boundary value problems for the Burgers equation in a rectangular domain, which in a sense can be taken as a model one. The fact is that such a problem often arises when studying the Burgers equation in domains with moving boundaries. Using the methods of functional analysis, priori estimates, and Faedo-Galerkin in Sobolev spaces and in a rectangular domain, we show the correctness of the initial boundary value problem for the Burgers equation with nonlinear boundary conditions of the Neumann type.

Keywords: Burgers equation, boundary value problem, Sobolev classes, rectangular domain, Galerkin methods, priori estimates.

Introduction

The study of the Burgers equation has a long history, some of which is given in [1–4], as well as in monographs [5] and [6].

In works [1] and [2] in Sobolev spaces, the correctness of the boundary value problem for the Burgers equation was established. In this case, the domain of independent variables degenerated according to a nonlinear law, and homogeneous Dirichlet conditions were set on the boundary.

The infiltration of the wetting front into a porous medium is a classical problem with a free boundary. Historically, the first example is the Green-Ampt model for water flow in soils [7]. There is a huge variety of situations (chemically reacting media, deformable media, capillarity effects, mass transfer, mixture flows, media with a complex structure, pollution, reclamation, soil freezing, production of composite materials, brewing, etc.).

Nonlinear Burgers equations and their modifications are also suitable models of fluid motion in porous media [8–13].

The range of application of boundary value problems for parabolic equations in a domain with a boundary that changes over time is quite wide. Such problems arise in the study of thermal processes in electrical contacts [14], the processes of ecology and medicine [15], in solving some problems of hydromechanics [16], thermomechanics in thermal shock [17], and so on.

In this paper, in Sobolev classes, we study the solvability of the initial boundary value problem for the Burgers equation in a rectangular domain with nonlinear boundary conditions of the Neumann type.

In Section 1, the statement of the boundary value problem under study is given, and the main result of the work is formulated. We study the questions of unique solvability of two auxiliary boundary value problems for the Burgers equation in rectangular and non-rectangular domains, which are used in the proof of the main results of the work. Sections 2–5 are devoted to the first auxiliary problem. In these sections, the correctness of this problem in Sobolev classes is established by the methods of a priori estimates and Faedo-Galerkin. In sections 4–5, Theorem 1.1, the main result of the work is proved. The work is completed by a brief conclusion.

[*] Corresponding author.
E-mail: bekaaskar@mail.ru

1 Problem statement and main result

In the domain $Q_{yt} = \{y,t|\ y \in (0,1),\ t \in (0,T)\}$, we consider the following initial boundary value problem:

$$\partial_t w + \alpha(t) w \partial_y w - \beta(t)\partial_y^2 w + \gamma(y,t)\partial_y w = g, \tag{1.1}$$

$$\left[\frac{\alpha(t)}{3} w^2 - \beta(t)\partial_y w\right]\Big|_{y=0} = 0,\ \left[\frac{\alpha(t)}{3} w^2 - \beta(t)\partial_y w\right]\Big|_{y=1} = 0,\ \ 0 < t < T, \tag{1.2}$$

$$w(y,0) = 0,\ \ 0 < y < 1. \tag{1.3}$$

where the given continuous functions $\alpha(t)$, $\beta(t)$, $\gamma(y,t)$ satisfy the following conditions

$$\alpha'(t) \le 0,\ \beta'(t) \le 0,\ \alpha_1 \le \alpha(t) \le \alpha_2,\ \beta_1 \le \beta(t) \le \beta_2,\ |\gamma(y,t)| \le \gamma_1,\ |\partial_y \gamma(y,t)| \le \gamma_1,\ \forall\, t \in [0,T], \tag{1.4}$$

with given positive constants

$$\alpha_i,\ \beta_i,\ i = 1,2,\ \gamma_1;\ \alpha(t),\ \beta(t) \in C^1([0,T]),\ \partial_y\gamma(y,t) \in C(\bar{Q}_{yt}). \tag{1.5}$$

Theorem 1.1 (Main result). Let $g \in L_2(Q_{yt})$ and conditions (1.4)–(1.5) be satisfied. Then boundary value problem (1.1)–(1.3) has a unique solution

$$w \in H^{2,1}(Q_{yt}) \equiv L_2(0,T; H^2(0,1)) \cap H^1(0,T; L_2(0,1)).$$

To apply the Faedo-Galerkin method, we need to solve the following spectral problem:

$$-Y''(y) = \lambda^2 Y(y),\ y \in (0,1), \tag{1.6}$$

$$Y'(0) + \lambda^2 Y(0) = 0, \tag{1.7}$$

$$Y'(1) - \lambda^2 Y(1) = 0, \tag{1.8}$$

obtained by applying the variable separation method $(u(y,t) = F(t)Y(y))$ from the following problem

$$\partial_t u - \partial_y^2 u = 0,\ y \in (0,1),\ t \in (0,T),$$

$$\partial_t u - \partial_x u\big|_{y=0} = 0,\ \ \partial_t u + \partial_x u\big|_{y=1} = 0,$$

$$u(y,0) = u_0(y).$$

2 Solving spectral problem (1.6)-(1.8)

We seek the general solution to equation (1.6) in the form

$$Y(y) = C_1 \exp\{i\lambda y\} + C_2 \exp\{-i\lambda y\},\ i = \sqrt{-1}. \tag{2.1}$$

Satisfying (2.1) to boundary conditions (1.7)–(1.8), we obtain

$$Y_{01}(y) = 1,\ \lambda_{01} = 0,\ \tan\frac{\lambda_{01}}{2} = -\lambda_{01},$$

$$Y_{2n-1}(y) = \cos\frac{\lambda_{2n-1}(1-2y)}{2},\ \lambda_{2n-1} = (2n-1)\pi + \varepsilon_{2n-1},\ \tan\frac{\lambda_{2n-1}}{2} = -\lambda_{2n-1},\ n \in \mathbb{N}, \tag{2.2}$$

$$Y_{02}(y) = \sin\frac{\lambda_{02}(1-2y)}{2},\ \lambda_{02} \approx \frac{2\pi}{5},\ \cot\frac{\lambda_{02}}{2} = \lambda_{02},$$

$$Y_{2n}(y) = \sin\frac{\lambda_{2n}(1-2y)}{2},\ \lambda_{2n} = 2n\pi + \varepsilon_{2n},\ \cot\frac{\lambda_{2n}}{2} = \lambda_{2n},\ n \in \mathbb{N}. \tag{2.3}$$

It is easy to see that the solutions of equations

$$\tan\frac{\lambda_{2n-1}}{2} = -\lambda_{2n-1},\ n \in \mathbb{N}, \qquad \text{and} \qquad \cot\frac{\lambda_{2n}}{2} = \lambda_{2n},\ n \in \mathbb{N},$$

are, respectively, close to points $(2n-1)\pi$ and $2n\pi,\ n \in \mathbb{N}$, and with the growth of n they approach arbitrarily close from the right to the corresponding specified points $(2n-1)\pi$ and $2n\pi,\ n \in \mathbb{N}$, i.e. $\varepsilon_n \to 0+$ at $n \to \infty$ (see Figure 2.1–2.2). If we introduce the notation $2x = (1-2y)\pi$, then we get: $x \in (-\pi/2, \pi/2)$.

By the Paley-Wiener theorem ([18], chapter V, 86, example), the system of functions (2.2) and (2.3) is complete in $L_2(0,1)$, since the system of functions:

$$\frac{\sqrt{2}\cos x}{\sqrt{\pi}},\ \frac{\sqrt{2}\sin 2x}{\sqrt{\pi}},\ \frac{\sqrt{2}\cos 3x}{\sqrt{\pi}},\ \frac{\sqrt{2}\sin 4x}{\sqrt{\pi}}, \ldots,$$

which is complete in $L_2(-\pi/2, \pi/2)$, will differ little from it. For the latter system, it is sufficient to make the replacement $x_1 = x + \pi/2$. We get the system of sines:

$$\frac{\sqrt{2}\sin x_1}{\sqrt{\pi}},\ \frac{\sqrt{2}\sin 2x_1}{\sqrt{\pi}},\ \frac{\sqrt{2}\sin 3x_1}{\sqrt{\pi}},\ \frac{\sqrt{2}\sin 4x_1}{\sqrt{\pi}}, \ldots,$$

which is complete in $L_2(0,\pi)$.

Note that the system of functions (2.2) and (2.3) is not orthogonal in $L_2(0,1)$.

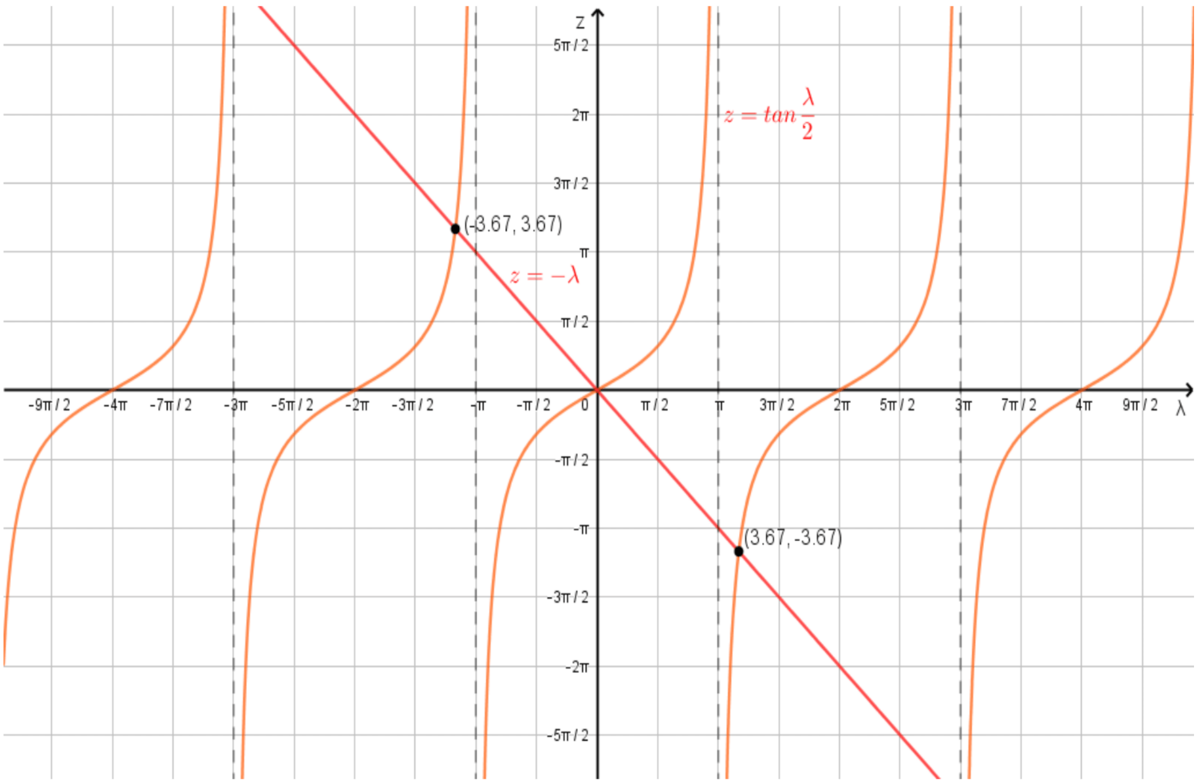

Figure 2.1. Graphs of functions $z = -\lambda,\ z = \tan\frac{\lambda}{2}$

Remark 2.1. The applicability of the Paley-Wiener theorem ([18], chapter V, 86, example) follows from the relations:

$$\lambda_1 \approx 3.673,\ \ \lambda_1 - \pi \approx 0.533,\ \ M\pi = |\lambda_1 - \pi| < 0.54 < \ln 2 \approx 0.693,\ \ \theta = \exp\{M\pi\} - 1 < 1.$$

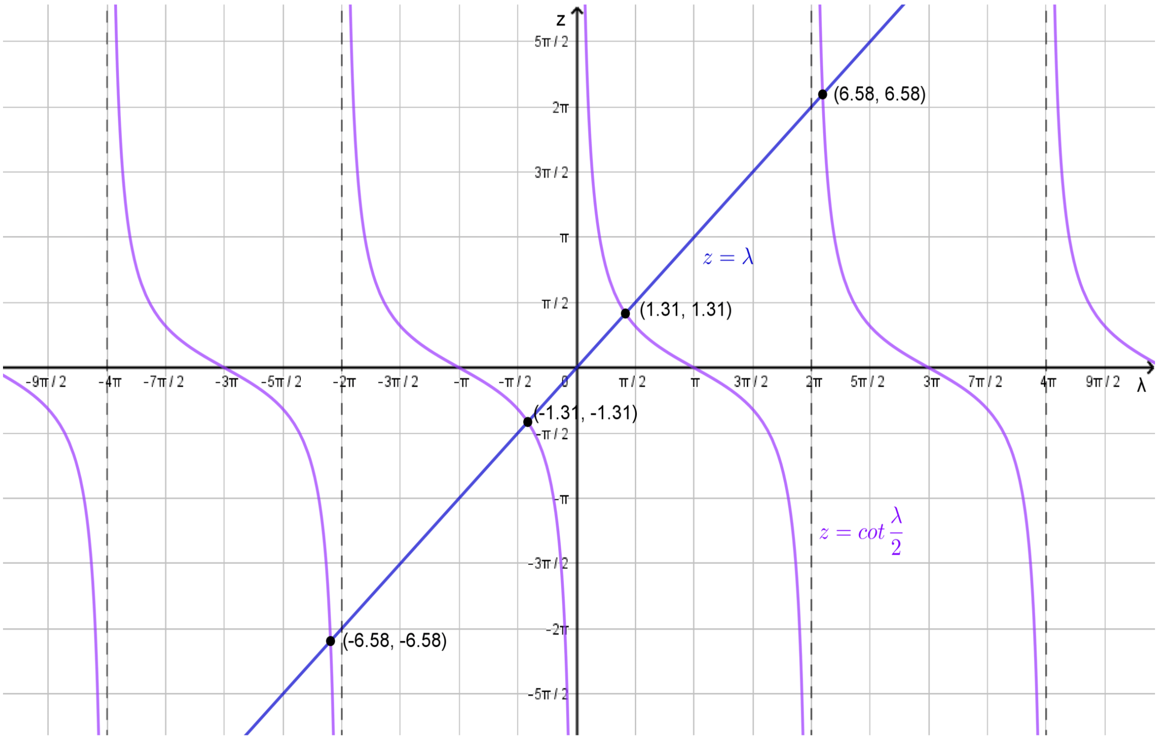

Figure 2.2. Graphs of functions $z=\lambda,\ z=\cot\frac{\lambda}{2}$

3 Setting and solving the approximate problem

We multiply equation (1.1) scalarly in $L_2(0,1)$ by function $v\in H^1(0,1)$. As a result, taking into account initial (1.3) and boundary conditions (1.2) we will have a weak statement of problem (1.1)–(1.3):

$$\int_0^1 \partial_t wvdy+\alpha(t)\int_0^1 w\partial_y wvdy+\beta(t)\int_0^1 \partial_y w\partial_y vdy+\int_0^1 \gamma(y,t)\partial_y wvdy-$$

$$-\frac{\alpha(t)}{3}w^2(1,t)v(1,t)+\frac{\alpha(t)}{3}w^2(0,t)v(0,t)=\int_0^1 gvdy,\ \ \forall\, v\in H^1(0,1), \tag{3.1}$$

$$w(y,0)=0,\ \ y\in(0,1). \tag{3.2}$$

We introduce the following approximate solution

$$w_n(y,t)=\sum_{j=1}^n c_j(t)Y_j(y),\ \ w_n(y,0)=\sum_{j=1}^n c_j(0)Y_j(y). \tag{3.3}$$

Next, we will satisfy this solution to an approximate version of problem (3.1)–(3.2):

$$\int_0^1 \partial_t w_nY_jdy+\alpha(t)\int_0^1 w_n\partial_y w_nY_jdy+\beta(t)\int_0^1 \partial_y w_n\partial_y Y_jdy+\int_0^1 \gamma(y,t)\partial_y w_nY_jdy-$$

$$-\frac{\alpha(t)}{3}w_n^2(1,t)Y_j(1)+\frac{\alpha(t)}{3}w_n^2(0,t)Y_j(0)=\int_0^1 gY_jdy, \tag{3.4}$$

$$w_n(y,0)=0,\ \ y\in(0,1), \tag{3.5}$$

for all $j=0,1,...,n$, and $t\in[0,T]$.

Lemma 3.1. Problem (3.4)–(3.5) has a unique solution $w_n(y,t)$.

Proof. Since the system of functions $Y_1(y), Y_2(y), ...$ is a basis in $L_2(0,1)$, we have

$$\det\{W_n\}=\|(Y_k(y),Y_j(y))\|_{k,j=1}^n\neq 0,\ \forall\ \text{finite}\ n;$$

W_n is a Gram matrix, $(\cdot,\cdot)$ is the scalar product in $L_2(0,1)$, $A_n=(\partial_y Y_k(y),\partial_y Y_j(y))_{k,j=1}^n$,

$$w_n^2(1,t)Y_j(1,t)-w_n^2(0,t)Y_j(0,t)=[\sum_{k=1}^n c_k(t)Y_k(1)]^2Y_j(1)-[\sum_{k=1}^n c_k(t)Y_k(0)]^2Y_j(0).$$

Further, if we introduce the notation

$$G_n(t)=\{g_0(t),...,g_n(t)\},\ P_n(t)=\{p_0(t),...,p_n(t)\},\ H_n(t)=\{h_0(t),...,h_n(t)\},$$

$$C_n(t)=\{c_1(t),...,c_n(t)\},$$

where

$$g_j(t)=\int_0^1 gY_j(y)dy,\ \ p_j(t)=-\alpha(t)\int_0^1 w_n\partial_y w_nY_j(y)dy-\int_0^1\gamma(y,t)\partial_y w_n(y,t)Y_j(y)dy,$$

$$h_j(t)=\frac{\alpha(t)}{3}[\sum_{k=1}^n c_k(t)Y_k(1)]^2Y_j(1)-[\sum_{k=1}^n c_k(t)Y_k(0)]^2Y_j(0),$$

for all $j=0,1,...,n$, then problem (3.4)–(3.5) is equivalent to the following Cauchy problem for a finite system of nonlinear ordinary differential equations

$$C_n'(t)=W_n^{-1}\left[-\beta(t)A_nc(t)+P_n(t)+H_n(t)+G_n(t)\right],\ \ C_n(0)=0. \tag{3.6}$$

Note that functions $p_j(t)$, $h_j(t)$ are well-defined, and function $g_j(t)$ is square integrable (by virtue of $g\in L_2(Q)$). Theorefore, the Cauchy problem (3.6) is uniquely solvable on some interval $[0,T']$, where $T'\leq T$. However, according to the priori estimates established below, we find that this solution $C_n(t)$ continues to a finite time T.

Thus, we find the functions $C_n(t)=\{c_j(t),\ j=0,1,...,n\}$ as a solution to the Cauchy problem (3.6) for each fixed finite n, and together with them the only approximate solution $w_n(y,t)$ to problem (3.4)–(3.5). Lemma 3.1 is completely proved.

4 A priori estimates

Lemma 4.1. There exists a positive constant K_1 independent of n, such that for all $t\in[0,T]$ the following estimate takes place

$$\|w_n(y,t)\|^2_{L_2(0,1)}+\beta_1\int_0^t\|\partial_y w_n(y,\tau)\|^2_{L_2(0,1)}d\tau\leq K_1.$$

Proof. Multiplying (3.4) by $c_j(t)$, summing the result over j from 1 to n and using the equality

$$\int_0^1 w_n(y,t)\partial_y w_n(y,t)w_n(y,t)dy=\frac{1}{3}w_n^3(1,t)-\frac{1}{3}w_n^3(0,t),$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1|w_n(y,t)|^2dy+\beta(t)\int_0^1|\partial_y w_n(y,t)|^2dy=$$

$$= -\int_0^1 \gamma(y,t)\partial_y w_n(y,t) w_n(y,t)dy + \int_0^1 g(y,t)w_n(y,t)dy. \tag{4.1}$$

Now, by integrating (4.1) with respect to t from 0 to t and using Cauchy's inequality

$$-\int_0^1 \gamma(y,t)\partial_y w_n(y,t) w_n(y,t)dy \leq \frac{\beta_1}{2}\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \frac{\gamma_1^2}{2\beta_1}\|w_n(y,t)\|^2_{L_2(0,1)},$$

$$\int_0^1 g(y,t)w_n(y,t)dy \leq \frac{1}{2}\|g(y,t)\|^2_{L_2(0,1)} + \frac{1}{2}\|w_n(y,t)\|^2_{L_2(0,1)},$$

we get

$$\|w_n(y,t)\|^2_{L_2(0,1)} + \beta_1 \int_0^t \|\partial_y w_n(y,\tau)\|^2_{L_1(0,1)}d\tau \leq$$

$$\leq \left(\frac{\gamma_1^2}{\beta_1} + 1\right)\int_0^t \|w_n(y,\tau)\|^2_{L_2(0,1)}d\tau + \int_0^T \|g(y,\tau)\|^2_{L_2(0,1)}d\tau. \tag{4.2}$$

From (4.2) follows

$$\|w_n(y,t)\|^2_{L_2(0,1)} \leq \left(\frac{\gamma_1^2}{\beta_1} + 1\right)\int_0^t \|w_n(y,\tau)\|^2_{L_2(0,1)}d\tau + \int_0^T \|g(y,\tau)\|^2_{L_2(0,1)}d\tau.$$

By applying the Gronwall's inequality, we obtain the estimate for $\|w_n(y,t)\|^2_{L_2(0,1)}$. By using this estimate in (4.2), we establish the required estimate for Lemma 4.1.

Embedding $H^1(0,1) \hookrightarrow C([0,1])$ from Lemma 4.1 we directly obtain:

Corollary 4.1. There exists a positive constant K_1' independent of n, such that for all $t \in [0,T]$ the following inequality holds

$$\int_0^t |w_n(0,\tau)|^2 d\tau + \int_0^t |w_n(1,\tau)|^2 d\tau \leq 2B\int_0^t \|w_n(y,t)\|^2_{H^1(0,1)} \leq K_1',$$

where B is a constant of the embedding $H^1(0,1) \hookrightarrow C([0,1])$.

Lemma 4.2. For a positive constant K_2 independent of n, for all $t \in (0,T]$ the following inequality takes place:

$$\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \beta_1 \int_0^t \|\partial_y^2 w_n(y,\tau)\|^2_{L_2(0,1)}d\tau \leq K_2. \tag{4.3}$$

Proof. Taking into account equality

$$\sum_{j=1}^n c_j \lambda_j^2 Y_j(y) = -\sum_{j=1}^n c_j \partial_y^2 Y_j(y) = -\partial_y^2 w_n(y,t),$$

which follows from (1.6) and (3.3), and multiplying equality (3.4) by $c_j\lambda_j^2$ and summing over j from 1 to n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \beta(t)\|\partial_y^2 w_n(y,t)\|^2_{L_2(0,1)} =$$

$$= \alpha(t)\Big(w_n(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\Big) + \Big(\gamma(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\Big) -$$

$$- \big(g(y,t), \partial_y^2 w_n(y,t)\big) + \partial_t w_n(y,t)\partial_y w_n(y,t)\Big|_{y=0}^{y=1} =$$

$$= \alpha(t)\left(w_n(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\right) + \left(\gamma(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\right) -$$

$$- \left(g(y,t), \partial_y^2 w_n(y,t)\right) + \frac{\alpha(t)}{9\beta(t)}\partial_t [w_n(y,t)]^3\Big|_{y=0}^{y=1}. \tag{4.4}$$

We will use the relations (similar ones are true for the term with $w_n(0,t)$)

$$\int_0^t \frac{\alpha(t)}{9\beta(t)}\partial_t [w_n(1,t)]^3\, dt = \frac{\alpha(t)}{9\beta(t)}[w_n(1,t)]^3 - \int_0^t \frac{\alpha'(t)\beta(t) - \alpha(t)\beta'(t)}{9[\beta(t)]^2}[w_n(1,t)]^3\, dt \leq$$

$$\leq \frac{\alpha_2}{9\beta_1}|w_n(1,t)|^3 + C_1 \int_0^t |w_n(1,t)|^3\, dt, \quad \text{where } 9C_1\beta_1^2 = \max_{0\leq t\leq T}|\alpha'(t)\beta(t) - \alpha(t)\beta'(t)|.$$

Let us establish the following estimate

$$\frac{\alpha_2}{9\beta_1}|w_n(1,t)|^3 \leq \|w_n(y,t)\|^3_{L_\infty(0,1)} \leq \frac{\alpha_2}{9\beta_1}\|w_n(y,t)\|^{3/2}_{H^1(0,1)}\|w_n(y,t)\|^{3/2}_{L_2(0,1)} =$$

$$= \frac{\alpha_2}{9\beta_1}\|w_n(y,t)\|^{3/2}_{H^1(0,1)}\|w_n(y,t)\|^{1/2}_{L_2(0,1)}\|w_n(y,t)\|_{L_2(0,1)}.$$

In the previous relation, we used the interpolation inequality from ([19], Theorems 5.8–5.9, p.140–141). Now, applying Young's inequality $(p^{-1} + q^{-1} = 1)$:

$$|AB| = \left|\left(a^{1/p}A\right)\left(a^{1/q}\frac{B}{a}\right)\right| \leq \frac{a}{p}|A|^p + \frac{a}{qa^q}|B|^q, \tag{4.5}$$

where

$$A = \|w_n(y,t)\|^{3/2}_{H^1(0,1)}, \quad B = \frac{\alpha_2}{9\beta_1}\|w_n(y,t)\|_{L_2(0,1)}\|w_n(y,t)\|^{1/2}_{L_2(0,1)}, \quad a = \frac{1}{6}, \quad p = \frac{4}{3}, \quad q = 4,$$

from here, we get

$$\frac{\alpha_2}{9\beta_1}|w_n(1,t)|^3 \leq \frac{1}{8}\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \left[\frac{1}{8} + \frac{2\alpha_2^4}{3^5\beta_1^4}\|w_n(y,t)\|^4_{L_2(0,1)}\right]\|w_n(y,t)\|^2_{L_2(0,1)} \leq$$

$$\leq \frac{1}{8}\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + D_1, \tag{4.6}$$

where the constant D_1 is determined according to the estimates of Lemma 4.1 and Corollary 4.1.

Similarly to the previous one, we obtain

$$\frac{\alpha_2}{9\beta_1}|w_n(0,t)|^3 \leq \frac{1}{8}\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \left[\frac{1}{8} + \frac{2\alpha_2^4}{3^5\beta_1^4}\|w_n(y,t)\|^4_{L_2(0,1)}\right]\|w_n(y,t)\|^2_{L_2(0,1)} \leq$$

$$\leq \frac{1}{8}\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + D_0, \tag{4.7}$$

where the constant D_0 is determined according to the estimates of Lemma 4.1 and Corollary 4.1.

First of all, we consider the estimates of the nonlinear summands from 4.4. To begin with, we have

$$\left|\left(w_n(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t)\right)\right| \leq \|w_n(y,t)\|_{L_4(0,1)}\|\partial_y w_n(y,t)\|_{H^1(0,1)}\|\partial_y w_n(y,t)\|_{L_4(0,1)} \leq$$

$$\leq \|w_n(y,t)\|_{L_4(0,1)}\|\partial_y w_n(y,t)\|_{H^1(0,1)}\|\partial_y w_n(y,t)\|_{L_\infty(0,1)}. \tag{4.8}$$

Further, consifeing the interpolation inequality from ([19], Theorems 5.8–5.9, p.140–141)

$$\alpha_2\|\partial_y w_n(y,t)\|_{L_4(0,1)} \leq C\|\partial_y w_n(y,t)\|^{1/2}_{H^1(0,1)}\|\partial_y w_n(y,t)\|^{1/2}_{L_2(0,1)}, \ \forall\, \partial_y w_n(y,t) \in H^1(0,1),$$

from (4.8), we obtain

$$\alpha_2 \left| \Big(w_n(y,t)\partial_y w_n(y,t), \partial_y^2 w_n(y,t) \Big) \right| \le$$

$$\le C\|w_n(y,t)\|_{L_4(0,1)}\|\partial_y w_n(y,t)\|^{3/2}_{H^1(0,1)}\|\partial_y w_n(y,t)\|^{1/2}_{L_2(0,1)} \le$$

$$\le \frac{\beta_1}{8}\|\partial_y^2 w_n(y,t)\|^2_{L_2(0,1)} + \left[\frac{\beta_1}{8} + C_2\|w_n(y,t)\|^4_{L_4(0,1)}\right]\|\partial_y w_n(y,t)\|^2_{L_2(0,1)}. \tag{4.9}$$

Here, we have used Young's inequality (4.5), where

$$A = \|\partial_y w_n(y,t)\|^{3/2}_{H^1(0,1)}, \;\; B = C\,\|w_n(y,t)\|_{L_4(0,1)}\,\|\partial_y w_n(y,t)\|^{1/2}_{L_2(0,1)}, \;\; a = \frac{\beta_1}{6}, \;\; p = \frac{4}{3}, \;\; q = 4.$$

Further, for the last two summands from (4.4) we will have:

$$\gamma_1 \left| \Big(\partial_y w_n(y,t), \partial_y^2 w_n(y,t) \Big) \right| \le \frac{\beta_1}{8}\|\partial_y^2 w_n(y,t)\|^2_{L_2(0,1)} + C_3\|\partial_y w_n(y,t)\|^2_{L_2(0,1)}, \tag{4.10}$$

$$\left| \big(g(y,t), \partial_y^2 w_n(y,t) \big) \right| \le \frac{\beta_1}{4}\|\partial_y^2 w_n(y,t)\|^2_{L_2(0,1)} + C_4\|g(y,t)\|^2_{L_2(0,1)}. \tag{4.11}$$

Taking into account inequalities (4.6)–(4.11), integrating (4.4) from 0 to t, we get

$$\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \beta_1 \int_0^t \|\partial_y^2 w_n(y,\tau)\|^2_{L_2(0,1)} d\tau \le A_4\|g(y,t)\|^2_{L_2(Q)} +$$

$$+ \int_0^t A_5(\tau)\|\partial_y w_n(y,\tau)\|^2_{L_2(0,1)} d\tau + 2C_0 \int_0^t |w_n(0,t)|^3\,dt + 2C_1 \int_0^t |w_n(1,t)|^3\,dt + 2(D_0 + D_1), \tag{4.12}$$

where

$$A_4 = 2C_4, \;\; A_5(t) = \frac{1}{2} + \frac{\beta_1}{4} + 2C_2\|w_n(y,t)\|^4_{L_4(0,1)} + 2C_3.$$

Let us estimate the last two integral summands from (4.12). By (4.6)–(4.7), we have

$$2C_0 \int_0^t |w_n(0,t)|^3\,dt \le \frac{C_0}{4}\int_0^t \|\partial_y w_n(y,t)\|^2_{L_2(0,1)}\,dt + 2D_0 T,$$

$$2C_1 \int_0^t |w_n(0,t)|^3\,dt \le \frac{C_1}{4}\int_0^t \|\partial_y w_n(y,t)\|^2_{L_2(0,1)}\,dt + 2D_1 T,$$

Thus, (4.12) takes the form:

$$\|\partial_y w_n(y,t)\|^2_{L_2(0,1)} + \beta_1 \int_0^t \|\partial_y^2 w_n(y,\tau)\|^2_{L_2(0,1)} d\tau \le A_4\|g(y,t)\|^2_{L_2(Q)} +$$

$$+ \int_0^t \left[A_5(\tau) + \frac{C_0 + C_1}{4}\right]\|\partial_y w_n(y,\tau)\|^2_{L_2(0,1)} d\tau + 2(D_0 + D_1)(1 + T), \tag{4.13}$$

From inequality (4.13) in the same way as in the proof of Lemma 4.1, we obtain the desired estimate (4.3). Lemma 4.2 is completely proved.

Lemma 4.3. For positive a constant K_3 independent of n, at all $t \in (0,T]$ the following inequality holds:

$$\|\partial_t w_n(y,t)\|^2_{L_2(Q_{yt})} \le K_3. \tag{4.14}$$

Proof. Let us write down equation (1.1) for the approximate solution $w_n(y,t)$:

$$\partial_t w_n + \alpha(t) w_n \partial_y w_n - \beta(t)\partial_y^2 w_n + \gamma(y,t)\partial_y w_n = g. \tag{4.15}$$

From equation (4.15), we obtain

$$\|\partial_t w_n\|_{L_2(Q_{yt})} \leq \alpha_2 \|w_n \partial_y w_n\|_{L_2(Q_{yt})} + \beta_2 \|\partial_y^2 w_n\|_{L_2(Q_{yt})} + \gamma_1 \|\partial_y w_n\|_{L_2(Q_{yt})} + \|g\|_{L_2(Q_{yt})}. \tag{4.16}$$

According to embedding $H^1(0,1) \hookrightarrow L_\infty(0,1)$ inequality $\|w_n\|_{L_\infty(0,1)} \leq B\|w_n\|_{H^1(0,1)}$ holds. Hence, considering Lemmas 4.1 and 4.2, we obtain

$$\|w_n \partial_y w_n\|^2_{L_2(Q_{yt})} \leq \int_0^T \|w_n(y,t)\|^2_{L_\infty(0,1)} \|\partial_y w_n(y,t)\|^2_{L_2(0,1)}\, dt \leq$$

$$\leq B\|\partial_y w_n(y,t)\|^2_{L_\infty(0,T;L_2(0,1))} \int_0^T \|w_n(y,t)\|^2_{H^1(0,1)}\, dt \leq BK_2K_1(1+T), \tag{4.17}$$

where B is a constant of embedding $H^1(0,1) \hookrightarrow L_\infty(0,1)$, K_1 and K_2 are constants from Lemmas 4.1 and 4.2, respectively.

Estimate (4.14) follows from (4.16), (4.17) and from the statement of Lemmas 4.1 and 4.2, Lemma 4.3 is completely proved.

5 Unique solvability of initial boundary value problem (1.1)–(1.3)

Lemmas 4.1–4.3 show that the sequence of Galerkin approximations $\{w_n(y,t),\ n = 1,2,3,...\}$ is bounded in space $L_\infty(0,T;H^1(0,1)) \cap L_2(0,T;H^2(0,1))$, and the sequence $\{\partial_t w_n(y,t), n = 1,2,3,...\}$ is bounded in $L_2(0,T;L_2(0,1))$.

Thus, we can extract a weakly convergent subsequence (we preserve the notation of the index n for the subsequence):

$$w_n(y,t) \to w(y,t) \text{ weakly in } L_2(0,T;H^2(0,1)) \cap H^1(0,T;L_2(0,1)), \tag{5.1}$$

$$w_n(y,t) \to w(y,t) \text{ strongly in } L_2(0,T;L_2(0,1)) \text{ and almost everywhere in } Q_{yt}, \tag{5.2}$$

Lemma 5.1. Let conditions (1.4)–(1.5) be satisfied and $g \in L_2(Q_{yt})$. Then initial boundary value problem (1.1)–(1.3) has a weak solution in space $H^{2,1}(Q_{yt})$.

Proof. Let $\varphi(t) \in \mathcal{D}((0,T))$, i.e. from the class of infinitely differentiable finite functions. We introduce the notation $v_j(y,t) = \varphi(t)Y_j(y)$, where $Y_j(y) \in H^1(0,1)$. Now, by multiplying integral identity (3.4) by the function $\varphi(t) \in \mathcal{D}((0,T))$ and integrating the result obtained with respect to t from 0 to T, we obtain

$$\int_0^T\int_0^1 \left[\partial_t w_n + \alpha(t) w_n \partial_y w_n - \beta(t)\partial_y^2 w_n + \gamma(y,t)\partial_y w_n\right] v_j\, dy\, dt +$$

$$+ \int_0^T \left[\beta(t)\partial_y w_n(1,t) - \frac{\alpha(t)}{3} w_n^2(1,t)\right] v_j(1,t)\, dt +$$

$$+ \int_0^T \left[-\beta(t)\partial_y w_n(0,t) + \frac{\alpha(t)}{3} w_n^2(0,t)\right] v_j(0,t)\, dt =$$

$$= \int_0^T\int_0^1 g v_j\, dy\, dt, \ \ \forall\, \varphi(t) \in \mathcal{D}((0,T)),\ \forall\, j = 1,...,n. \tag{5.3}$$

Since $\mathcal{D}((0,T);H^1(0,1))$ is dense in $L_2(0,T;H^1(0,1))$, then integral identity (5.3) can be rewritten as

$$\int_0^T\int_0^1 \left[\partial_t w_n + \alpha(t)w_n\partial_y w_n - \beta(t)\partial_y^2 w_n + \gamma(y,t)\partial_y w_n\right] v\,dy\,dt+$$

$$+\int_0^T \left[\beta(t)\partial_y w_n(1,t) - \frac{\alpha(t)}{3}w_n^2(1,t)\right] v(1,t)\,dt+$$

$$+\int_0^T \left[-\beta(t)\partial_y w_n(0,t) + \frac{\alpha(t)}{3}w_n^2(0,t)\right] v(0,t)\,dt=$$

$$=\int_0^T\int_0^1 gv\,dy\,dt,\quad \forall\, v(y,t)\in L_2(0,T;H^1(0,1)). \tag{5.4}$$

In integral identity (5.4), we pass to the limit as $n\to\infty$. In the expressions corresponding to the linear summands of equation (1.1) and boundary conditions (1.2), passing to the limit is carried out according to relation (5.1). As for the nonlinear summands, here, we have the following:

$$\int_0^T\int_0^1 \alpha(t)w_n(y,t)\partial_y w_n(y,t)v(y,t)\,dy\,dt = \int_0^T \alpha(t)\int_0^1 [w_n(y,t)-w(y,t)]\partial_y w_n(y,t)v(y,t)\,dy\,dt+$$

$$+\int_0^T \alpha(t)\int_0^1 w(y,t)\partial_y w_n(y,t)v(y,t)\,dy\,dt \to \int_0^T \alpha(t)\int_0^1 w(y,t)\partial_y w(y,t)v(y,t)\,dy\,dt, \tag{5.5}$$

since according to (5.2) and (5.1) the following limit relation holds

$$\int_0^T \alpha(t)\int_0^1 [w_n(y,t)-w(y,t)]\partial_y w_n(y,t)v(y,t)\,dy\,dt \to 0.$$

Further, according to (5.4) and (5.2) similarly to the previous one, we will have

$$\int_0^T w_n(1,t)w_n(1,t)v(1,t)\,dt = \int_0^T [w_n(1,t)-w(1,t)]w_n(1,t)v(1,t)\,dt+$$

$$+\int_0^T w(1,t)w_n(1,t)v(1,t)dt \to \int_0^T w^2(1,t)v(1,t)\,dt, \tag{5.6}$$

$$\int_0^T w_n(0,t)w_n(0,t)v(0,t)dt = \int_0^T [w_n(0,t)-w(0,t)]w_n(0,t)v(0,t)\,dt+$$

$$+\int_0^T w(0,t)w_n(0,t)v(0,t)dt \to \int_0^T w^2(0,t)v(0,t)\,dt. \tag{5.7}$$

So, by passing to the limit at $n\to\infty$ in integral identity (5.4), taking into account limit relations (5.5)–(5.7), as well as in initial condition (3.3), we get

$$\int_0^T\int_0^1 \left[\partial_t w + \alpha(t)w\partial_y w - \beta(t)\partial_y^2 w + \gamma(y,t)\partial_y w\right] v\,dy\,dt+$$

$$+\int_0^T \left[\beta(t)\partial_y w(1,t) - \frac{\alpha(t)}{3} w^2(1,t)\right] v(1,t)\,dt +$$

$$+\int_0^T \left[-\beta(t)\partial_y w(0,t) + \frac{\alpha(t)}{3} w^2(0,t)\right] v(0,t)\,dt =$$

$$= \int_0^T\int_0^1 gv\,dy\,dt,\ \ \forall\, v(y,t) \in L_2(0,T;H^1(0,1)). \tag{5.8}$$

$$\int_0^1 w(y,0)\psi(y)\,dy = 0,\ \ \forall\, \psi \in L_2(0,1). \tag{5.9}$$

Note that integral identity (5.8) is also valid for any test function $v(y,t) \in L_2(0,T;H_0^1(0,1)) \subset L_2(0,T;H^1(0,1))$, i.e. we have

$$\int_0^T\int_0^1 \left[\partial_t w + \alpha(t) w\partial_y w - \beta(t)\partial_y^2 w + \gamma(y,t)\partial_y w - g\right] v\,dy\,dt = 0,\ \ \forall\, v(y,t) \in L_2(0,T;H_0^1(0,1)). \tag{5.10}$$

Further, returning to (5.8) and taking into account that traces $v(1,t)$ and $v(0,t)$ from $L_2(0,T)$ of test function $v \in L_2(0,T;H^1(0,1))$ are independent of each other and are arbitrary, in this case the following identities

$$\int_0^T \left[\beta(t)\partial_y w(1,t) - \frac{\alpha(t)}{3} w^2(1,t)\right] \psi_1(t)\,dt = 0,\ \ \forall\, \psi_1(t) \in L_2(0,T), \tag{5.11}$$

$$\int_0^T \left[-\beta(t)\partial_y w(0,t) + \frac{\alpha(t)}{3} w^2(0,t)\right] \psi_0(t)\,dt = 0,\ \ \forall\, \psi_0(t) \in L_2(0,T), \tag{5.12}$$

follow from (5.8), that is, the integrands in square brackets from (5.10)–(5-12) define zero functionals over spaces $L_2(0,T;H_0^1(0,1))$ and $L_2(0,T)$, and belong to spaces $0 \in L_2(0,T;H^{-1}(0,1)) \subset \mathcal{D}'(Q_{yt})$ and $0 \in L_2(0,T) \subset \mathcal{D}'((0,T))$. Thus, from (5.10)–(5.12) we obtain that the weak limit function $w(y,t)$ satisfies equation (1.1) and boundary conditions (1.2), and from (5.9), it follows that it satisfies initial condition (1.3). This completes the proof of Lemma 5.1.

Lemma 5.2. Under the conditions of Lemma 5.1 the solution $w \in H^{2,1}(Q_{yt})$ of initial boundary value problem (1.1)–(1.3) is unique.

Proof. Let initial boundary value problem (1.1)–(1.3) have two different solutions $w^{(1)}(y,t)$ and $w^{(2)}(y,t)$. Then their difference $w(y,t) = w^{(1)}(y,t) - w^{(2)}(y,t)$ will satisfy the following homogeneous problem:

$$\partial_t w + \alpha(t) w\partial_y w^{(1)} + \alpha(t) w^{(2)}\partial_y w - \beta(t)\partial_y^2 w = 0, \tag{5.13}$$

$$\left[\frac{\alpha(t)}{3} w\left(w^{(1)} + w^{(2)}\right) - \beta(t)\partial_y w\right]\Big|_{y=0} = 0, \tag{5.14}$$

$$\left[\frac{\alpha(t)}{3} w\left(w^{(1)} + w^{(2)}\right) - \beta(t)\partial_y w\right]\Big|_{y=1} = 0. \tag{5.15}$$

According to Lemmas 4.1 and 4.2 we have

$$w^{(i)}(y,t) \in L_\infty(0,T;H^1(0,1)) \cap L_2(0,T;H^2(0,1)),\ i = 1,2. \tag{5.16}$$

Multiplying equation (5.13) by function $w(y,t)$ scalarly in $L_2(0,1)$ and taking into account (5.14)–(5.16), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(y,t)\|^2_{L_2(0,1)} + \beta_1\|\partial_y w(y,t)\|^2_{L_2(0,1)} \le \frac{\alpha(t)}{3}|w(1,t)|^2\left[w^{(1)}(1,t) + w^{(2)}(1,t)\right] +$$

$$+\frac{\alpha(t)}{3}|w(0,t)|^2\left[w^{(1)}(0,t)+w^{(2)}(0,t)\right]-\alpha(t)\int\limits_0^1\left[w\partial_y w^{(1)}+w^{(2)}\partial_y w\right]\,dy. \tag{5.17}$$

Now, we estimate the right-hand side of (5.17). According to (5.16) and by Lemma 4.1, we have:

$$\frac{\alpha(t)}{3}\left[w^{(1)}(1,t)+w^{(2)}(1,t)\right]|w(1,t)|^2\le$$
$$\le\frac{\alpha_2}{3}\left[\|w^{(1)}(1,t)\|_{L_\infty(0,T)}+\|w^{(2)}(1,t)\|_{L_\infty(0,T)}\right]|w(1,t)|^2\le C_1|w(1,t)|^2, \tag{5.18}$$
$$\frac{\alpha(t)}{3}\left[w^{(1)}(0,t)+w^{(2)}(0,t)\right]|w(0,t)|^2\le$$
$$\le\frac{\alpha_2}{3}\left[\|w^{(1)}(0,t)\|_{L_\infty(0,T)}+\|w^{(2)}(0,t)\|_{L_\infty(0,T)}\right]|w(0,t)|^2\le C_2|w(0,t)|^2, \tag{5.19}$$

Further, we have

$$\alpha(t)\int\limits_0^1\left[w^2\partial_y w^{(1)}+w^{(2)}w\partial_y w\right]\,dy=\alpha(t)\left[|w(1,t)|^2w^{(1)}(1,t)-|w(0,t)|^2w^{(1)}(0,t)\right]+$$
$$+\alpha(t)\int\limits_0^1\left[-2w^{(1)}w\partial_y w+w^{(2)}w\partial_y w\right]\,dy\le C_3|w(1,t)|^2+C_4|w(0,t)|^2+$$
$$+\frac{\alpha_2^2}{\beta_1}\left[2\|w^{(1)}\|_{L_\infty(Q_{yt})}+\|w^{(2)}\|_{L_\infty(Q_{yt})}\right]^2\|w\|^2_{L_2(0,1)}+\frac{\beta_1}{4}\|\partial_y w\|^2_{L_2(0,1)}\le$$
$$\le C_3|w(1,t)|^2+C_4|w(0,t)|^2+C_5\|w(y,t)\|^2_{L_2(0,1)}+\frac{\beta_1}{4}\|\partial_y w\|^2_{L_2(0,1)}. \tag{5.20}$$

We need the following estimates

$$(C_1+C_3)|w(1,t)|^2\le(C_1+C_3)\|w(y,t)\|^2_{L_\infty(0,1)}\le$$
$$\le(C_1+C_3)B\|w(y,t)\|_{H^1(0,1)}\|w(y,t)\|_{L_2(0,1)}\le$$
$$\le\frac{\beta_1}{8}\|\partial_y w(y,t)\|^2_{L_2(0,1)}+\left[\frac{\beta_1}{8}+\frac{2(C_1+C_3)^2B^2}{\beta_1}\right]\|w(y,t)\|^2_{L_2(0,1)}, \tag{5.21}$$
$$(C_2+C_4)|w(0,t)|^2\le(C_2+C_4)\|w(y,t)\|^2_{L_\infty(0,1)}\le$$
$$\le(C_2+C_4)B\|w(y,t)\|_{H^1(0,1)}\|w(y,t)\|_{L_2(0,1)}\le$$
$$\le\frac{\beta_1}{8}\|\partial_y w(y,t)\|^2_{L_2(0,1)}+\left[\frac{\beta_1}{8}+\frac{2(C_2+C_4)^2B^2}{\beta_1}\right]\|w(y,t)\|^2_{L_2(0,1)}, \tag{5.22}$$

where B is the norm of the embedding operator $H^1(0,1)\hookrightarrow L_\infty(0,1)$.

Based on relations (5.17)–(5.22) we establish

$$\frac{d}{dt}\|w(y,t)\|^2_{L_2(0,1)}+\beta_1\|\partial_y w(y,t)\|^2_{L_2(0,1)}\le$$
$$\le\left[\frac{\beta_1}{2}+\frac{4B^2}{\beta_1}\left((C_1+C_3)^2+(C_2+C_4)^2\right)+2C_5\right]\|w(y,t)\|^2_{L_2(0,1)},\ \ \forall t\in(0,T].$$

Hence, applying Gronwall's inequality, we obtain:

$$\|w(y,t)\|^2_{L_2(0,1)}\equiv 0,\ \forall t\in(0,T].$$

This means that $w^{(1)}(y,t)\equiv w^{(2)}(y,t)$ in $L_2(Q_{yt})$, i.e. the solution to initial boundary value problem (1.1)–(1.3) can be only one. Lemma 5.2 is completely proved.

From the statements of Lemmas 5.1 and 5.2 follows the validity of Theorem 1.1.

Thus, we have proved the main result of our work — Theorem 1.1.

Conclusion

In this paper, we have established a theorem on the unique solvability in Sobolev classes of a Neumann-type boundary value problem for the Burgers equation in a rectangular domain. The established results can be useful in the problems of modeling (a) nonlinear thermal fields in high voltage contact devices, (b) nonlinear processes of diffusion and propagation of foreign inclusions in the flows of water and atmospheric areas, etc.

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М.Т. Жиенәлиев[1], М.Ғ. Ерғалиев[1], Ә.Ә. Әсетов[2], А.Қ. Қалибекова[1]

[1] *Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[2] *Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан*

Тікбұрышты облыстағы Бюргерс теңдеуі үшін бір бастапқы шекаралық есеп туралы

Тікбұрышты облыстағы Бюргерс теңдеуінің кейбір бастапқы шекаралық есептері қарастырылған, бір мағынада оны моделді ретінде қабылдауға болады. Шындығында, мұндай мәселе көбінесе қозғалатын шекаралары бар облыстардағы Бюргерс теңдеулерін зерттеу кезінде туындайды. Айтылғанды растау үшін, мыналарға жүгінуге болады: [1] және [2] жұмыстарға. Функционалдық талдау әдістерін, Фаэдо-Галеркин және априорлық бағалау әдістерін қолдана отырып, Соболев кеңістігінде және тікбұрышты облыста Нейманн типіндегі сызықтық емес шекаралық шарттармен берілген Бюргерс теңдеуі үшін бастапқы шекаралық есептің дұрыс қойылғандығы көрсетілген.

Кілт сөздер: Бюргерс теңдеуі, шекаралық есеп, Соболевтік кеңістік, тікбұрышты облыс, Галеркин әдісі, априорлық бағалаулар.

М.Т. Дженалиев[1], М.Г. Ергалиев[1], А.А. Асетов[2], А.К. Калибекова[1]

[1] *Институт математики и математического моделирования, Алматы, Казахстан;*
[2] *Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан*

Об одной начальной граничной задаче для уравнения Бюргерса в прямоугольной области

В статье рассмотрена некоторая начально-граничная задача для уравнения Бюргерса в прямоугольной области, которую в известном смысле можно принять за модельную. Дело в том, что такая проблема часто возникает при изучении уравнения Бюргерса в областях с движущимися границами. В подтверждение сказанного можно сослаться на работы [1 и 2]. С помощью методов функционального анализа, априорных оценок и Фаэдо–Галеркина в пространствах Соболева и в прямоугольной области авторами показана корректность начально-граничной задачи для уравнения Бюргерса с нелинейными граничными условиями типа Неймана.

Ключевые слова: уравнение Бюргерса, граничная задача, пространство Соболева, прямоугольная область, метод Галеркина, априорные оценки.

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B.J. Kadirkulov[1,*], M.A. Jalilov[2]

[1] *Tashkent State University of Oriental Studies, Tashkent, Uzbekistan;*
[2] *Ferghana State University, Ferghana, Republic of Uzbekistan*
(E-mail: kadirkulovbj@gmail.com, alimuhammad9978@mail.ru)

On a nonlocal problem for a fourth-order mixed-type equation with the Hilfer operator

The present work is devoted to the study of the solvability questions for a nonlocal problem with an integro-differential conjugation condition for a fourth-order mixed-type equation with a generalized Riemann-Liouville operator. Under certain conditions on the given parameters and functions, we prove the theorems of uniqueness and existence of the solution to the problem. In the paper, given example indicates that if these conditions are violated, the formulated problem will have a nontrivial solution. To prove uniqueness and existence theorems of a solution to the problem, the method of separation of variables is used. The solution to the problem is constructed as a sum of an absolutely and uniformly converging series in eigenfunctions of the corresponding one-dimensional spectral problem. The Cauchy problem for a fractional equation with a generalized integro-differentiation operator is studied. A simple method is illustrated for finding a solution to this problem by reducing it to an integral equation equivalent in the sense of solvability. The authors of the article also establish the stability of the solution to the considered problem with respect to the nonlocal condition.

Keywords: mixed-type equation, nonlocal boundary value problem, the existence and uniqueness of a solution, fractional differentiation operator, the Hilfer operator, the Mittag-Leffler function.

1.Introduction and Problem Statement

Let $\Omega = \{(x,t) : 0 < x < 1, -a < t < b\}$, $\Omega_1 = \Omega \cap (t > 0)$, $\Omega_2 = \Omega \cap (t < 0)$, where a, b are positive real numbers. Considering this domain for the mixed-type equation

$$0 = \begin{cases} \dfrac{\partial^4 u}{\partial x^4} + D^{\alpha,\gamma} u, \ t > 0, \\ \dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^2 u}{\partial t^2}, \ t < 0, \end{cases} \tag{1}$$

the following nonlocal problem is considered.

Problem A. To find the function $u(x,t)$, which belongs to the class

$$t^{1-\gamma} D^{\alpha,\gamma} u, t^{1-\gamma} \frac{\partial^k u}{\partial x^k} \in C(\bar{\Omega}_1), \frac{\partial^k u}{\partial x^k} \in C(\bar{\Omega}_2), u_{tt} \in C(\Omega_2), u_{xxxx} \in C(\Omega_1 \cup \Omega_2), \tag{2}$$

where $k = \overline{0,2}$, satisfies equation (1) in the domain $\Omega_1 \cup \Omega_2$, the boundary conditions

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, t \in [-a, 0] \cup (0, b], \tag{3}$$

$$u_t(x,-a) = D^{\alpha,\gamma} u(x,b) + \varphi(x), \ 0 \le x \le 1, \tag{4}$$

and the gluing conditions

$$\lim_{t\to+0} J_{0+}^{1-\gamma} u(x,t) = \lim_{t\to-0} u(x,t), \ \lim_{t\to+0} J_{0+}^{1-\alpha} \frac{d}{dt} J_{0+}^{1-\gamma} u(x,t) = \lim_{t\to-0} u_t(x,t). \tag{5}$$

*Corresponding author.
E-mail: kadirkulovbj@gmail.com

Here $\varphi(x)$ is a given sufficiently smooth function, $D^{\alpha,\gamma}, 0 < \alpha \leq \gamma \leq 1$ is a generalized fractional differentiation operator (the definition of this operator is provided below).

For a function $\varphi(t)$, given on (a,b), $-\infty < a < b < \infty$, the expression

$$I_{a+}^{\alpha}\varphi(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{a}^{x} (x-t)^{\alpha-1}\varphi(t)dt, \; x \in (a,b)$$

is said to be the fractional Riemann–Liouville integral of the order $\alpha > 0$ [1, Vol. 1, p. 25]. Here $\Gamma(\alpha)$ is the Euler gamma function. Let $n-1 < \alpha \leq n$, $n \in N$. The fractional Riemann–Liouville derivative of a function $\varphi(t)$ of the order α is defined by the formula [1, Vol. 1, p. 27]:

$$D_{a+}^{\alpha}\varphi(x) = \frac{d^n}{dx^n} I_{a+}^{n-\alpha}\varphi(x), x \in (a,b).$$

The fractional Caputo derivative of a function $\varphi(t)$ of the order α is defined as follows [1, Vol. 1, p. 34]:

$${}_*D_{a+}^{\alpha}\varphi(x) = I_{a+}^{n-\alpha}\varphi^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)} \int\limits_{a}^{x} \frac{\varphi^{(n)}(t)dt}{(x-t)^{\alpha-n+1}}.$$

For $\alpha = n \in N$, these derivatives are reduced to the derivatives of integer order [1, Vol. 1, p. 27, 34]:

$$D_{a+}^{n}\varphi(x) = {}_*D_{a+}^{n}\varphi(x) = \frac{d^n\varphi}{dx^n}.$$

The generalized Riemann–Liouville derivative of the fractional order α, $n-1 < \alpha \leq n$, $n \in N$ and type β, $0 \leq \beta \leq 1$ (otherwise, the Hilfer fractional deivative) of a function $f(t)$ is defined by the formula

$$D_{a+}^{\alpha,\beta}\varphi(x) = I_{a+}^{\beta(n-\alpha)} \frac{d^n}{dx^n} I_{a+}^{(1-\beta)(n-\alpha)}\varphi(x).$$

Hence, it follows that for $\beta = 0$ the Hilfer fractional deivative coincides with the Riemann–Liouville operator $(D_{a+}^{\alpha,0} = D_{a+}^{\alpha})$, and in the case of $\beta = 1$, we obtain the fractional Caputo derivative, that is $D_{a+}^{\alpha,1} = {}_*D_{a+}^{\alpha}$.

Thus, the operator $D^{\alpha,\gamma}$ is a continuous interpolation of the well-known Riemann-Liouville and Caputo differentiation operators of fractional order.

Further, for the convenience of notation, we will use another notation for the Hilfer fractional derivative, i.e., $D^{\alpha,\gamma} = D_{a+}^{\alpha,\beta}$ where $\gamma = \alpha + \beta n - \alpha\beta$ and $\alpha \leq \gamma \leq n$.

The generalized operator $D^{\alpha,\gamma}$ was first introduced by Hilfer [2]. Applying the integral transforms of Fourier, Laplace, and Mellin, he investigated the Cauchy problem for the diffusion equation with a generalized operator $D^{\alpha,\gamma}$, the solution of which was represented in terms of the Fox H-functions.

In [3], boundary value problems were investigated for the fractional diffusion equation with a time generalized fractional Riemann-Liouville derivative. To solve the problem, in the finite domain, the Laplace method of separation of variables and transform was used. The solution was obtained in the form of an infinite series containing the Mittag-Leffler functions, and the asymptotic behavior of the solution was found at infinity. In an infinite domain with respect to a spatial variable, using the Fourier-Laplace transform method, the Cauchy problem was solved, and the fundamental solution to the Cauchy problem was found.

In [4], analytical and numerical solutions of boundary value problems were investigated for the fractional diffusion equation with fractional Hilfer time derivative and spatial fractional Riesz-Feller derivative. The Laplace and Fourier transform methods were applied to solve the problem, and solutions were presented using the Mittag-Leffler functions and Fox H-functions. The numerical solution of the problem was also considered by approximating fractional derivatives with fractional Grunwald-Letnikov derivatives.

In [5], the properties of the Hilfer operator were investigated in a special functional space, and an operational method was developed for solving fractional differential equations with this operator in the same space. Elaborating the results of [5], the authors of [6] developed an operational method for solving fractional differential equations containing a finite linear combination of the Hilfer operators.

In [7], the source identification problem was investigated for the generalized diffusion equation with the generalized integro-differentiation operator. We also note the work [8], where inverse problems were studied for a generalized fourth-order parabolic equation with the operator $D^{\alpha,\gamma}$.

Note that various models of practical problems using fractional calculus are constructed in [1], [9-12]. More detailed information, as well as a bibliography on the Hilfer fractional derivative and its properties, can be found in the monograph [13], where the theory of fractional integro-differentiation, including the Hilfer fractional derivative, is systematically presented.

Nonlocal problems are arisen in the study of various problems of mathematical biology, forecasting soil moisture, physics, and plasma problems. More detailed information on nonlocal problems can be found in the monograph [14]. With regard to nonlocal problems for mixed-type equations, significant results in this direction were obtained by K.B. Sabitov and his students [15–18]. Note that a nonlocal condition of type (3) takes place when simulating the problems of flow around airfoils by a flow of subsonic velocity with a supersonic zone [15]. It should also be noted the papers [19], [20], where nonlocal problems for equations of mixed type with the generalized in time fractional Riemann-Liouville derivative were studied.

In this paper, we study a nonlocal boundary value problem for a mixed-type equation with a Hilfer operator of fractional order, which is a further development and generalization of the results from [16], [21].

2. On the solution of the Cauchy problem for a fractional-order equation with the Hilfer operator

Consider the Cauchy problem for a fractional-order differential equation with the operator $D^{\alpha,\gamma}$

$$\begin{cases} D^{\alpha,\gamma}u(t) = \lambda u(t) + f(t),\, t \in (0,\ell), \\ \lim\limits_{t\to+0} \dfrac{d^k}{dt^k} J_{0+}^{n-\gamma} u(t) = u_k, k = 0, 1, ..n-1, \end{cases} \tag{6}$$

where $f(t)$ is a given function, $u_k = const$.

It should be noted that in [7], the Laplace method was used to solve this problem, and in [5], the solution of a more general problem in a special functional space was found applying the operational calculus. Here, in contrast to these works, we will show one simple way to solve the problem, which makes it possible to obtain the solution to this problem in an explicit form.

The following takes place:

Lemma 1. Let $t^{1-\gamma}f(t) \in C[0,\ell]$. Then a solution to problem (6) exists, is unique, belongs to the class $t^{1-\gamma}D^{\alpha,\gamma}u(t) \in C[0,\ell]$, and is represented in the form

$$u(t) = \sum_{k=0}^{n-1} u_k t^{\gamma+k-n} E_{\alpha,\gamma+k+1-n}(\lambda t^\alpha) + \int\limits_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) f(\tau)d\tau. \tag{7}$$

Here $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function, which has the form [22; 117], [1, Vol 1; 269]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\, k+\beta)},\;\; z,\alpha,\beta \in C,\;\mathrm{Re}(\alpha) > 0.$$

Proof. Taking into account the definition of the Hilfer operator, one can write the fractional-order differential equation corresponding to problem (6) in the following form:

$$J_{0+}^{\gamma-\alpha} D_{0+}^{\gamma} u(t) = \lambda u(t) + f(t).$$

Further, applying the operator J_{0+}^{α} to both sides of this equation, taking into account the linearity of J_{0+}^{α}, and also the formula [23; 75]

$$J_{0+}^{\delta} D_{0+}^{\delta} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^{\delta+k-n}}{\Gamma(\delta+k+1-n)} \lim_{x\to 0} \frac{d^k}{dt^k} J_{0+}^{n-\delta} u(t), \delta \in (n-1,n],$$

we obtain

$$u(t) - \sum_{k=0}^{n-1} \frac{t^{\gamma+k-n} u_k}{\Gamma(\gamma+k+1-n)} = \lambda J_{0+}^{\alpha} u(t) + J_{0+}^{\alpha} f(t). \tag{8}$$

Thus, we have reduced the solution of problem (6) to the solution of the Volterra integral equation of the second kind of the form (8).

Further, using Theorem 3.1 from [22; 123], we represent a solution of equation (8) in the form

$$u(t)=\sum_{k=0}^{n-1}\frac{t^{\gamma+k-n}u_k}{\Gamma(\gamma+k+1-n)}+J_{0+}^{\alpha}f(t)+$$

$$+\lambda\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})\left[\sum_{k=0}^{n-1}\frac{\tau^{\gamma+k-n}u_k}{\Gamma(\gamma+k+1-n)}+J_{0+}^{\alpha}f(\tau)\right]d\tau. \tag{9}$$

Denote

$$I_1(t)=\sum_{k=0}^{n-1}\frac{u_k}{\Gamma(\gamma+k+1-n)}\left(t^{\gamma+k-n}+\lambda\cdot\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})\tau^{\gamma+k-n}d\tau\right),$$

$$I_2(t)=J_{0+}^{\alpha}f(t)+\lambda\cdot\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})J_{0+}^{\alpha}f(\tau)d\tau.$$

Changing variables by the formula $s=t-\tau$, using formulas [22]

$$E_{\alpha,\mu}(z)=\frac{1}{\Gamma(\mu)}+z\cdot E_{\alpha,\mu+\alpha}(t),\ \alpha>0,\ \mu>0, \tag{10}$$

$$\frac{1}{\Gamma(\nu)}\int\limits_0^z(z-t)^{\nu-1}E_{\alpha,\beta}(\lambda t^{\alpha})t^{\beta-1}dt=z^{\beta+\nu-1}\cdot E_{\alpha,\beta+\nu}(\lambda z^{\alpha}),\nu>0,\ \beta>0, \tag{11}$$

the first integral can be easily reduced to the form

$$I_1(t)=\sum_{k=0}^{n-1}u_kt^{\gamma+k-n}E_{\alpha,\gamma+k+1-n}(\lambda t^{\alpha}). \tag{12}$$

Further, transform the second term in the expression for $I_2(t)$ to the form

$$\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})J_{0+}^{\alpha}f(\tau)d\tau=$$

$$=\frac{1}{\Gamma(\alpha)}\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})d\tau\int\limits_0^{\tau}(\tau-s)^{\alpha-1}f(s)ds=$$

$$=\frac{1}{\Gamma(\alpha)}\int\limits_0^t f(s)ds\int\limits_s^t(t-\tau)^{\alpha-1}(\tau-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})d\tau.$$

Taking into account formula (11), we represent the inner integral in the form

$$\int\limits_s^t(t-\tau)^{\alpha-1}(\tau-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})d\tau=\Gamma(\alpha)(t-\tau)^{2\alpha-1}E_{\alpha,2\alpha}(\lambda(t-\tau)^{\alpha}).$$

Further, considering formula (10), represent $I_2(t)$ in the form

$$I_2(t)=\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})f(\tau)d\tau. \tag{13}$$

We obtain formula (7) from (9), (12), and (13). The uniqueness of the solution to the problem follows from the method for constructing the solution, and its smoothness follows from the representation of the solution (7), as well as from the results of [5]. Lemma 1 is proved.

3. The existence, uniqueness and stability of a solution to Problem A.

To solve the problem, we will apply the spectral method. We will look for a solution to Problem A in the form of $u(x,t) = X(x) \cdot T(t)$. Substituting this expression into equation (1) and boundary conditions (3), we obtain the following spectral problem:

$$X^{IV}(x) - \lambda^4 X(x) = 0, X(0) = X(1) = X''(0) = X''(1) = 0.$$

The problem under consideration is self-adjoint, has a complete system of eigenfunctions in $L_2(0,1)$ of the form

$$X_n(x) = \sqrt{2}\sin\lambda_n x, \tag{14}$$

which forms a basis in $L_2(0,1)$. Here $\lambda_n = \pi n$, $n \in N$.

3.1. Uniqueness of the solution to problem A.

Let $u\,(x,t)$ be a solution to Problem A. Consider the following functions:

$$u_n^+(t) = \sqrt{2}\int\limits_0^1 u(x,t)\sin\lambda_n x dx, n = 1,2,... \tag{15}$$

$$u_n^-(t) = \sqrt{2}\int\limits_0^1 u(x,t)\sin\lambda_n x dx, n = 1,2,.... \tag{16}$$

Applying the operator $D^{\alpha,\gamma}$ to both sides of equality (15) with respect to t at $t \in (0;b)$ and differentiating equality (16), with respect to t twice at $t \in (-a;0)$, and also taking into account equation (1), we obtain

$$D^{\alpha,\gamma}u_n^+(t) = \sqrt{2}\int\limits_0^1 D^{\alpha,\gamma}u(x,t)\sin\lambda_n x dx = -\sqrt{2}\int\limits_0^1 u_{xxxx}(x,t)\sin\lambda_n x dx, \tag{17}$$

$$\frac{d^2u_n^-(t)}{dt^2} = \sqrt{2}\int\limits_0^1 u_{tt}(x,t)\sin\lambda_n x dx = -\sqrt{2}\int\limits_0^1 u_{xxxx}(x,t)\sin\lambda_n x dx. \tag{18}$$

In the integrals from the right-hand sides of equalities (17) and (18), integrating by parts four times, taking into account boundary conditions (2), we obtain the differential equations

$$D^{\alpha,\gamma}u_n^+(t) + \lambda_n^4 u_n^+(t) = 0, t > 0, \tag{19}$$

$$\frac{d^2}{dt^2}u_n^-(t) + \lambda_n^4 u_n^-(t) = 0, t < 0, \tag{20}$$

the general solutions of which have the form

$$u_n^{\pm}(t) = \begin{cases} A_n t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 t^{\alpha}),\ t > 0, \\ B_n\sin\lambda_n^2 t + L_n\cos\lambda_n^2 t,\ t < 0, \end{cases} \tag{21}$$

where $A_n, B_n, L_n, n = 1,2,...$ are arbitrary constants.

Taking into account conditions (4) and (5), we obtain from (15), (16) that functions $u_n^{\pm}(t)$ must satisfy the following conditions:

$$\lim_{t\to+0} J_{0+}^{1-\gamma}u_n^+(t) = \lim_{t\to-0} u_n^-(t),\ \lim_{t\to+0} J_{0+}^{1-\alpha}\left(\frac{d}{dt}J_{0+}^{1-\gamma}u_n^+(t)\right) = \lim_{t\to-0}\frac{du_n^-(t)}{dt}, \tag{22}$$

$$\frac{du_n^-(-a)}{dt} = D^{\alpha,\gamma}u_n^+(b) + \varphi_n \tag{23}$$

where

$$\varphi_n = \sqrt{2}\int\limits_0^1 \varphi(x)\sin\lambda_n x dx, n = 1, 2, ...$$

Further, satisfying functions (21) and conditions (22)–(23), we obtain the following system for finding the constants $A_n,\ B_n,\ L_n$:

$$\begin{cases} A_n = L_n, B_n = -\lambda_n^2 A_n, \\ L_n\lambda_n^2 \sin\lambda_n^2 a + B_n\lambda_n^2\cos\lambda_n^2 a + \lambda_n^4 A_n b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha) = \varphi_n. \end{cases} \quad (24)$$

The system, having the unique solution of the form

$$L_n = A_n, B_n = -\lambda_n^2 A_n,\ A_n = \frac{\varphi_n}{\lambda_n^2\Delta_n(a,b)}, \quad (25)$$

provided that for all $n \in N$ the equality

$$\Delta_n(a,b) \neq 0, \Delta_n(a,b) = \sin\lambda_n^2 a - \lambda_n^2\cos\lambda_n^2 a + \lambda_n^2 b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha) \quad (26)$$

holds.

Substituting (25) into (21), we finally obtain

$$u_n^{\pm}(t) = \begin{cases} \dfrac{\varphi_n}{\lambda_n^2\Delta_n(a,b)} t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 t^\alpha),\ t > 0, \\ \dfrac{\varphi_n}{\lambda_n^2\Delta_n(a,b)}\left(\cos\lambda_n^2 t - \lambda_n^2\sin\lambda_n^2 t\right),\ t \leq 0. \end{cases} \quad (27)$$

Using (27), it is easy to prove the uniqueness of the solution to *Problem A*. Indeed, let condition (26) be satisfied and $\varphi(x) \equiv 0$. Then $\varphi_n = 0$, and formulas (15), (16), and (27) imply

$$\int\limits_0^1 t^{1-\gamma}u(x,t)\sin\lambda_n x dx = 0, t \in [0,b],\ n = 1, 2, ...,$$

$$\int\limits_0^1 u(x,t)\sin\lambda_n x dx = 0, t \in [-a,0],\ n = 1, 2, ...$$

Further, taking into account the completeness of system (14) in $L_2(0,1)$, we conclude that $u(x,t) = 0$ almost everywhere on $[0,1]$ at any $t \in [-a,b]$. Since $t^{1-\gamma}u(x,t) \in C(\bar{\Omega}_1),\ u(x,t) \in C(\bar{\Omega}_2)$, we have $t^{1-\gamma}u(x,t) \equiv 0$ in $\overline{\Omega}$, that is, problem A has the unique solution in the class under consideration.

Thus, the following statement holds.

Theorem 1. If there exists a solution to Problem A, then, it is unique if and only if conditions (26) are satisfied for all $n \in N$.

Now let us consider the case when condition (26) is violated. Let $\Delta_m(a,b) = 0$ for some $a,\ b,\ \gamma \in (0,1]$, and $n = m$. Then the homogeneous Problem A (where $\varphi(x) \equiv 0$) has the nontrivial solution

$$V_m^{\pm}(x,t) = \sqrt{2}v_m^{\pm}(t)\sin\lambda_m x, \quad (28)$$

$$v_m^{\pm}(t) = \begin{cases} t^{\gamma-1}E_{\alpha,\gamma}(-\lambda_m^4 t^\alpha), t > 0, \\ \cos\lambda_m^2 t - \lambda_m^2\sin\lambda_m^2 t, t < 0. \end{cases}$$

Now, represent the expression $\Delta_n(a,b)$ in the form:

$$\Delta_n(a,b) = \sqrt{1+\lambda_n^4}\sin\left(\lambda_n^2 a - \rho_n\right) + \lambda_n^2 b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha), \quad (29)$$

where $\rho_n = \arcsin\left(\lambda_n^2/\sqrt{1+\lambda_n^4}\right)$ and $\rho_n \to \frac{\pi}{2}$ at $n \to +\infty$. From this, it can be seen that the expression $\Delta_n(a,b)$ vanishes only if

$$a = \frac{1}{\lambda_n^2}\left[(-1)^{k+1}\arcsin\frac{\lambda_n^2 b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha)}{\sqrt{1+\lambda_n^4}} + \pi k + \rho_n\right],\ n = 1, 2, ...$$

Since $\Delta_n(a,b)$ is the denominator of the fraction, for sufficiently large n, the expression $\Delta_n(a,b)$ can become sufficiently small, i.e., the problem of "small denominators" has arisen. Therefore, to substantiate the existence of a solution to this problem, it is necessary to show the existence of numbers a and b such that, for sufficiently large n, $\Delta_n(a,b)$ will be separated from zero.

Lemma 2. Let $\gamma \in (0,1]$, b be any positive real number, a be an irrational number such that either $a\pi \in N$ or $\pi a = \frac{p}{q} \in Q\backslash N$ where $p, q \in N, (p,q) = 1$ and q is an odd number. Then for large n there exists a positive constant B_0 such that the estimate

$$|\Delta_n(a,b)| \geq B_0 n^2 > 0 \tag{30}$$

is valid.

Proof. Let $\pi a = p \Leftrightarrow a = \frac{p}{\pi}, p \in N$. Then, we have from (26)

$$|\Delta_n(a,b)| = \lambda_n^2\left|(-1)^{a+1} + b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha)\right| \geq \lambda_n^2\left(1 - b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha)\right)$$

at all n and $b > 0$.

Here and below, we use the following properties of the Mittag-Leffler function:

1. For $\alpha, \beta \in (0,1], \alpha \leq \beta$, the function $E_{\alpha,\beta}(-z)$ is completely monotone on $(0,\infty)$ [1, Vol 1, p. 280].

2. Let $\alpha \in (0,2)$, β be a real constant, and $\arg z = \pi$. Then the inequality

$$|E_{\alpha,\beta}(z)| \leq \frac{M}{1+|z|} \tag{31}$$

takes place, where M is a positive constant independent of z [22; 136].

Then, it follows from (29) that there exists $n_0 \in N$ such that for all $n > n_0$ the inequality

$$\lambda_n^2\left(1 - b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha)\right) \geq B_1 n^2 > 0 \tag{32}$$

holds and hence $\Delta_n(a,b) \geq B_1 n^2 > 0$.

Let now $a = \frac{p}{q\pi}$, $p, q \in N$, $(q,p) = 1$, q be an odd number. Divide $n^2 p$ by q with a remainder: $n^2 p = sq + r$ where $s, r, \in N \cup \{0\}$, $0 \leq r < q$. Then expression (29) takes the form

$$\Delta_n(a,b) = \sqrt{1+\lambda_n^4}(-1)^{s+1}\cos\left(\frac{\pi r}{q} + \varepsilon_n\right) + \lambda_n^2 b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha) \tag{33}$$

where $\varepsilon_n = \arcsin\left(1/\sqrt{1+\lambda_n^4}\right) > 0$ and $\varepsilon_n \to 0$ at $n \to +\infty$.

Let $r = 0$. Then we have the case considered above. Let $r > 0$. Then $1 \leq r \leq q-1$, $q \geq 2$, and for large n $0 < \frac{\pi}{q} + \varepsilon_n \leq \frac{\pi r}{q} + \varepsilon_n \leq \pi - \frac{\pi}{q} + \varepsilon_n < \pi$.

Hence, it follows that if $q = 2l$, $l \in N$, then for $r = l$, we obtain that $\frac{\pi r}{q} + \varepsilon_n \to \frac{\pi}{2}$ at $n \to +\infty$, and if $q = 2l+1$, then $\frac{\pi r}{q} \neq \frac{\pi}{2}$ at any r from $[1, q-1]$. Since $\varepsilon_n \to 0$ and $b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha) \to 0$ at $n \to +\infty$, there exists a constant $n_1 > 0$ such that

$$\left|\cos\left(\frac{\pi r}{q} + \varepsilon_n\right)\right| - b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha) \geq \frac{1}{2}\left|\cos\frac{\pi r}{q}\right| - b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha) \geq B_2,$$

where $0 < B_2 < \frac{1}{2}\left|\cos\frac{\pi r}{q}\right|$. Then, taking into account these estimates, we obtain from (33) for $n > n_1$:

$$|\Delta_{a,b}| \geq \lambda_n^2\left(\cos\left(\frac{\pi r}{q} + \varepsilon_n\right) - b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha)\right) \geq$$

$$\geq \lambda_n^2\left(\frac{1}{2}\left|\cos\frac{\pi r}{q}\right| - b^{\gamma-1}E_{\alpha,\gamma}(-\lambda_n^4 b^\alpha)\right) \geq \pi^2 B_2 n^2. \tag{34}$$

It follows from (32) and (34) that (30) is valid for large n.

Lemma 2 is proved.

Note that the idea of the proof of Lemma 2 was borrowed from [18].

3.2. The existence of a solution to Problem A.

Let us turn to the proof of the existence of a solution to Problem A. The proof of the following lemma follows easily from (26), (30) and estimate (31).

Lemma 3. Let conditions (26) and (30) be satisfied. Then

1) if $t \in [0, b]$, then

$$t^{1-\gamma}\left|u_n^+(t)\right| \le \frac{B_3}{n^4}|\varphi_n|,\ \ t^{1-\gamma}\left|D^{\alpha,\gamma}u_n^+(t)\right| \le B_4\left|\varphi_n\right|,$$

2) if $t \in [-a, 0]$, then

$$\left|u_n^-(t)\right| \le \frac{B_5}{n^2}|\varphi_n|, \left|\frac{du_n^-(t)}{dt}\right| \le B_6|\varphi_n|, \left|\frac{d^2u_n^-(t)}{dt^2}\right| \le B_7 n^2|\varphi_n|,$$

here and below B_k, $k = \overline{1,7}$ are positive constants. Since system (14) is complete and forms a basis in $L_2(0,1)$, we look for a solution to Problem A in Ω in the form

$$u(x,t) = \begin{cases} \sqrt{2}\sum\limits_{n=1}^{\infty} u_n^+(t)\sin\lambda_n x,\ \ (x,t)\in\Omega_1, \\ \sqrt{2}\sum\limits_{n=1}^{\infty} u_n^-(t)\sin\lambda_n x,\ \ (x,t)\in\Omega_2, \end{cases} \tag{35}$$

where $u_n^{\pm}(t)$ are unknown functions. It is not difficult to see that, substituting function (35) into (1) and satisfying conditions (3)-(5) with respect to the sought functions, we obtain problem (19), (20), (22), (23), the solution of which has the form (27).

Thus, the solution to the problem can be represented in the form (35), where $u_n^{\pm}(t)$ are determined by formulas (27). Now, it remains to prove the legality of all these actions. For this, we formally compose the series from (35), using term-by-term differentiation

$$D^{\alpha,\gamma}u(x,t) = \sum_{n=1}^{\infty} D^{\alpha,\gamma}u_n^+(t)X_n(x),\ t>0, \tag{36}$$

$$\frac{\partial^k u(x,t)}{\partial x^k} = \sum_{n=1}^{\infty} u_n^+(t)\frac{d^k X_n(x)}{dx^k},\ \ k=\overline{1,4},\ t>0, \tag{37}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \sum_{n=1}^{\infty} \frac{d^2u_n^-(t)}{dt^2}X_n(x),\ t<0, \tag{38}$$

$$\frac{\partial^k u(x,t)}{\partial x^k} = \sum_{n=1}^{\infty} u_n^-(t)\frac{d^k X_n(x)}{dx^k},\ \ k=\overline{1,4},\ t<0. \tag{39}$$

Taking into account Lemmas 2, 3 one can easily see that series (38), (39), and series

$$\sum_{n=1}^{\infty} t^{1-\gamma}D^{\alpha,\gamma}u_n^+(t)X_n(x),\ \sum_{n=1}^{\infty} t^{1-\gamma}u_n^+(t)\frac{d^k X_n(x)}{dx^k}, k=\overline{0,4}\ t>0, \tag{40}$$

which are obtained from (36) and (37) by term-by-term multiplication by , are majorized by the series

$$\sum_{n=1}^{\infty} n^2|\varphi_n|. \tag{41}$$

Therefore, we investigate the convergence of series (41). Taking into account the relation

$$\varphi_n = \frac{1}{(\pi n)^3}\varphi_n^{(3)} = -\frac{\sqrt{2}}{(\pi n)^3}\int\limits_0^1 \varphi'''(x)\cos\lambda_n x dx,$$

as well as using the Cauchy-Schwarz and Bessel inequality, we have

$$\sum_{n=1}^{\infty} n^2|\varphi_n| \le \sum_{n=1}^{\infty}\frac{1}{n}|\varphi_n^{(3)}| \le \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\cdot\left(\sum_{n=1}^{\infty}|\varphi_n^{(3)}|^2\right)^{1/2} \le C\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)},$$

where $\sum\limits_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. Then, by the Weierstrass theorem, series (38)–(40) converge absolutely and uniformly in domains $\bar{\Omega}_1$ and $\bar{\Omega}_2$, respectively.

There fore it follows that the function $u(x,t)$, defined by series (36), belongs to class (2), and also satisfies the conditions (3)–(5).

Now, let $\Delta_n(a,b) = 0$ at some and $n = k_1, ..., k_s$, $1 \le k_1 < k_1 < ... < k_s, s \in N.$. Then the fulfillment of the orthogonality conditions is necessary and sufficient for the solvability of system (24)

$$\varphi_n = \sqrt{2}\int\limits_0^1 \varphi(x)\sin\lambda_n x dx = 0, n = k_1, ...k_s. \quad (42)$$

In this case, the solution to Problem is defined as the sum of the series

$$u(x,t) = \sqrt{2}\left[\sum_{n=1}^{k_1-1} + \sum_{n=k_1+1}^{k_2-1} + ... + \sum_{n=k_s+1}^{\infty} +\right] u_n^{\pm}(t)\sin\lambda_n x + \sum_m C_m V_m^{\pm}(t), \quad (43)$$

where in the last sum $m = k_1, ...k_s$, C_m are arbitrary constants, functions $V_m^{\pm}(t)$ are determined from formula (28).

Thus, the following statement holds.

Theorem 2. Let $\varphi(x) \in C^2[0,1], \varphi^{(3)}(x) \in L_2(0,1)$, $\varphi^{(2k)}(0) = \varphi^{(2k)}(1) = 0$, $k = \overline{0,1}$. Then Problem A is uniquely solvable if only if conditions (26) and (30) hold, and the solution is determined by series (35). If $\Delta_n(a,b) = 0$ for some a, b, γ, and $n = k_1, ..., k_s$, moreover condition (30) takes place, then Problem A is solvable only if orthogonality conditions (42) take place. In this case, the solution is determined by series (43).

3.3. Stability of the solution to Problem A.

Now, let us establish the stability of the solution to problem A with respect to its nonlocal condition (4). Let

$$||u(x,t)||_{C(\overline{\Omega})} = ||t^{1-\gamma}u(x,t)||_{C(\bar{\Omega}_1)} + ||u(x,t)||_{C(\bar{\Omega}_2)},$$

$$||u(x,t)||_{L_2(\Omega)} = ||t^{1-\gamma}u(x,t)||_{L_2(\Omega_1)} + ||u(x,t)||_{L_2(\Omega_2)},$$

where

$$||v(x,t)||_{C(\overline{\Omega})} = \max_{\overline{\Omega}}|v(x,t)|,$$

$$||f(x)||_{C[0,1]} = \max_{[0,1]}|f(x)|,$$

$$||v(x,t)||_{L_2(\Omega)} = \left(\iint\limits_{\Omega}|v(x,t)|^2 dxdt\right)^{1/2}, ||\varphi(x)||_{L_2(0,1)} = \left(\int\limits_0^1|\varphi(x)|^2 dx\right)^{1/2}.$$

Theorem 3. Let the conditions of Theorem 2 be satisfied. Then the solution to Problem A satisfies the estimate

$$||u(x,t)||_{C(\overline{\Omega})} \le C||\varphi(x)||_{C[0,1]}, \quad (44)$$

$$||u(x,t)||_{L_2(\Omega)} \le C||\varphi(x)||_{L_2(0,1)}. \tag{45}$$

In what follows, C will mean an arbitrary constant, the value of which is not of interest to us.

Proof. Let (x,t) be an arbitrary point from the domain $\bar{\Omega}_2$. Then using formulas (27), (35), on the base of Lemma 3 and the Cauchy-Bunyakovsky inequality, we obtain the estimate:

$$|u(x,t)| \le \sqrt{2}\sum_{n=1}^{\infty}|u_n^-(t)| \le \sqrt{2}B_5\sum_{n=1}^{\infty}\frac{1}{n^2}|\varphi_n|,$$

it easily follows from this that

$$||u(x,t)||_{C(\bar{\Omega}_2)} \le C||\varphi(x)||_{C[0,1]}. \tag{46}$$

In the same way, in the case of $(x,t) \in \bar{\Omega}_1$, we obtain the estimate

$$||t^{1-\gamma}u(x,t)||_{C(\bar{\Omega}_1)} \le C||\varphi(x)||_{C[0,1]}. \tag{47}$$

Estimate (44) follows from (46) and (47). Now we prove estimate (45). Since system (14) is orthonormal in $L_2(0,1)$, we get from (35), using the Parseval equality,

$$||u(x,t)||^2_{L_2(\Omega_2)} = \left(\sum_{n=1}^{\infty}u_n(t)X_n(x), \sum_{m=1}^{\infty}u_m(t)X_m(x)\right)_{L_2(\Omega_2)} = \sum_{n=1}^{\infty}\|u_n(t)\|^2_{L_2(-a,0)}.$$

Hence, based on Lemma 3, we get

$$||u(x,t)||_{L_2(\Omega_2)} \le C||\varphi(x)||_{L_2(0,1)}. \tag{48}$$

In a similar way we obtain the estimate

$$||t^{1-\gamma}u(x,t)||_{L_2(\Omega_1)} \le C||\varphi(x)||_{L_2(0,1)}. \tag{49}$$

Estimate (45) follows from (48) and (49). Therefore, solution (35) continuously depends on the function $\varphi(x)$.

Theorem 3 is proved.

4. Some examples

Example 1. Consider Problem A at $\gamma = 1$. Then $D^{\alpha,\gamma} = D^{\alpha,1} = {}_CD^{\alpha}$ and equation (1) has the form

$$0 = \begin{cases} \dfrac{\partial^4 u}{\partial x^4} + {}_CD^{\alpha}u, \ t > 0, \\ \dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^2 u}{\partial t^2}, \ t < 0, \end{cases} \tag{50}$$

we obtain Problem A for equation (50) with the Caputo operator ${}_CD^{\alpha}$, the solution of which has the form

$$u(x,t) = \begin{cases} \displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2}\varphi_n}{\lambda_n^2\Delta_n(a,b)}E_{\alpha,1}(-\lambda_n^4 t^{\alpha})\sin\lambda_n x, \ (x,t) \in \Omega_1, \\ \displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2}\varphi_n}{\lambda_n^2\Delta_n(a,b)}\left(\cos\lambda_n^2 t - \lambda_n^2\sin\lambda_n^2 t\right)\sin\lambda_n x, \ (x,t) \in \Omega_2. \end{cases}$$

Example 2. Consider the case of $\gamma = \alpha = 1$. Then $D^{\alpha,\gamma} = D^{1,1} = \frac{d}{dt}$ and equation (1) has the form

$$0 = \begin{cases} \dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial u}{\partial t}, \ t > 0, \\ \dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^2 u}{\partial t^2}, \ t < 0, \end{cases}$$

we obtain the integer partial differential equation. Since $E_{1,1}(z)=e^z$, the solution of the obtained problem has the form

$$u(x,t)=\begin{cases}\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2}\varphi_n}{\lambda_n^2\Delta_n(a,b)}e^{-\lambda_n^4 t^{\alpha}}\sin\lambda_n x,\ (x,t)\in\Omega_1,\\ \displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2}\varphi_n}{\lambda_n^2\Delta_n(a,b)}\left(\cos\lambda_n^2 t-\lambda_n^2\sin\lambda_n^2 t\right)\sin\lambda_n x,\ (x,t)\in\Omega_2.\end{cases}$$

Conclusion

In the paper, we established a criterion for the existence and uniqueness of a regular solution to a nonlocal problem for a fourth-order mixed-type differential equation with the Hilfer operator in a rectangular domain. For this, we used the spectral method, which allowed to construct a solution to the nonlocal problem (1)–(5) in the form of a Fourier series. Next, we proveed the stability of the obtained solution with respect to the problem data. Moreover, we provided a simple way to solve the Cauchy problem for a fractional differential equation with a generalized Riemann - Liouville derivative.

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Б.Ж. Кадиркулов[1], М.А. Жалилов[2]

[1] *Ташкент мемлекеттік шығыстану университеті, Ташкент, Өзбекстан;*
[2] *Фергана мемлекеттік университеті, Фергана, Өзбекстан*

Төртінші ретті аралас типті теңдеу үшін бір бейлокалды есеп жайында

Мақалада Риман-Лиувиллдің жалпыланған операторымен аралас төртінші ретті теңдеу үшін интегро-дифференциалды түйіндес шартты бір бейлокалды есептің шешілу мәселелерін зерттеуге арналған. Белгілі бір шарттарда берілген параметрлер мен функциялар үшін қойылған есептің шешімінің жалғыз және бар болу теоремалары дәлелденді. Бұл шарттар бұзылған кезде қойылған есептің нөлдік емес шешімі бар болатыны жайында мысал келтірілген. Есептің шешімінің жалғыз және бар болу теоремаларын дәлелдеу үшін айнымалыларды бөлу әдісі қолданылды. Шешімнің өзі бірөлшемді спектрлік есепке сәйкес абсолютті және бірқалыпты жинақталатын қатар түрінде алынған. Коши есебі жалпыланған интегро-дифференциалдау операторы бар бөлшек теңдеу үшін зерттелді, осы есепті эквивалентті интегралды теңдеудің шешімділігі мағынасында шешімін табудың қарапайым әдісі көрсетілген. Бейлокалды шарт бойынша қарастырылған есептің тұрақтылығы алынған.

Кілт сөздер: аралас типті теңдеу, бейлокалды шеттік есеп, шешімнің бар және жалғыз болуы, бөлшекті интегро-дифференциалдау операторы, Хилфер операторы, Миттаг-Леффлер функциясы.

Б.Ж. Кадиркулов[1], М.А. Жалилов[2]

[1] *Ташкентский государственный университет востоковедения, Ташкент, Узбекистан;*
[2] *Ферганский государственный университет, Фергана, Узбекистан*

Об одной нелокальной задаче для уравнения смешанного типа четвертого порядка с оператором Хилфера

Статья посвящена исследованию вопросов разрешимости одной нелокальной задачи с интегро-дифференциальным условием сопряжения для уравнения смешанного типа четвертого порядка с обобщенным оператором Римана–Лиувилля. При определенных условиях на заданные параметры и функции доказаны теоремы единственности и существования решения поставленной задачи. Приведен пример, показывающий, что при нарушении этих условий, сформулированная задача будет иметь нетривиальное решение. Для доказательства теорем единственности и существования решения поставленной задачи использован метод разделения переменных. Само решение построено в виде суммы абсолютно и равномерно сходящегося ряда по собственным функциям соответствующей одномерной спектральной задачи. Изучена задача Коши для дробного уравнения с обобщенным оператором интегро-дифференцирования, показан простой способ нахождения решения этой задачи путем сведения её к эквивалентному в смысле разрешимости интегральному уравнению. Авторами также установлена устойчивость решения рассматриваемой задачи по нелокальному условию.

Ключевые слова: уравнение смешанного типа, нелокальная краевая задача, существование и единственность решения, оператор дробного интегро-дифференцирования, оператор Хилфера, функция Миттаг-Леффлера.

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A.A. Kalybay[1,*], A.O. Baiarystanov[2]

[1]*KIMEP University, Almaty, Kazakhstan;*
[2]*L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan*
(E-mail: kalybay@kimep.kz, oskar_ 62@mail.ru)

Differential inequality and non-oscillation of fourth order differential equation

The oscillatory theory of fourth order differential equations has not yet been developed well enough. The results are known only for the case when the coefficients of differential equations are power functions. This fact can be explained by the absence of simple effective methods for studying such higher order equations. In this paper, the authors investigate the oscillatory properties of a class of fourth order differential equations by the variational method. The presented variational method allows to consider any arbitrary functions as coefficients, and our main results depend on their boundary behavior in neighborhoods of zero and infinity. Moreover, this variational method is based on the validity of a certain weighted differential inequality of Hardy type, which is of independent interest. The authors of the article also find two-sided estimates of the least constant for this inequality, which are especially important for their applications to the main results on the oscillatory properties of these differential equations.

Keywords: fourth order differential equation, oscillation, non-oscillation, variational principle, weighted inequality, space.

1 Introduction

Let $I = (0, \infty)$. Let v be a positive function twice differentiable on I and u be a non-negative function continuous on I.

We consider the following fourth order differential equation

$$(v(t)y''(t))'' - \lambda u(t)y(t) = 0, \ t \in I, \tag{1}$$

where $\lambda > 0$ is a real number.

The oscillatory properties of equation (1) have not yet been studied sufficiently. The obtained results are mainly the case where v or u are power functions. There are also results where equation (1) has been studied by its reduction to a Hamiltonian system and application of the Riccati technique using unknown fundamental solutions of the system. The development of the oscillation theory of equation (1) is given in the works [1–3], and references therein. For more details, we also refer to the monograph [4].

One more method to investigate the oscillatory properties of (1) is the variational method. This method is based on the fact that non-oscillation of equation (1) is equivalent to the validity of a certain second order differential inequality, which allows to obtain non-oscillation conditions in terms of the functions v and u. However, the known results on this differential inequality are not suitable for using them by this method. In this paper, under some assumptions on the function v in neighborhoods of zero and infinity, we find suitable characterizations for the validity of this second order differential inequality, and then apply them to obtain non-oscillation conditions of equation (1).

Let us note that the study of differential equations of fourth and higher orders by the variational method began in the works [5] and [6] under assumptions on the function v different from those presented here. More precisely, characterizations of the corresponding inequality depend on the number of zero boundary conditions at each endpoint of the interval, where this inequality is considered. This number, in turn, depends on assumptions on v. In the work [6], the corresponding second order inequality is studied on the interval $I_T = (T, \infty)$, $T > 0$,

*Corresponding author.
E-mail: kalybay@kimep.kz

under assumptions on v providing the existence of exactly two boundary conditions both at T. In this paper, we consider the interval I and assumptions on v are such that they also provide the existence of exactly two boundary conditions but one condition is at zero while the second is at infinity. In addition, the equation considered in [6] contains the differential operation $D_r^2 y(t) = \frac{d}{dt} r(t) \frac{dy(t)}{dt}$, where r is a positive function sufficiently time continuously differentiable on I. This operation becomes $y''(t)$ for $r = 1$.

This paper is organized as follows. Section 2 contains all the auxiliary statements necessary to prove the main results. Section 3 establishes the validity of the required second order differential inequality. In Section 4, on the basis of this inequality we find non-oscillation conditions of equation (1). Section 5 contains an example.

2 Auxiliary statements

Let $0 \leq a < b \leq \infty$. From the work [7], we have the following lemma.

Lemma A. (i) The inequality

$$\int\limits_a^b u(x) \left(\int\limits_a^x f(t)\, dt \right)^2 dx \leq C \int\limits_a^b v(t) f^2(t) dt \tag{2}$$

holds if and only if

$$A^+ = \sup_{a<z<b} \int\limits_z^b u(x) dx \int\limits_a^z v^{-1}(t)\, dt < \infty,$$

in addition, $A^+ \leq C \leq 4A^+$, where C is the best constant in (2).

(ii) The inequality

$$\int\limits_a^b u(x) \left(\int\limits_x^b f(t)\, dt \right)^2 dx \leq C \int\limits_a^b v(t) f^2(t) dt \tag{3}$$

holds if and only if

$$A^- = \sup_{a<z<b} \int\limits_a^z u(x) dx \int\limits_z^b v^{-1}(t)\, dt < \infty,$$

in addition, $A^- \leq C \leq 4A^-$, where C is the best constant in (3).

Denote by $W_{2,v}^2 \equiv W_{2,v}^2(I)$ the space of functions $f : I \to \mathbb{R}$ twice differentiable on the interval I, for which the norm

$$\|f\|_{W_{2,v}^2} = \|f''\|_{2,v} + |f'(1)| + |f(1)| \tag{4}$$

is finite, where $\|g\|_{2,v} = \left(\int\limits_0^\infty v(t) g^2(t) dt \right)^{\frac{1}{2}}$.

Let $C_0^\infty(I)$ be the set of finitely supported functions infinitely differentiable on the interval I. By the conditions on the function v we have that $C_0^\infty(I) \subset W_{2,v}^2(I)$. Denote by $\mathring{W}_{2,v}^2 \equiv \mathring{W}_{2,v}^2(I)$ the closure of the set $C_0^\infty(I)$ with respect to norm (4).

For $f \in W_{2,v}^2$ we assume that $\lim\limits_{t\to 0^+} f(t) = f(0)$ and $\lim\limits_{t\to\infty} f'(t) = f'(\infty)$.

Denote by $W_{2,v}^2(0,1)$ and $W_{p,v}^2(1,\infty)$ the contraction sets of functions from $W_{2,v}^2$ on the intervals $(0,1]$ and $[1,\infty)$, respectively.

Assume that

$$P_0(0,1) = \{C\chi_{(0,1)}(t) : \ C \in \mathbb{R}\},$$

$$P_1(1,\infty) = \{C\chi_{(1,\infty)}(t)t : \ C \in \mathbb{R}\},$$

$$L_0 W = \{f \in W_{p,v}^2 : \ f(0) = 0\},$$

$$R_1 W = \{f \in W_{p,v}^2 : \ f'(\infty) = 0\}.$$

From the work [8], we have one more statement.

Lemma B. *(i)* If $v^{-1} \notin L_1(0,1)$ and $t^2 v^{-1}(t) \in L_1(0,1)$, then $\mathring{W}^2_{2,v}(0,1) = L_0 W$; in addition, $W^2_{2,v}(0,1) = \mathring{W}^2_{2,v}(0,1) \dot{+} P_0(0,1)$.
(ii) If $v^{-1} \in L_1(1,\infty)$ and $t^2 v^{-1}(t) \notin L_1(1,\infty)$, then $\mathring{W}^2_{2,v}(1,\infty) = R_1 W$; in addition, $W^2_{2,v}(1,\infty) = \mathring{W}^2_{2,v}(1,\infty) \dot{+} P_1(1,\infty)$.
Here the sign $\dot{+}$ means the direct sum of subspaces.

3 Differential inequality

Function $y : I \to \mathbb{R}$ is called a solution of equation (1) if it is four times continuously differentiable on the interval and satisfies equation (1) for all $t > 0$.

Equation (1) is called oscillatory at infinity (at zero) if for any $T > 0$ there exists a solution of this equation having more than one double zero to the right (to the left) of T. Otherwise, equation (1) is called non-oscillatory.

Let us consider the following second order differential inequality

$$\lambda \int_T^\infty u(t)|f(t)|^2 dt \le C_T \int_T^\infty v(t)|f''(t)|^2 dt, \quad f \in \mathring{W}^2_{2,v}(T,\infty). \tag{5}$$

In the work [9], on the basis of the variational principle [10] there was established the following lemma.

Lemma C. Let C_T be the least constant in (5). The equation (1) is non-oscillatory at infinity if and only if for some $T > 0$ we have that $0 < C_T \le 1$.

If we consider the inequality

$$\lambda \int_0^T u(t)|f(t)|^2 dt \le C_T \int_0^T v(t)|f''(t)|^2 dt, \quad f \in \mathring{W}^2_{2,v}(0,T), \tag{6}$$

we can write the statement similar to Lemma C for non-oscillation of equation (1) at zero.

Lemma C yields that non-oscillation of equation (1) depends on the constant C_T in (5) and (6). Therefore, we need to find the value of C_T or at least estimate it from above and below.

We investigate equation (1) under the following conditions at zero and infinity:

$$\int_0^1 v^{-1}(t)dt = \infty, \quad \int_0^1 t^2 v^{-1}(t)dt < \infty, \quad \int_1^\infty v^{-1}(t)dt < \infty, \quad \int_1^\infty t^2 v^{-1}(t)dt = \infty. \tag{7}$$

Under these assumptions we consider the inequality

$$\lambda \int_0^\infty u(t)|f(t)|^2 dt \le C_0 \int_0^\infty v(t)|f''(t)|^2 dt, \quad f \in \mathring{W}^2_{2,v}(I). \tag{8}$$

Let

$$E_1 = \sup_{z>0} \int_z^\infty u(t)dt \int_0^z s^2 v^{-1}(s)\, ds,$$

$$E_2 = \sup_{z>0} \int_0^z t^2 u(t)dt \int_z^\infty v^{-1}(s)\, ds,$$

$$E = \lambda \max\{E_1, E_2\}.$$

Theorem 1. Let (7) hold. Then inequality (8) holds if and only if $E < \infty$; in addition, $E \le C_0 \le 8E$, where C_0 is the best constant in (8).

Proof. From condition (7) and Lemma B, it follows that

$$\mathring{W}^2_{2,v}(I) = \{f \in W^2_{2,v} : f(0) = 0;\ f'(\infty) = 0\} \equiv LRW.$$

Hence, for $f \in \mathring{W}^2_{2,v}(I)$ we have $f(t) = \int\limits_0^t f'(s)ds$ and $f'(s) = -\int\limits_s^\infty f''(x)dx$. Then $f(t) = -\int\limits_0^t \int\limits_s^\infty f''(x)dxds = -\int\limits_0^t \int\limits_s^t f''(x)dxds - t\int\limits_t^\infty f''(x)dx = -\int\limits_0^t xf''(x)dx - t\int\limits_t^\infty f''(x)dx$. Using this relation, we get

$$\lambda \int\limits_0^\infty u(t)|f(t)|^2 dt = \int\limits_0^\infty u(t) \left| \int\limits_0^t xf''(x)dx + t\int\limits_t^\infty f''(x)dx \right|^2 dt$$

$$\leq 2\lambda \int\limits_0^\infty u(t) \left| \int\limits_0^t xf''(x)dx \right|^2 dt + 2\lambda \int\limits_0^\infty t^2 u(t) \left| \int\limits_t^\infty f''(x)dx \right|^2 dt. \tag{9}$$

The latter gives that if

$$\int\limits_0^\infty u(t) \left| \int\limits_0^t xf''(x)dx \right|^2 dt \leq C_1 \int\limits_0^\infty v(t)|f''(t)|^2 dt \tag{10}$$

and

$$\int\limits_0^\infty t^2 u(t) \left| \int\limits_t^\infty f''(x)dx \right|^2 dt \leq C_2 \int\limits_0^\infty v(t)|f''(t)|^2 dt \tag{11}$$

with the least constants C_1 and C_2, respectively, then $C_0 \leq 2\lambda \max\{C_1, C_2\}$, where C_0 is the least constant in (8). From Lemma A we have that $C_1 \leq 4E_1$ and $C_2 \leq 4E_2$. Therefore,

$$C_0 \leq 8E. \tag{12}$$

Now, assuming $f'' \geq 0$ in (9), we get that if (8) holds, then (10) and (11) also hold and $C_0 \geq \lambda \max\{C_1, C_2\}$. From Lemma A, it follows that $E_1 \leq C_1$ and $E_2 \leq C_2$, which together with (12) yields that $E \leq C_0 \leq 8E$. The proof of Theorem 1 is complete.

4 Non-oscillation of equation (1)

Theorem 2. Let (7) hold. Then equation (1) is non-oscillatory at infinity and zero if

$$\sup_{z>0} \int\limits_z^\infty u(t)dt \int\limits_0^z s^2 v^{-1}(s)\, ds \leq \frac{1}{8\lambda}, \tag{13}$$

$$\sup_{z>0} \int\limits_0^z t^2 u(t)dt \int\limits_z^\infty v^{-1}(s)\, ds \leq \frac{1}{8\lambda}. \tag{14}$$

Proof. From assumption (7) it follows that the function v^{-1} is non-singular at the point $T > 0$, i.e., for any finite $N > T$ we have $\int\limits_T^N v^{-1}(t)\, dt < \infty$. Therefore, for any $f \in \mathring{W}^2_{2,v}(T,\infty)$ we get $f(T) = f'(T) = 0$ and

$$\mathring{W}^2_{2,v}(T,\infty) = \{f \in W^2_{2,v}(T,\infty) : f(T) = f'(T) = f'(\infty) = 0\} \equiv L^2RW.$$

We expand the function $f \in L^2RW$ by zero on the interval $(0,T)$, i.e., we assume that $f(t) = 0$ for $0 < t < T$. This gives that $f \in \mathring{W}^2_{2,v}(I)$. Therefore, $LRW \supset L^2RW$. Then

$$C_0 = \sup_{f \in LRW} \frac{\int\limits_0^\infty u(t)|f(t)|^2 dt}{\int\limits_0^\infty v(x)|f''(x)|^2 dx} \geq \sup_{f \in L^2RW} \frac{\int\limits_0^\infty u(t)|f(t)|^2 dt}{\int\limits_0^\infty v(x)|f''(x)|^2 dx}$$

$$= \sup_{f\in L^2RW} \frac{\int\limits_T^\infty u(t)|f(t)|^2dt}{\int\limits_T^\infty v(x)|f''(x)|^2dx} = C_T. \quad (15)$$

From (13) and (14), it follows that $E \le \frac{1}{8\lambda}$. Hence, by Theorem 1 we have that $0 < C_0 \le 1$. Therefore, due to (15), we get that $0 < C_T \le 1$. Thus, by Lemma C, it follows that equation (1) is non-oscillatory at infinity.

Now, we turn to non-oscillation of equation (1) at zero. In this case, at the point $T > 0$ we have that $f(T) = f'(T) = 0$ for $f \in \mathring{W}^2_{2,v}(0,T)$ and

$$\mathring{W}^2_{2,v}(0,T) = \{f \in W^2_{2,v}(0,T) : f(0) = f(T) = f'(T) = 0\} \equiv LR^2W.$$

We expand the function $f \in \mathring{W}^2_{2,v}(0,T)$ by zero on the interval (T,∞) and get $LRW \supset LR^2W$. Arguing as above in (15), we establish that from $0 < C_0 \le 1$ it follows $0 < C_T \le 1$. By Theorem 1 from (13) and (14) we have $0 < C_0 \le 1$. Thus, Lemma C written for inequality (6) yields that equation (1) is non-oscillatory at zero. The proof of Theorem 2 is complete.

5 Example

As an example, let us consider the following differential equation

$$(t^\alpha y''(t))'' - \lambda t^{-\beta} y(t) = 0,\ t \in I, \quad (16)$$

where $\lambda > 0$ is a real number. Assume that $1 < \alpha, \beta < 3$. It is easy to see that the function $v^{-1}(t) = t^{-\alpha}$ satisfies condition (7). Then

$$E_1(z) = \int\limits_z^\infty t^{-\beta}dt \int\limits_0^z s^{2-\alpha}\,ds = \frac{z^{1-\beta}}{\beta-1}\cdot\frac{z^{3-\alpha}}{3-\alpha} = \frac{z^{4-\alpha-\beta}}{(\beta-1)(3-\alpha)}.$$

$E_1 = \sup\limits_{z>0} E_1(z) < \infty$ if and only if $4-\alpha-\beta = 0$ that is $\beta = 4-\alpha$. Then $E_1 = \frac{1}{(\beta-1)(3-\alpha)} = \frac{1}{(3-\alpha)^2}$. Similarly, we can find that $E_2 = \frac{1}{(\alpha-1)^2}$. By Theorem 2 equation (16) is non-oscillatory at infinity and zero if $E_1 = \frac{1}{(3-\alpha)^2} \le \frac{1}{8\lambda}$ and $E_2 = \frac{1}{(\alpha-1)^2} \le \frac{1}{8\lambda}$. Therefore, equation (16) is non-oscillatory at infinity and zero if $\lambda \le \frac{1}{8}\min\{(\alpha-1)^2, (3-\alpha)^2\}$. Thus, we can write the following proposition.

Proposition. Let $1 < \alpha < 3$ and $\beta = 4-\alpha$. Then equation (16) is non-oscillatory at infinity and zero if $\lambda \le \frac{1}{8}\min\{(\alpha-1)^2, (3-\alpha)^2\}$.

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А.А. Қалыбай[1], А.О. Байарыстанов[2]

[1] *КИМЭП университеті, Алматы, Қазақстан;*
[2] *Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Нұр-Сұлтан, Қазақстан*

Дифференциалдық теңсіздік және төртінші ретті дифференциалдық теңдеудің тербелімсіздігі

Төртінші ретті дифференциалдық теңдеулердің тербелімділік теориясы жеткілікті түрде дамымаған. Нәтижелер дифференциалдық теңдеулер коэффициенттері дәрежелік функциялары болған жағдайда ғана белгілі болады. Бұл фактіні жоғары дәрежелі теңдеулерді зерттеудің қарапайым тиімді әдістерінің болмауымен түсіндіруге болады. мақалада төртінші ретті дифференциалдық теңдеулер класының тербелмелі қасиеттері вариациялық әдіспен зерттелген. Ұсынылған вариациялық әдіс теңдеулер коэффициенттері кез келген функция болуы ретінде қарастыруға мүмкіндік береді және негізгі нәтижелер олардың нөлге және шексіздікке жақын шекаралық әрекеттеріне байланысты. Сонымен қатар, бұл вариациялық әдіс тәуелсіз қызығушылық тудыратын Харди типті салмақты дифференциалдық теңсіздігінің негізділігіне талқыланған. Осы теңсіздік үшін ең кіші константаның екі жақты бағалауы табылған, бұл олардың осы дифференциалдық теңдеулердің тербелмелік қасиеттерінің негізгі нәтижелеріне қолданылуы үшін ерекше маңызды.

Кілт сөздер: төртінші ретті дифференциалдық теңдеу, тербелімділік, тербелімсіздік, вариациялық принцип, салмақты теңсіздік, кеңістік.

А.А. Калыбай[1], А.О. Байарыстанов[2]

[1] *Университет КИМЭП, Алматы, Казахстан;*
[2] *Евразийский национальный университет имени Л.Н. Гумилева, Нур-Султан, Казахстан*

Дифференциальное неравенство и неосцилляторность дифференциального уравнения четвертого порядка

Теория осцилляций дифференциальных уравнений четвертого порядка недостаточно хорошо изучена. Известны результаты только для случая, когда коэффициенты дифференциальных уравнений являются степенными функциями. Этот факт можно объяснить отсутствием простых эффективных методов для изучения уравнений высокого порядка. В статье исследованы осцилляционные свойства одного класса дифференциальных уравнений четвертого порядка вариационным методом. Представленный вариационный метод позволяет рассматривать любые произвольные функции в качестве коэффициентов, а основные результаты зависят от их граничного поведения в окрестностях нуля и бесконечности. Более того, этот вариационный метод основан на выполнении некоторого весового дифференциального неравенства типа Харди, представляющего самостоятельный интерес. Авторами найдены двусторонние оценки наименьшей константы для этого неравенства, которые особенно важны для их приложений к основным результатам по осцилляторности рассматриваемых дифференциальных уравнений.

Ключевые слова: дифференциальное уравнение четвертого порядка, осцилляторность, неосцилляторность, вариационный принцип, весовое неравенство, пространство.

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N.T. Orumbayeva*, T.D. Tokmagambetova, Zh.N. Nurgalieva

Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan
(E-mail: OrumbayevaN@mail.ru, tenggesh.tokmagambetova@gmail.com, nurgalieva.zhanel@mail.ru)

On the solvability of a semi-periodic boundary value problem for the nonlinear Goursat equation

In this paper, by means of a change of variables, a nonlinear semi-periodic boundary value problem for the Goursat equation is reduced to a linear gravity problem for hyperbolic equations. Reintroducing a new function, the obtained problem is reduced to a family of boundary value problems for ordinary differential equations and functional relations. When solving a family of boundary value problems for ordinary differential equations, the parameterization method is used. The application of this approach made it possible to establish the coefficients of the unique solvability of the semi-periodic problem for the Goursat equation and to propose constructive algorithms for finding an approximate solution.

Keywords: semi-periodic boundary value problem, second order boundary value problem, Goursat equation, boundary value problem, algorithm, approximate solution.

Introduction

On $\Omega = [0, X] \times [0, T]$ a boundary value problem for the nonlinear Goursat equation is considered

$$\frac{\partial^2 z}{\partial x \partial t} = 2\sqrt{f(x,t)\frac{\partial z}{\partial x}\frac{\partial z}{\partial t}}, \tag{1}$$

$$z(x,0) = 0, \tag{2}$$

$$\frac{\partial z(0,t)}{\partial t} = \psi^2(t), \tag{3}$$

$$z(x,0) = z(x,T), \tag{4}$$

where $f(x,t)$-given function depending on x and t.

Let $C(\Omega, R^n)$ be the spaces of functions $u : \Omega \to R^n$ which are continuous on Ω, with the rate $\|u\|_0 = \max\limits_{(x,t)\in\Omega} |u(x,t)|$.

In this paper, we study a remarkable equation, the importance of which was first noted by Goursat [1–2].

The function $z(x,t) \in C(\Omega, R^n)$ with partial derivatives $\frac{\partial z(x,t)}{\partial x} \in C(\overline{\Omega}, R^n)$, $\frac{\partial z(x,t)}{\partial t} \in C(\overline{\Omega}, R^n)$, $\frac{\partial^2 z(x,t)}{\partial x \partial t} \in C(\Omega, R^n)$ is called the classical solution to the problem (1)-(4), if it satisfies the system (1) with all $(x,t) \in \Omega$ and boundary conditions (2)–(4).

To find the solution of the problem (1)–(4), we make differential substitutions, introduce the functions $u = u(x,t)$, $g = g(x,t)$ by formulas: $u = \sqrt{\frac{\partial z}{\partial t}}$, $g = \sqrt{\frac{\partial z}{\partial x}}$. Differentiating these ratios, respectively, by t and x excluding z using the equation (1), we get the system $\frac{\partial u}{\partial x} = g\sqrt{f(x,t)}$, $\frac{\partial g}{\partial t} = u\sqrt{f(x,t)}$. Excluding g, we arrive at a linear equation for the function $u = u(x,t)$:

$$\frac{\partial^2 u}{\partial x \partial t} = a(x,t)\frac{\partial u}{\partial x} + f(x,t)u, \tag{5}$$

$$u(0,t) = \psi(t), \tag{6}$$

$$u(x,0) = u(x,T), \tag{7}$$

*Corresponding author.
E-mail: OrumbayevaN@mail.ru

$$z(x,t) = \int_0^t u^2(x,\eta)d\eta, \tag{8}$$

where $a(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\ln f(x,t)$.

Such problems these were investigated in the works [3–6].

We introduce a new unknown function $w(x,t) = \frac{\partial u(x,t)}{\partial x}$, and the problem (5)–(8) can be written in the form

$$\frac{\partial w}{\partial t} = a(x,t)w + f(x,t)u(x,t), \quad (x,t) \in \Omega, \tag{9}$$

$$w(x,0) = w(x,T), \quad x \in [0,X], \tag{10}$$

$$u(x,t) = \psi(t) + \int_0^x w(\xi,t)d\xi, \quad t \in [0,T], \tag{11}$$

$$z(x,t) = \int_0^t u^2(x,\eta)d\eta. \tag{12}$$

Here, the problem of finding a solution to a semi-periodic boundary value problem for hyperbolic equations (5)–(8) reduced to a family of periodic boundary value problems for a system of ordinary differential equations (9), (10) and functional relationships (11), (12).

The problems (5)–(8) and (9)–(12) are equivalent in the sense that if a pair of functions $(u^*(x,t), z^*(x,t))$ are the solution to the problem (5)–(8), then three $(w^*(x,t) = \frac{\partial u^*(x,t)}{\partial x}, u^*(x,t), z^*(x,t))$ is the solution of the problem (9)–(12) and vice versa, if three $(w^*(x,t), u^*(x,t), z^*(x,t))$- is the solution of the problem(9)–(12), then a pair of functions $(u^*(x,t), z^*(x,t))$ is the solution of the problem (5)–(8).

Algorithms for finding the solution of a semi-periodic boundary value problem for the nonlinear Goursat equation

To solve problem (9)–(12), we apply the method of a parametrization [7].

For the step $h > 0 : Nh = T$ we partition $[0,T) = \bigcup_{i=1}^{N}[(i-1)h, ih)$, $N = 1,2,....$ In this case, the area Ω is divided into N parts. By $w_i(x,t), u_i(x,t), z_i(x,t)$ we denote, respectively, the restrictions of the function $w(x,t), u(x,t), z(x,t)$ on $\Omega_i = [0,X] \times [(i-1)h, ih), i = \overline{1,N}$. Then problem (9)–(12) be equivalent to the boundary value problem [8–14]

$$\frac{\partial w_i}{\partial t} = a(x,t)w_i + f(x,t)u_i(x,t), \quad (x,t) \in \Omega_i, \tag{13}$$

$$w_1(x,0) - \lim_{t\to T-0} w_N(x,t) = 0, \tag{14}$$

$$\lim_{t\to sh-0} w_s(x,t) = w_{s+1}(x,sh), s = \overline{1,N-1}, \tag{15}$$

$$u_i(x,t) = \psi(t) + \int_0^x w_i(\xi,t)d\xi, \tag{16}$$

$$z_i(x,t) = \int_0^t (\psi(\eta) + \int_0^x w_i(\xi,\eta)d\xi)^2 d\eta, \tag{17}$$

where (15) is the condition of gluing functions $w(x,t)$ in the internal lines of the partition. By $\lambda_i(x)$ we get the value of the function $w_i(x,t)$ at $t = (i-1)h$, i.e. $\lambda_i(x) = w_i(x,(i-1)h)$ and denote $v_i(x,t) = w_i(x,t) - \lambda_i(x)$, $\quad i = \overline{1,N}$. We obtain an equivalent boundary value problem for the unknown functions $\lambda_i(x)$:

$$\frac{\partial v_i}{\partial t} = a(x,t)v_i + a(x,t)\lambda_i(x) + f(x,t)u_i(x,t), \quad (x,t) \in \Omega_i, \tag{18}$$

$$\lambda_i(x,(i-1)h) = 0, x \in [0,X], i = \overline{1,N}, \tag{19}$$

$$\lambda_1(x) - \lambda_N(x) - \lim_{t\to T-0} v_N(x,t) = 0, x \in [0,X], \tag{20}$$

$$\lambda_s(x) + \lim_{t\to sh-0} v_s(x,t) - \lambda_{s+1}(x) = 0, x \in [0,X], s = \overline{1,N-1}, \tag{21}$$

$$u_i(x,t) = \psi(t) + \int_0^x v_i(\xi,t)d\xi + \int_0^x \lambda_i(\xi)d\xi, (x,t) \in \Omega_i, i = \overline{1,N}. \tag{22}$$

$$z_i(x,t) = \int_0^t \left(\psi(\eta) + \int_0^x v_i(\xi,\eta)d\xi + \int_0^x \lambda_i(\xi)d\xi\right)^2 d\eta. \tag{23}$$

Problems (13)–(17) and (18)–(23) are equivalent in the sense that if the triple $\{w_i(x,t), u_i(x,t), z_i(x,t)\}$, $i = \overline{1,N}$ is the solution to the problem (13)–(17), then the system $\{\lambda_i(x) = w_i(x,(i-1)h), v_i(x,t) = w_i(x,t) - -w_i(x,(i-1)h), u_i(x,t), z_i(x,t)\}$, $i = \overline{1,N}$, will be the solution to the problem (18)–(23) and vice versa, if $\{\lambda_i(x), v_i(x,t), u_i(x,t), z_i(x,t)\}$, $i = \overline{1,N}$ - the solution to the problem (18)–(23), then the system $\{\lambda_i(x) + v_i(x,t),\ u_i(x,t), z_i(x,t)\}$, $i = \overline{1,N}$ will be the solution to the problem (7)–(10).

Problem (18)–(19) for fixed $\lambda_i(x)$, $u_i(x,t)$ is a one-parameter family of Cauchy problems for systems of ordinary differential equations, where $x \in [0,X]$ which is equivalent to the integral equation

$$v_i(x,t) = \int_{(i-1)h}^t a(x,\eta)v_i(x,\eta)d\eta + \int_{(i-1)h}^t a(x,\eta)d\eta \cdot \lambda_i(x) + \int_{(i-1)h}^t f(x,\eta)u_i(x,\eta)d\eta. \tag{24}$$

Instead of $v_i(x,t)$ we substitute the corresponding right-hand side (24) and repeating this process l ($l = = 1, 2, ...$) times we get

$$v_i(x,t) = D_{li}(x,t)\lambda_i(x) + G_{li}(x,t,v_i) + F_{li}(x,t,u_i), i = \overline{1,N}, \tag{25}$$

where

$$D_{l,i}(x,t) = \sum_{j=0}^{l-1} \int_{(i-1)h}^t a(x,\eta_1)\ldots \int_{(i-1)h}^{\eta_j} a(x,\eta_{j+1})d\eta_{j+1}\ldots d\eta_1,$$

$$F_{l,i}(x,t,u_i) = \int_{(i-1)h}^t f(x,\eta_1)u_i(x,\eta_1)d\eta_1 +$$

$$+\sum_{j=1}^{l-1} \int_{(i-1)h}^t a(x,\eta_1)... \int_{(i-1)h}^{\eta_{j-1}} a(x,\eta_j) \int_{(i-1)h}^{\eta_j} f(x,\eta_{j+1})u_i(x,\eta_{j+1})d\eta_{j+1}d\eta_j...d\eta_1,$$

$$G_{l,i}(x,t,v_i) = \int_{(i-1)h}^t a(x,\eta_1)... \int_{(i-1)h}^{\eta_{l-2}} a(x,\eta_{l-1}) \int_{(i-1)h}^{\eta_{l-1}} a(x,\eta_l)v_r(x,\eta_l)d\eta_l d\eta_{l-1}...d\eta_1, \eta_0 = t, i = \overline{1,N}.$$

Passing to the limit as $t \to ih - 0$, in (25) we find $\lim\limits_{t\to ih-0} v_i(x,t), i = \overline{1,N}, x \in [0,X]$, by replacing them into (20)-(21), for unknown functions $\lambda_i(x), i = \overline{1,N}$ we obtain the system of functional equations:

$$Q_l(x,h)\lambda(x) = -F_l(x,h,u) - G_l(x,h,v), \tag{26}$$

where

$$Q_l(x,h) = \begin{pmatrix} I & 0 & \ldots & 0 & -[I + D_{l,N}(x,Nh)] \\ I + D_{l,1}(x,h) & -I & \ldots & 0 & 0 \\ 0 & I + D_{l,2}(x,2h) & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I + D_{l,N-1}(x,(N-1)h) & -I \end{pmatrix},$$

$$F_l(x,h,u) = (-F_{l,N}(x,Nh,u_N), F_{l,1}(x,h,u_1), ..., F_{l,N-1}(x,(N-1)h,u_{N-1})),$$

$$G_l(x,h,v) = (-G_{l,N}(x,Nh,v_N), G_{l,1}(x,h,v_1), ..., G_{l,N-1}(x,(N-1)h,v_{N-1})),$$

where I - the identity matrix of dimension n.

To find a system of four functions $\{\lambda_i(x), v_i(x,t),\ z_i(x,t), u_i(x,t)\}, i = \overline{1,N}$, we have a closed system consisting of the equations (26), (25), (22) и (23). Assuming invertibility of the matrix $Q_l(x,h)$, with all $x \in [0,X]$, from the equation (26), where $v_i(x,t) = 0, u_i(x,t) = \psi(t)$, we find $\lambda^{(0)}(x) = (\lambda_1^{(0)}(x), \lambda_2^{(0)}(x), ..., \lambda_N^{(0)}(x))'$:

$$\lambda^{(0)}(x) = -[Q_l(x,h)]^{-1}\{F_l(x,h,\psi) + G_l(x,h,0)\}.$$

Using the equation (25), at $\lambda_i(x) = \lambda_i^{(0)}(x)$ we find functions $\{v_i^{(0)}(x,t)\}$, $i = \overline{1,N}$, т.е. $v_i^{(0)}(x,t) = D_{l,i}(x,t)\lambda_i^{(0)}(x) + F_{l,i}(x,t,\psi) + G_{l,i}(x,t,0)$.

The functions $u_i^{(0)}(x,t), i = \overline{1,N}$, are defined from the relations

$$u_i^{(0)}(x,t) = \psi(t) + \int\limits_0^x v_i^{(0)}(\xi,t)d\xi + \int\limits_0^x \lambda_i^{(0)}(\xi)d\xi, \quad (x,t) \in \Omega_i,$$

$$z_i^{(0)}(x,t) = \int\limits_0^t (\psi(\eta) + \int\limits_0^x v_i^{(0)}(\xi,\eta)d\xi + \int\limits_0^x \lambda_i^{(0)}(\xi)d\xi)^2 d\eta, (x,t) \in \Omega_i.$$

For the initial approximation of problem (18)-(23) we take the system $(\lambda_i^{(0)}(x), v_i^{(0)}(x,t), u_i^{(0)}(x,t), z_i^{(0)}(x,t))$, $i = \overline{1,N}$ and construct successive approximations on the following algorithm :

Step 1. A) Assuming that $u_i(x,t) = u_i^{(0)}(x,t)$, $i = \overline{1,N}$, , we find the first approximations of $\lambda_i(x)$, $v_i(x,t)$ by finding a solution to the problem (18)–(19). Taking $\lambda^{(1,0)}(x) = \lambda^{(0)}(x)$, $v_i^{(1,0)}(x,t) = v_i^{(0)}(x,t)$, we find the system of couples $\{\lambda_i^1(x), v_i^1(x,t)\}$, $i = \overline{1,N}$, as the limit of the sequence $\lambda_i^{(1,m)}(x), v_i^{(1,m)}(x,t)$, which is defined as follows:

Step 1.1. Assuming the invertibility of the matrix $Q_l(x,h)$, at all $x \in [0,X]$, from equation (26), where $v_i(x,t) = v_i^{(1,0)}(x,t)$, we find $\lambda^{(1,1)}(x) = (\lambda_1^{(1,1)}(x), \lambda_2^{(1,1)}(x), ..., \lambda_N^{(1,1)}(x))'$:

$$\lambda^{(1,1)}(x) = -[Q_l(x,h)]^{-1}\{F_l(x,h,u^{(0)}) + G_l(x,h,v^{(1,0)})\}.$$

By replacing the found $\lambda_i^{(1,1)}(x), i = \overline{1,N}$ in (25) we find

$$v_i^{(1,1)}(x,t) = D_{l,i}(x,t)\lambda_i^{(1,1)}(x) + F_{l,i}(x,t,u^{(0)}) + G_{l,i}(x,t,v^{(1,0)}).$$

Step 1.2. From equation (26), where $v_i(x,t) = v_i^{(1,1)}(x,t)$, we define

$$\lambda^{(1,2)}(x) = -[Q_l(x,h)]^{-1}\{F_l(x,h,u^{(0)}) + G_l(x,h,v^{(1,1)})\}.$$

Using as expression (25) again, , we find the functions $\{v_i^{(1,2)}(x,t)\}, i = \overline{1,N}$:

$$v_i^{(1,2)}(x,t) = D_{li}(x,t)\lambda_i^{(1,2)}(x) + F_{li}(x,t,u^{(0)}) + G_{li}(x,t,v^{(1,1)}).$$

On step $(1,m)$ we obtain the system of couples $\{\lambda_i^{(1,m)}(x), v_i^{(1,m)}(x,t)\}, i = \overline{1,N}$. Suppose that the solution of problem (18)-(21) is a sequence of systems of couples $\{\lambda_i^{(1,m)}(x), v_i^{(1,m)}(x,t)\}$ converge as $m \to \infty$ goes to continuous, respectively, on $x \in [0,X], (x,t) \in \Omega_i$ functions $\lambda_i^{(1)}(x)$, $v_i^{(1)}(x,t)$, $i = \overline{1,N}$.

B) Functions $u_i^{(1)}(x,t), i = \overline{1,N}$, are defined from the relations:

$$u_i^{(1)}(x,t) = \psi(t) + \int\limits_0^x v_i^{(1)}(\xi,t)d\xi + \int\limits_0^x \lambda_i^{(1)}(\xi)d\xi, (x,t) \in \Omega_i,$$

$$z_i^{(1)}(x,t)=\int\limits_0^t(\psi(\eta)+\int\limits_0^x v_i^{(1)}(\xi,\eta)d\xi+\int\limits_0^x\lambda_i^{(1)}(\xi)d\xi)^2d\eta,(x,t)\in\Omega_i.$$

Step 2. A) Assuming that $u_i(x,t)=u_i^{(1)}(x,t), i=\overline{1,N}$ we find the second approximations of $\lambda_i(x)$, $v_i(x,t)$ by finding a solution to the problem (18)-(21).

Taking $\lambda^{(2,0)}(x)=\lambda_i^{(1)}(x)$, $v_i^{(2,0)}(x,t)=v_i^{(1)}(x,t)$ we find the system of couples $\{\lambda_i^{(2)}(x),v_i^{(2)}(x,t)\}, i=\overline{1,N}$ as the limit of the sequence $\lambda_i^{(2,m)}(x),v_i^{(2,m)}(x,t)$, which is defined as follows:

Step 2.1. Assuming invertibility of the matrix $Q_l(x,h)$, at all $x\in[0,X]$, from equation (26), where $v_i(x,t)=v_i^{(2,0)}(x,t)$, we find $\lambda^{(2,1)}(x)=(\lambda_1^{(2,1)}(x),\lambda_2^{(2,1)}(x),...,\lambda_N^{(2,1)}(x))'$:

$$\lambda^{(2,1)}(x)=-[Q_l(x,h)]^{-1}\{F_l(x,h,u^{(1)})+G_l(x,h,v^{(2,0)})\}.$$

By replacing the found $\lambda_i^{(2,1)}(x), i=\overline{1,N}$ in (25) we find

$$v_i^{(2,1)}(x,t)=D_{li}(x,t)\lambda_i^{(2,1)}(x)+F_{li}(x,t,u^{(1)})+G_{li}(x,t,v^{(2,0)}).$$

Step 2.2. From equation (26), where $v_i(x,t)=v_i^{(2,1)}(x,t)$ we define

$$\lambda^{(2,2)}(x)=-[Q_l(x,h)]^{-1}\{F_l(x,h,u^{(1)})+G_l(x,h,v^{(2,1)})\}.$$

Using the expression (25), we find functions $\{v_i^{(2,2)}(x,t)\}, i=\overline{1,N}$:

$$v_i^{(2,2)}(x,t)=D_{li}(x,t)\lambda_i^{(2,2)}(x)+F_{li}(x,t,u^{(1)})+G_{li}(x,t,v^{(2,1)}).$$

On the step $(2,m)$ we obtain the system of couples $\{\lambda_i^{(2,m)}(x),v_i^{(2,m)}(x,t)\}, i=\overline{1,N}$. Suppose that the solution of problem (18)-(21) is a sequence of systems of couples $\{\lambda_i^{(1,m)}(x),v_i^{(1,m)}(x,t)\}$ which as $m\to\infty$ converges to $x\in[0,X],(x,t)\in\Omega_i$ functions $\lambda_i^{(2)}(x)$, $v_i^{(2)}(x,t)$, $i=\overline{1,N}$.

B) The functions $u_i^{(2)}(x,t), i=\overline{1,N}$, are defined from the relations:

$$u_i^{(2)}(x,t)=\psi(t)+\int\limits_0^x v_i^{(2)}(\xi,t)d\xi+\int\limits_0^x\lambda_i^{(2)}(\xi)d\xi,(x,t)\in\Omega_i,$$

$$z_i^{(2)}(x,t)=\int\limits_0^t(\psi(\eta)+\int\limits_0^x v_i^{(2)}(\xi,\eta)d\xi+\int\limits_0^x\lambda_i^{(2)}(\xi)d\xi)^2d\eta,(x,t)\in\Omega_i.$$

Sufficient conditions for the convergence of algorithms for finding its solution

The conditions of the following statement ensure the feasibility and convergence of the proposed algorithm, as well as the unique solvability of the problem (18)–(23).

Theorem 1. Let for some $h>0: Nh=T$, $N=1,2,...$ и $l,l=1,2,...$, $(nN\times nN)$ - the matrix $Q_l(x,h)$ be reversible for all $x\in[0,X]$ let the following inequalities be satisfied:

$$1)\|[Q_l(x,h)]^{-1}\|\le\gamma_l(x,h);$$

$$2)q_l(x,h)=\frac{(a(x)h)^l}{l!}\Big[1+\gamma_l(x,h)\sum_{j=1}^{l}\frac{(a(x)h)^j}{j!}\Big]\le\mu<1,$$

Then there exists a unique solution to problem (18)-(23) and the following estimates are valid

$$a)\max_{i=\overline{1,N}}\|\lambda_i^*(x)-\lambda_i^{(k)}(x)\|+\max_{i=\overline{1,N}}\sup_{t\in[(i-1)h,ih)}\|v_i^*(x,t)-v_i^{(k)}(x,t)\|\le$$

$$\leq z(x)f(x)\sum_{j=k}^{\infty}\frac{1}{j!}(\int_0^x z(\xi)f(\xi)d\xi)^j\int_0^x g(\xi)d\xi\max_{t\in[0,T]}\|\psi(t)\|,$$

$$b)\ \max_{i=\overline{1,N}}\sup_{t\in[(i-1)h,ih)}\|u_i^*(x,t)-u_i^{(k)}(x,t)\|\leq$$

$$\leq\int_0^x\max_{i=\overline{1,N}}\|\lambda_i^*(\xi)-\lambda_i^{(k)}(\xi)\|+\max_{i=\overline{1,N}}\sup_{t\in[(i-1)h,ih)}\|v_i^*(\xi,t)-v_i^{(k)}(\xi,t)\|d\xi,$$

$$c)\ \max_{i=\overline{1,N}}\sup_{t\in[(i-1)h,ih)}\|z_i^*(x,t)-z_i^{(k)}(x,t)\|\leq$$

$$\leq t\int_0^x\max_{i=\overline{1,N}}\|\lambda_i^*(\xi)-\lambda_i^{(k)}(\xi)\|d\xi+\int_0^t\int_0^x\max_{i=\overline{1,N}}\sup_{t\in[(i-1)h,ih)}\|v_i^*(\xi,\eta)-v_i^{(k)}(\xi,\eta)\|d\xi d\eta,$$

where $k=1,2,...,\quad a(x)=\max_{t\in[0,T]}\|a(x,t)\|,\quad f(x)=\max_{t\in[0,T]}\|f(x,t)\|,\quad b_1(x)=\gamma_l(x,h)h\sum_{j=0}^{l-1}\frac{(a(x)h)^j}{j!},$

$$b_2(x)=\Big[1+\gamma_l(x,h)\sum_{j=1}^{l}\frac{(a(x)h)^j}{j!}\Big]h\sum_{j=0}^{l-1}\frac{(a(x)h)^j}{j!},\quad b_3(x)=\gamma_l(x,h)\frac{(a(x)h)^l}{l!},$$

$$\theta(x)=\frac{1+b_3(x)}{1-q_l(x,h)}q_l(x,h)+b_3(x),\quad \rho(x)=\frac{1+b_3(x)}{1-q_l(x,h)}b_2(x)+b_1(x),$$

$$d(x)=\rho(x)f(x)\int_0^x[f(\xi)+1][b_1(\xi)+b_2(\xi)]d\xi+\theta(x)b_2(x)[f(x)+1].$$

By virtue of the equivalence of the problems (1.1)–(1.3) и (1.11)–(1.15) from Theorem 1 follows *Theorem 2.* Let the assumptions of Theorem 1 be satisfied. Then problem (1)–(4) as a unique solution $u^*(x,t)$ and the evaluation is performed

$$\max\{\|u^*\|_0,\|\frac{\partial u^*}{\partial x}\|_0\}\leq M_\nu(x,h)\max_{t\in[0,T]}\|\psi(t)\|,$$

where $M_\nu(x,h)=\max\{1+\int_0^x\tilde{\rho}(\xi)d\xi,\ \tilde{\rho}(\xi)\}$,

$$\tilde{\rho}(x)=z(x)\sigma(x)\exp(\int_0^x\rho(\xi)\sigma(\xi)d\xi)\int_0^x g(\xi)d\xi+\theta(x)+[\sigma(x)+1][b_1(x)+b_2(x)].$$

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Н.Т. Орумбаева, Т.Д. Токмагамбетова, Ж.Н. Нургалиева

Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан

Сызықтық емес Гурс теңдеуі үшін жартылайпериодтық шеттік есептің шешімділігі туралы

Мақалада айнымалыларды ауыстыру арқылы Гурс теңдеуі үшін сызықты емес жартылайпериодты шеттік есебі гиперболалық теңдеулер үшін сызықтық гравитация есебіне келтірілген. Жаңа функцияны қайта енгізу арқылы алынған есеп қарапайым дифференциалдық теңдеулер мен функционалдық қатынастарға арналған шекаралық есептер тобына түседі. Қарапайым дифференциалдық теңдеулер үшін шекаралық есептер тобын шешкен кезде параметрлеу әдісі қолданылды. Бұл тәсілді пайдалану Гурс теңдеуі үшін периодтық есептің біржақты шешілу коэффициенттерін анықтауға және жуық шешімді іздеудің конструктивті алгоритмдерін ұсынуға мүмкіндік береді.

Кілт сөздер: жартылайпериодты шеттік есеп, екінші ретті шеттік есеп, Гурс теңдеуі, шеттік есеп, алгоритм, жуық шешім.

Н.Т. Орумбаева, Т.Д. Токмагамбетова, Ж.Н. Нургалиева

Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан

О разрешимости полупериодической краевой задачи для нелинейного уравнения Гурса

В статье с помощью замены переменных нелинейная полупериодическая краевая задача для уравнения Гурса сведена к линейной задаче гравитации для гиперболических уравнений. Повторно введена новая функция, а полученная задача сведена к семейству краевых задач для обыкновенных дифференциальных уравнений и функциональных соотношений. При решении семейства краевых задач для обыкновенных дифференциальных уравнений использован метод параметризации. Применение данного подхода позволило установить коэффициенты однозначной разрешимости полуперидической задачи для уравнения Гурса и предложить конструктивные алгоритмы поиска приближенного решения.

Ключевые слова: полупериодическая краевая задача, краевая задача второго порядка, уравнение Гурса, краевая задача, алгоритм, приближенное решение.

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V.M. Savchin*, L.T. Huyen

Peoples' Friendship University of Russia, Moscow, Russia
(E-mail: savchin-vm@rudn.ru, luuthihuyen250393@gmail.com)

Nonpotentiality of a diffusion system and the construction of a semi-bounded functional

The wide prevalence and the systematic variational principles are used in mathematics and applications due to a series of remarkable consequences among which the possibility to establish the existence of the solutions of the initial equations, and the determination of stable approximations of the solutions of the considered equations by the so-called variational methods. In this connection, it is natural for a given system of equations to investigate the problem of the existence of its variational formulations. It can be considered as the inverse problem of the calculus of variations. The main goal of this work is to study this problem for a diffusion system of partial differential equations. A key object is the criterion of potentiality. On its ground, the nonpotentiality of the operator of the given boundary value problem with respect to the classical bilinear form is proved. This system does not admit a matrix variational multiplier of the given form. Thus, the diffusion system cannot be deduced from the classical Hamilton's principle. We posed the question that whether there exists a functional semi-bounded on solutions to the boundary value problem. We have done the algorithm of the constructive determination of such a functional. The main value of constructed functional action will be in applications of direct variational methods.

Keywords: nonpotential operators, diffusion system, semi-bounded functionals, variational multiplier.

Introduction

We consider the following system of partial differential equations (PDE) [1, 2]:

$$\tilde{N}^1(u) \equiv \sum_{i,j=1}^{n} a^{ij}(x,t,u^1)\frac{\partial^2 u^1}{\partial x^i \partial x^j} + f\left(x,t,u^1,\frac{\partial u^1}{\partial x^k}\right) - \frac{\partial u^1}{\partial t} = F^1(x,t,u^1,u^2),$$

$$\tilde{N}^2(u) \equiv \frac{\partial u^2}{\partial t} = F^2(x,t,u^1,u^2), \tag{1}$$

$$(x,t) = (x^1,...,x^n,t) \in Q_T = \Omega \times (0,T),$$

where the components u^1, u^2 of the vector u are unknown functions, the domain $\Omega \subset \mathbb{R}^n$ is bounded by the smooth surface $\partial\Omega$, $F^i : Q_T \times \mathbb{R}^2 \to \mathbb{R}(i = 1,2)$ are given differentiable functions, $f : Q_T \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a given twice differentiable function.

Denoting $F = (F^1,F^2), \tilde{N} = (\tilde{N^1},\tilde{N^2}), N = \tilde{N} - F$, we set

$$D(N) = \{(u^1,u^2) : u^1 \in C^{2,1}(\overline{Q}_T); u^2 \in C^1(\overline{Q}_T), u^i|_{t=0} = u^i_0(x^1,...,x^n),$$

$$u^i|_{t=T} = u^i_1(x^1,...,x^n), u^i|_{\partial\Omega\times(0,T)} = \psi^i(x,t)(i = 1,2)\}, \tag{2}$$

where $\psi^i(x,t), u^i_j(x) \in C(\overline{\Omega})(i = 1,2; j = 0,1)$ are given continuous functions, $\overline{\Omega} = \Omega \cup \partial\Omega$, $\overline{Q}_T = \overline{\Omega} \times [0,T]$.

*Corresponding author.
E-mail: savchin-vm@rudn.ru

In [3], it was proved a maximum principle (comparison theorem), existence and uniqueness theorems, and also convergence of the method of successive approximations for the directional derivative problem for the diffusion system.

In [4], it was illustrated that a comparison theorem is impossible for a multicomponent diffusion system unless further conditions are placed on the monotonicity of the functions involved. The method requires the construction of a counter-example.

In [5; 172], there were recently presented results of investigation of singularly perturbed reaction-advection-diffusion problems, which are based on a further development of the asymptotic comparison principle.

In [6], the separation method was used in order to obtain sufficient conditions for the solvability of the main inverse problem in the class of first-order Ito stochastic differential systems with random perturbations from the class of Wiener processes and diffusion degenerate with respect to a part of variables.

The problem of existence of Hamilton's principle for (1), (2) has not been investigated before. In modern interpretation [7], it can be considered as an inverse problem of the calculus of variations (IPCV).

The main aim of the paper is to investigate the existence of a solution of IPCV-Hamilton's principle for problem (1), (2).

Nonpotentiality of diffusion system

Let U, V be normed linear spaces over the field of real numbers $\mathbb{R}$, $U \subseteq V$; 0_U and 0_V be the zero elements in U and V respectively; N: $D(N) \subseteq U \to R(N) \subseteq V$ be an arbitrary twice Gâteaux differentiable operator with the domain $D(N)$ and the range $R(N)$.

We set N'_u as the first Gâteaux derivative of N at the point $u \in D(N)$ defined by the formula [7]

$$N'_u h = \frac{d}{d\varepsilon} N(u + \varepsilon h)|_{\varepsilon=0} = \delta N(u, h).$$

The mapping $\Phi(u; \cdot, \cdot) : V \times U \to \mathbb{R}$, being linear in each argument and depending on the parameter $u \in U$, is called a local bilinear form.

$\Phi'_u(h; v, g)$ is defined by

$$\Phi'_u(h; v, g) = \frac{d}{d\varepsilon} \Phi(u + \varepsilon h; v, g)|_{\varepsilon=0}.$$

Φ is called a nonlocal bilinear form if it does not depend on the parameter u, that is, $\Phi(u; \cdot, \cdot) \equiv \langle \cdot, \cdot \rangle$. Then $\Phi'_u(h; v, g) \equiv 0$.

It is said that $\langle \cdot, \cdot \rangle : V \times U \to \mathbb{R}$ is a nondegenerate nonlocal bilinear form if

1) the condition $\langle v, g \rangle = 0 \ \forall g \in U$ implies that $v = 0_V$;

2) the condition $\langle v, g \rangle = 0 \ \forall v \in V$ implies that $g = 0_U$.

Definition 1. [8] The operator $N : D(N) \subseteq U \to V$ is said to be potential on the set $D(N)$ with respect to the local bilinear form $\Phi(u; \cdot, \cdot) : V \times U \to \mathbb{R}$ if there exists a functional $F_N : D(F_N) = D(N) \to \mathbb{R}$ such that $\delta F_N[u, h] = \Phi(u; N(u), h) \forall u \in D(N), \forall h \in D(N'_u)$. Here F_N is called the potential of the operator N.

For the following explanation, we need the next theorem.

Theorem 1. [9] Let $N : D(N) \subseteq U \to V$ be a Gâteaux differentiable operator on the convex set $D(N)$ and the local bilinear form $\Phi(u; \cdot, \cdot) : V \times U \to \mathbb{R}$ be such that for any fixed elements $u \in D(N)$, and $g, h \in D(N'_u)$ the function $\varphi(\varepsilon) \equiv \Phi\left(u + \varepsilon h; N(u + \varepsilon h), g\right) \in C^1[0, 1]$. Then for the potentiality of the operator N on $D(N)$ with respect to Φ , it is necessary and sufficient that

$$\begin{gathered} J_{N,h,g}(u) \equiv \Phi(u; N'_u h, g) + \Phi'_u(h; N(u), g) = \\ = \Phi(u; N'_u g, h) + \Phi'_u(g; N(u), h) \quad \forall u \in D(N), \forall g, h \in D(N'_u). \end{gathered} \tag{3}$$

In this case

$$F_N[u] = \int_0^1 \Phi(u(\lambda); N(u(\lambda)), u - u_0) d\lambda + F_N[u_0],$$

where $u(\lambda) \equiv u_0 + \lambda(u - u_0)$; u_0 – an arbitrary fixed element from $D(N)$.

Condition (3) is called the criterion of the potentiality of the operator N with respect to the local bilinear form Φ.

Remark 1. If Φ is a nonlocal bilinear form, then (3) becomes

$$\langle N'_u h, g\rangle = \langle N'_u g, h\rangle \quad \forall u \in D(N), \forall g, h \in D(N'_u). \tag{4}$$

Let us introduce the classical bilinear form by

$$\Phi_1(v, g) = \langle v, g\rangle = \int_{Q_T} \sum_{i=1}^{2} v^i(x,t) g^i(x,t) dxdt. \tag{5}$$

Theorem 2. Operator (1) is not potential on set (2) with respect to nonlocal bilinear form (5).

Proof. From (1), we find the Gâteaux derivative

$$N'_u = \begin{pmatrix} a_1^1 & -\frac{\partial F^1}{\partial u^2} \\ -\frac{\partial F^2}{\partial u^1} & \frac{\partial}{\partial t} - \frac{\partial F^2}{\partial u^2} \end{pmatrix},$$

where

$$a_1^1 = \sum_{i,j=1}^{n} \left(\frac{\partial^2 u^1}{\partial x^i \partial x^j} \frac{\partial a^{ij}}{\partial u^1} + a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \right) + \frac{\partial f}{\partial u^1} + \sum_{k=1}^{n} \frac{\partial f}{\partial u^1_{x^k}} \frac{\partial}{\partial x^k} - \frac{\partial}{\partial t} - \frac{\partial F^1}{\partial u^1},$$

and $u^1_{x^k} = \frac{\partial u^1}{\partial x^k}$, $a^{ij} \equiv a^{ij}(x,t,u^1)$.

In accordance with conditions (2), we have

$$D(N'_u) = \{(h^1, h^2) : h^1 \in C^{2,1}(\overline{Q}_T), h^2 \in C^1(\overline{Q}_T), h^i|_{t=0} = 0,$$
$$h^i|_{t=T} = 0, h^i|_{\partial\Omega\times(0,T)} = 0 (i = 1, 2)\}.$$

Denoting by N'^*_u the adjoint operator to N'_u, we find

$$N'^*_u = \begin{pmatrix} a_1^{1*} & -\frac{\partial F^2}{\partial u^1} \\ -\frac{\partial F^1}{\partial u^2} & -\frac{\partial}{\partial t} - \frac{\partial F^2}{\partial u^2} \end{pmatrix},$$

where

$$\begin{aligned} a_1^{1*} &= \sum_{i,j=1}^{n} \left(\frac{\partial^2 u^1}{\partial x^i \partial x^j} \frac{\partial a^{ij}}{\partial u^1} + a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \frac{\partial^2 a^{ij}}{\partial x^j \partial x^i} + \frac{\partial a^{ij}}{\partial x^i} \frac{\partial}{\partial x^j} + \frac{\partial a^{ij}}{\partial x^j} \frac{\partial}{\partial x^i} \right) + \\ &+ \tfrac{\partial f}{\partial u^1} - \sum_{k=1}^{n} \left(\frac{\partial^2 f}{\partial u^1_{x^k} \partial x^k} + \frac{\partial f}{\partial u^1_{x^k}} \frac{\partial}{\partial x^k} \right) + \frac{\partial}{\partial t} - \frac{\partial F^1}{\partial u^1}, \end{aligned}$$

$$D(N'^*_u) = \{(v^1, v^2) : v^1 \in C^{2,1}(\overline{Q}_T), v^2 \in C^1(\overline{Q}_T), v^i|_{t=0} = 0, v^i|_{t=T} = 0, v^i|_{\partial\Omega\times(0,T)} = 0 (i = 1, 2)\}.$$

Let us prove that operator (1) does not satisfy criterion (4). For that, we find

$$\begin{aligned} \Phi_1(N'_u h, g) &= \int_{Q_T} \left\{ \left(a_1^1 h^1 - \frac{\partial F^1}{\partial u^2} h^2 \right) g^1 - \left[\frac{\partial F^2}{\partial u^1} h^1 + \left(\frac{\partial F^2}{\partial u^2} - \frac{\partial}{\partial t} \right) h^2 \right] g^2 \right\} dxdt \\ &= \int_{Q_T} \left\{ \left[\sum_{i,j=1}^{n} \left(\frac{\partial^2 u^1}{\partial x^i \partial x^j} \frac{\partial a^{ij}}{\partial u^1} h^1 + a^{ij} \frac{\partial^2 h^1}{\partial x^i \partial x^j} \right) + \right.\right. \\ &+ \tfrac{\partial f}{\partial u^1} h^1 + \sum_{k=1}^{n} \frac{\partial f}{\partial u^1_{x^k}} \frac{\partial h^1}{\partial x^k} - \frac{\partial h^1}{\partial t} - \frac{\partial F^1}{\partial u^1} h^1 - \left. \frac{\partial F^1}{\partial u^2} h^2 \right] g^1 - \\ &- \left. \left(\tfrac{\partial F^2}{\partial u^1} h^1 + \tfrac{\partial F^2}{\partial u^2} h^2 - \tfrac{\partial h^2}{\partial t} \right) g^2 \right\} dxdt. \end{aligned}$$

Integrating by parts, we obtain

$$\Phi_1(N'_u h, g) = \int_{Q_T} \Bigg\{ \sum_{i,j=1}^{n} \Bigg[\frac{\partial a^{ij}}{\partial u^1} \frac{\partial^2 u^1}{\partial x^i \partial x^j} h^1 g^1 + D_{x^i} \left(a^{ij} \frac{\partial h^1}{\partial x^j} g^1 \right) - \frac{\partial a^{ij}}{\partial x^i} \frac{\partial h^1}{\partial x^j} g^1 - -a^{ij} \frac{\partial h^1}{\partial x^j} \frac{\partial g^1}{\partial x^i} \Bigg] + \frac{\partial f}{\partial u^1} h^1 g^1 + \sum_{k=1}^{n} \Bigg[D_{x^k} \left(\frac{\partial f}{\partial u^1_{x^k}} h^1 g^1 \right) - -\frac{\partial^2 f}{\partial u^1_{x^k} \partial x^k} h^1 g^1 - \frac{\partial f}{\partial u^1_{x^k}} h^1 \frac{\partial g^1}{\partial x^k} \Bigg] - D_t(h^1 g^1) + h^1 \frac{\partial g^1}{\partial t} - -\frac{\partial F^1}{\partial u^1} h^1 g^1 - \frac{\partial F^1}{\partial u^2} h^2 g^1 - \frac{\partial F^2}{\partial u^1} h^1 g^2 - \frac{\partial F^2}{\partial u^2} h^2 g^2 + +D_t(h^2 g^2) - h^2 \frac{\partial g^2}{\partial t} \Bigg\} dxdt \;\; \forall h, g \in D(N'_u),$$

where $D_{x^i} = \dfrac{\partial}{\partial x^i}$, $D_t = \dfrac{\partial}{\partial t}$.

By virtue of the Divergence theorem and the condition $h \in D(N'_u)$, we have

$$\int_0^T \int_\Omega \left[D_{x^i} \left(a^{ij} \frac{\partial h^1}{\partial x^j} g^1 \right) \right] dx^1 ... dx^n dt = = \int_0^T \left[\int_{\partial\Omega} a^{ij} \frac{\partial h^1}{\partial x^j} g^1 dx^1 ... dx^{i-1} dx^{i+1} ... dx^n \right] dt = 0.$$

Similarly, we have

$$\int_0^T \int_\Omega \left[D_{x^k} \left(\frac{\partial f}{\partial u^1_{x^k}} h^1 g^1 \right) \right] dx^1 ... dx^n dt = 0,$$

$$\int_{Q_T} \Big[- D_t(h^1 g^1) + D_t(h^2 g^2) \Big] dxdt = \int_\Omega \Big(- h^1 g^1 |_{t=0}^{t=T} + h^2 g^2 |_{t=0}^{t=T} \Big) dx = 0.$$

Integrating by parts and applying the above results, we get

$$\Phi_1(N'_u h, g) = \int_{Q_T} \Bigg\{ \Bigg[\sum_{i,j=1}^{n} \left(\frac{\partial^2 u^1}{\partial x^i \partial x^j} \frac{\partial a^{ij}}{\partial u^1} g^1 + a^{ij} \frac{\partial^2 g^1}{\partial x^i \partial x^j} + \frac{\partial^2 a^{ij}}{\partial x^i \partial x^j} g^1 + + \frac{\partial a^{ij}}{\partial x^i} \frac{\partial g^1}{\partial x^j} + \frac{\partial a^{ij}}{\partial x^j} \frac{\partial g^1}{\partial x^i} \right) + \frac{\partial f}{\partial u^1} g^1 - \sum_{k=1}^{n} \left(\frac{\partial^2 f}{\partial u^1_{x^k} \partial x^k} g^1 + + \frac{\partial f}{\partial u^1_{x^k}} \frac{\partial g^1}{\partial x^k} \right) + \frac{\partial g^1}{\partial t} - \frac{\partial F^1}{\partial u^1} g^1 - \frac{\partial F^2}{\partial u^1} g^2 \Bigg] h^1 - - \left(\frac{\partial F^1}{\partial u^2} g^1 + \frac{\partial F^2}{\partial u^2} g^2 + \frac{\partial g^2}{\partial t} \right) h^2 \Bigg\} dxdt = \int_{Q_T} \left\{ \left(a_1^{1*} g^1 - \frac{\partial F^2}{\partial u^1} g^2 \right) h^1 - \left[\frac{\partial F^1}{\partial u^2} g^1 + \left(\frac{\partial F^2}{\partial u^2} + \frac{\partial}{\partial t} \right) g^2 \right] h^2 \right\} dxdt. \tag{6}$$

On the other hand, we have

$$\Phi_1(N'_u g, h) = \int_{Q_T} \left\{ \left(a_1^1 g^1 - \frac{\partial F^1}{\partial u^2} g^2 \right) h^1 - \left[\frac{\partial F^2}{\partial u^1} g^1 + \left(\frac{\partial F^2}{\partial u^2} - \frac{\partial}{\partial t} \right) g^2 \right] h^2 \right\} dxdt. \tag{7}$$

In (6) the coefficient at h^2 is $-\left[\frac{\partial F^1}{\partial u^2} g^1 + \left(\frac{\partial F^2}{\partial u^2} + \frac{\partial}{\partial t} \right) g^2 \right]$ and in (7) it is $-\left[\frac{\partial F^2}{\partial u^1} g^1 + \left(\frac{\partial F^2}{\partial u^2} - \frac{\partial}{\partial t} \right) g^2 \right]$. It follows that $\Phi_1(N'_u h, g)$ is not indentically equal to $\Phi_1(N'_u g, h)$. Thus, criterion (4) is not satisfied.

Let us investigate the existence of the matrix variational multiplier for operator (1).

Definition 2. An invertible linear operator $M : D(M) \subset R(N) \to V$ is called a variational multiplier for the operator $N : D(N) \subset U \to V$ if the operator $\hat{N} = MN$ is potential on the set $D(N)$ with respect to the given bilinear form.

Theorem 3. There is no matrix variational multiplier of the kind

$$M = \begin{pmatrix} m_{11}(x,t) & m_{12}(x,t) \\ m_{21}(x,t) & m_{22}(x,t) \end{pmatrix} \tag{8}$$

for operator $N(1)$.

Proof. Suppose that there exists a matrix variational multiplier of form (8) and $detM \neq 0$. Then the operator $\hat{N}(u) = MN(u)$ is potential with respect to the classical bilinear form (5).

Denoting $m_{pq} \equiv m_{pq}(x,t), p, q = 1, 2$ we get

$$\begin{aligned} \Phi_1(\hat{N'}_u h, g) &= \int_{Q_T} \sum_{p=1}^{2} \left[m_{p1} \left(a_1^1 h^1 - \frac{\partial F^2}{\partial u^1} h^2 \right) g^p + m_{p2} \left(\frac{\partial h^2}{\partial t} - \frac{\partial F^2}{\partial u^1} h^1 - \frac{\partial F^2}{\partial u^2} h^2 \right) g^p \right] dxdt = \\ &= \int_{Q_T} \sum_{p=1}^{2} \Bigg\{ m_{p1} \Bigg[\sum_{i,j=1}^{n} \left(\frac{\partial^2 u^1}{\partial x^i \partial x^j} \frac{\partial a^{ij}}{\partial u^1} h^1 + a^{ij} \frac{\partial^2 h^1}{\partial x^i \partial x^j} \right) + \frac{\partial f}{\partial u^1} h^1 + \\ &\quad + \sum_{k=1}^{n} \frac{\partial f}{\partial u^1_{x^k}} \frac{\partial h^1}{\partial x^k} - \frac{\partial h^1}{\partial t} - \frac{\partial F^1}{\partial u^1} h^1 - \frac{\partial F^2}{\partial u^1} h^2 \Bigg] g^p + \\ &\quad + m_{p2} \left(\frac{\partial h^2}{\partial t} - \frac{\partial F^2}{\partial u^1} h^1 - \frac{\partial F^2}{\partial u^2} h^2 \right) g^p \Bigg\} dxdt. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} \Phi_1\left(\hat{N'}_u h, g\right) &= \int_{Q_T} \sum_{p=1}^{2} \Bigg\{ \sum_{i,j=1}^{n} \Bigg[m_{p1} \frac{\partial a^{ij}}{\partial u^1} \frac{\partial^2 u^1}{\partial x^i \partial x^j} h^1 g^p + D_{x^j} \left(m_{p1} a^{ij} \frac{\partial h^1}{\partial x^i} g^p \right) - \\ &\quad - \frac{\partial m_{p1}}{\partial x^j} a^{ij} \frac{\partial h^1}{\partial x^i} g^p - m_{p1} \frac{\partial a^{ij}}{\partial x^j} \frac{\partial h^1}{\partial x^i} g^p - m_{p1} a^{ij} \frac{\partial h^1}{\partial x^i} \frac{\partial g^p}{\partial x^j} \Bigg] + \\ &\quad + m_{p1} \frac{\partial f}{\partial u^1} h^1 g^p + \sum_{k=1}^{n} \Bigg[D_{x^k} \left(m_{p1} \frac{\partial f}{\partial u^1_{x^k}} h^1 g^p \right) - \\ &\quad - \frac{\partial m_{p1}}{\partial x^k} \frac{\partial f}{\partial u^1_{x^k}} h^1 g^p - m_{p1} \frac{\partial^2 f}{\partial u^1_{x^k} \partial x^k} h^1 g^p - m_{p1} \frac{\partial f}{\partial u^1_{x^k}} h^1 \frac{\partial g^p}{\partial x^k} \Bigg] - \\ &\quad - D_t(m_{p1} h^1 g^p) + \frac{\partial m_{p1}}{\partial t} h^1 g^p + m_{p1} h^1 \frac{\partial g^p}{\partial t} - \\ &\quad - m_{p1} \frac{\partial F^1}{\partial u^1} h^1 g^p - m_{p1} \frac{\partial F^2}{\partial u^1} h^2 g^p - m_{p2} \frac{\partial F^2}{\partial u^1} h^1 g^p - m_{p2} \frac{\partial F^2}{\partial u^2} h^2 g^p + \\ &\quad + D_t(m_{p2} h^2 g^p) - \frac{\partial m_{p2}}{\partial t} h^2 g^p - m_{p2} h^2 \frac{\partial g^p}{\partial t} \Bigg\} dxdt. \end{aligned}$$

Applying the above results, we get

$$\begin{aligned} \Phi_1(\hat{N'}_u h, g) &= \int_{Q_T} \sum_{p=1}^{2} \Bigg[h^1 \Bigg(A_1 g^p + B_1 \frac{\partial g^p}{\partial x^k} + C_1 \frac{\partial g^p}{\partial x^i} + D_1 \frac{\partial g^p}{\partial x^j} + \\ &\quad + E_1 \frac{\partial^2 g^p}{\partial x^j \partial x^i} + m_{p1} \frac{\partial g^p}{\partial t} \Bigg) + h_2 \left(G_1 g^p - m_{p2} \frac{\partial g^p}{\partial t} \right) \Bigg] dxdt, \end{aligned}$$

where

$$A_1=\sum_{i,j=1}^n\left(m_{p1}\frac{\partial a^{ij}}{\partial u^1}\frac{\partial^2 u^1}{\partial x^i\partial x^j}+\frac{\partial^2 m_{p1}}{\partial x^j\partial x^i}a^{ij}+\frac{\partial m_{p1}}{\partial x^j}\frac{\partial a^{ij}}{\partial x^i}+\right.$$

$$\left.+\tfrac{\partial m_{p1}}{\partial x^i}\tfrac{\partial a^{ij}}{\partial x^j}+m_{p1}\tfrac{\partial^2 a^{ij}}{\partial x^j\partial x^i}\right)+m_{p1}\tfrac{\partial f}{\partial u^1}-\sum_{k=1}^n\left(\frac{\partial m_{p1}}{\partial x^k}\frac{\partial f}{\partial u^1_{x^k}}+\right.$$

$$\left.+m_{p1}\tfrac{\partial^2 f}{\partial u^1_{x^k}\partial x^k}\right)+\tfrac{\partial m_{p1}}{\partial t}-m_{p1}\tfrac{\partial F^1}{\partial u^1}-m_{p2}\tfrac{\partial F^2}{\partial u^1},$$

$$B_1=-m_{p1}\sum_{k=1}^n\frac{\partial f}{\partial u^1_{x^k}},C_1=\sum_{i,j=1}^n\left(\frac{\partial m_{p1}}{\partial x^j}a^{ij}+m_{p1}\frac{\partial a^{ij}}{\partial x^j}\right),$$

$$D_1=\sum_{i,j=1}^n\left(\frac{\partial m_{p1}}{\partial x^i}a^{ij}+m_{p1}\frac{\partial a^{ij}}{\partial x^i}\right),E_1=m_{p1}\sum_{i,j=1}^n a^{ij},$$

$$G_1=-\tfrac{\partial m_{p2}}{\partial t}-m_{p1}\tfrac{\partial F^2}{\partial u^1}-m_{p2}\tfrac{\partial F^2}{\partial u^2}.$$

On the other hand, we have

$$\Phi_1\left(\hat{N}'{}_u g,h\right)=\int_{Q_T}\sum_{p=1}^2\left[m_{p1}\left(a_1^1 g^1-\frac{\partial F^2}{\partial u^1}g^2\right)h^p+\right.$$

$$\left.+m_{p2}\left(\tfrac{\partial g^2}{\partial t}-\tfrac{\partial F^2}{\partial u^1}g^1-\tfrac{\partial F^2}{\partial u^2}g^2\right)h^p\right]dxdt$$

$$=\int_{Q_T}\sum_{p=1}^2 h^p\left[A_2g^1+B_2\frac{\partial g^1}{\partial x^k}+E_1\frac{\partial^2 g^1}{\partial x^i\partial x^j}-m_{p1}\frac{\partial g^1}{\partial t}-\right.$$

$$\left.-\left(m_{p1}\tfrac{\partial F^2}{\partial u^1}+m_{p2}\tfrac{\partial F^2}{\partial u^2}\right)g^2+m_{p2}\tfrac{\partial g^2}{\partial t}\right]dxdt,$$

where

$$A_2=m_{p1}\left(\sum_{i,j=1}^n\frac{\partial a^{ij}}{\partial u^1}\frac{\partial^2 u^1}{\partial x^i\partial x^j}+\frac{\partial f}{\partial u^1}-\frac{\partial F^1}{\partial u^1}\right)-m_{p2}\frac{\partial F^2}{\partial u^1},B_2=m_{p1}\frac{\partial f}{\partial u^1_{x^k}}.$$

Hence,

$$\Phi_1\left(\hat{N}'{}_u h,g\right)-\Phi_1\left(\hat{N}'{}_u g,h\right)=\int_{Q_T}\left\{h^1\left[(A_1-A_2)g^1+\left(A_1+\right.\right.\right.$$
$$+m_{11}\frac{\partial F^2}{\partial u^1}+m_{12}\frac{\partial F^2}{\partial u^2}\Big)g^2+\Big(B_1-B_2\Big)\frac{\partial g^1}{\partial x^k}+B_1\frac{\partial g^2}{\partial x^k}+$$
$$+C_1\left(\frac{\partial g^1}{\partial x^i}+\frac{\partial g^2}{\partial x^i}\right)+D_1\left(\frac{\partial g^1}{\partial x^j}+\frac{\partial g^2}{\partial x^j}\right)+E_1\frac{\partial^2 g^2}{\partial x^j\partial x^i}+2m_{11}\frac{\partial g^1}{\partial t}\tag{9}$$
$$+\Big(m_{21}-m_{12}\Big)\frac{\partial g^2}{\partial t}\Big]+h_2\Big[(G_1-A_2)g^1-\Big(m_{12}-m_{21}\Big)\frac{\partial g^1}{\partial t}+$$
$$\left.\left.+\left(G_1+m_{21}\frac{\partial F^2}{\partial u^1}+m_{22}\frac{\partial F^2}{\partial u^2}\right)g^2-2m_{22}\frac{\partial g^2}{\partial t}-B_2\frac{\partial g^1}{\partial x^k}-E_1\frac{\partial^2 g^1}{\partial x^i\partial x^j}\right]\right\}dxdt.$$

According to criterion (4), it must be

$$\Phi_1\left(\hat{N}'_u h,g\right)-\Phi_1\left(\hat{N}'_u g,h\right)=0\ \forall u\in D(N),\ \forall h,g\in D(N'_u).$$

By virtue of the arbitrariness of the functions h^k $(k=1,2)$ from (9) we obtain

$$(A_1 - A_2)g^1 + \left(A_1 + m_{11}\frac{\partial F^2}{\partial u^1} + m_{12}\frac{\partial F^2}{\partial u^2}\right)g^2 + \left(B_1 - B_2\right)\frac{\partial g^1}{\partial x^k} +$$
$$+B_1\frac{\partial g^2}{\partial x^k} + C_1\left(\frac{\partial g^1}{\partial x^i} + \frac{\partial g^2}{\partial x^i}\right) + D_1\left(\frac{\partial g^1}{\partial x^j} + \frac{\partial g^2}{\partial x^j}\right) + E_1\frac{\partial^2 g^2}{\partial x^j \partial x^i} +$$
$$+2m_{11}\frac{\partial g^1}{\partial t} + \left(m_{21} - m_{12}\right)\frac{\partial g^2}{\partial t} = 0,$$
$$(G_1 - A_2)g^1 - \left(m_{12} - m_{21}\right)\frac{\partial g^1}{\partial t} + \left(G_1 + m_{21}\frac{\partial F^2}{\partial u^1} +\right.$$
$$\left.+m_{22}\frac{\partial F^2}{\partial u^2}\right)g^2 - 2m_{22}\frac{\partial g^2}{\partial t} - B_2\frac{\partial g^1}{\partial x^k} - E_1\frac{\partial^2 g^1}{\partial x^i \partial x^j} = 0.$$

From here, by virtue of the arbitrariness of the functions g^k $(k = 1, 2)$, we get

$$m_{p1} = 0,$$
$$m_{21} - m_{12} = 0,$$
$$m_{22} = 0.$$

In total, $m_{pq}(x, t) = 0$ $(p, q = 1, 2)$. Therefore, $M = 0$ (zero matrix). It contradicts to what we have supposed above.

The construction of a semibounded functional

We have already proved that operator (1) is not potential with respect to nonlocal bilinear form (5) and there is no matrix variational multiplier of the given type. For the following exposition, we need the next theorem.

Consider an arbitrary equation

$$N(u) = 0_V, u \in D(N) \subseteq U \subseteq V, \tag{10}$$

where the operator N, in general case, is nonpotential with respect to the fixed nonlocal bilinear form $\Phi_1(\cdot, \cdot) \equiv \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$.

Theorem 4. [8] Let: 1) $N : D(N) \subseteq U \to V$ be a twice Gâteaux differentiable operator on the convex set $D(N)$; 2) $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be a given nonlocal bilinear form; 3) $C : D(C) \supseteq R(N) \to V$ be an arbitrary invertible linear symmetric operator, such that for any fixed elements $u \in D(N)$ and $g, h \in D(N'_u)$ the function $\varphi(\varepsilon) \equiv \left\langle N(u + \varepsilon h), CN'_{u+\varepsilon h} g\right\rangle \in C^1[0, 1]$. Then the operator N is potential on $D(N)$ with respect to the following local bilinear form

$$\Phi(u; v, g) = \langle v, CN'_u g\rangle .$$

Herewith

$$F_N[u] = \frac{1}{2}\langle N(u), CN(u)\rangle . \tag{11}$$

The proof is given in [8].

Note that $\delta F_N[u, h] = \Phi(u; N(u), h) = \langle N(u), CN'_u h\rangle$.

Denoting the adjoint operator of N'_u by N'^*_u and assuming that $R(C) \subseteq D(N'^*_u)$, it follows from the last equality that $\delta F_N[u, h] = \langle N'^*_u CN(u), h\rangle$ $\forall u \in D(N)$, $\forall h \in D(N'_u)$.

Assuming that $D(N'_u) = U$ and $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is a nonsingular continuous (in every argument) nonlocal bilinear form, we get $\delta F_N[u, h] = 0, u \in D(N), \forall h \in D(N'_u)$ if and only if

$$N_1(u) \equiv N'^*_u CN(u) = 0_V, u \in D(N). \tag{12}$$

Thus, the operator N_1 is potential on $D(N)$ with respect to the nonlocal bilinear form Φ_1.

If N'^*_u is an invertible operator, then problems (12) and (10) are equivalent in the following sense: if $\tilde{u}$ is a solution to one of them, then $\tilde{u}$ is a solution to the other ,i.e., $N(\tilde{u}) = 0_V \Leftrightarrow N_1(\tilde{u}) = 0_V$. In this case the functional (11) provides an indirect variational statement of problems (10).

If the operator C is positive definite with respect to nonlocal bilinear form $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{R}$, i.e., $\langle v, Cv\rangle \geq k\,\|v\|\,\forall v\in D(C)$, where $k>0$, then $F_N[u]\geq 0\;\forall u\in D(N)$ and $F_N[\tilde{u}]=0 \Leftrightarrow \tilde{u}$ is a solution of (10). Thus, in this case, formula (11) specifies a semi-bounded functional whose mininum is attained on the solutions to problem (10).

Note that functional (11) was obtained in another way in [10] when solving one of the statements of the inverse problem of the calculus of variations.

Let us define the operator C on $R(N)$ by the formula

$$(Cv)^j(x,t)=\int_{Q_T} K(x,t,y,\tau)\phi^j(x,t)\phi^j(y,\tau)v^j(y,\tau)dyd\tau \;(j=1,2), \tag{13}$$

where

$$K(x,t,y,\tau)\equiv K = exp\left(\sum_{i=1}^{n} x^i y^i + t\tau\right). \tag{14}$$

$\phi^i (i=1,2)$ are arbitrary functions of the class $C^{2,1}(\overline{Q}_T)$ such that $\phi^i(x,t)\neq 0((x,t)\in Q_T)$ and $\phi^i|_{t=0}=0$, $\phi^i|_{t=T}=0$, $\phi^i|_{\partial\Omega\times(0,T)}=0$ $(i=1,2)$. With this choice of functions ϕ^1,ϕ^2 we have $Cv\in D(N_u'^{*})$.

It is also easy to see that operator (13) is symmetric on R(N). Let us show that it is positive definite. For this, we find the expansion of function (14) in the Maclaurin series

$K=\sum_{|\alpha|=0}^{\infty}\frac{1}{(\alpha!)^2}\left(x^1\right)^{\alpha_1}...\left(x^n\right)^{\alpha_n}t^{\alpha_{n+1}}\left(y^1\right)^{\alpha_1}...\left(y^n\right)^{(\alpha_n)}\tau^{\alpha_{n+1}}$.

Here $\alpha=(\alpha_1,...,\alpha_{n+1})$; α_i $(i=\overline{1,n+1})$ are nonnegative integers; $|\alpha|=\sum\limits_{i=1}^{n+1}\alpha_i$, $\alpha!=\alpha_1!...\alpha_{n+1}!$.

Using the obtained series we find

$$\begin{aligned}\Phi_1(v,Cv) &= \int_{Q_T}\sum_{j=1}^{2} v^j(x,t)\int_{Q_T}K(x,t,y,\tau)\phi^j(x,t)\phi^j(y,\tau)v^j(y,\tau)dyd\tau dxdt\\ &= \sum_{j=1}^{2}\sum_{|\alpha|=0}^{\infty}\frac{1}{(\alpha!)^2}\int_{Q_T}(x^1)^{\alpha_1}..(x^n)^{\alpha_n}t^{\alpha_{n+1}}\phi^j(x,t)v^j(x,t)dxdt\\ &\times\int_{Q_T}(y^1)^{\alpha_1}...(y^n)^{\alpha_n}\tau^{\alpha_{n+1}}\phi^j(y,\tau)v^j(y,\tau)dyd\tau\\ &\equiv\sum_{j=1}^{2}\sum_{|\alpha|=0}^{\infty}\frac{1}{(\alpha!)^2}(M^{\alpha_1...\alpha_{n+1}j})^2\;\geq 0.\end{aligned}$$

We note that all the moments $M^{\alpha^1...\alpha^{n+1}j}$ vanish simultaneously if and only if $v^j=0(j=1,2)$ in Q_T [11]. Therefore, if $v\neq 0_V$ then $\Phi_1(v,Cv)>0$.

Thus, the operator C of form (13) is a positive definite and invertible.

Denoting by $K\equiv K(x,t,y,\tau)$, from (1) and (13) we get

$$\begin{aligned}(CN(u))^1(x,t) &= \int_{Q_T}\Bigg\{-\sum_{i,j=1}^{n}\frac{\partial u^1(y,\tau)}{\partial y^j}\phi^1(x,t)D_{y^i}\left[K\phi^1(y,\tau)a^{ij}(y,\tau,u^1)\right]+\\ &+K\phi^1(x,t)\phi^1(y,\tau)f\left(y,\tau,u^1,\frac{\partial u^1}{\partial y^k}\right)+u^1(y,\tau)\phi^1(x,t)\\ &D_\tau(K\phi^1(y,\tau))-K\phi^1(x,t)\phi^1(y,\tau)F^1(y,\tau,u^1,u^2)\Bigg\}dyd\tau,\\ (CN(u))^2(x,t) &= \int_{Q_T}[-u^2(y,\tau)\phi^2(x,t)D_\tau(K\phi^2(y,\tau))-\\ &-K\phi^2(x,t)\phi^2(y,\tau)F^2(y,\tau,u^1,u^2)]dyd\tau.\end{aligned} \tag{15}$$

Using formulas (1),(5),(11),(15) we find the required functional in the form

$$F_N[u]\;=\frac{1}{2}\int_{Q_T}\int_{Q_T}\Bigg\{L_1+L_2\Bigg\}dyd\tau dxdt,$$

where

$$L_1 = \left[\sum_{i,j=1}^{n} a^{ij}(x,t,u^1) \frac{\partial^2 u^1}{\partial x^i \partial x^j} + f\left(x,t,u^1,\frac{\partial u^1}{\partial x^k}\right) - \frac{\partial u^1(x,t)}{\partial t} - \right.$$

$$\left. -F^1(x,t,u^1,u^2) \right] \left[-\textstyle\sum_{i,j=1}^{n} \frac{\partial u^1(y,\tau)}{\partial y^j} \phi^1(x,t) D_{y^i} \left[K\phi^1(y,\tau) a^{ij}(y,\tau,u^1) \right] + \right.$$

$$+K\phi^1(x,t)\phi^1(y,\tau) f\left(y,\tau,u^1,\tfrac{\partial u^1}{\partial y^k}\right) + u^1(y,\tau)\phi^1(x,t) D_\tau (K\phi^1(y,\tau)) -$$

$$\left. -K\phi^1(x,t)\phi^1(y,\tau) F^1(y,\tau,u^1,u^2) \right],$$

$$L_2 = \left[\frac{\partial u^2(x,t)}{\partial t} - F^2(x,t,u^1,u^2) \right] \left[-u^2(y,\tau)\phi^2(x,t) D_\tau (K\phi^2(y,\tau)) - \right.$$

$$\left. -K\phi^2(x,t)\phi^2(y,\tau) F^2(y,\tau,u^1,u^2) \right].$$

Integrating by parts, we get

$$F_N[u] \quad = \tfrac{1}{2} \textstyle\int_{Q_T} \int_{Q_T} \left\{ H_1 + H_2 \right\} dy d\tau dx dt, \tag{16}$$

where

$$H_1 = \sum_{i,j=1}^{n} \frac{\partial u^1(x,t)}{\partial x^j} D_{x^i} \left\{ a^{ij}(x,t,u^1) \left[\sum_{i,j=1}^{n} \frac{\partial u^1(y,\tau)}{\partial y^j} \phi^1(x,t) D_{y^i} \right. \right.$$

$$\left[K\phi^1(y,\tau) a^{ij}(y,\tau,u^1) \right] + K\phi^1(x,t)\phi^1(y,\tau) f\left(y,\tau,u^1,\tfrac{\partial u^1}{\partial y^k}\right) +$$

$$\left. \left. +u^1(y,\tau)\phi^1(x,t) D_\tau (K\phi^1(y,\tau)) - K\phi^1(x,t)\phi^1(y,\tau) F^1(y,\tau,u^1,u^2) \right] \right\} +$$

$$+ \left[f\left(x,t,u^1,\frac{\partial u^1}{\partial x^k}\right) + u^1(x,t) D_t - F^1(x,t,u^1,u^2) \right]$$

$$\left[-\textstyle\sum_{i,j=1}^{n} \frac{\partial u^1(y,\tau)}{\partial y^j} \phi^1(x,t) D_{y^i} \left[K\phi^1(y,\tau) a^{ij}(y,\tau,u^1) \right] + K\phi^1(x,t) \right.$$

$$\phi^1(y,\tau) f\left(y,\tau,u^1,\tfrac{\partial u^1}{\partial y^k}\right) + u^1(y,\tau)\phi^1(x,t) D_\tau (K\phi^1(y,\tau)) -$$

$$\left. -K\phi^1(x,t)\phi^1(y,\tau) F^1(y,\tau,u^1,u^2) \right],$$

$$H_2 = u^2(x,t) u^2(y,\tau) D_t (\phi^2(x,t) D_\tau (K\phi^2(y,\tau))) +$$
$$+u^2(x,t)\phi^2(y,\tau) D_t (K\phi^2(x,t)) F^2(y,\tau,u^1,u^2) + u^2(y,\tau)\phi^2(x,t) D_\tau$$
$$(K\phi^2(y,\tau)) F^2(x,t,u^1,u^2) + K\phi^2(x,t)\phi^2(y,\tau) F^2(y,\tau,u^1,u^2) F^2(x,t,u^1,u^2).$$

Theorem 5. The functional of form (16) is semi-bounded on the solutions of problem (1),(2).
The theorem is proved above.

Remark 2. Functional (16) : 1) is bounded below on set (2); 2) takes a minimum value only on the solutions of problem (1), (2); 3) contains derivatives of unknown functions of lesser order, than the system of equations (1), (2); 4) the set of its stationary points contains the solution set of problem (1), (2).

Conslusions and future directions

The results of this paper can be summarized as follows.

(i) We studied the potentiality of the operator of the boundary value problem for a system of partial differential equations for diffusion. We showed that it is not potential with respect to the classical bilinear form. It means that the considered system cannot be obtained from Hamilton's variational principle.

(ii) The problem of the existence of a matrix variational multipler for (1) was investigated. We illustrated that there is no a matrix variational multiplier with elements depending on x and t.

(iii) We posed the question that whether there exists a functional semi-bounded on solutions of the given boundary value problem.

We have done the algorithm of the constructive determination of such a functional.

The main value of constructed functional (16) will be in applications of direct variational methods and its numerical performance.

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В.М. Савчин, Л.Т. Хуен

Ресей халықтар достығы университеті, Мәскеу, Ресей

Бір диффузиялық жүйе операторының бейпотенциалдығы және жартылайшекаралы функционалды құрастыру

Математика мен қосымшаларда вариациялық принциптердің кең таралуы және жүйелі қолданылуы бірқатар керемет салдарларға байланысты, олардың арасында бастапқы теңдеулер шешімдерінің бар екендігі және вариациялық әдістермен қарастырылатын шешімдердің тұрақты жуықтауларын анықтау мүмкіндігі бар. Осыған байланысты берілген теңдеулер жүйесі үшін оның вариациялық тұжырымдарының болуы туралы мәселені зерттеу заңды. Оны вариация есептеудің кері есебі деп қарастыруға болады. Бұл жұмыстың басты мақсаты — дербес туындылы дифференциалдық теңдеулердің диффузиялық жүйесі үшін осы есепті зерттеу. Негізгі объект — бұл потенциал критерийі. Оның негізінде берілген шекаралық есеп операторының классикалық белгісіз формаға қатысты бейпотенциалдығы дәлелденді. Бұл жүйенің берілген форманың матрицалық вариациялық көбейткішін қабылдамайтындығы көрсетілген. Осылайша, берілген диффузиялық жүйені классикалық Гамильтон принципінен шығаруға болмайды. Берілген шекаралық есепті шешудің функционалды жартылай байланысы бар ма деген сұрақ қойылған. Осындай функционалды конструктивті анықтау алгоритмі жасалған. Құрылған функционалдың негізгі мәні тікелей вариациялық әдістерді қолдануда болатынында.

Кілт сөздер: бейпотенциалды операторлар, диффузиялық жүйе, жартылай шектелген функциялар, вариациялық көбейткіш.

В.М. Савчин, Л.Т. Хуен

Российский университет дружбы народов, Москва, Россия

Непотенциальность оператора одной системы диффузии и построение полуограниченного функционала

Широкое распространение и систематическое использование вариационных принципов в математике и приложениях объясняется рядом замечательных следствий, среди которых возможность установить существование решений исходных уравнений и определение устойчивых приближений их решений так называемыми вариационными методами. В связи с этим для заданной системы уравнений естественно исследовать вопрос о существовании ее вариационных формулировок. Ее можно рассматривать как обратную задачу вариационного исчисления. Основная цель настоящей работы — исследование этой задачи для системы уравнений в частных производных диффузии. Ключевой объект — критерий потенциальности. На его основании доказана непотенциальность оператора данной краевой задачи относительно классической билинейной формы. Показано, что эта система не допускает матричный вариационный множитель данного вида. Таким образом, заданная система диффузии не может быть выведена из классического вариационного принципа Гамильтона. Поставлен вопрос о том, существует ли функционал, полуограниченный на решениях данной краевой задачи. Изложен алгоритм конструктивного определения такого функционала. Основная ценность построенного функционала заключается в применении прямых вариационных методов.

Ключевые слова: непотенциальные операторы, система диффузии, полуограниченные функционалы, вариационный множитель.

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S. Shaimardan[1,*], N.S. Tokmagambetov[1,2]

[1]*L.N. Gumilyev Eurasian National University, Nur-Sultan, Kazakhstan;*
[2]*Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan*
(E-mail: shaimardan.serik@gmail.com, nariman.tokmagambetov@gmail.com)

On the solutions of some fractional q-differential equations with the Riemann-Liouville fractional q-derivative

This paper is devoted to explicit and numerical solutions to linear fractional q-difference equations and the Cauchy type problem associated with the Riemann-Liouville fractional q-derivative in q-calculus. The approaches based on the reduction to Volterra q-integral equations, on compositional relations, and on operational calculus are presented to give explicit solutions to linear q-difference equations. For simplicity, we give results involving fractional q-difference equations of real order $a > 0$ and given real numbers in q-calculus. Numerical treatment of fractional q-difference equations is also investigated. Finally, some examples are provided to illustrate our main results in each subsection.

Keywords: Cauchy type q-fractional problem, existence, uniqueness, q-derivative, q-calculus, fractional calculus, Riemann–Liouville fractional derivative, q-fractional derivative.

Introduction

During the last three decades, fractional differential equations have attracted great attention and have been wide range used in real world phenomena related to physics, chemistry, biology, signal-and image processing. Moreover, they are equipped with social sciences such as food supplement, climate and economics, see e.g. [1–9]. Hence, there has been a significant development in ordinary and partial differential equations involving fractional derivatives and a huge amount of papers, and also some books devoted to this subject in various spaces have appeared, see e.g. the monographs of T. Sandev and Z. Tomovski [7], A.A. Kilbas et al. [8], R. Hilfer [9], K.S. Miller and the B. Ross [10], the papers [11–19] and the references therein.

The origin of the q-difference calculus can be traced back to the works in [20, 21] by F. Jackson and R.D. Carmichael [22] from the beginning of the twentieth century. For more interesting theory results and scientific applications of the q-difference calculus, we cite the monographs [23, 24, 25] and the references therein. Recently, the fractional q-difference calculus has been proposed by W. Al-salam [26] and R.P. Agarwal [27] and P.M. Rajkovic', S.D. Marinkovic, and M.S. Stankovic [28]. Recently, many researchers got much interested in looking at fractional q-differential equations (FDEs) as new model equations for many physical problems. For example, some researchers obtained q-analogues of the integral and differential fractional operators properties, such as the q-Laplace transform and q-Taylor's formula [29], q-Mittage Leffler function [27] and so on.

We also pronounce that up to now, much attention has been focused on the fractional q-difference equations. There have been some papers dealing with the existence and uniqueness, or multiplicity of solutions to linear fractional q-difference equations by the use of some well-known fixed point theorems. For some recent developments on the subject, see e.g. [30–33] and the references therein. In Section 2 of this paper, we construct explicit solutions to linear fractional q-differential equations with the Riemann-Liouville fractional q-derivative $D^{\alpha}_{q,a+}f$ of order $\alpha > 0$ given by Definition 2, in the space $C^{\alpha}_{q,n-\alpha}[0,a]$, denned in (8). The main result, in this Section, is Theorem 1, but in order to prove this result we need to prove two results (Theorem 1 and 3) of independent interest.

The paper is organized as follows: the main results are presented and proved in subsection 2.1 and subsection 2.3, and the announced examples are given in subsection 2.2, 2.3, and 2.5. In order to not disturb these presentations, we include in Section 1 some necessary Preliminaries.

[*]Corresponding author.
E-mail: shaimardan.serik@gmail.com

1 Preliminaries

First, we start by recalling some elements of q-calculus, for more information see e.g. the books [23], [25], and [33]. Throughout this paper, we assume that $0<q<1$ and $0<a<b<\infty$.

Let $\alpha\in\mathbb{R}$. Then a q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q=\frac{1-q^{\alpha}}{1-q},$$

where $\lim\limits_{q\to1}\frac{1-q^{\alpha}}{1-q}=\alpha$.

We introduce for $n\in\mathbb{N}$:

$$(a;q)_0=1,\ (a;q)_n=\prod_{k=0}^{n-1}\left(1-q^k a\right),\ (a;q)_\infty=\lim_{n\to\infty}(a,q)_n,\ (a;q)_\alpha=\frac{(a;q)_\infty}{(q^\alpha a;q)_\infty}.$$

The q-analogue of the power function $(a-b)_q^{\alpha}$ is defined by

$$(a-b)_q^{\alpha}{=}a^{\alpha}\frac{(b/a;q)_\infty}{(q^\alpha b/a;q)_\infty}.$$

Notice that $(a-b)_q^{\alpha}=a^{\alpha}(b/a;q)_\alpha$.

For any two real numbers α and β, we have

$$(a-b)_q^{\alpha}\,(a-q^{\alpha}b)_q^{\beta}=(a-b)_q^{\alpha+\beta}\,. \tag{1}$$

The q-analogue of the binomial coefficients $[n]_q!$ are defined by

$$[n]_q!=\begin{cases}1, & \text{if n} = 0,\\ [1]_q\times[2]_q\times\cdots\times[n]_q, & \text{if n} \in \text{N},\end{cases}$$

The gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x)=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},$$

for any $x>0$. Moreover, it yields that $\Gamma_q(x)[x]_q=\Gamma_q(x+1)$.

The q-analogue differential operator $D_qf(x)$ is

$$D_qf(x)=\frac{f(x)-f(qx)}{x(1-q)},$$

and the q-derivatives $D_q^n(f(x))$ of higher order are:

$$D_q^0(f(x))=f(x),\quad D_q^n(f(x))=D_q\left(D_q^{n-1}f(x)\right),\quad (n=1,2,3,\dots)$$

The q-integral (or Jackson integral) $\int\limits_0^a f(x)d_qx$ is defined by

$$\int\limits_0^a f(x)d_qx=(1-q)a\sum_{m=0}^{\infty}q^m f(aq^m)$$

and

$$\int\limits_a^b f(x)d_qx=\int\limits_0^b f(x)d_qx-\int\limits_0^a f(x)d_qx,$$

for $0 < a < b$. Notice that

$$\int_a^b D_q f(x) d_q x = f(b) - f(a).$$

For any $t, s > 0$ the definition of q-Beta function is that:

$$B_q(t,s) = \frac{\Gamma_q(t)\,\Gamma_q(s)}{\Gamma_q(t+s)} = \int_0^1 x^{t-1}(qx;q)_{s-1} d_q x. \tag{2}$$

The (Mittag-Leffler) q-function $E_{\alpha,\beta}(z;q)$ is defined by

$$E_{\alpha,\beta,a}\left[zx^{\alpha}(a/x;q)_{\alpha};q\right] = \sum_{k=0}^{\infty} \frac{z^k x^{k\alpha}(a/x;q)_{k\alpha}}{\Gamma_q(\alpha k+\beta)} \tag{3}$$

and

$$E_{\alpha,m,l}\left[z;q\right] = \sum_{k=0}^{\infty} c_k z^k \tag{4}$$

where c_0 and $c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]} (k \in \mathbb{N})$.

A q-analogue of the classical exponential function e^x is

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]!}. \tag{5}$$

Moreover, the multiple q-integral $\left(I_{q,a+}^n f\right)(x)$ is

$$\begin{aligned}\left(I_{q,a+}^n f\right)(x) &= \int_a^x \int_a^t \int_a^{t_{n-1}} \ldots \int_a^{t_2} d_q t_1 d_q t_2 \ldots d_q t_{n-1} d_q t \\ &= \frac{1}{\Gamma_q(n)} \int_a^x (x-qt)_q^{n-1} f(t) d_q t.\end{aligned}$$

Definition 1. The Riemann-Liouville q-fractional integrals $I_{q,a+}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\left(I_{q,a+}^{\alpha} f\right)(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x;q)_{\alpha-1} f(t) d_q t.$$

*Definition 2.*The Riemann-Liouville fractional q-derivative $D_{q,a+}^{\alpha} f$ of order $\alpha > 0$ is defined by

$$\left(D_{q,a+}^{\alpha} f\right)(x) := \left(D_{q,a+}^{[\alpha]} I_{q,a+}^{[\alpha]-\alpha} f\right)(x).$$

Notice that

$$\left(I_{q,a+}^{\alpha} x^{\lambda}(a/x;q)_{\lambda}\right)(x) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} x^{\alpha+\lambda}(a/x)_q^{\alpha+\lambda}, \tag{6}$$

for $\lambda \in (-1, \infty)$.

For $1 \le p < \infty$ we define the space $L_q^p = L_q^p[a,b]$ by

$$L_q^p[a,b] := \left\{ f : \left(\int_a^b |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\}.$$

Let $\alpha > 0$, $\beta > 0$ and $1 \leq p < \infty$. Then the q–fractional integration has the following semigroup property

$$\left(I^{\alpha}_{q,a+} I^{\beta}_{q,a+} f\right)(x) = \left(I^{\alpha+\beta}_{q,a+} f\right)(x), \tag{7}$$

for all $x \in [a,b]$ and $f(x) \in L^p_q[a,b]$.

Let $0 < a < b < \infty$ and $0 \leq \lambda \leq 1$. Then we introduce the space $C_{q,\lambda}[a,b]$ of functions f given on $[a,b]$, such that the functions with the norm

$$\|f\|_{C_{q,\lambda}[a,b]} := \max_{x\in[a,b]} \left| x^{\lambda} \left(qa/x;q\right)_{\lambda} f(x)\right| < \infty.$$

The space $C^{\alpha}_{q,n-\alpha}[0,a]$ defined for $n - q < \alpha \leq n$, $n \in \mathbb{N}$ by

$$C^{\alpha}_{q,n-\alpha}[0,a] := \{f(x) : f(x) \in C_{q,n-\alpha}[a,b], \left(D^{\alpha}_{q,a+} f\right)(x) \in C_{q,n-\alpha}[a,b]\}. \tag{8}$$

2 On the solutions of some fractional q-differential equations with the Riemann-Liouville fractional q-derivative

2.1 The Cauchy type problem for the fractional q-differential equation

First, we consider the Cauchy type problem for the fractional q-differential equation in the following form:

$$\left(D^{\alpha}_{q,0+} y\right)(x) - \lambda y(x) \quad = \quad f\left(x\right), \quad 0 < x \leq a, \alpha > 0; \lambda \in \mathbb{R}, \tag{9}$$

with the initial conditions:

$$\left(D^{\alpha-k}_{q,0+} y\right)(0+) = b_k, \ \ b_k \in \mathbb{R}, \ \ k = 0,1,2,\ldots,n = -[-\alpha]. \tag{10}$$

Next we construct the explicit solutions to linear fractional q-differential equations. In the classical case, several authors have considered such problems even in linear cases, see e.g. [8, Section 4] and the references therein.

Theorem 1. (See [34, Theorem 8.1]) Let $n-1 < \alpha \leq n; n \in \mathbb{N}$, G be an open set in $\mathbb{R}$ and $f(.,.) : (0,a] \times G \to \mathbb{R}$ be a function such that $F(x,y(x)) = f(x) + \lambda y(x) \in L^1_q[0,a]$ for any $y \in G$. If $y(x) \in L^1_q[0,a]$, then $y(t)$ satisfies a.e. the relations (9)-(10) if and only if $y(x)$ satisfies a.e. the integral equation

$$y(x) := \sum_{k=0}^{n-1} \frac{b_k}{[k]_q} x^{\alpha-k} + \left(I^{\alpha}_{q,0} f(t,y(t))\right)(x), \forall x \in (0,c]. \tag{11}$$

Theorem 2. Let $n - 1 < \alpha < n (n \in \mathbb{N})$ and let $0 \leq \gamma < 1$ be such that $\gamma \geqq \alpha$. Also, let $\lambda \in \mathbb{R}$ and $g(x) \in C^{\lambda}_q[0,b]$.If $f_0(x,y(x)) = \lambda y(x) + f(x)$, then the Cauchy problem (9)-(10) has unique solution $y(x) \in C^{\alpha,n-1}_{\gamma}[a,b]$ and this solution is given by

$$\begin{aligned} y(x) \quad &:= \quad \sum_{k=1}^{n} b_k x^{\alpha-k} E_{\alpha,\alpha-k+1,0}\left[\lambda x^{\alpha};q\right] \\ &+ \quad \int\limits_0^x x^{\alpha-1}\left(qt/x;q\right)_{\alpha-1} E_{\alpha,\alpha,t}\left[\lambda x^{\alpha}(q^{\alpha}t/x;q)_{\alpha};q\right] f(t) d_q t. \end{aligned} \tag{12}$$

Proof. First, we solve the Volterra q-integral equation (11), and apply the method of successive approximations by setting

$$y_0(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k}$$

and

$$\begin{aligned} y_i(x) \quad &= \quad y_0(x) + \frac{\lambda x^{\alpha-1}}{\Gamma_q\left(\alpha\right)} \int_0^x (qt/x;q)_{\alpha-1} y_{i-1}(t) d_q t \\ &+ \quad \frac{x^{\alpha-1}}{\Gamma_q\left(\alpha\right)} \int_0^x (qt/x;q)^{\alpha-1}_q f(t) d_q t. \end{aligned} \tag{13}$$

Using Definition 1 and (6), (13), we find $y_1(x)$:

$$y_1(x) = y_0(x) + \lambda \left(I^{\alpha}_{q,0+} y_0\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x)$$

that is,

$$\begin{aligned} y_1(x) &= \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k} + \lambda \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} \left(I^{\alpha}_{q,0+} t^{\alpha-k}\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k} + \lambda \sum_{k=1}^{n} \frac{b_k x^{2\alpha-k}}{\Gamma_q(2\alpha-k+1)} + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=1}^{n} b_k \sum_{m=1}^{2} \frac{\lambda^{m-1} x^{m\alpha-k}}{\Gamma_q(\alpha m-k+1)} + \left(I^{\alpha}_{q,0+} f\right)(x). \end{aligned} \tag{14}$$

Similarly, using Definition 1 and (6), (7), (14) we have for $y_2(x)$ that

$$\begin{aligned} y_2(x) &= y_0(x) + \lambda \left(I^{\alpha}_{q,0+} y_1\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k} + \frac{\lambda}{\Gamma_q(\alpha)} \sum_{k=1}^{n} b_k \sum_{m=1}^{2} \frac{\lambda^{m-1}}{\Gamma_q(\alpha m-k+1)} \left(I^{\alpha}_{q,0+} t^{m\alpha-k}\right)(x) \\ &+ \lambda \left(I^{\alpha}_{q,0+} I^{\alpha}_{q,0+} f(t)\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k} + \lambda \sum_{k=1}^{n} b_k \sum_{m=1}^{2} \frac{\lambda^{m-1}}{\Gamma_q(\alpha(m+1)-k+1)} x^{\alpha(m+1)-k} \\ &+ \lambda \left(I^{2\alpha}_{q,0+} f(t)\right)(x) + \left(I^{\alpha}_{q,0+} f\right)(x) \\ &= \sum_{k=1}^{n} \frac{b_k}{\Gamma_q(\alpha-k+1)} x^{\alpha-k} + \lambda \sum_{k=1}^{n} b_k \sum_{m=1}^{2} \frac{\lambda^{m-1}}{\Gamma_q(\alpha(m+1)-k+1)} x^{\alpha(m+1)-k} \\ &+ \frac{\lambda x^{2\alpha-1}}{\Gamma(2\alpha)} \int_0^x f(t)\,(qt/x;q)_{2\alpha-1}\, d_q t + \left(I^{\alpha}_{q,0+} f\right)(x). \end{aligned}$$

Thus,

$$\begin{aligned} y_2(x) &= \sum_{k=1}^{n} b_k \sum_{m=1}^{3} \frac{\lambda^{m-1} x^{\alpha m-k}}{\Gamma_q(\alpha m-k+1)} \\ &+ \int_0^x \left[\sum_{m=1}^{2} \frac{\lambda^{m-1} x^{\alpha m-1} (qt/x;q)_{\alpha m-1}}{\Gamma_q(\alpha m)}\right] f(t) d_q t. \end{aligned}$$

Continuing this process, we derive the following relation for $y_i(x)$:

$$\begin{aligned} y_i(x) &= \sum_{k=1}^{n} b_k \sum_{m=1}^{i+1} \frac{\lambda^{m-1} x^{\alpha m-k}}{\Gamma_q(\alpha m-k+1)} \\ &+ \int_0^x \left[\sum_{m=1}^{i} \frac{\lambda^{m-1} x^{\alpha m-1} (qt/x;q)_{\alpha m-1}}{\Gamma_q(\alpha m)}\right] f(t) d_q t \\ &= \sum_{k=1}^{n} b_k \sum_{m=0}^{i} \frac{\lambda^{m} x^{\alpha(m+1)-k}}{\Gamma_q(\alpha(m+1)-k+1)} \\ &+ \int_0^x \left[\sum_{m=0}^{i-1} \frac{\lambda^{m} x^{\alpha(m+1)-1} (qt/x;q)_{\alpha(m+1)-1}}{\Gamma_q(\alpha(m+1))}\right] f(t) d_q t. \end{aligned}$$

Taking the limit as $i \to \infty$ and using (1), we obtain the following explicit solution $y(x)$ to the q-integral equation (11):

$$\begin{aligned} y(x) &= \sum_{k=1}^{n} b_k x^{\alpha-k} \sum_{m=0}^{\infty} \frac{\lambda^m x^{\alpha m}}{\Gamma_q\left(\alpha m+\alpha-k+1\right)} \\ &+ \int_0^x x^{\alpha-1}(qt/x;q)_{\alpha-1}\left[\sum_{m=0}^{\infty}\frac{\lambda^m x^{\alpha m}(qt/x;q)_{\alpha m}}{\Gamma_q(\alpha m+\alpha)}\right] f(t)d_qt. \end{aligned}$$

On the basis of Theorem 1 and (3) an explicit solution to the Volterra q-integral equation (11) and hence, to the Cauchy type problem (9)-(10).

2.2 Miscellaneous Examples

In this subsection, we present some examples and discuss these examples in connection with the results obtained in Theorem 2. Our examples are q-analogues of examples given in [8, Examples 3.1-3.2].

Example 1. Let $0 < \alpha < 1$ and $\lambda, b \in \mathbb{R}$. Then, the solution to the Cauchy type problem in the following form:

$$\left(D^{\alpha}_{q,0+}y\right)(x) - \lambda y(x) = f(x), \quad \left(D^{\alpha-1}_{q,0+}y\right)(0+) = b$$

has the explicit solution

$$y(x) = bt^{\alpha-1}E_{\alpha,\alpha,0}\left[\lambda t^{\alpha};q\right] + x^{\alpha-1}\int_0^x (qt/x;q)_{\alpha-1}E_{\alpha,\alpha,t}\left[\lambda x^{\alpha}(q^{\alpha}t/x;q)_{\alpha};q\right]f(t)d_qt.$$

Hence, we can rewrite as follows:

$$\left(D^{\alpha}_{q,0+}y\right)(x) - \lambda y(x) = 0, \quad \left(D^{\alpha-1}_{q,0+}y\right)(0+) = b,$$

and the solution of this problem

$$y(x) = bt^{\alpha-1}E_{\alpha,\alpha,0}\left[\lambda t^{\alpha};q\right].$$

In particular, for $\alpha = 1/2$ the Cauchy type problem

$$\left(D^{1/2}_{q,0+}y\right)(x) - \lambda y(x) = f(x), \left(I^{1/2}_{q,0+}y\right)(0+) = b$$

has the solution given by

$$\begin{aligned} y(x) &= \frac{b}{t^{\frac{1}{2}}}E_{1/2,1/2,0}\left[\lambda t^{1/2};q\right] \\ &+ x^{1/2}\int_0^x (qt/x;q)_{1/2}E_{1/2,1/2,t}\left[\lambda x^{1/2}(q^{1/2}t/x;q)_{1/2};q\right]f(t)d_qt \end{aligned}$$

and the solution to the problem

$$\left(D^{1/2}_{q,0+}y\right)(x) - \lambda y(x) = 0, \quad \left(I^{1/2}_{q,0+}y\right)(0+) = b$$

is given by

$$y(x) = \frac{b}{t^{1/2}}E_{1/2,1/2,0}\left[\lambda t^{1/2}\right].$$

Example 2. We assume that $1 < \alpha < 2$ and $\lambda, b, d \in \mathbb{R}$. Then the Cauchy type problem in the following form:

$$\left(D^{\alpha}_{q,0+}y\right)(x) - \lambda y(x) = f(x), \quad \left(D^{\alpha-1}_{q,0+}y\right)(0+) = b, \quad \left(D^{\alpha-2}_{q,0+}y\right)(0+) = d,$$

and its solution has the form:

$$\begin{aligned} y(x) &= bt^{\alpha-1}E_{\alpha,\alpha,0}\left[\lambda t^{\alpha};q\right] + dt^{\alpha-2}E_{\alpha,\alpha-1,0}\left[\lambda t^{\alpha};q\right] \\ &+ x^{\alpha-1}\int_0^x (qt/x;q)_{\alpha-1}E_{\alpha,\alpha,t}\left[\lambda x^{\alpha}(q^{\alpha}t/x;q)_{\alpha};q\right]f(t)d_qt. \end{aligned}$$

Particularly, the solution to the problem

$$\left(D^{\alpha}_{q,0+}y\right)(x)-\lambda y(x)=0, \left(D^{\alpha-1}_{q,0+}y\right)(0+)=b,$$
$$\left(D^{\alpha-2}_{q,0+}y\right)(0+)=d,$$

is given by

$$y(x) \quad = \quad bt^{\alpha-1}E_{\alpha,\alpha,0}\left[\lambda t^{\alpha};q\right]+dt^{\alpha-2}E_{\alpha,\alpha-1,0}\left[\lambda t^{\alpha};q\right].$$

2.3 General homogeneous fractional q-differential equation

In subsection, we consider in the following more general homogeneous fractional q-differential equation than (9):

$$\left(D^{\alpha}_{q,0+}y\right)(x)-\lambda x^{\beta}y(x)=0, \quad 0<x\leq a<\infty, \alpha>0, \lambda\in\mathbb{R}, \tag{15}$$

with initial date

$$\left(D^{\alpha-k}_{q,0+}y\right)(0+)=b_k, b_k\in\mathbb{R}, k=0,1,2,\ldots,n=-[-\alpha]. \tag{16}$$

Theorem 3. Let $\alpha>0$, $n=-[-\alpha]$, $\lambda\in\mathbb{R}$ and $\beta\geq 0$. Then the Cauchy type problem (15)-(16) has a unique solution $y(x)$ in the space $C^{\alpha}_{q,n-\alpha}[0,a]$ and this solution is given by

$$y(x) \quad = \quad \sum_{j=1}^{n}\frac{b_j}{\Gamma_q(\alpha-j+1)}t^{\alpha-j}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta-j)}{\alpha}}\left[\lambda t^{\alpha+\beta};q\right]. \tag{17}$$

Proof. Let $\beta>-\alpha$. Then basic on Theorem 1 the problem (15)-(16) is equivalent in the space $C^{\alpha}_{q,n-\alpha}[0,a]$ to the Volterra q-integral equation of the second kind in the following form:

$$y(x)=\sum_{j=1}^{n}\frac{b_j}{\Gamma_q\left(\alpha-j+1\right)}t^{\alpha-j}+\frac{\lambda x^{\alpha-1}}{\Gamma_q\left(\alpha\right)}\int_0^x t^{\beta}\left(qt/x;q\right)_{\alpha-1}y(t)d_qt. \tag{18}$$

Similarity, we again apply the method of successive approximations to solve this q-integral equation (18). We assume that $y_0(x)=\sum\limits_{j=1}^{n}\frac{b_j}{\Gamma_q(\alpha-j+1)}x^{\alpha-j}$ and

$$y_m(x)=y_0(x)+\frac{\lambda x^{\alpha-1}}{\Gamma_q\left(\alpha\right)}\int_0^x t^{\beta}\left(qt/x;q\right)_{\alpha-1}y_{m-1}(t)d_qt. \tag{19}$$

Using the same arguments as above, by using (2), (7), and (19) we find $y_1(x)$:

$$\begin{aligned}
y_1(x) &= y_0(x)+\frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x t^{\beta}\left(qt/x;q\right)_{\alpha-1}y_0(x)d_qt \\
&= y_0(x)+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=1}^{n}\frac{b_jx^{\alpha-1}}{\Gamma_q(\alpha-j+1)}\int_0^x t^{\alpha+\beta-j}\left(qt/x;q\right)_{\alpha-1}d_qt \\
&= y_0(x)+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=1}^{n}\frac{b_jx^{\alpha-1}}{\Gamma_q(\alpha-j+1)}\int_0^1 (xy)^{\alpha+\beta-j}\left(qxy/x;q\right)_{\alpha-1}xd_qy \\
&= y_0(x)+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=1}^{n}\frac{b_jx^{2\alpha+\beta-j}}{\Gamma_q(\alpha-j+1)}\int_0^1 y^{\alpha+\beta-j}\left(qy;q\right)_{\alpha-1}d_qy \\
&= \sum_{j=1}^{n}\frac{b_j}{\Gamma_q(\alpha-j+1)}x^{\alpha-j}+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=1}^{n}\frac{b_jx^{2\alpha+\beta-j}}{\Gamma_q(\alpha-j+1)}B_q\left(\alpha+\beta-j+1,\alpha\right) \\
&= \sum_{j=1}^{n}\frac{b_j}{\Gamma_q(\alpha-j+1)}x^{\alpha-j} \\
&+ \lambda\sum_{j=1}^{n}\frac{b_jx^{2\alpha+\beta-j}}{\Gamma_q(\alpha-j+1)}\frac{\Gamma_q(\alpha+\beta-j+1)}{\Gamma_q(2\alpha+\beta-j+1)}.
\end{aligned} \tag{20}$$

Similarly, for $m = 2$ using (2), (19) and taking (20) into account, we derive

$$\begin{aligned} y_2(x) &= y_0(x) + \lambda \left(I^{\alpha}_{q,0+} y_1\right)(x) \\ &= y_0(x) + \lambda \sum_{j=1}^{n} \frac{b_j}{\Gamma_q(\alpha - j + 1)} \left(I^{\alpha}_{q,0+} t^{\alpha - j}\right)(x) \\ &+ \lambda^2 \sum_{j=1}^{n} \frac{b_j \Gamma_q(\alpha + \beta - j + 1)}{\Gamma_q(\alpha - j + 1)\,\Gamma_q(2\alpha + \beta - j + 1)} \left(I^{\alpha}_{q,0+} t^{2\alpha + \beta - j}\right)(x) \\ &= \sum_{j=1}^{n} \frac{b_j t^{\alpha - j}}{\Gamma_q(\alpha - j + 1)} \left[1 + c_1 \left(\lambda t^{\alpha+\beta}\right) + c_2 \left(\lambda t^{\alpha+\beta}\right)^2\right], \end{aligned}$$

where

$$\begin{aligned} c_1 &= \frac{\Gamma_q(\alpha + \beta - j + 1)}{\Gamma_q(2\alpha + \beta - j + 1)}, \\ c_2 &= \frac{\Gamma_q(\alpha + \beta - j + 1)\Gamma_q(2\alpha + 2\beta - j + 1)}{\Gamma_q(2\alpha + \beta - j + 1)\Gamma_q(3\alpha + 2\beta - j + 1)}. \end{aligned}$$

Continuing this process for $m \in \mathbb{N}$, we have $y_m(x)$:

$$y_m(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma_q(\alpha - j + 1)} t^{\alpha - j} \left[1 + \sum_{k=1}^{m} c_k \left(\lambda t^{\alpha+\beta}\right)^k\right], \tag{21}$$

where

$$c_k = \prod_{r=1}^{k} \frac{\Gamma_q[r(\alpha + \beta) - j + 1]}{\Gamma_q[r(\alpha + \beta) + \alpha - j + 1]}, \qquad k \in \mathbb{N}.$$

Taking the limit as $m \to \infty$ tow site of (21), we obtain the following explicit solution $y(x)$ to the Cauchy type problem (15)-(16):

$$y(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma_q(\alpha - j + 1)} t^{\alpha - j} \left[1 + \sum_{k=1}^{\infty} c_k \left(\lambda t^{\alpha+\beta}\right)^k\right].$$

According to the relations (4), we rewrite this solution (18) in terms of the generalized Mittag-Leffler q-function $E_{\alpha,m,l}\left[z; q\right]$.

2.4 Further Examples

In subsection, we present some examples and discuss them in connection with the results obtained in Section 2.3.

Example 3. Let $0 < \alpha < l$, $\beta > -\alpha$ and $\lambda \in \mathbb{R}$ and $b \in \mathbb{R}$. Then the solution to the Cauchy type problem

$$\left(D^{\alpha}_{q,0+} y\right)(x) - \lambda t^{\beta} y(x) = 0, \left(D^{\alpha-1}_{q,0+} y\right)(0+) = b,$$

is given by

$$y(x) = \frac{b t^{\alpha - 1}}{\Gamma_q(\alpha)} E_{\alpha, 1 + \frac{\beta}{\alpha}, 1 + \frac{(\beta - 1)}{\alpha}} \left[\lambda t^{\alpha+\beta}; q\right].$$

In particular, for $\alpha = 1/2$ the Cauchy type problem in the following form

$$\left(D^{1/2}_{q,0+} y\right)(x) - \lambda t^{\beta} y(x) = 0, \left(D^{-1/2}_{q,0+} y\right)(0+) = b,$$

has a unique solution given by

$$y(x) \quad = \quad \frac{b}{\Gamma_q(1/2)} t^{-\frac{1}{2}} E_{\frac{1}{2},1+2\beta,2\beta-1}\left[\lambda t^{\beta+\frac{1}{2}}; q\right].$$

Example 4. The solution to the Cauchy type problem

$$\left(D^{\alpha}_{q,0+}y\right)(x) - \lambda t^{\beta} y(x) = 0, \left(D^{\alpha-1}_{q,0+}y\right)(0+) = b,$$

$$\left(D^{\alpha-2}_{q,0+}y\right)(0+) = d,$$

with $b, d \in \mathbb{R}$, $1 < \alpha < 2$, $\beta \in \mathbb{R}$ $(\beta > -\alpha)$ and $\lambda \in \mathbb{R}$ has the form

$$\begin{aligned} y(x) \quad &= \quad \frac{b}{\Gamma_q(\alpha)} t^{\alpha-1} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta-1)}{\alpha}}\left[\lambda t^{\alpha+\beta}; q\right] \\ &+ \quad \frac{d}{\Gamma_q(\alpha)} t^{\alpha-2} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta-2)}{\alpha}}\left[\lambda t^{\alpha+\beta}; q\right]. \end{aligned}$$

2.5 The Cauchy Problems for Ordinary q-Differential Equations

In this subsection, we use the results of subsection 3.1 when $\alpha \in \mathbb{N}$, and we derive the explicit solutions to the Cauchy problems for ordinary q-differential equations of order n on $[0, a]$.

Let $\lambda, b_k \in \mathbb{R}, n, k \in \mathbb{N}$ such that $k \leq n$. Then we consider the ordinary q-differential equation:

$$D^{(n)}_q(x) - \lambda y(x) = f(x), \tag{22}$$

with inial data

$$D^{(n-k)}_q(0+) = b_k, \tag{23}$$

which is a particular case of the Cauchy problem (9)-(10) with $\alpha \in \mathbb{N}$. Therefore, from (12) we derive the solution to (22)-(23) in the following form:

$$\begin{aligned} y(x) : \quad &= \quad \sum_{j=1}^{n} b_j t^{n-j} E_{n,n-j+1,0}\left[(\lambda t)^n; q\right] + \\ &+ \quad x^{n-1} \int_0^x (qt/x; q)_{n-1} E_{n,n,t}\left[\lambda x^n (q^n t/x; q)_n; q\right] f(t) d_q t. \end{aligned} \tag{24}$$

which is the unique explicit solution of the Cauchy problem (24) in the space $C^n_{q,0}[0, a]$.

Example 5. Let $\alpha = 1$ and $b \in \mathbb{R}$. Then the solution to the Cauchy type problem in the following form

$$\left(D_q y\right)(x) - \lambda y(x) = f(x); \quad y(0) = b,$$

has a unique solution given by

$$\begin{aligned} y(x) \quad &= \quad bE_{1,1,0}\left[\lambda t; q\right] \\ &+ \quad \int_0^x xE_{1,1,t}\left[\lambda(qt/x); q\right] f(t) d_q t. \end{aligned}$$

Form (3) and (5) it follows that

$$\begin{aligned} y(x) \quad &= \quad b\sum_{k=0}^{\infty} \frac{\lambda^k t^k}{\Gamma_q(k+1)} + \\ &+ \quad \int_0^x \sum_{k=0}^{\infty} \frac{(\lambda(x-qt))^k}{\Gamma_q(k+1)} f(t) d_q t, \\ &= \quad be_q^{\lambda t} + \int_0^x e_q^{\lambda(x-qt)} f(t) d_q t. \end{aligned}$$

Example 6. Let $b, d \in \mathbb{R}$. Then the solution to the Cauchy type problem

$$\left(D_q^2 y\right)(x) - \lambda y(x) = f(x), \;\; y(0) = b, \;\; \left(D_q y\right)(x) = d,$$

is given by

$$\begin{aligned} y(x) &= btE_{2,2,0}\left[\lambda t^2; q\right] + dE_{2,1,0}\left[\lambda t^2; q\right] \\ &+ x\int_0^x (qt/x;q)E_{2,2,t}\left[x^2\lambda(q^2t/x;q)_2\right] f(t)d_qt, \end{aligned}$$

In particular, for $b, d \in \mathbb{R}$ and $f(x) = 0$ the solution to the problem

$$\left(D_q^2 y\right)(x) - \lambda y(x) = 0, \;\; y(0+) = b, \;\; \left(D_q y\right)(x) = d,$$

has the form

$$y(x) = btE_{2,2,0}\left[\lambda t^2; q\right] + dE_{2,1,0}\left[\lambda t^2; q\right].$$

Form (2) and (8) it follows that

$$\begin{aligned} y(x) &= bt\sum_{k=0}^{\infty}\frac{\lambda^k t^{2k}}{\Gamma_q(2k+2)} + d\sum_{k=0}^{\infty}\frac{\lambda^k t^{2k}}{\Gamma_q(2k+1)} \\ &= bsin_q(\sqrt{\lambda}t) + dcos_q(\sqrt{\lambda}t), \end{aligned}$$

where $sin_q(\sqrt{\lambda}t) = \sum\limits_{k=0}^{\infty}\frac{(\sqrt{\lambda}t)^{2k+1}}{\Gamma_q(2k+2)}$ and $cos_q(\sqrt{\lambda}t) = \sum\limits_{k=0}^{\infty}\frac{(\sqrt{\lambda}t)^{2k}}{\Gamma_q(2k+1)}$.

For $\lambda, b_k \in \mathbb{R}, n, k \in \mathbb{N}$ and $\beta \geq 0$ we consider the Cauchy problem in the following form:

$$y^{(n)}(x) - \lambda t^{\beta} y(x) = f(x); \;\; y^{(n-k)}(0+) = b_k,$$

which is a particular case of the problem (16)-(17) with $\alpha = n$. Using (4) we get the solution in the following form:

$$y(x) = \sum_{j=1}^{n}\frac{b_j}{\Gamma_q(n-j+1)}t^{n-j}E_{n,1+\frac{\beta}{n},1+\frac{(\beta-j)}{n}}\left[\lambda t^{n+\beta}; q\right].$$

Example 7. We assume that $\beta > -1$ and $b \in \mathbb{R}$. Then the solution to the Cauchy problem

$$\left(D_q y\right)(x) - \lambda t^{\beta} y(x) = f(x); \;\; y(0) = b$$

has the form

$$y(x) = bE_{1,1+\beta,\beta}\left[\lambda t^{1+\beta}; q\right].$$

Example 8. Let $b, d \in \mathbb{R}$ and $\beta > -2$. Then the solution to the Cauchy problem

$$\left(D_q^2 y\right)(x) - \lambda t^{\beta} y(x) = f(x), \;\; y(0+) = b, \;\; \left(D_q y\right)(0) = d,$$

is given by

$$y(x) = btE_{2,1+\frac{\beta}{2},\frac{(\beta+1)}{2}}\left[\lambda t^{2+\beta}; q\right] + dE_{2,1+\frac{\beta}{2},\frac{(\beta)}{2}}\left[\lambda t^{2+\beta}; q\right].$$

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С. Шаймардан[1], Н.С. Токмагамбетов[1,2]

[1]*Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Нұр-Сұлтан, Қазақстан;*
[2]*Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан*

Кейбір Риман-Лиувиль бөлшек q-туындылы q-бөлшек дифференциалдық теңдеулердің шешімдері туралы

Мақала бөлшек-сызықтық q-айырымдық теңдеулері мен бөлшек q-Риман–Лиувилл туындысымен байланысты Коши типтес есептерді нақты және сандық шешуге арналған. q-Вольтер интегралдық теңдеулеріне, композициялық қатынастарға және сызықтық q-айырымдық теңдеулерінің нақты шешімдерін алу үшін операциялық есептеуге редукцияға негізделген тәсілдер ұсынылған. Қарапайым бөлу үшін нақты $a > 0$ ретті q-бөлшек айырымдық теңдеулерін және q-есептеулеріндегі нақты сандарды қамтитын нәтижелер берілген. Сондай-ақ, бөлшек q-айырымдық теңдеулерінің сандық өңделуі зерттелді. Сонымен әр бөлімде негізгі нәтижелерді көрсететін бірнеше мысалдар келтірілген.

Кілт сөздер: Коши типтес q-бөлшек есеп, бар болуы, бірегейлігі, q-туынды, q-есептеу, бөлшек есептеу, Риман–Лиувилл бөлшек туындысы, q-бөлшек туынды.

С. Шаймардан[1], Н.С. Токмагамбетов[1,2]

[1]*Евразийский национальный университет имени Л.Н. Гумилева, Нур-Султан, Казахстан;*
[2]*Карагандинский университет имени академика Е.А.Букетова, Караганда, Казахстан*

О решениях некоторых q-дробных дифференциальных уравнений с дробными q-производными Римана–Лиувилля

Статья посвящена явному и численному решению дробно-линейных q-разностных уравнений и задачи типа Коши, связанной с дробной q-производной Римана–Лиувилля в q-исчислении. Представлены подходы, основанные на редукции к q-интегральным уравнениям Вольтерра, композиционным соотношениям и операционному исчислению, для получения явных решений линейных q-разностных уравнений. Для простоты авторами приведены результаты, включающие дробные q-разностные уравнения действительного порядка $a > 0$ и заданные действительные числа в q-исчислении. Также исследована численная обработка дробных q-разностных уравнений. В итоге, в каждом подразделе представлены некоторые примеры, иллюстрирующие полученные основные результаты.

Ключевые слова: q-дробная задача типа Коши, существование, единственность, q-производная, q-вычисление, дробное исчисление, дробная производная Римана–Лиувилля, q-дробная производная.

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O.N. Stanzhytskyi[1], A.T. Assanova[2], M.A. Mukash[3,*]

[1] *Taras Shevchenko National University of Kyiv, Kyiv, Ukraine;*
[2] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[3] *K.Zhubanov Aktobe Regional State University, Aktobe, Kazakhstan*
(E-mail: ostanzh@gmail.com, assanova@math.kz, mukashma1983@gmail.com)

Averaging method and two-sided bounded solutions on the axis of systems with impulsive effects at non-fixed times

The averaging method, originally offered by Krylov and Bogolyubov for ordinary differential equations, is one of the most widespread and effective methods for the analysis of nonlinear dynamical systems. Further, the averaging method was developed and applied for investigating of various problems. Impulsive systems of differential equations supply as mathematical models of objects that, during their evolution, they are subjected to the action of short-term forces. Many researches have been devoted to non-fixed impulse problems. For these problems, the existence, stability, and other asymptotic properties of solutions were studied and boundary value problems for impulsive systems were considered. Questions of the existence of periodic and almost periodic solutions to impulsive systems also were examined. In this paper, the averaging method is used to study the existence of two-sided solutions bounding on the axis of impulse systems of differential equations with non-fixed times. It is shown that a one-sided, bounding, asymptotically stable solution to the averaged system generates a two-sided solution to the exact system. The closeness of the corresponding solutions of the exact and averaged systems both on finite and infinite time intervals is substantiated by the first and second theorems of N.N. Bogolyubov.

Keywords: small parameter, averaging method, impulsive effects, stability, equilibrium position.

Introduction

Impulsive systems of differential equations supply as mathematical models of objects that, in the course of their evolution, they are subjected to the action of short-term forces. A fairly complete theory of such systems is presented in the monograph [1]. In our article, we will use the notation and some facts from this monograph. A study of real problems with state-dependent impulsive effects can be found, for example, in [2–4].

Plenty of studies have been done on non-fixed impulsive problems. For these problems, the existence, stability, and other asymptotic properties of solutions were studied in [5, 6], and [7–9], also boundary value problems for impulsive systems were analyzed. Questions of the existence of periodic and almost periodic solutions of impulsive systems were considered in [10–12]. These problems are closely related to the existence of two-sided bounded solutions on the axis for impulsive systems. It should to said that even for systems with impulsive effects at fixed times

$$\dot{x}(t) = X(t, x), \qquad t \neq t_i,$$

[*] Corresponding author.
E-mail: mukashma1983@gmail.com

$$\Delta x\Big|_{t=t_i} = I_i(x)$$

this is a rather difficult problem. The point is that, unlike ordinary differential equations, there are no theorems on the continuation of solutions to the left for impulsive systems. Indeed, the continuation of the solution to the left at the moments of impulsive effect requires global unique solvability of the nonlinear equation with impulsive effects

$$x(t_i + 0) = x(t_i) + I_i(x(t_i))$$

with respect to $x(t_i)$. It is well known that theorems on the existence of inverse mappings, in the case of a space dimension greater than 1, are only local in nature, which their application does not permit to continue the solution to the left.

In this article, to extend solutions to the left, we have used the averaging method. This method is one of the most widespread and effective methods for the analysis of nonlinear dynamical systems. The averaging method, originally offered by Krylov and Bogolyubov for ordinary differential equations, was later developed and practiced in various problems [13–17]. The closeness of the corresponding solutions of the exact and averaged systems both on finite and infinite time intervals is substantiated by the first and second theorems of N.N. Bogolyubov.

The work consists of an introduction, a main part, where the main results and examples from mathematical biology are formulated and confirmed.

1. The main part

In this paper, we consider a system of differential equations with impulsive effects at non-fixed times and a small parameter of the following form

$$\dot{x}(t) = \varepsilon X(t, x), \qquad t \neq t_i(x), \tag{1}$$

$$\Delta x\Big|_{t=t_i(x)} = \varepsilon I_i(x)$$

$$x(0) = x_0$$

where $\varepsilon > 0$ is a small parameter, $t_i(x) < t_{i+1}(x)$ $(i = 1, 2, \dots)$ moments of impulsive effects, functions X and I_i d are n - dimensional vector of functions.

We put $U_a = \{x \in R^d : |x| \leq a\}$. Suppose the following conditions are met:

1. The functions $X(t, x)$ and $I_i(x)$ are continuous in the set $Q = \{t \geq 0, x \in U_a\}$, bounded by a constant $M > 0$, and in x satisfy the Lipschitz condition with a constant $L > 0$;

2. Uniformly in t, x for $t \geq 0, x \in U_a$, there exist finite limits

$$X_0(x) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} X(s, x) ds,$$

$$I_0(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{t<t_i(x)<T} I_i(x),$$

3. Solution $y = y(t), y(0) = x(0)$ of the averaged system

$$\dot{y} = \varepsilon[X_0(y) + I_0(y)] \tag{2}$$

is defined for $t \geq 0$ and lies in U_a together with some neighborhood ρ and is uniformly asymptotically stable;

4. The moments of the impulsive effect $t_i(x)$ are continuous and their functions satisfy in U_a uniformly in $i \in N$, and the surfaces $t = t_i(x)$ satisfy the separation condition, that is

$$\min_{x\in U_a} t_{i+1}(x) < \min_{x\in U_a} t_i(x) (i = 1, 2, \dots)$$

Suppose that there is a constant $C > 0$ such that for all $t > 0$ and $x \in U_a$

$$i(t, x) \leq Ct$$

where $i(t,x)$ is the number of pulses on $(0,t)$.

It is also assumed that the solutions of system (1) intersect each surface $t = t_i(x)$ at most once, that is, there is no beating. The conditions for the absence of beating are well studied, for example, in [1, Lemmas 3.1, 3.2].

Theorem 1. Let Conditions 1-4 be satisfied. Then, for an arbitrary $\eta > 0$, one can specify ε_0 such that $\varepsilon < \varepsilon_0$ for $t \geq 0$, the inequality

$$|x(t) - y(t)| < \eta$$

where $x(t)(x(0) = y(0) = x_0)$ is a solution to the exact system (1).

Proof. Solution $y = y(t)$ of the averaged system

$$\frac{dy}{d\tau} = X_0(y) + I_0(y), \tag{3}$$

$x(0) = y(0) = x_0$ is uniformly asymptotically stable, then for arbitrary $\eta > 0$ there exists $\delta > 0$ such that for any other solution $y_1(\tau)$ of system (3) from the inequality

$$|y(\tau) - y_1(\tau)| < \frac{\eta}{2} \quad for \quad \tau \geq \tau_0, \tag{4}$$

and the fulfillment of the limiting relation

$$\lim_{\tau\to\infty} |y(\tau) - y_1(\tau)| = 0 \tag{5}$$

and δ does not depend on τ_0. In this case, we can assume that $\delta < \eta < \rho$.

Let $U_\delta(\tau_0)$ δ denote the neighborhood of the point $y(\tau_0)$. By virtue of the definition of uniform asymptotically stability, limit relation (5) is uniform in τ_0 for all neighborhoods $U_\delta(\tau_0)$. Let us show that uniformly in $x_0 \in U_\delta(\tau_0)$.

Let it not be so. Then, there are exists of a number $\mu > 0$ such that $U_\delta(\tau_0)$ one can indicate a converging sequence of points x_n and a sequence of numbers τ_n such that

$$|x_n - y(\tau_0)| < \delta, |y(\tau_n, x_n) - y(\tau_n)| \geq \mu, \tau_n \to \infty \tag{6}$$

where $y(\tau_n, x_n)$ is a solution to system (3) satisfying the condition $y(\tau_0, x_n) = x_n$. Let $\lim_{n\to\infty} x_n = x^0$. Then $\lim_{\tau\to\infty} |y(\tau, x^0) - y(\tau)| = 0$ and we can specify $T > 0$ such that the inequality

$$|y(\tau, x^0) - y(\tau)| < \frac{\sigma(\mu)}{2} \tag{7}$$

for all $\tau \geq T + \tau_0$, where $\sigma(\mu)$ is a constant guaranteeing the inclusion of solutions $y(\tau, x)$ of system (3), starting in the neighborhood $U_{\sigma(\mu)}(0)$ of the point $y(0)$, in $\frac{\mu}{2}$ - the neighborhood of the solution $y(t)$ for all $\tau \geq 0$.

Due to the continuous dependence on the initial data, it can be specified an $N > 0$ such that the inequality

$$|y(\tau, x_n) - y(\tau, x^0)| < \frac{\sigma(\mu)}{2} \tag{8}$$

for all $\tau \in [\tau_0, \tau_0 + T]$ and $n > N$. In this case, we can assume that $\tau_n \geq T + \tau_0$ for $n > N$. Inequalities (7) and (8) imply the inequality

$$|y(\tau_0 + T, x_n) - y(\tau_0 + T)| \leq \sigma(\mu).$$

Therefore, $|y(\tau_n, x_n) - y(\tau_n)| < \frac{\mu}{2}$ for $\tau_n \geq T + \tau_0$, and thus, for $\tau = \tau_n$, the inequality $|y(\tau_n, x_n) - y(\tau_n)| < \frac{\mu}{2}$, which contradicts one of the inequalities (6). Let us choose T so that for $\tau_n \geq T + \tau_0$ the inequality $|y(\tau_0) - y_1(\tau_0)| < \frac{\delta}{2}$ holds if $|y(\tau_0) - y_1(\tau_0)| < \delta$. In view of the above, the choice of T does not depend on either τ_0 or the initial data of the solutions $y_1(\tau_0)$. According to Samoilenko's theorem [16; 113], by averaging impulsive systems, behind the indicated δ and T one can find ε_0 such that for $\varepsilon < \varepsilon_0$ the solution $x(\frac{\tau}{\varepsilon}, x_0)$ of the exact system (1) is defined on the interval $[0, T]$, lies on the domain U_a together with some neighborhood and satisfies the inequality

$$|x(\frac{\tau}{\varepsilon}, x_0) - y(\tau_0)| < \frac{\delta}{2} \quad for \quad \tau \in [0, T]. \tag{9}$$

Thus, the estimate required in the theorem holds on $[0, T]$, and a solution to the exact system exists on this segment.

Let us further consider the solution to the averaged system $y_T(\tau, \varepsilon)$ such that $y_T(\tau, \varepsilon) = x(\frac{T}{\varepsilon}, x_0)$. By virtue of estimate (9), the following estimates hold:

$$|y_T(\tau, \varepsilon) - y(\tau)| < \frac{\eta}{2} \ \ for \ \ \tau \geq T, \tag{10}$$

$$|y_T(2T, \varepsilon) - y(2T)| < \frac{\delta}{2}. \tag{11}$$

Again, by virtue of the Samoilenko theorem, taking into account the uniformity in t of the limit in conditions 2, the solution $x(\frac{\tau}{\varepsilon}, x_0)$ of system (3) is extendable to $[T, 2T]$, lies in U_a together with some neighborhood, and the inequality

$$|x(\frac{\tau}{\varepsilon}, x_0) - y_T(\tau, \varepsilon)| < \frac{\delta}{2} \ \ for \ \ \tau \in [T, 2T] \ \ and \ \ \varepsilon < \varepsilon_0.$$

From (10), (11), and the last inequality, we have the estimate

$$|x(\frac{\tau}{\varepsilon}, x_0) - y(\tau)| \leq |x(\frac{\tau}{\varepsilon}, x_0) - y_T(\tau, \varepsilon)| + |y_T(\tau, \varepsilon) - y(\tau)| < \frac{\delta}{2} + \frac{\eta}{2} < \eta, \tau \in [T, 2T]$$

and for $\tau = 2T$ the estimate $|x(\frac{2T}{\varepsilon}, x_0) - y(2T)| < \delta$. Continuing this process, we obtain the validity of the statement of the theorem.

Consider the impulse system (1) and assume that now the impulses are defined on the entire axis, that is, $t_i(x)$ defined for $i \in Z, i = \pm 1, \pm 2, \ldots$.

Theorem 2. Let the functions $X(t, x)$ and $I_i(x)$ be defined in the domain $Q = \{t \in R, x \in U_a\}$ $(U_a = x \in R^d : |x| \leq a)$ and in these areas

1. The functions $X(t, x)$ and $I_i(x)$ are continuous in terms of a set of variables, bounded by a constant $M > 0$, and in x satisfy the Lipschitz condition with a constant $L > 0$;

2. Uniformly in t, x for $t \in R, x \in U_a$, there exist finite limits

$$X_0(x) = \lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} X(s, x) ds,$$

$$I_0(x) = \lim_{T \to \infty} \frac{1}{T} \sum_{t < t_i(x) < T} I_i(x),$$

3. The averaged system (2) has an asymptotically stable equilibrium position x_0 in the region U_a;

4. The moments of the impulsive effect $t_i(x)$ are continuous functions in U_a uniformly in $i \in N$, and the surfaces $t = t_i(x)$ satisfy the separation condition, that is

$$\min_{x \in U_a} t_{i+1}(x) < \min_{x \in U_a} t_i(x) (i = 1, 2, \ldots)$$

5. The surfaces $t = t_{-1}(x)$ and $t = t_1(x)$ do not intersect with the hyperplane $t = 0$.

Then, for an arbitrary $\eta > 0$, one can specify ε_0 such that $\varepsilon < \varepsilon_0$ the exact system for $t \in R$ has a solution $x(t)$ defined for $t \in R$ and the estimate

$$|x(t) - x_0)| < \eta. \tag{12}$$

Proof. We fix an arbitrary $\eta > 0$ and construct a solution to the exact system satisfying estimate (12). Since the solution x_0 is asymptotically stable, for a given $\eta > 0$ one can specify $\delta > 0$ and $T > 0$ such that for any solution $y(t)$ of system (2) the estimates

$$|y(\tau) - x_0| < \frac{\eta}{2} \ \ for \ \ \tau \geq \tau_0,$$

$$|y(\tau) - x_0| < \frac{\delta}{4} \ \ for \ \ \tau \geq \tau_0 + T,$$

if only $|y(\tau_0) - x_0| < \delta$, and δ and T are independent of τ_0. In this case, we can assume $\delta < \eta$.

Let z_0 be an arbitrary point from the δ-neighborhood of x_0. Consider a solution $x(\frac{\tau}{\varepsilon})$ of the exact system that goes out at $\tau = -T$ from the point z_0. According to Samoilenko's theorem, given δ and T, one can choose ε_0 such that, for $\varepsilon \leq \varepsilon_0$, the inequality

$$|x(\frac{\tau}{\varepsilon}) - y(\tau)| < \frac{\delta}{4} \ \ for \ \ \tau \in [-T, 0],$$

where $y(\tau)$ is the solution to the averaged system with the condition $y(-T) = z_0$. From (4)-(6) it follows that $|x(\frac{\tau}{\varepsilon}) - x_0| < \eta, \tau \in [-T, 0]$ and $|x(0) - x_0| < \frac{\delta}{2}$. Therefore, for $\varepsilon < \varepsilon_0$, all solutions of the exact system that begin with $\tau = -T$ in the δ-neighborhood of x_0, without leaving its η-neighborhood, fall, for $\tau = 0$, in the $\frac{\delta}{2}$-neighborhood of this point.

By analogous reasoning, by virtue of condition 2 of this theorem, it can be shown that, for $\varepsilon < \varepsilon_0$, the solutions of the exact system beginning with $\tau = -nT$ in the δ-neighborhood of x_0 do not leave for $\tau \in [-nT, -(n-1)T]$ from its η-neighborhoods, and for $\tau = -(n-1)T$ fall into the $\frac{\delta}{2}$-neighborhood of the point x_0 for any natural n. We denote by $S_n(\varepsilon)$ the set of values of solutions of the exact system at the point $\tau = 0$, which for $\tau = -nT$ lie in the δ-neighborhood of the point x_0. By what was said above and the uniqueness theorem, this set is not empty for any natural number n and $\varepsilon < \varepsilon_0$, and the inclusion $S_n(\varepsilon) \subseteq S_{(n-1)}(\varepsilon)$ holds. Note that by virtue of condition 5, as follows from [1], the solutions at the point $t = 0$ continuously depend on the initial data. Therefore, the sets $S_n(\varepsilon)$ are closed, and hence, their intersection is nonempty.

Let $z_0(\varepsilon)$ be a point general to all $S_n(\varepsilon)$. Now, for $\varepsilon < \varepsilon_0$, consider the solution of the exact system, which for $\tau = 0$ leaves the point $z_0(\varepsilon)$. By its construction, at the points $-nT$, it belongs to the δ-neighborhood of the point x_0 for any natural n. Therefore, this solution is unboundedly extendable to the left, and for any $\tau \leq 0$, estimate (12) is valid for $\varepsilon < \varepsilon_0$. The extendibility of the solution to the right and the validity of estimate (12) it follows from Theorem 1. The theorem is proved.

From Theorem 3 [18; 479], it follows that under the conditions of Theorem 1, the averaged system (2) has an asymptotically stable equilibrium position. Hence, by virtue of Theorem 2, it follows that the original impulsive system (1), defined on the entire axis, has a two-sided bounded solution on the entire axis for sufficiently small values of the parameter ε. Thus, the following corollary is true.

Consequence. Under conditions 1, 2, 4, 5 of Theorem 2 and condition 3 of Theorem 1, the impulse system (1) defined on the entire axis, for sufficiently small values of the parameter ε, has a two-sided bounded solution on the entire axis.

Thus, a one-sided, bounded, asymptotically stable solution to the averaged system (2) generates a two-sided solution to the exact impulsive system (1).

Example. Impulse model of dark reactions of photosynthesis. Consider the following impulsive system of differential equations

$$\begin{cases} \dot{x} = \varepsilon(x^2 - (1+j)xy + j) \\ \dot{y} = \varepsilon(\frac{1}{7}\mu(7x^2 - y^2 - 6xy), \ t \neq t_i(x,y) \\ \Delta x\Big|_{t=t_i(x,y)} = f_i(x,y) \\ \Delta y\Big|_{t=t_i(x,y)} = g_i(x,y), i = \pm 1, \pm 2, \dots . \end{cases} \tag{13}$$

where ε, j, μ are positive parameters.

System (13) without impulsive effect at $t \geq 0$ is a well-known mathematical model of dark processes of photosynthesis in plants. There $x(t)$ is the normal concentration at time t of fructose, and $y(t)$ is the normal concentration at time t of glucose. This model was first suggested by D.S. Chernavsky (1967) [19], and is one of the first models that describes oscillatory processes in living nature. It turns out that at certain relationships between the parameters in this model, self-oscillation modes arise (that is, there are stable periodic solutions.

In the absence of impulses, system (13) has a unique stationary solution $x_0 = 1, y_0 = 1$. Obviously, in a neighborhood of this point, the right-hand sides of the differential part of system (13) satisfy the conditions of Theorem 2. Let also conditions 4 and 5 of this theorem with respect to impulsive effects be satisfied in this neighborhood. Suppose also that for each natural n the condition

$$|\sum_{-nt_i(x)<n} I_i(x,y)| \leq C$$

with a constant C independent of n. Here $I_i(x,y) = \begin{pmatrix} f_i(x,y) \\ g_i(x,y) \end{pmatrix}$.

Then, obviously, condition 2 of Theorem 2 is satisfied. Moreover, $I_i(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and therefore, the averaged system has the form

$$\begin{cases} \dot{x} = \varepsilon(x^2 - (1+j)xy + j) \\ \dot{y} = \varepsilon(\frac{1}{7}\mu(7x^2 - y^2 - 6xy). \end{cases} \tag{14}$$

Let us check the fulfillment of condition 3 of Theorem 2 for it. It is well-known [16] that for a given model, the case when the parameters of the system are related by the relation

$$\mu = \frac{7}{8}(1-j). \tag{15}$$

These are the so-called bifurcation relations, upon passing through which self-oscillation regimes arise. Therefore, we will consider this particular case. Linearizing system (14) in the vicinity of the equilibrium position $x_0 = 1, y_0 = 1$ and writing down its first approximation matrix, it is easy to see that the characteristic equation for its eigenvalues has the form

$$\lambda^2 + \lambda(\frac{8}{7}\mu + j - 1) + \frac{16}{7}\mu j = 0,$$

and its roots

$$\lambda_{1,2} = -\frac{1}{2}(\frac{8}{7}\mu + j - 1) \pm \frac{1}{2}\sqrt{(\frac{8}{7}\mu + j - 1)^2 - 4\frac{16}{7}\mu j}, \tag{16}$$

From biological considerations [12] it follows that $j < 1$. Therefore, if $\mu > \frac{8}{7}(1-j)$, then the singular point (1,1) of system (14), by virtue of the stability theorem in the first approximation, is an asymptotically stable equilibrium position of system (14). Thus, condition 3 of Theorem 2 is satisfied. Consequently, for sufficiently small values of the parameter ε, the impulse system (13) has a solution bounded on the axis lying in a neighborhood of the point (1,1). Now, let us investigate the most important case for applications (15). In this case, roots (16) will be purely imaginary and the stability theorem in the first approximation does not apply.

To study stability in this case, we use the stability index theorem (Lyapunov exponent) [17]. In this case, roots (16) have the form $\pm i\omega$, where

$$\omega = \sqrt{2j(1-j)}.$$

First, we find the matrix S of the transition from the matrix of the first approximation system to its Jordan form

$$\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.$$

In our case, the matrix S is found from the matrix equation

$$S^{-1}\begin{pmatrix} 1-j & -1-j \\ \frac{8}{7}\mu & -\frac{8}{7}\mu \end{pmatrix} S = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.$$

Now, it is made the replacement in system (14)

$$\begin{pmatrix} x \\ y \end{pmatrix} = S\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

We get the system

$$\begin{cases} \dot{z_1} = \varepsilon\big(z_2\sqrt{2j(1-j)} + z_1 z_2(1+j)\sqrt{2j(1-j)} + z_1^2 j(1+j)\big) \\ \dot{z_2} = \varepsilon\big(-\frac{2(1+j)}{\sqrt{2j(1-j)}}z_1 + \frac{1}{2}z_1 z_2 j(1-j) - z_1^2\frac{(1-j)(\frac{j^2}{2}+j)}{\sqrt{2j(1-j)}} + z_2^2\frac{2j(1-j)^2}{\sqrt{2j(1-j)}}\big). \end{cases}$$

Following [13], the Lyapunov rate in our case has the form

$$I = -\omega^2(1-j)(\frac{1}{2}j+1) - 2j(1+j^2)\omega - \frac{\omega^3(\frac{1}{2}j+1)}{4} + \frac{1}{8}\omega j(1-j)^2 < 0.$$

Therefore, according to the well-known theorem on the stability index [17], the equilibrium position (1,1) of the averaged system (14) is also asymptotically stable in this case; thus, the original system (13) for sufficiently small values of the parameter e has a two-sided bounded solution on the entire axis, lying in a neighborhood of the point (1,1).

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А. Станжицкий[1], А. Асанова[2], М. Мұқаш[3]

[1] *Тарас Шевченко атындағы Киев ұлттық университеті, Киев, Украина;*
[2] *Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[3] *Қ. Жұбанов атындағы Ақтөбе өңірлік университеті, Ақтөбе, Қазақстан*

Бекітілмеген уақыт мезетіндегі импульсті жүйенің ось бойындағы екіжақты, шектелген шешімдері және орташалау әдісі

Крылов пен Боголюбов жәй дифференциалдық теңдеулер үшін ұсынған орташалау әдісі сызықсыз динамикалық жүйелерді талдаудың ең танымал және тиімді әдісі болып табылады. Одан әрі орташалау әдісі әртүрлі мәселелерді зерттеуге қолданылды. Дифференциалдық теңдеулердің импульстық жүйелері өз эволюциясы барысында қысқа мерзімді күштің әсеріне ұшырайтын нысандардың математикалық модельдерін көрсетіні жақсы белгілі. Бекітілмеген импульсі бар есептерге көптеген зерттеулер арналған. Бұл есептер үшін шешімдердің бар болуы, орнықтылығы және де басқа асимптотикалық қасиеттері зерттелген және импульстік жүйелер үшін шеттік есептер қарастырылған. Оған қоса импульстік жүйелердің периодты және периодты дерлік шешімдерінің бар болуы мәселелері зерттелген. Мақалада орташалау әдісі бекітілмеген уақыт мезетіндегі дифференциалдық теңдеулер үшін импульстік жүйенің ось бойындағы екіжақты, шектелген шешімдерінің бар болуын зерттеу үшін қолданылған. Орташаланған жүйеге біржақты, шектелген асимптотикалық орнықты шешім дәл жүйеге екіжақты шешім тудыратыны көрсетілген. Дәл жүйе мен орташаланған жүйенің сәйкес шешімдерінің ақырлы да, ақырсыз да уақыт аралықтарындағы жақындығы Н.Н. Боголюбовтың бірінші және екінші теоремалары арқылы негізделген.

Кілт сөздер: кіші параметр, орташалау әдісі, импульстік әсер, орнықтылық, тепе-теңдік күй.

А. Станжицкий[1], А. Асанова[2], М. Мукаш[3]

[1] *Киевский национальный университет имени Тараса Шевченко, Киев, Украина;*
[2] *Институт математики и математического моделирования, Алматы, Казахстан;*
[3] *Актюбинский региональный университет имени К.Жубанова, Актобе, Казахстан*

Метод усреднения и двусторонние, ограниченные на оси решения импульсных систем с нефиксированными моментами времени

Метод усреднения, предложенный Крыловым и Боголюбовым для обыкновенных дифференциальных уравнений, является одним из самых распространенных и эффективных методов анализа нелинейных динамических систем. Далее метод усреднения был развит и применен для исследования различных проблем. Как известно, импульсные системы дифференциальных уравнений представляют собой математические модели объектов, которые в ходе своей эволюции подвергаются действию краткосрочной силы. Задачам с нефиксированным импульсом посвящено много исследований. Для них изучены существование, устойчивость и другие асимптотические свойства решений и рассмотрены краевые задачи для импульсных систем. Кроме того, рассматривались вопросы существования периодических и почти периодических решений импульсных систем. В статье метод усреднения использован для исследования существования двусторонних, ограниченных на оси решений импульсных систем дифференциальных уравнений с нефиксированными моментами времени. Показано, что односторонее, ограниченное, асимптотически устойчивое решение усредненной системы порождает двустороннее решение точной системы. Близость соответствующих решений точной и усредненной систем как на конечных, так и на бесконечных интервалах времени обоснована первой и второй теоремами Н.Н. Боголюбова.

Ключевые слова: малый параметр, метод усреднения, импульсное воздействие, устойчивость, положение равновесия.

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A.R. Yeshkeyev*

Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan
(E-mail: aibat.kz@gmail.com)

On Jonsson varieties and quasivarieties

In this paper, new objects of research are identified, both from the standpoint of model theory and from the standpoint of universal algebra. Particularly, the Jonsson spectra of the Jonsson varieties and the Jonsson quasivarieties are considered. Basic concepts of 3 types of convexity are given: locally convex theory, $\varphi(x)$-convex theory, J-$\varphi(x)$-convex theory. Also, the inner and outer worlds of the model of the class of theories are considered. The main result is connected with the question of W. Forrest, which is related to the existential closedness of an algebraically closed variety. This article gives a sufficient condition for a positive answer to this question.

Keywords: Jonsson theory, semantic model, Jonsson spectrum, Jonsson variety, Jonsson quasivariety.

This article is related to the new concepts of Jonsson varieties and quasivarieties. The interest in such classes of structures lies in the fact that today the study of Jonsson theories are related exclusively to the problems of the classical model theory and some of its modern sections. However, it seems to us, the use of the concepts of varieties and quasivarieties will expand the area of application of the apparatus of Jonsson theories and, accordingly, the content of the results on the study of Jonsson theories also within the framework of universal algebra. Moreover, the well-known, already classical results on the description of model-theoretic questions, both varieties, and quasivarieties, can be considered from the standpoint of studying Jonsson's theories.

We give well-known definitions and results related to the description of various classes of varieties and quasivarieties.

If an axiomatizable class K is closed concerning subsystems, from it is called universal, if it is closed for homomorphic images, then it is called positive, and if it is closed to filtered products, then it is called Horn. We will call a theory T universal, positive, or Horn if it has a system of axioms consisting, respectively, of universal, positive, or Horn sentences.

The following theorem can already be related to the classical Model Theory.

Theorem 1. (Tarski, Lyndon, Keisler) [1; 351]. Let K be some axiomatizable class of algebraic systems. Then:

a) K is a universal if and only if the theory $Th(K)$ is universal;

b) K is positive if and only if the theory $Th(K)$ is positive;;

c) K is Horn if and only if the theory $Th(K)$ is Horn.

In what follows, the class of systems, if the context does not imply otherwise, we will understand the "axiomatizable class of systems".

Most of the classes of systems considered in algebra have some of the listed closedness properties. If the class K is a positive Horn universal, then K is called a variety. Examples of varieties are the classes of all semigroups, all groups, Abelian groups, Boolean rings, nilpotent groups of steps $\leq s$. Universal Horn classes are called quasivarieties. Quasivarieties are, for example, the class of torsion-free groups from any variety of groups and the class of rings of characteristic 0. An example of a positive Horn class that is neither positive nor universal is the class of atomless Boolean algebras.

Thus, we can formulate the well-known classical result:

Theorem 2. (Birkhoff) [2; 237] For a non-empty class K of algebraic systems to be a variety, it is necessary and sufficient that the following conditions are satisfied:

1) the Cartesian product of an arbitrary sequence of K-systems is a K-system;

2) any subsystem of an arbitrary K-system is a K-system;

*Corresponding author.
E-mail: aibat.kz@gmail.com

3) any homomorphic image of an arbitrary K-system is a K-system;
i.e. it is necessary and sufficient that the class K be hereditary, multiplicatively, and homomorphically closed.

From this, we see, for example, that groups do not form varieties in signature with one multiplication; fields do not form varieties in the ring signature, and so on.

In the well-known monograph of A.I. Maltsev [2], one can find the following definitions of varieties and quasivarieties.

Varieties and quasivarieties of algebras are classes of algebras that can be specified using a collection of identities or, respectively, quasi-identities (conditional identities). Since identities and quasi-identities are $\forall$-formulas, varieties, and quasivarieties are particular types of universally axiomatizable classes of algebras.

Now, we define what the unit structure means.

For each signature Ω there is a unique (up to isomorphism) unit system $\mathfrak{U}_e$ consisting of only one element e for which

$$F(e, ...e) = e, \;\; P(e, ..., e) = \mathrm{T} \;\; (F, P \in \Omega).$$

Definition 1. [2] A subclass M of algebraic systems of signature σ of some class K is called hereditary in K if each K-subsystem of an arbitrary M-system is an M-system.

Theorem 3. [2; 271]. For the class of systems R to be a quasivariety, it is necessary and sufficient that the class R is:

1) locally closed;
2) multiplicatively closed;
3) contained a unitary system.

Consequence 1. [2; 272] The class of systems R is a quasivariety if and only if R is

1) closed with respect to filtered products;
2) a hereditary;
3) contained a unitary system.

In [3–4], the question was raised about study of the class of Jonsson theories of a certain fixed class of structures concerning the concept of cosemanticness, which is an equivalence relation. Such classes of Jonsson theories form the Jonsson spectrum of a fixed class of structures of arbitrary signature. The main idea of this article is to study the Jonsson spectra of such classes of structures that are varieties or quasivarieties, the theory of whose classes is Jonsonian. On the one hand, this approach is new and naturally generalizes all higher formulations of problems; on the other hand, it narrows the class of varieties or quasivarieties with the additional condition of joint embedding and amalgam.

Let us give the following definitions.

Let K be the class of structures of countable signature σ. Let's introduce the notation:

$$\forall\exists(K) = Th(K) \cup \{\varphi \mid \varphi \text{ is a } \forall\exists\text{-sentence and } \; \varphi \cup Th(K) \; \text{ is a consistent}\}.$$

Definition 2. We call a K Jonsson variety if:

1) K is a variety in the usual sense [2; 269];
2) $\forall\exists(K)$ is a Jonsson theory.

Definition 3. We call a K Jonsson quasivariety if:

1) K is a quasivariety in the usual sense [2; 269];
2) $\forall\exists(K)$ is a Jonsson theory.

Consider the $JSpV(K)$ be Jonsson spectrum of the Jonsson varieties of class K, where K is the Jonsson variety:

$$JSpV(K) = \{T/T \text{ is a Jonsson theory, } T = Th_{\forall\exists}(N), \; N \subseteq K, \; N \text{ is a subvariety of } K\}.$$

Consider the $JSpQV(K)$ be Jonsson spectrum of the Jonsson quasivarieties of the class K, where K is the Jonsson quasivariety:

$$JSpQV(K) = \{T/T \text{ is a Jonsson theory, } T = Th_{\forall\exists}(N), \; N \subseteq K, \; N \text{ is a subquasivariety of } K\}.$$

Then $JSpQV(K)/_{\bowtie}$ is denoting the factor set of the Jonsson spectrum of Jonsson quasivariety of the class K by the relation $\bowtie$.

It is clear that $JSpQV(K) \subseteq JSp(\mathcal{A})$ for any model $\mathcal{A} \in K$, where $JSp(\mathcal{A})$ is the Jonsson spectrum [3–4] of the model A.

We say that class K_1 $JSpQV$-cosemantic to class K_2 ($K_1 \underset{JSpQV}{\bowtie} K_2$) if $JSpQV(K_1)/_{\bowtie} = JSpQV(K_2)/_{\bowtie}$.

Theorem 4. Let $[T] \in JSpV(K)$. K is a Jonsson variety of abelian groups, then $C_{[T]} \in E_{[T]}$ its semantic model of the center $[T]^*$, and $C_{[T]}$ be a divisible group and its standard Shmelev group is representable in the form $\oplus_p \mathbb{Z}_{p^\infty}^{(\alpha_p)} \oplus \mathbb{Q}^{(\beta)}$, where $\alpha_p, \beta \in \omega^+$, $2^\omega = |C_{[T]}|$.

We call the pair $(\alpha_p, \beta)^A_{C_{[T]}}$ the Jonsson invariant of the abelian group A if the standard group of the Shmelian group A is representable as $\oplus_p \mathbb{Z}_{p^\infty}^{(\alpha_p)} \oplus \mathbb{Q}^{(\beta)}$, where $C_{[T]}$ is semantic model of $[T] \in JSpV(A)/_{\bowtie}$.

Let K be the class of Jonsson variety of abelian groups. We define the following set $\{(\alpha_p, \beta)^K_{C_{[T]}} : [T] \in JSpV(K)/_{\bowtie}$, for all prime $p\}$ as the Jonsson invariant of the factor-set of Jonsson variety $JSpV(K)/_{\bowtie}$ and denote it by $JInv(JSpV(K)/_{\bowtie})$.

Let's introduce the notation: $\mathbb{C}_K = \{C_{[T]} \mid [T] \in JSpV(K)\}$.

Theorem 5. Let K_1 and K_2 be an arbitrary classes of Jonsson variety of abelian groups, then the following conditions are equivalent:

1) $K_1 \underset{JSpV}{\bowtie} K_2$;

2) $JInv(JSpV(K_1)/_{\bowtie}) = JInv(JSpV(K_2)/_{\bowtie})$.

Proof. The proof follows from the above definitions and from the theorem [5].

Let cl be the closure operator of some pregeometry defined on all subsets of semantic model of considered Jonsson theory.

Let $[T] \in JSpQV(\mathcal{A})/_{\bowtie}$. The center of Jonsson class $[T]$ will be called an elementary theory $[T]^*$ of its semantic model $\mathcal{C}_{[T]}$, i. e. $[T]^* = \mathrm{Th}(\mathcal{C}_{[T]})$ and $[T]^* = \mathrm{Th}(\mathcal{C}_\Delta)$ for every $\Delta \in [T]$.

Denote by $E_{[T]} = \bigcup\limits_{\Delta \in [T]} E_\Delta$ the class of all existentially closed models of the class $[T] \in JSpQV(\mathcal{A})/_{\bowtie}$. Note that $\bigcap\limits_{\Delta \in [T]} E_\Delta \neq \varnothing$, since at least for every $\Delta \in [T]$ we have $\mathcal{C}_{[T]} \in E_\Delta$.

Let $\mathcal{A}, \mathcal{B} \in K$, where K is a variety (a quasivariety).

Definition 4. We say that a model $\mathcal{A}$ is $JSpV$-cosemantic to a model $\mathcal{B}$ ($\mathcal{A} \underset{JSpV}{\bowtie} \mathcal{B}$) if $JSpV(\mathcal{A})/_{\bowtie} = JSpV(\mathcal{B})/_{\bowtie}$. Accordingly, we say that a model $\mathcal{A}$ is $JSpV$-cosemantic to a model $\mathcal{B}$ regarding Γ, where $\Gamma \subseteq L$ and write it $\mathcal{A} \overset{\Gamma}{\underset{JSpV}{\bowtie}} \mathcal{B}$ if $JSpV_\Gamma(\mathcal{A})/_{\bowtie} = JSpV_\Gamma(\mathcal{B})/_{\bowtie}$.

Definition 5. We say that a model $\mathcal{A}$ is $JSpQV$-cosemantic to a model $\mathcal{B}$ ($\mathcal{A} \underset{JSpQV}{\bowtie} \mathcal{B}$) if $JSpQV(\mathcal{A})/_{\bowtie} = JSpQV(\mathcal{B})/_{\bowtie}$. Accordingly, we say that a model $\mathcal{A}$ is $JSpQV$-cosemantic to a model $\mathcal{B}$ regarding Γ, where $\Gamma \subseteq L$ and write it $\mathcal{A} \overset{\Gamma}{\underset{JSpQV}{\bowtie}} \mathcal{B}$ if $JSpQV_\Gamma(\mathcal{A})/_{\bowtie} = JSpQV_\Gamma(\mathcal{B})/_{\bowtie}$.

Definition 6. The class $[T] \in JSpQV(\mathcal{K})/_{\bowtie}$ is called elementarily closed regarding to K, if $\forall [T]' \in JSpQV(\mathcal{K})/_{\bowtie}$: $[T]' \neq [T] \Rightarrow E_{[T]} \cap E_{[T]'} = \emptyset$.

Definition 7. The class $[T] \in JSpQV(\mathcal{K})/_{\bowtie}$ is called locally convex regarding to K, if $Th_{\forall\exists}(\bigcap\limits_{\Delta \in [T]} E_\Delta)$ is a Jonsson theory and reflexcially convex if $Th_{\forall\exists}(K) \in JSpQV(\mathcal{K})$.

Definition 8. The class $[T] \in JSpQV(\mathcal{K})/_{\bowtie}$ is called companion-convex regarding to K if the theory $\nabla = Th_{\forall\exists}(\bigcap\limits_{\Delta \in [T]} E_\Delta)$ is a Jonsson theory and has a model companion.

We can define the completeness of the class $[T]$ regarding Jonsson variety K (Jonsson quasivariety K) as follows (Definition 7), and all four types of completeness are independent of each other and can combine. An interesting problem is the transfer of results from the Jonsson spectrum to the Jonsson variety (Jonsson quasivariety), when the completeness of the Jonsson theory is replaced by the following types of completeness and their combinations.

Let $[T] \in JSpV(K)$ ($[T] \in JSpQV(K)$), $\Gamma \subseteq L$.

Definition 9. The class $[T]$ is called a Γ_i-complete class regarding to K, if the following i-conditions are true:

i_1) $\forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\mathcal{A} \overset{\Gamma}{\underset{JSpV}{\bowtie}} \mathcal{B}$ ($\mathcal{A} \overset{\Gamma}{\underset{JSpQV}{\bowtie}} \mathcal{B}$);

i_2) $\forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\mathcal{A} \overset{\Gamma}{\underset{JSpV}{\bowtie}} \mathcal{B}$ ($\mathcal{A} \overset{\Gamma}{\underset{JSpQV}{\bowtie}} \mathcal{B}$) and $\forall \Delta \in [T]$, $\Delta - \Gamma$-complete;

i_3) $\forall \varphi \in \Gamma$, $\forall \Delta \in [T]$, $\Delta \vdash \varphi$ or $\Delta \vdash \neg\varphi \Leftrightarrow \forall \mathcal{A}, \mathcal{B} \in Mod\Delta$, $\forall \Delta \in [T]$, $\mathcal{A} \equiv_\Gamma \mathcal{B}$;

i_4) $\forall \varphi \in \Gamma$, $\forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\mathcal{A} \models \varphi \Leftrightarrow \mathcal{B} \models \varphi$.

The concepts of completeness and model completeness do not coincide, but as shown by [6], in the case of a perfect Jonsson theory, these concepts coincide for the Jonsson theory under consideration. Therefore, in going over to the problem of the Jonsson spectrum, we must take into account that in the case of an imperfect class, these concepts do not coincide.

A Jonsson variety K (a Jonsson quasivariety K) will satisfy some model-theoretic notion P if each class $[T] \in JSpV(K)$ ($[T] \in JSpQV(K)$) satisfies the P property. Next, we define some particular cases of the property P for the Jonsson variety K (the Jonsson quasivariety K) through an arbitrary class $[T] \in JSpV(K)$ ($[T] \in JSpQV(K)$):

Definition 10. The class $[T]$ is model complete if and only if $\forall \Delta \in [T]$, Δ is model complete.

A Jonsson variety K (a Jonsson quasivariety K) is model complete, if any class $[T] \in JSpV(K)$ ($[T] \in JSpQV(K)$) is model complete if and only if $\forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\forall$ monomorphism $f : \mathcal{A} \to \mathcal{B}$ is elementary if and only if $\forall \varphi \in L, \exists \psi \in \forall \cap \exists : [T] \vdash (\varphi \sim \psi)$ $[T] \vdash (\varphi \sim \psi) \leftrightharpoons \forall \Delta \in [T], \Delta \vdash (\varphi \sim \psi)$.

Lemma 1. If $\Delta \in [T]$ and Δ is imperfect, then $\exists \mathcal{B} \in E_\Delta$, $\mathcal{B} \in E_{\Delta'}$ for some $\Delta' \in [T]$.

Definition 11. Two Jonsson variety (Jonsson quasyvariety) K_1, K_2 existentially mutually model complete to each other ($K_1 \leftrightarrow K_2$), if for any $[T]_1 \in K_1, [T]_2 \in K_2$ follows that the classes $[T]_1$, $[T]_2$ existentially mutually model complete to each other ($[T]_1 \leftrightarrow [T]_2$), i.e. $\forall \mathcal{B} \in E_{[T]_1}$, $\exists \mathcal{B}' \in E_{[T]_2} : \mathcal{B} \overset{\cong}{\to} \mathcal{B}'$ and the converse is true.

Lemma 2. $K_1 \leftrightarrow K_2 \Leftrightarrow Th_\forall(K_1) = Th_\forall(K_2)$.

Let us consider some properties of the Jonsson spectrum at fixed completeness (a special case of Definition 9 (i_2)).

Let K be a Jonsson variety (a Jonsson qvazyvariety).

$[T] \in JSpV(K)$ ($[T] \in JSpQV(K)$).

$Mod[T] = \{\mathcal{A} \in Mod\sigma_K / \mathcal{A} \models T_i, \forall T_i \in [T]\}$.

$Mod(JSpV(\mathcal{A})) = \{\mathcal{B} \in Mod\sigma_K / \mathcal{B} \models T_j, \forall T_j \in JSpV(\mathcal{A})\}$ ($Mod(JSpQV(\mathcal{A})) = \{\mathcal{B} \in Mod\sigma_K / \mathcal{B} \models T_j, \forall T_j \in JSpQV(\mathcal{A})\}$).

$Mod(JSpV(\mathcal{A})/_{\bowtie}) = \{\mathcal{B} \in Mod\sigma_K / \mathcal{B} \models [T], \forall [T] \in JSpV(\mathcal{A})/_{\bowtie}\}$ ($Mod(JSpQV(\mathcal{A})/_{\bowtie}) = \{\mathcal{B} \in Mod\sigma_K / \mathcal{B} \models [T], \forall [T] \in JSpQV(\mathcal{A})/_{\bowtie}\}$).

From above definitions, we can conclude the fact about elementarity: the $E_{[T]}$ is elementary class if and only if $[T]$ has a model companion.

Definition 12. $[T]$ has a model companion if any E_{T_i} will be an elementary class, $T_i \in [T]$.

Lemma 3. $K_1 \underset{JSpQV}{\bowtie} K_2 \Leftrightarrow JSpQV(\mathcal{A})/_{\bowtie} = JSpQV(\mathcal{B})/_{\bowtie}$, $\forall \mathcal{A} \in E_{K_1}, \forall \mathcal{B} \in E_{K_2}$.

Proof. The proof follows from Definitions 10-12.

$JSpV(K)$ ($JSpQV(K)$) is a perfect, i.e. for any $[T] \in JSpV(K)$ ($[T] \in JSpQV(K)$) is true that the $[T]$ is perfect iff $C_{[T]}$ is saturated.

Let $JSpV(K)$ ($JSpQV(K)$) be a perfect, then for any $[T] \in JSpV(K)$ ($[T] \in JSpQV(K)$) the following conditions hold:

1)$[T]$ is complete iff $\forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\mathcal{A} \underset{JSp}{\bowtie} \mathcal{B}$.

2) $E_{[T]} = \bigcup\limits_{i \in I} E_{T_i}$.

3) The $[T]$ is $\forall\exists$-complete if and only if $\forall \mathcal{A}, \mathcal{B} \in E_{[T]}$, $\mathcal{A} \overset{\forall\exists}{\underset{JSp}{\bowtie}} \mathcal{B}$.

4) $A \overset{\forall\exists}{\underset{JSp}{\bowtie}} \mathcal{B}$ iff $JSpQV_{\forall\exists}(\mathcal{A})/_{\bowtie}$ iff $JSpQV_{\forall\exists}(\mathcal{B})/_{\bowtie}$.

5) $JSpQV_{\forall\exists}(\mathcal{A})/_{\bowtie} = \{T \mid T$ is $\forall\exists$-complete Jonsson theory$, \mathcal{A} \models T\}$.

Definition 13. The class of the theory $[T]$ called existential prime, if:

1) it has a prime algebraic model and the class of all algebraically prime models; it is denoted by AP,

2) the class ($E_{[T]}$) of theory $[T]$ has non-empty intersection with an AP class, i.e., $[T]_{AP} \cap E_{[T]} \neq \emptyset$.

Definition 14. A set X is called a theoretical set, if X is Jonsson set, $\varphi(C) = X$ and the sentence $\forall \bar{x} \exists \bar{y} \psi(\bar{x}, \bar{y})$ defines some Jonsson theory.

A class $[T] \in JSp(M)/_{\bowtie}$ is called J-$\varphi(x)$-convex if all theories from class $[T]$ are J-$\varphi(x)$-convex.

Definition 15. The class $[T]$ will be called $\varphi(x)$-convex if it is strong convex in the classical sense [1] and for any existentially closed model N_i of this class, there is a theoretical set A_{N_i}, such that $cl(A_{N_i}) = N_i$, $\varphi(N_i) = A_{N_i}$ and $\bigcap\limits_i N_i = M \in E_{[T]}$.

Definition 16. The class $[T]$ with $AP_{[T]} \neq \varnothing$ will be called J-$\varphi(\bar{x})$-convex if it is convex in the classical sense [1] and for any existentially closed model N_i of this class, there is the Jonsson set A_{N_i}, such that $cl(A_{N_i}) = N_i$,

$\varphi(N_i) = A_{N_i}$ and $\bigcap_i N_i = M$, $M \in E_{[T]} \cap AP_{[T]}$, $N_i \prec_{\Sigma_1} C_{[T]}$, where $AP_{[T]}$ is the set of all algebraically prime models of the class $[T]$.

The following definition defines the inner world $(IW_{[T]}(A))$ of the model A of the class $[T]$ when $A \in E_{[T]}$.

Definition 17. Let $[T] \in JSpV(K)$ $([T] \in JSpQV(K))$. $IW_{[T]}(A) = \{A' \in E_{[T]} |\ f$ is isomorphism, $f : A' \to A,\ A \in E_{[T]}\}$ is called the inner world of the model A for $[T]$.

The following definition defines the outer world $(OW_{[T]}(A))$ of the model A of the class $[T]$ when $A \in E_{[T]}$.

Definition 18. $[T] \in JSpV(K)$ $([T] \in JSpQV(K))$. $OW_{[T]}(A) = \{B \in E_{[T]}$: there exist $A' \cong A,\ A' \subseteq B\}$ is called the outer world of the model A for $[T]$.

Just the world of the existentially closed model A will be the following set

$$W_{[T]}(A) = IW_{[T]}(A) \cup OW_{[T]}(A).$$

Note that the above definitions can connect two different existentially closed models in the case of a convex theory. As the following theorem is true.

Theorem 6. Let T be the perfect, strong convex Jonsson theory. Then for any models $A, B \in E_T$ the following is true:

1) $OW_T(A) \cap OW_T(B) \neq \emptyset$,

2) $IW_T(A) \cap IW_T(B) \neq \emptyset$.

Proof. By virtue of the perfectness of the theory T, $E_T = ModT^*$. All existentially closed models of the theory T are models of the center of the theory T, therefore, property 1) is true due to the fact that models A and B are existentially closed submodels of the semantic model C, where C – semantic model of the theory T. Due to the strongly convexity of the theory T, the intersection of any two models is not empty. Condition 2) is trivial and is performed due to the joint embedding property (JEP) of the theory T. In particular, the model C satisfies these conditions due to the T^+ universality of the model C.

Suppose L is an algebraic language and A is an algebra of type L. If we attach the elements of A as constants to L, then the new language will be denoted by $L(A)$. Let K be a class of algebras of type L and $A \in K$. We say that A is existentially closed in the class K if every existential sentence ψ in the language $L(A)$ which is true in some extention $A \subseteq B \in K$, is also true in A. If we restrict ourself to the positive existential sentences, then we obtain the definition of an algebraiclly closed algebra in the class K. Equivalently, A is existentially closed in K, if and only if every finite set of equations and inequations with coefficients from A, which is solvable in some $B \in K$ containing A, already has a solution in A itself. Similarly A is algebraically closed in K, if and only if every finite set of equations with coefficients from A, which is solvable in some $B \in K$ containing A, already has a solution in A.

The question from [7] which we can formulate as "For what varieties V of algebras with AP is it the case that every algebra A which is algebraically closed (in the sense of groups) is existentially closed?" We would like to obtain a complete characterization of such varieties.

In this work, we found the sufficient conditions to above question under additional JEP-property.

Theorem 7. Let $JSpV(K)$ be a Jonsson spectrum of the Jonsson variety K, then if K is perfect Jonsson class, for any semantic model which algebraically closed of the class $[T] \in JSp(K)$. This model belongs to $E_{[T]}$ also.

Proof. Suppose the opposite. There exists a model $A \in Mod([T])$, $[T] \in JSpV(K)$ such that A is algebraically closed, but $A \notin E_{[T]}$. Then there exists a sentence $\theta = \exists \overline{x} \neg \varphi(\overline{x})$ such that any $B \in Mod([T])$ such that $B \supseteq A$ and $B \models \theta$ follows that $A \nvDash \theta$, then $A \models \neg\theta$, then $A \models \forall \overline{x} \varphi(\overline{x})$. Since $\forall \Delta \in [T]$, Δ-Jonsson theory, there exists $B' \in E_{[T]}$ such that $A \to B'$, $B' \to C$, C is a semantic model of the class $[T]$. If $B \in E_{[T]}$, then $B \equiv_{\forall\exists} B'$ and the sentence θ in a particular case is a $\forall\exists$-sentence. If $B \notin E_{[T]}$, then there exists $B'' \in E_{[T]}$, such that $B \to B''$ и $B'' \to B'$.

Also, for all who have an interest in the particular case of the above-considered materials, one can find out in the following resources [8–12].

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А.Р. Ешкеев

Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан

Йонсондық көптүрлілік және квазикөптүрлілік туралы

Мақалада модельдер теориясы тұрғысынан да, әмбебап алгебра тұрғысынан да жаңа зерттеу объектілері анықталды. Атап айтқанда, йонсондық көптүрлілік пен йонсондық квазикөптүрліліктің йонсондық спектрлері қарастырылған. Дөңестіліктің 3 түрінің негізгі түсініктері берілген: локальді дөңес теория, $\varphi(x)$-дөңес теория, J-$\varphi(x)$-дөңес теория. Сонымен қатар, теориялар класының ішкі және сыртқы әлемдері зерттелген. Негізгі нәтиже В.Форрестің сұрағымен байланысты, ол алгебралық тұйық көптүрліліктің экзистенциалды тұйықталуына қатысты. Мақалада авторы бұл сұраққа оң жауап беру үшін жеткілікті шартты келтірген.

Кілт сөздер: йонсондық теория, семантикалық модель, йонсондық спектр, йонсондық көптүрлілік, йонсондық квазикөптүрлілік.

А.Р. Ешкеев

Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан

О йонсоновских многообразиях и квазимногообразиях

В статье определены новые объекты исследования, как с позиции теории моделей, так и с позиции универсальной алгебры. В частности, рассмотрены йонсоновские спектры йонсоновских многообразий и йонсоновских квазимногообразий. Даны основные понятия трех видов выпуклости: локально выпуклая теория, $\varphi(x)$-выпуклая теория, J-$\varphi(x)$-выпуклая теория. Кроме того, изучены внутренние и внешние миры модели класса теорий. Основной результат связан с вопросом У. Форреста, который относится к экзистенциальной замкнутости алгебраически замкнутого многообразия. Автором статьи приведено достаточное условие для положительного ответа на данный вопрос.

Ключевые слова: йонсоновская теория, семантическая модель, йонсоновский спектр, йонсоновское многообразие, йонсоновское квазимногообразие.

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110th anniversary of the outstanding scientist academician ORYMBEK AKHMETBEKOVICH ZHAUTYKOV (1911–1989)

Orymbek Akhmetbekovich Zhautykov was born in May, 1911 in the Kounrad District (now the Aktogay District) of the Karaganda Region. From 1920 to 1930, he studied first in the village school and then in the schools of the first and second stages in Karkaralinsk. In 1934, he graduated from the Faculty of Physics and Mathematics of the Abai Kazakh Pedagogical Institute and, as an excellent graduate, was appointed to the institute as an assistant. In succession, he was promoted to senior lecturer, associate professor, head of the department, Dean of Faculty of Physics and Mathematics, and deputy director of the institute for science and academic affairs. Zhautykov's scientific career started in Leningrad. In 1939, he entered the graduate school of the Leningrad State University. His scientific supervisor was a famous mathematician, Professor I.P. Natanson. Zhautykov's research interests were shaped under the influence of such prominent mathematicians as V.I. Smirnov, L.V. Kantorovich, N.P. Erugin, N.A. Artemiev. The outbreak of the Great Patriotic War interrupted his studies in the graduate school. In 1941, O.A. Zhautykov started a fruitful scientific collaboration with K.P. Persidsky, who came from Kazan to Alma-Ata. Professor Persidsky, the successor of scientific ideas of the outstanding Russian mathematician A.M. Lyapunov, organized the scientific seminar on the theory of stability. O.A. Zhautykov actively participated in the seminar. In 1944, he successfully defended his candidate dissertation entitled "Certain questions in the theory of stability of motion in the sense of Liapunov". The thesis presented an extension of the Liapunov and Chetaev theorems on the instability of the trivial solutions to systems of ordinary differential equations, and a number of results on the stability of solutions of associated systems. In early 1945, as part of a delegation of Kazakh scientists, O.A. Zhautykov arrived in Moscow to approve the structure and staff of the Academy of Sciences of the Kazakh SSR based on the existing branch of the Academy of Sciences of the USSR. During this period, in Moscow and Leningrad, he met with academicians I.M. Vinogradov, V.I. Smirnov, I.G. Petrovsky and other scientists. They discussed the problems and subjects of the future Sector of Mathematics and Mechanics, the establishment of which was planned as a part of the future Academy of Sciences of the Kazakh SSR and was warmly supported by those mathematicians. The Sector of Mathematics and Mechanics was established on March 1, 1945. At first, O.A. Zhautykov worked there as a senior researcher, and since 1951 he headed the Sector of Mathematics and Mechanics. During these years, he paid great attention to the training of highly-qualified scientific and pedagogical staff for the republic. On his initiative and with his active participation, many young graduates of Kazakhstani universities, especially Kazakh State University named after S.M. Kirov and the Kazakh Pedagogical Institute named after Abai, were sent to the central research institutions and universities of the republic. Scientific knowledge and directions received by young people in leading research centers built the basis for the further development of Kazakhstan mathematics. Many of them later became famous scientists and created their own scientific schools. O.A. Zhautykov's research was mainly focused on the theory of infinite systems of differential equations. He proved the existence of periodic solutions to infinite systems of differential equations and generalized Poincare's classical theorem on the analyticity of a solution with respect to a parameter. Developing the classical ideas of Poisson and Hamilton-Jacobi to countable canonical systems, O.A. Zhautykov proved the validity of the principle of least action for systems with an infinite number of degrees of freedom. O.A. Zhautykov made an important contribution to the theory of partial differential equations of the first order. He developed a method allowing obtaining the representation of solutions in the case of a countable number of independent variables. Developing the ideas of Academician I.G. Petrovsky, O.A. Zhautykov investigated the question of the well-posedness of the Cauchy problem for infinite systems of first-order partial differential equations with two independent variables. He established conditions for the existence of a solution to the Cauchy problem for a countable system of first-order partial differential equations with a finite number of independent variables of general form. Extending the averaging principle of N.N. Bogolyubov in nonlinear mechanics to a countable system of differential equations, O.A. Zhautykov proved a generalized theorem on the integral continuous dependence of solutions to a parameter. In 1961, O.A. Zhautykov defended his doctoral dissertation entitled "Research on the theory of countable systems of differential equations". He paid close attention to approximate methods for solving differential equations and their use in applied problems. O.A. Zhautykov justified the applicability of the method of operational calculus to find the exact and approximate solutions of infinite systems of differential equations. His research on the development of the truncation method, the method of a small parameter, and the averaging method made it possible to solve problems in the theory of oscillations of systems with an infinite number of degrees of freedom and many problems for infinite systems of ordinary differential and integro-differential equations. O.A. Zhautykov made a significant contribution to the study of the stability of integral manifolds of infinite systems of differential equations. He generalized the Lyapunov reduction principle and substantiated the use of the Laplace transform in constructing solutions to countable systems. A number of Zhautykov's works are devoted to the application of the methods of functional analysis to the study of problems of oscillations in distributed systems. Many researchers use his studies, devoted to the vibrations of a rectilinear rod with account for the energy dissipation in the material, as an application of functional analysis methods to vibration problems for elastic systems. O.A. Zhautykov was the first to consider boundary value problems for systems of differential equations with a countable number of parameters. Such problems often arise in control theory, when transferring a controlled object to a certain position. The peculiarity of the control of systems with an infinite number of degrees of freedom is that the extremal principle does not hold for them without additional conditions. Based on the linearization of nonlinear systems of differential equations, O.A. Zhautykov established necessary optimality conditions for such systems. This allowed the problem of optimal control of distributed parameters to be reduced to a problem for an infinite system of differential equations. O.A. Zhautykov developed a constructive method for studying boundary value problems for ordinary differential equations. This method was applied to conduct a comprehensive analysis of the behavior of periodic solutions to equations with a small parameter in critical cases. In 1974, O.A. Zhautykov jointly with K.G. Valeev published the monograph "Infinite Systems of Differential Equations". The value of this monograph was that it collected the latest achievements in the theory of infinite systems of differential equations, and many of them belonged to the authors. The existence and uniqueness of theorems for linear and nonlinear infinite systems, theorems on continuous dependence of a solution on a parameter, and theorems on extensibility of solutions were first presented in the monograph. Qualitative questions of infinite systems of differential equations with delayed argument were also comprehensively investigated. The book met with wide recognition far beyond the USSR. In 1976, O. Zhautykov was awarded the State Prize of the Kazakh SSR in the field of science and engineering. Many international scientists have cited O.A. Zhautykov's research findings in the study of initial and boundary value problems for differential equations with delayed argument. His theorems on the averaging and truncating countable systems of differential equations, as well as their applications to solving oscillation problems for elastic systems described by fourth-order partial differential equations, are presented in the monographs of Academician Yu.A. Mitropolskiy "Averaging method in nonlinear mechanics"(Kiev: Naukova dumka, 1971) and "Asymptotic methods for solving partial differential equations"(Kiev: Vishcha shkola, 1979, co-author B.I. Moseenkov). O.A. Zhautykov 's contribution to the development of mathematical science is fully reflected in the collections "Mathematics in the USSR during Forty Years 1917-1957 "Mathematics in the USSR during Fifty Years 1917-1967 "Mechanics in the USSR during Fifty Years 1917-1967 in the four-volume book "The History of Domestic Mathematics in the book "The Biographical Dictionary of Scientists in the Field of Mathematics". In 1962, O.A. Zhautykov was elected a full member of the Academy of Sciences of the Kazakh SSR for his fundamental research in the theory of differential equations and for his significant contribution to the development of mathematical science. Academician O.A. Zhautykov actively participated in numerous congresses, conferences, and symposia on topical issues of mathematics and mechanics, held in the Soviet

Union and abroad. In 1974, in recognition of his exceptional merit, O.A. Zhautykov was awarded the title of Honored Scientist of the Kazakh SSR. Along with intensive scientific activity, Academician O.A. Zhautykov paid constant attention to the education of the next generation of researchers in mathematics and mechanics. Fifteen candidate dissertations were defended under his supervision. He devoted more than fifty years to pedagogical activity delivering engaging and deeply meaningful lectures to students of Kazakh Pedagogical Institute, Kazakh State University, Kazakh Polytechnic Institute, and Kazakh Women's Pedagogical Institute. In 1958, O.A. Zhautykov published the first textbook on mathematical analysis in the Kazakh language, which became a paramount event in the history of Kazakhstani higher education. His example and experience contributed to the publishing of textbooks in national languages in other Soviet republics. O.A. Zhautykov was a prominent expert in the history and methodology of mathematics, a consistent popularizer of mathematical knowledge. In 1978, he wrote the book "Mathematics and Scientific and Technological Progress" where mathematical problems that significantly influenced the development of science and technology were presented in simple terms. O.A. Zhautykov published the first textbook on ordinary differential equations in the Kazakh language (in two parts, 1950 and 1952), essays about outstanding Russian mathematicians (1956), the books "From mental arithmetic to machine mathematics"(1959), "The history of the development of mathematics from ancient times to the early XVII century"(1967), and the textbook for teachers "Introduction to higher mathematics"(1984). Academician O.A. Zhautykov was a tireless and productive scientist. He published about 200 scientific and methodological works, textbooks, and articles. The efforts of Orymbek Akhmetbekovich Zhautykov to develop mathematical science in the republic, his tireless concern for young highly-qualified staff, and his great reputation among mathematicians contributed to the establishment of the Institute of Mathematics and Mechanics of the Academy of Sciences of the Kazakh SSR (on the basis of the Sector of Mathematics and Mechanics) in 1965. From 1969 to 1985, O.A. Zhautykov headed the Department of Physical and Mathematical Sciences, holding the position of academician-secretary and being a member of the Presidium of the Academy of Sciences of the Kazakh SSR. For many years, he led the Joint Scientific Council, and then a specialized Council for the defense of candidate dissertations. He was the chair of the problem council in mathematics at the Department of Physical and Mathematical Sciences, chairman of the methodological seminar at the Institute of Mathematics and Mechanics, and chairman of the scientific and methodological council at the board of the republican society "Knowledge" for the promotion of physical and mathematical knowledge. He was the editor of a number of thematic collections ("Differential Equations and Their Applications "Functional Analysis and Mathematical Physics"), a member of the editorial board, and then deputy editor-in-chief of the journal "News of the Academy of Sciences of the Kazakh SSR". Physics and Mathematics Series a member of the editorial board of the journal "Bulletin of the Academy of Sciences of the Kazakh SSR". A number of monographs were published under his editorship. Realizing that today's schoolchildren will fill tomorrow's university classrooms, O.A. Zhautykov paid special attention to the enhancement of physical and mathematical education in Kazakhstani schools. He delivered numerous lectures and presentations on educational problems for republican teachers. O.A. Zhautykov put a lot of effort into organizing the Republican Physics and Mathematics School in Alma-Ata, which now bears his name. The students of this school listened to his popular lectures on elementary mathematics. Today, many graduates of the school have become famous scientists, occupy government positions and work fruitfully for the benefit of independent Kazakhstan. At Zhautykov's initiative, the Junior Academy of Sciences for schoolchildren was organized in Almaty. He was the honorary president of the Academy for many years. An outstanding scientist, great teacher, talented scientific organizer, academician of the Academy of Sciences of the Kazakh SSR, doctor of physical and mathematical sciences, professor, laureate of the State Prize of the Kazakh SSR, Orymbek Akhmetbekovich Zhautykov passed away on May 16, 1989. For great merit in the creation and development of mathematical science, the education of scientific and pedagogical personnel, and in the enhancement of physical and mathematical education in Kazakhstan, O.A. Zhautykov was awarded the Order of the October Revolution, two Orders of the Badge of Honor, Certificate of Honor of the Supreme Council of the Kazakh SSR, and many medals and certificates. The Council of Ministers of the Kazakh SSR adopted a resolution to perpetuate the memory of the scientist. Republican Physics and Mathematics School in Alma-Ata and secondary school No. 1 in Karkaralinsk were named after Orymbek Akhmetbekovich Zhautykov. A memorial plaque was installed in the house where he lived. In January 2005, within the walls of the Republican Physics and Mathematics School named after O.A. Zhautykov, the First International Zhautykov Olympiad in mathematics and physics was held. About 200 schoolchildren from 15 countries participated in the Olympiad. Since then, seventeen International Zhautykov Olympiads in mathematics, physics, and computer science have been successfully held. This year, from January 8 to 13, the 17th International Zhautykov Olympiad was first organized in an online format. The Olympiad was attended by 1006 schoolchildren from 21 countries, representing 146 teams from Kazakhstan, Australia, Azerbaijan, Armenia, Belarus, Bulgaria, Georgia, Denmark, India, Indonesia, Iran, Kyrgyzstan, Mongolia, Russia, Romania, Serbia, Tajikistan, Turkmenistan, Turkey, Uzbekistan, and Ukraine. The scientific ideas and directions of O.A. Zhautykov have been successfully developed by his students and followers. One of his well-known students is Doctor of Physical and Mathematical Sciences, Professor Sartabanov Zhaishylyk Almagambetovich, who has been successfully working at the Aktobe Regional University named after K. Zhubanov for many years. Zh.A. Sartabanov and his students extend the methods and scientific results obtained by O.A. Zhautykov to new and important classes of partial differential equations. A talented student of Academician Zhautykov was Doctor of Physical and Mathematical Sciences, Professor Dulat Syzdykbekovich Dzhumabaev. He created his own mathematical school, which implements the fundamental ideas of Orymbek Akhmetbekovich Zhautykov in combination with the Dzhumabaev parameterization method. His numerous students successfully work at leading universities of Kazakhstan and the Institute of Mathematics and Mathematical Modeling. Unfortunately, Professor Dzhumabaev passed away in 2020. In 2014, the Scientific Library of the Academy of Sciences with the support of the family released a unique book "Zhautykov Orymbek Akhmetbekovich, Academician of the Academy of Sciences of the Kazakh SSR" in the Scientific and biographical series "Prominent figures in Kazakhstan science". The book reflects the life and work of O.A. Zhautykov and contains his biographical information, literature about him, documents from home archives (letters, memoirs of contemporaries, his poems, individual reviews, photographs), as well as chronological and alphabetical lists of his research papers. In the book's preface, the son of the academician - Doctor of Physical and Mathematical Sciences, Professor Bolat Orynbekovich Zhautykov writes: "The book offered to your attention is neither a memoir nor a biography, but is a collection of documents, letters, essays, drafts concerning the life and scientific work of the Academician of the Academy of Sciences of the Kazakh SSR, Professor, Doctor of Physical and Mathematical Sciences Orymbek Akhmetbekovich Zhautykov. It has been 100 years since his birth, and for more than 20 years now, he has not been with us. The further his time goes away, the stronger the need to characterize him, his versatile creative activity. As it seems to us, materials preserved in the family archive make it possible to do this in the most adequate way. Unfortunately, the collection does not reflect his pedagogical activity, which accompanied him all his life. The documents and letters are arranged in chronological order. Some letters from one respondent are placed next to each other, despite the fact that they are separated by some significant time interval. All correspondence from the pre-war period has not survived. It should be noted that Orymbek Akhmetbekovich considered the creation of the Institute of Mathematics in the system of the Academy of Sciences of the Kazakh SSR as one of his most significant achievements. Therefore, the book contains copies of letters and draft resolutions prepared by Orymbek Akhmetbekovich for the decision-making administrative bodies of the Republic and the governance of the Academy of Sciences, as well as letters from other correspondents related to the upcoming opening of the institute. Within the framework of that system of science administration in the USSR, the already adopted resolution on the establishment of the institute had to be "pushed" through the high offices in Moscow, which, overcoming various bureaucratic difficulties, Orymbek Akhmetbekovich successfully implemented. The collection presents the biographical sketches of people with whom Orymbek Akhmetbekovich corresponded and had friendly relations. These sketches give an idea of the scopes of their personalities, as well as the wide range of his correspondents."This year, 2021, marks the 30th anniversary of Kazakhstan's independence. On this significant date, we honor iconic personalities of the Kazakh land, who made an outstanding contribution to science and education in our country. In the year of the 110th anniversary of Academician O.A. Zhautykov, we are proud to pay tribute to his memory and respect for his invaluable contribution to the formation and development of Kazakhstan's mathematical science and higher education.

Editorial board of the journal
«Bulletin of the Karaganda University. Mathematics series»

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INFORMATION ABOUT AUTHORS

Aniyarov, A.A. — PhD, Leading Researcher, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.

Assanova, A.T. — Doctor of Physical and Mathematical Sciences, Professor, Chief scientific researcher of the Department of Differential Equations of the Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.

Assetov, A.A. — PhD Student, Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan.

Attaev, A.Kh. — Candidate of Physical and Mathematical Sciences, Docent, Head of department computeraided design mixed systems and control, Institute of Applied Mathematics and Automation of Kabardino-Balkarian Scientific Centre of the Russian Academy of Sciences, Nalchik, Russia.

Baiarystanov, A.O. — Candidate of physical and mathematical sciences, Associate Professor, L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan.

Bayraktar, B.R. — PhD of mathematical sciences, Assistant professor, Doctor, Uludag University, Department of Mathematics and Science Education, Bursa, Turkey.

Bekmaganbetov, K.A. — Doctor of physical and mathematical sciences, Associate professor, Head of the Department of Mathematics and Informatics, M.V. Lomonosov Moscow State University, Kazakhstan Branch, Nur-Sultan, Kazakhstan.

Bekmakhanbet, G.M. — Master's student of the Department of Mathematics, Khoja Ahmet Yasawi International Kazakh-Turkish University, Turkestan, Kazakhstan.

Berdyshev, A.S. — Doctor of physical and mathematical sciences, Professor, Head of the Department of Mathematics and Mathematical Modeling, Abai Kazakh National Pedagogical University, Almaty, Kazakhstan.

Bitimkhan, S. — Candidate of physical and mathematical sciences, Docent of the Department of Mathematical Analysis and Differential Equations, Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan.

Bobodzhanova, M.A. — Candidate of physical and mathematical sciences, Associate Professor of the Department of Higher Mathematics, Moscow Power Engineering Institute, National Research University, Moscow, Russia.

Daşdemir A. — Doctor of applied mathematics, Assistant professor, Kastamonu University, Kastamonu, Turkey.

Jalilov, M.A. — Teacher, Ferghana State University, Ferghana, Uzbekistan.

Jenaliyev, M.T. — Doctor of Physical and Mathematical Sciences, Professor, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.

Jumabayev, S.A. — PhD, Professor, Institute of Management, Academy of Public Administration under the President of Kazakhstan, Nur-Sultan, Kazakhstan.

Kadirkulov, B.J. — Doctor of science (Phys. and Math.), Assistant professor, Head of department of Mathematics and Information technologies, Tashkent State University of Oriental Studies, Tashkent, Uzbekistan.

Kalibekova, A.K. — Master of Science, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.

Kalimbetov, B.T. — Doctor of physical and mathematical sciences, Professor, Director of the Research Institute of Natural Sciences, Khoja Ahmet Yasawi International Kazakh-Turkish University, Turkestan, Kazakhstan.

Kalybay, A.A. — Candidate of physical and mathematical sciences, Associate Professor, KIMEP University, Almaty, Kazakhstan.

Kervenev, K.Ye. — Master, Senior Lecturer, Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan.

Kussainov, R.K. — Master of physics, Researcher, Institute of Mathematics and Mathematical Modeling, Almaty; Senior Lecturer, Shakarim University of Semey, Semey, Kazakhstan.

Huyen L.T. — S.M. Nikol'skii Mathematical Institute, RUDN University, Moscow, Russia.

Mukash, M.A. — PhD student of the Aktobe Regional University after K.Zhubanov, Aktobe, Kazakhstan.

Nurakhmetov, D.B. — PhD, Leading Researcher, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan.

Nurgalieva, Zh.N. — 2nd-year student of Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan.

Orumbayeva, N.T. — Candidate of physical and mathematical sciences, Associate professor, Professor of the Department of Mathematical Analysis and Differential Equations, Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan.

Ryskan, A.R. — PhD, Senior Lecturer, Abai Kazakh National Pedagogical University, Almaty, Kazakhstan.

Savchin, V.M. — Doctor of Physical and Mathematical Sciences, Professor, S.M. Nikol'skii Mathematical Institute, RUDN University, Moscow, Russia.

Shaimardan, S. — PhD, Senior teacher of the Department of Fundamental and Applied Mathematics Faculty, L.N. Gumilyov National Eurasian University, Nur-Sultan, Kazakhstan.

Stanzhytskyi, O.N. — Doctor of Physical and Mathematical Sciences, Professor, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine.

Tokmagambetov, N.S. — Master of Science, Senior teacher, Karagandy University of the name of academician E.A. Buketov, Karaganda; PhD doctoral student of L.N. Gumilyov National Eurasian University, Nur-Sultan, Kazakhstan.

Tokmagambetova, T.D. — Doctoral student of the Department of Mathematical Analysis and Differential Equations, Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan.

Toleugazy, Ye. — PhD, Senior Lecturer, M.V. Lomonosov Moscow State University, Kazakhstan Branch, Nur-Sultan, Kazakhstan.

Yergaliyev, M.G. — PhD, Institute of Mathematics and Mathematical Modeling, Al-Farabi Kazakh National University, Almaty, Kazakhstan.

Yeshkeyev, A.R. — Doctor of Physical and Mathematical Sciences, Professor of the Department of Algebra, Mathematical Logic and Geometry named after Professor T.G. Mustafin, Karagandy University of the name of academician E.A. Buketov, Karaganda, Kazakhstan.

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