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Bounded and multiperiodic solutions of the system of partial integro-differential equations

The system of partial integro-differential equations with an operator of differentiation with respect to directions of vector field is considered. The considering integro-differential equation does not contain space variables. The matricant is constructed that satisfies the linear matrix equation and some of its properties and estimates are obtained that are related to multiperiodicity in time variables. An integral representation of the multiperiodic solution of this integro-differential system through the resolvent of the resolving kernel is given, recurrent relations are obtained for finding them. Some properties of iterated kernels and resolvent are established, corresponding estimates are found. On the basis of the necessary and sufficient condition of periodicity, multiperiodic solutions of a linear integro-differential equation and additional properties of solutions are found. Sufficient conditions for the existence of a bounded and unique multiperiodic solution on all independent variables of the characteristics system of integro-differential equations with a differentiation operator are established.

Keywords: matricant, resolvent, kernel, multiperiodicity, integro-differential, Dirichlet, vector field.

Introduction

We consider the partial integro-differential equation

$$D_c u(\tau,t) = A(\tau,t)u(\tau,t) + \int\limits_{\tau}^{\tau+\theta} K(\tau,t,s,\sigma)u(s,\sigma)ds + f(\tau,t), \qquad (1)$$

where $u(\tau,t)$ is an unknown vector-function, $(\tau,t) \in R \times R^m$; $D_c = \frac{\partial}{\partial\tau} + \left\langle c, \frac{\partial}{\partial t}\right\rangle$ is differential operator, m is vector c constant, $\left\langle c, \frac{\partial}{\partial t}\right\rangle$ - scalar product of vectors c and $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_m}\right)$; $\sigma = \sigma_0 + cs = t - c\tau + cs$ is the characteristic of the differentiation operator D_c by the directions of the vector field $\frac{dt}{d\tau} = c$, $s \in R = (-\infty;+\infty)$; $(n \times n)$ are matrices $A(\tau,t)$ and $K(\tau,t,s,\sigma)$, n is vector-function $f(\tau,t)$.

At the present stage of development of the theory of integro-differential equations, of considerable interest is the development of methods for the qualitative study of the solvability of the problems under consideration. Research of the theory integro-differential equations involved many authors. As you know, V. Volterra used integro-differential equations in problems of hereditary elasticity [1], justified the existence of periodic fluctuations in biological associations, created a general theory of functionals [2]. For systems of integro-differential equations, which form the basis of the mathematical theory of oscillatory processes in natural science and engineering, are devoted to the considerable amount of work, we note [3–5]. In [6], the distribution of the results of M. Urabe to systems of integro-differential equations was considered, the representations of the solution of integro-differential equations through the resolvent of the kernel are investigated [7; 158], the role of these equations at describing processes with aftereffects [8] is indicated. Integro-differential equations find applications in problems of the theory heredity [9; 109], hereditary elasticity of the model, metal creep at high temperatures [10]. Note that the rheological processes are described with integro-differential equations [10], [11; 136, 166]. The monograph [12] is devoted to research of almost periodic solutions system of equations with quasiperiodic right-hand sides. In [13, 14] almost periodic solutions of integro-differential transfer type equations were investigated. Questions of the existence and construction of multiperiodic and pseudoperiodic solutions system of integro-differential equations containing a spatial variable are considered in [15]. By various problems of the theory of periodic and almost periodic oscillations were outlined in [16, 17].

The purpose of the work is to establish sufficient conditions to existence and uniqueness of the multiperiodic solution of integro-differential equation (1) with operator of differentiation with respect to direction of vector field.

Suppose that the conditions of (θ,ω)-periodicity, continuity by $\tau \in R$ and continuous differentiability by $t \in R^m$:

$$A(\tau+\theta, t+q\omega) = A(\tau,t) \in C^{(0,e)}_{\tau,t}\left(R\times R^m\right), q\in Z^m, \tag{2}$$

$$K(\tau+\theta, t+q\omega, s,\sigma) = K(\tau,t,s,\sigma) \in C^{(0,e,0,e)}_{\tau,t,s,\sigma}\left(R\times R^m\times R\times R^m\right), q\in Z^m, \tag{3}$$

$$K(\tau+\theta, t+q\omega, s+\theta, t+q\omega - c(\tau+\theta)+c(s+\theta)) = K(\tau,t,s,t-c\tau+cs), q\in Z^m,$$

$$f(\tau+\theta, t+q\omega) = f(\tau,t) \in C^{(0,e)}_{\tau,t}\left(R\times R^m\right), q\in Z^m. \tag{4}$$

Here m is vector $e=(1,\dots,1)$, the degree of smoothness by $t=(t_1,\dots,t_m)$; Z^m is set of integer vectors $q=(q_1,\dots,q_m)$ and $q\omega=(q_1\omega_1,\dots,q_m\omega_m)$ is multiple period with multiplicity q period $\omega=(\omega_1,\dots,\omega_m)$ and periods $\omega_0=\theta$, $\omega_1,\dots,\omega_m$ - are rationally incommensurable positive constants. Conditions (2) - (4) are called conditions (P).

Main results

We consider the homogeneous integro-differential equation

$$D_c u(\tau,t) = A(\tau,t)u(\tau,t) + \int\limits_{\tau}^{\tau+\theta} K(\tau,t,s,\sigma)u(s,\sigma)ds, \tag{5}$$

corresponding to the nonhomogeneous equation (1).

Taking into account $\sigma=\sigma_0+cs$ by the known technique [13], constructing the matricant of equation $D_c w = A(\tau,t)w$ we define the matrix $W(\tau_0,\tau,t)$ on the basis of integral matrix equation

$$W(\tau_0,\tau,t) = E + \int\limits_{\tau_0}^{\tau} A(s,t-c(\tau-s))W(\tau_0,s,t-c(\tau-s))ds \tag{6}$$

with the unit n-matrix E. Note, by virtue conditions (P), the matrix $W(s,\tau,t)$ is (θ,θ,ω)-periodic by s,τ,t: $W(s+\theta,\tau+\theta,t+q\omega) = W(s,\tau,t) \in C^{(0,0,e)}_{s,\tau,t}\left(R\times R\times R^m\right), q\in Z^m$. We sought the solution of the integral equation (6) in the form of series

$$W(\tau_0,\tau,t) = \sum_{m=0}^{\infty} W_m(\tau_0,\tau,t), \tag{7}$$

members of which we find from the recurrence relations: $W_0(\tau_0,\tau,t)=E$,

$$W_1(\tau_0,\tau,t) = \int\limits_{\tau_0}^{\tau} A(s,t-c(\tau-s))W_0(\tau_0,s,t-c(\tau-s))ds = \int\limits_{\tau_0}^{\tau} A(s,t-c(\tau-s))ds,\dots,$$

$$W_m(\tau_0,\tau,t)=\int\limits_{\tau_0}^{\tau}A(s,t-c(\tau-s))W_{m-1}(\tau_0,s,t-c(\tau-s))ds, m=1,2,\ldots.$$

We show that series (7) converges absolutely and uniformly. Take into account $A(\tau,t)$ matrix boundedness we have $\|W_0(\tau_0,\tau,t)\|=1$,

$$\|W_1(\tau_0,\tau,t)\|\le\int\limits_{\tau_0}^{\tau}\|A(s,t-c(\tau-s))\|ds\le\alpha|\tau-\tau_0|,\ldots,\|W_m(\tau_0,\tau,t)\|\le\alpha^m\frac{|\tau-\tau_0|^m}{m!},$$

where $\|A(\tau,t)\|=\sup\limits_{(\tau,t)\in R\times R^m}|A(\tau,t)|\le\alpha,\alpha=const.$

For the series (7) the majorant is series $1+\alpha M+\frac{\alpha^2M^2}{2!}+\ldots+\frac{\alpha^mM^m}{m!}+\ldots.$
Thus, the matrix series (7) converges absolutely and uniformly at $|\tau-\tau_0|\le M$, $M=const>0$. We assume that the matricant $W(\tau_0,\tau,t)$ for all $\tau\ge\tau_0$ and $t\in R^m$ satisfies condition

$$\|W(\tau_0,\tau,t)\|\le\Gamma e^{-\rho(\tau-\tau_0)},\Gamma\ge1,\rho>0. \tag{8}$$

By a direct verification we can make sure that the matricant $W(\tau_0,\tau,t)$ has the possesses multiperiodicity property with respect by τ_0,τ and t: $W(\tau_0+\theta,\tau+\theta,t+q\omega)-W(\tau_0,\tau,t)=0,q\in Z^m$.

Now using the replacement

$$u(\tau,t)=W(\tau_0,\tau,t)\upsilon(\tau,t) \tag{9}$$

equation (5) is reduced to the form

$$D_c\upsilon(\tau,t)=\int\limits_{\tau}^{\tau+\theta}Q(\tau_0,\tau,t,s,t-c(\tau-s))\upsilon(s,t-c(\tau-s))ds \tag{10}$$

with the kernel $Q(\tau_0,\tau,t,s,\sigma)=W^{-1}(\tau_0,\tau,t)K(\tau,t,s,\sigma)W(\tau_0,s,\sigma)$, where $\sigma=t-c(\tau-s)$.
We note, based on conditions (P), the kernel $Q(\tau_0,\tau,t,s,t-c(\tau-s))$ has the property:

$$Q(\tau_0,\tau+\theta,t+q\omega,s+\theta,t+q\omega-c(\tau+\theta)+c(s+\theta))=Q(\tau_0,\tau,t,s,t-c(\tau-s)),q\in Z^m.$$

For the matricant $V(\tau_0,\tau,t)$ of equation (10), we have the matrix integral equation

$$V(\tau_0,\tau,t)=E+\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))V(\tau_0,s,t-c(\tau-s))dsd\eta. \tag{11}$$

We sought the solution of the integral equation (11) as follows:

$$V(\tau_0,\tau,t)=\sum_{m=0}^{\infty}V_m(\tau_0,\tau,t) \tag{12}$$

with initial approach $V_0(\tau_0,\tau_0,t)=E$. $V_m(\tau_0,\tau,t)$ we find from the relation

$$V_m(\tau_0,\tau,t)=\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))V_{m-1}(\tau_0,s,t-c(\tau-s))dsd\eta,m=1,2,\ldots. \tag{13}$$

From (13) at $m=1$ follows $V_1(\tau_0,\tau,t)=\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\times$

$$\times V_0(\tau_0,s,t-c(\tau-s))dsd\eta=\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta,$$

where $Q_1(\tau_0,\eta,t,s,t)=Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$.

Applying the Dirichlet permutation rule based on the product of two matrices $Q(\tau_0, \eta_1, t, s_1, t) \left(\int\limits_{\tau_0}^{\eta} \int\limits_{\eta}^{\eta+\theta} Q(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s)) ds_1 d\eta_1 \right) \equiv Q_2(\tau_0, \eta, t, s, t)$ by integrating by $\tau = \eta$ at $t := t - c(\tau - \eta)$ from τ_0 to τ, we obtain the matrix $V_2(\tau_0, \tau, t)$:

$$V_2(\tau_0, \tau, t) = \int\limits_{\tau_0}^{\tau} \int\limits_{\eta}^{\eta+\theta} Q_2(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s)) ds d\eta,$$

where $Q_2(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s)) =$

$$= \int\limits_{\eta}^{\tau} \int\limits_{\eta_1}^{\eta_1+\theta} Q(\tau_0, \eta_1, t - c(\tau - \eta_1), s_1, t - c(\tau - s_1)) Q_1(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s)) ds_1 d\eta_1.$$

Further, continuing this process, we obtain

$$V_m(\tau_0, \tau, t) = \int\limits_{\tau_0}^{\tau} \int\limits_{\eta}^{\eta+\theta} Q_m(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s)) ds d\eta, \tag{14}$$

where the iterated kernels are determined from the recurrence relations

$$Q_m(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s)) = \int\limits_{\eta}^{\tau} \int\limits_{\eta_{m-1}}^{\eta_{m-1}+\theta} Q(\tau_0, \eta_{m-1}, t - c(\tau - \eta_{m-1}), s_{m-1}, t - c(\tau - s_{m-1})) \times$$

$$\times Q_{m-1}(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s)) ds_{m-1} d\eta_{m-1} =$$

$$= \int\limits_{\eta}^{\tau} \int\limits_{\eta}^{\eta+\theta} \dots \int\limits_{\tau_0}^{\eta_m} \int\limits_{\eta}^{\eta+\theta} Q(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s)) Q(\tau_0, \eta_1, t - c(\tau - \eta_1), s_1, t - c(\tau - s_1)) \times \dots \times$$

$$\times Q(\tau_0, \eta_{m-1}, t - c(\tau - \eta_{m-1}), s_{m-1}, t - c(\tau - s_{m-1})) ds_1 d\eta_1 ds_2 d\eta_2 \dots ds_{m-1} d\eta_{m-1}. \tag{15}$$

Lemma 1. For iterated kernels the estimates are valid:

$$\|Q_m(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s))\| \le Q_0^m \frac{\theta^{m-1} (\tau - \eta)^{m-1}}{(m-1)!}. \tag{16}$$

Proof. On the grounds (15) we have:

$$\|Q_1(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s))\| \le Q_0, Q_0 = const,$$

$$\|Q_2(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s))\| \le \int\limits_{\eta}^{\tau} \int\limits_{\eta_1}^{\eta_1+\theta} \|Q(\tau_0, \eta_1, t - c(\tau - \eta_1), s_1, t - c(\tau - s_1))\| \times$$

$$\times \|Q(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s))\| ds_1 d\eta_1 \le Q_0^2 \theta (\tau - \eta), \dots.$$

Further

$$\|Q_m(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s))\| \le \int\limits_{\eta}^{\tau} \int\limits_{\eta_{m-1}}^{\eta_{m-1}+\theta} \|Q(\tau_0, \eta_{m-1}, t - c(\tau - \eta_{m-1}), s_{m-1}, t - c(\tau - s_{m-1})) \times$$

$$\times Q_{m-1}(\tau_0, \eta, t - c(\tau - \eta), s, t - c(\tau - s))\| ds_{m-1} d\eta_{m-1} \le Q_0^m \frac{\theta^{m-1} (\tau - \eta)^{m-1}}{(m-1)!}.$$

Taking into account (16) from (14) we get

$$\|V_m(\tau_0,\tau,t)\| \le Q_0^m \frac{\theta^m(\tau-\tau_0)^m}{m!}, m=1,2,\dots. \tag{17}$$

From (17) it follows that the series (12) converges absolutely and uniformly with respect to τ and t for $\tau \ge \tau_0$.

Note that all iterated $Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$ kernels have the property:

$$Q_m(\tau_0+\theta,\eta,t+q\omega-c((\tau+\theta)-\eta),s,t+q\omega-c((\tau+\theta)-s)) = Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)).$$

On the basis of conditions (P), it can be shown that the matrix $V(\tau_0,\tau,t)$ is (θ,θ,ω)-periodic by τ_0,τ,t: $V(\tau_0+\theta,\tau+\theta,t+q\omega) = V(\tau_0,\tau,t) \in C^{(0,0,e)}_{\tau_0,\tau,t}\left(R\times R\times R^m\right), q\in Z^m$.

For the series (12), using (14) we have:

$$V(\tau_0,\tau,t) = E + \int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}\sum_{m=1}^{\infty} Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta. \tag{18}$$

The series $\sum\limits_{m=1}^{\infty} Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$ converges absolutely and uniformly to the continuous function $R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$, called the resolvent of kernel.

Resolved kernel satisfies the integral equation

$$R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)) = Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))+$$
$$+\int\limits_{\eta}^{\tau}\int\limits_{\eta}^{\eta+\theta} Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta. \tag{19}$$

We note, at $\eta=\tau$, the resolvent becomes the kernel

$$R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)) = Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)).$$

We find the solution of integral equation (19) in the form

$$R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)) = Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))+$$
$$+\int\limits_{\eta}^{\tau}\int\limits_{\eta_1}^{\eta_1+\theta} Q(\tau_0,\eta_1,t-c(\tau-\eta_1),s_1,t-c(\tau-s_1))Q_1(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds_1d\eta_1+$$
$$+\int\limits_{\eta}^{\tau}\int\limits_{\eta_2}^{\eta_2+\theta} Q(\tau_0,\eta_2,t-c(\tau-\eta_2),s_2,t-c(\tau-s_2))Q_2(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds_2d\eta_2+\dots+$$
$$+\int\limits_{\eta}^{\tau}\int\limits_{\eta_{m-1}}^{\eta_{m-1}+\theta} Q(\tau_0,\eta_{m-1},t-c(\tau-\eta_{m-1}),s_{m-1},t-c(\tau-s_{m-1}))\times$$
$$\times Q_{m-1}(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds_{m-1}d\eta_{m-1}+\cdots. \tag{20}$$

Further, based on the estimates (16), we obtain $\|R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\| \le$

$$\le Q_0\left(1+Q_0\theta(\tau-\eta)+\frac{Q_0^2\theta^2(\tau-\eta)^2}{2!}+\dots+\frac{Q_0^{m-1}\theta^{m-1}(\tau-\eta)^{m-1}}{(m-1)!}\right).$$

Therefore, $\|R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\| \le Q_0e^{Q_0\theta(\tau-\eta)}$,
where $\|Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\| \le Q_0$.

Applying the resolvent (20) to (18) we have

$$V(\tau_0,\tau,t) = E + \int_{\tau_0}^{\tau}\int_{\eta}^{\eta+\theta} R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta. \quad (21)$$

It is directly possible to show that $R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$ has continuous partial derivatives with respect to τ and t.

We note, the following properties of the resolvent:
1) We show that $R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$ satisfies equation (10). Indeed, applying the operator D_c to both sides of equation (19) we find:

$$D_cR(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)) = D_cQ(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))+$$

$$+\int_{\eta}^{\eta+\theta} Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds+$$

$$+\int_{\eta}^{\tau}\int_{\eta}^{\eta+\theta} D_c[Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))]dsd\eta =$$

$$= \int_{\eta}^{\eta+\theta} Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds.$$

Thus, the resolvent satisfies the integro-differential equation. Note that by the way we used some of the properties of operator D_c, which was to be proven.
2) Resolvent $R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$ is the multiperiodic function:

$$R(\tau_0+\theta,\eta,t+q\omega-c((\tau+\theta)-\eta),s,t+q\omega-c((\tau+\theta)-s)) = R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)), q\in Z^m.$$

Lemma 2. For the function $V(\tau_0,\tau,t)$, the following estimate takes place:

$$\|V(\tau_0,\tau,t)\| \le e^{Q_0\theta(\tau-\tau_0)}. \quad (22)$$

Proof. Using (13), (16), (21) we get

$$\|V(\tau_0,\tau,t)\| \le 1 + \int_{\tau_0}^{\tau}\int_{\eta}^{\eta+\theta} \|R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\|dsd\eta \le$$

$$\le 1 + \int_{\tau_0}^{\tau}\int_{\eta}^{\eta+\theta} Q_0e^{Q_0\theta(\tau-\eta)}dsd\eta \le e^{Q_0\theta(\tau-\tau_0)}.$$

That which was to be shown.

Further, taking into account (9), (21) we have

$$U(\tau_0,\tau,t) = W(\tau_0,\tau,t)\left(E + \int_{\tau_0}^{\tau}\int_{\eta}^{\eta+\theta} R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta\right). \quad (23)$$

Using (8) and (22) we estimate (23)

$$\|U(\tau_0,\tau,t)\| \le \|W(\tau_0,\tau,t)\|\|V(\tau_0,\tau,t)\| \le \Gamma e^{\chi(\tau-\tau_0)}, \chi = Q_0\theta-\rho<0. \quad (*)$$

Find a function $u(\tau, t)$, that satisfies for all $\tau > \tau_0$ and $t \in R^m$ to integro-differential equation (1) and initial condition

$$u(\tau_0, t) = \varphi(t) \in C_t^e(R^m). \tag{24}$$

Solution to the Cauchy problem (1), (24) is sought in the form

$$u(\tau, t) = U(\tau_0, \tau, t)\varphi(t - c(\tau - \tau_0)) + \int_{\tau_0}^{\tau} U(s, \tau, t)f(s, t - c(\tau - s))ds. \tag{25}$$

In (25) assuming that the vector function is $\varphi(t)$ any from $C_t^e(R^m)$, using the necessary and sufficient condition for the multiperiodicity of Umbetzhanov-Sartabanov

$$u(\tau_0 + \theta, t) = u(\tau_0, t) \in C_{\tau,t}^{(0,e)}(R \times R^m) \tag{26}$$

we search among the solutions (25) for the multiperiodic solution of system (1).

Using the necessary and sufficient condition for periodicity (26) to solve (25), we have

$$\varphi(t) = U(\tau_0, \tau_0 + \theta, t)\varphi(t - c((\tau_0 + \theta) - s))+$$

$$+ \int_{\tau_0}^{\tau_0+\theta} U(s, \tau_0 + \theta, t)f(s, t - c((\tau_0 + \theta) - s))ds. \tag{27}$$

Introducing the designation $\psi(t) = \int_{\tau_0}^{\tau_0+\theta} U(s, \tau_0 + \theta, t)f(s, t - c((\tau_0 + \theta) - s))ds$, making a shift by period θ and taking into account the θ-periodicity of the matricant $U(\tau_0, \tau, t - c(\tau - s))$ and vector-functions $f(\tau, t - c(\tau - s))$ we find $\psi(t) = \int_{\tau_0-\theta}^{\tau_0} U(s, \tau_0, t)f(s, t - c(\tau_0 - s))$. Putting $\varphi_0(t) = \psi(t)$, solving equation (27) by the method of successive approximations, we have:

$$\varphi_m(t) = U(\tau_0, \tau_0 + \theta, t - c((\tau_0 + \theta) - s))\varphi_{m-1}(t - c(\tau_0 - s)) + \psi(t), m = 1, 2, \ldots. \tag{28}$$

Along the way, using the convolution type formula for $m = 1$ from (28) we get

$$\varphi_1(t) = U(\tau_0, \tau_0 + \theta, t)\varphi_0(t - c(\tau_0 - s)) + \psi(t) = U(\tau_0, \tau_0 + \theta, t) \int_{\tau_0-\theta}^{\tau_0} U(s, \tau, t)f(s, t - c(\tau_0 - s))ds+$$

$$+ \int_{\tau_0-\theta}^{\tau_0} U(s, \tau, t)f(s, t - c(\tau_0 - s))ds = \int_{\tau_0-2\theta}^{\tau_0-\theta} U(s, \tau, t)f(s, t - c(\tau_0 - s))ds+$$

$$+ \int_{\tau_0-\theta}^{\tau_0} U(s, \tau, t)f(s, t - c(\tau_0 - s))ds = \int_{\tau_0-2\theta}^{\tau_0} U(s, \tau, t)f(s, t - c(\tau_0 - s))ds.$$

Further, using the method of complete mathematical induction, assuming at $k = m$, we have $\varphi_m(t) = \int_{\tau_0-m\theta}^{\tau_0} U(s, \tau, t)f(s, t - c(\tau_0 - s))ds$. In the last integral, replacing the integration variable. Using the θ-periodicity of vector-function $f(\tau, t - c(\tau - s))$ with respect to the variable τ we set $\varphi_k(t) = \int_{\tau_0-k\theta}^{\tau_0} U(s, \tau, t)f(s, t-$ $-c(\tau_0 - s))ds$, $k = 0, 1, 2, \ldots$. Thus we have

$$\varphi^*(t) = \sum_{k=0}^{\infty} \int_{\tau_0-k\theta}^{\tau_0} U(s, \tau, t)f(s, t - c(\tau - s))ds =$$

$$= \int_{-\infty}^{\tau_0} U(s,\tau,t) f(s,t-c(\tau-s))ds. \qquad (29)$$

Substituting (29) into (25), by the way using the group property, we obtain:

$$u(\tau,t) = U(\tau_0,\tau,t) \int_{-\infty}^{\tau_0} U(s,\tau,t) f(s,t-c(\tau-s))ds+$$

$$+\int_{\tau_0}^{\tau} U(s,\tau,t) f(s,t-c(\tau-s))ds = \int_{-\infty}^{\tau} U(s,\tau,t)) f(s,t-c(\tau-s))ds.$$

In this way,

$$u^*(\tau,t) = \int_{-\infty}^{\tau} U(s,\tau,t) f(s,t-c(\tau-s))ds. \qquad (30)$$

The convergence of the improper integral in the right-hand side of (30) is ensured by the boundedness of the vector function $f(\tau, t-c(\tau-s))$.

We note some properties of the vector-function $u^*(\tau,t)$:
1) function $u^*(\tau,t)$ satisfies the integro-differential equation (1) and at $\tau \to \tau_0 + 0$ it turns into $\varphi^*(t)$;
2) it is the multiperiodic function by τ and t with a period vector (θ, ω).

Indeed, consider

$$u^*(\tau+\theta, t+q\omega) = \int_{-\infty}^{\tau+\theta} U(s,\tau+\theta,t+q\omega) f(s,t+q\omega-c((\tau+\theta)-s))ds =$$

$$= \int_{-\infty}^{\tau} U(s+\theta,\tau+\theta,t+q\omega) f(s+\theta,t+q\omega-c((\tau+\theta)-(s+\theta)))ds =$$

$$= \int_{-\infty}^{\tau} U(s,\tau,t) f(s,t-c(\tau-s))ds = u^*(\tau,t).$$

That which was to be demonstrated.
3) for functions $u^*(\tau,t)$ the inequality is fair

$$\|u^*(\tau,t)\| \le \Gamma f_0 \left|\chi\right|^{-1}, \qquad (31)$$

where $\|f(\tau,t-c(\tau-s))\| = \sup\limits_{(\tau,t)\in R\times R^m} |f(\tau,t-c(\tau-s))| \le f_0, f_0 = const.$
4) the solution $u^*(\tau,t)$ is unique.

Result obtained is formulated as the theorem.

Theorem. If conditions (P) and (*) are satisfied, then there is a unique multiperiodic solution $u^*(\tau,t)$ to system (1), defined as (30) and satisfying condition (31).

References

1 Volterra V. Lecons sur les equations integrals et les equations integro-differentielles / V. Volterra. — Paris, 1913. — 165 p.

2 Вольтерра В. Теория функционалов, интегральных и интегро-дифференциальных уравнений / В. Вольтерра. — М.: Наука, 1982. — 303 с.

3 Быков Я.В. О некоторых задачах теории интегро-дифференциальных уравнений / Я.В. Быков. — Фрунзе: Киргиз. гос. ун-т, 1957. — 328 с.

4 Massera I.L. Linear differential equations and functional analysis / I.L. Massera, I.I. Schaffer // I. Annals of math. — 1958. — 67, № 3. — P. 517-573.

5 Langenhop C.E. Note on Almost Periodic solutions of Nonlinear Differential Equations / C.E. Langenhop // Jour. Math. Phys. — 1959. — № 38. — C. 126–129.

6 Самойленко А.М. Метод Бубнова-Галеркина построения периодических решений интегро-дифференциальных уравнений типа Вольтера / А.М. Самойленко, О.Д. Нуржанов // Диф. уравнения. — 1979. — 15, № 8. — С. 1503–1517.

7 Барбашин Е.А. Введение в теорию устойчивости / Е.А. Барбашин. — М.: Наука, 1967. — 224 с.

8 Иманалиев М.И. Нелинейные интегро-дифференциальные уравнения с частными производными / М.И. Иманалиев. — Бишкек: Илим, 1992. — 112 с.

9 Нахушев А.М. Уравнения математической биологии / А.М. Нахушев. — М.: Высш. шк., 1995. — 301 с.

10 Работнов Ю.Н. Элементы наследственной механики твердых тел / Ю.Н. Работнов. — М.: Наука, 1977. — 384 с.

11 Ильюшин А.А. Основы математической теории термовязкоупругости / А.А. Ильюшин, Б.Е. Победря. — М.: Наука, 1970. — 278 с.

12 Харасахал В.Х. Почти периодические решения обыкновенных дифференциальных уравнений / В.Х. Харасахал. — Алма-Ата: Наука, 1970. — 200 с.

13 Умбетжанов Д.У. Почти периодические решения эволюционных уравнений / Д.У. Умбетжанов. — Алма-Ата: Наука, 1990. — 184 с.

14 Умбетжанов Д.У. О существовании почти многопериодического решения одной системы интегро-дифференциальных уравнений в частных производных / Д.У. Умбетжанов, А.Б. Бержанов // Изв. АН КазССР. Сер. физ.-мат. — 1983. — № 5. — С. 11–15.

15 Сартабанов Ж.А. Псевдопериодические решения одной системы интегро-дифференциальных уравнений / Ж.А. Сартабанов // Укр. матем. журн. — 1989. — 41, № 1. — С. 125–130.

16 Абдикаликова Г.А. Построение почти периодического решения одной квазилинейной параболической системы / Г.А. Абдикаликова // Изв. МОН НАН РК. Сер. физ.-мат. — 2001. — № 3. — С. 3–8.

17 Kulzhumiyeva A.A. On multiperiodic integrals of a linear system with the differentiation operator in the direction of the main diagonal in the space of independent variables / A.A. Kulzhumiyeva, Zh.A. Sartabanov // Eurasian Mathematical Journal. — 2017. — 8, № 1. — P. 67–75.

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Дербес туындылы интегро-дифференциалдық теңдеулер жүйесінің шенелген және көппериодты шешімдері

Векторлық өріс бағыттары бойынша дифференциалдау операторлы дербес туындылы интегро-дифференциалдық теңдеулер жүйесі зерттелді. Қарастырылатын интегро-дифференциалдық теңдеуге кеңістік айнымалылары енбейді. Сызықты матрицалық теңдеуге қанағаттандыратын матрицант тұрғызылды және уақыт айнымалылары бойынша көппериодтылықпен байланысты оның кейбір қасиеттері мен бағалаулары алынды. Қарастырылатын интегро-дифференциалдық жүйенің көппериодты шешімінің шешуші ядроның резольвентасы арқылы интегралдық түрі келтірілген, оларды табу үшін рекурренттік қатынастар алынған. Итерацияланған ядролар мен резольвентаның кейбір қасиеттері орнатылған, сәйкес бағалаулар тағайындалған. Периодтылықтың қажетті және жеткілікті шарты негізінде сызықты интегро-дифференциалдық теңдеудің көппериодты шешімдері табылған, сонымен қатар шешімнің қосымша қасиеттері анықталған. Дифференциалдау операторлы интегро-дифференциалдық теңдеулер жүйесінің сипаттамаларында шенелген, барлық тәуелсіз айнымалылары бойынша көппериодты шешімінің бар және жалғыз болуының жеткілікті шарттары орнатылған.

Кілт сөздер: матрицант, резольвента, ядро, көппериодтылық, интегро-дифференциалды, Дирихле, векторлық өріс.

Г.М. Айтенова, Ж.А. Сартабанов, Г.А. Абдикаликова, А. Керимбеков

Ограниченные и многопериодические решения системы интегро-дифференциальных уравнений в частных производных

Исследована система интегро-дифференциальных уравнений в частных производных с оператором дифференцирования по направлениям векторного поля. Рассматриваемое интегро-дифференциальное уравнение не содержит пространственных переменных. Построен матрицант, удовлетворяющий линейному матричному уравнению, и получены некоторые его свойства и оценки, связанные с многопериодичностью по временным переменным. Приведено интегральное представление многопериодического решения данной интегро-дифференциальной системы через резольвенту разрешающего ядра, получены рекуррентные соотношения для их нахождения. Установлены некоторые свойства итерированных ядер и резольвенты, найдены соответствующие оценки. На основе необходимого и достаточного условия периодичности найдены многопериодические решения линейного интегро-дифференциального уравнения, а также выявлены дополнительные свойства решений. Установлены достаточные условия существования ограниченного и единственного многопериодического по всем независимым переменным решения на характеристиках системы интегро-дифференциальных уравнений с оператором дифференцирования.

Ключевые слова: матрицант, резольвента, ядро, многопериодичность, интегро-дифференциальное, Дирихле, векторное поле.

References

1 Volterra, V. (1913). Lecons sur les equations integrals et les equations integro-differentielles. Paris.

2 Volterra, V. (1982). *Teoriia funktsionalov, intehralnykh i intehrodifferentsialnykh uravnenii [Theory of functionals, integral and integro-differential equations].* Moscow: Nauka [in Russian].

3 Bykov, Ia.V. (1957). *O nekotorykh zadachakh teorii intehrodifferentsialnykh uravnenii [On some problems of the theory of integro-differential equations].* Frunze: Kirhizskii hosudarstvennyi universitet [in Russian].

4 Massera, I.L., & Schaffer, I.I. (1958). Linear differential equations and functional analysis. *I. Annals of math., 67, 3,* 517–573.

5 Langenhop, C.E. (1959). Note on Almost Periodic solutions of Nonlinear Differential Equations. *Jour. Math. Phys., 38,* 126–129.

6 Samoilenko, A.M., & Nurzhanov, O.D. (1979). Metod Bubnova-Halerkina postroeniia periodicheskikh reshenii intehrodifferentsialnykh uravnenii tipa Voltera [Bubnov-Galerkin method for constructing periodic solutions of integro-differential equations of Volterra type]. *Differentsialnye Uravneniia – Differential equations, 15, 8,* 1503–1517 [in Russian].

7 Barbashin, E.A. (1967). *Vvedenie v teoriiu ustoichivosti [Introduction to the theory of sustainability].* Moscow: Nauka [in Russian].

8 Imanaliev, M.I. (1992). *Nelineinye intehrodifferentsialnye uravneniia s chastnymi proizvodnymi [Nonlinear partial differential integro-differential equations].* Bishkek: Ilim [in Russian].

9 Nakhushev, A.M. (1995). *Uravneniia matematicheskoi biolohii [Equations of mathematical biology].* Moscow: Vysshaia shkola [in Russian].

10 Rabotnov, Iu.N. (1977). *Elementy nasledstvennoi mekhaniki tverdykh tel [Elements of hereditary solid mechanics].* Moscow: Nauka [in Russian].

11 Iliushin, A.A., & Pobedria, B.E. (1970). *Osnovy matematicheskoi teorii termoviazkoupruhosti [Fundamentals of the mathematical theory of thermo-viscous-elasticity].* Moscow: Nauka [in Russian].

12 Kharasakhal, V.Kh. (1970). *Pochti periodicheskie resheniia obyknovennykh differentsialnykh uravnenii [Almost periodic solutions of ordinary differential equations].* Alma-Ata: Nauka [in Russian].

13 Umbetzhanov, D.U. (1990). *Pochti periodicheskie resheniia evoliutsionnykh uravnenii [Almost periodic solutions of evolutionary equations].* Alma-Ata: Nauka [in Russian].

14 Umbetzhanov, D.U., & Berzhanov, A.B. (1983). O sushchestvovanii pochti mnohoperiodicheskoho resheniia odnoi sistemy intehrodifferentsialnykh uravnenii v chastnykh proizvodnykh [On the existence of an almost multiperiodic solution of one system of integro-differential partial differential equations]. *Izvestiia AN KazSSR. Seriia fiziko-matematicheskaia – News of Academy of Sciences of the Kazakh SSR. Ser. Phys. Mat., 5,* 11–15 [in Russian].

15 Sartabanov, Zh.A. (1989). Psevdoperiodicheskie resheniia odnoi sistemy intehrodifferentsialnykh uravnenii [Pseudoperiodic solutions of one system of integro-differential equations]. *Ukrainskii Matematicheskii zhurnal – Ukr. Mat. Journal, 41, 1,* 125–130 [in Russian].

16 Abdikalikova, G.A. (2001). Postroenie pochti periodicheskoho resheniia odnoi kvazilineinoi paraboliches-koi sistemy [Construction of the almost periodic solution of one quasilinear parabolic system]. *Izvestiia MON NAN RK. Seriia fiziko-matematicheskaia – News of MON NAS RK. Ser. Phys. Mat., 3,* 3–8 [in Russian].

17 Kulzhumiyeva, A.A., & Sartabanov, Zh.A. (2017). On multiperiodic integrals of a linear system with the differentiation operator in the direction of the main diagonal in the space of independent variables. *Eurasian Mathematical Journal, 8, 1,* 67–75.

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Approaching of the solution of a static compressible medium to the solution of an incompressible medium

A well-known analogy of the flow of viscous incompressible fluid and incompressible elastic medium. According to this analogy, the solution of the equations of the elasticity theory with the Poisson's ratio $\nu = 0,5$ and for any fixed shear modulus μ can be interpreted as a motion of a viscous incompressible fluid with viscosity μ. Thus, we can consider the usual static linear elasticity task with Hooke's law at $\lambda \to \infty$, as a mathematical model of approaching to incompressible medium. In this paper, we obtained the asymptotic $\lambda \to \infty$. Estimation of the proximity of the solution of an elastic static problem with Hooke's law to the solution of incompressible medium (Stokes problem). The final estimate allows to use well-known difference schemes and algorithms for an elastic compressible medium to solve incompressible medium. In this paper, an estimate of the proximity of the solutions of these problems is proved at $\lambda \to \infty$, i.e. $\underset{\lambda\to\infty}{u\to\overline{u}^H}$ $\underset{\lambda\to\infty}{\lambda\,\mathrm{div}\,\overline{u}\to -p}$ $\underset{\lambda\to\infty}{\sigma\to\sigma^H}$. To substantiate this fact in [1–3], various methods for the first boundary value problem were investigated. For the static problem of the theory of elasticity, there is currently a whole series of papers devoted to numerical implementation using difference schemes. In paper [4], the estimate $O(\lambda^{-\alpha})$ where $k = \frac{1}{2}$ was obtained, in the proposed paper the estimate $O(\lambda^{-1})$, and in further work we will show that this estimate is best possible in order.

Keywords: incompressible medium, Hooke's law, stresses, deformations, displacements, Lamé coefficients.

A well-known analogy of the flow of viscous incompressible fluid and incompressible elastic medium. According to this analogy, any solution of the equations of the elasticity theory with the Poisson's ratio $\nu = 0,5$ $\left(\nu = \frac{\lambda}{2(\lambda+\mu)}\right)$ to any shear modulus μ can be interpreted as a motion of a viscous incompressible fluid with a viscosity μ (Stokes problem) [5].

In a bounded simply connected domain $D \subset R^3$ with a sufficiently smooth boundary γ we seek a solution to the problem of the theory of elasticity for an incompressible medium that satisfies the equilibrium equation

$$\mu\triangle\overline{u} - \nabla p + \overline{f} = 0, \;\; x \in D, \tag{1}$$

the condition of incompressibility of medium

$$\mathrm{div}\,\overline{u} = 0, x \in D, \tag{2}$$

by the correlation of the displacement-strain

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right), \;\; i,k = 1,2,3, \tag{3}$$

to equations of state of medium

$$\sigma_{ik} = -\delta_{ik}p + 2\mu\varepsilon_{ik}, \tag{4}$$

where σ_{ik} is components of the stress tensor, δ_{ik} is the Kronecker symbol, p is function of pressure, and to boundary conditions

$$\sum_{k=1}^{3}\sigma_{ik}n_k = 0, \;\; x \in \gamma, \tag{5}$$

the task (1)–(4) with boundary conditions of the first kind, i.e. when

$$u = 0, \;\; x \in \gamma, \tag{6}$$

was investigated by various techniques. Its solution was considered as the limit in a certain sense at $\lambda \to \infty$ solutions $\overline{u}^{\lambda}$ the problem of the theory of elasticity for a compressible medium.

$$\mu \triangle \overline{u}^{\lambda} + (\lambda + \mu)\nabla div \overline{u}^{\lambda} + \overline{f} = 0, x \in D, \tag{7}$$

where the components of the strain and stress tensors are related to Hooke's law

$$\sigma_{ij} = \delta_{ij}\lambda\theta + 2\mu\varepsilon_{ij}, \;\; i,j = 1,2,3, \tag{8}$$

where $\theta = \sum\limits_{i=1}^{3} \varepsilon_{ii}$, $\lambda > 0$, $\mu > 0$ the Lam*é* constants, the static problem of the theory of elasticity for an incompressible material (1)–(4), (6) studied in [6–7]. For it, a difference scheme is constructed from the first order of accuracy. Let us turn to the study of the behavior of the solution of problem (3), (5), (7), (8). This problem is not always solvable [8]. The conditions for its solvability are that the main vector and the main moment of the bulk forces are zero.

$$\int\limits_{D} \overline{f} dx = 0, \;\; \int\limits_{D} \overline{f} \times \overline{r} dx = 0, \tag{9}$$

in the case of fulfillment of conditions (9), problems (3), (5), (7), (8) are not uniquely solvable. To single out its only solution, additional conditions are needed.

$$\int\limits_{D} \overline{u} dx = 0, \;\; \int\limits_{D} \operatorname{rot} \overline{u} dx = 0, \tag{10}$$

further, we will assume that for solving the problem (3), (5), (7), (8) the conditions (9), (10) are fulfilled. We first carry out auxiliary arguments. Let problem be solved.

$$\operatorname{div} \overline{\mathrm{v}} = p, \;\; x \in D, \;\; \overline{\mathrm{v}} = \overline{\varphi}, \;\; x \in \gamma. \tag{11}$$

Lemma 1. Following [10, 11], let it be

$$\gamma \in C^{2+m}, \;\; p \in W_2^m(D), \;\; \varphi \in W_2^{m+\frac{1}{2}}(\gamma), \tag{12}$$

moreover we suppose that $(p,1)_D = 0$, $(\overline{\varphi}, \overline{n})_{\gamma} = \int\limits_{\gamma} \overline{\varphi} \;\; \overline{n} \;\; ds$. Then there exists an additive and homogeneous operator $\mathrm{v} = \mathrm{v}(p,u)$ solving the problem (11), (12) and there is fair estimate

$$\|\mathrm{v}\|_{W_2^{m+1}(D)} \le M_m \left(\|p\|_{W_2^m(D)} + \|\varphi\|_{W_2^{m+\frac{1}{2}}(\gamma)} \right).$$

Lemma 2. Let conditions (12) be fulfilled, then there exists a homogeneous operator $\mathrm{v} = \mathrm{v}(p,u)$ that the evaluation takes place

$$\|\mathrm{v}\|_{W_2^{m+1}(D)} \le M_m \left[\|p\|_{W_2^m(D)} + \|\varphi\|_{W_2^{m+\frac{1}{2}}(\gamma)} \right],$$

where v satisfies the following problem

$$\operatorname{div} \overline{\mathrm{v}} = p \cdot [\mu_n(D)]^{-1}, \;\; (\overline{\varphi}, \overline{n})_{\gamma}, \;\; in \;\; D,$$

$$\overline{\mathrm{v}} = \overline{\varphi} + [\mu_{n-1}(\gamma)]^{-1} \cdot (p,1)_D \cdot \overline{n}, \;\; on \;\; \gamma,$$

$\mu_n(D)$ is n-dimensional Lebesgue measure of the domain D, $\mu_{n-1}(\gamma)$ is $n-1$-dimensional Lebesgue measure of its boundary.

Proof. Let Ψ_1 - this sequel to D, that the assessment is made

$$\|\Psi_1\|_{W_2^{m+1}(D)} \le M_m \|\Psi\|_{W_2^{m+\frac{1}{2}}(\gamma)}.$$

As Ψ_1, you can take the solution of the following problem:

$$\Psi_1 = 0, \;\; in \;\; D,$$

$$\Psi_1 = \varphi, \;\; on \;\; \gamma.$$

Let Ψ_2 be similar extension of n to D. Finally, let be

$$\overline{\Psi} = \overline{\Psi}_1 + [\mu_{n-1}(\gamma)]^{-1} \cdot (p,1)_D \cdot \overline{\Psi}_2.$$

Then the vector function $\overline{z} = \overline{\mathrm{v}} - \overline{\Psi}$ is a solution of problem

$$\operatorname{div} \overline{z} = p - \operatorname{div} \overline{\Psi} + [\mu_n(D)]^{-1} (\overline{\varphi}, \overline{n})_\gamma,$$

$$\overline{z} = 0, \;\; on \;\; \gamma.$$

Thus, for the vector function $\overline{z}(x)$, the conditions of the Lemma are already satisfied, as required. Now, for any scalar function is $p(x) \in L_z(D)$.

The formula

$$\|p\|_{W_2^{-1}(D)} = \sup_{q \in W_2^1(D)} \frac{|(p,q)_D|}{\|q\|_{W_2^1(D)}}$$

for all $q \neq 0$ defines the norm of a linear functionality over the space $W_2^1(D)$. A formula

$$\|P\|_{\overset{0}{W}{}_2^{-1}(D)} = \sup_{q \in \overset{0}{W}{}_2^1(D)} \frac{|(p,q)_D|}{\|q\|_{\overset{0}{W}{}_2^1(D)}}$$

defines the norm of a linear functionality over a space $\overset{0}{W}{}_2^{-1}(D)$. Whence it follows that

$$\|p\|_{\overset{0}{W}{}_2^{-1}(D)} \leq \|p\|_{W_2^{-1}(D)} \leq \|p\|_{L_2(D)}.$$

Consequence of Lemma 2. Let v be a solution to problem

$$\operatorname{div} \overline{\mathrm{v}} = p, \;\; in \;\; D. \tag{13}$$

$$\overline{\mathrm{v}} = [\mu_{n-1}(\gamma)]^{-1} (p,1)_D \cdot \overline{n}, \;\; on \;\; \gamma, \tag{14}$$

For an arbitrary function $p(x) \in L_2(D)$. Then the estimate is true

$$\|\overline{\mathrm{v}}\|_{W_2^1(D)} \leq c\|p\|_{L_2(D)}.$$

Let us turn to the problem (3), (5), (7), (8). The solution of this problem satisfies the integral identity

$$E(\overline{u}, \overline{\mathrm{v}}) + \lambda \int\limits_D \operatorname{div} \overline{u} \cdot \operatorname{div} \overline{\mathrm{v}} dx = \int\limits_D \overline{f} \cdot \overline{\mathrm{v}} dx. \tag{15}$$

For all $\mathrm{v} \in W_2^1(D)$ where

$$E(\overline{u}, \overline{\mathrm{v}}) = \frac{1}{2} \mu \int\limits_D \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial \mathrm{v}_i}{\partial x_j} + \frac{\partial \mathrm{v}_j}{\partial x_i} \right) dx.$$

Let in (15) $\overline{\mathrm{v}} = \overline{u}$, then

$$E(\overline{u}, \overline{u}) + \lambda \|\operatorname{div} \overline{u}\|^2 = \int\limits_D \overline{f} \overline{u} dx, \tag{16}$$

Consistently evaluating the right-hand side of equality (16) we will have

$$\left| \int\limits_D \overline{f} \overline{u} dx \right| \leq \|\overline{f}\|_{W_2^{-1}(D)} \cdot \|u\|_{W_2^1(D)} \leq \delta \|u\|^2_{W_2^1(D)} + c_\delta \|f\|^2_{W_2^{-1}(D)},$$

$\delta > 0$, $c_\delta > 0$ are constants. Further, taking into account the Korn inequality [2], we obtain

$$\|u\|^2_{W^1_2(D)} + \lambda \|\operatorname{div} \overline{u}\|^2 \le c\|f\|^2_{W^{-1}_2(D)}.$$

Let v from (13) be the solution of the problem (13), (14), setting that in (15) $\operatorname{div} \overline{\mathrm{v}} = p$, we have

$$\lambda \int\limits_D \operatorname{div} \overline{u} \cdot p dx \le c\|p\|_{L_2(D)} + \|f\|_{W^{-1}_2(D)} \cdot \|p\|_{L_2(D)} \le c\|p\|_{L_2(D)}, \tag{17}$$

in (17) we set $p = \operatorname{div} u$, whence it follows that

$$\lambda \|\operatorname{div} \overline{u}\|_{L_2(D)} \le c < \infty.$$

Thus, we have obtained the following estimate

$$\|u\|_{W^1_2(D)} + \lambda \|\operatorname{div} \overline{u}\|_{L_2(D)} \le c < \infty. \tag{18}$$

Let us pass in (15) to the limit $\lambda \to \infty$. Since, by virtue of estimate (18), we have the relations $\overline{u} \to \overline{u}_0$ weakly in $W^1_2(D)$ at $\lambda \to \infty$, $\lambda \operatorname{div} \overline{u} \to p$ weakly in $L_2(D)$ at $\lambda \to \infty$. From this we obtain that $\overline{u}_0$ and p satisfy the integral identity

$$E(\overline{u}_0, \overline{\mathrm{v}}) - \int\limits_D p \operatorname{div} \overline{\mathrm{v}} dx = \int\limits_D \overline{f}\overline{\mathrm{v}} dx.$$

For an arbitrary vector function $\overline{\mathrm{v}} \in W^1_2(D)$. That is, we will have in the limit for $\lambda = \infty$ a generalized solution of the boundary value problem

$$\mu \triangle \overline{u}_0 - \nabla p + \overline{f} = 0, \;\; x \in D, \tag{19}$$

$$\operatorname{div} \overline{u}_0 = 0, \;\; \sigma^0_{ik} = \mu \left(\frac{\partial u_{oi}}{\partial x_k} + \frac{\partial u_{ok}}{\partial x_i} \right) - \delta_{ik} p, \;\; i, k = 1, 2, 3,$$

$$\sum_{k=1}^{3} \sigma^0_{ik} n_k = 0, \gamma.$$

Next, we estimate the rate of convergence of the solution and problem (3), (5), (7), (8) to the solution $\overline{u}_0$, p of the problem (19).

Denote by $\overline{\mathrm{w}} = \overline{u} - \overline{u}_0$, $p - \lambda \operatorname{div} \overline{u} = \pi$. By virtue of (5), (7), (8) and (19) we obtain

$$E(\overline{\mathrm{w}}, \overline{\mathrm{v}}) - (\pi, \operatorname{div} \overline{\mathrm{v}})_D = 0. \tag{20}$$

Whence it follows that

$$\|\overline{\mathrm{w}}\|^2_{W^1_2(D)} \le \|\pi\|_{L_2(D)} \cdot \|\operatorname{div} \overline{\mathrm{v}}\| \le c \cdot \|\pi\|_{L_2(D)} \cdot \lambda^{-1}. \tag{21}$$

Let v be the solution of the following problem

$$\operatorname{div} \overline{\mathrm{v}} = \pi, \;\; x \in D, \tag{22}$$

$$\overline{\mathrm{v}} = \|\mu_{n-1}(\gamma)\|^{-1} (\pi, 1)_D \cdot \overline{n}, \;\; on \;\; \gamma. \tag{23}$$

And suppose that in (20) the vector function $\overline{\mathrm{v}}$ satisfies (22), (23), then using the consequence of Lemma 2 we obtain

$$\|\pi\|^2 \le c\|\mathrm{w}\|_{W^1_2(D)} \cdot \|\overline{\mathrm{v}}\|_{W^1_2(D)} \le c\|\mathrm{w}\|_{W^1_2(D)} \cdot \|\pi\|_{L_2(D)}. \tag{24}$$

As a result, taking into account (24) there is an assessment

$$\|\pi\|_{L_2(D)} \le c\|\mathrm{w}\|_{W^1_2(D)}. \tag{25}$$

Referring to the estimate (21) then to (25) as a result, we obtain

$$\|\mathrm{w}\|^2_{W^1_2(D)} + \|\pi\|^2_{L_2(D)} \le c \cdot \lambda^{-2}, \tag{26}$$

so we have proved.

Theorem. Let $f \in W_2^{-1}(D)$, then the estimate (26) is valid.

Comment. Here, in the course of the argument, the existence and uniqueness of theorem for the generalized solution of problem (24) is proved. In [11], an estimate of proximity

$$\|\pi\|_{L_2(D)} \leq c \cdot \lambda^{-1/2},$$

was obtained, here from (26) follows

$$\|\pi\|_{L_2(D)} \leq c \cdot \lambda^{-1}.$$

References

1 Кобельков Г.М. Об эквивалентных нормировках подпространств / Г.М. Кобельков // Analysis mathematica, Budapest. — 1977. — Т. 3. — С. 177–186.

2 Михлин С.Т. Спектр пучка операторов теории упругости / С.Т. Михлин. — УМН. — 1973. — Т. 28, Вып. 3(171). — С. 43–82.

3 Babusca I Lectures on the mathematical foundations of the finite-element method with applications / I Babusca, A.K. Aziz. — N.-Y., 1972. — P. 345.

4 Степаненко В.М. Аппроксимация решения задачи теории упругости для несжимаемого материала / В.М. Степаненко. — Т. 9, № 6. — Новосибирск: Б.И., 1978. — С. 136–145.

5 Гольдштейн Р.В. Применение интегральных уравнений для численного решения задач теории упругости и пластичности / Сб. «Численные методы механики сплошной среды» / Р.В. Гольдштейн. — 1978. — Т. 9, № 5. — С. 37–69.

6 Кобельков Г.М. Об одном итерационном методе решения разностных задач теории упругости / Г.М. Кобельков. — Докл. АН СССР. — 1977. — Т. 233, № 5. — С. 33–38.

7 Кузнецов Ю.А. Решение уравнения Гельмольца методом фиктивных областей / Ю.А. Кузнецов, А.М. Мацокин // Сб. «Вычислительные методы линейной алгебры». ВЦ СОАН ССС, 1972. — С. 127–144.

8 Фикера Г. Теоремы существования в теории упругости / Г. Фикера. — М.: Мир, 1974. — С. 159.

9 Ладыженская О.А. О некоторых задачах векторного анализа и обобщенных постановках краевых задач для уравнения Навье-Стокса / О.А. Ладыженская, В.А. Солонников // Записки науч. сем. ЛОМИ. 1976. — Т. 59. — С. 81–116.

10 Ладыженская О.А. Линейные и квазилинейные уравнения эллиптического типа / О.А. Ладыженская, Н.Н. Уральцева. — М.: Наука, 1973. — С. 120.

11 Bibly B.A. The change of shape of a viscous ellipsoidal region embedded in a slowly deforming matrix hawing a different viscosity / B.A. Bibly, I.D. Eshelby, A.K. Kundu // Technophisics. — 1975. — Vol. 28, № 4. — P. 265–274.

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Статикалық сығылатын ортаның шешімін сығылмайтын ортаны шешуге жақындату

Тұтқыр сығылмайтын сұйықтықтың және сығылмайтын серпімді ортаның ағынының ұқсастығы белгілі. Бұл ұқсастыққа сәйкес Пуассон коэффициентіндегі серпімділік теориясының теңдеулерінің шешімі $v = 0,5$ және кез келген тіркелген модулінде μ тұтқырлықпен тұтқыр қысылмайтын сұйықтықтың қозғалысы μ ретінде түсіндірілуі мүмкін. Осылайша, Гук заңымен сызықтық серпімділіктің әдеттегі статикалық есебін $\lambda \to \infty$ сызық қысылмайтын ортаға жақындаудың математикалық моделі ретінде қарастыруға болады. Мақалада асимптотикалық $\lambda \to \infty$ алынды. Гук заңымен тығыз статикалық есептің шешілмейтін ортаны шешуге жақындығын бағалауға (Стокс есебі) негіз бар. Соңғы бағалау сығылмайтын ортаны шешу үшін серпімді қысылған ортаға арналған белгілі айырымдық схемалар мен алгоритмдерді пайдалануға мүмкіндік береді. Авторлар осы міндеттерді шешу жақындығын дәлелдеді $\lambda \to \infty$, с. с. $\underset{\lambda\to\infty}{u\to\overline{u}^H}$ $\underset{\lambda\to\infty}{\lambda\,\mathrm{div}\,\overline{u}\to-p}$ $\underset{\lambda\to\infty}{\sigma\to\sigma^H}$. Бұл фактіні негіздеу үшін [1–3] бірінші шеттік есептің әртүрлі әдістерімен зерттелгенін көруге болады. Серпімділік теориясының статикалық есебі үшін қазіргі уақытта әртүрлі схемалардың көмегімен сандық іске асыруға арналған жұмыстардың тұтас циклі бар. [4] жұмыста $k = \frac{1}{2}$ болғандағы $O(\lambda^{-\alpha})$ бағалауы алынды, онда $O(\lambda^{-1})$ бағалауы бар, алдағы уақытта да бұл баға рет-ретімен жақсартылмайтыны көрсетіледі.

Кілт сөздер: қысылмайтын орта, Гук заңы, кернеу, орын ауыстыру, Ламе коэффициенттері.

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Приближение решения статической сжимаемой среды к решению несжимаемой среды

Известна аналогия течения вязкой несжимаемой жидкости и несжимаемой упругой среды. Согласно этой аналогии, решение уравнений теории упругости при коэффициенте Пуассона $v = 0,5$ и при любом фиксированном модуле сдвига μ может быть интерпретировано как движение вязкой несжимаемой жидкости с вязкостью μ. Таким образом, можно рассматривать обычную статическую задачу линейной упругости с законом Гука при $\lambda \to \infty$ как математическую модель приближения к несжимаемой среде. В статье была получена асимптотическая по $\lambda \to \infty$ оценка близости решения упругой статической задачи с законом Гука к решению несжимаемой среды (задача Стокса). Конечная оценка позволяет использовать известные разностные схемы и алгоритмы для упругой сжимаемой среды для решения несжимаемой среды. В работе доказана оценка близости решений этих задач при $\lambda \to \infty$, т.е. $\underset{\lambda\to\infty}{u\to\overline{u}^H}$ $\underset{\lambda\to\infty}{\lambda\,\mathrm{div}\,\overline{u}\to-p}$ $\underset{\lambda\to\infty}{\sigma\to\sigma^H}$. Для обоснования этого факта в [1–3] были исследованы различные приемы для первой краевой задачи. Для статической задачи теории упругости в настоящее время имеется целый цикл работ, посвященных численной реализации с помощью разностных схем. В [4] получена оценка $O(\lambda^{-\alpha})$, где $k = \frac{1}{2}$, в предлагаемой работе имеет место оценка $O(\lambda^{-1})$, и в дальнейшем будет показано, что это оценка неулучшаема по порядку.

Ключевые слова: несжимаемая среда, закон Гука, напряжения, деформации, перемещения, коэффициенты Ламе.

References

1 Kobelkov, G.M. (1977). Ob ekvivalentnykh normirovkakh podprostranstv [On equivalent normalization of subspaces]. *Analysis mathematica, Vol. 3,* 177–186. Budapest [in Russian].

2 Mikhlin, S.T. (1973). Spektr puchka operatorov teorii upruhosti [The spectrum of the pencil of operators of the theory of elasticity]. *UMN, Vol. 28, 3(171),* 43–82 [in Russian].

3 Babusca, I., & Aziz, A.K. (1972). *Lectures on the mathematical foundations of the finite-element method with applications.* N.-Y.

4 Stepanenko, V.M. (1978). Approksimatsiia resheniia zadachi teorii upruhosti dlia neszhimaemoho materiala [Approximation of the solution of the elasticity theory problem for incompressible material]. Vol. 9, No. 6, 136–145. Novosibirsk [in Russian].

5 Goldstein, R.V. (1978). Primenenie intehralnykh uravnenii dlia chislennoho resheniia zadach teorii upruhosti i plastichnosti [Application of integral equations for the numerical solution of problems of the theory of elasticity and plasticity]. *Chislennye metody mekhaniki sploshnoi sredy – Numerical methods of continuum mechanics, Vol. 9, No. 5,* 37–69 [in Russian].

6 Kobelkov, G.M. (1977). Ob odnom iteratsionnom metode resheniia raznostnykh zadach teorii upruhosti [On an iterative method for solving difference problems in the theory of elasticity]. *Doklady AN SSSR – Reports of the USSR Academy of Sciences, Vol. 233, No. 5,* 33–38 [in Russian].

7 Kuznetsov, Yu.A., & Mazokin, A.M. (1972). Reshenie uravneniia Helmoltsa metodom fiktivnykh oblastei [Solution of the Helmoltz equation by the method of fictitious domains]. Sbornik Vychislitelnye metody lineinoi alhebry. VTS SOAN SSS [in Russian].

8 Ficker, G. (1974). *Teoremy sushchestvovaniia v teorii upruhosti [The existence theorems in the theory of elasticity].* Moscow: Mir [in Russian].

9 Ladyzhenskaya, O.A, & Solonnikov, V.A. (1976). O nekotorykh zadachakh vektornoho analiza i obobshchennykh postanovkakh kraevykh zadach dlia uravneniia Nave-Stoksa [On some problems of vector analysis and generalized formulations of boundary value problems for the Navier-Stokes equation]. *Zapiski nauchnoho seminara LOMI – Notes of the LOMI scientific seminar, Vol. 59,* 81–116 [in Russian].

10 Ladyzhenskaya, O.A., & Uraltseva, N.N. (1973). *Lineinye kvazilineinye uravneniia ellipticheskoho tipa [Linear and quasilinear equations of elliptic type].* Moscow: Nauka [in Russian].

11 Bibly, B.A., Eshelby, I.D., & Kundu, A.K. (1975). The change of shape of a viscous ellipsoidal region embedded in a slowly deforming matrix hawing a different viscosity. *Technophisics, Vol. 28, No. 4,* 265–274.

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A conjugation problem for the heat equation in the field where the boundary moves in linear order

Boundary-value problems for parabolic equations in domains with moving boundaries are fundamentally different from the classical parabolic equations. Due to the dependence of the region size on time, the methods of separation of variables and integral transformations are not applicable to this type of problems in general case, since remaining within the framework of classical methods of mathematical physics, it is not possible to coordinate the solution of the heat conduction equation with the motion of the boundary of the heat transfer region. The solution of this problem has been the subject of research of many domestic and foreign mathematicians [1–8]. A large number of works are devoted to boundary-value problems in non-degenerate domains; they considered the existence of classical solutions by the method of thermal potentials for both the heat conduction equation and for more general parabolic equations. But if the region degenerates at the initial moment of time, then the method of successive approximations for solving integral equations cannot be applied. Since at the degeneration of the domain integral operators become special, that is, when they affect the constant and the upper limit tends to zero, they do not tend to zero. Integral equations of this kind were obtained in [8] in the study of the thermal field of liquid contact bridges and an asymptotic solution was found that can be used to solve practical problems. This paper is devoted to the study of the first boundary value problem for the heat conduction equation with a discontinuous coefficient in the domain that degenerates at the initial moment of time when the boundary moves by linear law. An explicit form of the solution of this problem is obtained, afterwards that can be applied for a numerical approximations.

Keywords: parabolic equations, first boundary value problem, thermal potential method, Jacobi polynomials.

Formulation of the problem: formulation of the problem: it is required to find the function $u_1(x,t), u_2(x,t)$ satisfying following equations:

$$\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x^2}, \quad ((x,t) \in D^-(-\alpha_1 t < x < 0, \quad t > 0)), \tag{1}$$

$$\frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2}, \quad ((x,t) \in D^+(0 < x < \alpha_2 t, \quad t > 0)) \tag{2}$$

boundary conditions:

$$u_1(-\alpha_1 t, t) = \varphi_1(t), \quad u_2(\alpha_2 t, t) = \varphi_2(t), \tag{3}$$

and conjugation conditions:

$$u_1\Big|_{x=-0} = u_2\Big|_{x=+0}, \quad k_1 \frac{\partial u_1}{\partial x}\Big|_{x=-0} = k_2 \frac{\partial u_2}{\partial x}\Big|_{x=+0}, \tag{4}$$

where $\alpha_i > 0, k_i > 0, (i = 1,2)$.

With the help of a suitable substitute [9]: $u_i(x,t) = v_i(x_1,t_1)\frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}}$, where $x_1 = \frac{x}{t}, t_1 = -\frac{1}{t}$ the problem (1)–(4) reduces to the problem without initial conditions in the area with a fixed boundary: required to find functions $v_1(x_1,t_1), v_2(x_1,t_1)$ satisfying following equations

$$\frac{\partial v_1}{\partial t_1} = \frac{\partial^2 v_1}{\partial x_1^2}, \quad ((x_1,t_1) \in G^-(-\alpha_1 < x_1 < 0, \quad -\infty < t_1 < -\frac{1}{T})), \tag{5}$$

$$\frac{\partial v_2}{\partial t_1} = \frac{\partial^2 v_2}{\partial x_1^2}, \quad ((x_1, t_1) \in G^+(0 < x_1 < \alpha_2, \quad -\infty < t_1 < -\frac{1}{T})) \tag{6}$$

boundary conditions:

$$v_1(-\alpha_1, t_1) = \psi_1(t_1), \quad v_2(\alpha_2, t_1) = \psi_2(t_1), \tag{7}$$

and conjugation conditions:

$$v_1\Big|_{x_1=-0} = v_2\Big|_{x_1=+0}, \quad k_1\frac{\partial v_1}{\partial x_1}\Big|_{x_1=-0} = k_2\frac{\partial v_2}{\partial x_1}\Big|_{x_1=+0}, \tag{8}$$

where $\psi_i(t_1) = \varphi_i\left(-\frac{1}{t_1}\right)\frac{e^{-\frac{\alpha_i^2}{4t_1}}}{\sqrt{-t_1}}$, $(i = 1, 2)$.

A problem without initial conditions (5)–(8) can be solved as follows [10]: required to find functions $w_1(x_1, t_1)$, $w_2(x_1, t_1)$ satisfying equations

$$\frac{\partial w_1}{\partial t_1} = \frac{\partial^2 w_1}{\partial x_1^2}, \quad ((x_1, t_1) \in G^-(-\alpha_1 < x_1 < 0, \ t_0 < t_1 < -\frac{1}{T})) \tag{9}$$

$$\frac{\partial w_2}{\partial t_1} = \frac{\partial^2 w_2}{\partial x_1^2}, \quad ((x_1, t_1) \in G^+(0 < x_1 < \alpha_2, \ t_0 < t_1 < -\frac{1}{T})) \tag{10}$$

initial conditions:

$$w_1(x_1, t_0) = f_1(x_1, t_0), \quad w_2(x_1, t_0) = f_2(x_1, t_0) \tag{11}$$

boundary conditions:

$$w_1(-\alpha_1, t_1) = \psi_1(t_1), \quad w_2(\alpha_2, t_1) = \psi_2(t_1), \tag{12}$$

and conjugation conditions:

$$w_1\Big|_{x_1=-0} = w_2\Big|_{x_1=+0}, \quad k_1\frac{\partial w_1}{\partial x_1}\Big|_{x_1=-0} = k_2\frac{\partial w_2}{\partial x_1}\Big|_{x_1=+0}, \tag{13}$$

where $-\infty < t_0 < -\frac{1}{T} < 0$.

Explicit solution of the problem (9)–(13) found in the [7]:

$$w_1(x_1, t_1) = \int_{-\alpha_1}^{0} G_{11}(x_1, \xi, t_1 - t_0) f_1(\xi, t_0) d\xi + \int_{0}^{\alpha_2} G_{12}(x_1, \xi, t_1 - t_0) f_2(\xi, t_0) d\xi +$$
$$+ \int_{t_0}^{t_1} \frac{\partial G_{11}(x_1, -\alpha_1, t_1 - \tau_1)}{\partial \xi}\psi_1(\tau_1) d\tau_1 - \int_{t_0}^{t_1} \frac{\partial G_{12}(x_1, \alpha_2, t_1 - \tau_1)}{\partial \xi}\psi_2(\tau_1) d\tau_1,$$

$$w_2(x_1, t_1) = \int_{-\alpha_1}^{0} G_{21}(x_1, \xi, t_1 - t_0) f_1(\xi, t_0) d\xi + \int_{0}^{\alpha_2} G_{22}(x_1, \xi, t_1 - t_0) f_2(\xi, t_0) d\xi +$$
$$+ \int_{t_0}^{t_1} \frac{\partial G_{21}(x_1, -\alpha_1, t_1 - \tau_1)}{\partial \xi}\psi_1(\tau_1) d\tau_1 - \int_{t_0}^{t_1} \frac{\partial G_{22}(x_1, \alpha_2, t_1 - \tau_1)}{\partial \xi}\psi_2(\tau_1) d\tau_1,$$

where

$$G_{11}(x_1, \xi, t_1) = G(x_1 - \xi, t_1) - G(2\alpha_1 + x_1 + \xi, t_1) +$$
$$+ \sum_{n=1}^{\infty}\sum_{k=1}^{n} \alpha_k^{(n)}(G(2(n-k+1)\alpha_1 + 2(k-1)\alpha_2 + x_1 - \xi, t_1) +$$
$$+ G(2(n-k+1)\alpha_1 + 2(k-1)\alpha_2 - x_1 + \xi, t_1) -$$
$$- G(2(n-k+2)\alpha_1 + 2(k-1)\alpha_2 + x_1 + \xi, t_1) - G(2(n-k)\alpha_1 + 2(k-1)\alpha_2 - x_1 - \xi, t_1)$$

$$G_{12}(x_1, \xi, t_1) = \mu_2 \sum_{n=1}^{\infty}\sum_{k=1}^{n} \beta_k^{(n)}(G(2(n-k+1)\alpha_1 + 2k\alpha_2 + x_1 - \xi, t_1) +$$

$$+G(2(n-k)\alpha_1+2(k-1)\alpha_2-x_1+\xi,t_1)-G(2(n-k+1)\alpha_1+2(k-1)\alpha_2+x_1+\xi,t_1)-$$
$$-G(2(n-k)\alpha_1+2k\alpha_2-x_1-\xi,t_1))$$

$$G_{21}(x_1,\xi,t_1)=\mu_1\sum_{n=1}^{\infty}\sum_{k=1}^{n}\gamma_k^{(n)}(G(2(n-k)\alpha_1+2(k-1)\alpha_2+x_1-\xi,t_1)+$$
$$+G(2(n-k+1)\alpha_1+2k\alpha_2-x_1+\xi,t_1)-$$
$$-G(2(n-k+1)\alpha_1+2(k-1)\alpha_2+x_1+\xi,t_1)-G(2(n-k)\alpha_1+2k\alpha_2-x_1-\xi,t_1))$$

$$G_{22}(x_1,\xi,t_1)=G(x_1-\xi,t_1)-G(2l_2-x_1-\xi,t_1)+$$
$$+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\delta_k^{(n)}(G(2(n-k)\alpha_1+2k\alpha_2+x_1-\xi,t_1)+G(2(n-k)\alpha_1+2k\alpha_2-x_1+\xi,t_1)+$$
$$-G(2(n-k)\alpha_1+2(k-1)\alpha_2+x_1+\xi,t_1)-G(2(n-k)\alpha_1+2(k+1)\alpha_2-x_1-\xi,t_1))$$

$$\alpha_k^{(n)}=\begin{cases}(-\lambda)^n, \quad k=1, \quad n\ge 1\\ (-1)^{n+k}\sum_{m=0}^{M}(-1)^m C_{n-k+1}^{m+1}C_{k-2}^{m}\lambda^{n-2m-2}\mu^{2m+2}, 2\le k\le n,\end{cases}$$
$$M=\min(k-2,n-k)$$

$$\beta_k^{(n)}=\gamma_k^{(n)}=(-1)^{n+k}\sum_{m=0}^{K}(-1)^m C_{k-1}^{m}C_{n-k}^{m}\lambda^{n-2m-1}\mu^{2m}, \quad 1\le k\le n$$
$$K=\min(k-1,n-k)$$

$$\delta_k^{(n)}=\begin{cases}\lambda^n, \quad k=n, \quad n\ge 1\\ (-1)^{n+k+1}\sum_{m=0}^{N-1}(-1)^m C_{n-k-1}^{m}C_{k}^{m+1}\lambda^{n-2m-2}\mu^{2m+2}, \quad 1\le k\le n-1,\end{cases}$$
$$N=\min(k,n-k)$$

$$\mu^2=\mu_1\mu_2, \mu_i=\frac{2k_i}{k_1+k_2}, \quad (i=1,2), \quad \lambda=\frac{k_1-k_2}{k_1+k_2}; \quad G(y,\tau)=\frac{1}{2\sqrt{\pi\tau}}e^{-\frac{y^2}{4\tau}}$$

By the definition of Jacobi polynomials [11]:

$$P_k(\theta;\alpha,\beta)=\sum_{m=0}^{k}C_{\alpha+k}^{m}C_{\beta+k}^{k-m}\left(\frac{\theta-1}{2}\right)^{k-m}\left(\frac{\theta+1}{2}\right)^{m},$$

then the coefficients $\alpha_k^{(n)}$, $\beta_k^{(n)}$, $\gamma_k^{(n)}$, $\delta_k^{(n)}$ can be expressed in terms of Jacobi polynomials. If we introduce the following notation $\mu^2-\lambda^2=\theta$ and by virtue of the fact $\mu^2+\lambda^2=1$, then we have

$$\alpha_k^{(n)}=\begin{cases}(-1)^n\left(\frac{1-\theta}{2}\right)^{\frac{n-2k+1}{2}}, k=1, n\ge 1,\\ (-1)^n\frac{n-k+1}{k-1}\left(\frac{1+\theta}{2}\right)\left(\frac{1-\theta}{2}\right)^{\frac{n-2k+2}{2}}P_{k-2}(\theta;n-2k+2,1), k-2\le n-k,\\ \left(\frac{1+\theta}{2}\right)\left(\frac{1-\theta}{2}\right)^{\frac{2k-n-2}{2}}P_{n-k}(\theta;2k-n-2,1), k-2>n-k,\end{cases} \tag{14}$$

$$\delta_k^{(n)}=\begin{cases}\left(\frac{1-\theta}{2}\right)^{\frac{n}{2}}, k=n, n\ge 1,\\ \frac{k}{n-k}\left(\frac{1+\theta}{2}\right)\left(\frac{1-\theta}{2}\right)^{\frac{2k-n}{2}}P_{n-k-1}(\theta;2k-n,1), k-1>n-k-1,\\ (-1)^n\left(\frac{1+\theta}{2}\right)\left(\frac{1-\theta}{2}\right)^{\frac{n-2k}{2}}P_{k-1}(\theta;n-2k,1), k-1\le n-k-1,\end{cases} \tag{15}$$

$$\beta_k^{(n)}=\gamma_k^{(n)}=\begin{cases}\left(\frac{1-\theta}{2}\right)^{\frac{2k-n-1}{2}}P_{n-k}(\theta;2k-n-1,0), k-1>n-k,\\ (-1)^{n+1}\left(\frac{1-\theta}{2}\right)^{\frac{n-2k+1}{2}}P_{k-1}(\theta;n-2k+1,0), k-1\le n-k\end{cases} \tag{16}$$

Now we state some auxiliary result from [12].

Lemma. Let $h(\theta)=\left(\frac{1-\theta}{2}\right)^{\alpha}$ be a weight function defined on the interval $[-1,1]$, $\alpha>-1$. If $P_k(\theta;\alpha,0)$ is a sequence of the corresponding orthogonal Jacobi polynomials, then the function $\left(\frac{1-\theta}{2}\right)^{\frac{\alpha}{2}}|P_k(\theta;\alpha,0)|$ attains its maximum in $[-1,1]$ at the point $\theta=-1$.

By lemma and formulas (14) - (16), we obtain the following estimates

$$\{|\beta_k^{(n)}|,|\gamma_k^{(n)}|\}<1,\quad \{|\alpha_k^{(n)}|,|\delta_k^{(n)}|\}<2.$$

By applying the last estimates it is easy to verify that

$$\lim_{t_0\to-\infty}\int_{-\alpha_1}^{0}G_{i1}(x_1,\xi,t_1-t_0)f_1(\xi,t_0)d\xi=0,\quad \lim_{t_0\to-\infty}\int_{0}^{\alpha_2}G_{i2}(x_1,\xi,t_1-t_0)f_2(\xi,t_0)d\xi=0, i=1,2.$$

Further, when $\tau_0\to-\infty$ the solution of the problem without initial conditions (5)–(8) is obtained [10].

$$v_1(x_1,t_1)=\int_{-\infty}^{t_1}\frac{\partial G_{11}(x_1,-\alpha_1,t_1-\tau_1)}{\partial\xi}\psi_1(\tau_1)d\tau_1-\int_{-\infty}^{t_1}\frac{\partial G_{12}(x_1,\alpha_2,t_1-\tau_1)}{\partial\xi}\psi_2(\tau_1)d\tau_1,$$

$$v_2(x_1,t_1)=\int_{-\infty}^{t_1}\frac{\partial G_{21}(x_1,-\alpha_1,t_1-\tau_1)}{\partial\xi}\psi_1(\tau_1)d\tau_1-\int_{-\infty}^{t_1}\frac{\partial G_{22}(x_1,\alpha_2,t_1-\tau_1)}{\partial\xi}\psi_2(\tau_1)d\tau_1.$$

Going back to the original variables we obtain an explicit solution to the problem (1)–(4):

$$u_1(x,t)=\int_0^t g_{11}(x,t-\tau)\varphi_1(\tau)d\tau+\int_0^t g_{12}(x,t-\tau)\varphi_2(\tau)d\tau,$$

$$u_2(x,t)=\int_0^t g_{21}(x,t-\tau)\varphi_1(\tau)d\tau+\int_0^t g_{22}(x,t-\tau)\varphi_2(\tau)d\tau,$$

where

$$g_{11}(x,t-\tau)=-2\frac{\partial G(x+\alpha_1 t,t-\tau)}{\partial x}e^{\frac{\alpha_1(\alpha_1 t+\alpha_1\tau+2x)}{4}}-$$
$$-2\sum_{n=1}^{\infty}\sum_{k=1}^{n}\alpha_k^{(n)}\left(\frac{\partial G((2n-2k+3)\alpha_1 t+2(k-1)\alpha_2 t+x,t-\tau)}{\partial x}\cdot\right.$$
$$e^{\frac{((2n-2k+3)\alpha_1+2(k-1)\alpha_2)^2t+\alpha_1^2\tau+2(2n-2k+3)\alpha_1x+4(k-1)\alpha_2x}{4}}+\frac{\partial G((2n-2k+1)\alpha_1 t+2(k-1)\alpha_2 t-x,t-\tau)}{\partial x}\cdot$$
$$\left.e^{\frac{((2n-2k+1)\alpha_1+2(k-1)\alpha_2)^2t+\alpha_1^2\tau-2(2n-2k+1)\alpha_1x-4(k-1)\alpha_2x}{4}}\right)$$

$$g_{12}(x,t-\tau)=2\mu_2\sum_{n=1}^{\infty}\sum_{k=1}^{n}\beta_k^{(n)}\left(\frac{\partial G(2(n-k+1)\alpha_1 t+(2k-1)\alpha_2 t+x,t-\tau)}{\partial x}\cdot\right.$$
$$e^{\frac{(2(n-k+1)\alpha_1+(2k-1)\alpha_2)^2t+\alpha_2^2\tau+4(n-k+1)\alpha_1x+2(2k-1)\alpha_2x}{4}}+\frac{\partial G(2(n-k)\alpha_1 t+(2k-1)\alpha_2 t-x,t-\tau)}{\partial x}\cdot$$
$$\left.e^{\frac{(2(n-k)\alpha_1+(2k-1)\alpha_2)^2t+\alpha_2^2\tau-4(n-k)\alpha_1x-2(2k-1)\alpha_2x}{4}}\right)$$

$$g_{21}(x,t-\tau)=-2\mu_1\sum_{n=1}^{\infty}\sum_{k=1}^{n}\gamma_k^{(n)}\left(\frac{\partial G((2n-2k+1)\alpha_1 t+2(k-1)\alpha_2 t+x,t-\tau)}{\partial x}\cdot\right.$$
$$e^{\frac{((2n-2k+1)\alpha_1+2(k-1)\alpha_2)^2t+\alpha_1^2\tau+2(2n-2k+1)\alpha_1x+4(k-1)\alpha_2x}{4}}+\frac{\partial G((2n-2k+1)\alpha_1 t+2k\alpha_2 t-x,t-\tau)}{\partial x}\cdot$$
$$\left.e^{\frac{((2n-2k+1)\alpha_1+2k\alpha_2)^2t+\alpha_1^2\tau-2(2n-2k+1)\alpha_1x-4k\alpha_2x}{4}}\right)$$

$$
\begin{aligned}
&g_{22}(x,t-\tau) = 2\frac{\partial G(\alpha_2 t - x, t-\tau)}{\partial x} e^{\frac{\alpha_2(\alpha_2 t + \alpha_2\tau - 2x)}{4}} - \\
&-2\sum_{n=1}^{\infty}\sum_{k=1}^{n}\delta_k^{(n)}\left(\frac{\partial G(2(n-k)\alpha_1 t + (2k-1)\alpha_2 t + x, t-\tau)}{\partial x}\cdot\right. \\
&e^{\frac{(2(n-k)\alpha_1+(2k-1)\alpha_2)^2 t + \alpha_2^2\tau + 4(n-k)\alpha_1 x + 2(2k-1)\alpha_2 x}{4}} + \frac{\partial G(2(n-k)\alpha_1 t + (2k+1)\alpha_2 t - x, t-\tau)}{\partial x}\cdot \\
&\left. e^{\frac{(2(n-k)\alpha_1+(2k+1)\alpha_2)^2 t + \alpha_2^2\tau - 4(n-k)\alpha_1 x - 2(2k+1)\alpha_2 x}{4}}\right)
\end{aligned}
$$

References

1 Gevrey M. Sur les equations aux derives partielles du type parabolique / M. Gevrey // J. Math. Pures et Appl. — 1913. — 9 — № 4. — P. 305–471.

2 Petrovsky I.G. Zur ersten Randwertaufgabe der Warmeleitungsgleichungen // Compos. Math. — 1935. — 1 — № 3. — P. 383–419.

3 Камынин Л.И. О решении IV и V краевых задач для одномерного параболического уравнения второго порядка в криволинейной области / Л.И. Камынин // Журн. вычисл. математики и мат. физики. — 1969. — Т. 9, № 3. — С. 558–572.

4 Михайлов В.П. Теорема существования и единственности решения одной задачи для параболического уравнения в области с особыми точками на границе / В.П. Михайлов // Тр. МИАН СССР. — 1967. — Т. 91. — С. 47–58.

5 Кондратьев В.А. Краевые задачи для параболических уравнений в замкнутых областях / В.А. Кондратьев // Тр. Москов. мат. общ-ва. — 1966. — Т. 15. — С. 400–451.

6 Ким Е.И. Решение в малом задачи с нелинейным граничным условием для уравнения теплопроводности в расширяющейся области / Е.И. Ким, А.А. Кавокин // Изв. АН КазССР. Сер. физ-мат. — 1977. — № 1. — С. 41–46.

7 Ким Е.И. Решение задачи теории теплопроводности с разрывным коэффициентом и вырождающимися подвижными границами / Е.И. Ким, У.К. Койлышов // Изв. АН КазССР. Сер. физ.-мат. — 1984. — № 3. — С. 35–39.

8 Харин С.Н. Тепловые процессы в электрических контактах и связанные с ними сингулярные интегральные уравнения: дис. ... канд. физ-мат. наук / С.Н. Харин. — Алма-Ата, 1968. — 131 с.

9 Миллер У. Симметрия и разделение переменных / У. Миллер, мл.; пер. с англ. Г.П. Бабенко. — М.: Мир, 1981. — 342 с.

10 Тихонов А.Н. Уравнения математической физики / А.Н. Тихонов, А.А. Самарский. — М.: Наука, 1977. — 736 с.

11 Сеге Г. Ортогональные многочлены / Г. Сеге; пер. с англ. В.С. Виденского. — М.: Гос. изд-во физ.-мат. лит., 1962. — 500 с.

12 Суетин П.К. Классические ортогональные многочлены / П.К. Суетин. — М.: Наука, 1979. — 419 с.

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Шекарасы сызықты заңмен қозғалатын облыста жылуөткізгіштік теңдеу үшін бір түйіндес есеп

Шекарасы қозғалмалы облыстағы параболалық типті теңдеулер үшін шекаралық есептер әдеттегі классикалық есептерден айтарлықтай ерекшеленеді. Облыстың өлшемі уақытқа тәуелді болғандықтан, бұл типтегі есептерге жалпы жағдайда Фурьенің айнымалыларды жіктеу және интегралдық түрлендіру әдістерін қолдануға болмайды, себебі математикалық физиканың классикалық әдістерін қолдансақ, жылуөткізгіш теңдеудің шешімі жылутасымалдау облысының шекарасының қозғалысымен келіспейді. Бұл мәселенің шешімі көптеген отандық және шетелдік математиктердің зерттеу объектісі болып табылды [1–8]. Шекарасы қозғалмалы облыстағы жылуөткізгіш есептер көптеген авторлардың жұмыстарында қарастырылған. Көптеген жұмыстар азғындалмаған облыстағы шекаралық есептерге арналған, оларда жылуөткізгіш теңдеу үшін жылу потенциалдар әдісі арқылы классикалық шешімнің бар болу сұрағы зерттелген. Облыс азғындалған жағдайда тізбектеп жуықтау әдісі интегралдық теңдеудің шешімін табуға жарамайды. Мұндай типтегі интегралдық теңдеу сұйық контактілі жылу өрістерін зерттеу барысында [8] жұмыста алынған және онда практикалық есептердің шешімі ретінде қарастыруға болатын асимптотикалық шешімі табылған. Берілген жұмыс шекарасы сызықты заң бойынша қозғалатын, бастапқы уақыт сәтінде азғындалатын облыста жылуөткізгіштік теңдеу үшін бір түйіндес есептің шешімін зерттеуге арналған. Қойылған есептің сандық әдіспен шешуге қолдануға болатын айқын шешімі алынған.

Кілт сөздер: жылуөткізгіш теңдеу, түйіндес есеп, қозғалмалы шекара, азғындалатын облыс, айқын шешім.

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Об одной задаче сопряжения для уравнения теплопроводности в области при движении границы по линейному закону

Краевые задачи для уравнений теплопроводности в областях с движущейся границей принципиально отличны от классических. Вследствие зависимости размера области от времени, к этому типу задач в общем случае не применимы методы разделения переменных и интегральных преобразований, так как, оставаясь в рамках классических методов математической физики, не удается согласовать решение уравнения теплопроводности с движением границы области теплопереноса. Решение этой проблемы являлось предметом исследования многих отечественных и зарубежных математиков [1–8]. Большое число работ посвящены краевым задачам в невырождающихся областях, в них рассматривались вопросы существования классических решений методом тепловых потенциалов для уравнений параболического типа. Когда область вырождается в начальный момент времени, то метод последовательных приближений решения интегральных уравнений невозможно применить. Так как при вырождении области интегральные операторы становятся особыми, т.е. при их воздействии на постоянную и стремлении верхнего предела к нулю, они не стремятся к нулю. Интегральные уравнения такого рода были получены в работе [8], при изучении теплового поля жидких контактных мостиков, и найдено асимптотическое решение, которое можно использовать для решения практических задач. Данная работа посвящена исследованию одной задачи сопряжения для уравнения теплопроводности в области, вырождающейся в начальный момент времени, когда граница движется по линейному закону. Получен явный вид решения данной задачи, который впоследствии можно применять и для численного решения.

Ключевые слова: уравнения теплопроводности, задача сопряжения, подвижная граница, вырождающаяся область, явное решение.

References

1 Gevrey, M. (1913). Sur les equations aux derives partielles du type parabolique. *J. Math. Pures et Appl., 9, 4,* 305–471.

2 Petrovsky, I.G. (1935). Zur ersten Randwertaufgabe der Warmeleitungsgleichungen. *Compos. Math., 1, 3,* 383–419.

3 Kamynin, L.I. (1969). O reshenii IV i V kraevykh zadach dlia odnomernoho parabolicheckoho uravneniia vtoroho poriadka v krivolineinoi oblacti [About IV and V solutions a one-dimensional boundary value problems for second order parabolic equation in a curvilinear region]. *Zhurnal vychislitelnoi matematiki i matematicheskoi fiziki – Journal of Calculated Math. and Math. Physiks E., 9, 3,* 558–572 [in Russian].

4 Mikhailov, V.P. (1967). Teorema sushchestvovaniia i edinstvennosti resheniia odnoi zadachi dlia parabolicheskoho uravneniia v oblasti s osobymi tochkami na hranitse [The existence and uniqueness theorem for solving a single problem for a parabolic equation in a domain with singular points on the boundary]. *Trudy MIAN SSSR – Proceedings MIAN USSR, 91,* 47–58 [in Russian].

5 Kondratyev, V.A. (1966). Krayevye zadachi dlia parabolicheskikh uravnenii v zamknutykh oblastiakh [Boundary value problems for parabolic equations in closed domains]. *Trudy Moskovskoho matematicheskoho obshchestva – Works of the Moscow Mathematical Society, 15,* 400–451 [in Russian].

6 Kim, E.I., & Kavokin, A.A. (1977). Reshenie v malom zadachi s nelineinym hranichnym usloviem dlia uravneniia teploprovodnosti v rasshiriaiushcheisia oblasti [Solution in a small problem with a nonlinear boundary condition for the heat equation in an expanding domain]. *Izvestie AN KazSSR. Seriia fiziko-matematicheskaiia – News of the Kazakh SSR. Ser. phys.-math., 1,* 41–46 [in Russian].

7 Kim, E.I., & Koilyshov, U.K. (1984). Reshenie zadachi teorii teploprovodnotsti s razryvnym koeffitsientom i vyrozhdaiushchimisia podvizhnymi hranitsami [Solution of the of heat conduction with a discontinuous coefficient and degenerate moving boundaries]. *Izvestie AN KazSSR. Seriia fiziko-matematicheskaiia – News of the Kazakh SSR. Ser. phys.-math.,* 35–39 [in Russian].

8 Kharin, S.N. (1968). Teplovye protsessy v elektricheskikh kontaktakh i svyazannye s nimi sinhulyarnye intehralnye uravneniia [Thermal processes in electrical contacts and associated singular integral equations]. *Candidate's thesis.* Almaty [in Russian].

9 Miller, W.Jr. (1981). *Simmetriia i razdelenie peremennykh [Symmetry and Separation of variables].* (G.P. Babenko, Trans.) Moscow: Mir [in Russian].

10 Tikhonov, A.N., & Samarskii, A.A. (1977). *Uravneniia matematicheckoi fiziki [Equations of Mathematical Physics].* Moscow: Nauka [in Russian].

11 Szego, G. (1962). *Ortohonalnye mnohochleny [Orthogonal polynomials].* (V.S. Videnski, Trans.). Moscow [in Russian].

12 Suetin, P.K. (1979). *Klassicheskie ortohonalnye mnohochleny [Classical orthogonal polynomials].* Moscow: Nauka [in Russian].

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On the integral equation of an adjoint boundary value problem of heat conduction

An integral equation is considered, to which a nonhomogeneous first boundary value problem with an adjoint heat conduction operator is reduced. The problem is set in an infinite plane angle, that is, a boundary of the domain moves with a constant velocity, and the domain degenerates to a point at the initial moment of time. The incompressibility of the integral operator for the equation under study is shown. Using the relations for an independent variable, the equation under study is equivalently reduced to a certain simplified equation. With the help of replacements for independent variables, the equation is reduced to an integral equation with a difference kernel. By applying the Laplace transform, the obtained equation is reduced to an ordinary first-order differential equation (linear). Its solution is found. By using the inverse Laplace transform, a solution of the nonhomogeneous integral equation under study is obtained in the form of a convergent series in some domain.

Keywords: heat conduction, nonhomogeneous singular integral equation, adjoint boundary value problem, Laplace transform.

Introduction

In the study of some nonlocal internal-boundary problems for a parabolic equation, spectrally loaded parabolic equations, problems with a moving boundary and inverse problems for parabolic equations, etc. there is a need to study singular integral equations of the form:

$$\psi(t) - \frac{1}{2\,a\sqrt{\pi}} \int_t^{\infty} \left[\frac{\tau + t}{(\tau - t)^{\frac{3}{2}}} \; \exp\left\{ -\frac{(\tau + t)^2}{4a^2(\tau - t)} \right\} + \right.$$

$$\left. + \frac{1}{(\tau - t)^{\frac{1}{2}}} \; \exp\left\{ -\frac{\tau - t}{4a^2} \right\} \right] \psi(\tau)\, d\tau = f(t), \qquad (t > 0). \tag{1}$$

The boundary value problems in the case of temperature heating are reduced to such equations (the first boundary value problem):

$$\frac{\partial v}{\partial t} = -a^2 \frac{\partial^2 v}{\partial x^2},$$

with boundary conditions:

$$v(x,\ t)|_{x=0} = \upsilon^*(t), \quad v(x,\ t)|_{x=t} = \omega^*(t), \quad v(x,\ t)|_{t=\infty} = 0.$$

1. Incompressibility of an integral operator and reducing the integral equation to an equation with a difference kernel

For the kernel of equation (1):

$$K(\tau, t) = \frac{1}{2\,a\sqrt{\pi}} \left[\frac{\tau + t}{(\tau - t)^{\frac{3}{2}}} \; \exp\left\{ -\frac{(\tau + t)^2}{4a^2(\tau - t)} \right\} + \frac{1}{(\tau - t)^{\frac{1}{2}}} \; \exp\left\{ -\frac{\tau - t}{4a^2} \right\} \right], \tag{2}$$

we have:

$$\lim_{t\to\infty}\int_t^{\infty} K(\tau,t)\,d\tau = \lim_{t\to\infty}\left(2e^{-\frac{2t}{a^2}}+1\right) = 1_{+0}.$$

Hence, the characteristic part of equation (1) is the second term of the kernel (2).

Using relations:

$$\tau+t = 2\tau-(\tau-t), \qquad \frac{(\tau+t)^2}{4a^2\,(\tau-t)} = \frac{\tau\, t}{a^2\,(\tau-t)} + \frac{\tau-t}{4a^2},$$

equation (1) will be rewritten as:

$$\psi(t) - \int_t^{\infty} \frac{1}{2a\sqrt{\pi}}\left\{\frac{2\tau}{(\tau-t)^{3/2}}\exp\left\{-\frac{\tau t}{a^2(\tau-t)}\right\}+\right.$$

$$\left.+\frac{1}{\sqrt{\tau-t}}\left(1-\exp\left\{-\frac{\tau t}{a^2(\tau-t)}\right\}\right)\right\}\cdot\exp\left\{-\frac{\tau-t}{4a^2}\right\}\,\psi(\tau)\,d\tau = f(t).$$

It is enough to find a solution to the «simplified» equation:

$$\psi(t) - \int_t^{\infty} k^*(t,\tau)\,\psi(\tau)\,d\tau \;= g(t), \tag{3}$$

where

$$k^*(t,\tau) = \frac{1}{2a\sqrt{\pi}}\left\{\frac{2\tau}{(\tau-t)^{3/2}}\exp\left\{-\frac{\tau t}{a^2(\tau-t)}\right\} + \frac{1}{\sqrt{\tau-t}}\left(1-\exp\left\{-\frac{\tau t}{a^2(\tau-t)}\right\}\right)\right\},$$

$$g(t) = \exp\left\{-\frac{t}{4a^2}\right\}\cdot f(t).$$

We consider the integral equation (3):

$$\psi(t) - \frac{1}{2a\sqrt{\pi}}\int_t^{\infty}\left\{\frac{2\tau}{(\tau-t)^{3/2}}\exp\left\{-\frac{\tau t}{a^2(\tau-t)}\right\}+\right.$$

$$\left.+\frac{1}{\sqrt{\tau-t}}\left(1-\exp\left\{-\frac{\tau t}{a^2(\tau-t)}\right\}\right)\right\}\psi(\tau)\,d\tau = g(t). \tag{4}$$

Integral equation (4) is reduced to an equation with a difference kernel by means of replacements:

$$t = \frac{1}{t_1}, \qquad \tau = \frac{1}{\tau_1}$$

and notation:

$$y\,(t_1) = \frac{1}{t_1^{3/2}}\cdot\psi\left(\frac{1}{t_1}\right), \qquad g_1\,(t_1) = \frac{1}{t_1^{1/2}}\cdot g\left(\frac{1}{t_1}\right).$$

As a result, we obtain the equation:

$$t_1\cdot y_1(t_1) - \frac{1}{2a\sqrt{\pi}}\int_0^{t_1}\frac{1}{(t_1-\tau_1)^{1/2}}\left(1-\exp\left\{-\frac{1}{a^2(t_1-\tau_1)}\right\}\right)y(\tau_1)\,d\tau_1 -$$

$$-t_1\cdot\frac{1}{2a\sqrt{\pi}}\int_0^{t_1}\frac{2}{(t_1-\tau_1)^{3/2}}\exp\left\{-\frac{1}{a^2(t_1-\tau_1)}\right\}y(\tau_1)\,d\tau_1 = g_1\,(t_1) \tag{5}$$

2. Solution of a homogeneous equation with a difference kernel

Applying the Laplace transform to the equation (5) we obtain the operator equation:

$$-\overline{y}^{'}(p)-\frac{1}{2a\sqrt{p}}\left(1-\exp\left(-\frac{2\sqrt{p}}{a}\right)\right)\overline{y}(p)+\left\{\exp\left(-\frac{2\sqrt{p}}{a}\right)\overline{y}(p)\right\}^{'}_{p}=\bar{G}_1(p).$$

After simple transformations we finally get:

$$\overline{y}^{'}(p)+\frac{1}{2a\sqrt{p}}\frac{ch\frac{\sqrt{p}}{a}}{sh\,\frac{\sqrt{p}}{a}}\overline{y}(p)=-\frac{\bar{G}_1(p)}{1-\exp\left(-\frac{2\sqrt{p}}{a}\right)}. \tag{6}$$

The solution of the differential equation (6) is the following function:

$$\overline{y}(p)=\frac{C}{sh\frac{\sqrt{p}}{a}}-\frac{1}{2\,sh\frac{\sqrt{p}}{a}}\int_p^{\infty}\bar{G}_1(q)\,\exp\left(\frac{\sqrt{q}}{a}\right)\,dq. \tag{7}$$

The solution of the homogeneous equation corresponding to (6) is the following function:

$$\overline{y}_{hom}(p)=\frac{C}{sh\frac{\sqrt{p}}{a}}. \tag{8}$$

(1-st term on the right side of the expression (7)).

To (8) we apply the inverse Laplace transform:

$$y_{hom}(t_1)=-C\,\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\,\left(\frac{\nu}{2};a^2t_1\right)\right]_{\nu=0}, \tag{9}$$

where

$$\widehat{\theta}_0\,(\nu;t)=\frac{1}{\sqrt{\pi x}}\left\{\sum_{n=0}^{\infty}\exp\left(-\frac{1}{x}\left(\nu+n+\frac{1}{2}\right)^2\right)-\sum_{n=-1}^{-\infty}n\cdot\exp\left(-\frac{1}{x}\left(\nu+n+\frac{1}{2}\right)^2\right)\right\}$$

is the modified theta function, and

$$-\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\,\left(\frac{\nu}{2};x\right)\right]_{\nu=0}=$$

$$=\frac{1}{2\sqrt{\pi}x^{\frac{3}{2}}}\left\{\sum_{n=0}^{+\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4x}\right)-\sum_{n=-1}^{-\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4x}\right)\right\}=$$

$$=\frac{1}{\sqrt{\pi}x^{\frac{3}{2}}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4x}\right).$$

A particular solution of the differential equation (6) is the function:

$$\overline{y}_{part}(p)=-\frac{1}{2\,sh\frac{\sqrt{p}}{a}}\int_p^{\infty}\bar{G}_1(q)\,\exp\left(\frac{\sqrt{q}}{a}\right)\,dq, \tag{10}$$

where

$$\bar{G}_1(q)=\int_0^{\infty}e^{-qt}g_1\,(t)\,dt.$$

By virtue of replacements:

$$t=\frac{1}{t_1},\quad \tau=\frac{1}{\tau_1}$$

and designations:

$$y\,(t_1)=\frac{1}{t_1^{3/2}}\,\psi\left(\frac{1}{t_1}\right),$$

from (9), we obtain the solution of the homogeneous equation corresponding to the integral equation (4):

$$\psi_{hom}\left(t\right)=\frac{C}{a^{3}\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{(2n+1)^{2}}{4a^{2}}t\right).$$

Then, the solution of the homogeneous equation, corresponding to the original an integral equation (1), has the form:

$$\psi_{hom}\left(t\right)=\frac{C}{a^{3}\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{n^{2}+n}{a^{2}}t\right). \qquad (11)$$

The following theorem is proved:

Theorem 1. The integral equation

$$\psi(t)-\frac{1}{2\,a\sqrt{\pi}}\int_{t}^{\infty}\left[\frac{\tau+t}{(\tau-t)^{\frac{3}{2}}}\ \exp\left\{-\frac{(\tau+t)^{2}}{4a^{2}(\tau-t)}\right\}+\right.$$

$$\left.+\frac{1}{(\tau-t)^{\frac{1}{2}}}\ \exp\left\{-\frac{\tau-t}{4a^{2}}\right\}\right]\psi(\tau)\,d\tau=0, \qquad (t>0)$$

in the class of essentially bounded functions at $t\geq t_0>0$ has the solution

$$\psi\left(t\right)=\frac{C}{a^{3}\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{n^{2}+n}{a^{2}}t\right),$$

moreover, the norm of an integral operator acting in classes of continuous functions is equal to 3.

3. Solving the nonhomogeneous equation with a difference kernel

Next, we proceed to solving the corresponding nonhomogeneous equation.

As:

$$\bar{G}_1(q)=\int_0^{\infty}e^{-qt}g_1\left(t\right)\,dt,$$

then (10) can be rewritten as

$$\overline{y}_{part}(p)=-\frac{1}{2\,sh\frac{\sqrt{p}}{a}}\int_p^{\infty}\exp\left(\frac{\sqrt{q}}{a}\right)\,dq\int_0^{\infty}e^{-qt_2}g_1\left(t_2\right)\,dt_2=$$

$$=-\frac{1}{2\,sh\frac{\sqrt{p}}{a}}\int_0^{\infty}g_1\left(t_2\right)\,dt_2\int_p^{\infty}\exp\left(-qt_2+\frac{\sqrt{q}}{a}\right)\,dq. \qquad (12)$$

In (12) we calculate the inner integral:

$$\int_p^{\infty}\exp\left(-qt_2+\frac{\sqrt{q}}{a}\right)\,dq=\left\|\ \frac{\sqrt{q}}{a}=z;\quad q=a^2z^2;\quad dq=2a^2z\,dz\ \right\|=$$

$$=2a^2\int_{\frac{\sqrt{p}}{a}}^{\infty}z\cdot\exp\left(-t_2a^2z^2+z\right)\,dz=\left\|\ \xi=a\sqrt{t_2}z-\frac{1}{2a\sqrt{t_2}}\ \right\|=$$

$$=2a^2\left[\frac{1}{a^2t_2}\exp\left(\frac{1}{4a^2t_2}\right)\left\{\frac{1}{2a\sqrt{t_2}}\int_{\sqrt{t_2p}-\frac{1}{2a\sqrt{t_2}}}^{\infty}e^{-\xi^2}\,d\xi+\int_{\sqrt{t_2p}-\frac{1}{2a\sqrt{t_2}}}^{\infty}\xi\cdot e^{-\xi^2}\,d\xi\right\}\right]=$$

$$=\frac{\sqrt{\pi}}{2at^{\frac{3}{2}}}\exp\left(\frac{1}{4a^2t_2}\right)\cdot erfc\left(\sqrt{t_2}\sqrt{p}-\frac{1}{2a\sqrt{t_2}}\right)+\frac{1}{t_2}\exp\left(\frac{1}{4a^2t_2}\right)\cdot\exp\left\{-\left(\sqrt{t_2}\sqrt{p}-\frac{1}{2a\sqrt{t_2}}\right)^2\right\}.$$

We introduce the notation:

$$\widehat{A}\,(t_2;p)=\frac{\sqrt{\pi}}{2at_2^{\frac{3}{2}}}\exp\left(\frac{1}{4a^2t_2}\right)\cdot erfc\left(\sqrt{t_2}\sqrt{p}-\frac{1}{2a\sqrt{t_2}}\right)+$$

$$+\frac{1}{t_2}\exp\left(\frac{1}{4a^2t_2}\right)\cdot\exp\left\{-\left(\sqrt{t_2}\sqrt{p}-\frac{1}{2a\sqrt{t_2}}\right)^2\right\}. \tag{13}$$

Then, taking into account the notation (13), the function (12) takes the form:

$$\overline{y}_{part}(p)=-\frac{1}{2\,sh\frac{\sqrt{p}}{a}}\int_0^\infty\widehat{A}\,(t_2;p)\;g_1\,(t_2)\;dt_2=-\int_0^\infty\widehat{G}\,(t_2;p)\;g_1\,(t_2)\;dt_2=$$

$$=-\int_0^\infty\left(\widehat{G}_1\,(t_2;\sqrt{p})+\widehat{G}_1\,(t_2;\sqrt{p})\right)\,g_1\,(t_2)\;dt_2, \tag{14}$$

where

$$\widehat{G}\,(t_2;p)=\widehat{G}_1\,(t_2;\sqrt{p})+\widehat{G}_1\,(t_2;\sqrt{p})\,, \tag{15}$$

and

$$\widehat{G}_1\,(t_2;\sqrt{p})\;=\frac{\sqrt{\pi}}{4at_2^{\frac{3}{2}}}\,sh\frac{\sqrt{p}}{a}\exp\left(\frac{1}{4a^2t_2}\right)\cdot erfc\left(\sqrt{t_2}\sqrt{p}-\frac{1}{2a\sqrt{t_2}}\right);$$

$$\widehat{G}_2\,(t_2;\sqrt{p})\;=\frac{1}{2t_2\,sh\frac{\sqrt{p}}{a}}\exp\left(\frac{1}{4a^2t_2}\right)\cdot\exp\left\{-\left(\sqrt{t_2}\sqrt{p}-\frac{1}{2a\sqrt{t_2}}\right)^2\right\}.$$

We consider the latest equalities:

$$\widehat{G}_1\,(t_2;\sqrt{p})\;=\frac{\sqrt{\pi}}{2at_2^{\frac{3}{2}}}\left(e^{\frac{\sqrt{p}}{a}}-e^{-\frac{\sqrt{p}}{a}}\right)\exp\left(\frac{1}{4a^2t_2}\right)\cdot erfc\left(\sqrt{t_2}\sqrt{p}-\frac{1}{2a\sqrt{t_2}}\right)=$$

$$=\frac{\sqrt{\pi}}{2at_2^{\frac{3}{2}}}\cdot e^{\frac{1}{4a^2t}}\cdot e^{-\frac{\sqrt{p}}{a}+t_2p}\cdot erfc\left(\sqrt{t_2}\cdot\sqrt{p}-\frac{1}{2a\sqrt{t_2}}\right)\cdot e^{-t_2p}\cdot\frac{1}{1-e^{-\frac{2\sqrt{p}}{a}}}= \tag{16}$$

$$=\frac{1}{2}\,\widehat{G}_1^{(1)}\,(t_2,\sqrt{p})\cdot e^{-t_2p}\cdot\frac{1}{1-e^{-\frac{2\sqrt{p}}{a}}}.$$

If in expression

$$\widehat{G}_1^{(1)}\,(t_2,\sqrt{p})=\frac{\sqrt{\pi}}{at_2^{\frac{3}{2}}}\cdot e^{\frac{1}{4a^2t}}\cdot e^{-\frac{\sqrt{p}}{a}+t_2p}\cdot erfc\left(\sqrt{t_2\cdot p}-\frac{1}{2a\sqrt{t_2}}\right)$$

from (16) to replace $\sqrt{p}$ with p, then from it is known that

$$\widehat{G}_1^{(1)}\,(t_2,p)_{\bullet}=^{\bullet}\frac{1}{at_2^2}\exp\Big\{-\frac{t_1^2}{4t_2}+\frac{t_1}{2at_2}\Big\}. \tag{17}$$

To find $\hat{G}_1\left(t_2,\sqrt{p}\right)$, we use the relation considering that $\hat{F}\,(p)_{\bullet}=^{\bullet}f\,(\tau)$:

$$\widehat{F}\,(\sqrt{p})_{\bullet}=^{\bullet}\frac{1}{2\sqrt{\pi}t_1^{\frac{3}{2}}}\int_0^\infty\tau\,\exp\Big\{-\frac{\tau^2}{4t_1}\Big\}\,f\,(\tau)\,d\tau. \tag{18}$$

Then from (17) we get:

$$\widehat{G}_1^{(1)}\,(t_2,\sqrt{p})_{\bullet}=^{\bullet}G_1^{(1)}\,(t_2,t_1)\,. \tag{19}$$

Taking into account (18), the function-original (19) can be rewritten as:

$$G_1^{(1)}\,(t_2,t_1)=\frac{1}{2\sqrt{\pi}\,t_1^{\frac{3}{2}}}\int_0^\infty\tau\,\exp\Big\{-\frac{\tau^2}{4t_1}\Big\}\,\frac{1}{at_2^2}\,\exp\Big\{-\frac{\tau^2}{4t_2}+\frac{\tau}{2at_2}\Big\}d\tau=$$

$$= \frac{1}{2a\sqrt{\pi}}\,\frac{1}{t_1^{\frac{3}{2}}}\cdot\frac{1}{t_2^2}\int_0^\infty \tau\ \exp\left\{-\tau^2\frac{t_1+t_2}{4t_1t_2}+\tau\,\frac{1}{2at_2}\right\}d\tau = \frac{1}{2a\sqrt{\pi}t_2^2}\,\frac{1}{t_1^{\frac{3}{2}}}\times$$

$$\times\frac{2t_1t_2}{(t_1+t_2)}\ \exp\left\{\frac{t_1t_2}{4a^2t_2^2 2\,(t_1+t_2)}\right\}D_{-2}\left(-\frac{\sqrt{2t_1t_2}}{2at_2\sqrt{t_1+t_2}}\right) = \tag{20}$$

$$= \frac{1}{a\sqrt{\pi}\,t_2\,\sqrt{t_1}\,(t_1+t_2)}+\frac{1}{2a^2\,t_2^{\frac{3}{2}}\,(t_1+t_2)^{\frac{3}{2}}}\ \exp\left\{\frac{t_1}{8a^2t_2\,(t_1+t_2)}\right\}erfc\left(-\frac{\sqrt{t_1}}{2a\sqrt{t_2}\sqrt{t_1+t_2}}\right).$$

Let's go back to the relationship (16):

$$\widehat{G}_1\,(t_2,\sqrt{p}) = \frac{1}{2}\,\widehat{G}_1^{(1)}\,(t_2,\sqrt{p})\cdot e^{-t_2\cdot p}\,\frac{1}{1-e^{-\frac{2\sqrt{p}}{a}}}. \tag{21}$$

We use the following property of the Laplace transform (Time shifting):

$$e^{-\alpha p}\,F\,(p)_{\bullet} =^{\bullet} f\,(t-\alpha)\,,$$

here $F\,(p)_{\bullet} =^{\bullet} f\,(t)$.

From here

$$\hat{G}_1^{(1)}\,(t_2,p)\cdot e^{-t_2p}{}_{\bullet} =^{\bullet} G_1^{(1)}\,(t_2,t_1-t_2)\,. \tag{22}$$

We write the expression $G_1^{(1)}\,(t_2,t_1-t_2)$ from (22) explicitly using the formula (20):

$$G_1^{(1)}\,(t_2,t_1-t_2) = \frac{1}{a\sqrt{\pi}t_2\sqrt{t_1-t_2}\,t_1}+\frac{\exp\left(\frac{t_1-t_2}{8a^2t_2t_1}\right)}{2a^2t_1^{\frac{3}{2}}t_2^{\frac{3}{2}}}\,erfc\left(\frac{-\sqrt{t_1-t_2}}{2a\sqrt{t_2}\sqrt{t_1}}\right). \tag{23}$$

It should be noted that $G_1^{(1)}\,(t_2,t_1-t_2)\neq 0$ when $t_1>t_2$.

Next we will find the original of the last factor (image) in the ratio (21):

$$\frac{1}{1-e^{-\frac{2\sqrt{p}}{a}}} = 1+\sum_{n=1}^{\infty}\exp\left(-\frac{2n}{a}\sqrt{p}\right)_{\bullet} =^{\bullet} \delta\,(t_1)+\frac{1}{a\sqrt{\pi}t^{\frac{3}{2}}}\sum_{n=1}^{\infty}n\cdot\exp\left(-n^2\cdot\frac{1}{a^2t_1}\right). \tag{24}$$

Using the convolution theorem, relations (23) and (24) we obtain:

$$G_1\,(t_2,t_1) = \frac{1}{2}\int_0^{t_1-t_2}G_1^{(1)}\,(t_2,t_1-t_2-\tau)\cdot\left[\delta\,(\tau)+\frac{1}{a\sqrt{\pi}\tau^{\frac{3}{2}}}\sum_{n=1}^{\infty}n\cdot\exp\left(-n^2\cdot\frac{1}{a^2t_1}\right)\right]d\tau =$$

$$= \frac{1}{2}\int_{t_2}^{t_1}G_1^{(1)}\,(t_2,\tau-t_2)\left[\delta\,(t_1-\tau)+\frac{1}{a\sqrt{\pi}}\sum_{n=1}^{\infty}n\cdot\exp\left(-n^2\frac{1}{a^2\,(t_1-\tau)}\right)\right]d\tau =$$

$$= \frac{1}{2}G_1^{(1)}\,(t_2,t_1-t_2)+\frac{1}{2a\sqrt{\pi}}\sum_{n=1}^{\infty}\int_{t_2}^{t_1}n\cdot\exp\left(-n^2\frac{1}{a^2\,(t_1-\tau)}\right)\cdot G_1^{(1)}\,(t_2,\tau-t_2)\ d\tau. \tag{25}$$

Then the original of the obtained solution (14) has the form:

$$y_{part}\,(t_1) = -\int_0^{t_1}G\,(t_2,t_1)\,g_1\,(t_2)\,dt_2 = -\left(\int_0^{t_1}G_1\,(t_2,t_1)\,g_1\,(t_2)\,dt_2+\int_0^{t_1}G_2\,(t_2,t_1)\,g_1\,(t_2)\,dt_2\right), \tag{26}$$

where

$$G\,(t_2,t_1) = G_1\,(t_2,t_1)+G_2\,(t_2,t_1)$$

is the original of the function (15).

Now we find $G_2\,(t_2,t_1)$.

$$\widehat{G}_2\,(t_2;\sqrt{p})\ = \frac{1}{2t_2\,sh\frac{\sqrt{p}}{0}}\exp\left(\frac{1}{4a^2t_2}\right)\cdot\exp\left\{-\left(\sqrt{t_2}\sqrt{p}-\frac{1}{2a\sqrt{t_2}}\right)^2\right\} =$$

$$= \frac{1}{e^{\frac{\sqrt{p}}{a}} - e^{-\frac{\sqrt{p}}{a}}} \cdot \frac{1}{t_2} \cdot \exp\left(\frac{1}{4a^2 t_2} - t_2 p + \frac{\sqrt{p}}{a} - \frac{1}{4a^2 t_2}\right) = \frac{1}{t_2} e^{-t_2 p} \cdot \frac{1}{1 - e^{-\frac{2\sqrt{p}}{a}}}.$$

Then we have:

$$\widehat{G}_2\left(t_2, p\right) = \frac{1}{t_2} \sum_{n=0}^{\infty} e^{-t_2 p} \cdot e^{-\frac{2n\sqrt{p}}{a}} = \frac{1}{t_2} e^{-t_2 p} \left\{1 + \sum_{n=1}^{\infty} e^{-\frac{2n\sqrt{p}}{a}}\right\} \bullet =^{\bullet}$$

$$\bullet =^{\bullet} \frac{1}{t_2} \left\{\delta\left(t_1 - t_2\right) + \sum_{n=1}^{\infty} n \cdot \frac{e^{-\frac{n^2}{a^2 (t_1 - t_2)}}}{a\sqrt{\pi}\left(t_1 - t_2\right)^{\frac{3}{2}}}\right\} = G_2\left(t_2, t_1\right).$$

and $\frac{1}{t_2}\left\{\delta\left(t_1 - t_2\right) + \sum_{n=1}^{\infty} n \cdot \frac{e^{-\frac{n^2}{a^2 (t_1 - t_2)}}}{a\sqrt{\pi}(t_1 - t_2)^{\frac{3}{2}}}\right\} = 0$; at $t_2 > t_1$.

Then the second term in the solution (26) takes the form:

$$y_{part}^{(2)}\left(t_1\right) = -\int_0^{t_1} \left\{\delta\left(t_1 - t_2\right) + \sum_{n=1}^{\infty} n \cdot \frac{e^{-\frac{n^2}{a^2 (t_1 - t_2)}}}{a\sqrt{\pi}\left(t_1 - t_2\right)^{\frac{3}{2}}}\right\} \frac{1}{t_2} g_1\left(t_2\right) dt_2 =$$

$$= -\frac{1}{t_1} g_1\left(t_1\right) - \frac{1}{a\sqrt{\pi}} \sum_{n=1}^{\infty} \int_0^{t_1} \frac{n}{\left(t_1 - t_2\right)^{\frac{3}{2}}} \cdot e^{-\frac{n^2}{a^2 (t_1 - t_2)}} \cdot \frac{1}{t_2} \cdot g_1\left(t_2\right) dt_2.$$

Finally taking into account (25) and (26) we have:

$$y_{part}\left(t_1\right) = -\frac{1}{t_1} g_1\left(t_1\right) - \frac{1}{a\sqrt{\pi}} \sum_{n=1}^{\infty} \int_0^{t_1} \frac{n}{\left(t_1 - t_2\right)^{\frac{3}{2}} t_2} \exp\left(-\frac{n^2}{a^2\left(t_1 - t_2\right)}\right) g_1\left(t_2\right) dt_2 -$$

$$-\frac{1}{2} \int_0^{t_1} G_1\left(t_2, t_1\right) g_1\left(t_2\right) dt_2. \tag{27}$$

Then the solution of equation (5) has the form:

$$y(t_1) = y_{hom}(t_1) + y_{part}(t_1), \tag{28}$$

where $y_{part}(t_1)$ is determined by formula (27) and the solution of the corresponding homogeneous equation was determined above:

$$y_{hom}(t_1) = -C \left[\frac{\partial}{\partial \nu} \widehat{\theta}_0 \left(\frac{\nu}{2}; a^2 t_1\right)\right]_{\nu=0}, \tag{29}$$

and

$$-\left[\frac{\partial}{\partial \nu} \widehat{\theta}_0 \left(\frac{\nu}{2}; x\right)\right]_{\nu=0} = \frac{1}{\sqrt{\pi} x^{\frac{3}{2}}} \sum_{n=0}^{\infty} (2n+1) \exp\left(-\frac{(2n+1)^2}{4x}\right).$$

Let's go back to the old variables.

Earlier, the replacement $t = \dfrac{1}{t_1}$ and the following designations were introduced:

$$y\left(t_1\right) = \frac{1}{t_1^{\frac{3}{2}}} \psi\left(\frac{1}{t_1}\right), \qquad g_1\left(t_1\right) = \frac{1}{\sqrt{t_1}} g\left(\frac{1}{t_1}\right).$$

Therefore, from (23) we have (here: $t_2 = \dfrac{1}{\tau}, \quad t_1 = \dfrac{1}{t}, \quad \tau = \dfrac{1}{\tau_1}$):

$$G_1^{(1)}\left(t_2, t_1 - t_2\right) = G_1^{(1)}\left(\frac{1}{\tau}, \frac{1}{t} - \frac{1}{\tau}\right) = \frac{t^{\frac{3}{2}} \tau^{\frac{3}{2}}}{a\sqrt{\pi}\sqrt{\tau - t}} + \frac{t^{\frac{3}{2}} \tau^{\frac{3}{2}} \exp\left(\frac{\tau - t}{8a^2}\right)}{2a^2} erfc\left(\frac{-\sqrt{\tau - t}}{2a}\right).$$

From the last formula for (25) we get:

$$G_1(t_2,t_1)=G_1\left(\frac{1}{\tau},\frac{1}{t}\right)=\frac{t^{\frac{3}{2}}\tau^{\frac{3}{2}}}{2a\sqrt{\pi}\sqrt{\tau-t}}+\frac{t^{\frac{3}{2}}\tau^{\frac{3}{2}}\exp\left(\frac{\tau-t}{8a^2}\right)}{4a^2}erfc\left(\frac{-\sqrt{\tau-t}}{2a}\right)+$$

$$+\frac{1}{2a^2\sqrt{\pi}}\sum_{n=1}^{\infty}\int_{\tau}^{t}n\cdot\exp\left(-n^2\frac{\tau_1 t}{a^2(\tau_1-t)}\right)\times$$

$$\times\left(\frac{\tau_1^{\frac{3}{2}}\tau^{\frac{3}{2}}}{\sqrt{\pi}\sqrt{\tau-\tau_1}}+\frac{\tau_1^{\frac{3}{2}}\tau^{\frac{3}{2}}\exp\left(\frac{\tau-\tau_1}{8a^2}\right)}{2a}erfc\left(\frac{-\sqrt{\tau-\tau_1}}{2a}\right)\right)\frac{d\tau_1}{\tau_1^2}.$$

Then the solution (27) can be rewritten in the form:

$$y_{part}(t_1)=-\frac{1}{t_1}g_1(t_1)-\frac{1}{a\sqrt{\pi}}\sum_{n=1}^{\infty}\int_0^{t_1}\frac{n}{(t_1-t_2)^{\frac{3}{2}}t_2}\exp\left(-\frac{n^2}{a^2(t_1-t_2)}\right)g_1(t_2)\,dt_2-$$

$$-\frac{1}{2}\int_0^{t_1}G_1(t_2,t_1)\cdot g_1(t_2)\,dt_2$$

$$t^{\frac{3}{2}}\psi_{part}(t)=-t^{\frac{3}{2}}g(t)-\frac{1}{a\sqrt{\pi}}\sum_{n=1}^{\infty}\int_t^{\infty}\frac{\tau\cdot n}{\left(\frac{1}{t}-\frac{1}{\tau}\right)^{\frac{3}{2}}}\exp\left(-\frac{n^2}{a^2\left(\frac{1}{t}-\frac{1}{\tau}\right)}\right)\sqrt{\tau}g(\tau)\frac{d\tau}{\tau^2}-$$

$$-\frac{1}{2}\int_t^{\infty}\frac{t^{\frac{3}{2}}\tau^{\frac{3}{2}}}{2a\sqrt{\pi}\sqrt{\tau-t}}\cdot\sqrt{\tau}g(\tau)\frac{d\tau}{\tau^2}-\frac{1}{2}\int_t^{\infty}\frac{t^{\frac{3}{2}}\tau^{\frac{3}{2}}\exp\left(\frac{\tau-t}{8a^2}\right)}{4a^2}erfc\left(\frac{-\sqrt{\tau-t}}{2a}\right)\cdot\sqrt{\tau}g(\tau)\frac{d\tau}{\tau^2}-$$

$$-\frac{1}{2}\int_t^{\infty}\frac{1}{2a^2\sqrt{\pi}}\sum_{n=1}^{\infty}\int_{\tau}^{t}n\exp\left(-n^2\frac{\tau_1 t}{a^2(\tau_1-t)}\right)\left(\frac{\tau_1^{\frac{3}{2}}\tau^{\frac{3}{2}}}{\sqrt{\pi}\sqrt{\tau-\tau_1}}+\right.$$

$$\left.+\frac{\tau_1^{\frac{3}{2}}\tau^{\frac{3}{2}}\exp\left(\frac{\tau-\tau_1}{8a^2}\right)}{2a}erfc\left(\frac{-\sqrt{\tau-\tau_1}}{2a}\right)\right)\frac{d\tau_1}{\tau_1^2}\sqrt{\tau}g(\tau)\frac{d\tau}{\tau^2}.$$

After simplifications we get:

$$\psi_{part}(t)=-g(t)-\frac{1}{a\sqrt{\pi}}\sum_{n=1}^{\infty}\int_t^{\infty}\frac{\tau\,n}{(\tau-t)^{\frac{3}{2}}}\cdot\exp\left(-\frac{n^2 t\,\tau}{a^2(\tau-t)}\right)g(\tau)\,d\tau-$$

$$-\frac{1}{4a\sqrt{\pi}}\int_t^{\infty}\frac{1}{\sqrt{\tau-t}}g(\tau)\,d\tau-\frac{1}{8a^2}\int_t^{\infty}\exp\left(\frac{\tau-t}{8a^2}\right)erfc\left(\frac{-\sqrt{\tau-t}}{2a}\right)\cdot g(\tau)\,d\tau-$$

$$-\frac{1}{4a^2\sqrt{\pi}t^{\frac{3}{2}}}\int_t^{\infty}\sum_{n=1}^{\infty}\int_{\tau}^{t}\frac{n}{\sqrt{\tau_1}}\exp\left(-n^2\frac{\tau_1 t}{a^2(\tau_1-t)}\right)\left(\frac{1}{\sqrt{\pi}\sqrt{\tau-\tau_1}}+\right.$$

$$\left.+\frac{\exp\left(\frac{\tau-\tau_1}{8a^2}\right)}{2a\sqrt{\tau_1}}erfc\left(\frac{-\sqrt{\tau-\tau_1}}{2a}\right)\right)d\tau_1\,g(\tau)\,d\tau$$

or

$$\psi_{part}(t)=-g(t)-\frac{1}{a\sqrt{\pi}}\sum_{n=1}^{\infty}\int_t^{\infty}\frac{\tau\cdot n}{(\tau-t)^{\frac{3}{2}}}\exp\left(-\frac{n^2 t\,\tau}{a^2(\tau-t)}\right)g(\tau)\,d\tau-$$

$$-\frac{1}{4a\sqrt{\pi}}\int_t^{\infty}\frac{1}{\sqrt{\tau-t}}g(\tau)\,d\tau-\frac{1}{8a^2}\int_t^{\infty}\exp\left(\frac{\tau-t}{8a^2}\right)erfc\left(\frac{-\sqrt{\tau-t}}{2a}\right)\cdot g(\tau)\,d\tau-$$

$$-\frac{1}{4a^2\sqrt{\pi}t^{\frac{3}{2}}}\int_t^\infty g(\tau)\int_\tau^t\left(\frac{1}{\sqrt{\pi}\sqrt{\tau_1}\sqrt{\tau-\tau_1}}+\frac{1}{2a\tau_1}\exp\left(\frac{\tau-\tau_1}{8a^2}\right)erfc\left(\frac{-\sqrt{\tau-\tau_1}}{2a}\right)\right)\times$$

$$\times\sum_{n=1}^{\infty}n\cdot\exp\left(-n^2\frac{\tau_1 t}{a^2(\tau_1-t)}\right)d\tau_1\cdot d\tau. \qquad (30)$$

Then the solution of the integral equation (3) taking into account the obtained expressions (29) or (11) (see Theorem 1) and (30) has the explicit form:

$$\psi(t)=\frac{C}{a^3\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{n^2+n}{a^2}t\right)-g(t)-$$

$$-\frac{1}{a\sqrt{\pi}}\sum_{n=1}^{\infty}\int_t^\infty\frac{\tau\, n}{(\tau-t)^{\frac{3}{2}}}\exp\left(-\frac{n^2 t\,\tau}{a^2(\tau-t)}\right)\cdot g(\tau)\,d\tau-$$

$$-\frac{1}{4a\sqrt{\pi}}\int_t^\infty\frac{1}{\sqrt{\tau-t}}g(\tau)\,d\tau-\frac{1}{8a^2}\int_t^\infty\exp\left(\frac{\tau-t}{8a^2}\right)erfc\left(\frac{-\sqrt{\tau-t}}{2a}\right)\cdot g(\tau)\,d\tau-$$

$$-\frac{1}{4a^2\sqrt{\pi}t^{\frac{3}{2}}}\int_t^\infty g(\tau)\int_\tau^t\left(\frac{1}{\sqrt{\pi}\sqrt{\tau_1}\sqrt{\tau-\tau_1}}+\frac{1}{2a\tau_1}\exp\left(\frac{\tau-\tau_1}{8a^2}\right)erfc\left(\frac{-\sqrt{\tau-\tau_1}}{2a}\right)\right)\times$$

$$\times\sum_{n=1}^{\infty}n\,\exp\left(-n^2\frac{\tau_1 t}{a^2(\tau_1-t)}\right)d\tau_1\,d\tau. \qquad (31)$$

4. Main results

Theorem 2. The solution of the integral equation (1) with the singular kernel (2) has an explicit form defined by the formula (31).

Remark. Singular homogeneous integral equations were considered in works [1–4]. Their kernels were also incompressible, but kernels had an another form. In this connection, the weight classes of the solution existence differ from the class of the solution existence for the equation considered in this work. We also note that boundary value problems for a spectrally loaded parabolic equation reduce to this kind of singular integral equations, when the load line moves according to the law $x = t$ [5–10] and problems for essentially loaded equation of heat conduction [11–15].

In works [16, 17] it is shown that the homogeneous Volterra integral equation of the second kind, to which the homogeneous boundary value problem of heat conduction in the degenerating domain is reduced, has a nonzero solution.

In works [18, 19] boundary value problems for heat equation in angular domains with special boundary conditions are studied. The problems are reduced to singular integral equations of Volterra type of the second kind, similar to the equation (1).

References

1 Kosmakova, M.T., Akhmanova, D.M., Iskakov, S.A., Tuleutaeva, Zh.M., & Kasymova, L.Zh. (2019). Solving one pseudo-Volterra integral equation. *Bulletin of the Karaganda University. Mathematics series, 1 (93),* 72–77. DOI: 10.31489/2019M1/72-77

2 Jenaliyev, M., Amangaliyeva, M., Kosmakova, M., & Ramazanov, M. (2014). About Dirichlet boundary value problem for the heat equation in the infinite angular domain. *Boundary Value Problems, 213,* 1–21. DOI: 10.1186/s13661-014-0213-4

3 Amangaliyeva, M.M., Dzhenaliev, M.T., Kosmakova, M.T., & Ramazanov M.I. (2015). On one homogeneous problem for the heat equation in an infinite angular domain. *Siberian Mathematical Journal, Vol. 56, No. 6*, 982–995. DOI: 10.1134/S0037446615060038

4 Jenaliyev, M., Amangaliyeva, M., Kosmakova, M., & Ramazanov, M. (2015). On a Volterra equation of the second kind with «incompressible» kernel. *Advances in Difference Equations, 71,* 1–14. DOI: 10.1186/s13662-015-0418-6

5 Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). On a particular second kind Volterra integral equation with a spectral parameter. *Siberian Mathematical Journal, 52, 1,* 1–10. DOI: 10.1134/S0037446606010010

6 Amangaliyeva, M.M., Akhmanova, D.M., Dzhenaliev, M.T., & Ramazanov, M.I. (2011). Boundary value problems for a spectrally loaded heat operator with load line approaching the time axis at zero or infinity. *Differential Equations, 47, 2,* 231–243. DOI: 10.1134/S0012266111020091

7 Kosmakova, M.T. (2016). On an integral equation of the Dirichlet problem for the heat equation in the degenerating domain. *Bulletin of the Karaganda University. Mathematics series, 1 (81),* 62–67.

8 Dzhenaliev, M.T., & Ramazanov, M.I. (2006). On the boundary value problem for the spectrally loaded heat conduction operator. *Siberian Mathematical Journal, 47, 3,* 433–451. DOI: 10.1007/s11202-006-0056-z

9 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: I. *Differential Equations, 43, 4,* 513-524. DOI: 10.1134/S0012266107040106

10 Dzhenaliev, M.T., & Ramazanov, M.I. (2007). On a boundary value problem for a spectrally loaded heat operator: II. *Differential Equations, 43, 6,* 806-812. DOI: 10.1134/S0012266107060079

11 Ramazanov, M.I., Kosmakova, M.T., Romanovsky, V.G., Zhanbusinova, B.H., & Tuleutaeva, Z.M. (2018). Boundary value problems for essentially-loaded parabolic equation. *Bulletin of the Karaganda University. Mathematics series, 4 (92),* 79–86. DOI: 10.31489/2018M4/79-86

12 Muvasharkhan Jenaliyev, M., & Ramazanov, M. (2016). On a homogeneous parabolic problem in an infinite corner domain. *AIP Conference Proceedings, 1759,* 020085. DOI: 10.1063/1.4959699

13 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). Uniqueness and non-uniqueness of solutions of the boundary value problems of the heat equation. *AIP Conference Proceedings, 1676,* 020028. DOI: 10.1063/1.4930454

14 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2014). On the spectrum of Volterra integral equation with the «incompressible» kernel. *AIP Conference Proceedings, 1611,* 127–132. DOI: 10.1063/1.4893816

15 Kosmakova, M.T., Orumbayeva, N.T., Medeubaev, N.K., & Tuleutaeva, Zh.M. (2018). Problems of Heat Conduction with Different Boundary Conditions in Noncylindrical Domains. *AIP Conference Proceedings, 1997,* UNSP 020071-1. DOI: 10.1063/1.5049065

16 Akhmanova, D.M., Ramazanov, M.I., & Yergaliyev. M.G. (2018). On an integral equation of the problem of heat conduction with domain boundary moving by law of t = x(2). *Bulletin of the Karaganda University. Mathematics series, 1 (89),* 15–19. DOI: 10.31489/2018M1/15-19

17 Kosmakova, M.T., Ramazanov, M.I., Tokesheva, A.S., & Khairkulova, A.A. (2016). On the non-uniqueness of solution to the homogeneous boundary value problem for the heat conduction equation in an angular domain. *Bulletin of the Karaganda University. Mathematics series, 4 (84),* 80–87. DOI: 10.31489/2016M4 /80-87

18 Jenaliyev, M.T., Iskakov, S.A., & Ramazanov, M.I. (2017). On a parabolic problem in an infinite corner domain. *Bulletin of the Karaganda University. Mathematics series, 1 (85),* 28–35. DOI: 10.31489/2017 M1/28-35

19 Jenaliyev, M.T., Iskakov, S.A., Ramazanov, M.I., & Tuleutaeva, Z.M. (2018). On the solvability of the first boundary value problem for the loaded equation of heat conduction. *Bulletin of the Karaganda University. Series Mathematics, 1 (89),* 33–41.

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Жылуөткізгіштіктің түйіндес есебінің бір интегралдық теңдеуі жайлы

Жылуөткізгіштіктің түйіндес операторлы біртекті емес бірінші шеттік есебі келтірілетін интегралдық теңдеу қарастырылды. Есеп шексіз жазық бұрышта қойылған, яғни облыстың шекарасы тұрақты жылдамдықпен қозғалады және облыс уақыттың бастапқы мезгілінде нүктеге айналады. Зерттелудегі теңдеудің интегралдық операторының сығылмайтыны көрсетілген. Тәуелсіз айнымалы үшін қатынастарды қолданып, зерттеліп отырған теңдеу қандайда бір эквивалентті ықшам теңдеуге келтірілді. Тәуелсіз айнымалылар үшін ауыстырулар көмегімен теңдеу айырымдық ядросы бар интегралдық теңдеуге келтірілді. Лаплас түрлендіруін қолдану арқылы алынған теңдеу қарапайым бірінші ретті дифференциалдық теңдеуге (сызықтық) келтірілді. Оның шешуі табылды. Лапластың кері түрлендіруінің көмегімен зерттелетін біртекті емес интегралды теңдеудің қандай да бір облыста жинақты қатар түріндегі шешуі алынды.

Кілт сөздер: жылуөткізгіштік, біртекті емес сингулярлы интегралдық теңдеу, түйіндес шеттік есеп, Лаплас түрлендіруі.

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Об одном интегральном уравнении сопряженной задачи теплопроводности

Рассмотрено интегральное уравнение, к которому сводится неоднородная первая краевая задача с сопряженным оператором теплопроводности. Задача поставлена в бесконечном плоском угле, т.е. граница области движется с постоянной скоростью, и область вырождается в точку в начальный момент времени. Показана несжимаемость интегрального оператора исследуемого уравнения. Используя соотношения для независимой переменной, исследуемое уравнение эквивалентно сводится к некоторому упрощенному уравнению. С помощью замен для независимых переменных уравнение сводится к интегральному уравнению с разностным ядром. Применением преобразования Лапласа полученное уравнение сведено к обыкновенному дифференциальному уравнению первого порядка (линейному). Найдено его решение. С помощью обратного преобразования Лапласа получено решение исследуемого неоднородного интегрального уравнения в виде сходящегося ряда в некоторой области.

Ключевые слова: теплопроводность, неоднородное сингулярное интегральное уравнение, сопряженная граничная задача, преобразование Лапласа.

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On the C^*-algebra valued G-metric space related with fixed point theorems

There are many fixed point results in the various kinds of metric spaces such as b-metric spaces, quaternion metric spaces, G-metric spaces, uniform spaces, non-commutative Banach spaces etc. In this work, we consider the one of modern methods as C^*-algebra G-metric space with fixed point theory to solve problems above mentioned. We prove the fixed point theorems for a mapping under the contractive conditions in C^*-algebra G-metric space. Besides, we establish the not only existence but also uniqueness theorem of fixed point in the such space. Also, we provide several examples to put support behind our main result.

Keywords: G-metric space, fixed point theory, C^*-algebra, banach contraction principle.

Introduction

Chandok et all. [1] introduced the concept of C^*-algebra valued partial metric and had illustrated examples significant fixed point results with provided examples. Dixmier [2] prepared his book includes in general C^*-Algebra with outstanding structures and theorems such as von Neumann, Stone-Weierstrass, Repr esentation and Duality theorems etc.

Kalaivani and Kalpana [3] considered the notion of C^*-algebra valued b- metric spaces and obtained new contraction mapping gives an application to linear equation systems using such space. Kamran and his collabrates [4] generalized the new Banach contraction principle in C^*-algebra-valued b-metric and obtained some result for an integral equation as application in a C^*-algebra-valued b-metric space. Kang et all. [5] worked on the Complex valued G-metric spaces and also demonstrated and new contraction principle in such space. Zhenhua, Jiang and Sun [6] established the notion of $C*$-algebra-valued metric spaces and authenticated several fixed point with new idea in the set of a unital C^*-algebra's positive elements. Later, Omran and Salama [7] got some new resuts for common coupled fixed point theorems on C^*-algebra-valued metric spaces basic definitions and facts about structures of C^*-algebra and Fixed Point Theory.

G. Kalpana and C. Kalaivani [8] established coincidence fixed point theorems for two mappings in C^*-algebra-valued S-metric spaces which satisfy new contractive conditions. They demonstrated a common fixed point theorem for Banach contraction principle in this space. A. Zada et al. [9] considered graph theory and studied on common fixed points in the C^*-valued metric space endowed with the graph G under G-contractive condition. S. Radenovic and his collegues [10] used a method to reducing coupled fixed point results to the respective ones for mappings with one variable in the framework of b-metric spaces. They also demonstrated that each C^*-algebra-valued b-metric space is a cone b-metric space over normal cone with normal constant equal to 1. The purpose of the paper Malhotraa & Bansal [11] was to study on the the existence of the common coupled fixed point as well as uniqueness in complete b-metric spaces by considering diverse contractive conditions. Concept of generalized metric spaces (i.e. G-metric spaces) was introduced by Mustafa & Sims [12] in and valuable results for fixed point theory in G-metric spaces were obtained by these authors. In their papers, Ozer & Omran [13–15] have obtained useful and various types of fixed point theorems play a central role in the C^*-algebra valued metric spaces or C^*-algebra valued b-metric spaces. Shen, Jiang and Zhenhua [16] have proved new fixed-point theorems for self-mappings with contractive by introducing the notion of the complete C^*-algebra-valued G-metric space and also provided an application of the theory for a type of differential equations.

Similar work was done in [17] too.

Definition 1.1. Let X be a non empty set and $G : X \times X \times X \to \mathbb{R}^+$, be a function satisfying the following properties:

$$(G1) G(e, f, g) = 0 \qquad if \quad e = f = g,$$

$$(G_2)\, G(e, e, f) > 0 \qquad \text{for all } \ e, f \in X, \text{ with } e \neq f,$$

$$(G_3)\, G(e, e, f) \leq G(e, f, g) \qquad \text{for all } \ e, f, g \ \in X \text{ with } f \neq g,$$

$(G4)$ $G(e, f, g) = G(e, g, f) = G(f, e, g) = G(f, g, e) = G(g, e, f) = G(g, f, e)$ (symmetry in all variables)

$$(G_5)\, G(e, f, g) \leq G(e, k, k) + G(k, f, g) \qquad \text{for all } \ k, e, f, g \in X. \text{ (rectangle inequality)}$$

Then the function G is called as a Generalized Metric, more specifically a G-metric on X and the pair (X, G) is called a G-metric space.
Lemma 1.2 . Let $\mathcal{A}$ be a C^*- algebra and $a \in \mathcal{A}^+$. Then,
1) There is a unique element $b \in \mathcal{A}^+$ such that $b^2 = a$.
2) The set $\mathcal{A}^+ = \ a^*a \ \mid \ a \ \in \mathcal{A}$ with a conjugate-linear involution $*\colon \mathcal{A} \to \ \mathcal{A}$
3) If $a, \ b \ \in \mathcal{A}$ and $0 \leq a \leq b$, then $\| \ a \ \| \leq \|b\|$.
4) If $a \in \mathcal{A}^+$ and $\| \ a \ \| < \ \frac{1}{2}$ then $(1 - a)$ invertible and $\left\| \ \frac{a}{1-a} \ \right\| < 1$.
On following references, the C^*- algebra valued metric space is defined as follows:
Definition 1.3. Let X be a nonempty set and the mapping d : X × X → $\mathcal{A}$ satisfies:
1) $0 \leq d(a, b)$ for all $a, b \in X$ and $0 = d(a, b) \Leftrightarrow a = b$
2) $d(a, b) = d(b, a)$ for all $a, b \in X$
3) $d(a, b) \leq d(a, c) + d(c, b)$ for all $a, b, c \in X$
Then d is called a C^*-algebra valued metric on X and $(X, \mathcal{A}, d)$ is called C^*-algebra valued metric space.
Now, we introduce to C^*-algebra valued G- metric space as follows:
Definition 1.4. Let X be a nonempty set, and $G_{\mathcal{A}} : X \times X \times X \to \mathcal{A}$ be a mapping satisfying the following properties:

$$(G1) \ G_{\mathcal{A}}(p, q, r) = 0 \ if \ p = q = r,$$

$$(G_2)\, 0 < G_{\mathcal{A}}(p, p, q)\,, \qquad \forall \ p, q \in X, \quad \text{with} \qquad p \neq q,$$

$(G_3)\, G_{\mathcal{A}}(p, p, q) \leq G_{\mathcal{A}}(p, q, r)\,, \ \ \forall \ p, q, r \in X \quad$ with $\quad p \neq r, (G_4)\, G_{\mathcal{A}}(p, q, r) = G_{\mathcal{A}}(p, r, q) = G_{\mathcal{A}}(q, p, r) \ldots$ (symmetry in all variables)

$$(G_5)\, G_{\mathcal{A}}(p, q, r) \leq G_{\mathcal{A}}(p, k, k) + G_{\mathcal{A}}(k, q, r)\,, \ \forall \ k, p, q, r \in X. \ \ \text{(rectangle inequality)}$$

Then the function $G_{\mathcal{A}}$ is called a C^*-algebra valued G- metric and the triple $(X, \mathcal{A}, G_{\mathcal{A}})$ is called a C^*-algebra valued G- metric space.
Example 1.5. Let X=$\mathbb{R}$ and defined $G : X \times X \times X \to \mathcal{A}^+$ by

$$G(r, s, t) = \|r - s\|\, I + \ \|s - t\|\, I + \ \|t - r\|\, I,$$

for all $r, s, t \ \in X$and I is the unit element in $\mathcal{A}$. Then, $(X, \mathcal{A}, G)$ is a C^*-algebra valued G- metric space.
Definition 1.6. Let $(X, \mathcal{A}, G)$ be C*-algebra valued G- metric space and let $\{a_n\}$ be a sequence in X. We say that $\{a_n\}$ is convergent with respect to $\mathcal{A}$ and $a \in X$ if for any $e > 0$, there exist $\exists \ n, m \in \mathbb{N}$ such that $G(a, a_m, a_n) < e$. We can also use different presentations for that as follows:
$a_n \to a$ or $\lim_{n\to\infty} G(a, a_n, a_m \ ) = 0$ or $\lim_{n\to\infty} a_n \ = a$
Definition 1.7. Let $(X, \mathcal{A}, G)$ be a C^*-algebra valued G- metric space and let $\{c_n\}$ be a sequence in X. The sequence c_n is called as Cauchy sequence respect to $\mathcal{A}$ if, for $e > 0$ there exists a positive integer $n^* \in N$ such that $G(c_l, c_m, c_n) < e, \ \forall \ n, l, m \geq n^*$ or $G(c_l, c_m, c_n) \to 0$ as $l, m, n \to \infty$ or $\|G(c, c_m, c_n)\| \ \to 0$ with $l, m, n \to \infty$.

We say $(X, \mathcal{A}, G)$ is complete C^*-algebra valued G- metric space if every sequence with respect to $\mathcal{A}$ is convergent.

Main Theorem and Results

Proposition 2.1. Let $(X, \mathcal{A}, G)$ be a C^*-algebra valued G- metric space. Then for any $c \in X$, the followings are equivalent:

1. c_n is a Cauchy sequence convergent to $c \in X$.
2. $\|G(c_n, c_n, c)\| \to 0$ as $n \to \infty$.
3. $\|G(c_m, c, c)\| \to 0$ as $m \to \infty$.
4. $\|G(c_m, c_n, c)\| \to 0$ as $m, n \to \infty$.

The proof can be done from the Definition 1.3.

Proposition 2.2. Let $(X, \mathcal{A}, G)$ be C^*-algebra valued G-metric space. Then the function $G(x, y, z)$ continues in all triple of values.

Proof. Suppose that let $\{x_k\}$, $\{y_m\}$, $\{z_n\}$ be convergent to $x, y, z \in X$ respectively. By (G_5), we have

$$G(x, y, z) \le G(y, y_m, y_m) + G(y_m, x, z)$$

$$G(x, z, y_m) \le G(x, x_k, x_k) + G(x_k, y_m, z)$$

$G(x, z, y_m) \le G(x, x_k, x_k) + G(x_k, y_m, z)$ and $G(x, z_k, y_m) \le G(z, z_n, z_n) + G(z_n, z_m, x_k)$

So,

$$G(x, y, z) - G(x_k, y_m, z_n) \le G(y, y_m, y_m) + G(x, x_k, x_k) + G(z, z_n, z_n)$$

$$\Rightarrow \| G(x, y, z) - G(x_k, y_m, z_n)\| \le \| G(y, y_m, y_m)\| + \| G(x, x_k, x_k)\| + \| G(z, z_n, z_n)\|,$$

since $\{x_k\}$, $\{y_m\}$, $\{z_n\}$ are convergent, this yields that

$\| G(x, y, z) - G(x_k, y_m, z_n)\| \to 0$ for $k, m, n \to \infty$.

Now, we prove the contraction principle C^*-algebra valued G- metric space as follows:

Theorem 2.3. Let $(X, \mathcal{A}, G)$ be a complete C^*-algebra valued G- metric space. Let $T: X \to X$ be a contraction mappng on X such that

$$G(Tx, Ty, Tz) \le a^* G(x, y, z)\, a, \tag{1}$$

for all $x, y, z \in X$ where $a \in \mathcal{A}$ and $\| a\| < 1$. So, contraction mapping T has a unique fixed point.

Proof. Suppose that T satisfies (1). Let x_0 be an arbitrary point in X and define the sequence x_n as follows:

$$x_1 = Tx_0, x_2 = Tx_1, \ldots, x_m = T^n x_0.$$

Then by (1), we get

$$G(x_{n+1}, x_{n+1}, x_n) = G\ (Tx_n, Tx_n, Tx_{n-1}) \le a^* G(x_n, x_n, x_{n-1})\, a.$$

Again by (1), we get

$$\begin{aligned} G(x_{n+1}, x_{n+1}, x_n) &\le (a^*)^2 G(x_{n-1}, x_{n-1}, x_{n-2})\, a^2 \\ &\le \ldots \\ &\le \ldots \\ &\le (a^*)^n G(x_1, x_1, x_0)\, a^n . \end{aligned}$$

Using Lemma 1.2, we have

$$\| G(x_{n+1}, x_{n+1}, x_n) \| \le \| (a^*)^n G(x_1, x_1, x_0)\, a^n \|$$

$$\le \| (a^*)^n \| . \| G(x_1, x_1, x_0) \| . \|a^n\|,$$

since $\| (a^*)^n \| = \| a^*.a^* \ldots .a^* \| \le \| a^* \| . \| a^* \| . \ldots . \| a^* \|$, we get

$$\| a^* \|^n . \| a \|^n . \| G(x_1, x_1, x_0) \| = \| a \|^{2n} . \| G(x_1, x_1, x_0) \|$$

due to $\| a^* \| = \| a\|$.

Now, for $n > m$

$$G(x_n, x_n, x_m) \le G(x_{n-1}, x_{n-1}, x_{n-2}) + G(x_{n-2}, x_{n-2}, x_{n-3}) + G(x_{n-3}, x_{n-3}, x_{n-4})$$
$$\le G(x_{m+1}, x_{m+1}, x_m)$$

Therefore,

$$\| G(x_n, x_n, x_m) \| \le \left(\| a \|^{2n} + \| a \|^{2n-2} + \dots + \| a \|^{2m} \right) \| G(x_1, x_1, x_0) \| .$$

For $n \to \infty$,

$$\| G(x_n, x_n, x_m) \| \le \| a \|^{2m} . \left(1 + \| a \| + \| a \|^{2} + \dots \right) . \| G(x_1, x_1, x_0) \| .$$

$$\le \frac{\| a \|^{2n}}{1 - \| a \|} . \| G(x_1, x_1, x_0) \| \to 0$$

as $n, \ m \to \infty$. Thus, x_n is a Cauchy sequence in X with respect to $\mathcal{A}$ (by Proposition 2.1).
Next we prove that $x = Tx$. Assume that $Tx \neq x$, then by Proposition 2.1, we have

$$G\ (Tx_n, Tx, Tx) \le G\ (x_{n+1}, Tx, Tx) \le a^* G(x_n, x, x)\, a.$$

Using Lemma 1.2, we obtain

$$0 \le \left\| G\ (x_{n+1}, Tx, Tx) \right\| \le \| a^* \| . \| G(x, x, x) \| . \| a \|$$
$$\le \| a \|^{2} . \| G(x_n, x, x) \| .$$

Since x_n is a Cauchy sequence then $\| G(x_n, x, x) \| \to 0$ as $n \to \infty$. Therefore, $\left\| G\ (x_{m+1}, Tx, Tx) \right\| \to 0$, x_n convergences to x.

Thus $\left\| G\ (x_{n+1}, Tx, Tx) \right\| \to \| G\ (x, Tx, Tx) \|$, since x_n is a Cauchy sequence and $\| G\ x, Tx, Tx) \| \to 0$ by proposition 2.2.
We prove that $x = Tx$. To see that the fixed point is unique suppose that $x \neq y$ such that $Tx = x$, $Ty = y$. Then by (1) we have,

$$0 \le G(x, y, y) = G(Tx, Ty, Ty) \le a^* G(x, y, y)\, a.$$

Therefore,

$$0 \le a^* G(x, y, y)\, a \le \| a^* \| . \| G(x, y, y) \| . \| a \| .$$

Put $\| a \|^{2} = k \in [0, 1]$

$$0 \le \| G\ (x, y, y) \| \le k. \| G\ (x, y, y) \| .$$

This is contradiction and we get

$$\| G\ (x, y, y) \| = 0.$$

So, $G\ (x, y, y) = 0$ and this gives that $x = y$.
Example 2.4. Let $=\mathbb{R}$, $\mathcal{A}$=$\mathrm{M}_3(\mathbb{R})$ with $\|(p, q, r)\|$=max $|r|$, $|q|$, $|p|$
(or $\mathcal{A}$=$\mathbb{R}^3$ with $\|(p, q, r)\|$=$\left(|p|^2 + |q|^2 + |r|^2 \right)^{\frac{1}{2}}$) and $G(p, q, r) = dig\ (|p - q|\ ,\ a\,|q - r|\ ,\ |p - r|)$ where $p, q, r \in X$, $0 < a$, are constant. It is easy to check that $(X, \mathrm{M}_3(\mathbb{R}), G)$ is completed C*-algebra valued G- metric space.
Proof. 1) It is clear that $G(p, q, r) > 0$ holds.

$$2)\ \ if\ \ p = q = r \Rightarrow G(p, q, r) = dig \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

$$G(p, p, q) = dig \begin{pmatrix} |p - p| & 0 & 0 \\ 0 & a\,|p - q| & 0 \\ 0 & 0 & |p - q| \end{pmatrix} = 0_{\mathrm{M}_3(\mathbb{R})}$$

with $p \neq q$.

$$G(p,p,q) = dig\begin{pmatrix} |p-p| & 0 & 0 \\ 0 & a|p-q| & 0 \\ 0 & 0 & |p-q| \end{pmatrix} = dig\begin{pmatrix} 0 & 0 & 0 \\ 0 & a|p-q| & 0 \\ 0 & 0 & |p-q| \end{pmatrix}$$

$$= |p-q|\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \end{pmatrix} > 0, \ (\ p \neq q, \ 0 < a,)$$

$$3)G(p,q,r) = dig\begin{pmatrix} |p-q| & 0 & 0 \\ 0 & |q-r| & 0 \\ 0 & 0 & |p-r| \end{pmatrix}$$

$$G(p,q,r) = \begin{pmatrix} |p-q| & 0 \\ 0 & a|p-q| \end{pmatrix}, \ a = 1$$

$$G(p,q,r) > 0, p = q = r \Rightarrow G(p,q,r) = 0$$

and

$G(p,p,q) = \begin{pmatrix} |p-p| & 0 \\ 0 & a|p-q| \end{pmatrix} > 0$ if $u \neq v$.

$$G(p,p,q) = \begin{pmatrix} |p-p| & 0 \\ 0 & 1|p-q| \end{pmatrix} < \begin{pmatrix} |p-q| & 0 \\ 0 & 1|p-q| \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & |p-q| \end{pmatrix}$$

and since $G(p,q,r) = G(r,p,q) = \begin{pmatrix} |r-p| & 0 \\ 0 & |p-q| \end{pmatrix}$.

So,

$$\begin{pmatrix} 0 & 0 \\ 0 & |p-q| \end{pmatrix} < \begin{pmatrix} |r-p| & 0 \\ 0 & |p-q| \end{pmatrix}$$

4) $G(p,q,r) = G(P(p,q,r))$ for all $P(p,q,r)$ points.

$$5)G(p,q,r) = \begin{pmatrix} |r-p| & 0 \\ 0 & |p-q| \end{pmatrix} = \begin{pmatrix} |r-k+k-p| & 0 \\ 0 & |q-k+k-r| \end{pmatrix}$$

$$\leq \begin{pmatrix} |r-k|+|k-p| & 0 \\ 0 & |q-k|+|k-r| \end{pmatrix} = \begin{pmatrix} |k-p| & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} |r-k| & 0 \\ 0 & |q-k| \end{pmatrix}$$

So,

$$G(p,k,k) + G(r,k,q) = \ G(p,k,k) + G(k,q,r), \quad \forall \ k,p,q,r \in X.$$

Example 2.5.
$X = L^{\infty}(E)$ and $H = L^2(E)$ where E is Lebesgue space. by B($\mathcal{H}$) is the set of all bounded linear operators on $\mathcal{H}$. B($\mathcal{H}$) is C^* algebra with operator norm

$$\|\mathrm{T}\| = \ \sup \ a \in X, \ \| < Ta, \ Ta > \|.$$

Define $G(f,g,h) = \ p_{|f-g|} + p_{|g-h|}$, $f,g,h \ \in X$ where $p_h : \ \mathcal{H} \to \mathcal{H}$ defined by $p_h(f) = h \Diamond f$ composit of these operators where $h \ \in \mathcal{H}$ and $f \in \ H = L^2(E)$, p_h is the multiplacative operator.

Example 2.6. Let $X = [-1,1]$ and $\mathcal{A}$=$\mathbb{C}$ be complex.
$G(u,v,w) = \ \|u-v\|.k + \|v-w\|.k + \ \ \|w-u\|.k$ since $k > 0_{\mathcal{A}}$, defined $T: \ X \to X$ such that $Tx = \ x/2$.

$$G(Tu,Tv,Tw) = \ \|u/2 - v/2\|.I + \|v/2 - w/2\| I + \ \ \|w/2 - w/2\|.I$$

$a = 1/2$, $\|a\| = 1/2 < 1$ and $\imath$f we chose $k = I$ then $\imath$t is clear that $\|I/2\| = 1/2.I = \ \frac{1}{2}$.
Then, T has a unique fixed point.
Remark. In the above example (Example 2.1) $X = L^{\infty}(E)$, $H \ = \ L^2(E)$ and $\mathcal{A}$ =B($\mathcal{H}$). $(X, \mathcal{A}\ , G)$ defined above, let $G(f,g,h) = \ p_{|f-g|} + p_{|g-h|}$ and $T: \ L^{\infty}(E) \to (E)$ by

$$\|G(Ta,Tb,Tc)\ \| = \sup \mathrm{h} \in L^2(E), \ < p_{|\mathrm{Ta}-\mathrm{Tb}|}h, \ \mathrm{a} > + \ \sup \mathrm{h} \in L^2(E), \ < p_{|\mathrm{Tb}-\mathrm{Tc}|}h, \ \mathrm{a} >$$

inner product in Hilbert space. Set,

$$A = k.I, \|A\| < 1/2.$$

(Following Zhenhu et all [10]),
We get

$$\|G(Ta, Tb, Tc)\| \leq \|k\| (\|a-b\|_\infty + \|b-c\|_\infty) \leq \|k\| \|G(a,b,c)\|$$

and so, T has a unique fixed point.

Conclusion

There is a good deal of applications for fixed point theory in many different fields of mathematics and engineering which include applied mathematics, dynamical systems, operation systems, Partial differential equations (PDE), ıntegral equations, computer sciences and also other scientific fields.

In this paper, we introduced the C*-Algebra valued G- metric space and established several important interesting/significant theorems for fixed point theory.

Results obtained in this paper useful for non commutative spaces to measure and operation system depended on PDE 's to solve.

We hope that our results help the researchers to augment and also promote their studies on fixed point theory to carry out a general framework for the applications in life.

References

1 Chandok, S., Kumar, D., & Park, C. C*-algebra valued partial metric space and fixed point theorems. *ias.ac.in.* Retrieved from https://www.ias.ac.in/public/Volumes/pmsc/forthcoming/PMSC-D-18-00162.-pdf

2 Dixmier, J. (1977). C*-Algebras". North-Holland Publ. Co., Amsterdam, New York, Oxford.

3 Kalaivani, C. & Kalpana, G. (2018). A Fixed Point in C*-algebra-valued S_b-metric spaces. *Taga Journal, 14,* 1442–1448.

4 Kamran, T., Postolach, M., Ghiura, A., Batul, S., & Ali, R. (2016). The Banach contraction principle in C*-algebra valued b-metric spaces with application. *Fixed Point Theory and Applications, 2016:10.*

5 Kang, S.M., Singh, B., Gupta, V., & Kumar, S. (2013). Contraction Principle in Complex Valued G-Metric Spaces. *Int. Journal of Math. Analysis, 7(52),* 2549–2556.

6 Zhenhua, M., Jiang, L., & Sun, H. (2014). C*-algebra-valued metric spaces and related fixed point theorems. *Fixed Point Theory and Applications, 2014:206.*

7 Omran, S. & Moustafa M. Salama (2018). Common Coupled Fixed Point in C*-algebras Valued Metric Spaces. *International Journal of Applied Engineering Research ISSN 0973-4562, 13(8),* 5899–5903.

8 Kalpana, G., & Kalaivani, C. (2018). COMMON FIXED POINT THEOREMS IN C*-ALGEBRA-VALUED S-METRIC SPACE. *International Journal of Pure and Applied Mathematics, Vol. 119, No. 15,* 367–375.

9 Zada, A., Saifullah, S. & Ma, Z. (2016). COMMON FIXED POINT THEOREMS FOR G-CONTRACTION IN C*-ALGEBRA VALUED METRIC SPACES. *International Journal of Analysis and Applications, Vol. 11, No. 1,* 23–27.

10 Radenovic, S., Vetro, P., Nastasi, A., & Quan, L.T. (2017). Coupled Fixed Point Theorems in C*-Algebra-Valued b-Metric Spaces, *SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR, SER. A: APPL. MATH. INFORM. AND MECH., Vol. 9(1),* 81–90.

11 Malhotra, N., & Bansal, B. (2015). Some Coupled Common Fixed Point Theorems for Generalized Contraction in b-Metric Spaces. *Journal of Nonlinear Analysis and Applications, 8,* 8–16.

12 Mustafa, Z. & Sims, B. (2006). A New Approach to a Generalize Metric Spaces. *J. Nonlinear Convex Anal., 7,* 289–297.

13 Özer, Ö., & Omran, S. (2016). Common Fixed Point Theorems in C*- Algebra Valued b-Metric Spaces. *AIP Conference Proceedings 1773,* 050005. DOI: 10.1063/1.4964975.

14 Özer, Ö., & Omran, S. (2017). On The Generalized C*-Valued Metric Spaces Related With Banach Fixed Point Theory. *International Journal of Advanced and Applied Sciences, 4(2),* 35–37.

15 Özer, Ö., & Omran, S. (2019). A Result On the Coupled Fixed Point Theorems in C*-algebra Valued b-Metric Spaces. *Italian Journal of Pure and Applied Mathematics, No. 42,* 722–730.

16 Shen, C., Jiang, L., & Zhenhua, M. (2018). C*-Algebra-Valued G-Metric Spaces and Related Fixed-Point Theorems. *Journal of Function Spaces, Article ID 3257189,* 8.

17 Shehwar, D., & Kamran, T. (2015). C*-Valued G-contractions and fixed points. *Journal of Inequalities and Applications, 2015:304.* DOI 10.1186/s13660-015-0827-9.

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Қозғалмайтын нүкте жайлы теоремалармен байланысты C^*-алгебралық G-метрикалық кеңістік жайлы

b-Метрикалық кеңістік, кватерниондық метрикалық кеңістік, G- метрикалық, біркелкі, коммутативті емес банах және тағы басқа әртүрлі метрикалық кеңістіктерде нақтыланған нүктелі көптеген нәтижелер бар. Мақалада жоғарыда көрсетілген есептерді шешу үшін қозғалмайтын нүкте теориясымен G-метрикалық кеңістіктің C^*-алгебрасы сияқты қазіргі заманғы әдістердің бірі қарастырылды. G-кеңістігінің C^*-алгебрасында сығылмалы жағдайларда бейнелеу үшін қозғалмайтын нүкте жайлы теоремалар дәлелденді. Сонымен қатар тек қана бар болу теоремасын ғана емес, сондай-ақ осы кеңістіктегі қозғалмайтын нүктенің жалғыздығы жайлы теоремасын да нақтылады. Сонымен қатар негізгі нәтижені қолдаудың бірнеше мысалы келтірілді.

Кілт сөздер: G-метрикалық кеңістік, қозғалмайтын нүктелер теориясы, C^*-алгебра, Банахтың сығылмалы бейнелеулер қағидасы.

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О C^*-алгебраическом G-метрическом пространстве, связанном с теоремами о неподвижных точках

Существует множество результатов с фиксированной точкой в различных метрических пространствах, таких как b-метрические, кватернионные метрические, G-метрические, равномерные, некоммутативные банаховы пространства и т.д. В статье рассмотрен один из современных методов — C^*-алгебра G-метрического пространства с теорией неподвижной точки для решения указанных выше задач. Доказаны теоремы о неподвижной точке для отображения при сжимающих условиях в C^*-алгебре G-метрического пространства. Кроме того, установлена не только теорема существования, но и теорема единственности неподвижной точки в таком пространстве. Также приведены примеры поддержки нашего основного результата.

Ключевые слова: G-метрическое пространство, теория неподвижных точек, C^*-алгебра, принцип сжимающих отображений Банаха.

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Estimates of the norm of the convolution operator in anisotropic Besov spaces with the dominated mixed derivative

In this paper, we investigate the boundedness of the norm of the convolution operator in Sobolev spaces with the dominated mixed derivative and anisotropic Nikolsky-Besov spaces. For Sobolev spaces with the dominated mixed derivatives, an analogue of Young's inequality is obtained, namely, relations of the form

$$W_{\mathbf{p}}^{\boldsymbol{\gamma}} * W_{\mathbf{r}}^{\boldsymbol{\beta}} \hookrightarrow W_{\mathbf{q}}^{\boldsymbol{\alpha}} \tag{1}$$

are proved when the corresponding conditions on the parameters are satisfied. The main goal of the paper is to solve the following problems. Let f and g be functions from some classes of the Nikolsky-Besov space scale. We would like to find the Nikolsky-Besov space such that the convolution $f * g$ belongs to this space. Using relation (1) and the Nursultanov interpolation theorem for anisotropic spaces, an analogue of the O'Neil theorem was obtained for the Nikolsky-Besov space scale $B_{\mathbf{pq}}^{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha}$, $\mathbf{p}$, $\mathbf{q}$ are vector parameters. Relations of the form $B_{\mathbf{ps_1}}^{\boldsymbol{\gamma}} * B_{\mathbf{rs_2}}^{\boldsymbol{\beta}} \hookrightarrow B_{\mathbf{qs}}^{\boldsymbol{\alpha}}$ are obtained, with the corresponding ratios of vector parameters. The theorems obtained in this paper complement the results of Batyrov and Burenkov, where similar problems were considered in isotropic Nikolsky-Besov spaces, that is, in spaces where the parameters are scalars.

Keywords: convolution operator, anisotropic Sobolev and Besov spaces, interpolation.

Introduction

Let I be either a n-dimensional torus $\mathbb{T}^n = [0,1)^n$, or the Euclidean space $\mathbb{R}^n$. Let $f(x)$ and $g(x)$ be measurable functions on I with respect to the n-dimensional Lebesgue measure such that for almost all $x \in I$ there exists an integral

$$\int_I f(x-y)g(y)dy.$$

In this case, it is said that the convolution of these functions is defined

$$(f * g)(x) = \int_I f(x-y)g(y)dy. \tag{1.1}$$

The classical Young's inequality [1; 199] has the following form. Let

$$1 \le p, r, q \le \infty, \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}. \tag{1.2}$$

If $f \in L_p(I)$, $g \in L_r(I)$, then there exists almost everywhere on I the convolution $f * g$, belonging to the space $L_q(I)$ and the following inequality holds

$$\|f * g\|_{L_q(I)} \le \|f\|_{L_p(I)} \|g\|_{L_r(I)}. \tag{1.3}$$

We will write this statement as follows

$$L_p(I) * L_r(I) \hookrightarrow L_q(I).$$

This inequality plays an important role in harmonic analysis and in the theory of partial differential equations [1–3].

Note that if

$$1 < p, r, q < \infty, \quad \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}, \tag{1.4}$$

then for $g_0(x) = \dfrac{1}{|x|^{\frac{n}{r}}}$ the inequality holds

$$\|f * g_0\|_{L_q(I)} \leq C\|f\|_{L_p(I)}.$$

This inequality is called the Hardy-Littlewood-Sobolev inequality. It does not follow from Young's inequality, since $\|g_0\|_{L_r(I)} = \infty$. A generalization of inequality (1.3) obtained by O'Neil [4] (see also [5, 6]).

If (1.4) is true and $0 < s_1, s_2, s \leq \infty$, $\dfrac{1}{s} = \dfrac{1}{s_1} + \dfrac{1}{s_2}$, then

$$L_{ps_1} * L_{rs_2} \hookrightarrow L_{qs} \tag{1.5}$$

and in particular

$$L_p * L_{r\infty} \hookrightarrow L_q, \tag{1.6}$$

where L_{ps} is Lorentz space.

Note that in relation (1.5), condition (1.4) is essential. The limiting cases of the O'Neil inequality with condition (1.2) were considered in [7].

The O'Neil inequality for anisotropic Lorentz spaces was studied in [8–10]. In the case of $n \geq 2$ these results are extend the inequality (1.6). In the one-dimensional case, the O'Neil inequality was extended in [11, 12].

There are generalizations of the Young and O'Neil inequalities for various functional spaces: weighted L_p spaces, classical and Lorentz weighted spaces, Hardy spaces, Wiener spaces, Orlicz spaces; [5, 6, 8, 13–18], and references therein.

Convolution operators were studied in various spaces of smooth functions in [19–22].

V.I. Burenkov and B.E. Batyrov in [21] proved the following statement: let $-\infty < l_1, l_2, l_3 < \infty$, $0 < p_1, p_2, p_3 \leq \infty$, $0 < \theta_1, \theta_2, \theta_3 \leq \infty$. For any $f_1 \in B^{l_1}_{p_1\theta_1}(\mathbb{R}^n)$, $f_2 \in B^{l_2}_{p_2\theta_2}(\mathbb{R}^n)$ such that Ff_1 and Ff_2 are regular generalized functions and their (pointwise) product $Ff_1 \cdot Ff_2 \in S(\mathbb{R}^n)$, there exists a number $c_3 > 0$ such that

$$\|f_1 * f_2\|_{B^{l_3}_{p_3\theta_3}(\mathbb{R}^n)} \leq c_3\|f_1\|_{B^{l_1}_{p_1\theta_1}(\mathbb{R}^n)}\|f_2\|_{B^{l_2}_{p_2\theta_2}(\mathbb{R}^n)}, \tag{1.7}$$

holds if and only if the following conditions hold:

1) $p_3 \geq p_1$, $p_3 \geq p_2$;

2) $\dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{1}{p_3} - 1 \geq 0$;

and one of the conditions

3a) $l_3 < l_1 + l_2 - n\left(\dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{1}{p_3} - 1\right)$

or

3b) $l_3 = l_1 + l_2 - n\left(\dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{1}{p_3} - 1\right)$ и $\dfrac{1}{\theta_3} \leq \dfrac{1}{\theta_1} + \dfrac{1}{\theta_2}$,

where Ff is the Fourier transform of the function f:

$$(Ff)(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-ix\xi} f(\xi) d\xi.$$

For $p_2 = p_3$, $\theta_2 = \theta_3$, $0 < l_2 < l_3 < \infty$ inequality (1.7) and some of its generalizations follow from the results obtained in the works of K.K. Golovkin and V.A. Solonnikov [19, 20], and [23].

In this paper, we investigate the boundedness of the convolution operator in anisotropic Besov spaces with the dominated mixed derivative.

2. Anisotropic Besov spaces with dominated mixed derivative

Let $\boldsymbol{\alpha} \in \mathbb{R}^n$, $\mathbf{1} < \mathbf{p}= (p_1, \dots , p_n) <\boldsymbol{\infty}$, $\mathbf{0} < \mathbf{q}= (q_1, \dots , q_n) \leq\boldsymbol{\infty}$. Following [24–26], we define the space $B^{\boldsymbol{\alpha}}_{\mathbf{pq}}(\mathbb{T}^n)$ as the set of series $f = \sum\limits_{m\in\mathbb{Z}^n} a_m e^{2\pi i(m,x)}$ (generally speaking, divergent) for which

$$\|f\|_{B^{\boldsymbol{\alpha}}_{\mathbf{pq}}(\mathbb{T}_n)} = \left(\sum_{k_n=0}^{\infty} \dots \left(\sum_{k_1=0}^{\infty} \left(2^{\sum\limits_{j=1}^{n} \alpha_j k_j} \|\Delta_k(f)\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right)^{q_1}\right)^{\frac{q_2}{q_1}} \dots\right)^{\frac{1}{q_n}} < \infty$$

is finite, where $\Delta_k(f)(x) = \sum\limits_{\substack{2^{k_j-1}\leq|m_j|<2^{k_j}\\ j=1,\dots,n}} a_m e^{2\pi i(m,x)}$, $k \in \mathbb{Z}^n_+$, $(m, x) = \sum\limits_{i=1}^{n} m_i x_i$.

For $q = \infty$ the values of $\left(\sum\limits_{k\in\mathbb{Z}} b_k^q\right)^{\frac{1}{q}}$, $\left(\int\limits_{\mathbb{T}} f^q\right)^{\frac{1}{q}}$ are understood respectively as $\sup\limits_{k\in\mathbb{Z}} |b_k|$, $ess \sup\limits_{x\in\mathbb{T}} |f(x)|$.

We say that the series $f = \sum\limits_{k\in\mathbb{Z}^n} a_k e^{2\pi i(k,x)}$ is an element of the space $W^{\boldsymbol{\alpha}}_{\mathbf{p}}(\mathbb{T}^n)$ [24] if there is a function $f^{\boldsymbol{\alpha}} \in L_{\mathbf{p}}(\mathbb{T}^n)$ Fourier series of which coincides with the series $\sum\limits_{k\in\mathbb{Z}^n} \bar{k}^{\alpha} a_k e^{2\pi i(k,x)}$, here $\bar{k}^{\alpha} = \prod\limits_{j=1}^{n} \bar{k}_j^{\alpha}$, $\bar{k}_j = \max\{|k_j|, 1\}$, $j = 1, \dots , n$,

$$\|f\|_{W^{\boldsymbol{\alpha}}_{\mathbf{p}}(\mathbb{T}^n)} \overset{def}{=} \|f^{\alpha}\|_{L_{\mathbf{p}}(\mathbb{T}^n)}.$$

We define the concept of convolution for the elements of these spaces.

Let $f = \sum\limits_{k\in\mathbb{Z}^n} a_k e^{2\pi i(k,x)}$ and $g = \sum\limits_{k\in\mathbb{Z}^n} b_k e^{2\pi i(k,x)}$ be trigonometric series. By the convolution of these series we mean the series

$$f * g = \sum_{k\in\mathbb{Z}^n} a_k b_k e^{2\pi i(k,x)}. \tag{2.1}$$

Note that for the «good» functions f and g, the convolution defined by equality (2.1) coincides with the classical definition (1.1). If the functions f and g from the corresponding spaces in (1.3), then $f(x) \overset{L_{\mathbf{p}}}{=} \sum\limits_{k\in\mathbb{Z}^n} \hat{f}(k)e^{2\pi i(k,x)}$ and $g(x) \overset{L_{\mathbf{r}}}{=} \sum\limits_{k\in\mathbb{Z}^n} \hat{g}(k)e^{2\pi i(k,x)}$ and $(f * g)(x) = \int\limits_{\mathbb{T}^n} f(x - y)g(y)dy =^{L_{\mathbf{q}}} \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\hat{g}(k)e^{2\pi i(k,x)}$. Here, equalities are understood in the sense of the corresponding metrics.

We will need interpolation properties of anisotropic Sobolev and Besov spaces [24, 27]. Let $0 < \boldsymbol{\theta} = (\theta_1, \dots , \theta_n) < 1$, $E = \{\boldsymbol{\varepsilon} = (\varepsilon_1 \dots , \varepsilon_n) : \varepsilon_j \in \{0, 1\}, j = 1, \dots , n\}$ be the vertices of the n-dimensional unit cube, $\{A_{\boldsymbol{\varepsilon}}\}_{\boldsymbol{\varepsilon}\in E}$ be Banach spaces that are subspaces of some linear Hausdorff space. For the element $a \in \sum\limits_{\boldsymbol{\varepsilon}\in E} A_{\boldsymbol{\varepsilon}}$, we define the functional

$$K(t, a; A_{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon} \in E) = \inf_{a=\sum\limits_{\boldsymbol{\varepsilon}\in E} a_{\boldsymbol{\varepsilon}}} \sum_{\boldsymbol{\varepsilon}\in E} t^{\boldsymbol{\varepsilon}} \|a_{\boldsymbol{\varepsilon}}\|_{A_{\boldsymbol{\varepsilon}}},$$

where $t^{\boldsymbol{\varepsilon}} = t_1^{\varepsilon_1} \dots t_n^{\varepsilon_n}$.

By $A_{\boldsymbol{\theta}\mathbf{q}} = (A_{\boldsymbol{\varepsilon}}; \boldsymbol{\varepsilon} \in E)_{\boldsymbol{\theta}\mathbf{q}}$ we denote a linear subset of $\sum\limits_{\boldsymbol{\varepsilon}\in E} A_{\boldsymbol{\varepsilon}}$, for elements of which

$$\|a\|_{A_{\boldsymbol{\theta}\mathbf{q}}} = \left(\int_0^{\infty} \dots \left(\int_0^{\infty} |t_1^{\frac{1}{\theta_1}-1} \dots t_n^{\frac{1}{\theta_n}-1} K(t, a; A_{\varepsilon}, \varepsilon \in E)|^{q_1} \frac{dt_1}{t_1}\right)^{\frac{q_2}{q_1}} \dots \frac{dt_n}{t_n}\right)^{\frac{1}{q_n}} < \infty$$

is true.

Lemma 2.1 [28] Let T be a linear operator such that

$$T:\, A_{\boldsymbol{\varepsilon}} \to B_{\boldsymbol{\varepsilon}} \text{ with norm } M_{\boldsymbol{\varepsilon}},\, \boldsymbol{\varepsilon} \in E.$$

Then

$$T:\, (A_{\boldsymbol{\varepsilon}};\, \boldsymbol{\varepsilon} \in E)_{\boldsymbol{\theta}\mathbf{q}} \to (B_{\boldsymbol{\varepsilon}},\, \boldsymbol{\varepsilon} \in E)_{\boldsymbol{\theta}\mathbf{q}}$$

with the norm $\|T\| \le \max\limits_{\boldsymbol{\varepsilon}\in E} M_{\boldsymbol{\varepsilon}}$.

Theorem 2.1 ([24]) Let $\mathbf{1} \le \mathbf{p} = (p_1, \ldots, p_n) < \boldsymbol{\infty}$, $\mathbf{0} < \mathbf{r} = (r_1, \ldots, r_n)$, $\mathbf{q} = (q_1, \ldots, q_n) \le \boldsymbol{\infty}$, $\boldsymbol{\varepsilon} \in E$, $\boldsymbol{\alpha}_0 = (\alpha_1^0, \ldots, \alpha_n^0)$, $\boldsymbol{\alpha}_1 = (\alpha_1^1, \ldots, \alpha_n^1) \in \mathbb{R}^n$. Then

$$\left(B_{\mathbf{pr}}^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n);\, \boldsymbol{\varepsilon} \in E\right)_{\boldsymbol{\theta}\mathbf{q}} = B_{\mathbf{pq}}^{\boldsymbol{\alpha}}(\mathbb{T}^n),$$

$$\left(W_{\mathbf{p}}^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n);\, \boldsymbol{\varepsilon} \in E\right)_{\boldsymbol{\theta}\mathbf{q}} = B_{\mathbf{pq}}^{\boldsymbol{\alpha}}(\mathbb{T}^n),$$

where $\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}} = (\alpha_1^{\varepsilon}, \ldots, \alpha_n^{\varepsilon})$, $\mathbf{0} < \boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) < \mathbf{1}$, $\boldsymbol{\alpha} = (1 - \boldsymbol{\theta})\boldsymbol{\alpha}_0 + \boldsymbol{\theta}\boldsymbol{\alpha}_1$.

3. Main result

Lemma 3.1 Let $\mathbf{1} \le \mathbf{q}, \mathbf{p}, \mathbf{r} < \infty$, $\dfrac{\mathbf{1}}{\mathbf{q}} + \mathbf{1} = \dfrac{\mathbf{1}}{\mathbf{p}} + \dfrac{\mathbf{1}}{\mathbf{r}}$, $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^n$, $\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{\gamma}$. Suppose that $f \in W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)$, $g \in W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)$. Then $f * g \in W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)$ and

$$\|f * g\|_{W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \le \|f\|_{W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)}.$$

Proof. Let $f = \sum\limits_{k\in\mathbb{Z}^n} a_k e^{2\pi i(k,x)} \in W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)$, $g = \sum_{k\in\mathbb{Z}^n} b_k e^{2\pi i(k,x)} \in W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)$. According to the definition, there are functions $f^{\boldsymbol{\beta}} \in L_{\mathbf{p}}(\mathbb{T}^n)$, $g^{\boldsymbol{\gamma}} \in L_{\mathbf{r}}(\mathbb{T}^n)$ Fourier series of which coincide, respectively, with the $\sum\limits_{k\in\mathbb{Z}^n} \bar{k}^{\boldsymbol{\beta}} a_k e^{2\pi i(k,x)}$, $\sum\limits_{k\in\mathbb{Z}^n} \bar{k}^{\boldsymbol{\gamma}} b_k e^{2\pi i(k,x)}$.

From Young's inequality for Lebesgue spaces with mixed metric [29; 25] $(f^{\boldsymbol{\beta}} * g^{\boldsymbol{\gamma}}) \in L_{\mathbf{q}}(\mathbb{T}^n)$ and has the inequality

$$\|f^{\boldsymbol{\beta}} * g^{\boldsymbol{\gamma}}\|_{L_{\mathbf{q}}(\mathbb{T}^n)} \le \|f^{\boldsymbol{\beta}}\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \|g^{\boldsymbol{\gamma}}\|_{L_{\mathbf{r}}(\mathbb{T}^n)}.$$

Now we note that

$$(f^{\boldsymbol{\beta}} * g^{\boldsymbol{\gamma}})(x) \overset{L_{\mathbf{q}}}{=} \sum_{k\in Z^n} \bar{k}^{\boldsymbol{\beta}+\boldsymbol{\gamma}} a_k b_k e^{2\pi i(k,x)} = (f * g)^{\boldsymbol{\alpha}}(x),$$

which means that $(f * g) \in W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)$ and the inequality

$$\|f * g\|_{W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \le \|f\|_{W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)}$$

holds.

Theorem 3.1 Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^n$, $\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{\gamma}$, $\mathbf{1} \le \mathbf{q}, \mathbf{p}, \mathbf{r} < \boldsymbol{\infty}$, $\mathbf{1} + \dfrac{\mathbf{1}}{\mathbf{q}} = \dfrac{\mathbf{1}}{\mathbf{p}} + \dfrac{\mathbf{1}}{\mathbf{r}}$, $\mathbf{0} < \mathbf{h}, \boldsymbol{\eta}, \boldsymbol{\xi} \le \boldsymbol{\infty}$, $\dfrac{\mathbf{1}}{\mathbf{h}} = \dfrac{\mathbf{1}}{\boldsymbol{\eta}} + \dfrac{\mathbf{1}}{\boldsymbol{\xi}}$. Suppose that $f \in B_{\mathbf{p}\boldsymbol{\eta}}^{\boldsymbol{\beta}}(\mathbb{T}^n)$, $g \in B_{\mathbf{r}\boldsymbol{\xi}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)$. Then $f * g \in B_{\mathbf{qh}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)$ and

$$\|f * g\|_{B_{\mathbf{qh}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \le C \|f\|_{B_{\mathbf{p}\boldsymbol{\eta}}^{\boldsymbol{\beta}}(\mathbb{T}^n)} \|g\|_{B_{\mathbf{r}\boldsymbol{\xi}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)}.$$

Proof. Let $f \in W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)$, $g \in W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)$, then from Lemma 3.1 it follows that $(f * g) \in W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)$ and the following inequality

$$\|f * g\|_{W_{\mathbf{q}}^{\boldsymbol{\alpha}}(\mathbb{T}^n)} \le \|f\|_{W_{\mathbf{p}}^{\boldsymbol{\beta}}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)} \tag{3.1}$$

holds true.

Let $\boldsymbol{\alpha}_0 = (\alpha_1^0, \ldots, \alpha_n^0)$, $\boldsymbol{\alpha}_1 = (\alpha_1^1, \ldots, \alpha_n^1)$, $\boldsymbol{\beta}_0 = (\beta_1^0, \ldots, \beta_n^0)$, $\boldsymbol{\beta}_1 = (\beta_1^1, \ldots, \beta_n^1) \in \mathbb{R}^n$, $\alpha_i^0 \ne \alpha_i^1$, $\beta_i^0 \ne \beta_i^1$, $i = \overline{1, n}$. Let $\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}} = (\alpha_1^{\varepsilon_1}, \ldots, \alpha_n^{\varepsilon_n})$, $\boldsymbol{\beta}_{\boldsymbol{\varepsilon}} = (\beta_1^{\varepsilon_1}, \ldots, \beta_n^{\varepsilon_n})$, $\boldsymbol{\varepsilon} \in E$ such that $\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}} - \boldsymbol{\beta}_{\boldsymbol{\varepsilon}} = \boldsymbol{\gamma}$.

We rewrite inequality (3.1) for the $\boldsymbol{\alpha}_{\varepsilon}$ and $\boldsymbol{\beta}_{\boldsymbol{\varepsilon}}$ parameters

$$\|f * g\|_{W_{\mathbf{q}}^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n)} \le \|f\|_{W_{\mathbf{p}}^{\boldsymbol{\beta}_{\boldsymbol{\varepsilon}}}(\mathbb{T}^n)} \|g\|_{W_{\mathbf{r}}^{\boldsymbol{\gamma}}(\mathbb{T}^n)},\ \boldsymbol{\alpha}_{\boldsymbol{\varepsilon}} = \boldsymbol{\beta}_{\boldsymbol{\varepsilon}} + \boldsymbol{\gamma},\, \boldsymbol{\varepsilon} \in E.$$

For a fixed $g \in W^{\boldsymbol{\gamma}}_{\mathbf{r}}(\mathbb{T}^n)$ operator $A_g f = f * g$ acts boundedly from $W^{\boldsymbol{\beta}_{\boldsymbol{\varepsilon}}}_{\mathbf{p}}(\mathbb{T}^n)$ to $W^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}_{\mathbf{q}}(\mathbb{T}^n)$. Then, using the anisotoropic interpolation theorem (Theorem 2.1)

$$A_g : \left(W^{\boldsymbol{\beta}_{\boldsymbol{\varepsilon}}}_{\mathbf{p}}(\mathbb{T}^n),\, \boldsymbol{\varepsilon} \in E\right)_{\boldsymbol{\theta\xi}} \to \left(W^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}_{\mathbf{q}}(\mathbb{T}^n),\, \boldsymbol{\varepsilon} \in E\right)_{\boldsymbol{\theta\xi}},$$

we obtain that the operator acts boundedly

$$A_g : B^{\boldsymbol{\beta}}_{\mathbf{p}\boldsymbol{\xi}}(\mathbb{T}^n) \to B^{\boldsymbol{\alpha}}_{\mathbf{q}\boldsymbol{\xi}}(\mathbb{T}^n)$$

and

$$\|A_g\| \leq C\|g\|_{W^{\boldsymbol{\gamma}}_{\mathbf{r}}(\mathbb{T}^n)},$$

where $\boldsymbol{\alpha} = (1-\boldsymbol{\theta})\boldsymbol{\alpha}_0 + \boldsymbol{\theta}\boldsymbol{\alpha}_1$, $\boldsymbol{\beta} = (1-\boldsymbol{\theta})\boldsymbol{\beta}_0 + \boldsymbol{\theta}\boldsymbol{\beta}_1$ for any $\mathbf{0} < \boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) < \mathbf{1}$. Thus, we have obtained the inequality:

$$\|f * g\|_{B^{\boldsymbol{\alpha}}_{\mathbf{q}\boldsymbol{\xi}}(\mathbb{T}^n)} \leq C\|f\|_{B^{\boldsymbol{\beta}}_{\mathbf{p}\boldsymbol{\xi}}(\mathbb{T}^n)}\|g\|_{W^{\boldsymbol{\gamma}}_{\mathbf{r}}(\mathbb{T}^n)}, \tag{3.2}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ satisfy the conditions of the theorem.

In inequality (3.2) we assume that $\boldsymbol{\xi} = \infty$, then we have

$$\|f * g\|_{B^{\boldsymbol{\alpha}}_{\mathbf{q}\infty}(\mathbb{T}^n)} \leq C\|f\|_{B^{\boldsymbol{\beta}}_{\mathbf{p}\infty}(\mathbb{T}^n)}\|g\|_{W^{\boldsymbol{\gamma}}_{\mathbf{r}}(\mathbb{T}^n)}.$$

We fix $\boldsymbol{\beta} \in \mathbb{R}^n$. Let $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, $\boldsymbol{\gamma}_0$, $\boldsymbol{\gamma}_1 \in \mathbb{R}^n$ be arbitrary vector parameters satisfying the conditions $\boldsymbol{\alpha}_i = \boldsymbol{\beta} + \boldsymbol{\gamma}_i$, $i = 0,1$ и $\alpha^0_j \neq \alpha^1_j$, $\gamma^0_j \neq \gamma^1_j$, $j = \overline{1,n}$. Then for the parameters $\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}} = (\alpha^{\varepsilon_1}_1, \ldots \alpha^{\varepsilon_n}_n)$, $\boldsymbol{\gamma}_{\boldsymbol{\varepsilon}} = (\gamma^{\varepsilon_1}_1, \ldots \gamma^{\varepsilon_n}_n)$, $\boldsymbol{\varepsilon} \in E$ the inequality

$$\|f * g\|_{B^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}_{\mathbf{q}\infty}(\mathbb{T}^n)} \leq C\|f\|_{B^{\boldsymbol{\beta}}_{\mathbf{p}\infty}(\mathbb{T}^n)}\|g\|_{W^{\boldsymbol{\gamma}_{\boldsymbol{\varepsilon}}}_{\mathbf{r}}(\mathbb{T}^n)}.$$

holds.

Now for a fixed $f(x)$ we define a linear operator $B_f g = f * g$. Then B_f acts boundedly from $W^{\boldsymbol{\gamma}_{\boldsymbol{\varepsilon}}}_{\mathbf{r}}(\mathbb{T}^n)$ in $B^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}_{\mathbf{q}\infty}(\mathbb{T}^n)$ with an estimate of the norm

$$\|B_f\|_{W^{\boldsymbol{\gamma}_{\boldsymbol{\varepsilon}}}_{\mathbf{r}}(\mathbb{T}^n) \to B^{\boldsymbol{\alpha}_{\boldsymbol{\varepsilon}}}_{\mathbf{q}\infty}(\mathbb{T}^n)} \leq C\|f\|_{B^{\boldsymbol{\beta}}_{\mathbf{p}\infty}(\mathbb{T}^n)}.$$

Further, using Theorem 2.1 and Lemma 2.1, we have that the operator B_f is bounded from $B^{\boldsymbol{\gamma}}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n)$ to $B^{\boldsymbol{\alpha}}_{\mathbf{q}\boldsymbol{\xi}}(\mathbb{T}^n)$ and

$$\|f * g\|_{B^{\boldsymbol{\alpha}}_{\mathbf{q}\boldsymbol{\xi}}(\mathbb{T}^n)} \leq C\|f\|_{B^{\boldsymbol{\beta}}_{\mathbf{p}\infty}(\mathbb{T}^n)}\|g\|_{B^{\boldsymbol{\gamma}}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n)},$$

where $\boldsymbol{\alpha} = (1-\boldsymbol{\theta})\boldsymbol{\alpha}_0 + \boldsymbol{\theta}\boldsymbol{\alpha}_1$, $\boldsymbol{\gamma} = (1-\boldsymbol{\theta})\boldsymbol{\gamma}_0 + \boldsymbol{\theta}\boldsymbol{\gamma}_1$ for any $\mathbf{0} < \boldsymbol{\theta} = (\theta_0, \ldots, \theta_n) < \mathbf{1}$.

Similarly, we can obtain the inequality

$$\|f * g\|_{B^{\boldsymbol{\alpha}}_{\mathbf{q}\boldsymbol{\eta}}(\mathbb{T}^n)} \leq C\|f\|_{B^{\boldsymbol{\beta}}_{\mathbf{p}\boldsymbol{\eta}}(\mathbb{T}^n)}\|g\|_{B^{\boldsymbol{\gamma}}_{\mathbf{r}\infty}(\mathbb{T}^n)}.$$

Thus, for the bilinear convolution operator $T(f,g) = f * g$, we have

$$T : B^{\boldsymbol{\beta}_0}_{\mathbf{p}\infty}(\mathbb{T}^n) \times B^{\boldsymbol{\gamma}_0}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n) \to B^{\boldsymbol{\alpha}_0}_{\mathbf{q}\boldsymbol{\xi}}(\mathbb{T}^n),$$

$$T : B^{\boldsymbol{\beta}_1}_{\mathbf{p}\boldsymbol{\eta}}(\mathbb{T}^n) \times B^{\boldsymbol{\gamma}_1}_{\mathbf{r}\infty}(\mathbb{T}^n) \to B^{\boldsymbol{\alpha}_1}_{\mathbf{q}\boldsymbol{\eta}}(\mathbb{T}^n),$$

where the corresponding parameters satisfy the conditions of the theorem.

Next, applying the bilinear interpolation theorem (Theorem 4.4.1, [30, 125]), we have

$$T : \left(B^{\boldsymbol{\beta}_0}_{\mathbf{p}\infty}(\mathbb{T}^n), B^{\boldsymbol{\beta}_1}_{\mathbf{p}\boldsymbol{\eta}}(\mathbb{T}^n)\right)_{[\theta]} \times \left(B^{\boldsymbol{\gamma}_0}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n), B^{\boldsymbol{\gamma}_1}_{\mathbf{r}\infty}(\mathbb{T}^n)\right)_{[\theta]} \to \left(B^{\boldsymbol{\alpha}_0}_{\mathbf{q}\boldsymbol{\xi}}(\mathbb{T}^n), B^{\boldsymbol{\alpha}_1}_{\mathbf{q}\boldsymbol{\eta}}(\mathbb{T}^n)\right)_{[\theta]}.$$

Since the space $B^{\mathbf{s}}_{\mathbf{pq}}(\mathbb{T}^n)$ is a retract of $l^{\mathbf{s}}_{\mathbf{q}}(L_{\mathbf{p}})(\mathbb{T}^n)$, we have

$$\left(B^{\boldsymbol{\beta}_0}_{\mathbf{p}\mathbf{s}_0}(\mathbb{T}^n), B^{\boldsymbol{\beta}_1}_{\mathbf{p}\mathbf{s}_1}(\mathbb{T}^n)\right)_{[\theta]} = B^{\boldsymbol{\beta}}_{\mathbf{ps}}(\mathbb{T}^n),$$

where $\boldsymbol{\beta} = \boldsymbol{\beta}_0(1-\boldsymbol{\theta}) + \boldsymbol{\beta}_1\boldsymbol{\theta}$, $\dfrac{1}{\mathbf{s}} = \dfrac{1-\boldsymbol{\theta}}{\mathbf{s_0}} + \dfrac{\boldsymbol{\theta}}{\mathbf{s}_1}$.

This implies

$$T: B^{\boldsymbol{\beta}}_{\mathbf{p}\boldsymbol{\eta}}(\mathbb{T}^n)\times B^{\boldsymbol{\gamma}}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n)\to B^{\boldsymbol{\alpha}}_{\mathbf{qh}}(\mathbb{T}^n).$$

Finally,

$$\|f*g\|_{B^{\boldsymbol{\alpha}}_{\mathbf{qh}}(\mathbb{T}^n)}\le C\|f\|_{B^{\boldsymbol{\beta}}_{\mathbf{p}\boldsymbol{\eta}}(\mathbb{T}^n)}\|g\|_{B^{\boldsymbol{\gamma}}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n)},$$

where

$$\frac{\mathbf{1}}{\mathbf{h}}=\frac{\mathbf{1}}{\boldsymbol{\eta}}+\frac{\mathbf{1}}{\boldsymbol{\xi}},\ \mathbf{1}+\frac{\mathbf{1}}{\mathbf{q}}=\frac{\mathbf{1}}{\mathbf{p}}+\frac{\mathbf{1}}{\mathbf{r}},\ \boldsymbol{\alpha}=\boldsymbol{\beta}+\boldsymbol{\gamma}.$$

Taking into account the embeddings of spaces ([24], Theorem 4), we can obtain the following theorem.

Theorem 3.2 Let $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathbb{R}^n$, $\boldsymbol{\alpha}\le\boldsymbol{\beta}+\boldsymbol{\gamma}$, $1\le\mathbf{q},\mathbf{p},\mathbf{r}<\infty$, $\boldsymbol{\alpha}=\boldsymbol{\beta}+\boldsymbol{\gamma}+\mathbf{1}+\frac{\mathbf{1}}{\mathbf{q}}-\frac{\mathbf{1}}{\mathbf{p}}-\frac{\mathbf{1}}{\mathbf{r}}$, $\mathbf{0}<\mathbf{h},\boldsymbol{\eta},\boldsymbol{\xi}\le\boldsymbol{\infty}$. Suppose that $f(x)$ and $g(x)$ are measurable functions on $\mathbb{T}^n$ such that $f\in B^{\boldsymbol{\beta}}_{\mathbf{p}\boldsymbol{\eta}}(\mathbb{T}^n)$, $g\in B^{\boldsymbol{\gamma}}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n)$. Then $f*g\in B^{\boldsymbol{\alpha}}_{\mathbf{qh}}(\mathbb{T}^n)$ and

$$\|f*g\|_{B^{\boldsymbol{\alpha}}_{\mathbf{qh}}(\mathbb{T}^n)}\le C\|f\|_{B^{\boldsymbol{\beta}}_{\mathbf{p}\boldsymbol{\eta}}(\mathbb{T}^n)}\|g\|_{B^{\boldsymbol{\gamma}}_{\mathbf{r}\boldsymbol{\xi}}(\mathbb{T}^n)},$$

where $\frac{1}{\mathbf{h}}\le\frac{1}{\boldsymbol{\eta}}+\frac{1}{\boldsymbol{\xi}}$.

References

1 Bennett C. Interpolation of Operators / C. Bennett, R. Sharpley. Pure and Applied Mathematics 129, Boston, MA, Academic Press, INC. — 1988. — 469 p.

2 Brézis H. A note on limiting cases of Sobolev embeddings and convolution inequalities / H. Brézis, S. Wainger // Comm. Partial Differential Equations. — 1980. — 5. — No. 7. — P. 773–789.

3 Hörmander L. The analysis of linear partial differential operators I. Distribution theory and Fourier analysis. Reprint of the second edition / L. Hörmander. — Berlin: Classics in Mathematics, Springer-Verlag. — 2013. — 440 p.

4 O'Neil R. Convolution operators and $L(p,q)$ spaces / R. O'Neil // Duke Math. J. — 1963. — 30. — P. 129–142.

5 Yap, L.Y.H. Some remarks on convolution operators and $l(p,q)$ spaces / L.Y.H. Yap // Duke Math J. — 1969. — 36. — P. 647–658.

6 Hunt R.A. On $L(p,q)$ spaces / R.A. Hunt // Enseignement Math. — 1966. — 12. — No. 2. — P. 249-276.

7 Nursultanov E. Convolution inequalities in Lorentz spaces / E. Nursultanov, S. Tikhonov // J. Fourier Anal. Appl. — 2011. — 17. — P. 486–505.

8 Blozinski A.P. On a convolution theorem for $L(p,q)$ spaces / A.P. Blozinski // Trans. Amer. Math. Soc. — 1972. — 164. — P. 255–265.

9 Нурсултанов Е.Д. Мультипликаторы кратных рядов Фурье / Е.Д. Нурсултанов, Н.Т. Тлеуханова // Тр. МИ РАН им. В.А. Стеклова. — 1999. — 227. — P. 231–236.

10 Tleukhanova N.T. The O'Neil-type inequalities for anisotropic Lorentz spaces $L(\bar{p},\bar{q})$ / N.T. Tleukhanova, K.K. Sadykova // Eurasian Mathematical Journal. В печати.

11 Nursultanov E. Norm inequalities for convolution operators / E. Nursultanov, S. Tikhonov, N. Tleukhanova // C. R. Acad. Sci. Paris. — 2009. — I. — No. 347. — P. 1385–1388.

12 Nursultanov E. Norm convolution inequalities in Lebesgue spaces / E. Nursultanov, S. Tikhonov, N. Tleukhanova // Rev. Mat. Iberoam. — 2018. — 34. — No. 2. — P. 811–838.

13 Heil C. An introduction to weighted Wiener amalgams. In Wavelets and their applications / C. Heil // Allied Publishers, New Delphi. — 2003. — P. 183–216.

14 Kamińska A. On convolution operator in Orlicz spaces / A. Kamińska, J. Musielak // Rev. Mat. Univ. Complutense. — 1989. — 2. — P. 157–178.

15 Kerman R.A. Convolution theorems with weights / R.A. Kerman // Trans. Amer. Math. Soc. — 1983. — 280. — No. 1. — P. 207–219.

16 Kerman R. Convolution algebras with weighted rearrangement-invariant norm / R. Kerman, E. Sawyer // Studia Math. — 1994. — 108. — No. 2. — P. 103–126.

17 Nursultanov E. Weighted norm inequalities for convolution and Riesz potential / E. Nursultanov, S. Tikhonov // Potential Analysis. — 2015. — 42. — No. 2. — P. 435–456.

18 Sampson G. (L_p, L_q) mapping properties of convolution transforms / G. Sampson, A. Naparstek, V. Drobot // Studia Math. — 1976. — 55. — No. 1. — P. 41–70.

19 Головкин К.К. Оценки интегральных операторов в трансляционно-инвариантных нормах / К.К. Головкин, В.А. Солонников // Тр. МИ АН СССР. — 1964. — 70. — С. 47–58.

20 Головкин К.К. Оценки интегральных операторов в трансляционно-инвариантных нормах. II / К.К. Головкин, В.А. Солонников // Тр. МИ АН СССР. — 1966. — 92. — С. 5–30.

21 Батыров Б.Е. Об оценках сверток в пространствах Никольского-Бесова / Б.Е. Батыров, В.И. Буренков // Док. АН СССР. — 1993. — 330. — № 1. — С. 9–11.

22 Bui H. Weighted Young's inequality and convolution theorems on weighted Besov spaces / H. Bui // Math. Nachr. — 1994. — 170. — P. 25–37.

23 Головкин К.К. Об оценках операторов свертки / К.К. Головкин, В.А. Солонников // Зап. науч. сем. ЛОМИ. — 1968. — 7. — P. 6–86.

24 Nursultanov E.D. Interpolation theorems for anisotropic function spaces and their applications / E.D. Nursultanov // Dokl. Math. — 2004. — 69. — No. 1. — P. 16-–19.

25 Бекмаганбетов К.А. Теоремы вложения анизотропных пространств Бесова $B_{\mathbf{pr}}^{\alpha\mathbf{q}}([0,2\pi)^n)$ / К.А. Бекмаганбетов, Е.Д. Нурсултанов // Изв. РАН. Сер. Матем. — 2009. — 73. — № 4. — P. 3–16.

26 Bekmaganbetov K.A. Order of the orthoprojection widths of the anisotropic Nikolsky–Besov classes in the anisotropic Lorentz space / K.A. Bekmaganbetov, Ye. Toleugazy // Eurasian Math. J. — 2016. — 7. — No. 3. — P. 8–16.

27 Nursultanov E.D. Application of Interpolational Methods to the Study of Properties of Functions of Several Variables / E.D. Nursultanov // Mathematical Notes. — 2004. — 75. — No. 3. — P. 341–351.

28 Nursultanov E.D. On the coefficients of multiple Fourier series in L_p spaces / E.D. Nursultanov // Izvestiia RAN: Ser. Mat. — 2000. — 64. — No. 1. — P. 95–122.

29 Бесов О.В. Интегральные представления функций и теоремы вложения / О.В. Бесов, В.П. Ильин, С.М. Никольский. — М.: Наука, 1975. — 480 с.

30 Берг Й. Интерполяционные пространства. Введение / Й. Берг, Й. Лефстрем. — М.: Мир, 1980. — 264 с.

К.К. Садыкова, Н.Т. Тлеуханова

Аралас туындысы басым анизотропты Бесов кеңістігіндегі үйірткі операторының нормаларын бағалау

Мақалада үйірткі операторы нормасының аралас туындысы басым Соболев және анизотропты Никольский-Бесов кеңістіктеріндегі шенелуі зерттелді. Аралас туындысы басым Соболев кеңістігі үшін Юнг теңсіздігінің аналогы алынды, атап айтқанда, $W_p^\gamma * W_r^\beta \hookrightarrow W_q^\alpha$ түріндігі қатынас дәлелденді, мұнда катынас параметрлеріне сәйкесінше шарттар қойылған. Жұмыстың негізгі мақсаты келесі есеп болып табылады: айталық, f және g — Никольский-Бесов кеңістігі шкаласының қандай да бір функциялар класы болсын. Олардың $f * g$ үйірткісі қай кеңістікке жататынын анықтау қажет. (1) қатынас пен Нұрсұлтановтың анизотропты кеңістіктерге арналған интерполяциялық теоремаларын қолдана отырып, $B_{pq}^{\boldsymbol{\alpha}}$ Никольский-Бесов кеңістігінің шкаласы үшін О'Нейл теоремасының аналогы алынды, мұнда α, p, q — векторлық параметрлер. $B_{ps_1}^\gamma * B_{rs_2}^\beta \hookrightarrow B_{qs}^\alpha$ түріндігі қатынас дәлелденді, мұнда қатынас параметрлеріне сәйкесінше шарттар қойылған. Осы жұмыста алынған теоремалар изотропты Никольский-Бесов кеңістігіндегі, яғни скаляр параметрлі кеңістіктегі, Батыров пен Буренковтың ұқсас есептер қарастырған нәтижелерін толықтырады.

Кілт сөздер: үйірткі операторы, анизотропты Бесов кеңістігі, анизотропты Соболев кеңістігі, интерполяция.

К.К. Садыкова, Н.Т. Тлеуханова

Оценки нормы оператора свертки в анизотропных пространствах Бесова с доминирующей смешанной производной

В статье исследована ограниченность нормы оператора свертки в пространствах Соболева, с доминирующей смешанной производной, и анизотропных пространствах Никольского-Бесова. Для пространств Соболева с доминирующей смешанной производной получен аналог неравенства Юнга, а именно доказаны соотношения вида $W_p^\gamma * W_r^\beta \hookrightarrow W_q^\alpha$ при выполнении соответствующих условий на параметры. Основной целью работы является решение следующей задачи: пусть f и g — функции из некоторых классов шкалы пространств Никольского-Бесова. Нужно определить, к какому пространству принадлежит их свертка $f * g$. Используя соотношение (1) и интерполяционные теоремы Нурсултанова для анизотропных пространств, получен аналог теоремы О'Нейла для шкалы пространств Никольского-Бесова B_{pq}^α, где α, p, q — векторные параметры. Получены соотношения вида $B_{ps_1}^\gamma * B_{rs_2}^\beta \hookrightarrow B_{qs}^\alpha$, при соответствующих соотношениях векторных параметров. Полученные в данной работе теоремы дополняют результаты Батырова и Буренкова, где рассматривались подобные задачи в изотропных пространствах Никольского-Бесова, т.е. в пространствах, где параметры являются скалярами.

Ключевые слова: оператор свертки, анизотропные пространства Бесова, анизотропные пространства Соболева, интерполяция.

References

1 Bennett, C. & Sharpley, R. (1988). *Interpolation of Operators.* Pure and Applied Mathematics 129, Boston, MA, Academic Press, INC.

2 Brézis, H. & Wainger, S. (1980). A note on limiting cases of Sobolev embeddings and convolution inequalities. *Comm. Partial Differential Equations, 5, 7,* 773-789.

3 Hörmander, L. (2013). *The analysis of linear partial differential operators I. Distribution theory and Fourier analysis. Reprint of the second edition.* Berlin: Classics in Mathematics, Springer-Verlag.

4 O'Neil, R. (1963). Convolution operators and $L(p, q)$ spaces. *Duke Math. J., 30,* 129–142.

5 Yap, L.Y.H. (1969). Some remarks on convolution operators and $l(p, q)$ spaces. *Duke Math J., 36,* 647–658.

6 Hunt, R.A. (1966). On $L(p, q)$ spaces. *Enseignement Math., 12, 2,* 249–276.

7 Nursultanov, E. & Tikhonov, S. (2011). Convolution inequalities in Lorentz spaces. *J. Fourier Anal. Appl., 17,* 486-505.

8 Blozinski, A.P. (1972). On a convolution theorem for $L(p, q)$ spaces. *Trans. Amer. Math. Soc., 164,* 255–265.

9 Nursultanov, E.D. & Tleukhanova, N.T. (1999). Multiplikatory kratnykh riadov Fure [Multipliers of multiple Fourier series]. *Trudy MI RAN imeni V.A. Steklova – Proceedings of the Steklov MI RAS, 227,* 231–236 [in Russian].

10 Tleukhanova, N.T. & Sadykova, K.K. The O'Neil-type inequalities for anisotropic Lorentz spaces $L(\bar{p}, \bar{q})$. *Eurasian Mathematical Journal.*

11 Nursultanov, E. & Tikhonov, S. & Tleukhanova, N. (2009). Norm inequalities for convolution operators. *C. R. Acad. Sci. Paris, I, 347,* 1385–1388.

12 Nursultanov, E. & Tikhonov, S. & Tleukhanova, N. (2018). Norm convolution inequalities in Lebesgue spaces. *Rev. Mat. Iberoam., 34, 2,* 811–838.

13 Heil, C. (2003). An introduction to weighted Wiener amalgams. In Wavelets and their applications. *Allied Publishers, New Delphi,* 183–216.

14 Kamińska, A. & Musielak, J. (1989). On convolution operator in Orlicz spaces. *Rev. Mat. Univ. Complutense, 2,* 157–178.

15 Kerman, R.A. (1983). Convolution theorems with weights. *Trans. Amer. Math. Soc., 280, 1,* 207–219.

16 Kerman, R. & Sawyer, E. (1994). Convolution algebras with weighted rearrangement-invariant norm. *Studia Math., 108, 2,* 103–126.

17 Nursultanov, E. & Tikhonov, S. (2015). Weighted norm inequalities for convolution and Riesz potential. *Potential Analysis, 42, 2,* 435–456.

18 Sampson, G. & Naparstek, A. & Drobot, V. (1976). (L_p, L_q) mapping properties of convolution transforms. *Studia Math., 55, 1,* 41-70.

19 Golovkin, K.K. & Solonnikov, V.A. (1964). Otsenki intehralnykh operatorov v transliatsionno-invariantnykh normakh [Estimates of integral operators in translation-invariant norms]. *Trudy MI AN SSSR – Proceedings of MI AN USSR, 70,* 47–58 [in Russian].

20 Golovkin, K.K. & Solonnikov, V.A. (1966). Otsenki intehralnykh operatorov v transliatsionno-invariantnykh normakh. II [Estimates of integral operators in translation-invariant norms. II]. *Trudy MI AN SSSR – Proceedings of MI AN USSR, 92,* 5–30 [in Russian].

21 Batyrov, B.E. & Burenkov, V.I. (1993). Ob otsenkakh svertok v prostranstvakh Nikolskoho-Besova [On estimates of convolutions in Nikolsky-Besov spaces]. *Doklady AN SSSR – Reports of the AS USSR, 330, 1,* 9–11 [in Russian].

22 Bui, H. (1994). Weighted Young's inequality and convolution theorems on weighted Besov spaces. *Math. Nachr., 170,* 25–37.

23 Golovkin, K.K., & Solonnikov, V.A. (1968). Ob otsenkakh operatorov svertki [On estimates of convolution operators]. *Zapiski nauchnoho seminara LOMI – Notes of scientific seminars of the LBMI, 7,* 6–86 [in Russian].

24 Nursultanov, E.D. (2004). Interpolation theorems for anisotropic function spaces and their applications. *Dokl. Math., 69, 1,* 16–19.

25 Bekmaganbetov, K.A. & Nursultanov, E.D. (2009). Teoremy vlozheniia anizotropnykh prostranstv Besova $B_{\mathbf{pr}}^{\alpha\mathbf{q}}([0,2\pi)^n)$ [Embedding theorems for anisotropic Besov spaces $B_{\mathbf{pr}}^{\alpha\mathbf{q}}([0,2\pi)^n)$]. *Izvestiia RAN SSSR. Seriia Matematila – Proceedings of the RAS USSR. Ser. Math., 73, 4,* 3–16 [in Russian].

26 Bekmaganbetov, K.A. & Toleugazy, Ye. (2016). Order of the orthoprojection widths of the anisotropic Nikolsky–Besov classes in the anisotropic Lorentz space. *Eurasian Math. J., 7, 3,* 8–16.

27 Nursultanov, E.D. (2004). Application of Interpolational Methods to the Study of Properties of Functions of Several Variables. *Mathematical Notes, 75, 3,* 341–351.

28 Nursultanov, E.D. (2000). On the coefficients of multiple Fourier series in L_p spaces. *Izvestiia RAN: Ser. Mat., 64, 1,* 95–122.

29 Besov, O.V. & Il'in, V.P. & Nikolskii, S.M. (1975). *Intehralnye predstavleniia funktsii i teoremy vlozheniia [Integral presentation of functions and theorem of embanding].* Moscow: Nauka [in Russian].

30 Berg, I. & Löfstrem, I. (1980) *Interpoliatsionnye prostranstva. Vvedenie [Interpolation spaces. Introduction].* Moscow: Mir [in Russian].

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On construction of the comparison function of program motion in probable statement

In the class of ordinary differential equations the following modification of the inverse problem of differential systems was previously considered: to construct both a set of systems of differential equations and a set of comparison functions for the given program motion. In this article, the modification of the inverse problem is considered in Stochastic case. In this problem it is assumed that random perturbations are from the class of processes with independent increments. By the given program of motion, two sets are constructed: the set of first-order Itô stochastic differential equations and the set of comparison functions. It is proved that there is a stability in probability of the given program motion with respect to the constructed comparison functions. To solve the problem Lyapunov functions method is used. Using Lyapunov's second method makes it is possible to weaken the conditions imposed on the components of the constructed comparison functions, in contrast to the application of the Lyapunov characteristic numbers method for solving the inverse problem in the class of ordinary differential equations. The following cases are considered: 1) comparison functions obviously not depending on time; 2) the set of comparison vector functions depends on y and t; 3) the set of comparison vector functions has the form $Q(\lambda, t$, where $\lambda(y, t)$ describes an analytically given program motion; 4) the set of comparison vector functions has the form $C(t)\lambda$.

Keywords: stochastic differential equations, inverse problems, stability in probability, comparison function, program motion, random process.

Introduction

At present, possible formulations of the inverse problems for differential systems have been known. General methods for solving these problems in the class of ordinary differential equations have been well developed in [1–11]. Generalization of these problems to the class of partial differential equations is investigated in [12–14]. In the theory of inverse problems of differential systems, an important requirement is the requirement of stability of the given properties of motion [5]. This requirement is related to the system's operability and its non-compliance with perturbations. Therefore, solving of stability problem of the program motion is essential for the further development of the qualitative theory of inverse problems of differential systems and constructing of the systems of program motion.

In the theory of stability, the possible perturbed motions of the material system are compared with the unperturbed motion with respect to the corresponding values of the given kinematic indicators of motion at each time instant $t \geq t_0$. It is assumed that the kinematic indicators of motion can be described by the vector function $Q(y, t)$, called the comparison function and given in some domain of the space $G(y_1, ..., y_n)$ of the change in the phase coordinates of the given system. The study of the stability of unperturbed motion is reduced to establishing the behavior of the difference of the values of this vector function respectively on possible perturbed motion and on unperturbed motion of system

$$x = Q_{perturbed\, motion} - Q_{unperturbed\, motion},$$

for all $t \geq t_0$. In problems of construction of stable systems, the desired system parameters and additional control forces are also determined from the conditions imposed on this difference. In the established formulations of the stability problems, the comparison functions are given. Also the unperturbed motion and the equations of motion of the material system are given. Thus, the solving of the stability problem is reduced to determining the stability conditions for the given motion of the system with respect to the given comparison functions. However in many problems of the theory of stability it is useful problem to construct the comparison functions with respect to which there is a stability of given properties of motion of the mechanical system.

Galiullin A.S. posed the one of the inverse problems of dynamics namely the problem of construction of the set of comparison functions in the class of ordinary differential equations [1–3]:

It is required to construct the corresponding set of equations of motion for the material system

$$\dot{y} = Y(y,t), \tag{1.1}$$

by the given law of motion

$$\Lambda : y = \varphi(t),\ \ y \in R^n,\ \ \varphi \in C^1\{t \geq t_0\}. \tag{1.2}$$

Let the considering equations be the class of equations admitting the existence of the unique solution of the equation (1.1) with initial condition $y|_{t=t_0} = \varphi(t_0),\ \{t_0, \varphi(t_0)\}$. And it is required to construct the set of n-dimensional vector functions $Q(y)$. $Q(y)$ is holomorphic vector functions in some ε-neighborhood $\Lambda_\varepsilon = \{\|y - \varphi(t)\| < \varepsilon\}$ of the integral manifold $\Lambda(t)$ (1.2) for all $t \geq t_0$. There is stability in the sense of Lyapunov in relation to the components of $Q(y)$.

The problem of construction of the set of comparison functions is solved by the method of characteristic Lyapunov numbers in [1–3]. The solution of this problem determines the set of kinematic indicators of motion (conditions imposed on them), with respect to which the given motion properties of the system are stable.

The set of equations of motion of the system is constructed in the form

$$\dot{y} = \dot{\varphi}(t) + \Phi(y,t). \tag{1.3}$$

Here $\Phi(y,t)$ is some holomorphic vector function in the domain Λ_ε for all $t \geq t_0,\ \Phi(y,t)|_{y=\varphi(t)} = 0$. Further the equation of perturbed motion of the first approximation with respect to the vector function $Q(y)$ is compiled. The equation of perturbed motion is reduced to the system of linear differential equations by some linear transformation. It is required that the resulting system is correct and that its characteristic numbers are positive. Then, if the applied linear transformation is Lyapunov transformation, then there is stability of the given motion (1.2) with respect to the vector function Q(y). Following [2], the set of the required components of the vector functions Q (y) is determined from the conditions

$$C, \dot{C} \text{ are limited, } \det C^{-1} \neq 0 \text{ for all } t \geq t_0, \tag{1.4}$$

here $C = \|\psi^i_\nu\|^n_n,\ \psi^i_\nu = \dfrac{\partial Q_i(y)}{\partial y_\nu}|_{y=\varphi(t)},\ i, \nu = 1, ..., n$ and $C(t)$ is the Lyapunov transformation [15].

In the Preface of the J. Adomian's book «Stochastic Systems» (Moscow, 1987) the famous scientist Richard Bellman emphasizes: «It is very important to decide which model to build: deterministic model or stochastic model. Deterministic models are very useful. But stochastic models are more realistic. The difficulty is that the analysis of stochastic models by mathematical means is very complicated».

Stochastic differential equations of Itô type describe numerous and important in the application models of mechanical systems. These models take into account the effects of external random forces. In particular, models take into account the motion of an artificial satellite of the Earth under the action of the aerodynamic forces [16] or the fluctuation drift of the heavy gyroscope in a cardan suspension [17] and many others.

As an example showing the importance of taking into account random perturbations, we can cite the inverse problem of the dynamics of a spacecraft's flight. For example, the aerodynamic moments of a spacecraft always have random components [16] generated by density fluctuations of the planet's atmosphere. In addition, random changes in the moments of inertia cause thermoelastic vibrations of stabilizing rods, vibrations of liquids in cans, antennas and solar panels. And the study of the effect of random perturbations on the dynamics of the spacecraft is so important that ignoring these perturbations can significantly reduce the lifetime of the spacecraft [18].

Inverse problems in the class of stochastic differential systems are considered in [19–21]. Problems of stability in probability of the given program motion by Lyapunov function method are studied in [22, 23].

Let us consider the probabilistic formulation of the problem posed earlier in the class of ordinary differential equations [1–3], namely, with the additional assumption of the presence of random perturbations.

Statement of the problem

It is required to construct the corresponding set of equations of motion for the material system

$$\dot{y} = Y(y,t) + \sigma(y,t)\dot{\xi},\ \ \xi \in R^k, \tag{2.1}$$

by the given program of motion

$$\Lambda : \lambda \equiv y - \varphi(t) = 0, y \in R^n, \ \varphi \in C^1, \ \|\varphi\| \le l. \tag{2.2}$$

Here $\xi(t) = \omega(t) + \int_{R^n} c(y)P(t, dy)$ is random process with independent increments; $\omega(t)$ is Wiener process; $P(t, A)$ is Poisson process as a function of t and Poisson stochastic measure as a function of set A; $c(y)$ is vector function mapping R^n into the space of the process values $\xi(t)$ for all t.

Let the considering equations be the class of equations admitting the existence of a unique up to stochastic equivalence of solution of the equation (2.1) with initial condition $y|_{t=t_0} = \varphi(t_0)$. It is also required to construct a set of s-dimensional vector functions $Q(y)$ so that there is stability in probability of the program motion (2.2) with respect to $Q(y)$.

Following [3], the equation of perturbed motion of the material system, for which the given motion (2.2) is possible, is represented as

$$\dot{\lambda} = A(\lambda; y, t) + B(\lambda; y, t)\dot{\xi}. \tag{2.3}$$

Here $A(\lambda; y, t)$ is a vector function and $B(\lambda; y, t)$ is a $n \times k$-dimentional Erugin type matrix such that $A(0; y, t) \equiv 0$, $B(0; y, t) \equiv 0$.

Let $LV(\lambda(x, t), x, t)$ be generating operator of $\xi(t)$ [22].

In the future we need the following definitions:

Definition 1 [24]. A function $a(r)$ is called the function of Khan class $a(r) \in K$ if it is continuous and strictly increasing and satisfies the condition $a(0) = 0$.

Definition 2 [25]. The program manifold (2.2) of the equation (2.1) is called ρ-stable in probability if

$$\lim_{\rho(y_0, \Lambda(t_0)) \to 0} P_{x_0}\{\sup_{t>0} \rho(y^{y_0, t_0}(t), \Lambda(t)) > \varepsilon\} = 0.$$

1 A set of vector functions $Q(y)$ that obviously not depending on time

Theorem 1. Let there exist a Lyapunov function $V(\lambda; y, t)$ on the neighborhood $\Lambda_\varepsilon = \{\|y - \varphi(t)\| < \varepsilon\}$ of the integral manifold Λ satisfying the conditions

$$a(\|\lambda\|) \le V(\lambda; y, t) \le b(\|\lambda\|), a, b \in K, \tag{2.4}$$

$$LV \le -c(\|\lambda\|), c \in K. \tag{2.5}$$

Then the program motion $\lambda \equiv y - \varphi(t) = 0$ of system (2.3) is asymptotically ρ- stable in probability with respect to an arbitrary s- dimensional vector function $Q(y)$, which is continuous on the neighborhood Λ_ε for $1 \le s \le n$.

Proof. By definition of stability [3], we consider the difference $x = Q(y) - Q(\varphi(t))$. By the condition of the theorem, there is a Lyapunov function with properties (2.4), (2.5). This provides an asymptotically ρ-stability in probability of program motion $\lambda = 0$ [22], i.e.

$$\lim_{\rho(y_0, \Lambda(t_0)) \to 0} P_{y_0}\left\{\lim_{t\to\infty} \sup \rho(y^{y_0, t_0}(t), \Lambda(t)) = 0\right\} = 1. \tag{2.6}$$

And from the continuity of the vector function $Q(y)$ and conditions (2.6) we have

$$\lim_{\rho(y_0, \Lambda(t_0)) \to 0} P_{y_0}\left\{\lim_{t\to\infty} \sup \|Q(y(t, t_0, y_0)) - Q(\varphi(t))\| = 0\right\} = 1.$$

This means that the motion $\lambda \equiv y - \varphi(t) = 0$ of the system (2.3) is asymptotically stable with respect to the vector function $Q(y)$.

2 A set of vector functions $Q(y, t)$ depending on y and t

Theorem 2. Let there exist a Lyapunov function $V(\lambda; y, t)$ on the neighborhood Λ_ε of the integral manifold Λ with properties (2.4), (2.5).

Then the program motion $\lambda \equiv y - \varphi(t) = 0$ of system (2.3) is asymptotically ρ- stable in probability with respect to an arbitrary s- dimensional vector function $Q(y,t)$, which is continuous in y and t and satisfying the condition

$$\|x\| \leq \beta(\|\lambda\|), \beta \in K. \tag{2.7}$$

Here $x = Q(y,t) - Q(\varphi(t),t)$.

Proof. The existence of a function $V(\lambda; y, t)$ with properties (2.4), (2.5) implies an asymptotical uniform in $\{t_0, y_0\}$ ρ-stability in probability of motion $\lambda \equiv y - \varphi(t) = 0$ [22], i.e.

$$\lim_{\rho(y_0,\Lambda(t_0))\to 0} P_{y_0}\left\{\lim_{t\to\infty} \sup \rho(y^{y_0,t_0}(t), \Lambda(t)) = 0\right\} = 1.$$

And we get from (2.6) and (2.7) that

$$\lim_{\rho(y_0,\Lambda(t_0))\to 0} P_{y_0}\left\{\lim_{t\to\infty} \sup \|Q(y(t,t_0,y_0),t) - Q(\varphi(t),t)\| = 0\right\} = 1.$$

Consequently, there is an asymptotic stability in probability of the motion $\lambda \equiv y - \varphi(t) = 0$ of system (2.3) with respect to the vector function $Q(y,t)$.

3 Program motion $\lambda(y,t) = 0$ and a set of comparison vector functions $Q(\lambda,t)$

Let the program motion be given as

$$\Lambda(t) : \lambda(y,t) = 0. \tag{2.8}$$

Here $\lambda \in R^k$, $y \in R^n$, $k \leq n$.

Suppose that it takes place *rang* $\left\{\frac{\partial\lambda}{\partial y}\right\} = k$ for all $y \in \Lambda_h$, $t \geq t_0$ on a neighborhood $\Lambda_h(t) \in R^n$

$$\Lambda_h(t) : \|\lambda(y,t)\| \leq h, t \geq t_0. \tag{2.9}$$

The set of equations of the perturbed motion for which the given program (2.8) is one of the possible ones, can be represented as

$$\dot{\lambda} = A(\lambda; y, t) + B(\lambda; y, t)\dot{\xi}.$$

Here $A(\lambda; y, t)$ is a vector function and $B(\lambda; y, t)$ is a $n \times k$-dimentional Erugin type matrix such that $A(0; y, t) \equiv 0$, $B(0; y, t) \equiv 0$.

Consider the continuous s-dimensional vector functions $Q(\lambda, t)$ satisfying the condition

$$\|x\| \leq \beta(\|\lambda\|), \beta \in K. \tag{2.10}$$

Here $x = Q(\lambda(y,t)) - Q(0,t)$, $1 \leq s \leq n$.

Theorem 3. Let there exist a Lyapunov function $V(\lambda; y, t)$ on the neighborhood (2.9) of the integral manifold (2.8) satisfying the condition (2.4) and

$$LV \leq -c(\|\lambda\|), c \in K.$$

Then the integral manifold (2.8) is asymptotically stable in probability with respect to an arbitrary s-dimensional vector function $Q(\lambda, t)$, which is continuous in λ and t and satisfying condition (2.10) for $1 \leq s \leq n$.

The proof is similar to the proof of Theorem 2.

4 A set of n-dimensional vector functions of the form $C(t)\lambda$

Let the equation of perturbed motion (2.3) in the first approximation have the form

$$\dot{\lambda} = A_1(t)\lambda + A_2(\lambda, t) + B\dot{\xi}.$$

Let us consider the Lyapunov function $V(\lambda) = (\lambda, \lambda)$ and n- dimensional vector function $Q(y,t) = C(t)\lambda$. Here $\lambda \equiv y - \varphi(t)$. In this particular case $x = Q(y) - Q(\varphi(t))$ has the form

$$x = C(t)\lambda.$$

Also suppose that

1) the matrix $A_1^T(t) + A_1(t)$ is definitely negative and the vector function A_2 satisfies the condition $||A_2|| = o(||\lambda||)$;

2) the matrix

$$C(t) = \frac{\partial Q}{\partial y}\bigg|_{y=\varphi(t)} \text{ is continuous and limited for all } t \geq t_0. \tag{2.11}$$

Then from properties 1), 2) and Theorem 2 the following theorem holds.

Theorem 4. Let $A_1(t)$ and $C(t)$ are continuous matrices such that conditions 1) and 2) hold. Then the motion $\lambda \equiv y - \varphi(t) = 0$ of the system (2.3) is stable in probability with respect to an arbitrary vector functions $Q(y,t) = C(t)\lambda$.

Remark. In Theorem 4 instead of condition (1.4) the weaker condition (2.11) is required.

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References

1 Галиуллин А.С. Задачи построения функций сравнения в теории устойчивости движения / А.С. Галиуллин // Дифференциальные уравнения. — 1974. — 10. — № 8. — С. 1527–1529.

2 Галиуллин А.С. Методы решения обратных задач динамики / А.С. Галиуллин. — М.: РУДН, 1986. — 224 с.

3 Галиуллин А.С. Избранные труды: [В 2 т.]. — Т. I. / А.С. Галиуллин. — М.: РУДН, 2009. — 462 с.

4 Мухарлямов Р.Г. О построении систем дифференциальных уравнений движения механических систем / Р.Г. Мухарлямов // Дифференциальные уравнения. — 2003. — 39. — № 3. — С. 343–353.

5 Mukharlyamov R.G. Reduction of dynamical equations for the systems with constraints to given structure / R.G. Mukharlyamov // Journal of Applied Mathematics and Mechanics. Elsevier. — 2007. — 71. — P. 361–370.

6 Mukharlyamov R.G. Differential-algebraic equations of programmed motions of Lagrangian dynamical systems / R.G. Mukharlyamov // Mechanics of Solids. — 2011. — 46. — No. 4. — C. 534—543.

7 Mukharlyamov R.G. Control of system dynamics and constrains stabilization / R.G. Mukharlyamov, M.I. Tleubergenov // Communications in Computer and Information Science. — 2017. — 700. — P. 431–442.

8 Zhumatov S.S. Asymptotic Stability of Implicit Differential Systems in the Vicinity of Program Manifold / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2014. — 66. — No. 3. — P. 625–632.

9 Zhumatov S.S. Exponential Stability of a Program Manifold of Indirect Control Systems / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2010. — 62. — No. 6. — P. 907–915.

10 Zhumatov S.S. Stability of a Program Manifold of Control Systems with Locally Quadratic Relations / S.S. Zhumatov // Ukrainian Mathematical Journal. — 2009. — 61. — No. 3. — P. 500–509.

11 Llibre J. Inverse Problems in Ordinary Differential Equations and Applications / J. Llibre, R. Ramirez. — Switzerland: Springer International Publishing, 2016. — 266 p.

12 Budochkina S.A. An operator equation with the second time derivative and Hamilton-admissible equations / S.A. Budochkina, V.M. Savchin // Doklady Mathematics. — 2016. — 94. — No. 2. — P. 487–489.

13 Savchin V.M. Nonclassical Hamiltons actions and the numerical performance of variational methods for some Dissipative Problems / V.M. Savchin, S.A. Budochkina // Communications in Computer and Information Science. — 2016. — 678. — P. 624–634.

14 Savchin V.M. Invariance of functional and related Euler-Lagrange equations / V.M. Savchin, S.A. Budochkina // Russian Mathematics. — 2017. — 61. — No. 2. — P. 49–54.

15 Демидович Б.П. Лекции по математической теории устойчивости / Б.П. Демидович. — М.: Наука, 1967. — 472 с.

16 Сагиров П. Стохастические методы в динамике спутников / П. Сагиров // Механика. Период. сб. пер. иностр. ст. — 1974. — № 5(147). — С. 28—47. — № 6(148). — С. 3—38.

17 Синицын И.Н. О флуктуациях гироскопа в кардановом подвесе / И.Н. Синицын // Изв. АН СССР. Механика твердого тела. — 1976. — № 3. — С. 23–31.

18 Демин В.Г. Движение искусственного спутника в нецентральном поле тяготения / В.Г. Демин. — М., 1968. — 352 с.

19 Tleubergenov M.I. Main inverse problem for differential system with generate diffusion / M.I. Tleubergenov, G.T. Ibraeva // Ukrainian Mathematical Journal. — 2013. — 65. — 5. — P. 787–792.

20 Tleubergenov M.I. On the inverse stochastic reconstruction problem / M.I. Tleubergenov // Differential Equations. — 2014. — 50. — No. 2. — P. 274—278.

21 Tleubergenov M.I. Stochastic inverse problem with indirect control / M.I. Tleubergenov, G.T. Ibraeva // Differential Equations. — 2017. — 53. — No. 10. — P. 1387–1391.

22 Vassilina G.K. Solution of the Problem of Stochastic Stability of an Integral Manifold by the Second Liapunov Method / G.K. Vassilina, M.I. Tleubergenov // Ukrainian Mathematical Journal. — 2016. — 68. — No. 1. — P. 14–28.

23 Vasilina G.K. On the optimal stabilization of an integral manifold [Electronic resource] / G.K. Vassilina, M.I. Tleubergenov // Journal of Mathematical Sciences. — 2018. — 229. — No. 4. — P. 390–402. — Access mode: http://DOI 10.1007/s10958-018-3684-5.

24 Руш Н. Прямой метод Ляпунова в теории устойчивости / Н. Руш, П. Абетс, М. Лалуа. — М.: Мир, 1980. — 300 с.

25 Хасьминский Р.З. Устойчивость систем дифференциальных уравнений при случайных возмущениях их параметров / Р.З. Хасьминский. — М., 1969. — 368 с.

Г.Қ. Василина, М.Ы. Тілеубергенов

Ықтималды жағдайдағы бағдарламалық қозғалыстың салыстыру фунциясының құрылуы туралы

Жай дифференциалдық теңдеулер класында бұрын дифференциалдық жүйелердің кері есебінің келесі түрленуі қарастырылды: дифференциалдық теңдеулер жүйелерінің жиынтығын және осы бағдарламаланған қозғалыс үшін салыстырмалы функциялар жиынтығын құру. Мақалада кері есептің бұл түрі ықтималды жағдайда зерттелді. Бұл есепте кездейсоқ түрткілер тәуелсіз өсімшелі кездейсоқ үдерістер класынан деп есептелді. Берілген қозғалыс бағдарламасына сәйкес, екі жиын құрылады: Итоның бірінші ретті стохастикалық дифференциалдық теңдеулер жиынтығы және салыстыру функциялары жиыны. Құрастырылған салыстыру функцияларына қатысты берілген бағдарламаланған қозғалыстың ықтималдық бойынша орнықтылығы бар екені дәлелденді. Бұл есепті шешу үшін Ляпунов функцияларының әдісі қолданылды. Ляпуновтың екінші әдісінің пайдалануы жай дифференциалдық теңдеулер класындағы кері есептерді шешу үшін қажет. Ляпуновтың сипаттаушы сандары әдісін қолданудан айырмашылығы алынған салыстыру функцияларының құраушыларына енгізілген шарттарды әлсіретуге мүмкіндік береді. Келесі жағдайлар қарастырылған, яғни салыстыру векторлық функциялары: 1) уақыттан нақты тәуелді емес; 2) жиынтығы y және t-дан тәуелді; 3) жиынтығы $Q(\lambda, t)$ түрінде болады, мұнда $\lambda(y, t) = 0$ аналитикалық түрде берілген бағдарламалық қозғалысты сипаттайды; 4) жиынтығы $C(t)\lambda$ түрінде болады. Қарастырылған жағдайларда құрылған салыстыру функциясына қатысты бағдарламалық қозғалыстың ықтималдық бойынша асимптотикалық орнықтылығының жеткілікті шарттары алынған.

Кілт сөздер: стохастикалық дифференциалдық теңдеулер, кері есептер, ықтималдық бойынша орнықтылық, салыстыру функциясы, бағдарламалық қозғалыс, кездейсоқ процесс.

Г.К. Василина, М.И. Тлеубергенов

О построении функции сравнения программного движения в вероятностной постановке

В классе обыкновенных дифференциальных уравнений ранее была рассмотрена следующая модификация обратной задачи дифференциальных систем: построить как множество систем дифференциальных уравнений, так и множество функций сравнения по заданному программному движению. В статье данная модификация обратной задачи рассмотрена в вероятностной постановке. В исследуемой задаче предположены случайные возмущения из класса процессов с независимыми приращениями. По заданной программе движения строятся два множества: множество стохастических дифференциальных уравнений Ито первого порядка и множество функций сравнения. Доказано, что имеет место устойчивость по вероятности заданного программного движения относительно построенных функций сравнения. Для решения поставленной задачи применен метод функций Ляпунова. Использование второго метода Ляпунова позволяет ослабить условия, накладываемые на составляющие построенных функций сравнения, в отличие от применения метода характеристичных чисел Ляпунова для решения обратной задачи в классе обыкновенных дифференциальных уравнений. Изучены случаи, когда множество вектор-функций сравнения 1) не зависит явно от времени; 2) зависит от y и t; 3) имеет вид $Q(\lambda, t)$, где $\lambda(y, t) = 0$, описывает аналитически заданное программное движение; 4) имеет вид $C(t)\lambda$. В рассмотренных случаях получены достаточные условия асимптотической устойчивости по вероятности программного движения относительно построенной функции сравнения.

Ключевые слова: стохастические дифференциальные уравнения, обратные задачи, устойчивость по вероятности, функция сравнения, программное движение, случайный процесс.

References

1 Galiullin, A.S. (1974). Zadachi postroieniia funktsii sravneniia v teorii ustoichivosti dvizheniia [Problems of construction of comparison functions in stability theory of the motion]. *Differentsialnye uravneniia – Differential equations, 10, 8,* 1527–1529 [in Russian].

2 Galiullin, A.S. (1986). *Metody resheniia obratnykh zadach dinamiki [Methods for solving inverse problems of dynamics].* Moscow: Nauka [in Russian].

3 Galiullin, A.S. (2009). *Izbrannye trudy [Selected Works.]* (Vol. 1, 2). Moscow: RUDN [in Russian].

4 Muharlyamov, R.G. (2003). O postroenii sistem differentsialnykh uravnenii dvizheniia mekhanicheskikh sistem [On construction of differential equations systems of motion of the mechanical systems]. *Differentsialnye uravneniia – Differential equations, 39, 3,* 343–353 [in Russian].

5 Mukharlyamov, R.G. (2007). Reduction of dynamical equations for the systems with constraints to given structure. *Journal of Applied Mathematics and Mechanics, 71,* Elsevier, 361–370.

6 Mukharlyamov, R.G. (2011). Differential-algebraic equations of programmed motions of Lagrangian dynamical systems. *Mechanics of Solids, 46, 4,* 534–543.

7 Mukharlyamov, R.G., & Tleubergenov M.I. (2017) Control of system dynamics and constrains stabilization. *Communications in Computer and Information Science,* 700, 431–442.

8 Zhumatov, S.S. (2014). Asymptotic Stability of Implicit Differential Systems in the Vicinity of Program Manifold. *Ukrainian Mathematical Journal,* 66, 3, 625–632.

9 Zhumatov, S.S. (2010). Exponential Stability of a Program Manifold of Indirect Control Systems. *Ukrainian Mathematical Journal,* 62, 6, 907–915.

10 Zhumatov, S.S. (2009) Stability of a Program Manifold of Control Systems with Locally Quadratic Relations. *Ukrainian Mathematical Journal,* 61, 3, 500–509.

11 Llibre J., & Ramirez, R. (2016). *Inverse Problems in Ordinary Differential Equations and Applications.* Springer International Publishing Switzerland.

12 Budochkina, S.A., & Savchin, V.M. (2016). An operator equation with the second time derivative and Hamilton-admissible equations. *Doklady Mathematics, 94, 2,* 487–489.

13 Savchin, V.M., & Budochkina, S.A. (2016). Nonclassical Hamiltons actions and the numerical performance of variational methods for some Dissipative Problems. *Communications in Computer and Information Science, 678,* 624–634.

14 Savchin, V.M., & Budochkina, S.A. (2017). Invariance of functional and related Euler-Lagrange equations. *Russian Mathematics, 61, 2,* 49–54.

15 Demidovich, B.P. (1967). *Lektsii po matematicheskoi teorii ustoichivosti [Lectures on mathematical theory of stability].* Moscow: Nauka [in Russian].

16 Sagirov, P. (1974). Stokhasticheskiie metody v dinamike sputnikov [Stochastic methods in satellite dynamics] *Mekhanika. Periodicheskii sbornik perevodov inostrannykh statei – Mechanics. Periodic collection of translation of foreign articles, 5(147),* 28–47, No. 6(148), 3–38. Moscow [in Russian].

17 Sinitsyn, I.N. (1976). O fluktuatsiiakh hiroskopa v kardanovom podvese [On fluctuations of the gyroscope in a cardan suspension]. *Izvestiia AN SSSR. Mekhanika tverdoho tela – News of Academy of Sciences of the USSR. Solid mechanics, 3,* 23–31. Moscow [in Russian].

18 Demin, V.G. (1968). *Dvizhenie iskusstvennoho sputnika v netsentralnom pole tiahoteniia [Motion of an artificial satellite in a non-Central gravitational field].* Moscow: Nauka [in Russian].

19 Tleubergenov, M.I., & Ibraeva G.T. (2013). Main inverse problem for differential system with generate diffusion *Ukrainian Mathematical Journal, 65, 5,* 787–792.

20 Tleubergenov, M.I. (2014). On the inverse stochastic reconstruction problem. *Differential Equations, 50, 2,* 274–278.

21 Tleubergenov, M.I., & Ibraeva, G.T. (2017). Stochastic inverse problem with indirect control. *Differential Equations, 53, 10,* 1387–1391.

22 Vassilina, G.K., & Tleubergenov, M.I. (2016). Solution of the Problem of Stochastic Stability of an Integral Manifold by the Second Liapunov Method. *Ukrainian Mathematical Journal, 68, 1,* 14–28.

23 Vasilina, G.K., & Tleubergenov, M.I. (2018). On the optimal stabilization of an integral manifold. *Journal of Mathematical Sciences, 229, 4,* 390—402. Retrieved from http://DOI 10.1007/s10958-018-3684-5.

24 Rusch, N., Abets, P., & Lalua, M. (1980). *Priamoi metod Liapunova v teorii ustoichivosti [Lyapunov's direct method in stability theory].* Moscow: Mir [in Russian].

25 Khas'minskii, R.Z. (1969). *Ustoichivost sistem differentsialnykh uravnenii pri sluchainykh vozmushcheniiakh ikh parametrov [Stability of differential equations systems with random perturbations of their parameters].* Moscow: Nauka [in Russian].

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Small models of hybrids for special subclasses of Jonsson theories

This article presents the result related to the model-theoretic properties of special subsets of the semantic model of some fixed Jonsson theory. The specialty of these sets is due to their definability and closure. Further, we consider fragments of these sets and create a hybrid of these fragments from them. The theory is subject to conditions of strongly convexity and existentially coreness. As a result, the class of existentially closed and algebraically prime models in its non-empty intersection contains a core model. By module of the above conditions, the hybrid of the considered fragments has a model that contains a special core subset, definable closure of which gives a certain existentially closed model, which is an algebraic prime model of the theory under consideration.

Keywords: Jonsson theory, semantic model, hybrid, existentially prime theory, pregeometry, model companion.

One of the most important aspects of the classical model theory is the description of countable models of the considered theory. In this article, we propose to consider this topic in the framework of the intersection of two issues. Namely, with the help of special subsets of the semantic model of a fixed Jonsson theory, consider hybrids of fragments of these sets and further, using the principle of «Rheostat» for these hybrids, study some properties of small models of these hybrids. Thus, in the intersection there are issues related to the theme of hybrids and the principle of Rheostat when working with small models. These topics were considered in the following articles [1–5].

We give the necessary definitions of the basic concepts of this article.

Let given a countable language of the first order.

The following definition describes one of the basic concepts of this article.

Definition 1. A theory T is Jonsson if:

1) Theory T has infinite models;
2) Theory T is inductive;
3) Theory T has the joint embedding property (JEP);
4) Theory T has the property of amalgam (AP).

Examples of Jonsson theories are:

1) Group Theory;
2) Theory of Abelian groups;
3) Theory of fields of fixed characteristics;
4) Theory of Boolean algebras;
5) Theory of polygons over a fixed monoid;
6) Theory of modules over a fixed ring;
7) Theory of linear order.

The following definition of the universality and homogeneity of model allocates semantic invariant of any Jonsson theory, namely its semantic model. Moreover, it turned out that the saturation or non-saturation of this model significantly changes the structural properties of both the Jonsson theory itself and its class of models.

Definition 2. Let $\kappa \geq \omega$. Model M of theory T is called κ-universal for T, if each model T with the power strictly less κ isomorphically imbedded in M; κ-homogeneous for T, if for any two models A and A_1 of theory T, which are submodels of M with the power strictly less then κ and for isomorphism $f : A \to A_1$ for each extension B of model A, wich is a submodel of M and is model of T with the power strictly less then κ there is exist the extension B_1 of model A_1, which is a submodel of M and an isomorphism $g : B \to B_1$ which extends f.

Definition 3. Model C of Jonsson theory T is called semantic model, if it is ω^+-homogeneous-universal.

As can be seen from the definition of the Jonsson theory, this theory is not complete. But nevertheless, with the help of its semantic invariant (semantic model) we can always determine the center of Jonsson theory, which is a complete theory.

Definition 4. The center of Jonsson theory T is called an elementary theory of its semantic model. And denoted through T^*, i.e. $T^* = Th(C)$.

The following two facts speak about the «good» exclusivity of the semantic model.

Fact 1 [6; 160]. Each Jonsson theory T has k^+-homogeneous-universal model of power 2^k. Conversely, if a theory T is inductive and has infinite model and ω^+-homogeneous-universal model then the theory T is a Jonsson theory.

Fact 2 [6; 160]. Let T is a Jonsson theory. Two k-homogeneous-universal models M and M_1 of T are elementary equivalent.

It is well known from the course of model theory that a saturated model is always a homogeneous-universal model, the reverse is also true. But this definition of homogeneous-universal model [7; 299] is considered as a rule in the framework in the study of complete theory. In the framework of the study of Jonsson theory, we are dealing with a particular case of the definition of a homogeneous-universal model belonging to B. Jonsson [8]. The concept of a saturated model is the same in both cases. By virtue of a more general situation of homogeneous-universality in the case of Jonsson theory, we do not have a saturation criterion in terms of homogeneous-universal as in [7, 299]. Therefore, those Jonsson theories, the semantic model of which is saturated, allocate in a special subclass of class of all Jonsson theories, and such theories are called perfect. We give a definition of perfectness of Jonsson theory.

Definition 5. Jonsson theory T is called a perfect theory, if each semantic model of theory T is saturated model of T^*.

The first author of this article obtained a result describing the perfect Jonsson theory.

Theorem 1 [6, 158]. Let T is a Jonsson theory. Then the following conditions are equivalent:

1) Theory T is perfect;

2) Theory T^* is a model companion of theory T.

From the above list of Jonsson theories, the following examples 2), 3), 4), 6), 7) are examples of a perfect Jonsson theory. But, for example, group theory is not such, due to the fact that it does not have a model companion.

Let E_T be a class of all existentially closed models of Jonsson theory T.

This class of models in general case for an arbitrary theory can be empty. But the following result [9, 367] is well known, which says that any inductive theory has a nonempty class of existentially closed models. Since the Jonsson theory is a subclass of class of inductive theories, we can say that E_T is a non-empty class.

In the case of a perfect Jonsson theory, the class of models of center of this theory coincides with E_T. This follows from the following theorem.

Theorem 2 [6; 162]. If T be a perfect Jonsson theory then $E_T = ModT^*$.

Let L be a countable language of first order. Let T be Jonsson theory in the language L and its semantic model is C.

Let T be some Jonsson theory in a fixed language and C — its semantic model.

Definition 6. Let $X \subseteq C$. We will say that a set X is $\nabla - cl$-Jonsson subset of C, if X satisfies the following conditions:

1) X is ∇-definable set (this means that there is a formula from ∇, the solution of which in the C is the set X, where $\nabla \subseteq L$, that is ∇ is a view of formula, for example $\exists, \forall, \forall\exists$ and so on.);

2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator defining a pregeometry [10; 289] over C (for example $cl = acl$ or $cl = dcl$).

Lemma 1. Let T be Jonsson theory, E_T be the class of its existentially closed models. Then for any model $A \in E_T$ the theory $Th_{\forall\exists}(A)$ is a Jonsson theory.

Proof can be extract from [6].

Let X_1, X_2 be ∇-cl-Jonsson subset of C, where C is semantic model of theory T. Let $M_1 = cl(X_1)$, $M_2 = cl(X_2)$, where $M_1, M_2 \in E_T$. $Th_{\forall\exists}(M_1) = T_1$, $Th_{\forall\exists}(M_2) = T_2$. C_1 is semantic model of theory T_1, C_2 is semantic model of theory T_2.

We define the essence of the operation of the symbol $\boxdot$ for algebraic construction of models, which will play important role in the definition of hybrids. Let $\boxdot \in \{\cup, \cap, \times, +, \oplus, \prod_F, \prod_U\}$, where $\cup$-union, $\cap$-intersection, $\times$-Cartesian product, $+$-sum and $\oplus$-direct sum, $\prod_F$-filtered product and $\prod_U$-ultraproduct.

The following definition gives a hybrid of the first type [5] for two Jonsson theories.

Definition 7. A hybrid $H(T_1, T_2)$ of Jonsson theories T_1, T_2 is called the theory $Th_{\forall\exists}(C_1 \boxdot C_2)$, if it is Jonsson. Herewith, the algebraic construction $(C_1 \boxdot C_2)$ is called a semantic hybrid of the theories T_1, T_2.

Note the following fact:

Fact 3. For the theory $H(T_1, T_2)$ in order to be Jonsson enough to be that their semantic hybrid $(C_1 \boxdot C_2) \in E_T$.

Proof. This follows by Lemma 1.

By small models we mean different types of countable models of a fixed theory of a countable language. These types of models include the following: minimal, core, rigid, atomic, prime, existentially closed, and their various generalizations about this fixed theory. For example, the concept of an algebraically prime model is a generalization of a prime model. In [11], various types of generalization of the atomic model were considered, and in this paper we interpolate these generalizations using the «Rheostat» principle. Below is a list of definitions of various types of small models, as well as special cases of Jonsson sets different closures [12], which are special formula subsets of the semantic model of the considered theory. The definable closures of these subsets will define special types of models and for the corresponding mutually model consistent fixed fragments we will consider hybrids of these fragments and study small models.

When studying the model-theoretic properties of an inductive theory, so called existentially closed models [6] play an important role and for inductive theories, the class of existentially closed models is never empty. Recall it definition.

Definition 8. Model A of a theory T is said to be existentially closed if for any model B and any existential formula $\varphi(\overline{x})$ with constants of A we have $A \models \exists\overline{x}\varphi(\overline{x})$ provided that A is a submodel of B and $B \models \exists\overline{x}\varphi(\overline{x})$.

An analogue of a prime model (in the sense of a complete theory) for an inductive model, generally speaking, incomplete theory, is the concept of an algebraically prime model, which introduced A. Robinson [12].

Definition 9. A is an algebraically prime model of theory T, if A is a model of T and A may be isomorphically embedded in each model of the theory T.

Next definition of existentially prime theory is introduced by A.R. Yeshkeyev in the work [6].

Definition 10. The inductive theory T is called the existentially prime if: it has an algebraically prime model, the class of its AP (algebraically prime models) denote by AP_T; class E_T non trivial intersects with class AP_T, i.e. $AP_T \bigcap E_T \neq 0$.

The following definition of convex theory belongs to A. Robinson [13].

Definition 11. The theory T is said to be convex if for any its model A and for any family $\{B_i \mid i \in I\}$ of substructures of A, which are models of the theory T, the intersection $\bigcap_{i\in I} B_i$ is a model of T, provided it is non-empty. If in addition such an intersection is never empty, then T is said to be strongly convex.

The following definition is taken from J. Baldwin and D. Kueker's work [11].

Definition 12. B is a $(\Gamma_1, \Gamma_2,)$ - atomic model of T, if B is a model of T and for every n every n-tuple of elements of A satisfies some formula from B in Γ_1, which is complete for Γ_2-formulas.

The following definitions 13 and 14 about of various types of small models, as well as special cases of Jonsson sets [12].

Definition 13. A set A is said to be $(\nabla_1, \nabla_2) - cl$ atomic in the theory T, if

1) $\forall a \in A, \exists \varphi \in \nabla_1$ such that for any formula $\psi \in \nabla_2$ follows that φ is complete formula for ψ and $C \models \varphi(a)$;

2) $cl(A) = M, M \in E_T$, and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$ atomic model of theory T.

Definition 14. A set A is said to be $(\nabla_1, \nabla_2) - cl$-algebraically prime in the theory T, if

1) A is $(\nabla_1, \nabla_2) - cl$-atomic set in T;

2) $cl(A) = M, M \in AP_T$,

and obtained model M is said to be $(\nabla_1, \nabla_2) - cl$ algebraically prime model of theory T.

The following definition of model consistent theory from work [14] is well known.

Definition 15. T and T^* are mutually model consistent, i.e. every model of theory T is embeddable in a model of theory T^*, and vice versa.

To obtain the result, the following definition was introduced by the first author of this article.

Definition 16. The theory is called existentially core if this theory is existentially prime and among existentially closed algebraically prime models there is at least one core model.

In order to define needed for us special types of atomicity of certain models we have used the following idea. It is the so-called principle of «rheostat» [12]. Let us recall that idea.

Principle of «rheostat».

Let two countable models A_1, A_2 of some Jonsson theory T be given. Moreover, A_1 is an atomic model in the sense of [1], and X is $(\nabla_1, \nabla_2) - cl$ -algebraically prime set of theory T and $cl(X) = A_2$. Since $\nabla_1 = \nabla_2 = L$, then $A_1 \cong A_2$.

By the definition of (∇_1, ∇_2) - algebraic primeness of the set X, the model A_2 is both existentially closed and algebraically prime. Thus, the model A_2 is isomorphically embedded in the model A_1. Since by condition the model A_1 is countably atomic, then according to the Vaught's theorem, A_1 is prime, i.e. it is elementarily embedded in the model A_2. Thus, the models A_1, A_2 differ from each other only by the interior of the set X. This follows from the fact that any element of $a \in A_2 \backslash X$ implements some principal type, since $a \in cl(X)$. That is, all countable atomic models in the sense of [15] are isomorphic to each other, then by increasing X we find more elements that do not realize the principal type and, accordingly, $cl(X)$ is not an atomic model in the sense of [15]. Thus, the principle of rheostat is that, by increasing the power of the set X, we move away from the notion of atomicity in the sense of [15] and on the contrary, decreasing the power of the set X we move away from the notion of atomicity in the sense of [11].

Let $AAPC \in$ {atomic, algebraically prime, core}. Thus, by specifying the set X as $(\nabla_1, \nabla_2) - cl - AAPC$, (where APC is a semantic property), we can also specify atomicity in the sense [11] in relation to atomicity (A) in the sense of [15]. And accordingly, using of the principle of «rheostat» after the $AAPC$ property is defined, we obtain the corresponding concepts of atomic models, the role of which is played A_2 from the principle of «rheostat».

The main result is the following theorem.

Let ∇ from the definition 6, such that $\nabla \in \{\forall, \exists, \forall\exists\}$.

Theorem 1. Let T be perfect, strongly convex, existentially core, complete for $\forall\exists$-sentences Jonsson theory. X_1, X_2 are (∇, ∇)-cl-atomic sets of the theory T, where $dcl(X_1) = M_1$, $dcl(X_2) = M_2$, $M_1, M_2 \in E_T$. $Th_{\forall\exists}(M_1) = T_1, Th_{\forall\exists}(M_2) = T_2$. C_1, C_2 are semantic models of the theories T_1, T_2, correspondingly. T_1 is model consistent with T_2 and with T. Let $H(T_1, T_2) = Th_{\forall\exists}(C_1 \times C_2)$.

If M_1, M_2 are (Σ, Σ)-cl-atomic models, then there exist $M \in E_{H(T_1,T_2)}$ such that M is countable model and $M \in AP_{H(T_1,T_2)}$ and M is (∇, ∇)-cl-algebraically prime model.

Proof. The conditions in the head of the theorem are independent. That is there are perfect, strongly convex, existentially core, complete for $\forall\exists$-sentences in all possible different combinations among themselves. Therefore, all these conditions are fulfilled at the same time distinguishing a fairly good class, generally speaking incomplete theories, among the Jonsson theories.

It is well known from [16] that convex theories satisfy the following property: for any model of such a theory, its any non-empty subset generates a unique submodel of this theory. Namely, this model is the intersection of all models of this theory that contain a given set. If T is strongly convex, then the intersection of all T models contained in this T model is also a T model. This intersection is called the T core model.

If T satisfies the joint embedding property to the fact that this theory is strongly convex, then its core model is unique up to isomorphism. That is core model has the property that it is isomorphic to exactly one submodel of each T model, and is uniquely defined as the largest model with this property.

Using the principle of a rheostat, we can see that the larger the set X_1, X_2 in power, the more existentially closed models M_1, M_2 of the theory T differ from being atom models of this theory in the sense of [15]. But as T_1 and T_2 are model consistent with T, any model from $ModT_i$, where $i \in \{1, 2\}$ is isomorphically embedded into some $N \in E_T$, where $n \subseteq C$.

Consider the algebraic construction $C_1 \times C_2$, we denote it as $D = \{\langle a, b \rangle : a \in C_1, b \in C_2\}$. Since T is a Jonsson theory, then due to its axiomatizable the class of its models is closed with respect to isomorphic images, which means that we can conclude that without loss of generality $C_1, C_2 \in ModT$. Due to the strong convexity of the theory T, $C_1 \cap C_2 \neq \emptyset$ and let $C_1 \cap C_2 = F \in ModT$. Moreover, due to the existentially corecity of the theory T there is a core model F' of the theory T, which is an existentially closed submodel of F. Since T is an axiomatizable theory and $C_1, C_2 \in ModT$ models, we conclude that $C_1 \times C_2 \in ModT$. Due to the model consistency of T, T_1, T_2, we can conclude, without loss of generality, that $C_1 \times C_2 \in ModT_i$, where $i \in \{1, 2\}$. Then, by virtue of Theorem 3.2 b) of [11] and the above, the model is $F' \in E_{T_i}$, where $i \in \{1, 2\}$. Then, consider $Th_{\forall\exists}(C_1 \times C_2)$, due to the model consistency of T, T_1, T_2, and $\forall\exists$-completeness T is Jonsson theory, which is $H(T_1, T_2)$. It is easy to see that $H(T_1, T_2)$ is model consistency with T theory, then for any two N_1, N_2 - (Σ, Σ) - cl - atomic models there is a model $K \in E_{H(T_1,T_2)}$, which is (∇, ∇) - cl - algebraic prime, as K we can take F'.

All concepts that are not defined in this article and connected information with these statements can be extracted from [6].

References

1 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 80. — No. 4. —P. 47–51.

2 Poizat B. Positive Jonsson Theories / B. Poizat, A.R. Yeshkeyev // Logica Universalis. — 2018. — Vol. 12. — No. 1–2. — P. 101–127.

3 Yeshkeyev A.R. Properties of lattices of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 25–32.

4 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 33–39.

5 Yeshkeyev A.R. Properties of hybrids of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University. Mathematics series. — 2018. — Vol. 92. — No. 4. — P. 99–104.

6 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. — 370 с.

7 Кейслер Х.Дж. Теория моделей / Х.Дж. Кейслер, Ч.Ч. Чэн. — М.: Мир, 1977. — 614 с.

8 Jonsson B. Homogeneous universal relational systems / B. Jonsson. — Math. Scand, 1960. — P. 137–142.

9 Hodges W.A. Model Theory / W.A. Hodges. — Cambridge University Press, 1993. — P. 772.

10 Marker D. Model Theory: In introduction / D. Marker. — Springer-Verlag New York. Inc., 2002. — P. 342.

11 Baldwin J.T. Algebraically prime models / J.T. Baldwin, D.W. Kueker // Ann. Math. Logic. — 1981. — 20. — P. 289–330.

12 Yeshkeyev A.R. The atomic definable subsets of semantic model / A.R. Yeshkeyev, A.K. Issayeva, N.M. Mussina // Bulletin of the Karaganda University. Mathematics series. — 2019. — Vol. 94. — No. 2. — P. 84–91.

13 Robinson A. Introduction to Model Theory and to the Mathematics of Algebra / A. Robinson. — Amsterdam, 1963.

14 Барвайс Дж. Справочная книга по математической логике: [В 4 ч.]. — Ч. 1. Теория моделей / Дж. Барвайс; пер. с англ. — М.: Наука; Гл. ред. физ.-мат. лит., 1982. — 126 с.

15 Vaught R. Denumerable models of complete theories in Infinitistic Methode / R. Vaught // Pergamon. — London, 1961. — P. 303–321.

16 Kueker D.W. Core structures for theories / D.W. Kueker // Fundamenta Mathematicae LXXXIX (1973). — P. 154–171.

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Йонсондық теориялардың арнайы ішкі кластары гибридтерінің кішігірім модельдері

Мақалада кейбір бекітілген йонсондық теорияның семантикалық моделінің арнайы ішкі жиындардың модельді-теоретикалық қасиеттерімен байланысты нәтижесі берілген. Осы жиындардың арнайылығы олардың анықталуымен және тұйықтылығымен келісілген. Әрі қарай осы жиындардың фрагменттері қарастырылды және осы фрагменттерден гибрид жасалды. Теорияға қатты дөңестілік және экзистенциалды ядролық шарттар қойылды. Нәтижесінде экзистенциалды тұйық және алгебралық жай модельдер класы өздерінің құр емес қиылысуында ядролық модельді құрады. Жоғарыда көрсетілген шарттардың модулі бойынша қарастырып отырған фрагменттің арнайы ядролық ішкі жиынды құрайтын моделі бар, оның анықталған тұйықтамасы қарастырып отырған теорияның алгебралық жай моделі болатын кейбір экзистенциалды тұйық модельді берді.

Кілт сөздер: йонсондық теория, семантикалық модель, гибрид, экзистенциалды жай теория, модельді компаньон.

А.Р. Ешкеев, Н.М. Мусина

Малые модели гибридов специальных подклассов йонсоновских теорий

В статье представлен результат, связанный с теоретико-модельными свойствами специальных подмножеств семантической модели некоторой фиксированной йонсоновской теории. Специальность этих множеств обусловлена их определимостью и замыканием. Рассмотрены фрагменты этих множеств и из них создан гибрид фрагментов. На теорию накладываются условия сильно выпуклости и экзистенциально ядерности. В результате класс экзистенциально замкнутых и алгебраически простых моделей в своем непустом пересечении содержит ядерную модель. По модулю указанных выше условий гибрид рассматриваемых фрагментов имеет модель, которая содержит специальное ядерное подмножество, определимое замыкание которого дает некоторую экзистенциально замкнутую модель, являющуюся алгебраической простой моделью рассматриваемой теории.

Ключевые слова: йонсоновская теория, семантическая модель, гибрид, экзистенциально простая теория, модельный компаньон.

References

1 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University. Series Mathematics, Vol. 80, 4,* 47–51.

2 Poizat, B., & Yeshkeyev, A.R. (2018). Positive Jonsson Theories. *Logica Universalis, Vol. 12, 1–2,* 101–127.

3 Yeshkeyev, A.R., & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Series Mathematics, Vol. 79, 3,* 25–32.

4 Yeshkeyev, A.R., & Ulbrikht, O.I.(2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Series Mathematics, Vol. 79, 3,* 33–39.

5 Yeshkeyev, A.R., & Mussina, N.M. (2018). Properties of hybrids of Jonsson theories. *Bulletin of the Karaganda University. Series Mathematics, Vol. 92, 4,* 99–104.

6 Yeshkeyev, A.R., & Kasymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo KarHU [in Russian].

7 Keisler, H.J., & Chang, C.C. (1977). *Model Theory.* Elsiver science publishing company.

8 Jonsson, B. (1960). *Homogeneous universal relational systems.* Math. Scand., 137–142.

9 Hodges, W.A. (1993). *Model Theory.* Cambridge University Press.

10 Marker, D. (2002). *Model Theory: In introduction.* Springer-Verlag New York, Inc.

11 Baldwin, J.T. & Kueker, D.W. (1981). Algebraically prime models. *Ann. Math. Logic., 20,* 289–330.

12 Yeshkeyev, A.R., Issayeva, A.K., & Mussina, N.M. (2019). The atomic definable subsets of semantic model. *Bulletin of the Karaganda University. Series Mathematics, Vol. 94, 2.*

13 Robinson, A. (1963). Introduction to Model Theory and to the Mathematics of Algebra. Amsterdam.

14 Barwise, J. (1982). *Spravochnaia kniha po matematicheskoi lohike [Handbook of mathematical logic. Model theory].* (Part 1). Moscow: Nauka [in Russian].

15 Vaught, R. (1961) Denumerable models of complete theories in Infinitistic Methode. *Pergamon.* London.

16 Kueker, D.W. (1973). Core structures for theories. *Fundamenta Mathematicae LXXXIX,* 154–171.

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Connection of Jonsson theory with some Jonsson polygons theories

The main result of this article is connected with some Jonsson theory of polygons and is devoted to obtaining a description of the characteristic of such polygons using invariants related to them. With this limited consideration (the group is a strong particular case of a monoid) were obtained for such a theory of the property of perfection and Jonsson property. Thus, for some existential complete and perfect Jonsson theories, there is the same Jonsson theory of polygons, in which two Jonsson theories are syntactically similar.

Keywords: Jonsson theory, semantic model, perfect theory, polygon, model companion, syntactic and semantic similarity of Jonsson theories.

In this article we will consider the connection of the Jonsson theory with a certain Jonsson theory of polygons. In particular, the Jonsson theory will be existentially complete and perfect. These restrictions are connected with the fact that the Jonsson theories are generally not complete and existential completeness is the minimum possible requirement for formulas with quantifiers. On another hand, in the imperfect case, we have very few algebraic examples to achieve any more descriptive results. The most striking example of imperfect Jonsson theory is the theory of groups, for which it has been proven that it does not have a model companion. All our previous achievements in the field of the study of Jonsson theories have been associated with the technique of working on a model companion of the theory under consideration [1–4]. The main idea that prompted us to write this article was the idea of transferring the syntactic similarity of the complete theories from [5] to the syntactic similarity of Jonsson theories [6–8]. In [5], it was shown that any complete theory is syntactically similar to a certain polygon (i.e, the theory of this polygon). In English literature, the term polygon over the S monoid usually uses the term $S-acts$ [9–12]. In this article, we follow the terminology of Professor T.G. Mustafin, whom he first defined and formulated model-theoretical concepts and issues related to polygons [13–16].

We give a list of the necessary definitions of concepts and their necessary model-theoretical properties. The following definition belongs to T.G. Mustafin [5].

Definition 1. Let T_1 and T_2 be complete theories. We will speak, as T_1 and T_2 are syntactically similar, if $f : F(T_1) \to F(T_2)$ exists bijection such that:

1) restriction f to $F_n(T_1)$ is isomorphism of Boolean algebras $F_n(T_1)$ and $F_n(T_2)$, $n < \omega$;

2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$,$n < \omega$;

3) $f(v_1 = v_2) = (v_1 = v_2)$.

Let T be an arbitrary Jonsson theory, then $E(T) = \bigcup_{n<\omega} E_n(T)$, where $E_n(T)$ is a lattice of $\exists$-formulas with n free variables, T^* is a center of Jonsson theory T, i.e. $T^* = Th(C)$, where C is semantic model of Jonsson T theory in the sense of [14].

The following definition was introduced by first author of this article.

Definition 2. Let T_1 and T_2 are arbitrary Jonsson theories. We say, that T_1 and T_2 are Jonsson's syntactically similar, if exists a bijection $f : E(T_1) \longrightarrow E(T_2)$ such that:

1) restriction f to $E_n(T_1)$ is isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;

2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;

3) $f(v_1 = v_2) = (v_1 = v_2)$.

Definition 3. The Jonsson theory of T is called perfect if every semantic model of T is a saturated model of T^*.

Definition 4. By a polygon over a monoid S (or we called as $S-acts$) we mean a structure with only unary functions $\langle A; f_\alpha : \alpha \in S\rangle$ such that:

1) $f_e(a)$, $\forall a \in A$, where e is the unit of S;

2) $f_{\alpha\beta}(a) = f_\alpha(f_\beta(a))$, $\forall \alpha, \beta \in S$, $\forall a \in A$.

Definition 5. A theory of T is called model complete if, for any model of A and B of theory T, such that $A \subseteq B \Rightarrow A \preceq B$. Equivalently, every isomorphic embedding is an elementary embedding.

Consider the concepts of model completeness, model companion and model consistency theory T [17].

Definition 6. T and T_1 is mutually model consistent, i.e. any model of the theory T is embedded in the model of the theory T_1. And simultaneously the inverse statement is true.

Definition 7. Let T, T^* be L-theories. T^* is a model completion of T if:

(1) T and T^* are mutually model consistent;

(2) T^* is model complete;

(3) If $\mathfrak{M}$, then $T^* \cup D(\mathfrak{M})$ is complete.

T^* is a model companion of T if (1) and (2) hold.

Theorem 1 [5]. For every T there is a polygon theory T_Π that if the signature T is finite, then T_Π is a hull of T, and otherwise - almost a hull T.

Theorem 2 [18]. Let T be Jonsson theory. Then the following conditions are equivalent:

1) T is perfect;

2) T has a model companion.

In the article [19] polygons over S was considered, where S is a group. The main result of this article is obtaining the description of characterization of such polygons with the help of concerning invariants. Under this restricted consideration (a group is strong particular case of a monoid) was obtained the properties of perfectness and Jonssoness. By an occasion we paid attention from this work with the following lemma:

Lemma [19]. Let T be an α — Jonsson theory and all the completions of T admit the elimination of quantifiers. Then

1) T is perfect;

2) T is 0-Jonsson.

And as a consequences of this lemma for us the point 1 from the following theorem is important.

Theorem 3 [19]. 1) Each α -Jonsson theory of polygons is perfect and it is a Jonsson theory for all α, $0 \leq \alpha \leq \omega$.

It is not surprised because from the article [20] follows that even for regular polygon there are non model complete regular polygons. It means that there are sufficient many non saturated semantic models of polygons.

Earlier the first author of this work have got the link between both kind of similarities for existentially complete and perfect two Jonsson theories:

Theorem 4 [21]. Let T_1 and T_2 be $\exists$ — complete perfect Jonsson theories. Then following conditions are equivalent:

1) T_1 and T_2 are Jonsson's syntactically similar;

2) T_1^* and T_2^* are syntactically similar (in the sense of [5]).

Essentially that the centers of Jonsson theories are complete theories and we can proceed to transfer the elementary properties of centers onto Jonsson theories. Eventually it is sufficient to check implementation of similarity for Jonsson theories over lattices of existentially formulas.

The main result of this article is the following theorem.

Theorem 5. Let T be $\exists$ - complete perfect Jonsson theory $\exists$ - complete perfect then there is T' - $\exists$ - complete perfect Jonsson theory of polygons, such that:

Theory T will be J syntactically similar to theory T'.

Proof. For proof, we note the following facts:

1) If Jonsson theory T is perfect, its center $T^* = Th(C)$ is Jonsson theory, where C is a semantic model of T. The proof can be extracted from the fact that Jonsson theory is perfect if and only if its center T^* is a model companion of T.

2) If two complete theories T_1 and T_2 are syntactically similar to each other in the sense of [5], then they are semantically similar to each other in the sense of [5]. The proof can be extracted from [5].

3) Semantically similar theories in the sense of [5] are invariant regarding to the first order semantic properties, which include the formula properties of elements and a subset of models of these theories.

Let us recall the definition of semantic property [5].

Definition 8. A property (or a notion) of theories (or models, or elements of models) is called semantic if and only if it is invariant relative to semantic similarity.

For example from [5] it is known that:

Proposition 1 [5]. The following properties and notions are semantic:
(1) type;
(2) forking;
(3) λ-stability;
(4) Lascar rank;
(5) Strong type;
(6) Morley sequence;
(7) Orthogonality, regularity of types;
(8) $I(\aleph_\alpha, T)$ - the spectrum function.
4) Being a Jonsson theory is a semantic property. The proof follows from [5, 15].

Since for any complete theory from Theorem 1 there follows the existence of some complete theory of polygons, we can consider in particular some complete Jonsson theory T_1^* that satisfies the condition of Theorem 4, i.e T_1^* is the center of some $\exists$ - complete perfect Jonsson theory T_1. Then T_1^* is syntactically similar in the sense of [5] to some complete theory of polygons T_Π. By virtue of the above facts 1–4 and Theorems 1, 4, we can conclude that T_Π is a complete and model complete Jonsson theory. Then there is some Jonsson theory T'_Π that satisfies the condition of Theorem 4 and T_Π is the center of the theory T'_Π. Then, by virtue of Theorem 4, we can conclude that T'_Π is the desired Jonsson theory of polygons satisfying Theorem 5, q.e.d.

All undetermined concepts in this article can be extracted from [21].

References

1 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 80. — No. 4. — P. 47–51.

2 Poizat B. Positive Jonsson Theories / B. Poizat, A.R. Yeshkeyev // Logica Universalis. — 2018. — Vol. 12. — No. 1-2. — P. 101–127.

3 Yeshkeyev A.R. Properties of lattices of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 25–32.

4 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 33–39.

5 Mustafin T.G. On similarities of complete theories / T.G. Mustafin // Logic Colloquium '90. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic. — Helsinki, 1990. — P. 259–265.

6 Ешкеев А.Р. О подобиях в йонсоновских теориях / А.Р. Ешкеев // Материалы 10-й Межвуз. конф. по математике и механике. — Алматы, 2005. — Т. 1. — С. 182–184.

7 Ешкеев А.Р. Подобие теорий предметных областей / А.Р. Ешкеев // Знания – Онтологии — Теории: (ЗОНТ-09) ИМ СО РАН: материалы Всерос. конф. — Новосибирск, 2009. — С. 207–212.

8 Yeshkeyev A.R. On similarities of $\Delta - PJ$ — theories in enriched signature / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2011. Vol. 62, — No. 4. — P. 54–57.

9 Гоулд В. Теоретико-модельные свойства свободных, проективных и плоских S-полигонов / В. Гоулд, А.В. Михалев, Е.А. Палютин, А.А. Степанова // Фундаментальная и прикладная математика. — 2008. — Т. 14, № 7. — С. 63–110.

10 Бунина Е.И. Элементарные свойства категории полигонов над моноидом / Е.И. Бунина, А.В. Михалев // Сиб. фонд «Алгебра и логика». — 2006. — Т. 45, № 6. — С. 687–709.

11 Бунина Е.И. Элементарная эквивалентность моноидов эндоморфизмов свободных полигонов / Е.И. Бунина, А.В. Михалев // Чебышевский сб. — 2005. — Т. 6, № 4. — С. 49–63.

12 Михалев А.В. Теоретико-модельные свойства регулярных полигонов / Е.В. Овчинникова, Е.А. Палютин, А.А. Степанова // Фундамент. и прикл. матем. — 2004. — Т. 10, № 4. — С. 107–157.

13 Мустафин Т.Г. Введение в прикладную теорию моделей / Т.Г. Мустафин, Т.А. Нурмагамбетов. — Караганда: Изд-во КарГУ, 1987. — 94 с.

14 Mustafin Y. Quelques proprietes des theories de Jonsson / Y. Mustafin // The Journal of Symbolic Logic. — 2002. — Vol. 67. — No. 2. — P. 528–536.

15 Мустафин Т.Г. Обобщенные условия Йонсона и описание обобщенно-йонсоновских теорий булевых алгебр / Т.Г. Мустафин // Матем. тр. — Новосибирск, 1998. — Т. 1. — № 1. — С. 135–197.

16 Мустафин Т.Г. О стабильной теории полигонов / Т.Г. Мустафин // Теория моделей и ее применения. — Новосибирск: Наука; Сиб. отд-ние, 1988. — С. 92–108.

17 Барвайс Дж. Справочная книга по математической логике: [В 4 ч.]. — Ч. 1. Теория моделей / Дж. Барвайс; пер с англ. — М.: Наука; Гл. ред. физ.-мат. лит., 1982. — 126 с.

18 Ешкеев А.Р. Йонсоновские теории. — Караганда: Изд-во КарГУ, 2009. — 250 с.

19 Мустафин Т.Г. Описание йонсоновских теорий полигонов над группой / Т.Г. Мустафин, Е.С. Нурхайдаров // Исследования в теории алгебраических систем: сб. науч. тр. (межвуз.). — Караганда: Изд-во КарГУ, 1995. — С. 67–73.

20 Овчинникова Е.В. Не линейно упорядоченный моноид, над которым класс регулярных полигонов полон, но не модельно полон / Е.В. Овчинникова // Алгебра и теория моделей. 3: сб. тр. — Новосибирск: Изд-во НГТУ, 2001. — С. 83–98.

21 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. — 370 с.

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Кейбір йонсондық теория полигондарымен йонсондық теорияның байланысы

Мақаланың негізгі нәтижесі кейбір йонсондық теория полигонымен йонсондық теорияның байланысын және инварианттарға қатысты полигондардың көмегімен алынған сипаттамасы қарастырылды. Шектелген қарастыруда (группа моноидтың қатты дербес жағдайы болып табылады) теория үшін кемелділік және йонсондық қасиеттері алынды. Сонымен қатар экзистенциалды толық кемел йонсондық теориясы үшін кейбір экзистенциалды толық кемел йонсондық теорияның полигоны табылып, осы екі теорияның йонсондық синтаксистік ұқсас болатындығы көрсетілді.

Кілт сөздер: йонсондық теория, семантикалық модель, кемел теория, полигон, модельді компаньон, йонсондық теориялардың синтаксистік және семантикалық ұқсастылығы.

А.Р. Ешкеев, Г.А. Уркен

Связь йонсоновской теории с некоторой йонсоновской теорией полигонов

Основной результат данной статьи связан с некоторой йонсоновской теорией полигонов и посвящен получению описания характеристики таких полигонов с помощью относящихся к ним инвариантов. При этом ограниченном рассмотрении (группа является сильным частным случаем моноида) были получены для такой теории свойства совершенности и йонсоновости. Таким образом, для некоторых экзистенциально полных и совершенных йонсоновских теорий существует такая же йонсоновская теория полигонов, в которой две йонсоновские теории синтактически подобны.

Ключевые слова: йонсоновская теория, семантическая модель, совершенная теория, полигон, модельный компаньон, синтаксическое и семантическое подобия йонсоновских теорий.

References

1 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University. Mathematics series, Vol. 80, 4,* 47–51.

2 Poizat, B., & Yeshkeyev, A.R. (2018). Positive Jonsson Theories. *Logica Universalis, Vol. 12, 1-2,* 101–127.

3 Yeshkeyev, A.R., & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 25–32.

4 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 33–39.

5 Mustafin, T.G. (1990). On similarities of complete theories. *Logic Colloquium '90. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic.* Helsinki.

6 Yeshkeyev, A.R. (2005). O podobiiakh v ionsonovskikh teoriiakh [On similarities in Jonsson theories]. *Materialy 10-i Mezhvuzovskoi konferentsii po matematike i mekhanike, Vol. 1.* (pp. 182–184) [in Russian].

7 Yeshkeyev, A.R. (2009). Podobie teorii predmetnykh oblastei [The similarity of theories of subject areas]. *Znaniia — Ontolohii — Teorii: (ZONT-09) IM SO RAN: Materialy Vseros. konf.* (pp. 207–212). Novosibirsk [in Russian].

8 Yeshkeyev, A.R. (2011). On similarities of $\Delta - PJ$ — theories in enriched signature. *Bulletin of the Karaganda University. Mathematics series, Vol. 62, 2,* 54–57.

9 Gould, V., Mikhalev, A.V., Palyutin, E.A., & Stepanova, A.A. (2008). Teoretiko-modelnye svoistva svobodnykh, proektivnykh i ploskikh S-polihonov [Model-theoretic properties of free, projective, and flat S-acts]. *Fundamentalnaia i prikladnaia matematika, Vol. 14, 7,* 63–110 [in Russian].

10 Bunina, E.I., & Mikhalev, A.V. (2006). Elementarnye svoistva katehorii polihonov nad monoidom [Elementary properties of the category of polygons over a monoid]. *Sib. fond «Alhebra i lohika», Vol. 45, 6,* 687–709 [in Russian].

11 Bunina, E.I., & Mihalev, A.V. (2005). Elementarnaia ekvivalentnost monoidov endomorfizmov svobodnykh polihonov [Elementary equivalence of monoids of endomorphisms of free polygons]. *Chebishevskii sbornik, Vol. 6, 4,* 49–63 [in Russian].

12 Mikhalev, A.V., Ovchinnikova, E.V., Paliutin, E.A., & Stepanova, A.A. (2004). Teoretiko-modelnye svoistva rehuliarnykh polihonov [Model-theoretic properties of regular polygons]. *Fundamentalnaia i prikladnaia matematika, 10, 4,* 107–157 [in Russian].

13 Mustafin, T.G., & Nurmagambetov, T.A. (1987). Vvedenie v prikladnuiu teoriiu modelei [Introduction to applied Model Theory]. Karaganda: Izdatelstvo KarHU [in Russian].

14 Mustafin, Y. (2002). Quelques proprietes des theories de Jonsson. *The Journal of Symbolic Logic, Vol. 67, 2,* 528–536.

15 Mustafin, T.G. (1998). Obobshchennye uslovia Yonsona i opisaniie obobshchenno-ionsonovskikh teorii bulevykh alhebr [Generalized Jonsson's conditions and a description of generalized Jonsson theories of Boolean algebras]. *Matematicheskie trudy – Mathematical Works, Vol. 1, 1,* 135–197. Novosibirsk [in Russian].

16 Mustafin, T.G. (1988). O stabilnoi teorii polihonov [On the stable theory of polygons]. *Teoriia modelei i ee primeneniia.* Novosibirsk: Nauka; Sibirskaia otdelenie [in Russian].

17 Barwise J. (Ed.) (1982). *Spravochnaia kniha po matematicheskoi lohike. Chast 1. Teoriia modelei [Handbook of mathematical logic. Part 1. Model theory].* Moscow: Nauka [in Russian].

18 Yeshkeyev, A.R. (2009). Ionsonovskie teorii [Jonsson theory]. Karaganda: Izdatelsvo KarHU [in Russian].

19 Mustafin, T.G., & Nurkhaidarov, E.S. (1995). Opisanie ionsonovskikh teorii polihonov nad hruppoi [Description of the Jonsson polygon theories over the group]. *Issledovaniia v teorii alhebraicheskikh sistem. Sbornik nauchnykh trudov (mezhvuzovskii).* Karaganda: Izdatelstvo KarHU [in Russian].

20 Ovchinnikova, E.V. (2001). Ne lineino uporiadochennyi monoid, nad kotorym klass rehuliarnykh polihonov polon, no ne modelno polon [Non linearly ordered monoid over which the class of regular polygons is complete but not model is complete]. *Algebra i teoriia modelei.* Novosibirsk: Izdatelstvo NHTU [in Russian].

21 Yeshkeyev, A.R., & Kasymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo KarHU [in Russian].

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Bending of the rigidly fixed elastic-plastic plate

The number of plate structures include coatings, as well as box systems, working as a complete spatial system. The plate structures include systems elements of which are both rods and plates, for example, frame-panel structures with the bearing capacity of frame-panel buildings when considering the theory of bending. In the theory of elastic-plastic plates, the algorithm of calculation according to this theory, its basic formulas, the basic relations of the theory of elasticity are considered. This article discusses the bending of the rigidly fixed elastic-plastic plate. Calculations were made using the method of separation of variables. This method was implemented in the program MathCAD. Values of the laws of stress and modulus of materials, bending functions, shear components, deformation components, bending moment values, torsion and internal forces, stresses are obtained and plots of distribution functions are plotted. The values of the distribution functions of the rigidly fixed elastic-plastic plate are determined.

Keywords: plate, elastic, plastic, malleability, bending, deformation, stress.

Introduction

Plates of flagstones are called elastic cylindrical or prismatic bodies thickness of which is small comparing with two others in the size. Plate is widespread in building: panels, ferro-concrete or slab, the bases of the big buildings, flagstones, etc. Depending on foundation plates differ: round, pillow-like, triangular, quadrangular and polygonal. At present, plates are used in many technical branches: in building, in aircraft, in ship building, at assemblage of cars. Plates are the important designs in different branches of technics and in building. Therefore plate predesign is very important. To make or use any construction predesign is necessary. As the plate is the main element of many constructions predesign of bending and bearing ability is the important problem. To determine regularity of change of internal forces of elastic-plastic plates. In the calculation of the elastic, plastic, elastic-plastic plates with different bearing on the bending deflections and compare distributed functions [1, 2].

Main part

Consider a rigid-reinforced elastic-plastic plate of rectangular shape in a Cartesian coordinate system (l_1, l_2, h – plate dimensions in the direction of the coordinate axes) (Fig. 1) [3–11].

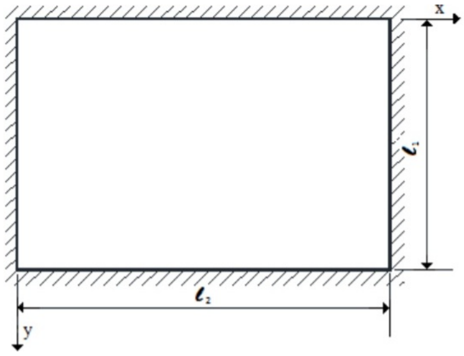


Figure 1. Rigidly fixed plate

Let it be acted q_0 on by a uniformly distributed load. The calculation is made using the method of separation of variables. This method will be implemented in the program MathCAD.

Given:

$$l_1 = 1,\ l_2 = 1,\ a_1 = 1,\ b_1 = 1,\ a_2 = 2,\ b_2 = 2,\ h = \frac{1}{2},\ E_0 = 2 \cdot 10^5,\ v = \frac{1}{4},\ q_0 = 1.$$

The boundary values of the parameters when rigidly fixed form will take the following form:

$$\alpha_0 = 0,\ \beta_0 = 0.$$

We calculate the dimensionless function of the plate bend:

$$f(x,\ y) = X(x) \cdot Y(y) = \frac{1}{24^2}\left(x^4 - 2x^3 + x^2\right) \cdot \left(y^4 - 2y^3 + y^2\right).$$

Bending parameters are defined as follows:

$$m = \frac{l_1}{l_2} = 1,\ I_1 = \frac{1}{720},\ I_2 = 0,\ J_1 = \frac{1}{720},\ J_2 = 0,$$

$$\alpha = \frac{1}{\frac{1}{m^2} \cdot J_1 + 2 \cdot I_2 \cdot J_2 + m^2 \cdot I_1} = 360.$$

Cylindrical plate stiffness:

$$D_0 = \frac{E_0 \cdot h^3}{12 \cdot (1 - v^2)} = \frac{20000}{9}.$$

The laws of stress and module functions of materials (Fig. 2):

$$\varphi(\xi) = \sin\left(\frac{\pi}{2} \cdot \xi\right),\ \ \psi(\xi) = \frac{\sin\left(\frac{\pi}{2} \cdot \xi\right)}{\xi},$$

where ξ - deformation parameter.

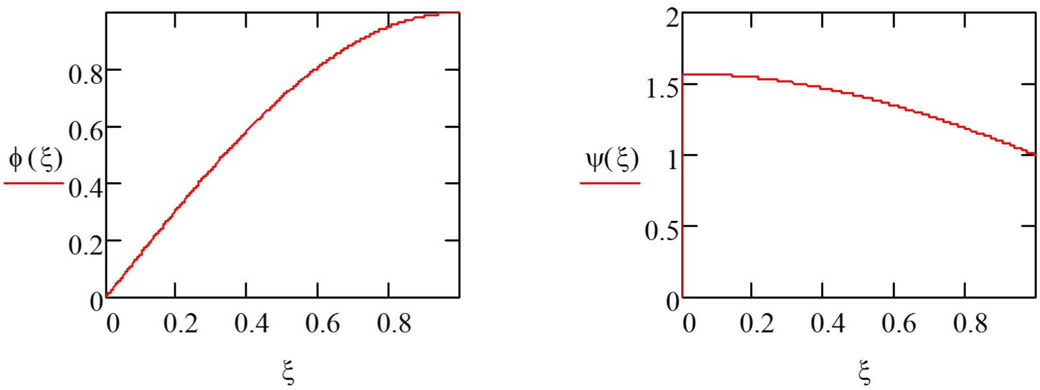


Figure 2. The laws of stress and module functions of materials

The value of the stress change laws and material functions of the module are shown in Table 1.

Table 1

The value of thestress change laws and functions of materials module

ξ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\varphi(\xi)$	0	0.156	0.309	0.454	0.588	0.707	0.809	0.891	0.951	0.988	1
$\psi(\xi)$	0	1.564	1.545	1.513	1.469	1.414	1.348	1.273	1.189	1.097	1

Using dimensionless function of bending plates, as follows: define the great bend feature $W(x, y, \xi)$ (Fig. 3):

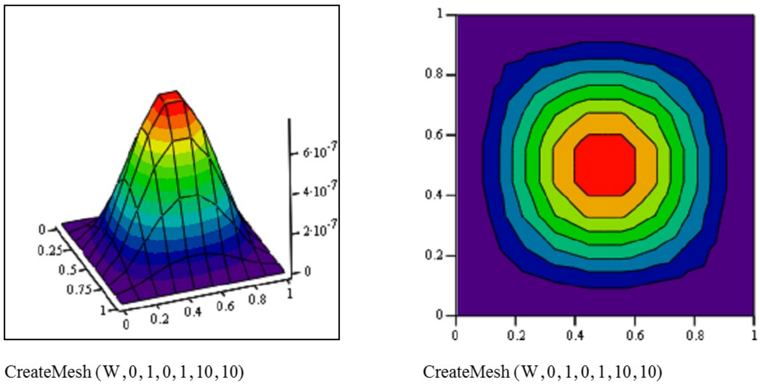


Figure 3. Bending function

Using the components of the shear, determine the strain components and their diagrams at $z = 0.5$, $\xi = 0.5$ (Fig. 4) [12]:

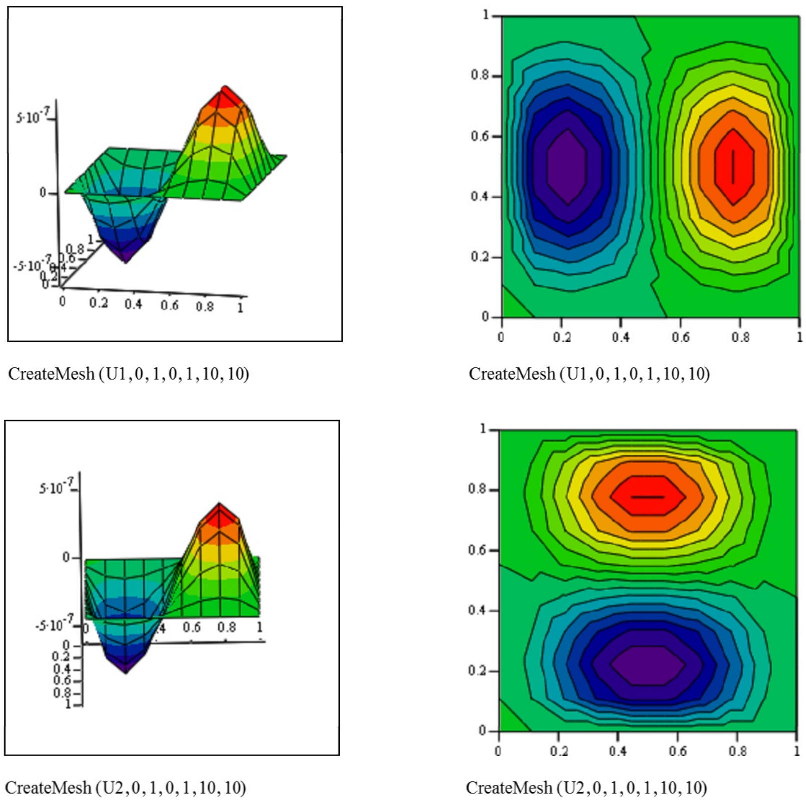
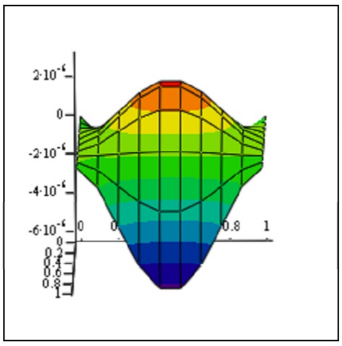
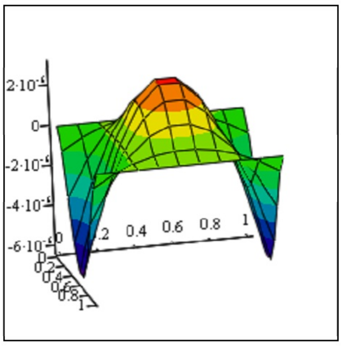


Figure 4. Shear components

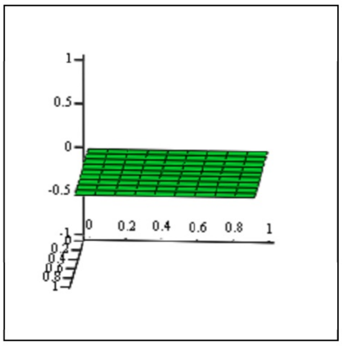
Using shear components, we define the deformation components and their diagrams at $z = 0.5$ (Fig. 5) [13]:



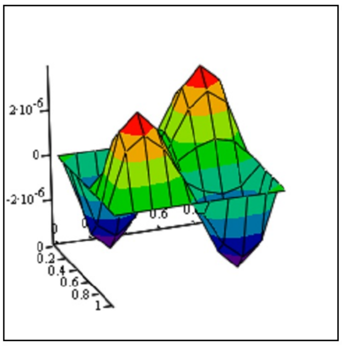
CreateMesh (ε1 , 0, 1, 0, 1, 10, 10)



CreateMesh (ε2 , 0, 1, 0, 1, 10, 10)



CreateMesh (ε3 , 0, 1, 0, 1, 10, 10)



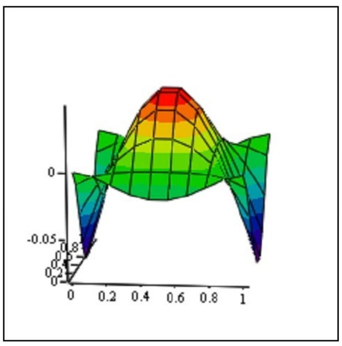
CreateMesh (γ12 , 0, 1, 0, 1, 10, 10)

Figure 5. Deformation components

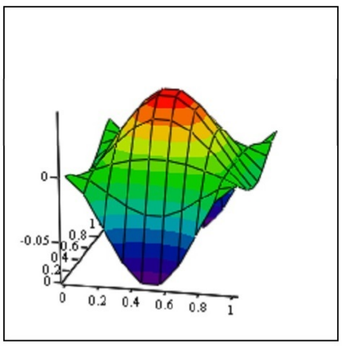
We calculate the internal forces (Fig. 6):

$$\overline{E}_0 = \frac{640000}{3};\ \overline{E}\,(\xi) = \frac{640000 \cdot \sin\left(\frac{\pi \cdot \xi}{2}\right)}{3 \cdot \xi};\ D\,(\xi) = \frac{20000 \cdot \sin\left(\frac{\pi \cdot \xi}{2}\right)}{9 \cdot \xi};$$

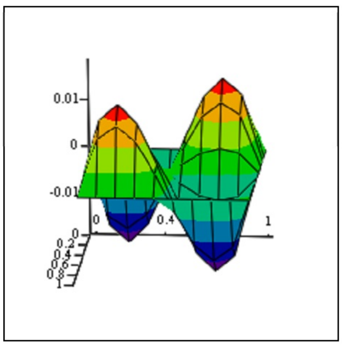
$$PU\,(x,\ y) = 1;\ q\,(x,\ y) = q_0 \cdot PU\,(x,\ y) = 1.$$



CreateMesh (M1, 0, 1, 0, 1, 10, 10)



CreateMesh (M2, 0, 1, 0, 1, 10, 10)



CreateMesh (M12, 0, 1, 0, 1, 10, 10)

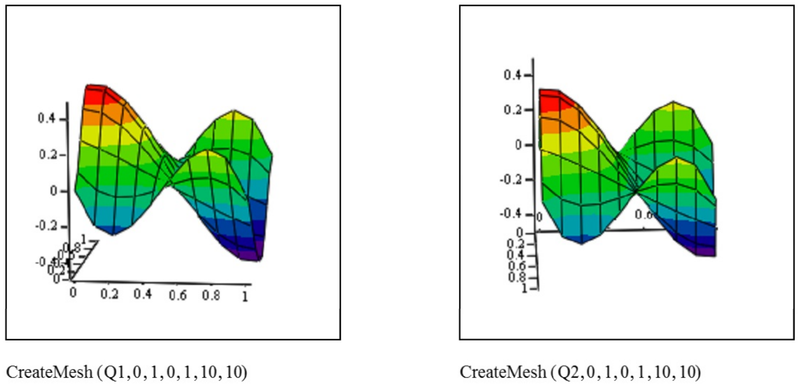


Figure 6. Moments of bending, torsion and internal forces

We find the components of stress, using the found internal forces [14–16]. At $x = 0.5$, $y = 0.5$ and $x = -0.5$, $y = -0.5$ and their diagrams are as follows (Fig. 7):

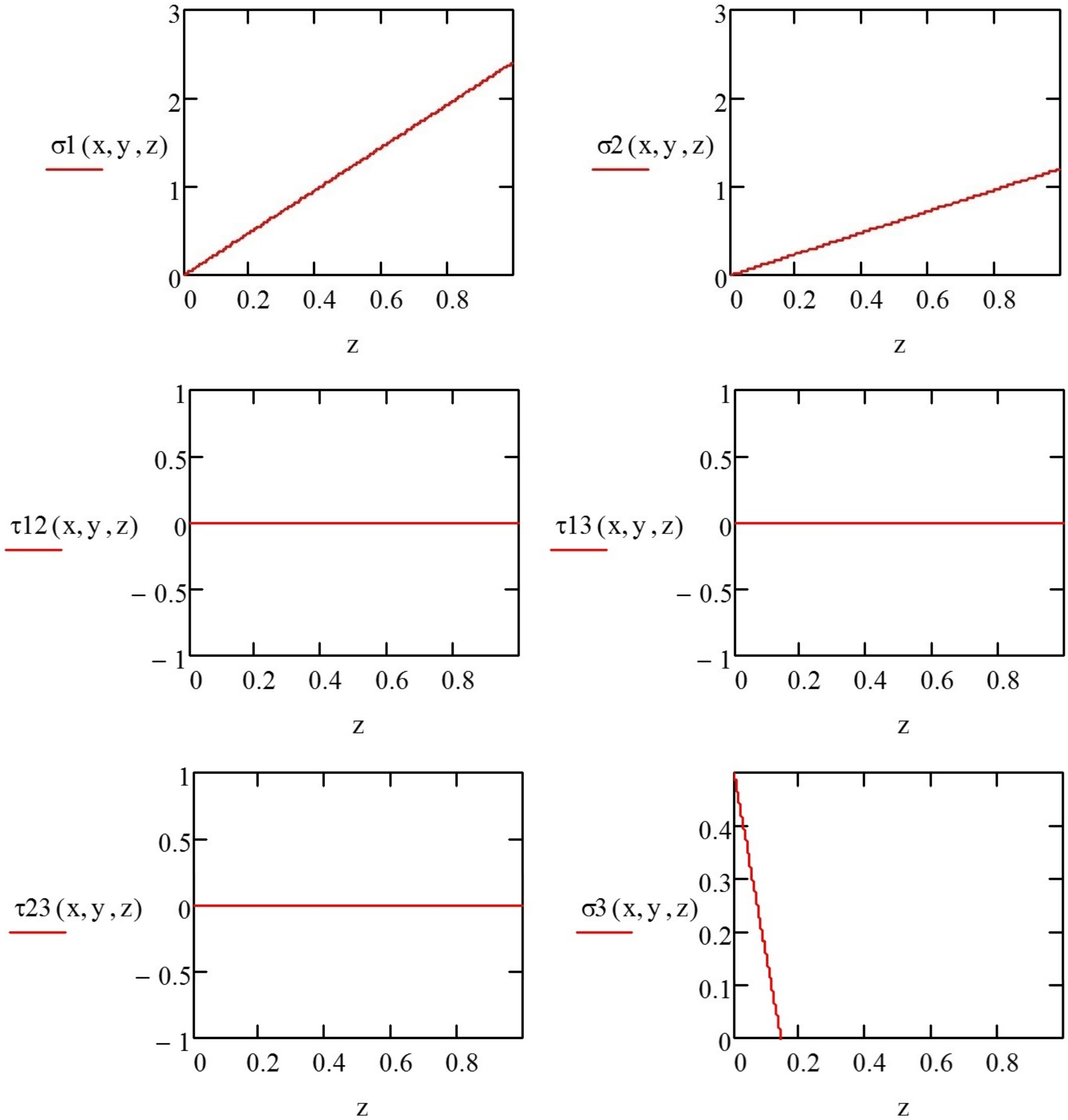


Figure 7. Stresses

Diagrams of the distribution functions (Fig. 8):

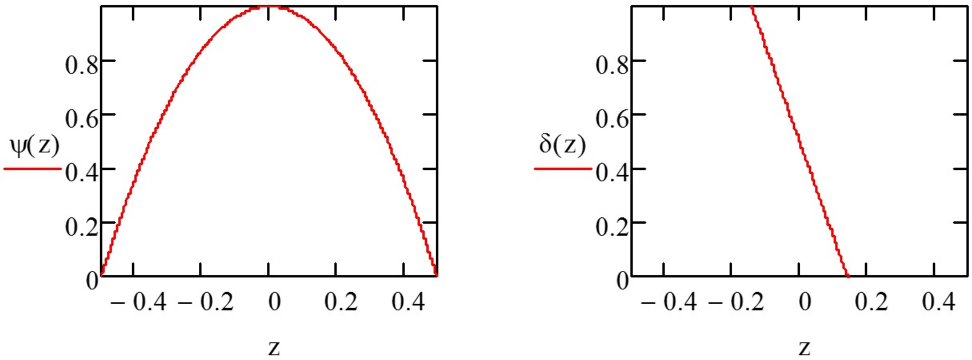


Figure 8. Distribution functions

The values of the distribution functions of the rigidly fixed elastic-plastic plate are given in Table 2 [17, 18].

Table 2

Values of distribution functions of the rigidly fixed elastic-plastic plate

$W\,(x,\ y,\ \xi)$	$U_1\,(x,\ y, z,\ \xi)$	$\varepsilon_1\,(x,\ y, z,\ \xi)$	$M_1\,(x,\ y)$	$M_{12}\,(x,\ y)$	$Q_1\,(x,\ y,\ \xi)$	$\sigma_1\,(x,\ y, z)$
0	0	0	-0.02	0	0	-0.469
$1.0068 \cdot 10^{-6}$	0	$-1.611 \cdot 10^{-5}$	$-3.922 \cdot 10^{-3}$	0	0.09	-0.094
$3.1820 \cdot 10^{-6}$	0	$-5.091 \cdot 10^{-5}$	0.015	0	0.12	0.365
$5.4815 \cdot 10^{-6}$	0	$-8.77 \cdot 10^{-5}$	0.033	0	0.105	0.783
$7.1595 \cdot 10^{-6}$	0	$-1.146 \cdot 10^{-4}$	0.045	0	0.06	1.07
$7.7685 \cdot 10^{-6}$	0	$-1.243 \cdot 10^{-4}$	0.049	0	0	1.172
$7.1595 \cdot 10^{-6}$	0	$-1.146 \cdot 10^{-4}$	0.045	0	-0.06	1.07
$5.4815 \cdot 10^{-6}$	0	$-8.77 \cdot 10^{-5}$	0.033	0	-0.105	0.783
$3.1820 \cdot 10^{-6}$	0	$-5.091 \cdot 10^{-5}$	0.015	0	-0.12	0.365
$1.0068 \cdot 10^{-6}$	0	$-1.611 \cdot 10^{-5}$	$-3.922 \cdot 10^{-3}$	0	-0.09	-0.094
0	0	0	-0.02	0	0	-0.469

Conclusions

Thus, this calculation was performed using elastic-plastic theory. The influence of the plate bending function in the form of polynomials in the elastic-plastic state is taken as a parameter. It is obtained analytically by applying the bending function of the beam taking into account the deformation. The law of change, which corresponds to the elastic-plastic material of stress, is used. In General, the calculation of the bending of a rigidly fixed elastic-plastic plate is shown.

It was found that many problems of the technique can be solved by the method of separating variables. Using this method simultaneously with the calculation of the plate bending, it is possible to solve the problem of free oscillations and other problems.

References

1 Liu M. Multiple fully stressed designs of steel frame structures with semi-rigid connections / M. Liu, S.A. Burns // International journal for numerical methods in engineering. — 2003. — Vol. 58, No. 6. — P. 821–838. DOI: 10.1002/nme.807.

2 Larsson D. In-plane modal testing of a free isotropic rectangular plate / D. Larsson // Experimental mechanics. — 1997. — Vol. 37, No. 3. — P. 339–343. DOI: 10.1007/BF02317428.

3 Турсунов К.А. Прямоугольный конечный элемент пластины / К.А. Турсунов, А.Е. Елешова // Тр. ун-та. Раздел Математика. Механика. — Караганда: Изд. КарГТУ, 2000. — Вып. 1(1). — С. 61–63.

4 Nuguzhinov Zh.S. Calculation of continuous beams taking into account the elastic compliance of supports / Zh.S. Nuguzhinov, S.K. Akhmediev, S.R. Zholmagambetov, O. Khabidolda // Bulletin of the Karaganda University. Mathematics series. — 2018. — No. 4 (92). — P. 139–147. DOI: 10.31489/2018M4/139-147

5 Yessenbayeva G.A. On the calculation of the rectangular finite element of the plate / G.A. Yessenbayeva, D.N. Yesbayeva, T.Kh. Makazhanova // Bulletin of the Karaganda University. Mathematics series. — 2018. — No. 2 (90). — P. 150–156. DOI: 10.31489/2018M2/150-156.

6 Kudaibergenov M.K. Stress-strain state horizontal coal seam of finite length / M.K. Kudaibergenov, A.L. Karchevsky, K.T. Iskakov // Bulletin of the Karaganda University. Mathematics series. — 2018. — No. 2 (90). — P. 133–142. DOI: 10.31489/2018M2/133-142.

7 Yessenbayeva G.A. On mathematical and analytical methods for solving problems on vibrations of membranes and plates / G.A. Yessenbayeva, D.N. Yesbayeva, D. Bauyrzhankyzy, D.A. Nurgali // Bulletin of the Karaganda University. Mathematics series. — 2018. — No. 1 (89). — P. 113-120.

8 He Z. Effect of vertex singularities on stress intensities near plate free surfaces / Z. He, A. Kotousov, F. Berto // Fatigue & Fracture of Engineering Materials & Structures. — 2015. — Vol. 38. — No. 7. — P. 860–869. DOI: 10.1111/ffe.12294.

9 Berto F. Three-dimensional effects in finite thickness plates weakened by rounded notches and holes under in-plane shear / F. Berto, C. Marangon // Fatigue & Fracture of Engineering Materials & Structures. — 2013. — Vol. 36. — No. 11. — P. 1139–1152. DOI: 10.1111/ffe.12098.

10 Campagnolo A. The effects of different boundary conditions on three-dimensional cracked discs under anti-plane loading / A. Campagnolo, F. Berto, P. Lazzarin // European Journal of Mechanics A-Solids. — 2015. — Vol. 50. — P. 76–86. DOI: 10.1016/j.euromechsol.2014.11.001.

11 Kotousov A. Effect of plate thickness on stress state at sharp notches and the strength paradox of thick plates / A. Kotousov // International Journal of Solids and Structures. — 2018. — Vol. 47. — No. 14–15. — P. 1916-1923. DOI: 10.1016/j.ijsolstr.2010.03.029.

12 Bambach M.R. Experimental techniques for testing unstiffened plates in compression and bending / M.R. Bambach, K.J.R. Rasmussen // Experimental mechanics. — 2004. — Vol. 44. — No. 1. — P. 91–96. DOI: 10.1177/0014485104039625.

13 Kotousov A. Fundamental solutions for the generalised plane strain theory / A. Kotousov, C.H. Wang // International Journal of Engineering Science. — 2002. — Vol. 40. — No. 15. — P. 1775–1790. DOI: 10.1016/S0020-7225(02)00041-1.

14 Kotousov A. Three-dimensional stress constraint in an elastic plate with a notch / A. Kotousov, C.H. Wang // International Journal of Solids and Structures. — 2002. — Vol. 39. — No. 16. — P. 4311–4326. DOI: 10.1016/S0020-7683(02)00340-2.

15 Lazzarin P. Three-dimensional stress fields due to notches in plates under linear elastic and elastic-plastic conditions / P. Lazzarin, M. Zappalorto, F. Berto // Fatigue & Fracture of Engineering Materials & Structures. — 2015. — Vol. 38. — No. 2. — P. 140–153. DOI: 10.1111/ffe.12138.

16 Nakamura T. 3-dimensional stress-field near the crack front of a thin elastic plate / T. Nakamura, D.M. Parks // Journal of Applied Mechanics-Transactions of the Asme. — 1988. — Vol. 55. — No. 4. — P. 805-813. DOI: 10.1115/1.3173725.

17 Beghini M. Measurement of coatings' elastic properties by mechanical methods: Part 1. Consideration on experimental errors / M. Beghini, L. Bertini, F. Frendo // Experimental mechanics. — 2001. — Vol. 41. — No. 4. — P. 293–304. DOI: 10.1007/BF02323922.

18 Seytmuratov A.Zh. Equations of vibration of a two-dimensionally layered plate strictly based on the decision of various boundary-value problems / A.Zh. Seytmuratov, B.M. Nurlanova, N.K. Medeubaev // Bulletin of the Karaganda University. Mathematics series. — 2017. — No. 3 (87). — P. 109–116. DOI: 10.31489/2017M3/109-116.

Л.С. Әсетова, Б.М. Нурланова, И.А. Самойлова

Жақтары қатты бекітілген серпімді-иілімді пластинаның иілуі

Пластиналық конструкциялар қатарына жабулар, сонымен қатар тұтас кеңістіктік жүйе ретінде жұмыс істейтін қорап тәріздес жүйелер жатады. Пластиналық конструкциялар қатарына өзектер де, пластиналар да элементтер болып табылатын жүйелер кіреді, мысалы, каркасты-панельдік ғимараттардың көтеруші қабілеті бар белдік конструкциялар. Серпімді-иілімді пластинаның иілу теориясын қарастырғанда осы теория бойынша есептеу алгоритмі, оның неізгі формулалары, серпімділік теорияның негізгі қатыстары қарастырылды. Мақалада жақтары қатты бекітілген серпімді-иілімді пластинаның иілуі зерттелді. Есептеу айнымалыларды бөлу әдісінің көмегімен жүргізілген. Бұл әдіс MathCAD бағдарламасы бойынша жүзеге асырылды. Кернеу мен материал модулінің функциясының өзгеру заңдарының мәндері, майысу функциясы, жылжулар компоненттері, деформация компоненттері, иілу, бұралу моменттері және ішкі күштер, кернеулердің мәндері алынды және үлестірім функцияларының эпюралары салынды. Жақтары қатты бекітілген серпімді-иілімді пластинаның таралу функцияларының мәндері анықталды.

Кілт сөздер: пластина, серпімді, икемді, иілімді, иілу, деформация, кернеу.

Л.С. Асетова, Б.М. Нурланова, И.А. Самойлова

Изгиб жестко закрепленной упруго-пластической пластины

К числу пластинных конструкций относятся покрытия, а также коробчатые системы, работающие как целостная пространственная система. В состав пластинных конструкций входят системы, элементами которых являются как стержни, так и пластины, например, при рассмотрении теории изгиба это каркасно-панельные конструкции с несущей способностью каркасно-панельных зданий. В теории упруго-пластического изгиба пластин рассмотрены алгоритм расчета, его основные формулы, основные соотношения теории упругости. В статье рассчитан изгиб жестко закрепленной упруго-пластической пластины. Расчеты производились с помощью метода разделения переменных, реализованного с использованием программы MathCAD. Авторами получены значения законов изменения напряжений и функции модуля материалов, функции изгиба, компонентов сдвига и деформации, значения моментов изгиба, кручения и внутренних сил, напряжений. Построены эпюры функций распределения. Определены значения функций распределения жестко закрепленной упруго-пластической пластины.

Ключевые слова: пластина, упругий, пластический, податливость, изгиб, деформация, напряжение.

References

1 Liu, M., & Burns, S.A. (2003). Multiple fully stressed designs of steel frame structures with semi-rigid connections. *International journal for numerical methods in engineering, Vol. 58, 6,* 821–838. DOI: 10.1002/nme.807.

2 Larsson, D. (1997). In-plane modal testing of a free isotropic rectangular plate. *Experimental mechanics, Vol. 37, 3,* 339–343. DOI: 10.1007/BF02317428.

3 Tursunov, K.A., & Eleshova, A.E. (2000). Priamouholnyi konechnyi element plastiny [The rectangular finite plate element]. *Trudy universiteta. Razdel Matematika. Mekhanika – Proceedings of the University. Section: Mathematics. Mechanics. No 1(1),* 61–63. Karaganda: Publ. KarHTU [in Russian].

4 Nuguzhinov, Zh.S., Akhmediev, S.K., Zholmagambetov, S.R., & Khabidolda O. (2018). Calculation of continuous beams taking into account the elastic compliance of supports. *Bulletin of the Karaganda University. Series Mathematics, 4 (92),* 139–147. DOI: 10.31489/2018M4/139-147.

5 Yessenbayeva, G.A., Yesbayeva, D.N., & Makazhanova, T.Kh. (2018). On the calculation of the rectangular finite element of the plate. *Bulletin of the Karaganda University. Mathematics series, 2 (90),* 150–156. DOI: 10.31489/2018M2/150-156.

6 Kudaibergenov, M.K., Karchevsky, A.L., & Iskakov, K.T. (2018). Stress-strain state horizontal coal seam of finite length. *Bulletin of the Karaganda University. Mathematics series, 2 (90),* 133–142. DOI: 10.31489 /2018M2/133-142.

7 Yessenbayeva, G.A., Yesbayeva, D.N., Bauyrzhankyzy, D., & Nurgali, D.A. (2018). On mathematical and analytical methods for solving problems on vibrations of membranes and plates. *Bulletin of the Karaganda University. Mathematics series, 1 (89),* 113–120.

8 He, Z., Kotousov, A., & Berto, F. (2015). Effect of vertex singularities on stress intensities near plate free surfaces. *Fatigue & Fracture of Engineering Materials & Structures, Vol. 38, 7,* 860–869. DOI: 10.1111/ffe.12294.

9 Berto, F., & Marangon, C. (2013). Three-dimensional effects in finite thickness plates weakened by rounded notches and holes under in-plane shear. *Fatigue & Fracture of Engineering Materials & Structures, Vol. 36, 11,* 1139–1152. DOI: 10.1111/ffe.12098.

10 Campagnolo, A., Berto, F., & Lazzarin, P. (2015). The effects of different boundary conditions on three-dimensional cracked discs under anti-plane loading. *European Journal of Mechanics A-Solids, Vol. 50,* 76–86. DOI: 10.1016/j.euromechsol.2014.11.001.

11 Kotousov, A. (2018). Effect of plate thickness on stress state at sharp notches and the strength paradox of thick plates. *International Journal of Solids and Structures, Vol. 47, 14-15,* 1916–1923. DOI: 10.1016/j.ijsolstr.2010.03.029.

12 Bambach, M.R., & Rasmussen, K.J.R. (2004). Experimental techniques for testing unstiffened plates in compression and bending. *Experimental mechanics, Vol. 44, 1,* 91–96. DOI: 10.1177/0014485104039625.

13 Kotousov, A., & Wang, C.H. (2002). Fundamental solutions for the generalised plane strain theory. *International Journal of Engineering Science, Vol. 40, 15,* 1775–1790. DOI: 10.1016/S0020-7225(02)00041-1.

14 Kotousov, A., & Wang, C.H. (2002). Three-dimensional stress constraint in an elastic plate with a notch. *International Journal of Solids and Structures, Vol. 39, 16,* 4311–4326. DOI: 10.1016/S0020-7683(02)00340-2.

15 Lazzarin, P., Zappalorto, M., & Berto, F. (2015). Three-dimensional stress fields due to notches in plates under linear elastic and elastic-plastic conditions. *Fatigue & Fracture of Engineering Materials & Structures, Vol. 38, 2,* 140–153. DOI: 10.1111/ffe.12138.

16 Nakamura, T., & Parks, D.M. (1988). 3-dimensional stress-field near the crack front of a thin elastic plate. *Journal of Applied Mechanics-Transactions of the Asme, Vol. 55, 4,* 805–813. DOI: 10.1115/1.3173725.

17 Beghini, M., Bertini, L., & Frendo, F. (2001). Measurement of coatings' elastic properties by mechanical methods: Part 1. Consideration on experimental errors. *Experimental mechanics, Vol. 41, 4,* 293–304. DOI: 10.1007/BF02323922.

18 Seytmuratov, A.Zh., Nurlanova, B.M., & Medeubaev, N.K. (2017). Equations of vibration of a two-dimensionally layered plate strictly based on the decision of various boundary-value problems. *Bulletin of the Karaganda University. Mathematics series, 33 (87),* 109–116. DOI: 10.31489/2017M3/109-116.

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Modified epoxy matrix with improved properties for protection of transport vehicles

It is substantiated that for improving the performance characteristics of vehicle parts, including their corrosion and wear resistance, it is advisable to use protective polymeric composite coatings. It is shown that in order to increase the indexes of physical and mechanical and thermophysical properties in the epoxy binder, it is necessary to introduce additives: modifiers, plasticizers, dispersed and fiber fillers. The introduction of modifiers into the epoxy binder is relevant, and effectively use them in the complex. The influence of modifier and hardener on the adhesion strength and destructive stresses during bending of the developed epoxy composite has been analyzed. The critical content of the components: the MBMA modifier is 0.25 . . . 0.50 pts.wt., hardener PEPA — 8 . . . 10 wt. per 100 pts.wt. of epoxy oligomer ED-20 was found by the method of mathematical planning of the experiment. The introduction of such ingredients into the epoxy binder increases the adhesion strength of matrix to $\sigma_a = 28.9 \ldots 31.3$ MPa and the fracture stresses during the flexion to $\sigma_{fl} = 51.2 \ldots 54.4$ MPa. The obtained results allow to create materials with improved physical and mechanical properties in the complex.

Keywords: composite, epoxy matrix, method of mathematical planning of the experiment, regression equation.

Introduction

The authors [1–6] have shown the expediency of using composite coatings on a polymeric basis to improve the performance of parts of transport. In this case, not only the choice of a polymer matrix, but also its modification by catalysts, plasticizers and other kinds of additives is of great importance. In this regard, the choice of the widely distributed epoxy resin ED-20 as a basis for the polymer binder is relevant. Coating on the basis of such epoxy oligomer is characterized by increased adhesion, physico-mechanical and thermophysical properties. This leads to an increase in the lifetime of the protective coatings of parts of transport equipment.

In the works [7–12] shown, that in order to improve the properties of coatings in the complex, which operate under conditions of aggressive media and temperature changes, it is advisable to introduce modifiers at the critical content. In this case, it is necessary to introduce active modifiers in the polymer, but also the hardener at the optimal content. This approach allows you to get a material with a homogeneous structure. It ensures maximum increase in the indexes of characteristics.

In this context, in order to reduce the number of experimental studies, it is proposed to conduct mathematical planning of the experiment. On the previous stage, the influence of the number of ingredients on the basic properties of epoxy CM was investigated. The critical content of the 2,4-diaminoazobenzene - 4'-carboxylic acid (MBMA) modifier, and a polyethylene polyamine hardener (PEPA) in the polymer matrix was established. However, from a practical standpoint, it is important to form composites with these components at different content in the epoxy resin. In this context, it is expedient and necessary to use the method of mathematical planning of the experiment. It allows to reduce the number of conducted studies and optimize the contents of the ingredients to obtain a matrix with the maximal values of the selected characteristics. Aim of work is to optimize the content of modifier and hardener for the protective coatings of the transport equipment by the method of mathematical planning of the experiment.

Results and discussion

The adhesive strength at normal break and fracture stresses during the flexion of epoxy composites at the different content of modifier and hardener (2,4-diaminoazobenzene - 4'-carboxylic acid and polyethylene polyamine respectively) was investigated. For standardization, as well as for simplification of calculations, each component (filler) is encoded by conditional units taking into account variations (Table 1).

Table 1

Levels of variables on conditional and natural scale

Components	Factor	Average level, q, pts. wt.	Variation step, Δq, pts. wt.	Values of variables (pts. wt.), corresponding to conditional units		
				-1	0	+1
Modifier MBMA	x_1	0.50	0.25	0.25	0.50	0.75
Hardener PEPA	x_2	10	2	8	10	12

According to the experiment planning scheme 9 experiments ($N = 9$) were conducted, each of which was repeated three times ($p = 3$) in order to exclude system errors (Table 2). In order that planning matrix to be orthogonal [13–15], the corrected values of x' level were entered, which were calculated by the formula:

$$x'_i = (x_i)^2 - \frac{\sum_{u-1}^{N} x_{iu}^2}{N}. \qquad (1)$$

The expanded matrix of planning of complete factor experiment (CFE) and its results are shown in Table 3.

Table 2

Scheme of experiment planning

№ of experiment (u)	x_0	x_1	x_2	$x_3 = x_1^2 - d$	$x_4 = x_2^2 - d$	$x_1 x_2$
1	1	-1	-1	0.33	0.33	+1
2	1	+1	-1	0.33	0.33	-1
3	1	-1	+1	0.33	0.33	-1
4	1	+1	+1	0.33	0.33	+1
5	1	0	0	-0.67	-0.67	0
6	1	+1	0	0.33	-0.67	0
7	1	-1	0	0.33	-0.67	0
8	1	0	+1	-0.67	0.33	0
9	1	0	-1	-0.67	0.33	0
$\sum_{\mathrm{u-1}}^{\mathrm{N}} x_{iu}^2$	9	6	6	2	2	4

Table 3

Results of study of adhesive strength at break and fracture stresses during the flexion of the epoxy matrix

№ of experiment	Content of components, q, pts. wt.		Adhesive strength at break, σ_a, MPa	Fracture stresses during the flexion, σ_{fl}, MPa
	x_1	x_2	y_1	y_2
1	0.25	8	28.2	48.4
2	0.75	8	26.6	50.0
3	0.25	12	27.8	51.0
4	0.75	12	26.4	49.3
5	0.50	10	28.9	48.8
6	0.75	10	27.9	50.2
7	0.25	10	31.3	51.2
8	0.50	12	27.2	48.6
9	0.50	8	25.5	54.4

The mathematical model $y = f(x_1, x_2)$ was formed as a regression equation:

$$y = b_0 + b_1 x_1 + b_2 x_2 + b_{11} x_1^2 + b_{22} x_2^2 + b_{12} x_1 x_2. \qquad (2)$$

The regression coefficients were determined by the formula:

$$b_i = \frac{\sum_{u=1}^{N} x_i y_i}{\sum_{u-1}^{N} x_{iu}^2}. \quad (3)$$

Received coefficients of regression equation are given in Table 4.

Table 4

The coefficients of regression equation

b_0	b_1	b_2	b_{11}	b_{22}	b_{12}
28.81	-1.07	0.18	0.83	-2.42	0.05

As a result, in the analysis of the adhesive strength at break, the following regression equation was determined:

$$y = 28.81 - 1.07x_1 + 0.18x_2 + 0.83x_1^2 - 2.42x_2^2 + 0.05x_1x_2.$$

For the statistical processing of experiment results, a test of reproducibility of experiments by the Cochran test was conducted:

$$G = \frac{S_{u\max}^2}{\sum_{u=1}^{N} S_u^2} \leq G_{(0{,}05;f_1;f_2)}, \quad (4)$$

where: S_{ui}^2 — dispersion of experiment results on combinations of few factor levels for $m = 3$; m — number of parallel experiments; S_{umax}^2 — the highest dispersion in design line;

Dispersions of adequacy were determined by the formula:

$$S_{ui}^2 = \frac{\sum_{i=1}^{m} (y_i - \overline{y_i})^2}{m-1}, \quad (5)$$

where: y_{im} — value, received from each parallel experiment; $\overline{y_i}$ — average value y, received in parallel experiments.

Mean square error was determined by formula:

$$\sigma^2 \{y\} = \frac{\sum_{i=1}^{N=9} \sigma^2 \{y\}_i}{N(m-1)}, \quad (6)$$

where $\sigma^2 \{y\}_i = \sum_{i=1}^{m=3} (y_i - \overline{y_i})^2$;

$$\sigma^2 \{y_{av}\} = \frac{a^2\{y\}}{N}, or S_{b_0}^2 = \frac{S_0^2}{N}. \quad (7)$$

Dispersion values are shown in Table 5.

Table 5

Values of dispersions of adequacy (S_{ui}^2) and mean square error ($\sigma^2 \{y\}_i$)

№	The dispersions of adequacy		The mean square error	
	conditional designation	value	conditional designation	value
1	S_{u1}^2	0.04	$\sigma^2 \{y\}_1$	0.08
2	S_{u2}^2	0.07	$\sigma^2 \{y\}_2$	0.14
3	S_{u3}^2	0.09	$\sigma^2 \{y\}_3$	0.18
4	S_{u4}^2	0.09	$\sigma^2 \{y\}_4$	0.18
5	S_{u5}^2	0.19	$\sigma^2 \{y\}_5$	0.38
6	S_{u6}^2	0.01	$\sigma^2 \{y\}_6$	0.02
7	S_{u7}^2	0.09	$\sigma^2 \{y\}_7$	0.18
8	S_{u8}^2	0.07	$\sigma^2 \{y\}_8$	0.14
9	S_{u9}^2	0.04	$\sigma^2 \{y\}_9$	0.08

Moreover:
$\sum_{i=1}^{N} S_{ui}^2 = 0.69$
$\sigma^2 \{y\} = S_0^2 = 0.077.$
Then the calculated value of the Cochran test at the 5 % level of significance:

$$G_{calc} = \frac{S_{u_{\max}}^2}{\sum_{i=1}^{N} S_{ui}^2}; \tag{8}$$

$G_{calc} = \frac{0.19}{0.69} = 0.275.$
Testing the experiment results by the Cochran test [16] for a fixed probability $\alpha = 0.05$ confirmed the reproducibility of the experiments. Dispersion of experiment results on combinations of few factor levels: $S_{u\max}^2 = 0.19$. Calculated value of Cochran test: $G_{calc} = 0.275$.
Table value of Cochran test: $G_{tab} = 0.478$.
That is, the condition (7) is fulfilled:

$$G_{calc} = 0.275 \le G_{tab} = 0.478.$$

Subsequently, the coefficients significance of regression equation was determined by analyzing the results according to the experimental design (Table 6).

Table 6

The experimental results of study of adhesive strength at break of materials

№ of experiment	Adhesive strength at break, σ_a, MPa			Average value, σ_a, MPa
	1	2	3	
1	28.0	28.4	28.2	28.2
2	26.5	26.9	26.4	26.6
3	27.8	28.1	27.5	27.8
4	26.4	26.1	26.7	26.4
5	28.7	29.4	28.6	28.9
6	27.9	27.8	28.0	27.9
7	31.3	31.0	31.6	31.3
8	27.5	27.1	27.0	27.2
9	25.3	25.5	25.7	25.5

Then the dispersions of regression coefficients (Table 7) were determined by the formula:

$$S_{b_i}^2 = \frac{S_0^2}{\sum_{u-1}^{N} x_{iu}^2}. \tag{9}$$

The significance of the regression coefficients was determined by the Student's test [17, 18]. Here with the table (t_m) and calculated criterion (t_{calc}) of Student's test (Table 7) were determined.
Depending on freeness: $f = N(n-1) = 9(3-1) = 18$ the Student's test value was calculated, which is $t_T = 2.1$.
Calculated values of Student's test (t_{calc}) and coefficients significance were determined:
$t_0, t_1, t_2, t_{11}, t_{22}, t_{12} > t$.
Moreover:

$$t_i = \frac{|b_i|}{S_{b_i}}. \tag{10}$$

Table 7

Dispersion of coefficients of regression (S_b^2)
and calculated values of Student's criterion (t_{calc})

№	Dispersion of coefficients of regression		Calculated values of Student's criterion	
	conditional designation	value	conditional designation	value
1	$S_{b_0}^2$	0.009	t_{0p}	301.09
2	$S_{b_1}^2$	0.013	t_{1p}	9.44
3	$S_{b_2}^2$	0.013	t_{2p}	1.62
4	$S_{b_{11}}^2$	0.038	t_{11p}	4.26
5	$S_{b_{22}}^2$	0.038	t_{22p}	12.34
6	$S_{b_{12}}^2$	0.019	t_{12p}	0.4

Calculated values of Student's criterion t_0, t_{t1}, t_{11}, t_{22} are larger than t_T, so it was considered that b_0, b_1, b_{11}, b_{22} coefficients of the regression equation are significant. Calculated values t_2, t_{12} are lower than t_T, so it was considered that b_2, b_{12} coefficients of the regression equation are insignificant. As a result of rejection of insignificant coefficients, the following regression equation was received:

$$y = 28.81 - 1.07\ x_1 + 0.83\ x_1^2 - 2.42\ x_2^2.$$

The adequacy of the model was checked by Fisher test [11]:

$$F_c = \frac{S_{u\ max}^2}{S_y^2} \leq F_{(0.05;\ f_{ad};\ f_y)}, \tag{11}$$

where $S_{u\ max}^2 = 0.19$ — calculated value of dispersion of adequacy (Table 5);

$$S_y^2 = \frac{\sum_{i=1}^{N} S_{ui}^2}{N}; \tag{12}$$

$S_y^2 = 0.077$ — mean square error;
So: $F_c = 2.49$.
$F_{(0.05;f_{ad};f_u)}$ — table value of Fisher test in 5 % significance level ($f_1 = N - (k+1) = 9 - (4+1) = 4$, $f_2 = N(n-1) = 9(3-1) = 18$). So: $F_t = 2.93$ [19].

Calculated value of Fisher test is less than table one, so the requirement (10) is fulfilled. It is possible to assume that equation adequately characterizes the composition.

Interpretation process of received mathematical model, as a rule, is not just determination of factors influence. A simple comparison of absolute value of linear coefficients does not determine the relative degree factors influence, since there are also quadratic squared terms and paired interactions. In a detailed analysis of the received adequate model, it is necessary to take into account the fact that for a quadratic model the degree of factor influence on the change of output value is not constant.

Dependencies that connect normalized and natural values of the variables are as follows:

$$x_i = \frac{q_i - q_{i0}}{\Delta q_i}, \tag{13}$$

where: q_i — value of i experiment factor, q_{i0} — value of zero level, Δq_i — variation interval.

Substituting these values in accordance with the formula (13) into the regression equation and transforming it, we receive the following regression equation with the natural values of the variables:

$$\sigma_a = -26.23 + 13.28\ q_1^2 - 0.605\ q_2^2 - 17.56\ q_1 + 12.1\ q_2.$$

Given equation in natural values allows only predicting the output value for any point in the middle of range of factor variations. However, with its help it is possible to construct graphs of dependence of output value (adhesive strength at break) from any factor (or two factors). Geometric interpretation of the response surface is shown on Fig. 1-3.

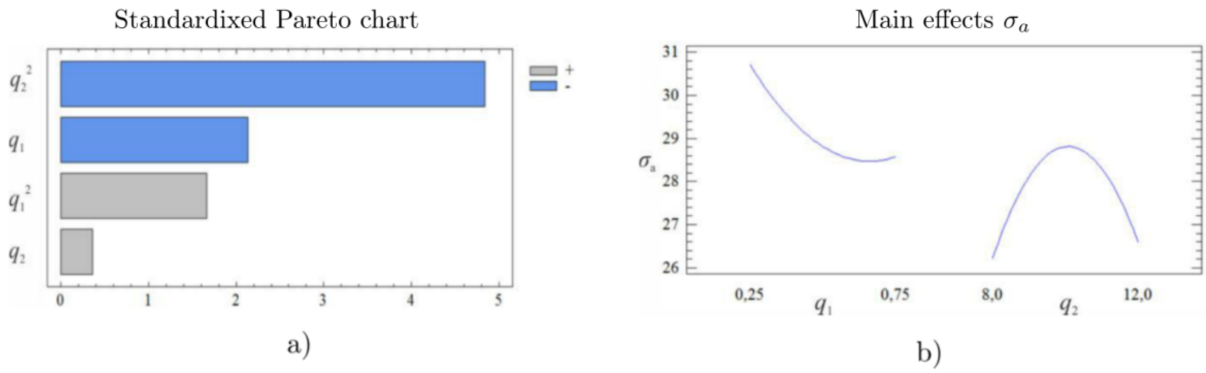

Figure 1. Standardized Pareto chart (a) and main effects (b)

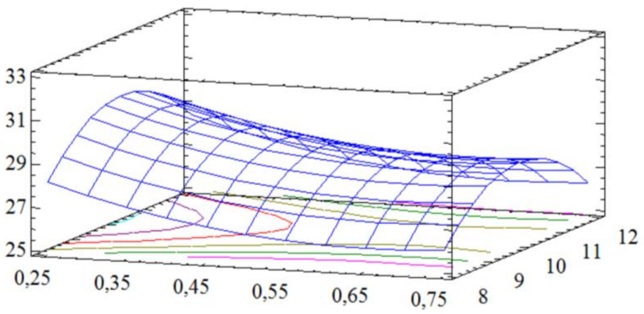

Figure 2. Estimated surface $\sigma_a = f(q_1, q_2)$

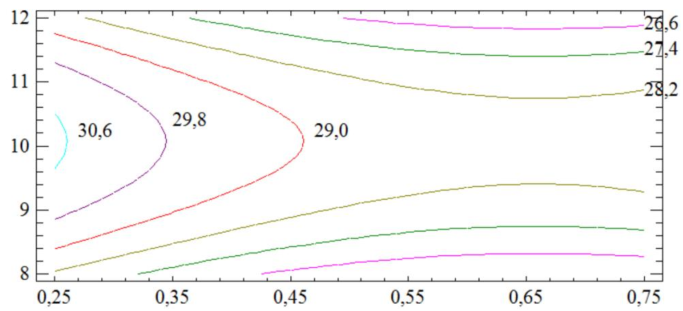

Figure 3. Contours of estimated response surface

On the basis of experimental studies, it has been found that both factors are significant. It should be noted that the effect of the content of the MBMA modifier on the indexes of adhesion strength at break is higher compared to the PEPA hardener (according to the Pareto chart). It was determined that optimal indexes of adhesion strength at break have a developed epoxy composite with a modifier and a hardener at the following contents: the MBMA modifier is $0.25 \ldots 0.50$ parts by mass, PEPA hardener — 10 parts by mass, ($\sigma_a = 28.9 \ldots 31.3$ MPa) by analyzing the calculated response surface. Similarly, to the above calculations scheme, the composition formula was optimized according to the fracture stresses during the flexion indexes. The encoding of natural components values and the experimental design scheme are chosen according to Table 1 and Table 2.

In the process of study results analysis of composites fracture stresses during the flexion, the following values of the regression coefficients were received (Table 8).

Table 8

The coefficients of regression equation of fracture stresses during the flexion

b_0	b_1	b_2	b_{11}	b_{22}	b_{12}
50.46	-0.18	-0.65	-0.58	0.22	-0.83

As a result, the following regression equation was found:

$$y = 50.46 - 0.18\ x_1 - 0.65\ x_2 - 0.58\ x_1^2 - 0.22\ x_2^2 + 0.83\ x_1 x_2.$$

For statistical processing of experiment results, a test of experiments reproducibility was conducted according to the Cochran test [20].

Dispersions values that were calculated by formula (5-7) are shown in Table 9.

Table 9

Values of dispersions of adequacy (S_{ui}^2) and mean square error ($\sigma^2\{y\}_i$)

№	The dispersions of adequacy		The mean square errors	
	conditional designation	value	conditional designation	value
1	S_{u1}^2	0.13	$\sigma^2\{y\}_1$	0.26
2	S_{u2}^2	0.21	$\sigma^2\{y\}_2$	0.42
3	S_{u3}^2	0.13	$\sigma^2\{y\}_3$	0.26
4	S_{u4}^2	0.07	$\sigma^2\{y\}_4$	0.14
5	S_{u5}^2	0.13	$\sigma^2\{y\}_5$	0.26
6	S_{u6}^2	0.13	$\sigma^2\{y\}_6$	0.26
7	S_{u7}^2	0.16	$\sigma^2\{y\}_7$	0.32
8	S_{u8}^2	0.07	$\sigma^2\{y\}_8$	0.14
9	S_{u9}^2	0.07	$\sigma^2\{y\}_9$	0.14

Moreover:

$$\sum_{i=1}^{N} S_{ui}^2 = 1.1$$

$$\sigma^2\{y\} = S_0^2 = 0.122.$$

Calculated value of the Cochran test at the 5 % significance level was determined by formula (8):

$$G_c = \frac{0.21}{1.1} = 0.191.$$

Testing the experiment results by the Cochran test [20] for a fixed probability $\alpha = 0.05$ confirmed the experiments reproducibility. Dispersion characterizing dispersal of the experiments results in combination of few factor levels: $S_{u\ max}^2 = 0.21$. Calculated value of Cochran test: $G_{calc} = 0.191$.

Table value of Cochran test: $G_{tab} = 0.478$.

So, the requirement is fulfilled:

$$G_{calc} = 0.191 \leq G_{tab} = 0.478.$$

At the next stage, the coefficients significance of regression equation is determined, analyzing the results according to the experimental design (Table 10).

Table 10

The experimental results of study of fracture stresses during the flexion of CM

№	Fracture stresses during the flexion, σ_{fl}, MPa			Average value, σ_{fl}, MPa
	1	2	3	
1	48.0	48.5	48.7	48.4
2	49.6	50.5	49.9	50.0
3	50.7	50.9	51.4	51.0
4	49.5	49.4	49.0	49.3
5	48.9	48.4	49.1	48.8
6	50.5	50.3	49.8	50.2
7	51.6	50.8	51.2	51.2
8	48.8	48.7	48.3	48.6
9	54.5	54.1	54.6	54.4

Subsequently, dispersion of regression coefficients is determined by formulas (9-10). The significance of regression coefficients is determined according to Student's criterion, which table value is $t_T = 2.1$. Calculated values of Student's criterion are shown in Table 11.

Table 11

Dispersion of coefficients of regression (S_b^2)
and calculated values of Student's criterion (t_p)

№	Dispersion of coefficients of regression		Calculated values of Student's criterion	
	conditional designation	value	conditional designation	value
1	$S_{b_0}^2$	0.014	t_{0p}	430.81
2	$S_{b_1}^2$	0.020	t_{1p}	1.28
3	$S_{b_2}^2$	0.020	t_{2p}	4.55
4	$S_{b_{11}}^2$	0.061	t_{11p}	2.36
5	$S_{b_{22}}^2$	0.061	t_{22p}	0.88
6	$S_{b_{12}}^2$	0.031	t_{12p}	4.7

Calculated values of Student's criterion t_0, t_2, t_{11}, t_{12}, are larger than t_T, so it was considered that b_0, b_2, b_{11}, b_{12} coefficients of the regression equation are significant. Calculated values t_1, t_{22} are lower than t_T, so it was considered that b_1, b_{22} coefficients of the regression equation are insignificant. As a result of rejection of insignificant coefficients, the following regression equation was received:

$$y = 50.46 - 0.65\ x_2 - 0.58\ x_1^2 + 0.83\ x_1\ x_2.$$

The adequacy of the model was checked by Fisher's test.

Calculated value of adequacy dispersion: $S_{u\ max}^2 = 0.21$ (Table 9).

The mean square error: $S_y^2 = 0.122$.

So: $F = 1.718$.

$F_{(0,05;f_W;f_u)}$ — table value of Fisher's test in 5 % significance level ($F_{(t)} = 2.93$).

Calculated value of Fisher's test is smaller than table on, so requirement (11) is fulfilled. Consequently, the equation adequately shows the composition formula.

After transformations in accordance with formula (13), the following regression equation with the natural values of variables was received:

$$\sigma_{fl} = 59.69 - 9.28\ q_1^2 + 1.66\ q_1\ q_2 - 7.32\ q_1 - 1.155\ q_2.$$

Geometric interpretation of response surface is shown on Fig. 4–6.

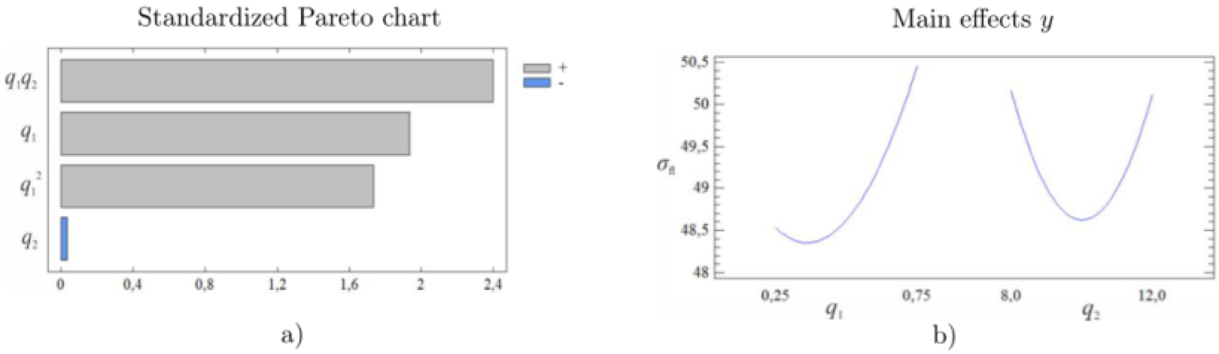

Figure 4. Standardized Pareto chart (a) and main effects (b)

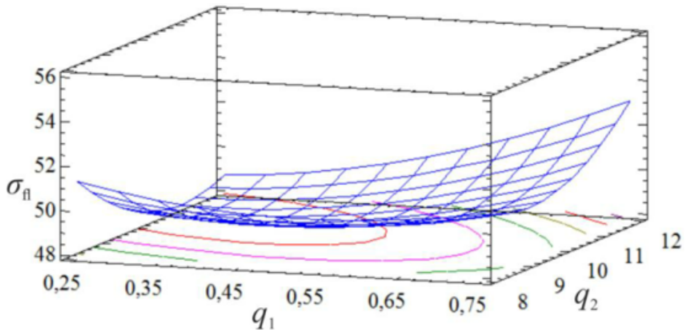

Figure 5. Estimated surface $\sigma_{fl} = f(q_1, q_2)$

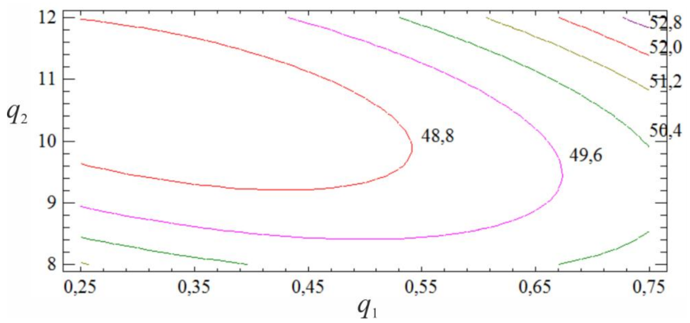

Figure 6. Contours of estimated response surface

The obtained results indicate that both factors of the regression equation are significant. It should be noted that the greatest influence on the output parameters of the composite have a pair interaction and linear and quadratic dependencies of the first factor. In the course of the analysis it was found that the indexes of fracture stresses during the flexion have the maximal values at the next content of components: the MBMA modifier is 0.25 . . . 0.50 pts. wt., PEPA hardener — 8 . . . 10 pts.wt. ($\sigma_{fl} = 51.2 \ldots 54.4$ MPa). With further increase in the content of additives, a deterioration of the parameters of fracture stresses during the flexion was observed. In our opinion, this is a consequence of the oversaturation of the ingredients in the polymer matrix, which adversely affects the physical and mechanical properties of the material.

Conclusions

The critical content of the components: the MBMA modifier is 0.25 . . . 0.50 pts.wt., hardener PEPA — 8 . . . 10 wt. per 100 pts.wt. of epoxy oligomer ED-20 was found by the method of mathematical planning of the experiment. The introduction of such ingredients into the epoxy binder increases the adhesion strength of matrix to $\sigma_a = 28.9 \ldots 31.3$ MPa and the fracture stresses during the flexion to $\sigma_{fl} = 51.2 \ldots 54.4$ MPa. It is analyzed that with increasing the content of the modifier it is possible to increase the parameters of fracture stresses during the flexion. These studies will be presented in the next works. The obtained results allow to create materials with improved physical and mechanical properties in the complex. The obtained materials should be used in the form of protective coatings to improve the performance and repair of parts of transport equipment.

References

1 Xinyu Li. Mathematical modeling and evolutionary algorithm-based approach for integrated process planning and scheduling / Li Xinyu, Gao Liang, Shao Xinyu, Zhang Chaoyong, Wang Cuiyu // Computers & Operations Research. — Vol. 37, Issue 4. — 2010. — P. 656–667.

2 Brailo M. The Investigation of Tribological Properties of Epoxy-Polyether Composite Materials for Using in the Friction Units of Means of Sea Transport / M. Brailo, A. Buketov, S. Yakushchenko, O. Sapronov, V. Vynar, O. Kobelnik // Materials Performance and Characterization. — 2018. — 7. — No. 1. — P. 275–299.

3 Buketov A. Development of Epoxy-Polyester Composite with Improved Thermophysical Properties for Restoration of Details of Sea and River Transport / A. Buketov, M. Brailo, S. Yakushchenko, A. Sapronova // Advances in Materials Science and Engineering. — 2018. — Article ID 6378782. — 6 p.

4 Buketov A. The formulation of epoxy-polyester matrix with improved physical and mechanical properties for restoration of means of sea and river transport / A.V. Buketov, M.V. Brailo, S.V. Yakushchenko, O.O. Sapronov, S.O. Smetankin // Journal of Marine Engineering & Technology. — 2018. — P. 1–6.

5 Buketov A.V. Dynamics of destruction of epoxy composites filled with ultra-dispersed diamond under impact conditions / A.V. Buketov, O.O. Sapronov, M.V. Brailo, P.O. Maruschak, S.V. Yakushchenko, S.V. Panin, V.D. Nigalatiy // Mechanics of Advanced Materials and Structures. — 2018. — P. 1–9.

6 Brailo M.V. Optimization of contents of two-component polydispersed filler by applying the mathematical design of experiment in forming composites for transport repairing / M.V. Brailo, A.V. Buketov, S.V. Yakushchenko, O.O. Sapronov, L. Dulebova // Bulletin of the Karaganda University. Mathematics series. — 2018. — №1 (89). — P. 93–104.

7 Sandler J. Development of a dispersion process for carbon nanotubes in an epoxy matrix and the resulting electrical properties / J. Sandler, M.S.P. Shaffer, T. Prasse, W. Bauhofer, K. Schulte, A.H. Windle // Polymer. — 1999. — 40. — № 21. — P. 5967–5971.

8 Akimov A.V. Development of polymer composites with improved thermophysical properties for shipbuilding and ship repair / A.V. Akimov, A.V. Buketov, O.O. Sapronov, M.V. Brailo, S.V. Yakushchenko, S.A. Smetankin // Composites: Mechanics, Computations, Applications: An International Journal. — 2019. — 10. — No. 2. — P. 117–134.

9 Manwar Hussain. Mechanical property improvement of carbon fiber reinforced epoxy composites by Al_2O_3 filler dispersion / Hussain Manwar, Nakahira Atsushi, Niihara Koichi // Materials Letters. — 1996. — 26, No. 3. — P. 185–191.

10 Buketov A. Enhancing performance characteristics of equipment of sea and river transport by using epoxy composites / A. Buketov, P. Maruschak, O. Sapronov, D. Zinchenko, V. Yatsyuk, S. Panin // Transport. — 2016. — 31. — P. 333–342.

11 Buketov A.V. Optimization of components in development of polymeric coatings for restoration of transport vehicles / A.V. Buketov, M.V. Brailo, D.P. Stukhlyak, S.V. Yakushchenko, O.O. Sapronov, V.V. Cherniavskyi, V.M. Husiev, D.A. Dmitriev, V.M. Yatsyuk, O.M. Bezbakh, R.Yu. Negrutsa // Bulletin of the Karaganda University. Mathematics series. — 2018. — No. 4(92). — P. 119–131.

12 Buketov A. Investigation of thermophysical properties of epoxy nanocomposites / A. Buketov, P. Maruschak, O. Sapronov, M. Brailo, O. Leshchenko, L. Bencheikh, A. Menou // Molecular Crystals and Liquid Crystals. — 2016. — 628:1. — P. 167–179.

13 Fang F. Hierarchical Orthogonal Matrix Generation and Matrix-Vector Multiplications in Rigid Body Simulations / F. Fang, J. Huang, G. Huber, J.A. McCammon, B. Zhang // SIAM Journal on Scientific Computing. — 2018. — 40. — P. A1345–A1361.

14 Бондарь А.Г. Планирование эксперимента в химической технологии (основное положение, примеры и задачи): учеб. для студ. высш. учеб. завед. / А.Г. Бондарь, Г.А. Статюха. — Киев: Вищ. шк., 1976. — 184 с.

15 Durán A.J. Orthogonal matrix polynomials satisfying second-order differential equations / A.J. Durán, F.A. Grünbaum // International Mathematics Research Notices. — 2004. — 2004. — P. 461–484.

16 Грушко И.М. Основы научных исследований: учеб. для техн. вузов / И.М. Грушко, В.В. Попов и др.; под ред. В.И. Крутова, В.В. Попова. — М.: Высш. шк., 1989. — 400 с.

17 Математические методы планирования эксперимента / под ред. В.В. Пененко. — Новосибирск: Наука, 1981. — 250 с.

18 Kim T.K. T test as a parametric statistic / T.K. Kim // Korean journal of anesthesiology. — 2015. — 68. — P. 540.

19 Chen Z. Is the weighted z test the best method for combining probabilities from independent tests? / Z. Chen // Journal of evolutionary biology. — 2011. — 24. — P. 926–930.

20 Wilrich P.T. Critical values of Mandel's h and k, the Grubbs and the Cochran test statistic / P.T. Wilrich // AStA Advances in Statistical Analysis. — 2013. — 97. — P. 1–10.

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Көлік құралдарын қорғау үшін жақсартылған қасиеттері бар модификацияланған эпоксидті матрица

Көлік құралдары бөлшектерінің пайдалану сипаттамаларын, оның ішінде олардың коррозияға қарсы қасиеттері мен тозуға төзімділігін арттыру үшін қорғаныштық полимерлік композиттік жабындарды пайдалану орынды екендігі негізделген. Физикалық-механикалық және жылу-физикалық қасиеттерінің көрсеткіштерін арттыру үшін эпоксидті байланыстырушы қоспаларды енгізу қажет: модификаторлар, пластификаторлар, дисперсиялық және талшықты толтырғыштар. Модификаторларды эпоксидті байланыстырғыш етіп енгізу өзекті болып табылады және де оларды кешенде пайдалану тиімді. Модификатор мен қатайтқыштың адгезиялық беріктікке және әзірленген эпоксидті композитті майыстыру кезінде кернеуді бұзатын әсері талданды. Эксперименттi математикалық жоспарлау әдісімен компоненттердің сыни құрамы анықталды: модификатор МБМА — $0,25\ldots 0,50$ масс. сағ, ПЕПА қатайтқыш — $8\ldots 10$ масс. с 100 мас. сағ эпоксидті олигомер ЭД-20. Мұндай ингредиенттерді эпоксидті байланыстырғыш енгізу матрицаның адгезиялық беріктігінің көрсеткіштерін $\sigma_a = 28.9\ldots 31,3$ МПа дейін арттыруға мүмкіндік береді және бүгу кезінде қиратушы кернеулер $\sigma_{fl} = 51,2\ldots 54,4$ МПа дейін. Алынған нәтижелер кешенде физика-механикалық қасиеттерінің жақсартылған көрсеткіштері бар материалдарды жасауға мүмкіндік берді.

Кілт сөздер: композит, эпоксидті матрица, эксперименттi математикалық жоспарлау әдісі, регрессия теңдеуі.

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Модифицированная эпоксидная матрица с улучшенными свойствами для защиты транспортных средств

Обосновано, что для повышения эксплуатационных характеристик деталей транспортных средств, в том числе их антикоррозийных свойств и износостойкости, целесообразно использовать защитные полимерные композитные покрытия. Показано, что для повышения показателей физико-механических и теплофизических свойств в эпоксидное связующее необходимо вводить добавки: модификаторы, пластификаторы, дисперсные и волокнистые наполнители. Актуальным является введение модификаторов в эпоксидное связующее, причем эффективно использовать их в комплексе. Проанализировано влияние модификатора и отвердителя на адгезионную прочность и разрушающие напряжения при изгибе разработанного эпоксидного композита. Методом математического планирования эксперимента установлено критическое содержание компонентов: модификатор МБМА — 0,25 … 0,50 масс.ч, отвердитель ПЕПА — $8\ldots 10$ масс.с на 100 масс.ч эпоксидного олигомера ЭД-20. Введение в эпоксидное связующее таких ингредиентов позволяет повысить показатели адгезионной прочности матрицы до $\sigma_a = 28.9 \ldots 31,3$ МПа и разрушающих напряжений при сгибании до $\sigma_{fl} = 51,2 \ldots 54,4$ МПа. Полученные результаты позволяют создать материалы с улучшенными в комплексе показателями физико-механических свойств.

Ключевые слова: композит, эпоксидная матрица, метод математического планирования эксперимента, уравнение регрессии.

References

1 Li, X., Gao, L., Shao, X., Zhang, C., & Wang, C. (2010). Mathematical modeling and evolutionary algorithm-based approach for integrated process planning and scheduling. *Computers & Operations Research, 37(4)*, 656-667.

2 Brailo, M., Buketov, A., Yakushchenko, S., Sapronov, O., Vynar, V., & Kobelnik, O. (2018). The investigation of tribological properties of epoxy-polyether composite materials for using in the friction units of means of Sea transport. *Materials Performance and Characterization, 7(1)*, 275-299.

3 Buketov, A., Brailo, M., Yakushchenko, S., & Sapronova, A. (2018). Development of Epoxy-Polyester Composite with Improved Thermophysical Properties for Restoration of Details of Sea and River Transport. *Advances in Materials Science and Engineering,* 6.

4 Buketov, A.V., Brailo, M.V., Yakushchenko, S.V., Sapronov, O.O., & Smetankin, S.O. (2018). The formulation of epoxy-polyester matrix with improved physical and mechanical properties for restoration of means of sea and river transport. *Journal of Marine Engineering & Technology*, 1-6.

5 Buketov, A.V., Sapronov, O.O., Brailo, M.V., Maruschak, P.O., Yakushchenko, S.V., Panin, S.V., & Nigalatiy, V.D. (2018). Dynamics of destruction of epoxy composites filled with ultra-dispersed diamond under impact conditions. *Mechanics of Advanced Materials and Structures*, 1-9.

6 Brailo, M.V., Buketov, A.V., Yakushchenko, S.V., Sapronov, O.O., & Dulebova, L. (2018). Optimization of contents of two-component polydispersed filler by applying the mathematical design of experiment in forming composites for transport repairing. *Bulletin of the Karaganda University. Mathematics series, 1(89),* 93-104.

7 Sandler, J., Shaffer, M.S.P., Prasse, T., Bauhofer, W., Schulte, K., & Windle, A.H. (1999). Development of a dispersion process for carbon nanotubes in an epoxy matrix and the resulting electrical properties. *Polymer, 40(21)*, 5967-5971.

8 Akimov, A.V., Buketov, A.V., Sapronov, O.O., Brailo, M.V., Yakushchenko, S.V., & Smetankin, S.A. (2019). Development of polymer composites with improved thermophysical properties for shipbuilding and ship repair. *Composites: Mechanics, Computations, Applications: An International Journal, 10(2),* 117-134.

9 Hussain, M., Nakahira, A., & Niihara, K. (1996). Mechanical property improvement of carbon fiber reinforced epoxy composites by Al_2O_3 filler dispersion. *Materials Letters, 26(3),* 185-191.

10 Buketov, A., Maruschak, P., Sapronov, O., Zinchenko, D., Yatsyuk, V., & Panin, S. (2016). Enhancing performance characteristics of equipment of sea and river transport by using epoxy composites. *Transport, 31(3),* 333-342.

11 Buketov, A.V., Brailo, M.V., Stukhlyak, D.P., Yakushchenko, S.V., Sapronov, O.O., Cherniavskyi, V.V., Husiev, V.M., Dmitriev, D.A., Yatsyuk, V.M., Bezbakh, O.M., & Negrutsa, R.Yu. (2018). Optimization of components in development of polymeric coatings for restoration of transport vehicles. *Bulletin of the Karaganda University. Mathematics series, 4(92),* 119-131.

12 Buketov, A., Maruschak, P., Sapronov, O., Brailo, M., Leshchenko, O., Bencheikh, L., & Menou, A. (2016). Investigation of thermophysical properties of epoxy nanocomposites. *Molecular Crystals and Liquid Crystals, 628(1),* 167-179.

13 Fang, F., Huang, J., Huber, G., McCammon, J.A., & Zhang, B. (2018). Hierarchical Orthogonal Matrix Generation and Matrix-Vector Multiplications in Rigid Body Simulations. *SIAM Journal on Scientific Computing, 40(3),* A1345-A1361.

14 Bondar, A.G., & Statiukha, G.A. (1976). *Planirovanie eksperimenta v khimicheskoi tekhnolohii [Planning an experiment in chemical technology].* Kiev: Vishcha shkola [in Russian].

15 Durán, A.J., & Grünbaum, F.A. (2004). Orthogonal matrix polynomials satisfying second-order differential equations. *International Mathematics Research Notices, 2004(10),* 461–484.

16 Grushko, I.M., & Popov, V.V. (1989). *Osnovy nauchnykh issledovanii: uchebnik dlia tekhnicheskikh vuzov [Fundamentals of research: a textbook for technical universities].* V.I. Krutikov (Eds.). Moscow: Vysshaia shkola [in Russian].

17 Penenko, V.V. (Eds.). (1981). *Matematicheskie metody planirovaniia eksperimenta [Mathematical methods of experimental design].* Novosibirsk: Nauka [in Russian].

18 Kim, T.K. (2015). T test as a parametric statistic. *Korean journal of anesthesiology, 68(6),* 540.

19 Chen, Z. (2011). Is the weighted z test the best method for combining probabilities from independent tests? *Journal of evolutionary biology, 24(4),* 926-930.

20 Wilrich, P.T. (2013). Critical values of Mandel's h and k, the Grubbs and the Cochran test statistic. *AStA Advances in Statistical Analysis, 97(1),* 1-10.

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Initial values of pores' pressure and stress in the problems for soil consolidation

According to the basic model of V. Florin a method was developed for determining the initial excess pore pressure p_0 at all points of the soil mass. Here the sum of the main stresses in the skeleton of the soil was found. Moreover, the additional pressure due to the application of external load, with an instant consolidation of the soil goes to zero. As a design scheme, a two-phase soil cylinder compaction with the radius r, the height h with a permeable bottom and walls is assumed. A uniformly distributed load with intensity q is applied on some upper part $a < R$ of the cylinder area. In connection with the symmetry of the problem under investigation with respect to z axis, it is investigated in the cylindrical coordinates. On the basis of this result, the solution of the problem for the concentrated force is determined.

Keywords: pore pressure, stress, consolidation, soil, compaction, porosity.

Now the country has ambitious tasks for the development of science and capital construction. The solution of these tasks requires the most rational use of material and financial resources allocated for this construction. In this regard, the problem of designing and building high-rise buildings and large hydraulic structures on water-saturated clay soils poses a number of tasks for its complete solution. Studies of the causes of the deformation of many buildings and structures built in various regions of Kazakhstan, in particular, the South Kazakhstan region and beyond, have shown that water saturated soils, formed by the increase in groundwater, often lie at the base of the structures. At the same time, the deformation of the soil, determined by the influence of external pressure on it, reached unacceptable values.

The correct approach to solving these issues in all cases can facilitate the construction, speed up the time frame for its implementation and reduce the cost of the structure itself. However, any errors in this regard can be fraught with consequences. Here the main thing is to set the task correctly, especially in cases of erection of large and important structures erected on water-saturated clay soils, taking into account all the necessary conditions in which they find themselves. To achieve the greatest effect in solving these problems in all cases, you should strive to assess and predict the rate of sediment of the foundations of structures.

The process of compressing the soil layer usually occurs gradually over time. Therefore, there is a final sediment and a sediment changing over time. In clay soil, the compression process under static load occurs mainly due to the displacement of clay particles with partial destruction of the natural structure of the soil and its connections. The low permeability of the clay soil mainly determines the rate of its compression. At the same time, the slow destruction of its bonds under load determines the slow change in precipitation over time. At the same time, the duration of the precipitation of a layer of clay soil depends on the thickness of the compacted massif and reaches its final value after a long period of time.

For the calculation of the consolidation of heterogeneous soil bases in the device vertical drains need to know: structural properties of clay soils; basic assumptions for studying soil compaction; arrangements of the vertical drains in the soil foundations; properties of the heterogeneity of weak soils; equations of the state of the skeleton of the soil; the basic resolving equations of mechanics of compacted inhomogeneous soils; boundary conditions of problems; typical settlement schemes. Wherein:

– the excess pore pressure equals zero at $t > 0$ on the surface of a vertical drain with a radius r_0, i.e.

$$p_0(r_0, 0, t) = 0; \tag{1}$$

– the movement of water does not occur in the zones of influence of a vertical drain with a radius R, i.e.

$$\frac{\partial p}{\partial r} = 0 \quad \text{at} \quad r = R; \tag{2}$$

– the excess pore pressure is zero on the horizontal surface of the soil mass, i.e.

$$p(r,0,t)=0; \tag{3}$$

– the lower horizontal boundary of the soil mass is impenetrable. Consequently,

$$\frac{\partial p}{\partial z}=0 \ \text{ at } \ z=h. \tag{4}$$

Expressions (1)–(4) are used as boundary conditions for solving problems of the theory of consolidation of earth masses. To determine the initial excess pore pressure p_0 at all points of the soil mass, we use the basic model of V.A. Florin. At the same time, bearing in mind the unchangeability of the porosity coefficient at the initial moment of application of an external load, we find that the sum of the main stresses in the soil skeleton $\theta^{(0)}=0$. In addition, the additional pressure p^* due to the application of external load, with the instant consolidation of the soil goes to zero. Considering all this in accordance with the expression $\theta^{(0)}=\theta^*=n(p_0-p^*)$, we get

$$p_0=\frac{\theta^*}{n}, \tag{5}$$

where n takes one of the values 1, 2, 3 depending on the dimensionality of the studied problems.

The initial stresses in the soil mass skeleton according to the basic design model are found using the following formula:

$$\sigma_{ij}^{(0)}=\sigma_{ij}^*-\delta_{ij}p_0, \tag{6}$$

where i,j — take values $1,2$ for the flat task; i,j - values $1,2,3$ for the spatial problem; index (0) means that the values of these quantities correspond to the moment of instantaneous application of the load; δ_{ij} is the Kronecker symbol.

Moreover, expressions (6) for a two-dimensional problem with regard to dependence (5) can be represented as follows:

$$\sigma_{11}^{(0)}=-\sigma_{22}^{(0)}=(\sigma_{11}^*-\sigma_{22}^*)/2; \quad \sigma_{12}^{(0)}=\sigma_{12}^*. \tag{7}$$

For a three-dimensional problem, it has the form

$$\sigma_{ij}^{(0)}=\sigma_{ij}^*-\delta_{ij}\theta^*/3. \tag{8}$$

Here, the additional shear stresses arising at the initial moment of instantaneous load application are equal to their final values. The corresponding normal stresses are much smaller than the final values, and at each point of the ground environment, according to expressions (7), are equal in magnitude and inverse in sign.

It should be noted that in the case of a flat problem and a flat boundary surface, the determination of the above initial stresses in the soil skeleton after finding the values of p_0 is not difficult. Indeed, if the values θ^* are known, then the calculation of the stresses $\sigma_{11}^*,\sigma_{22}^*$ and σ_{12}^* are easily done using the expressions:

$$\sigma_{11}^*=\left(\theta^*+y\frac{\partial\theta^*}{\partial y}\right)/2; \quad \sigma_{22}^*=\left(\theta^*-y\frac{\partial\theta^*}{\partial y}\right)/2; \quad \sigma_{12}^*=-y\frac{\partial\theta^*}{\partial x}/2. \tag{9}$$

Considering expressions (7) and (9) we find:

$$\sigma_{11}^{(0)}=-\sigma_{22}^{(0)}=y\frac{\partial\theta^*}{\partial y}; \quad \sigma_{12}^{(0)}=-y\frac{\partial\theta^*}{\partial x}. \tag{10}$$

Expressions (5)–(10) applied to the axially symmetric problem are of the form:

$$p_0=\frac{\sigma_{11}^*+\sigma_{22}^*}{3}; \quad \sigma_{11}^*=\sigma_{rr}^*; \quad \sigma_{22}^*=\sigma_{33}^*=\sigma_{zz}^*; \tag{11}$$

$$\sigma_{ij}^{(0)}=\sigma_{ij}^*-\delta_{ij}p_0; \tag{12}$$

$$\sigma_{rr}^{(0)}=-\sigma_{zz}^{(0)}=(\sigma_{rr}^*-\sigma_{zz}^{(0)})/2; \quad \sigma_{rz}^{(0)}=\sigma_{rz}^*. \tag{13}$$

As can be seen from (11)–(13), the determinations of the initial and final stresses depend on the distribution of the instantaneous pressures in the soil. If the continuous function $p_0(r,z,t)$ for the axially symmetric problem was previously found in some way, then the initial stresses in the soil skeleton in relation to a limited compaction region can be expressed as

$$\sigma_{rr}^{(0)} = -\sigma_{zz}^{(0)} = -\frac{z}{2}\frac{\partial p_0(r,z)}{\partial z}; \quad \sigma_{rz}^{(0)} = -z\frac{\partial p_0(r,z)}{\partial r}, \tag{14}$$

where $p_0(r,z,t)$ is the initial distribution of pore pressure; r is a radius of the soil cylinder; $\sigma_{ij}^{(0)}$ is the initial pressures at the studied point.

Thus, in order to calculate the values of pore pressure for any moment of time and coordinates, it is necessary to know the values of these quantities for the initial $p_0(r,z,0) = p_{ini}$ point of time $t = 0$.

We proceed to the determination of the pressure in the pore fluid for the initial moment of time. To do this, we consider the compaction of a two-phase soil cylinder of radius r, height h with a permeable bottom and walls. Let at the same time on some upper part of the area $a < R$ a uniformly distributed load with intensity q is applied.

In view of the symmetry of the problem under study with respect to the z axis, it is convenient to study it in cylindrical coordinates.

Therefore, in order to determine p_{ini}, one should solve the equation

$$\nabla^2 p_0 = \frac{\partial}{\partial r}\left(k_r\frac{\partial p_0}{\partial r}\right) + \frac{1}{r}\cdot\frac{\partial}{\partial r}(k_r p_0) + \frac{1}{r^2}\cdot\frac{\partial}{\partial \varphi}\left(k_\varphi\frac{\partial p_0}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(k_z\frac{\partial p_0}{\partial z}\right) = 0. \tag{15}$$

If the pressure in the pore fluid does not depend on the angle, then instead of equation (15) we have

$$k_r\left(\frac{\partial^2 p_0}{\partial r^2} + \frac{1}{r}\frac{\partial p_0}{\partial r}\right) + k_z\frac{\partial^2 p_0}{\partial z^2} = 0. \tag{16}$$

Here $p_0 = p_{ini}$; k_r, k_z are the constant filtration coefficients in horizontal and vertical directions.

The solution of equation (16) when $k_r = k_z$ applied to the problem under consideration should satisfy the following boundary conditions:

$$\begin{gathered}\frac{\partial p_0}{\partial r} = 0 \ \text{ at } \ r = R;\\ \frac{\partial p_0}{\partial z} = 0 \ \text{ at } \ z = 0;\\ p_0 = q - p_{str} \ \text{ at } \ z = h, r \le a;\\ p_0 = 0 \ \text{ at } \ z = h, r > a.\end{gathered} \tag{17}$$

Here p_{str} is the structural strength of compression.

This solution can be represented as follows:

$$p_0(r,z,0) = (q - p_{str})\left(\frac{a^2}{R^2} + \frac{a}{R}\right)\sum_{k=1}^{\infty}\frac{J_1(\mu_k\frac{a}{R})ch\frac{\mu_k}{R}z}{\mu_k J_k^2(\mu_k)ch\frac{\mu_k}{R}h}J_0\left(\mu_k\frac{r}{R}\right), \tag{18}$$

where μ_k is the countless positive roots of the transcendental equation of the form $J_1(\mu) = 0$; $J_0(\mu)$, $J_1(\mu)$ are the Bessel functions of the first kind, respectively, of the zero and first orders.

From the expression (18) when $z = h$ we get

$$p_0(r,h,0) = \frac{qa^2}{R^2} + \frac{2aq}{R}\sum_{k=1}^{\infty}\frac{J_1(\mu_k\frac{a}{R})}{\mu_k J_k^2(\mu_k)}J_0\left(\mu_k\frac{r}{R}\right). \tag{19}$$

The sum of the series (19) at $r \le a$ is equal to q, and at $r > a$ equals zero. Consequently, the obtained solution (18) fully satisfies all the boundary conditions (17).

The solution of the problem for a concentrated force is obtained from the equation (16) under the following boundary conditions:

$$\begin{gathered}\frac{\partial p_0}{\partial r} = 0 \ \text{ at } \ r = R;\\ \frac{\partial p_0}{\partial z} = 0 \ \text{ at } \ z = 0;\end{gathered} \tag{20}$$

$$p_0 = \infty \quad \text{at} \quad z = h, \quad r = 0;$$

$$p_0 = 0 \quad \text{at} \quad z = h, \quad r \neq 0.$$

This solution under the boundary conditions (20) has the form

$$p_0(r,z,0) = \frac{Q}{\pi R^2} + \frac{Q}{\pi R^2} \sum_{k=1}^{\infty} \frac{J_0(\mu_k \frac{r}{R}) ch \frac{\mu_k}{R} z}{J_0^2(\mu_k) ch \frac{\mu_k}{R} h}. \tag{21}$$

The expression (21) can be obtained directly from the solution for a uniformly distributed load (18), assuming $Q = \pi a^2 q = const$ always at value of a that tends to zero.

Consider the more general case, i.e. the distributed external load varies in the time and for the boundary conditions of the form:

$$\alpha^{(oc)} \frac{\partial p_0}{\partial x_k} + \beta^{(oc)} p_0(x_k) = \gamma^{(oc)} q, \qquad k = 1,2,3. \tag{22}$$

Here $\beta^{(oc)} = 0$, $\gamma^{(oc)} = 0$, at $x = r = R$ or $z = 0$ and $\alpha^{(oc)} = 0, \gamma^{(oc)} = \beta^{(oc)}$ at $x = r \leq a, z \to h$ or $r > a, z \to h$ at $\alpha^{(oc)} = \gamma^{(oc)} = 0$. Note that the $\alpha^{(oc)}, \beta^{(oc)}, \gamma^{(oc)}$ are the boundary condition parameters.

The solution of equation (16) with the boundary conditions (22) can be represented as follows

$$p_0(r,z) = \frac{2}{R^2} \sum_{k=1}^{\infty} \frac{J_0(\mu_k \frac{r}{R}) ch \frac{\mu_k}{R} z}{J_k^2(\mu_k) ch \frac{\mu_k}{R} h} \int_0^a q(r,h) J_0\left(\mu_k \frac{r}{R}\right) dr. \tag{23}$$

After finding the change in the instantaneous pressures in the soil, according to formulas (11)–(13), the initial and final stresses in the soil skeleton can already be determined. At the same time we have

$$-\sigma_{rr}^{(0)} = \sigma_{zz}^{(0)} = -\frac{z}{2} \frac{2}{R^3} \sum_{k=0}^{\infty} \frac{\mu_k J_0(\mu_k \frac{r}{R}) ch \frac{\mu_k}{R} z}{J_k^2(\mu_k) ch \frac{\mu_k}{R} h} \int_0^a q(r,h) J_0\left(\mu_k \frac{r}{R}\right) dr;$$

$$\sigma_{rz}^{(0)} = -z \frac{2}{R^3} \sum_{k=0}^{\infty} \frac{\mu_k J_1(\mu_k \frac{r}{R}) ch \frac{\mu_k}{R} z}{J_k^2(\mu_k) ch \frac{\mu_k}{R} h} \int_0^a q(r,h) J_0\left(\mu_k \frac{r}{R}\right) dr;$$

$$\sigma_{rr}^* = \left(1 + z \frac{\partial}{\partial z}\right) p_0; \quad \sigma_{zz}^* = \left(1 - z \frac{\partial}{\partial z}\right) p_0; \quad \sigma_{rz}^* = \sigma_{zr}^* = -z \frac{\partial p_0}{\partial x}.$$

Consider a few special cases related to the loading of the upper surface of a compacted soil cylinder.

Case 1. On a part of the outer area with the radius $a < R$, a uniformly distributed load with intensity q is applied. For this case, the expression (23) is reduced to the form:

$$p_0(r,z) = \frac{2a(q - p_{str})}{R^2} \sum_{k=1}^{\infty} \frac{J_0(\mu_k \frac{r}{R}) ch \frac{\mu_k}{R} z}{J_k^2(\mu_k) ch \frac{\mu_k}{R} h} J_0\left(\mu_k \frac{r}{R}\right). \tag{24}$$

Expression (24) makes it possible to determine the distribution of pore pressure in a compacted two-phase soil cylinder for an initial point in time. Moreover, for a three-phase soil environment, it looks as follows:

$$p_0^t(r,z) = \frac{1}{\omega_0^{'}} p_0(r,z).$$

Here $\omega_0^{'}$ is the number that takes into account the three-phase soil. Using relations (9)–(14) we find the initial and final stresses in the soil skeleton.

Calculation of vertical drains. Existing baseline calculations with vertical drains are also based on the theory of filtration consolidation. The calculations are to determine the degree of compaction of the soil of the base under the influence of an external load at any time.

The use of vertical sand drains usually allows to reduce the time for consolidation of soil foundations composed of weak water-saturated clay soils in the construction of transport, industrial and hydraulic structures.

For the calculation of the vertical sand drains, the soil compaction is considered around a single drain. To do this, in a soil array with planes that limit the scope of one drain from another, cut a prismatic block of water-saturated clay soils that the drains were located along the vertical axis of the block. Then, in order to calculate the stress-strain state, the prismatic block is replaced by a soil cylinder of the same volume with a drain along the vertical axis of the cylinder.

In this case, the calculation of vertical drains also reduces to solving the axially symmetric spatial problem of the theory of the multiphase soils consolidation. In particular, determining the pressures in the pore fluid for the initial moment of time consists in solving equation (16) under appropriate boundary conditions. Moreover, in calculating the problems of filtration consolidation for the case of application of vertical drains, the boundary conditions are taken on the basis of the existing classical calculations of filtration consolidation [1–4].

Therefore, the boundary conditions for solving the problem in the case of application of vertical drains for the initial moment of time has the form (22). And $\alpha^{(oc)} = \gamma^{(oc)} = 0$ at $r = r_0$, i.e. the excess pore pressure or pressure function on the surface of a vertical drain of a radius r_0 is equal to 0; $\alpha^{(oc)} = \gamma^{(oc)} = 1$ at $t = \tau_1$, i.e. at the time of application of the load, the pore pressure on the surface of the soil layer is equal $\frac{1}{3}\theta^* + p^*$; $\beta^{(oc)} = 0, \ \ \alpha^{(oc)} = \gamma^{(oc)} = 0$, at $r = R$, i.e. through the surface of the cylinder of the zone of influence of the vertical drain with a radius R as a result of symmetry, the flow of water does not occur; $\beta^{(oc)} = 0, \gamma^{(oc)} = 0$, at $z = 0$, i.e. the lower horizontal boundary of the soil mass is impenetrable, and due to the symmetry of the flow there is no movement of water through the surface.

The solution of the equation (16) under these boundary conditions can be represented as

$$p_0(r,z) = \frac{\pi^2}{2} \sum_{i=0}^{\infty} \frac{\mu_i^2 J_1\left(\mu_i \frac{R}{\sqrt{k_r}}\right) \frac{1}{k_r} \int_{\frac{r_0}{\sqrt{k_r}}}^{\frac{R}{\sqrt{k_r}}}}{ch(\mu_i \frac{h}{\sqrt{k_z}})[J_0^2(\mu_i \frac{r_0}{\sqrt{k_r}}) - J_1^2(\mu_i \frac{R}{\sqrt{k_r}})]} V_0\left(\mu_i \frac{r}{\sqrt{k_r}}\right) ch\mu_i \frac{z}{\sqrt{k_r}}, \tag{25}$$

$$V_0(\mu_i \bar{r}) = J_0(\mu_i \bar{r_0}) Y_0(\mu_i \bar{r}) - Y_0(\mu_i \bar{r_0}) J_0(\mu_i \bar{r}).$$

Using the expression (25), we find the pressure in the pore fluid of the vertical drain for the initial moment of time.

The solution of the problem for a uniformly distributed force is determined from relation (25). In this case, in (25), the value $q(r,h,t)$ is replaced by q and we perform the integrations. It should be noted that the problems of soil consolidation were studied in [5–15] with regard to physical nonlinearity and heterogeneity of the earth masses, respectively.

References

1 Терцаги К. Теория механики грунтов (под ред. Н.А. Цытовича) / К. Терцаги. — М.: Госстройиздат, 1961. — С. 3–214.

2 Герсеванов Н.М. Собрание сочинений / Н.М. Герсеванов. — Т. 1, 2. — М.: Стройвоенмориздат, 1948. — 375 с.

3 Флорин В.А. Основы механики грунтов / В.А. Флорин. — М.: Госстройиздат, 1961. — Т. 2. — 543 с.

4 Цытович Н.А. Механика грунтов / Н.А. Цытович. — М.: Изд. литературы по строительству, архитектуре и строительным материалам, 1963. — 633 с.

5 Abzhapbarov A. Problems of the theory of the consolidation solved in the special functions / A. Abzhapbarov, A. Dasibekov, P. Duisebayeva, A. Polatbek // AIP conference proceedings 1759, 020058 (2016); doi:10.1063/1.4959672.

6 Ashirbayev N. The features of a non-stationary state of stress in the elastic multisupport construction / N. Ashirbayev, Zh. Ashirbayeva, A. Abzhapbarov, M. Shomanbayeva // AIP conference proceedings 1759, 020039 (2016); doi:10.1063/1.4959653.

7 Безволев С.Г. Комплексная методика определения параметров инженерного расчета консолидации водонасыщенных вязких грунтов / С.Г. Безволев // Инженерная геология. — 2018. — 13. — № 4, 5. — С. 64–72.

8 Тер-Мартиросян З.Г. Опыт преобразования слабых водонасыщенных грунтов сваями конечной жесткости / З.Г. Тер-Мартиросян, А.З. Тер-Мартиросян, В.В. Сидоров // Вестн. МГСУ. — 2018. — 13. — № 3 (114). — С. 271–281.

9 Тер-Мартиросян А.З. Экспериментально-теоретические основы преобразования слабых водонасыщенных глинистых грунтов при поверхностном и глубинном уплотнении / А.З. Тер-Мартиросян, З.Г. Тер-Мартиросян // Инженерная геология. — 2015. — № 4. — С. 16–25.

10 Yunusov A.A. Multidimensional problems of soils' consolidation with modulus of deformation, variable in its depth / A.A. Yunusov, A. Dasibekov, B.N. Korganbaev // News of the national academy of sciences of the Republic of Kazakhstan. Physico-mathematical series. — 2018. — 1. — No. 317. — P. 75–86.

11 Campos L.M.B.C. On 36 forms of the acoustic wave equation in potential flows and inhomogeneous media / L.M.B.C. Campos // Applied Mechanics Reviews. — 2007. — 60. — P. 149–171.

12 Lu G. A multiphysics-viscoplastic cap model for simulating blast response of cemented tailings backfill / G. Lu, M. Fall, L. Cui // Journal of Rock Mechanics and Geotechnical Engineering. — 2017. — 9. — No. 3. — P. 551–564. doi: 10.1016/j.jrmge.2017.03.005.

13 Abdollahi F. Surface and Nonlocal Effects on Coupled In-Plane Shear Buckling and Vibration of Single-Layered Graphene Sheets Resting on Elastic Media and Thermal Environments using DQM / F. Abdollahi, A. Ghassemi // Journal of Mechanics. — 2018. — 34. — No. 6. — P. 847–862. doi:10.1017/jmech.2018.14

14 Chen Q. Progress in modeling of fluid flows in crystal growth processes / Q. Chen, Y. Jiang, J. Yan // Progress in natural science. — 2008. — 18. — No. 12. — P. 1465–1473.

15 Cuellar P. Pore pressure buildup and soil stress relaxation in monopile foundations of offshore wind converters / P. Cuellar, M. Baessler, S. Georgi, W. Rucker // Bautechnik. — 2012. — 89. — No. 9. — P. 585–593.

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Қысымның кеуек қуысындағы бастапқы шарты және кернеудің топырақ консолидациясындағы есебі

Топырақтың жылжу қасиеті В.А.Флорин берген түрінде жазылған серпімді жылжымалы теориясына бойсындырып, кеуек қуысындағы бастапқы қысымды p_0 анықтау әдістері келтірілген. Мақалада биіктігі h радиусы r болатын, едені мен жандары су өткізетін екіфазалы топырақтан тұратын цилиндр тығыздалуының алғашқы уақыттағы кернеуі зерттелген. Цилиндр жоғары бетіне ауданы $a < R$ бетіне қарқындылығы q болған тең жайылған күш қойылған. Зерттеліп отырған есептің z өсіне симметриялы болуына байланысты цилиндрлік координаталарда қарастырылған. Осындай қойылымында топырақтың кеуегіндегі суға түсетін басым күшімен қатар, оның қаңқасындағы бастапқы кездегі кернеуді анықтайтын есептеу өрнектері табылған.

Кілт сөздер: кеуек қуысындағы қысым, кернеу, консолидация, топырақ, тығыздық, кеуектілік, цилиндрлік координаттар, шекаралық шарттар.

А. Дасибеков, А. Абжапбаров, Н.К. Аширбаев, М.Т. Шоманбаева

Начальные значения порового давления и напряжений в задачах консолидации грунтов

Согласно основной модели В.А. Флорина, разработана методика определения начального избыточного порового давления p_0 во всех точках грунтового массива. При этом найдена сумма главных напряжений в скелете грунта. Кроме этого, дополнительное давление, обусловленное приложением внешней нагрузки, при мгновенной консолидации грунта обращается в нуль. В качестве расчетной схемы принято уплотнение двухфазного грунтового цилиндра радиусом r, высотой h с водопроницаемым дном и стенками. На некоторой верхней части площади цилиндра $a < R$ приложена равномерно-распределенная нагрузка с интенсивностью q. В связи с симметрией данной задачи относительно оси z она исследована в цилиндрических координатах. На основе этого результата определено решение задачи для сосредоточенной силы.

Ключевые слова: поровое давление, напряжение, консолидация, грунт, уплотнение, пористость.

References

1 Tertsagi, K. (1961). *Teoriia mekhaniki hruntov [Theory of soil mechanics].* Moscow: Hosstroiizdat [in Russian].

2 Gersevanov, N.M. (1948). *Sobranie sochinenii [Collected Works]* (Vols. 1, 2). Moscow: Stroivoenmorizdat [in Russian].

3 Florin, V.A. (1961). *Osnovy mekhaniki hruntov [Fundamentals of soil mechanics].* (Vol. 2). Moscow: Hosstroiizdat [in Russian].

4 Tsytovich, N.A. (1963). *Mekhanika hruntov [Soil mechanics].* Moscow: Izd. literatury po stroitelstvu, arkhitekture i stroitelnym materialam [in Russian].

5 Abzhapbarov, A., Dasibekov, A., Duisebayeva, P., & Polatbek, A. (2016). Problems of the theory of the consolidation solved in the special functions. *AIP conference proceedings 1759*, 020058; doi:10.1063/1.4959672.

6 Ashirbayev, N., Ashirbayeva, Zh., Abzhapbarov, A., & Shomanbayeva, M. (2016). The features of a non-stationary state of stress in the elastic multisupport construction. *AIP conference proceedings 1759*, 020039; doi:10.1063/1.4959653.

7 Bezvolev, S.G. (2018). Kompleksnaia metodika opredeleniia parametrov inzhenernoho rascheta konsolidatsii vodonasyshchennykh viazkikh hruntov [Comprehensive method for determining the engineering calculation parameters for the consolidation of water-saturated viscous soils]. *Inzhenernaia heolohiia – Engineering Geology World, 13, 4-5*, 64–72 [in Russian].

8 Ter-Martirosyan, Z.G., & Ter-Martirosyan, A.Z., & Sidorov, V.V. (2018). Opyt preobrazovaniia slabykh vodonasyshchennykh hruntov svaiami konechnoi zhestkosti [Experience of transformation of weak water-saturated soils using piles of finite stiffness]. *Vestnik MHSU – Proceedings of the Moscow State University of Civil Engineering, 13, 3 (114)*, 271–281 [in Russian].

9 Ter-Martirosyan, A.Z., & Ter-Martirosyan, Z.G. (2015). Eksperimentalno-teoreticheskie osnovy preobrazovaniia slabykh vodonasyshhennykh hlinistykh hruntov pri poverkhnostnom i hlubinnom uplotnenii [Experimental and theoretical basis of transformation of weak water-saturated clay soils at surface and deep compaction]. *Inzhenernaia heolohiia – Engineering Geology World, 4*, 16–25 [in Russian].

10 Yunusov, A.A., Dasibekov A., Korganbaev, B.N. (2018). Multidimensional problems of soils' consolidation with modulus of deformation, variable in its depth. *News of the NAS of the RK. Phys. – Mathem ser., Vol. 1 (317)*, 75–86.

11 Campos, L.M.B.C. (2007). On 36 forms of the acoustic wave equation in potential flows and inhomogeneous media. *Applied Mechanics Reviews, 60,* 149–171.

12 Lu, G., Fall, M., Cui, L. (2017). A multiphysics-viscoplastic cap model for simulating blast response of cemented tailings backfill. *Journal of Rock Mechanics and Geotechnical Engineering, 9(3)*, 551–564. doi: 10.1016/j.jrmge.2017.03.005.

13 Abdollahi, F., Ghassemi, A. (2018). Surface and Nonlocal Effects on Coupled In-Plane Shear Buckling and Vibration of Single-Layered Graphene Sheets Resting on Elastic Media and Thermal Environments using DQM. *Journal of Mechanics, 34(6)*, 847–862. doi:10.1017/jmech.2018.14

14 Chen, Q., Jiang, Y., Yan, J. (2008). Progress in modeling of fluid flows in crystal growth processes. *Progress in natural science, 18(12)*, 1465–1473.

15 Cuellar, P., Baessler, M., Georgi, S., Rucker, W. (2012). Pore pressure buildup and soil stress relaxation in monopile foundations of offshore wind converters. *Bautechnik, 89(9)*, 585–593.

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Conveyance of a particle by a vertical screw, which is limited by a coaxial fixed cylinder

Differential equations of relative particle motion along the periphery of a vertical screw have been deduced which rotates about its axis and is limited by a coaxial fixed cylinder. The equations have been solved by applying numerical methods and the trajectories of relative particle helical motion, that is to say, movement along the edge of a screw, which is common for a screw surface and for a limiting cylinder, have been built. The force of particle friction on the screw surface and on the surface of a cylindrical cover has been taken into consideration. The cases, when a helix angle, that is to say, the angle of screw edge inclination, is less than the angle of particle friction on the surface of a screw, when it is equal to a friction angle and when it is higher than a friction angle, have been considered. In addition, a specific case, when a screw is fixed and a particle moves downwards by gravity, which takes place in spiral chutes, has been analyzed. Kinematic characteristics-time curves have been presented.

Keywords: particle, vertical screw, angular velocity of rotation, cylindrical cover, differential equations, kinematic parameters.

Introduction

A screw surface is a universal structural element of various machines. The surface of a screw conoid, which is referred to as a screw, is the most common one. It is widely used in screw conveyers for transporting technological material. A separate element of the material is a body, movement on a running surface of which is very difficult to describe, because, in this case, the inertia forces that arise from its rotation must be taken into account. In some cases, these forces can be neglected or they can be ignored, if the size of a body is small and it is taken as a material particle [1] or in case of low angular velocities of a body, for example, when handling machinery is operated [2]. It is essential to study the movement of material particles on the surface of a screw, which rotates about its axis. Paper [3] considers their movement on a vertical screw beginning from the axis of rotation to the point when they reach a limiting cylinder. The conveyance of particles upwards takes place after they collide with a limiting cylindrical cover. The paper investigates the movement of a particle when interacting with two surfaces: the surface of a running screw and the surface of a fixed cylindrical cover. The nature of material particles is rather broad. They include not only mechanical particles but fluid or gas particles, which interact with running surfaces, as well [4]. The movement of mechanical particles on the surface of a running soil tilling disk is considered in paper [5], its movement on the surface of a cylinder is covered in paper [6] and its motion on the surface of a cone is investigated in paper [7]. The study of the movement of particles on screw surfaces is of practical importance because such surfaces are used in screw conveyers [8, 9], in separators [10] and other in the papers [11–18].

Material and method.

In paper [4] it is stated that, if a particle moves by gravity on the surface of a fixed vertical screw conoid, it moves away from its axis. It happens because of the fact that a particle moves along a curvilinear trajectory, which results in the appearance of a centrifugal force that causes particle movement to the periphery. If a screw conoid rotates about its axis, a particle performs a compound motion: it slides on the surface of a conoid, but in absolute motion it rotates about the axis of a conoid as well, which causes particle movement to the periphery. Thus, it reaches a cylindrical cover. The common line of a conoid and a cover (that is to say, the line of their intersection) is a screw line. A particle slides along this screw line having simultaneous contact with the surface of a conoid (a screw) and with the surface of a cylindrical cover. Let us begin the development of the mathematical model of particle movement by writing the parametric equation of a helical line:

$$x = R\cos\alpha; \qquad y = R\sin\alpha; \qquad z = -R\alpha\,\mathrm{tg}\ \beta, \tag{1}$$

where R — the radius of a limiting cylinder — a constant value; β — the angle of a helix — a constant value; α — the rotation angle of the point of a helix about its axis — an independent variable.

Two surfaces pass through a helix (1): a screw conoid (a screw) and a cylinder. When a particle contacts these two surfaces, reaction forces appear, which are directed normally to them. In order to find the directions of these normal lines, it is necessary to have surface equations.

The surface of a screw is made up by the plural of rectilinear generators, which are parallel to a horizontal plane. Their one end passes through a helix (1) and their another end is directed to the axis of a screw (Fig. 1, a). Based on the screw principle, let us write its parametric equations:

$$\begin{gathered} X = (R-u)\cos\alpha; \\ Y = (R-u)\sin\alpha; \\ Z = -R\alpha\,\mathrm{tg}\beta, \end{gathered} \tag{2}$$

where u — the length of a rectilinear generator of a screw — the second independent variable of the surface.

Counting the length of a generator begins from a helix. If $u = R - r$, where r — the radius of a screw shaft, the surface will be limited by two helixes (Fig. 1, a). At $u = 0$ the parametric equations of a helix are obtained (1). In order to differentiate surface equations from line equations, let us write them in lower case for a line and in upper case for a surface.

A vertical screw rotates about its axis with the angular velocity ω. A particle is on a helix (1) and, at the same time, it contacts a moving part of a screw surface and a fixed surface of a cylindrical cover (Fig. 1, b). The following forces are exerted on it: weight force mg (m — the mass of a particle, $g = 9.81 m/s^2$ — the acceleration of gravity), the reaction N of the conoid surface directed along the normal to its surface, the reaction N_R, directed along the normal to the surface of a cylinder and the reaction force F_f (Fig. 1, b). Suppose a particle moves upwards in absolute motion, then the friction force on a screw surface F_{fa} is directed oppositely to the direction of sliding, and the one on a cylinder surface F_{fc} — oppositely to the velocity of absolute motion (Fig. 1, b). The vectors of these forces will be tangential relative to the trajectory of relative motion (the helix, along which a particle is sliding) and absolute motion (the cylindrical line on the cover, which is not shown). They are non-collinear, but act in opposite directions.

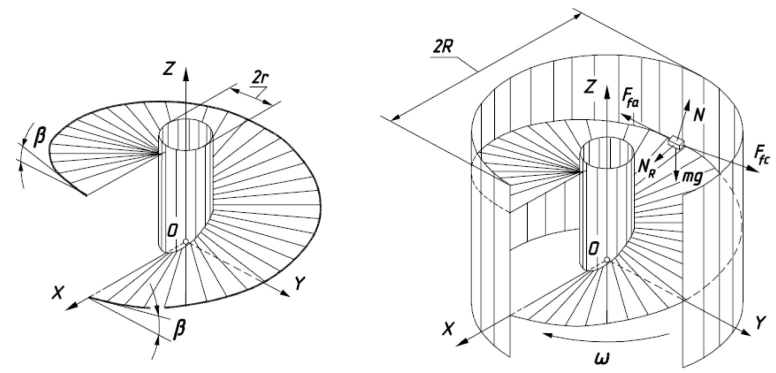


Figure 1. To the conveyance of a particle by a vertical screw:
a) one flight of a screw surface, limited by the outer edge — a helix with the inclination angle β;
b) the forces exerted upon a particle, with interacts with two surfaces:
the one of a screw and the one of a cylindrical cover

Let us derive a motion equation in the form of $m\overline{a} = \overline{F}$, where m –particle mass, $\overline{a}$ – the vector of absolute acceleration, $\overline{F}$– the resultant vector of the exerted forces listed above. Let us write the vector equation in projections onto OXYZ coordinates.

If we assume the variable α to be time-dependent t in the equations (1), that is to say, $\alpha = \alpha(t)$, this internal dependence will define the law of particle sliding along a helix, that is to say, the law of relative motion. Thus, under this condition, the equations (1) are the equations of relative motion.

A screw rotates about its axis with the angular velocity ω. During the time t it rotates by the angle $\alpha_a = -\omega t$ and moves along a helix for a certain distance, according to the equations (1). The direction of screw rotation is chosen so that a particle can move upwards in absolute motion, when it slides along a helix. Let us rotate the helix (1) about OZ axis by the angle $\alpha_a = -\omega t$, according to the known rotation equations:

$$\begin{gathered} x_a = R\cos\alpha\cos(-\omega t) - R\sin\alpha\sin(-\omega t) = R\cos(\omega t - \alpha); \\ y_a = R\cos\alpha\sin(-\omega t) + R\sin\alpha\cos(-\omega t) = -R\sin(\omega t - \alpha); \\ z_a = -R\alpha \mathrm{tg}\beta. \end{gathered} \quad (3)$$

The parametric equations (3) take into consideration two rotational motions: by the angle $\alpha = \alpha(t)$ in relative motion and by the angle $\alpha_a = -\omega t$ in translational motion, thus, they are the equations of absolute particle motion.

Let us differentiate the equations (3) with respect to the time t and we obtain the projections of the absolute velocity vector:

$$\begin{gathered} x_a' = -R(\omega - \alpha')\sin(\omega t - \alpha); \\ y_a' = -R(\omega - \alpha')\cos(\omega t - \alpha); \\ z_a' = -R\alpha' \mathrm{tg}\beta. \end{gathered} \quad (4)$$

Its value can be found as a geometric sum of the projections (4):

$$V_a = \sqrt{x_a'^2 + y_a'^2 + z_a'^2} = \frac{R}{\cos\beta}\sqrt{\omega(\omega - 2\alpha')\cos^2\beta + \alpha'^2}. \quad (5)$$

Let us find the projections of the unit vector, which sets the direction of particle absolute velocity, by dividing the equations (4) by the velocity value (5):

$$\begin{gathered} T_{Vax} = -\frac{(\omega - \alpha')\sin(\omega t - \alpha)\cos\beta}{A}; \\ T_{Vay} = -\frac{(\omega - \alpha')\cos(\omega t - \alpha)\cos\beta}{A}, \\ T_{Vaz} = -\frac{\alpha'\sin\beta}{A}, \end{gathered} \quad (6)$$

where $A = \sqrt{\omega(\omega - 2\alpha')\cos^2\beta + \alpha'^2}$.

Having differentiated the equation (4), we obtain the projections of the absolute acceleration w of a particle:

$$\begin{gathered} x_a'' = R\alpha''\sin(\omega t - \alpha) - R(\omega - \alpha')^2\cos(\omega t - \alpha); \\ y_a'' = R\alpha''\cos(\omega t - \alpha) + R(\omega - \alpha')^2\sin(\omega t - \alpha); \\ z_a'' = -R\alpha'' \mathrm{tg}\beta. \end{gathered} \quad (7)$$

In order to break the equation $m\overline{a} = \overline{F}$ in projections onto the axes of OXYZ coordinates, it is necessary to determine the directions of the forces, which are shown in Figure 1, b. In case of one of the forces — the friction force F_{fc} — its direction is found in the form of a unit vector (6), here, it is worth taking into account that the action of the force F_{fc} is directed oppositely to this vector. Let us find the direction of the friction force F_{fa}, which is directed oppositely to the vector of relative velocity. Let us find the projections of this vector by differentiating the equations (1):

$$x' = -R\alpha'\sin\alpha;$$

$$y' = R\alpha' \cos\alpha; \tag{8}$$
$$z' = -R\alpha' \operatorname{tg}\beta.$$

Let us find the value of absolute particle motion:

$$V = \sqrt{x'^2 + y'^2 + z'^2} = \frac{R\alpha'}{\cos\beta}. \tag{9}$$

Let us find the projections of a unit directing vector of relative velocity by dividing the projections (8) of this velocity by its absolute value (9):

$$T_{Vx} = -\cos\beta\sin\alpha; \qquad T_{Vy} = \cos\beta\cos\alpha; \qquad T_{Vz} = -\sin\beta. \tag{10}$$

The vector projections (10) are written without taking into consideration helix rotary motion. All the forces are projected onto fixed OXYZ coordinates. In order to exert the force F_{fa} at the point of particle location, it is necessary to rotate the projections (10) by the angle $(-\omega t)$ about Oz axis. Afterwards they take the following form:

$$\{\cos\beta\sin(\omega t - \alpha); \qquad \cos\beta\cos(\omega t - \alpha); \qquad -\sin\beta\}. \tag{11}$$

Let us find the direction of the reaction forces of a screw surface and of a cylindrical cover surface.

The direction of a normal to the surface is determined by a vector product of two vectors that pass through the point on the surface (the point where a particle is located) and the tangents to the coordinate lines, which pass through this point. Let us find a normal vector to the screw surface. The vectors, which are tangent to the coordinate lines of the surface, are partial derivatives of the equations (2) of a screw:

$$\begin{aligned} \frac{\partial X}{\partial\alpha} &= -(R-u)\sin\alpha; & \frac{\partial X}{\partial u} &= -\cos\alpha; \\ \frac{\partial Y}{\partial\alpha} &= (R-u)\cos\alpha; & \frac{\partial Y}{\partial u} &= -\sin\alpha; \\ \frac{\partial Z}{\partial\alpha} &= -R\operatorname{tg}\beta; & \frac{\partial Z}{\partial u} &= 0. \end{aligned} \tag{12}$$

Let us find a vector product of the vectors (12):

$$\begin{vmatrix} X & Y & Z \\ -(R-u)\sin\alpha & (R-u)\cos\alpha & -R\operatorname{tg}\beta \\ -\cos\alpha & -\sin\alpha & 0 \end{vmatrix} = \tag{13}$$
$$= \{-R\operatorname{tg}\beta\sin\alpha; \qquad R\operatorname{tg}\beta\cos\alpha; \qquad R-u\}.$$

The normal vector (13) is not a unit vector. Its position on the surface is determined by two internal coordinates u and α. It is necessary to know the normal to the surface of a screw on a helix at the point of particle location, that is to say, at $u = 0$. Let us substitute $u = 0$ into the vector equation (13) and reduce it to a unit one:

$$\{-\sin\beta\sin\alpha; \qquad \sin\beta\cos\alpha; \qquad \cos\beta\}. \tag{14}$$

Analogically, the normal to a cylindrical cover can be found. However, its projections can be found much easier. Figure 1, b shows that a normal to a cylinder is parallel to a horizontal plane and is directed toward its axis. Let us write the result:

$$\{-\cos\alpha; \qquad -\sin\alpha; \qquad 0\}. \tag{15}$$

For the foregoing reasons, let us rotate the unit vectors (14), (15) about the angle $(-\omega t)$. After this, the unit normal vector to the surface of a screw at the point of particle location can be written as:

$$\{\sin\beta\sin(\omega t - \alpha); \qquad \sin\beta\cos(\omega t - \alpha); \qquad \cos\beta\}. \tag{16}$$

The unit normal vector to a cylindrical cover at the point of particle location takes the following form:

$$\{-\cos(\omega t - \alpha); \qquad \sin(\omega t - \alpha); \qquad 0\}. \tag{17}$$

The defined directions of the exerted forces are the following: for the friction forces F_{fc} and F_{fa} — by the unit vectors (6) and (11) taken opposite in sign; for the surface reactions N of a screw and N_R of a cylindrical cover — by the unit vectors (16) and (17), respectively. The last force — the force of particle weight mg — is directed downwards, thus, the unit vector is set by the projections:

$$\{0\,; \qquad 0\,; \qquad -1\}. \tag{18}$$

The value of the friction forces F_{fa} and F_{fc} is determined by the value of surface reaction and the corresponding friction coefficient: $F_{fa} = fN, F_{fc} = f_R N_R$, where f and f_R — the coefficients of particle friction on the surface of a screw and on the surface of a cylindrical cover, respectively.

Let us break the vector equation $m\overline{a} = \overline{F}$ down in projections onto the axes of OXYZ coordinates, taking into consideration the directions (6), (11), (16), (17) and (18) of the corresponding forces $F_{fc} = f_R N_R$, $F_{fa} = fN, N, N_R$ and the weight force mg:

$$\begin{aligned} mx_a'' &= f_R N_R \frac{(\omega-\alpha')\sin(\omega t-\alpha)\cos\beta}{\sqrt{\omega(\omega-2\alpha')\cos^2\beta+\alpha'^2}} - fN\cos\beta\sin(\omega t-\alpha) + \\ &\quad + N\sin\beta\sin(\omega t-\alpha) - N_R\cos(\omega t-\alpha); \\ my_a'' &= f_R N_R \frac{(\omega-\alpha')\cos(\omega t-\alpha)\cos\beta}{\sqrt{\omega(\omega-2\alpha')\cos^2\beta+\alpha'^2}} - fN\cos\beta\cos(\omega t-\alpha) + \\ &\quad + N\sin\beta\cos(\omega t-\alpha) + N_R\sin(\omega t-\alpha); \\ mz_a'' &= f_R N_R \frac{\alpha'\sin\beta}{\sqrt{\omega(\omega-2\alpha')\cos^2\beta+\alpha'^2}} + fN\sin\beta + N\cos\beta - mg. \end{aligned} \tag{19}$$

Let us substitute the equations of absolute accelerations from (7) into (19) and we obtain the system of three differential equations with three unknown functions: $\alpha = \alpha(t), N = N(t)$ and $N_R = N_R(t)$. Let us solve the system of the equations for α'', N, N_R:

$$\alpha'' = \frac{g\cos\beta}{R}(\sin\beta - f\cos\beta) + f_R(\omega-\alpha')^2\cos\beta\frac{\omega\cos\beta(\cos\beta+f\sin\beta)-\alpha'}{\sqrt{\omega(\omega-2\alpha')\cos^2\beta+\alpha'^2}}. \tag{20}$$

$$N = m\cos\beta\left(g - \frac{f_R R\omega(\omega-\alpha')^2\sin\beta}{\sqrt{\omega(\omega-2\alpha')\cos^2\beta+\alpha'^2}}\right). \tag{21}$$

$$N_R = mR(\omega-\alpha')^2. \tag{22}$$

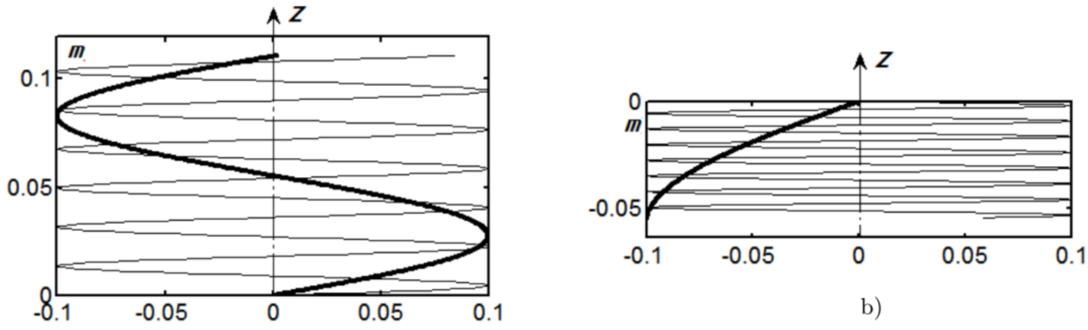
The differential equation (20) does not depend on (21) and (22), that is why it can be solved separately. The surface reactions (21) and (22) are found after the equation (20) is solved. In order to solve it, we need to apply numerical methods.

During its upward movement, a particle can move with various velocities. At the same angular velocity of screw rotation, the velocity of its upward movement z_a' will depend on the value of a helix angle β. After a transient process, that is to say, after the movement is stabilized, this velocity and the angular velocity of particle sliding α' will be steady. Thus, the angular acceleration $\alpha'' = 0$. The substitution of this value into the differential equation (20) enables us to find the value of α' at the preset screw parameters, the angular velocity of its rotation and friction coefficients.

Results

The investigation has been conducted for the cases of various values of the angle β. For example, at $\beta = 10^0,\ R = 0.1m,\ \omega = 15s^{-1},\ f = f_R = 0.3$ we obtain $\alpha' = 2.1s^{-1}$. At $\beta = 20^0$ and when other parameters are constant, we obtain: $\alpha' = -0.5s^{-1}$. Thus, there is a change in the direction of particle sliding. It means that at $\beta = 10^0$ a particle moves upwards (Fig. 2, a) and at $\beta = 20^0$ it moves downwards (Fig. 2, b). At $\alpha' = 0$ a particle does not slide, that is to say, a particle «sticks» and rotates around a circle in absolute motion. In this case, it is necessary to find the corresponding helix angle β. Let us substitute $\alpha' = 0$ into the equation (20), equate it to zero and solve it for the angle β:

$$\beta = \mathrm{Arctg}\frac{Rf_R\omega^2 - fg}{Rf_R f\omega^2 + g}. \tag{23}$$



a) b)

Figure 2. Relative and absolute (heavy line) trajectories of particle movement at $R = 0.1m$, $\omega = 15s^1$, $f = f_R = 0.3$ built according to the equations (1) and (3), at $\alpha = \alpha' \cdot t$ over the time $t = 3s$:
a) $\beta = 10^0$ – a particle moves upwards; b) $\beta = 20^0$ – a particle moves downwards

According to (23), when $R = 0.1m$, $\omega = 15s^{-1}$, $f = f_R = 0.3$, the boundary value of a helix angle is $\beta = 17.8^0$. If $\beta \geq 17.8^0$, the upward movement of a particle at the preset parameters is not possible. If the angular velocity of screw rotation is increased, the boundary value of β increases as well, but there is a certain limit. Let us find the boundary, which the expression (23) approaches to, at the unbounded increase of the angular velocity of rotation ω:

$$\lim_{\omega\to\infty} \operatorname{Arctg}\frac{Rf_R\omega^2 - fg}{Rf_Rf\omega^2 + g} = \operatorname{Arcctg} f. \tag{24}$$

Arctg f is the angle of particle friction on the surface of a screw and, in our case, it is equal to Arctg $0.3 = 16.7^0$. According to (24), the boundary value of the helix angle β is Arcctg $0.3 = 73.3^0$, that is to say, these two angles add up to a right angle. At $\beta \geq 73.3^0$ particle upward movement is not possible for any angular velocities of screw rotation.

If a certain angular velocity ω of screw rotation is set, we can find the boundary value of a helix angle by the formula (23). In this range (from zero to the boundary value) there is a specific angle β, which provides the maximum velocity of particle upward transportation. The task is to determine the peak value of the last expression in the equations (4) or (8), since the vertical components of relative and absolute velocities are equal.

Since the equation (20) for a stationary process at $\alpha'' = 0$ must be solved by means of numerical methods in order to determine the value of the angular velocity α' of particle sliding, the determination of the optimum angle β will be carried out in two stages. At the first stage, from the equation (20) we evaluate α' for the preset angles β from the acceptable range of their values. At the second stage, by substituting the determined values of α' and the corresponding angles β into the last expression in the equations (4) or (8), we determine the vertical component of the velocity of a particle. According to these data, it is possible to plot a dependency graph $z' = z'(\beta)$, which determines the optimum value of the angle β. Let us present the simplified variant of the calculation for $\omega = 15s^{-1}$ (the acceptable range of the values of the angle $\beta = 0 \ldots 17.8^0$):

$$\beta = 5^0; \qquad \alpha' = 3.53s^{-1}; \qquad z' = 0.03m/s;$$

$$\beta = 10^0; \qquad \alpha' = 2.08s^{-1}; \qquad z' = 0.037m/s;$$

$$\beta = 15^0; \qquad \alpha' = 0.73s^{-1}; \qquad z' = 0.02m/s.$$

The obtained results show that the highest velocity of particle upward conveyance is provided by the angle $\beta = 10^0$. According to the same method of calculation, let us find the optimum value of the angle β for the angular velocity $\omega = 25s^{-1}$: $\beta = 20^0$. Here, the vertical component of particle velocity is equal to $z' = 0.29m/s$. If we take into account that the acceptable range of the angle values for the velocity $\omega = 25s^{-1}$ is equal to $\beta = 0 \ldots 45.7^0$, it may be concluded that the optimum value of the angle β is close to the average value of the acceptable range. Paper [8] presents the formula for the determination of this angle in case when the angular velocity of screw rotation increases infinitely ($\omega \to \infty$):

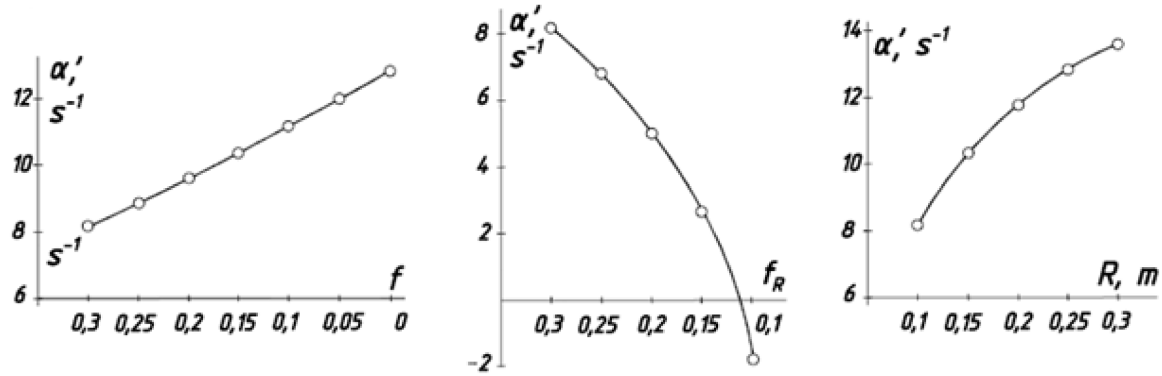
$$\beta = 45^0 - 0.5 \operatorname{arctg} f. \tag{25}$$

Let us find $\beta = 36.65^0$ from (25) for $f = 0$. This value is exactly two times less than the boundary one ($\beta = 73.3^0$), that is to say, it is precisely in the middle of the acceptable range.

Let us consider how the coefficients of friction f and f_R influence the velocity of particle upward movement. If there is a decrease in the coefficient f, the angular velocity α' of particle sliding increases, that is to say, the velocity of its upward movement increases. The maximum value of the upward movement is reached at $f = 0$, that is to say, when the surface of a screw is absolutely smooth. Figure 3, a presents a dependency graph $\alpha' = \alpha'(f)$ for $R = 0.1m, \omega = 25s^{-1}, f_R = 0.3, \beta = 20^0$.

The decrease of the coefficient f_R of particle friction on a cylindrical cover results in the decrease in the angular velocity α' of particle sliding (Fig. 3, b). The initial conditions are the same as the ones of the graph presented in Figure 3, a, except for the fact that the variable is not f but f_R. Its decrease leads to the fact that, at its certain value the inverse process begins and a particle starts downward movement. The boundary limit is the value f_R for $\alpha' = 0$. Having solved (23) for f_R, we obtain:

$$f_R = \frac{g(f + \mathrm{tg}\beta)}{R\omega^2(1 - f\mathrm{tg}\beta)}. \tag{26}$$



a) $R = 0.1m, f_R = 0.3$ b) $R = 0.1m, f = 0.3$; c) $f = 0.3, f_R = 0.3$

Figure 3. Graphs of the angular velocity of particle sliding α' depending on various factors at $\omega = 25s^{-1}, \beta = 20^0$

For the preset data from (26), we determine the boundary value of the friction coefficient $f_R : f_R = 0.12$.

If the angular velocity of screw rotation is increased, for instance, to $30s^{-1}$, particle upward movement is not possible. The boundary value of the coefficient of friction for this angular velocity is equal to $f_R = 0.08$. If there is an increase in the angular velocity of screw rotation, the boundary value of the coefficient decreases. Proceeding to the boundary, we obtain:

$$\lim_{\omega \to \infty} \frac{g(f + \mathrm{tg}\beta)}{R\omega^2(1 - f\mathrm{tg}\beta)} = 0. \tag{27}$$

Thus, if the surface of a cylindrical cover is absolutely smooth, particle upward movement is not possible at any angular velocities of screw rotation.

Figure 3, b presents a dependency graph $\alpha' = \alpha'(R)$ showing that the increase in the radius R, that is to say, the increase in the dimensions and the specific amount of metal per structure, results in the increase of the velocity of upward movement, however, this increase has its limit.

At the negative value of the angle β, the direction of the coiling of a helix changes. It means that at the same direction of the angular velocity of screw rotation, a particle slides not upwards but downwards on its surface. The transportation of a particle downwards has its peculiarities depending on the value of the angle β. If the sign is changed before the angle β in the equation (20), one of the expressions in the parenthesis will be written as $f \cos\beta - \sin\beta$. If the angle β is equal to the angle of friction, that is to say, $f = \mathrm{tg}\,\beta$, this expression becomes zero. In order for the equation (20) to become zero, it is enough to set one more expression in the parenthesis to zero: $\omega - \alpha' = 0$. It means that the angular velocity of particle sliding is equal in its value to the angular velocity of screw rotation. Since they are oppositely directed, their sum is equal to zero, that is to say, a particle does not rotate in absolute motion, it moves downwards along a straight line — a cylinder generator.

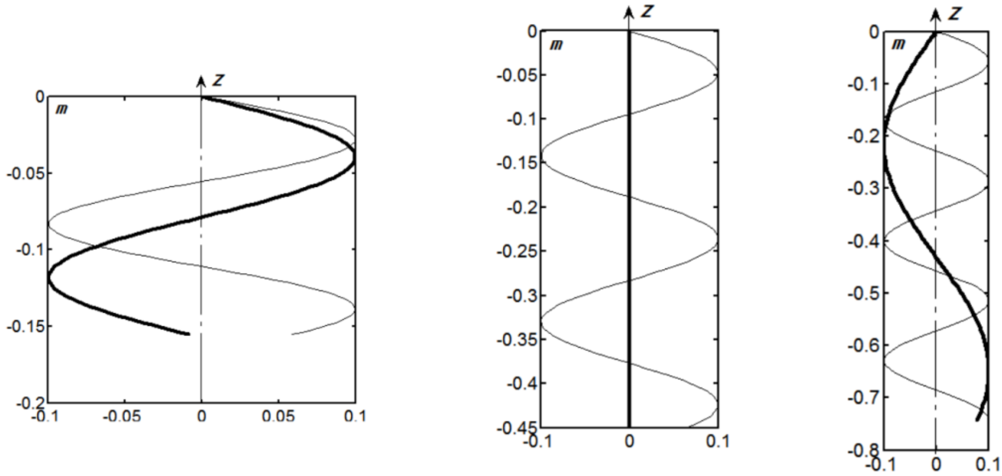
Let us find α' and z' for three angles: a small angle, a high angle and the angle that is equal to the friction angle β_f, the same as we did at particle upward movement for $R = 0.1m, \omega = 15s^{-1}, f = f_R = 0.3$.

$$\beta < \beta_f: \qquad \beta = -10^0; \qquad \alpha' = 8.8s^{-1}; \qquad z' = -0.16m/s;$$

$$\beta = \beta_f: \qquad \beta = -Arctgf = -16.7^0; \qquad \alpha' = 15s^{-1}; \qquad z' = -0.45m/s;$$

$$\beta > \beta_f: \qquad \beta = -20^0; \qquad \alpha' = 20.4s^{-1}; \qquad z' = -0.74m/s.$$

Figure 4 presents relative and absolute trajectories of particle movement in these cases. Front projections are presented in various scales.

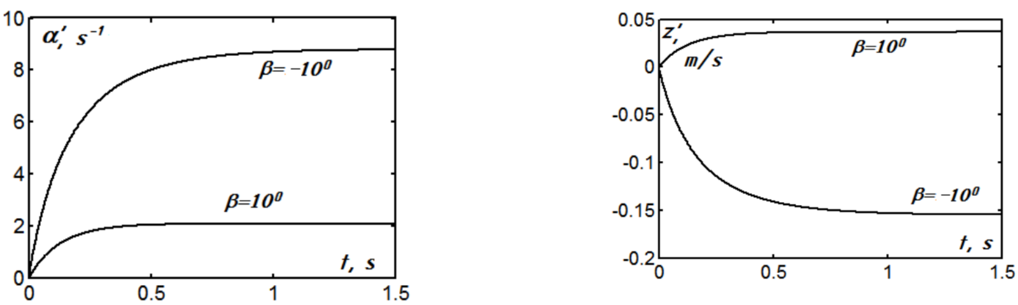


a b c

Figure 4. Relative and absolute (heavy line) trajectories of particle downward movement at $R = 0.1m$, $\omega = 15s^{-1}$, $f = f_R = 0.3$ built according to the equations (1) and (3) over the time $t = 1s$

The solution of the differential equation (20) applying numerical methods enables us to determine the kinematic characteristics of particle movement during a transient process. Figure 5 presents the dependency graphs $\alpha' = \alpha'(t)$ and $z' = z'(t)$, which show that the movement of a particle becomes practically steady over the time of 1.5 s. The angular velocities of particle sliding and the velocities of movement in a vertical direction reach the values that we have obtained before.

Eventually, the process of bulk material transportation will be different from the conveyance of a single particle.



a b

Figure 5. Graphs of the change of the kinematic characteristics of particle movement in a transient process at $R = 0.1m, \omega = 15m^{-1}, f = f_R = 0.3$ and various values of the angle β: a) the dependency graph $\alpha' = \alpha'(t)$; b) the dependency graph $z' = z'(t)$

However, it can be assumed that the quality pattern of material transportation will be similar to the conveyance of a single particle. In order to determine quantitative characteristics, which describe the efficiency of conveyance, separate investigations are required.

Conclusions

A particle, which gets onto the surface of a vertical screw that rotates about its axis, is thrown off to a coaxial fixed limiting cylinder under the action of centrifugal force. Its further movement takes place along the common line of these two surfaces — a helix — with its simultaneous sliding on both surfaces. Particle absolute motion in a vertical direction can be both upward and downward, depending on the helix angle β and on the angular velocity of screw rotation. There is a boundary value of the angle β, at which particle upward movement is not possible at any values of the angular velocity of screw rotation. In addition, there is the value of the angle, at which the velocity of upward movement is the highest.

The decrease in the coefficient of particle friction on the surface of a screw results in the increase of the velocity of its upward conveyance and the decrease in the coefficient of particle friction on the surface of a limiting cylinder results in the decrease of this velocity. If the surface of a screw is absolutely smooth, the velocity of particle upward movement is the highest and, if there is the same surface of a limiting cylinder, particle upward movement is not possible.

If there is a change in the coiling of a helix, particle conveyance is possible downwards only and the velocity is significantly higher.

References

1 Адамчук В.В. Дослідження відносного руху матеріальної частинки по похилій лопатці відцентрового апарата / В.В. Адамчук, В.М. Булгаков, С.Ф. Пилипака, Я. Франчак // Наук. вісн. Луган. нац. ун-ту. Технічні науки. — 2005. — № 29. — С. 32–56.

2 Адамчук В.В. Дослідження руху частинки по плоскому диску, який обертається навколо перпендикулярної осі, нахиленої до горизонту / В.В.Адамчук, В.М. Булгаков, Д.Г. Войтюк, С.Ф. Пилипака // Вісн. Львів. наці. аграр. ун-ту: агроінженерні дослідження. — 2008. — № 12 (2). — С. 189–197.

3 Gevko R.B. The investigation of the process of a screw conveyer safety device actuation / R.B. Gevko, O.M. Klendiy // INMATEH: Agricultural Engineering. — 2014. — Vol. 42. — No. 1. — P. 55–60.

4 Гевко Р. Поектування пневмо-механічного транспортера сипких матеріалів / Р. Гевко, В. Дзюра, Р. Романовський // Вісн. Терноп. нац. техн. ун-ту. – 2009. — Т. 14. — № 4. — С. 84–88.

5 Hevko R.B. Mathematical model of the pneumatic-screw conveyor screw mechanism operation / R.B. Hevko, V.O. Dzyura, R.M. Romanovsky // INMATEH: Agricultural Engineering. — 2014. — Vol. 44. — No. 3. — P. 103–110.

6 Hevko R.B. Investigation of a transfer branch of a flexible screw conveyer / R.B. Hevko, M.B. Klendii, O.M. Klendii // INMATEH: Agricultural Engineering. — 2016. — Vol. 48. — No. 1. — P. 29–34.

7 Hevko R.B. Development of design and investigation of operation processes of loading pipes of screw conveyors / R.B. Hevko, R.I. Rozum, O.M. Klendii // INMATEH: Agricultural Engineering. — 2016. — Vol. 50. — No. 3. — P. 89–96.

8 Hevko R.B. Feasibility study of the process of transpotration and stirring of mixture in continuous-flow conveyers / R.B. Hevko, B.O. Yazlyuk, M.V. Liubin, O.A. Tokarchuk, O.M. Klendii, V.R. Pankiv // INMATEH: Agricultural Engineering. — 2017. — Vol. 51. — No. 1. — P. 49–58.

9 Клендій М.Б. Розробка конструкції та обґрунтування параметрів лопатевого транспортера-змішувача / М.Б. Клендій // Наук. вісн. нац. аграр. ун-ту. — 2005. — Вип. 92. — С. 533–540.

10 Klendii M.B. Inverrelation between incidence ange and roll ange of concave disks of soil tillage implements / M.B. Klendii, O.M. Klendii // INMATEH: Agricultural Engineering. — 2016. — Vol. 49. — No. 2. — P. 13–20.

11 Klendii M.B. Analytical model of arranging soil-tilling concave disks for determination of geometrical and technical data / M.B. Klendii, S.F. Pylypaka // Науковий вісник НУБіП України. Техніка та енергетика АПК. — 2016. — № 241. — С. 140–150.

12 Pylypaka S.F. Particle motion over the surface of a rotary vertical axis helicoid / S.F. Pylypaka, M.B. Klendii, O.M. Klendii // INMATEH: Agricultural Engineering. — 2017. — Vol. 51. — No. 1. — P. 15–28.

13 Пилипака С.Ф. Траєкторії руху частинок по шорсткій похилій площині при їх боковій подачі / С.Ф. Пилипака, А.В. Несвідомін // Прикладна геометрія та інженерна графіка. — 2011. — № 87. — С. 36–41.

14 Василенко П.М. Теория движения частицы по шероховатым поверхностям сельскохозяйственных машин / П.М. Василенко // Киев: УАСХН, 1960. — 283 с.

15 Вітровий А.О. Силовий аналіз робочого органу гнучкого гвинтового конвеєра / А.О. Вітровий, Р.Б. Гевко // Зб. наук. ст. Луцьк. державн. техн. ун-ту. Сільськогосподарські машини. — 1998. — № 4. — С. 8–14.

16 Заика П.М. Избранные задачи земледельческой механiки / П.М. Заика. — Киев: УСХА, 1992. — 507 с.

17 Hevko B.M. Improvement of machine safety devices / B.M. Hevko, R.B. Hevko, O.M. Klendii, M.V. Buriak, Y.V. Dzyadykevych, R.I. Rozum // Acta Polytechnica, Journal of Advanced Engineering. — 2018. — Vol. 58. — No. 1. — P. 17-25.

18 Baranovsky V.M. Justification of rational parameters of a pneumoconveyor screw feeder / V.M. Baranovsky, R.B. Hevko, V.O. Dzyura, O.M. Klendii, M.B. Klendii, R.M. Romanovsky. // INMATEH: Agricultural Engineering. — 2018. — Vol. 54. — No. 1. — P. 15-24.

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Бөлшектерді тік шнекпен, осьпен бірге шектелген қозғалмайтын цилиндрмен тасымалдау

Бөлшектің тік шнектің шеткі бойынша салыстырмалы жылжуының дифференциалды теңдеулері құрастырылды, ол өз осінің айналасында айналатын және осьпен бірге қозғалмайтын цилиндрмен шектелген. Теңдеу сандық әдістермен есептелген және шнек беті мен шектеуші цилиндр үшін ортақ болып табылатын шнек жиегі – бұрандалы сызық бойынша бөлшектердің салыстырмалы қозғалысының траекториялары салынған. Шнек беті мен цилиндрлік қаптаманың беті бойынша бөлшектердің үйкеліс күші бөлек есепке алынған. Шнектің берілген бұрыштық жылдамдығы кезінде бөлшектердің көтерілуі мүмкін болмаған кезде бұрандалы желіні көтеру бұрышының шекті мәні табылды. Үйкеліс бұрыштарының және шектеуші цилиндр радиусының бөлшектердің көтеру жылдамдығына әсері анықталды. Уақыт функциясындағы кинематикалық сипаттамалардың графиктері келтірілген.

Кілт сөздер: салыстырмалы қозғалыс, тік шнек, бөлшектер, бұрыштық айналу жылдамдығы, цилиндрлік қаптау, дифференциалдық теңдеулер, кинематикалық параметрлер.

С.Ф. Пилипака, В.Н. Несвидомин, Н.Б. Клендий,
И.Л. Роговский, Т.А. Кресан, В.И. Троханяк

Транспортировка частицы вертикальным шнеком, ограниченным соосным неподвижным цилиндром

Составлены дифференциальные уравнения относительного перемещения частицы по периферии вертикального шнека, который вращается вокруг своей оси и ограничен соосным неподвижным цилиндром. Уравнение рассчитано численными методами, и построены траектории относительного движения частицы по винтовой линии – кромке шнека, которая является общей для поверхности шнека и ограничивающего цилиндра. Отдельно учтена сила трения частицы по поверхности шнека и поверхности цилиндрического кожуха. Найдено пороговое значение угла подъема винтовой линии, когда подъем частицы становится невозможным при заданной угловой скорости вращения шнека. Выявлено влияние углов трения и радиуса ограничивающего цилиндра на скорость подъема частицы. Приведены графики кинематических характеристик в функции времени.

Ключевые слова: относительное движение, вертикальный шнек, частица, угловая скорость вращения, цилиндрический кожух, дифференциальные уравнения, кинематические параметры.

References

1 Adamchuk, V.V., Bulhakov, V.M., Pylypaka, S.F., & Franchak, Yan. (2005). Doslidzhennia vidnosnoho rukhu materialnoi chastinki po pokhilii lopattsi vidtsentrovoho aparata [Investigation of relative material particle motion over an inclined blade of a centrifugal machine]. *Naukovii visnik Luhanskoho natsionalnoho universitetu. Seriia Tekhnichni nauki – National Bulletin of Luhansk National University. Technical Sciences, LNAU, Vol. 29,* 32–56 [in Ukrainian].

2 Adamchuk, V.V., Bulhakov, V.M., Voitiuk, D.H., & Pylypaka, S.F. (2008). Doslidzhennia rukhu chastinki po ploskomu disku, iakii obertatsia navkolo perpendikuliarnoi osi, nakhilenoi do horizontu [Investigation of particle motion over a flat disk, which rotates about a perpendicular axis that is inclined to a horizon]. *Visnik Lvivskoho natsionalnoho ahrarnoho universitetu: ahroinzhenerni doslidzhennia – Bulletin of Lviv National Agrarian University: agricultural engineering research. Lviv National Agrarian University, Vol. 12, 2,* 189-197 [in Ukrainian].

3 Gevko, R.B., & Klendiy, O.M. (2014). The investigation of the process of a screw conveyer safety device actuation. INMATEH: Agricultural Engineering, Vol. 42,1, 55-60.

4 Hevko, R.B., Dzyura, V.O., & Romanovsky, R.M. (2009). Poektuvannia pnevmo-mekhanichnoho transportera sipkikh materialiv [Designing of pneumo-mechanical conveyer for loose materials]. *Visnik Ternopilskoho natsionalnoho tekhnichnoho universitetu – Bulletin of I.Pyliui Ternopil State Technical University, Vol. 14, 4,* 84–88 [in Ukrainian].

5 Hevko, R.B., Dzyura, V.O., & Romanovsky, R.M. (2014). Mathematical model of the pneumatic-screw conveyor screw mechanism operation. *INMATEH: Agricultural engineering, Vol. 44, 3,* 103–110.

6 Hevko, R.B., Klendii, M.B., & Klendii, O.M. (2016). Investigation of a transfer branch of a flexible screw conveyer. *INMATEH: Agricultural engineering, Vol. 48, 1,* 29–34.

7 Hevko, R.B., Rozum, R.I., & Klendii, O.M. (2016). Development of design and investigation of operation processes of loading pipes of screw conveyors. *INMATEH: Agricultural engineering, Vol. 50, 3,* 89–96.

8 Hevko, R.B., Yazlyuk, B.O., Liubin, M.V., Tokarchuk, O.A., Klendii, O.M., & Pankiv, V.R. (2017). Feasibility study of the process of transpotration and stirring of mixture in continuous-flow conveyers. *INMATEH: Agricultural engineering, Vol. 51, 1,* 49–58.

9 Klendii, M.B. (2005). Rozrobka konstruktsii ta obr'runtuvannia parametriv lopatevoho transporterazmishuvacha [Design development and substantiation of the parameters of a bladed conveyer-mixer]. *Naukovii visnik Natsionalnoho ahrarnoho universitetu – Scientific Bulletin of National Agrarian University, Vol. 92, 2,* 533–540 [in Ukrainian].

10 Klendii, M.B., & Klendii, O.M. (2016). Inverrelation between incidence ange and roll ange of concave disks of soil tillage implements. *INMATEH: Agricultural Engineering, Vol. 49, 2,* 13–20.

11 Klendii, M.B., & Pylypaka, S.F. (2016). Analytical model of arranging soil-tilling concave disks for determination of geometrical and technical data. Naukovii visnik NUBiP Ukraini. *Tekhnika ta energetika APK, Vol. 241,* 140–150.

12 Pylypaka, S.F., Klendii, M.B., & Klendii, O.M. (2017). Particle motion over the surface of a rotary vertical axis helicoids. *INMATEH: Agricultural engineering, Vol. 51, 1,* 15–28.

13 Pylypaka, S.F., & Nesvidomin, A.V. (2011). Traektorii rukhu chastinok po shorstkii pokhilii ploshchini pri ikh bokovii podachi [Trajectories of particle motion over rough inclined planes at side feed]. *Prikladna heometriia ta inzhenerna hrafika – Applied Geometry and Engineering Graphics, Vol. 87,* 36–41 [in Ukrainian].

14 Vasilenko, P.M. (1960). *Teoriia dvizheniia chastitsy po sherokhovatym poverkhnostiam selskokhoziaistvennykh mashin [Theory of particle motion over rough surfaces of agricultural machinery].* Kiev: UASKhN [in Russian].

15 Vitrovyi, A.O., & Hevko, R.B. (1998). Silovii analiz robochoho orhanu hnuchkoho hvintovoho konveera [Power analysis of the operating element of a flexible screw conveyer]. *Zbirnik naukovikh statei Lutskoho derzhavnoho tekhnichnoho universitetu silskohospodarski mashini – Collection of Scientific Papers Agricultural machinery, Vol. 4*, 8–14 [in Ukrainian].

16 Zaika, P.M. (1992). *Izbrannye zadachi zemledelcheskoi mekhaniki [Selected tasks of agricultural mechanics].* Kiev: USKhA [in Russian].

17 Hevko, B.M., Hevko, R.B., Klendii, O.M., Buriak, M.V., Dzyadykevych, Y.V., & Rozum, R.I. (2018). Improvement of machine safety devices. *Acta Polytechnica, Journal of Advanced Engineering, Vol. 58, 1,* 17–25.

18 Baranovsky, V.M., Hevko, R.B., Dzyura, V.O., Klendii, O.M., Klendii, M.B., & Romanovsky R.M. (2018). Justification of rational parameters of a pneumoconveyor screw feeder. *INMATEH: Agricultural engineering, Vol. 54, 1,* 15–24.

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On the calculation of round plates for bending

The article is devoted to the study of bending of round plates, which are the basis for calculating many problems of mechanics. In the article the structure of this method is presented, its main components are highlighted; its types are characterized, as well as its classical approaches. In this article the research of the bending problem for round plates is carried out in particular cases. Methods of bending calculation of round plates like all analytical methods have a number of advantages, which are also noted in this article. The article is focused mainly on mechanics, engineers and technical specialists.

Keywords: round plate, plate deflection function, axisymmetric loads, maximum deflection of the round plate.

Plates are widely used now in various fields of technology: construction, engineering, aviation, shipbuilding, etc. During operation, the plates are subjected to temperature, chemical, force and other influences. These effects cause plate deformations. Therefore, the problem of calculating the plates are so diverse and have such a different character [1–6].

Many structural elements, such as the bottoms of pistons, tanks, apparatus, hatches, various kinds of covers; flanges; diaphragms, etc. [7], are round plates. The simplest form of deformation for such elements is their bending [8–12].

We consider the bending of a round plate and several axisymmetric loads: P is the concentrated force at the center of the plate, T is the ring load, q is the distributed load.

We denote by h the thickness of the plate, which can be constant or variable. The outer radius of the plate is denoted by R. Vertical linear displacements of the mid-plane points (along the z-axis) are called deflections and are denoted by the letter W.

For thin plates, the assumptions called Kirchhoff hypotheses are valid [13].

We will distinguish two directions in the plate:

– the radial direction (all parameters of this direction are denoted by the index «r»),

– the circumferential direction (all parameters of this direction will be marked with the index «φ»)

According to Kirchhoff's hypotheses for radial and circumferential deformations, we obtain

$$\varepsilon_r = z\frac{d\varphi}{dr}, \quad \varepsilon_\varphi = \varphi\frac{z}{r}.$$

Then from Hooke's law for the plane stress state ($\sigma_r \neq 0$, $\sigma_\varphi \neq 0$, $\sigma_z = 0$) it is followed

$$\sigma_r = \frac{Ez}{1-\nu^2}\left(\varphi' + \nu\frac{\varphi}{r}\right), \quad \sigma_\varphi = \frac{Ez}{1-\nu^2}\left(\nu\varphi' + \frac{\varphi}{r}\right). \tag{1}$$

The relationship between the deflection and the angle of rotation is written as

$$W' = -\varphi. \tag{3}$$

Normal stresses σ_r are grouped in the bending moment M_r, and σ_φ are grouped in the bending moment M_φ

$$M_r = D\left(\varphi' + \nu\frac{\varphi}{r}\right), \quad M_\varphi = D\left(\nu\varphi' + \frac{\varphi}{r}\right), \tag{3}$$

where D is cylindrical rigidity of the plate.

Tangential stresses from the platform of the unit width form a transverse force Q in the circumferential section. The normal stresses are easily expressed in terms of bending moments. It is sufficient to substitute expressions (3) in (1). Then we receive

$$\sigma_r = \frac{M_r}{J} z, \quad \sigma_\varphi = \frac{M_\varphi}{J} z,$$

where $J = \frac{h^3}{12}$ is the moment of inertia for a rectangular strip of unit width.

The signs rule of the bending moment for round plates: we assume that the bending moment is positive if the upper layer of the plate is stretched.

With axisymmetric bending, the problem of calculating a round plate is significantly simplified, since in all equations and formulas describing the bending of the plate, the derivatives with respect to the angular coordinate are equal to zero. Therefore, the differential equation of plate bending takes the following form

$$\frac{d^4W}{dr^4} + \frac{2}{r}\frac{d^3W}{dr^3} - \frac{1}{r^2}\frac{d^2W}{dr^2} + \frac{1}{r^3}\frac{dW}{dr} = \frac{q}{D}. \tag{4}$$

The equation (4) is a differential equation of Euler.

In the case of plate equilibrium under the action of efforts M_r, M_φ, Q and the acting load q on the center of the plate, it is possible to obtain the differential equation of the problem in a simpler form. Making two equilibrium equations, we obtain

$$r\varphi'' + \varphi' - \frac{\varphi}{r} = \frac{Q}{D}. \tag{5}$$

This is an ordinary second-order linear differential equation of second order with respect to a function for the angle of rotation, and the resolving equation for axisymmetric bending through the function of deflection is the equation of the fourth order. The general solution of the equation (5) takes the form

$$\varphi = \frac{C_1}{r} + C_2 r + \frac{1}{D}\int_{r_0}^{r} \frac{r^2 - s^2}{r} Q(s)ds.$$

The calculation of bending for circular plates with constant thickness

We consider the pure bending of a solid round plate by moments distributed along the hinge-supported contour. In this case, there is no transverse load. The transverse (cutting) force also equals zero: $Q = 0$ and we have $M_r = M_\varphi = m$, then the deflection of any point from equation (2) is equal to

$$W = C - \frac{m}{2D(1+\nu)}\left(r^2 - r_o^2\right).$$

The constant C is found from the boundary condition of the contour fixing.

In the case of hinged fastening the contour at $r = R$ we have $W = 0$. If C equals zero ($C = 0$) and $r_0 = R$, the condition of the hinged fastening is satisfied. Then the following formula is valid

$$W = \frac{m}{2D(1+\nu)}\left(r^2 - r_o^2\right).$$

The greatest deflection (at $r = 0$) is

$$W_{\max} = \frac{mR^2}{2D(1+\nu)}.$$

We consider a round plate under the action of a concentrated force P_0 at its center. In this case we have

$$Q(s) = \frac{P_0}{2\pi s}. \tag{6}$$

Substituting (6) into the formula of the particular solution (5), we get

$$\varphi_1 = \frac{P_0}{4\pi D}\left[r \ln\frac{r}{r_0} - \frac{1}{2r}\left(r^2 - r_0^2\right)\right].$$

Then the general solution can be written as

$$\varphi = \frac{C_1}{r} + C_2 r + \frac{P_0}{4\pi D}\left[r \ln\frac{r}{r_0} - \frac{1}{2r}\left(r^2 - r_0^2\right)\right]. \tag{7}$$

We will find C_1, C_2 from boundary conditions. If $r = 0$, $\varphi = 0$, then from (7) we determinate C_1

$$C_1 = -\frac{P_0}{8\pi D} r_0^2.$$

If $r = R$, $\varphi = 0$, then from (7) we find C_2

$$C_2 = \frac{P_0}{4\pi D}\left(\frac{1}{2} - \ln\frac{R}{r_0}\right).$$

As a result, the general solution takes the form

$$\varphi = \frac{P_0 r}{4\pi D}\ln\frac{r}{R}.$$

We calculate the bending moments

$$M_r = -\frac{P_0}{4\pi}\left[1 + (1+\nu)\ln\frac{r}{R}\right], \quad M_\varphi = -\frac{P_0}{4\pi}\left[\nu + (1+\nu)\ln\frac{r}{R}\right].$$

In the center of the plate, if $r = 0$ then $\ln\frac{r}{R} \to \infty$ and therefore, the values of bending moments tend to infinity. On the edge of the plate, if $r = R$ then $\ln\ 1 = 0$, and we have that the abcolute values of bending moments are equal to

$$M_r = \frac{P_0}{4\pi}, \quad M_\varphi = \frac{P_0 \nu}{4\pi}.$$

Infinitely large values of bending moments are only a consequence of the extreme schematization for the mathematical model of the problem (the concentrated force is applied at the point) [13]. In fact, this does not happen, the load is distributed over a small platform, and in a small neighborhood of the point of force application we have $M_r = M_\varphi$, as in all other cases of loading [14].

We consider the general differential equation of plate bending

$$\frac{d^4W}{dr^4} + \frac{2}{r}\frac{d^3W}{dr^3} - \frac{1}{r^2}\frac{d^2W}{dr^2} + \frac{1}{r^3}\frac{dW}{dr} = \frac{q}{D}. \tag{4}$$

The general solution of the differential equation (8) has the form

$$W(r) = C_1 + C_2 \ln r + C_3 r + C_4 r \ln r + W_1(r), \tag{8}$$

where $W_1(r)$ is a partial solution of the equation (4). To find a particular solution, we present equation (4) as

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dW}{dr}\right)\right)\right] = \frac{q(r)}{D}. \tag{9}$$

After integration (9) we obtain

$$W_1(r) = \frac{1}{D}\int_a^r \frac{1}{\xi}d\xi \int_b^\xi \eta d\eta \int_c^\eta \frac{1}{\zeta}d\zeta \int_d^\zeta q(\tau)\tau d\tau. \tag{10}$$

For the case when the plate is under the action of a uniformly distributed load $q = const$, after integration (10) we have

$$W_1(r) = \frac{qr^4}{64D}. \tag{11}$$

Included in (8) constant of integrations C_1, C_2, C_3 and C_4 are determined from the corresponding boundary conditions in each specific problem.

We consider the problem of bending for a round plate, rigidly pinched along the contour. This plate is under the action of a uniformly distributed load q. The deflection of the plate is determined by the expression (8), where the particular solution has the form (11).

In accordance with the physical meaning of the problem in the center of the plate at $r = 0$ deflection and internal forces must have finite values. To satisfy this condition, it is necessary to remove the partial integrals containing the natural logarithm in the general solution (8), putting the constant integrations C_2 and C_4 equal to zero. Thus, we obtain the following expression for the deflection of the plate

$$W(r) = \frac{q}{64D} \cdot r^4 + C_1 + C_3 r^2.$$

To determine the constants C_1 and C_3, we use the boundary conditions on the rigidly clamped plate contour

$$r = R; \quad W = 0; \quad \varphi = \frac{dW}{dr} = 0.$$

Using these conditions, we obtain two algebraic equations with respect to C_1 and C_3. Having solved these equations, we receive

$$C_1 = \frac{q \cdot R^4}{64D}, \quad C_2 = -\frac{q \cdot R^2}{32D}.$$

The final expression for the deflection is determined by the following formula

$$W = \frac{q \cdot r^4}{64D} - \frac{q \cdot R^2 \cdot r^2}{32D} + \frac{q \cdot R^4}{64D} = \frac{q}{64D} \left(R^2 - r^2\right)^2.$$

Then we get the expressions for the internal forces in the plate in the form

$$M_r = \frac{q}{16} \left(R^2 (1 + \nu) - r^2 (3 + \nu)\right),$$

$$M_\varphi = \frac{q}{16} \left(R^2 (1 + \nu) - r^2 (1 + 3\nu)\right),$$

$$Q = -\frac{qr}{2}.$$

In the center of the plate, the bending moments are equal to each other, the transverse force is zero, and the deflection has a maximum value equal to

$$W_{\max} = \frac{q \cdot R^4}{64D}.$$

If $r = R$ we find the moments on the plate contour

$$M_{r(r=R)} = -\frac{q \cdot R^2}{8}; \quad M_{\varphi(r=R)} = -\frac{q \cdot R^2}{8}\nu.$$

On the center of the plate, where $r = 0$, we have

$$M_r = M_\varphi = \frac{q \cdot R^2}{16} (1 + \nu).$$

The maximum stress on the plate contour is equal to

$$\sigma_{r\max} = -\frac{6M_r}{h^2} = \frac{3}{4} \frac{q \cdot R^2}{h^2}.$$

In accordance with the condition of rigidity [14]

$$\sigma_{r\max} = \frac{6 \cdot M_{r\max}}{h^2} \leq [\sigma]_{\text{bend.}} - 5\ \%,$$

we make the adjustment of section thickness h.

The round plate has a supported edge. Uniformly distributed in a circle of radius b the load P acts on the plate. In this case, the expression for the function $W(x, y)$ has different forms for $r > b$ and $r < b$

$$W_{r>b} = \frac{P}{8\pi D}\left[\left(r^2+b^2\right)\ln\frac{r}{a} + \frac{(3+\nu)a^2-(1-\nu)b^2}{2(1+\nu)a^2}\left(a^2-r^2\right)\right],$$

$$W_{r<b} = \frac{P}{8\pi D}\left[\left(r^2+b^2\right)\ln\frac{b}{a} + \frac{(3+\nu)a^2-(1-\nu)b^2}{2(1+\nu)a^2}\left(a^2-r^2\right)+r^2-b^2\right].$$

For a plate with a pinched edge at the same load P for the function $W(x, y)$, we have [15]

$$W_{r>b} = \frac{P}{8\pi D}\left[\left(r^2+b^2\right)\ln\frac{r}{a} + \frac{a^2+b^2}{2a^2}\left(a^2-r^2\right)\right],$$

$$W_{r<b} = \frac{P}{8\pi D}\left[\left(r^2+b^2\right)\ln\frac{b}{a} + \frac{a^2+b^2}{2a^2}\left(a^2-r^2\right)+r^2-b^2\right].$$

General case for bending of round plates

If the load on the plate or the conditions of its fixation are not axisymmetric, then the plate deflection depends on variables r, φ and must satisfy the differential equation

$$D\,\Delta\Delta\, W = q\,(x, y). \tag{12}$$

Obviously, we are looking for a solution for a round plate in polar coordinates. The equation (12) in polar coordinates has the form

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2}{\partial\varphi^2}\right)\left(\frac{\partial^2 W}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial W}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 W}{\partial\varphi^2}\right)=\frac{q(r,\varphi)}{D}$$

or in expanded form

$$\frac{\partial^4 W}{\partial r^4}+\frac{2}{r^2}\cdot\frac{\partial^4 W}{\partial r^2\partial\varphi^2}+\frac{1}{r^4}\cdot\frac{\partial^4 W}{\partial\varphi^4}+\frac{2}{r}\cdot\frac{\partial^3 W}{\partial r^3}-\frac{2}{r^3}\cdot\frac{\partial^3 W}{\partial r\partial\varphi^2}-$$
$$-\frac{1}{r^2}\frac{\partial^2 W}{\partial r^2}+\frac{4}{r^4}\cdot\frac{\partial^2 W}{\partial\varphi^2}+\frac{1}{r^3}\cdot\frac{\partial W}{\partial r}=\frac{q(r,\varphi)}{D}. \tag{13}$$

In the general case of round plates bending to obtain a solution, the deflection of the plate can be represented as a trigonometric Fourier series with respect to the angular coordinate φ

$$W(r,\varphi) = W_0(r) + \sum_{n=1}^{\infty}\left[\xi_n(r)\cos n\varphi + \eta_n(r)\sin n\varphi\right], \tag{14}$$

where the functions $W_0(r)$, $\xi_n(r)$ and $\eta_n(r)$ characterize the change in the plate deflection in the radial direction and they are to be determined.

An arbitrary load $q(r,\varphi)$ causing bending of the plate can also be decomposed into a trigonometric series, similar to the series (14)

$$q(r,\varphi) = q_0(r) + \sum_{n=1}^{\infty}\left[\lambda_n(r)\cos n\varphi + \mu_n(r)\sin n\varphi\right], \tag{15}$$

where

$$[q_0(r) = \frac{1}{2\pi}\cdot\int_0^{2\pi} q(r,\varphi)d\varphi, \quad \lambda_n(r) = \frac{1}{\pi}\cdot\int_0^{2\pi} q(r,\varphi)\cos n\varphi d\varphi,$$

$$\mu_n(r) = \frac{1}{\pi}\cdot\int_0^{2\pi} q(r,\varphi)\sin n\varphi d\varphi.$$

Substituting (14), (15) into (13) and comparing the coefficients of linearly independent functions 1, $\cos n\varphi$, $\sin n\varphi$ we obtain three equations to determine functions $W_0(r)$, $\xi_n(r)$ и $\eta_n(r)$

$$\frac{d^4 W_0}{dr^4}+\frac{2}{r}\frac{d^3 W_0}{dr^3}-\frac{1}{r^2}\frac{d^2 W_0}{dr^2}+\frac{1}{r^3}\frac{dW_0}{dr}=\frac{q_0}{D}, \tag{16}$$

$$W_{i,n}^{IV} + \frac{2}{r} \cdot W_{i,n}''' - \frac{(2n^2+1)}{r^2} \cdot W_{i,n}'' + \frac{(2n^2+1)}{r^3} \cdot W_{i,n}' + \frac{n^2(n^2-4)}{r^4} W = \frac{q_{i,n}}{D}, \tag{17}$$

where

$$W_{i,n} = W_{i,n}(r), \quad W_{1,n} = \xi_n, \quad W_{2,n} = \eta_n,$$

$$q_{i,n} = q_{i,n}(r), \quad q_{1,n} = \lambda_n, \quad q_{2,n} = \mu_n; \quad i = 1,\ 2.$$

In equation (16), which describes the plate deflection, the derivatives with respect to the angular coordinate vanish, since W_0 is a function of r as the coefficient of the Fourier series with respect to φ. Thus, the function W_0 describes the axisymmetric bending of a round plate. Note that the partial differential equation (16) coincides with the equation (4).

Equation (17) is an ordinary differential Euler equation. By replacing the variables $r = e^t$, equation (17) is reduced to a linear differential equation with constant coefficients

$$\tilde{W}_{i,n}^{IV} - 4\tilde{W}_{i,n}''' + 2(2-n^2)\tilde{W}_{i,n}'' + 4n^2\tilde{W}_{i,n}' + n^2(n^2-4)\tilde{W}_{i,n} = \frac{\tilde{q}_{i,n}}{D}, \tag{18}$$

where $\tilde{W}_{i,n} = \tilde{W}_{i,n}(t)$; $\tilde{q}_{i,n} = \tilde{q}_{i,n}(t)$, $i = 1,\ 2$. The characteristic equation for (18), obviously, has the form

$$\tau_{i,n}^4 - 4\tau_{i,n}^3 + 2(2-n^2)\tau_{i,n}^2 + 4n^2\tau_{i,n} + n^2(n^2-4) = 0. \tag{19}$$

In many cases, when solving the problem of bending for a round plate, the accuracy given by the formula (14) is sufficient if there is only the first term of the series (14). If $n = 1$, then the characteristic equation (19) has roots $\tau_{i,1,2} = 1$, $\tau_{i,3} = 3$, $\tau_{i,4} = -1$. In this case, the general solution of the differential equation (17) takes the form

$$W_{i,1}(r) = A_{i,1}r + B_{i,1}r\ln r + K_{i,1}r^3 + \frac{L_{i,1}}{r} + W_{i,1}^{(1)}(r),$$

where $A_{i,1}$, $B_{i,1}$, $K_{i,1}$, $L_{i,1}$ are constants of integration, $W_{i,1}^{(1)}$ is a partial solution of the equation (17). Constants of integration $A_{i,1}$, $B_{i,1}$, $K_{i,1}$, $L_{i,1}$ are usually determined from the given boundary conditions, and a partial solution $W_{i,1}^{(1)}$ depends on the type of load applied to the plate.

If $n = 1$ the plate deflection in the general case can be written in the form

$$W(r,\varphi) = W_0(r) + \left[A_{1,1}r + B_{1,1}r\ln r + K_{1,1}r + \frac{L_{1,1}}{r} + W_{1,1}^{(1)}(r)\right]\cos\varphi +$$

$$+ \left[A_{2,1}r + B_{2,1}r\ln r + K_{2,1}r + \frac{L_{2,1}}{r} + W_{2,1}^{(1)}(r)\right].$$

It should be noted that almost all the problems related to the study of stresses and strains in a plate are reduced to solving boundary value problems for one or several differential equations. The exact solution of these equations does not cause difficulties only in some elementary cases. In more complex cases, finding a solution in analytical form is associated with great mathematical difficulties.

In such cases, it is recommended to use approximate solution methods: variational methods (Ritz method, Galerkin method, Treffz method, Kantorovich method, etc.), which give an approximate analytical expression for the desired function and numerical methods (finite difference method, grid method, variational-difference method, finite element method, etc.), which determine the numerical values of the function for different values of the argument.

References

1 Awrejcewicz J. Theory of plates and shells: new trends and applications / J. Awrejcewicz, I.V. Andrianov // International Journal of Nonlinear Sciences and Numerical Simulation. — 2014. — 5(1). — P. 23–36. DOI: 10.13140/2.1.2244.5767

2 Delale F. Stress analysis of multilayered plates around circular holes / F. Delale // International Journal of Engineering Science. — 1984. — 22(1). — P. 57–75. DOI: 10.1016/0020-7225(84)90132-0

3 Voyiadjis G.Z. Elasto-Plastic and Damage Analysis of Plates and Shells / G.Z. Voyiadjis, P. Woelke. — Springer-Verlag Berlin Heidelberg, 2008. — 208 p. DOI:10.1007/978-3-540-79351-9-9

4 England A.H. Stiffness coefficients for inhomogeneous elastic plates / A.H. England // International Journal of Engineering Science. — 2009. — 47 (3). — P. 1446–1459. DOI: 10.1016/j.ijengsci.2009.10.002

5 Librescu L. A few remarks concerning several refined theories of anisotropic laminated plates / L. Librescu, J.N. Reddy // International Journal of Engineering Science. — 1989. — 27(5). — P. 515-527. DOI: 10.1016 / 0020-7225(89)90004-9

6 Andrianov I.V. Asymptotic Methods in the Theory of Plates with Mixed Boundary Conditions / I.V. Andrianov, J. Awrejcewicz, V.V. Danishevs'kyy, A.O. Ivankov. — Wiley, 2014. — 288 p. DOI: 10.1002 /9781118725184

7 Громовик А.И. Расчет круглых пластин: метод. указ. / А.И. Громовик. — Омск: СибАДИ, 2011. — 33 с.

8 Akhanova A.S. On the calculation of plates by the series representation of the deflection function / A.S. Akhanova, G.A. Yessenbayeva, N.K. Tursyngaliyev // Bulletin of the Karaganda University. Mathematics Series. — 2016. — № 2(82). — P. 15-22.

9 Yessenbayeva, G.A. On the calculation of rectangular plates by the variation method / G.A. Yessenbayeva, A.A. Smailova // Bulletin of the Karaganda University. Mathematics Series. — 2016. — № 2(82). — P. 56–62.

10 Животов А.Г. О расчете прямоугольных пластин методом тригонометрических рядов / А.Г. Животов, Г.А. Есенбаева // Вестн. Караганд. ун-та. Сер. Математика. — 2015. — № 3(79). — С. 44 — 49.

11 Косауров А.П. Анализ и особенности методов при расчете пластин и оболочек на изгиб / А.П. Косауров, П.В. Тимофеев. — М.: Фонд «Основание», 2013. — 17 с.

12 Yessenbayeva G.A. On the calculation of the rectangular finite element of the plate / G.A. Yessenbayeva, D.N. Yesbayeva, T.Kh. Makazhanova // Bulletin of the Karaganda University. Mathematics Series. — 2018. — № 2(90). — P. 150-156.

13 Завьялов В.Н. Основы строительной механики пластин / В.Н. Завьялов, Е.А. Мартынов, В.М. Романовский. — Омск: СибАДИ, 2012. — 116 с.

14 Тимошенко С.П. Теория упругости / С.П.Тимошенко, Дж. Гудьер. — М.: Наука, 1979. — 560 с.

15 Феодосьев В.И. Сопротивление материалов / В.И. Феодосьев. — М.: Изд-во МГТУ им. Н.Э. Баумана, 1999. — 592 с.

Г.А. Есенбаева, А.Т. Касимов, Б.А. Касимов

Иілу үшін дөңгелек пластиналарды есептеу туралы

Мақала дөңгелек пластиналардың иілуін зерттеу мәселесіне арналған, пластина механиканың көптеген есептеудің негізі болып табылады. Авторлар осы әдістің құрылымын берген, оның негізгі компоненттерін көрсеткен, сондай-ақ оның түрлері мен классикалық тәсілдерін сипаттаған. Мақалада жеке жағдайларда дөңгелек пластиналардың иілу мәселесі бойынша зерттеу жүргізілді. Дөңгелек пластиналардың иілуін есептеу әдістері, барлық аналитикалық әдістер сияқты, бірнеше артықшылықтарға ие. Негізінен, мақала механиктерге, инженерлерге және техникалық мамандықтары мамандарға бағытталған.

Кілт сөздер: дөңгелек пластина, пластинаның иілу функциясы, осьсимметриялық жүктемелер, дөңгелек пластинаның ең жоғары иілуі.

Г.А. Есенбаева, А.Т. Касимов, Б.А. Касимов

О расчете круглых пластин на изгиб

Статья посвящена вопросу исследования изгиба круглых пластин, являющихся основой расчета многих задач механики. Авторами представлена структура данного метода, выделены его основные компоненты, охарактеризованы виды, а также его классические подходы. В статье проведено исследование задачи об изгибе круглых пластин в частных случаях. Методы расчета изгиба круглых пластин, как и все аналитические методы, имеют ряд преимуществ, которые отмечены в статье. Главным образом, статья рассчитана на механиков, инженеров и специалистов технических специальностей.

Ключевые слова: круглая пластина, функция прогиба пластины, осесимметричные нагрузки, максимальный прогиб круглой пластины.

References

1 Awrejcewicz, J. & Andrianov, I.V. (2014). Theory of plates and shells: new trends and applications. *International Journal of Nonlinear Sciences and Numerical Simulation, 5(1),* 23–36. DOI: 10.13140/2.1.2244.5767

2 Delale, F. (1984). Stress analysis of multilayered plates around circular holes. *International Journal of Engineering Science, 22(1),* 57–75. DOI: 10.1016/0020-7225(84)90132-0

3 Voyiadjis, G.Z. & Woelke, P. (2008). *Elasto-Plastic and Damage Analysis of Plates and Shells.* Springer-Verlag Berlin Heidelberg. DOI:10.1007/978-3-540-79351-9-9

4 England, A.H. (2009). Stiffness coefficients for inhomogeneous elastic plates. *International Journal of Engineering Science, 47 (3),* 1446–1459. DOI: 10.1016/j.ijengsci.2009.10.002

5 Librescu, L. & Reddy, J.N. (1989). A few remarks concerning several refined theories of anisotropic laminated plates. *International Journal of Engineering Science, 27(5),* 515–527. DOI: 10.1016/0020-7225(89)90004-9

6 Andrianov, I.V., Awrejcewicz, J., Danishevs'kyy, V.V. & Ivankov, A.O. (2014). *Asymptotic Methods in the Theory of Plates with Mixed Boundary Conditions.* Wiley. DOI: 10.1002/9781118725184

7 Gromovik, A.I. (2011). *Raschet kruhlykh plastin: Metodicheskie ukazaniia [Calculation of round plates: Methodical instructions].* Omsk: SibADI [in Russian].

8 Akhanova, A.S., Yessenbayeva, G.A. & Tursyngaliyev, N.K. (2016). On the calculation of plates by the series representation of the deflection function. *Bulletin of the Karaganda University. Mathematics Series, 2(82),* 15–22.

9 Yessenbayeva, G.A. & Smailova, A.A. (2016). On the calculation of rectangular plates by the variation method. *Bulletin of the Karaganda University. Mathematics Series, 2(82),* 56–62.

10 Zhivotov, A.G. & Yessenbayeva, G.A. (2015). O raschete pryamouholnykh plastin metodom trihonometricheskikh riadov [On the calculation of rectangular plates by the method of trigonometric series]. *Bulletin of the Karaganda University. Mathematics Series, 3(79),* 44–49 [in Russian].

11 Kosaurov, A.P. & Timofeev, P.V. (2013). *Analiz i osobennosti metodov pri raschete plastin i obolochek na izhib [Analysis and features of methods for calculation of plates and shells for bending].* Moscow: Fond «Osnovanie» [in Russian].

12 Yessenbayeva, G.A., Yesbayeva, D.N. & Makazhanova, T.Kh. (2018). On the calculation of the rectangular finite element of the plate. *Bulletin of the Karaganda University. Mathematics Series, 2(90),* 150–156.

13 Zavyalov, V.N., Martynov, E.A. & Romanovsky, V.M. (2012). *Osnovy stroitelnoi mekhaniki plastin [Basics of structural mechanics of plates].* Omsk: SibADI [in Russian].

14 Timoshenko, S.P. & Goodier, J. (1979). *Teoriia upruhosti [Theory of elasticity].* Moscow: Nauka [in Russian].

15 Feodosyev, V.I. (1999). *Soprotivlenie materialov [Strength of materials].* Moscow: Izdatelstvo MHTU imeni N.E. Baumana [in Russian].

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On the finite element method for calculation of rectangular plates

The article is devoted to the study of bending problems for rectangular plates, which are of great applied importance and are found everywhere in various branches of science and technology. The calculation of plate bending is performed by the finite element method. In this article the structure of the method for calculating the deformed and stress state of a rectangular finite element of the plate is described, their main components are highlighted; the classical approach of calculating rectangular plates is characterized. The mathematical apparatus of the calculation is presented in the volume necessary for calculating the plates. This article is focused mainly on mechanics, physicists, engineers and technical specialists.

Keywords: finite element method, bending of a rectangular plate, finite element of a plate, plate deflection function, coordinate functions.

Plates are rightly considered the most universal and widespread elements in virtually all sectors of the economy. They are widely used in industry, in construction, in various branches of technology as structural elements and parts of multifarious devices, numerous mechanisms and machines for diverse purposes.

Among the plates of different shapes, a special place is occupied by a rectangular plate, since the rectangular plate has a universal shape and is the basis for the calculation of many plate structures [1, 2].

Separate rectangular plates are used in construction in the form of wall panels, wall beams, slabs and floor panels and coatings, foundation slabs, etc. Connected horizontal and vertical plates form the load-bearing system, which in relation to the buildings referred to as the wall system. An obliquely positioned plate can form the span load-bearing structures. The system of rectangular inclined plates, the middle surface of which is deployed on a plane, is called a fold. Folds are also widely used in construction and engineering.

During operation, the plates and plate structures are exposed to temperature, power, mechanical stress, wear, etc. All this causes, first of all, the bending of the plate. Therefore, the theory of plate bending is an important section of the general theory of plates.

To create new plates with specified performance characteristics and strength, it is necessary to investigate how temperature-time, mechanical (including oscillatory), chemical and other influences can cause destructive processes in the structure of the material. Therefore, the required qualities of the plates are usually provided by calculating the effect of such forces on the strength and on the characteristics of the material necessary for operation [3–6].

When calculating the plate strength for bending, it is necessary to have information about its stress-strain state. Many various analytical and approximate methods of plate theory are used for this purpose [7–10]. An exact solution in analytical form for such problems of plate bending is possible only in some particular cases of the geometrical type of the plate, the load and the conditions for its fixation on the supports; therefore, for engineering practice, approximate, but sufficiently accurate methods for solving the considered problem are of special importance.

It should be noted that when calculating the plates by analytical methods in the most general formulation: with arbitrary boundary conditions (including elastic), different types of load, complex shapes of plates, with cuts, projections, etc., we have to face with great mathematical difficulties, and in most cases to obtain an analytical solution is not possible. Such a problem can be solved by applying the very efficient finite element method, which is a numerical approximate method for plates, but which gives a sufficiently high accuracy of solutions.

The finite element method [11] is one of the numerical methods for solving the problems of solid deformable body mechanics. This method is effectively developed in recent years. The name of this method to some extent predetermines its essence: when using the finite element method, the calculated structure is mentally divided into separate elements, the stress-strain state of which is previously studied in detail and can be considered known. The finite element method is based on two main ideas: discretization of the object under study on a finite set of elements and piecewise-element approximation of the functions under study.

It is assumed that the elements are connected to each other in a finite number of points, called nodes. At these points, efforts are determined. These efforts characterize the interaction of individual elements, through which, ultimately, stresses and displacements of each element are calculated. Thus, the problem is discretized and reduced to solving a system of algebraic equations with respect to unknown forces or displacements of nodes.

Depending on which quantities are accepted as unknown, there are three classical approaches used in the finite element method: the force method, the displacement method, and the mixed method. Note that due to a number of advantages, the approach based on the idea of the displacement method is the most widespread in the finite element method.

The type of the finite element is determined depending on the type of the structure under consideration. So for rod systems, rods with different fastenings at the ends, which are element nodes, can be taken as the finite element. Thin-walled spatial systems consisting of plates and shells are divided into triangular, rectangular elements or elements of any other shape with nodes at angular points. Next, we consider a rectangular finite element of the plate.

The deformed state of the rectangular finite element of the plate is completely determined by the following parameters [12]

$$
\begin{gathered}
W\,(x,y)=\vec{q}^{\,T}\vec{V},\\
\alpha\,(x,y)=(\vec{q}^{\,1})^T\vec{V},\quad \beta\,(x,y)=(\vec{q}^{\,2})^T\vec{V},\\
\xi_1\,(x,y)=(\vec{q}^{\,11})^T\vec{V},\quad \xi_2\,(x,y)=(\vec{q}^{\,22})^T\vec{V},\quad \xi_{12}\,(x,y)=(\vec{q}^{\,12})^T\vec{V},\\
\vec{q}^{\,1}=\frac{\partial\,\vec{q}}{\partial\,x},\quad \vec{q}^{\,2}=\frac{\partial\,\vec{q}}{\partial\,y},\quad \vec{q}^{\,11}=\frac{\partial^2\,\vec{q}}{\partial\,x^2},\quad \vec{q}^{\,12}=\frac{\partial^2\,\vec{q}}{\partial\,x\partial\,y},\quad \vec{q}^{\,22}=\frac{\partial^2\,\vec{q}}{\partial\,y^2},
\end{gathered}
\tag{1}
$$

where $W(x,y)$ is the deflection function; $\alpha(x,y)$ is an angle of rotation (tilt) along the axis x; $\beta(x,y)$ is an angle of rotation (tilt) along the axis y; $\xi_1(x,y)$ is a curvature of the curve along the axis x; $\xi_2\,(x,y)$ is a curvature of the curve along the axis y; $\xi_{12}(x,y)$ is torsion of the curve; $\vec{q}$ is the vector of plate coordinate functions; $\vec{V}$ is the vector of nodal displacements for the finite element of the plate.

Taking into account the formulas of forces and (1)

$$
M_1=-D\left(\frac{\partial^2 W}{\partial\,x_1^2}+\nu\,\frac{\partial^2 W}{\partial\,x_2^2}\right),\quad M_2=-D\left(\frac{\partial^2 W}{\partial\,x_2^2}+\nu\,\frac{\partial^2 W}{\partial\,x_1^2}\right),\quad M_{12}=-D\,(1-\nu\,)\frac{\partial^2 W}{\partial\,x_1\partial\,x_2},
$$

$$
Q_1=-D\,\frac{\partial}{\partial\,x_1}(\,\nabla^2 W\,),\quad Q_2=-D\,\frac{\partial}{\partial\,x_2}(\,\nabla^2 W\,),\quad \nabla^2\,W=\frac{\partial^2 W}{\partial\,x_1^2}+\frac{\partial^2 W}{\partial\,x_2^2},
$$

where $x=\frac{x_1}{a}$, $y=\frac{x_2}{b}$, x,y are dimensionless coordinates; the stress state of the finite element of the plate is determined as

$$
\begin{gathered}
M_1=-D\,\left(\xi_1+\nu\,\xi_2\right)=-DA_1^T\vec{V},\quad A_1=\vec{q}^{\,11}+\nu\,\vec{q}^{\,22},\\
M_2=-D\,\left(\xi_2+\nu\,\xi_1\right)=-DA_2^T\vec{V},\quad A_2=\vec{q}^{\,22}+\nu\,\vec{q}^{\,11},\\
M_{12}=-D\,(1-\nu\,)\xi_{12}=-DB^T\vec{V},\quad B=\vec{q}^{\,12},\\
Q_1=-D\,C_1^T\vec{V},\quad C_1=\vec{q}^{\,111}+\vec{q}^{\,122},\\
Q_2=-D\,C_2^T\vec{V},\quad C_2=\vec{q}^{\,211}+\vec{q}^{\,222}.
\end{gathered}
\tag{2}
$$

On the basis of (1) and (2) it is easy to determine the deformation and force characteristics for the finite element of the plate at any point with a known vector of nodal displacements $\vec{V}$.

Considering values of functions [13]

$$
\begin{gathered}
f_1(x)=2x^3-3x^2+1,\quad f_2(x)=x^3-2x^2+x,\\
f_3(x)=3x^2-2x^3,\quad f_4(x)=x^3-x^2,\\
\varphi_1(y)=2y^3-3y^2+1,\quad \varphi_2(y)=y^3-2y^2+y,\\
\varphi_3(y)=3y^2-2y^3,\quad \varphi_4(y)=y^3-y^2,
\end{gathered}
$$

where $f_1(x), ..., f_4(x), \varphi_1(y), ..., \varphi_4(y)$ are coordinate functions for bending beams, and taking into account that the plate coordinate functions are equal to [14]

$$q_1(x,y) = f_1(x)V_1(y) + u_1(x)\varphi_1(y) - u_1(x)V_1(y),$$

$$q_2(x,y) = af_2(x)V_1(y), \quad q_3(x,y) = bu_1(x)\varphi_2(y),$$

$$q_4(x,y) = f_3(x)V_1(y) - u_2(x)\varphi_3(y) + u_2(x)V_2(y),$$

$$q_5(x,y) = af_4(x)V_1(y), \quad q_6(x,y) = bu_2(x)\varphi_2(y),$$

$$q_7(x,y) = f_3(x)V_2(y) + u_2(x)\varphi_3(y) - u_2(x)V_2(y),$$

$$q_8(x,y) = af_4(x)V_2(y), \quad q_9(x,y) = bu_2(x)\varphi_4(y),$$

$$q_{10}(x,y) = u_1(x)\varphi_3(y) - f_3(x)V_2(y) + u_2(x)V_2(y),$$

$$q_{11}(x,y) = af_2(x)V_2(y), \quad q_{12}(x,y) = bu_1(x)\varphi_4(y),$$

$$u_1(x) = 1 - x, \quad u_2(x) = x, \quad V_1(y) = 1 - y, \quad V_2(y) = y,$$

where u_1, u_2, V_1, V_2 are coordinate functions of the rod during torsion, we have values of derivatives from coordinate functions

$$q_1^{11} = \frac{6}{a^2}(2x-1)(1-y), \quad q_2^{11} = \frac{2}{a}(3x-2)(1-y), \quad q_3^{11} = 0,$$

$$q_4^{11} = \frac{6}{a^2}(1-2x)(1-y), \quad q_5^{11} = \frac{2}{a}(3x-1)(1-y), \quad q_6^{11} = 0,$$

$$q_7^{11} = \frac{6}{a^2}(1-2x)y, \quad q_8^{11} = \frac{2}{a}(3x-1)y, \quad q_9^{11} = 0,$$

$$q_{10}^{11} = -\frac{6}{a^2}(1-2x)y, \quad q_{11}^{11} = \frac{2}{a}(3x-2)y, \quad q_{12}^{11} = 0.$$

$$q_1^{22} = \frac{6}{b^2}(2y-1)(1-x), \quad q_2^{22} = 0, \quad q_3^{22} = \frac{2}{b}(3y-2)(1-x),$$

$$q_4^{22} = -\frac{6}{b^2}(1-2y)x, \quad q_5^{22} = 0, \quad q_6^{22} = \frac{2}{b}(3y-2)x,$$

$$q_7^{22} = \frac{6}{b^2}(1-2y)x, \quad q_8^{22} = 0, \quad q_9^{22} = \frac{2}{b}(3y-1)x,$$

$$q_{10}^{22} = \frac{6}{b^2}(1-2y)(1-x), \quad q_{11}^{22} = 0, \quad q_{12}^{22} = \frac{2}{b}(3y-1)(1-x),$$

$$q_1^{12} = -\frac{1}{ab}\left[\left(6x^2-6x+1\right) + \left(6y^2-6y\right)\right], \quad q_2^{12} = -\frac{1}{b}\left(3x^2-4x+1\right),$$

$$q_3^{12} = -\frac{1}{a}\left(3y^2-4y+1\right), \quad q_4^{12} = -\frac{1}{ab}\left[\left(6x-6x^2\right) + \left(6y-6y^2-1\right)\right],$$

$$q_5^{12} = -\frac{1}{b}\left(3x^2-2x\right), \quad q_6^{12} = \frac{1}{a}\left(3y^2-4y+1\right),$$

$$q_7^{12} = \frac{1}{ab}\left[\left(6x-6x^2-1\right) + \left(6y-6y^2\right)\right], \quad q_8^{12} = \frac{1}{b}\left(3x^2-2x\right),$$

$$q_9^{12} = \frac{1}{a}\left(3y^2-2y\right), \quad q_{10}^{12} = -\frac{1}{ab}\left[\left(6x-6x^2-1\right) + \left(6y-6y^2\right)\right],$$

$$q_{11}^{12} = \frac{1}{b}\left(3x^2-4x+1\right), \quad q_{12}^{12} = -\frac{1}{a}\left(3y^2-2y\right).$$

Here, the upper indices of q show the order of differentiation with respect to the corresponding coordinate, and the lower ones show the ordinal number. The functions of the distribution for transverse forces Q_1 and Q_2 can be written as

$$C_1^1 = \frac{12}{a^3}(1-y) - \frac{6}{ab^2}(2y-1), \quad C_1^2 = \frac{6}{a^2}(1-y), \quad C_1^3 = -\frac{2}{ab}(3y-2),$$

$$C_1^4 = -\frac{12}{a^3}(1-y) - \frac{6}{ab^2}(1-2y), \quad C_1^5 = \frac{6}{a^2}(1-y), \quad C_1^6 = \frac{2}{ab}(3y-2),$$
$$C_1^7 = -\frac{12}{a^3}y + \frac{6}{ab^2}(1-2y), \quad C_1^8 = \frac{6}{a^2}y, \quad C_1^9 = \frac{2}{ab}(3y-1),$$
$$C_1^{10} = \frac{12}{a^3}y - \frac{6}{ab^2}(1-2y), \quad C_1^{11} = \frac{6}{a^2}y, \quad C_1^{12} = -\frac{2}{ab}(3y-1),$$
$$C_2^1 = \frac{12}{b^3}(1-x) - \frac{6}{a^2b}(2x-1), \quad C_2^2 = -\frac{2}{ab}(3x-2), \quad C_2^3 = \frac{6}{b^2}(1-x),$$
$$C_2^4 = \frac{12}{b^3}x - \frac{6}{a^2b}(1-2x), \quad C_2^5 = -\frac{2}{ab}(3x-1), \quad C_2^6 = \frac{6}{b^2}x,$$
$$C_2^7 = -\frac{12}{b^3}x + \frac{6}{a^2b}(1-2x), \quad C_2^8 = \frac{2}{ab}(3x-1), \quad C_2^9 = \frac{6}{b^2}x,$$
$$C_2^{10} = -\frac{12}{b^3}(1-x) - \frac{6}{a^2b}(1-2x), \quad C_2^{11} = \frac{2}{ab}(3x-2), \quad C_2^{12} = \frac{6}{b^2}(1-x).$$

It is known, stress state is completely determined by internal efforts.

We consider an arbitrary i-th node of the plate. For this node we have three efforts: torsional and bending moments $2M_{12i}$, M_{1i}, M_{2i}.

The vector $\vec{M}$ of the nodal internal forces for the finite element takes the form

$$\vec{M} = \begin{pmatrix} \vec{M}_1 \\ \vec{M}_2 \\ \vec{M}_3 \\ \vec{M}_4 \end{pmatrix} = B\vec{V}, \quad t = 2(1-\nu), \tag{3}$$

where

$$\vec{M}_i = \begin{pmatrix} 2M_{12i} \\ M_{1i} \\ M_{2i} \end{pmatrix}, \quad i = 1,\ 2,\ 3,\ 4,$$

B is a matrix of efforts; $\vec{V}$ is a vector of nodal displacements.

We note that the values of derivatives in the nodes of the finite element were used in the preparation of the matrix B of efforts (3). The effort matrix B is shown below

$$B = \begin{pmatrix} \eta_{1,1} & \cdots & \eta_{1,12} \\ \cdots & \cdots & \cdots \\ \eta_{12,1} & \cdots & \eta_{12,12} \end{pmatrix},$$

where

$$\eta_{1,1} = \eta_{1,7} = \eta_{4,1} = \eta_{4,7} = \eta_{7,1} = \eta_{7,5} = \eta_{7,7} = \eta_{10,1} = \eta_{10,7} = -\frac{t}{ab},$$
$$\eta_{1,4} = \eta_{1,10} = \eta_{4,4} = \eta_{4,10} = \eta_{7,4} = \eta_{7,10} = \eta_{10,4} = \eta_{10,10} = \frac{t}{ab},$$
$$\eta_{1,2} = \eta_{4,5} = \eta_{10,2} = -\frac{t}{b}, \quad \eta_{1,11} = \eta_{4,8} = \eta_{7,8} = \eta_{10,11} = \frac{t}{b},$$
$$\eta_{1,3} = \eta_{4,3} = \eta_{7,12} = \eta_{10,12} = -\frac{t}{a}, \quad \eta_{1,6} = \eta_{4,6} = \eta_{7,9} = \eta_{10,9} = \frac{t}{a},$$
$$\eta_{2,1} = \eta_{5,4} = \eta_{8,7} = \eta_{11,10} = -6\left(\frac{1}{a^2} + \frac{\nu}{b^2}\right), \quad \eta_{3,1} = \eta_{6,4} = \eta_{9,7} = \eta_{12,10} = -6\left(\frac{1}{b^2} + \frac{\nu}{a^2}\right),$$
$$\eta_{2,2} = \eta_{11,11} = -\frac{4}{a}, \quad \eta_{5,5} = \eta_{8,8} = \frac{4}{a}, \quad \eta_{3,3} = \eta_{6,6} = -\frac{4}{b}, \quad \eta_{9,9} = \eta_{12,12} = \frac{4}{b},$$
$$\eta_{3,2} = \eta_{12,11} = -\frac{4\nu}{a}, \quad \eta_{6,5} = \eta_{9,8} = \frac{4\nu}{a}, \quad \eta_{2,3} = \eta_{5,6} = -\frac{4\nu}{b}, \quad \eta_{8,9} = \eta_{11,12} = \frac{4\nu}{b},$$
$$\eta_{2,4} = \eta_{5,1} = \eta_{8,10} = \eta_{11,7} = \frac{6}{a^2}, \quad \eta_{3,10} = \eta_{6,7} = \eta_{9,4} = \eta_{12,1} = \frac{6}{b^2},$$

$$\eta_{3,4} = \eta_{6,1} = \eta_{9,10} = \eta_{12,7} = \frac{6\nu}{a^2}, \quad \eta_{2,10} = \eta_{5,7} = \eta_{8,4} = \eta_{11,1} = \frac{6\nu}{b^2},$$

$$\eta_{2,5} = \eta_{11,8} = -\frac{2}{a}, \quad \eta_{5,2} = \eta_{8,11} = \frac{2}{a}, \quad \eta_{3,12} = \eta_{6,9} = -\frac{2}{b}, \quad \eta_{9,6} = \eta_{12,3} = \frac{2}{b},$$

$$\eta_{3,5} = \eta_{12,8} = -\frac{2\nu}{a}, \quad \eta_{6,2} = \eta_{9,11} = \frac{2\nu}{a}, \quad \eta_{2,12} = \eta_{5,9} = -\frac{2\nu}{b}, \quad \eta_{8,6} = \eta_{11,3} = \frac{2\nu}{b}.$$

The remaining elements of the matrix are equal to zero. The elements of this matrix depend on the sizes a and b of the finite element and also depend on the parameter t (3). The multiplier of this matrix is $(-D)$.

Using the expressions of transverse forces and bending moments

$$Q_1 = -D\vec{C}_1^T\vec{V}, \quad Q_1 = -D\vec{C}_2^T\vec{V}, \quad \tilde{M} = -D(1+\nu)A^T\vec{V},$$

we write the vector of transverse forces

$$Q = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} = -DC\vec{V}, \quad T_i = \begin{pmatrix} Q_{1i} \\ Q_{2i} \\ \tilde{M}_i \end{pmatrix}, \quad i = 1,\, 2,\, 3,\, 4.$$

Here C is the matrix of transverse forces, which is determined by the values of derivatives in the nodes of the finite element; D is the cylindrical rigidity of the plate; $\vec{V}$ is a vector of nodal displacements of the finite element.

$$\tilde{M} = M_x + M_y$$

is the generalized moment at which the value of the shear deflection $\tilde{W}(x,y)$ can be found by the following well-known formula [12]

$$\tilde{M}(x,y) = GF \cdot \tilde{W}(x,y),$$

where GF is shear rigidity. $\vec{C}_1, \vec{C}_2, \vec{A}$ are vectors of the coordinate functions, respectively Q_1, Q_2 and $\tilde{M}$; i is the number of a current node for the finite element. The transverse forces matrix C is presented below

$$C = \begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,12} \\ \cdots & \cdots & \cdots \\ \gamma_{12,1} & \cdots & \gamma_{12,12} \end{pmatrix},$$

where

$$\gamma_{1,4} = \gamma_{4,4} = \gamma_{7,7} = \gamma_{10,7} = -\frac{6}{a}\left(\frac{2}{a^2} + \frac{1}{b^2}\right), \quad \gamma_{1,1} = \gamma_{4,1} = \gamma_{7,10} = \gamma_{10,10} = \frac{6}{a}\left(\frac{2}{a^2} + \frac{1}{b^2}\right),$$

$$\gamma_{2,10} = \gamma_{5,7} = \gamma_{8,7} = \gamma_{11,10} = -\frac{6}{b}\left(\frac{2}{b^2} + \frac{1}{a^2}\right), \quad \gamma_{2,1} = \gamma_{5,4} = \gamma_{8,4} = \gamma_{11,1} = \frac{6}{b}\left(\frac{2}{b^2} + \frac{1}{a^2}\right),$$

$$\gamma_{3,1} = \gamma_{6,4} = \gamma_{9,7} = \gamma_{12,10} = -6\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

$$\gamma_{1,10} = \gamma_{4,10} = \gamma_{7,1} = \gamma_{10,1} = -\frac{6}{ab^2}, \quad \gamma_{1,7} = \gamma_{4,7} = \gamma_{7,4} = \gamma_{10,4} = \frac{6}{ab^2},$$

$$\gamma_{2,4} = \gamma_{5,1} = \gamma_{8,1} = \gamma_{11,4} = -\frac{6}{a^2 b}, \quad \gamma_{2,7} = \gamma_{5,10} = \gamma_{8,10} = \gamma_{11,7} = \frac{6}{a^2 b},$$

$$\gamma_{1,2} = \gamma_{1,5} = \gamma_{3,4} = \gamma_{4,2} = \gamma_{4,5} = \gamma_{6,1} = \gamma_{7,8} = \gamma_{7,11} = \gamma_{9,10} = \gamma_{10,8} = \gamma_{10,11} = \gamma_{12,7} = \frac{6}{a^2},$$

$$\gamma_{2,3} = \gamma_{2,12} = \gamma_{3,10} = \gamma_{5,6} = \gamma_{5,9} = \gamma_{6,7} = \gamma_{8,6} = \gamma_{8,9} = \gamma_{9,4} = \gamma_{11,3} = \gamma_{11,12} = \gamma_{12,1} = \frac{6}{b^2},$$

$$\gamma_{1,6} = \gamma_{2,11} = \gamma_{4,6} = \gamma_{5,5} = \gamma_{7,12} = \gamma_{8,5} = \gamma_{10,12} = \gamma_{11,11} = -\frac{4}{ab},$$

$$\gamma_{1,3} = \gamma_{2,2} = \gamma_{4,3} = \gamma_{5,8} = \gamma_{7,9} = \gamma_{8,8} = \gamma_{10,9} = \gamma_{11,2} = \frac{4}{ab},$$

$$\gamma_{1,9} = \gamma_{2,8} = \gamma_{4,9} = \gamma_{5,2} = \gamma_{7,3} = \gamma_{8,2} = \gamma_{10,3} = \gamma_{11,8} = -\frac{2}{ab},$$

$$\gamma_{1,12} = \gamma_{2,5} = \gamma_{4,12} = \gamma_{5,11} = \gamma_{7,6} = \gamma_{8,11} = \gamma_{10,6} = \gamma_{11,5} = \frac{2}{ab},$$

$$\gamma_{6,5} = \gamma_{9,8} = \frac{4}{a}, \quad \gamma_{9,9} = \gamma_{12,12} = \frac{4}{b},$$

$$\gamma_{3,5} = \gamma_{12,8} = -\frac{2}{a}, \quad \gamma_{6,2} = \gamma_{9,11} = \frac{2}{a}, \quad \gamma_{3,12} = \gamma_{6,9} = -\frac{2}{b}, \quad \gamma_{9,6} = \gamma_{12,3} = \frac{2}{b}.$$

The remaining elements of the transverse forces matrix are equal to zero.

Replacement of the original construction with a set of discrete elements allows unifying the calculation of various building objects: rod systems, thin-walled and massive structures and real structures that combine rods, plates, shells etc. This circumstance makes the finite element method very universal and explains its increased popularity.

Today, the finite element method is a powerful tool for engineering analysis and physical research through the creation of software packages such as ANSYS, MSC.NASTRAN, MSC.MARC, COSMOS, ABAQUS. These packages of computer programs implement the computational process of the finite element method, and also have a convenient interface for input of initial data, control of the calculation process and processing of calculation results [15].

References

1 Завьялов В.Н. Основы строительной механики пластин / В.Н. Завьялов, Е.А. Мартынов, В.М. Романовский. — Омск: СибАДИ, 2012. — 116 с.

2 Косауров А.П. Анализ и особенности методов при расчете пластин и оболочек на изгиб / А.П. Косауров, П.В. Тимофеев. — М.: Фонд «Основание», 2013. — 17 с.

3 Awrejcewicz J. Theory of plates and shells: new trends and applications / J. Awrejcewicz, I.V. Andrianov // International Journal of Nonlinear Sciences and Numerical Simulation. — 2014. — 5(1). — P. 23–36. DOI: 10.13140/2.1.2244.5767

4 Delale F. Stress analysis of multilayered plates around circular holes / F. Delale // International Journal of Engineering Science. — 1984. — 22(1). — P. 57–75. DOI: 10.1016/0020-7225(84)90132-0

5 Voyiadjis G.Z. Elasto-Plastic and Damage Analysis of Plates and Shells / G.Z. Voyiadjis, P. Woelke. — Springer-Verlag Berlin Heidelberg, 2008. — 208 p. DOI: 10.1007/978-3-540-79351-9-9

6 Librescu L. A few remarks concerning several refined theories of anisotropic laminated plates / L. Librescu, J.N. Reddy // International Journal of Engineering Science. — 1989. — 27(5). — P. 515–527. DOI: 10.1016/0020-7225(89)90004-9

7 Andrianov I.V. Asymptotic Methods in the Theory of Plates with Mixed Boundary Conditions / I.V. Andrianov, J. Awrejcewicz, V.V. Danishevs'kyy, A.O. Ivankov. — Wiley, 2014. — 288 p. DOI: 10.1002/9781118 725184

8 Akhanova A.S. On the calculation of plates by the series representation of the deflection function / A.S. Akhanova, G.A. Yessenbayeva, N.K. Tursyngaliyev // Bulletin of the Karaganda University. Mathematics series. — 2016. — № 2(82). — P. 15–22.

9 Yessenbayeva, G.A. On the calculation of rectangular plates by the variation method / G.A. Yessenbayeva, A.A. Smailova // Bulletin of the Karaganda University. Mathematics series. — 2016. — № 2(82). — P. 56–62.

10 Животов А.Г. О расчете прямоугольных пластин методом тригонометрических рядов / А.Г. Животов, Г.А. Есенбаева // Вестн. Караганд. ун-та. Сер. Математика. — 2015. — № 3(79). — С. 44–49.

11 Bass J. Adaptive finite element methods for a class of evolution problems in viscoplasticity / J. Bass, J.T. Oden // International Journal of Engineering Science. — 1987. — 25(6) — P. 623–653. DOI: 10.1016 /0020-7225(87)90053-X

12 Тимошенко С.П. Теория упругости / С.П. Тимошенко, Дж. Гудьер. — М.: Наука, 1979. — 560 с.

13 Турсунов К.А. Прямоугольный конечный элемент пластины / К.А. Турсунов, А.Е. Елешова // Тр. ун-та. Раз. 5: Математика. Механика. — 2000. — Вып. 1. — С. 61–63.

14 Yessenbayeva G.A. On the calculation of the rectangular finite element of the plate / G.A. Yessenbayeva, D.N. Yesbayeva, T.Kh. Makazhanova // Bulletin of the Karaganda University. Mathematics series. — 2018. — No. 2(90). — P. 150–156.

15 Фокин В.Г. Метод конечных элементов в механике деформируемого твердого тела / В.Г. Фокин. — Самара: Самар. гос. техн. ун-т, 2010. — 131 с.

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Тікбұрышты пластиналарды есептеу үшін соңғы элементтердің әдісі туралы

Мақала тікбұрышты пластиналардың иілу мәселелерін зерттеуге арналған. Пластиналардың үлкен қолданбалы маңызы бар және ғылым мен техниканың әртүрлі салаларында әр жерде кездеседі. Пластиналардың иілуін есептеу соңғы элементтер әдісімен орындалды. Авторлар пластинаның тікбұрышты соңғы элементінің деформацияланған және кернеулі күйін есептеу әдісінің құрылымын берген, оның негізгі компоненттерін анықтаған, тікбұрышты пластиналарды есептеудің классикалық тәсілін сипаттаған. Математикалық есептеу аппараты пластиналарды есептеу үшін қажетті көлемде ұсынылған. Мақала механиктерге, физиктерге, инженерлерге және техникалық мамандықтардың мамандарына бағытталған.

Кілт сөздер: соңғы элементтер әдісі, тікбұрышты пластинаның иілуі, пластинаның соңғы элементі, пластинаның иілу функциясы, координаттық функциялар.

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О методе конечных элементов при расчете прямоугольных пластин

Статья посвящена исследованию задач изгиба прямоугольных пластин, которые имеют большое прикладное значение и встречаются повсеместно в самых различных отраслях науки и техники. Расчет изгиба пластин выполнен методом конечных элементов. Авторами представлена структура метода расчета деформированного и напряженного состояния прямоугольного конечного элемента пластины, выделены его основные компоненты, охарактеризован классический подход расчёта прямоугольных пластин. Математический аппарат расчета предложен в необходимом для расчёта пластин объеме. Статья ориентирована на механиков, физиков, инженеров и специалистов технических специальностей.

Ключевые слова: метод конечных элементов, изгиб прямоугольной пластины, конечный элемент пластины, функция прогиба пластины, координатные функции.

References

1 Zavyalov, V.N., Martynov, E.A. & Romanovsky, V.M. (2012). *Osnovy stroitelnoi mekhaniki plastin [Basics of structural mechanics of plates].* Omsk: SibADI [in Russian].

2 Kosaurov, A.P. & Timofeev, P.V. (2013). *Analiz i osobennosti metodov pri raschete plastin i obolochek na izhib [Analysis and features of methods for calculation of plates and shells for bending].* Moscow: Fond «Osnovanie» [in Russian].

3 Awrejcewicz, J. & Andrianov, I.V. (2014). Theory of plates and shells: new trends and applications. *International Journal of Nonlinear Sciences and Numerical Simulation, 5(1),* 23–36. DOI: 10.13140/2.1.2244.5767

4 Delale, F. (1984). Stress analysis of multilayered plates around circular holes. *International Journal of Engineering Science, 22(1),* 57–75. DOI: 10.1016/0020-7225(84)90132-0

5 Voyiadjis, G.Z. & Woelke, P. (2008). *Elasto-Plastic and Damage Analysis of Plates and Shells.* Springer-Verlag Berlin Heidelberg. DOI:10.1007/978-3-540-79351-9-9

6 Librescu, L. & Reddy, J.N. (1989). A few remarks concerning several refined theories of anisotropic laminated plates. *International Journal of Engineering Science, 27(5),* 515–527. DOI: 10.1016/0020-7225(89)90004-9

7 Andrianov, I.V., Awrejcewicz, J., Danishevs'kyy, V.V. & Ivankov, A.O. (2014). *Asymptotic Methods in the Theory of Plates with Mixed Boundary Conditions.* Wiley. DOI: 10.1002/9781118725184

8 Akhanova, A.S., Yessenbayeva, G.A. & Tursyngaliyev, N.K. (2016). On the calculation of plates by the series representation of the deflection function. *Bulletin of the Karaganda University. Mathematics Series, 2(82),* 15–22.

9 Yessenbayeva, G.A. & Smailova, A.A. (2016). On the calculation of rectangular plates by the variation method. *Bulletin of the Karaganda University. Mathematics Series, 2(82),* 56–62.

10 Zhivotov, A.G. & Yessenbayeva, G.A. (2015). O raschete priamouholnykh plastin metodom trihonometricheskikh riadov [On the calculation of rectangular plates by the method of trigonometric series]. *Bulletin of the Karaganda University. Mathematics Series, 3(79),* 44–49 [in Russian].

11 Bass, J. & Oden, J.T. (1987). Adaptive finite element methods for a class of evolution problems in viscoplasticity. *International Journal of Engineering Science, 25(6),* 623–653. DOI: 10.1016/0020-7225(87)90053-X

12 Timoshenko, S.P. & Goodier, J. (1979). *Teoriia upruhosti [Theory of elasticity].* Moscow: Nauka [in Russian].

13 Tursunov, K.A. & Eleshova, A.E. (2000). Priamouholnyi konechnyi element plastiny [The rectangular finite plate element]. *Proceedings of the University. Section 5: Mathematics. Mechanics, 1,* 61–63 [in Russian].

14 Yessenbayeva, G.A., Yesbayeva, D.N. & Makazhanova, T.Kh. (2018). On the calculation of the rectangular finite element of the plate. *Bulletin of the Karaganda University. Mathematics Series, 2(90),* 150–156.

15 Fokin, V.G. (2010). *Metod konechnykh elementov v mekhanike deformiruemoho tviordoho tela [The finite element method in solid mechanics].* Samara: Samarskii hosudarstvennyi tekhnicheskii universitet [in Russian].

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