https://doi.org/10.31489/2025M3/150-163 *Research article*

Well-posed problems for the Laplace-Beltrami operator on a stratified set consisting of punctured circles and segments

B.E. Kanguzhin[1], M.O. Mustafina[2], O.A. Kaiyrbek[3,*]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *East Kazakhstan University named after Sarsen Amanzholov, Ust-Kamenogorsk, Kazakhstan;*
[3] *Al-Farabi Kazakh National University, Almaty, Kazakhstan*
(E-mail: kanguzhin53@gmail.com, mergulm011@gmail.com, kaiyrbek.zhalgas@gmail.com)

The Laplace-Beltrami operator is studied on a stratified set consisting of two punctured circles and an interval. A complete description of all well-posed boundary value problems for the Laplace-Beltrami operator on such a set is given. In the second part of the paper, a class of self-adjoint well-posed problems for the Laplace-Beltrami operator on the specified stratified set is identified. The obtained results can be considered as a generalization of known results on geometric graphs. In particular, the stratified set under consideration can be interpreted as graphs with loops. Studies on the spectral asymptotics of Sturm-Liouville operators on plane curves homotopic to a finite interval are also closely related to the present results paper. Since the punctured circle is diffeomorphic to a finite interval, the spectral methods applied to differential operators on a finite interval can be modified to study the spectral properties of differential operators on the punctured circle. The main results of this paper are obtained by modifications of methods that were previously used in the study of the asymptotic behavior of the eigenvalues of the Sturm-Liouville operator on a finite interval.

Keywords: graph, Laplace-Beltrami operator, unique solution, punctured circle, inhomogeneous system of equations, differential operators, eigenvalue, inhomogeneous equation, local coordinate.

2020 Mathematics Subject Classification: 05C07, 05C25, 05C30, 05C17.

1 *Stratified set* Ω *and functions over* Ω

We consider the stratified set Ω formed by two punctured circles C_1, C_2 and interval $l = (0,1)$ as well as two points A and B. In this case, Ω is a connected set (Fig. 1), the role of one-dimensional strata is played by C_1, C_2, l, and the role of zero-dimensional strata is played by single-point sets $\{A\}$ and $\{B\}$.

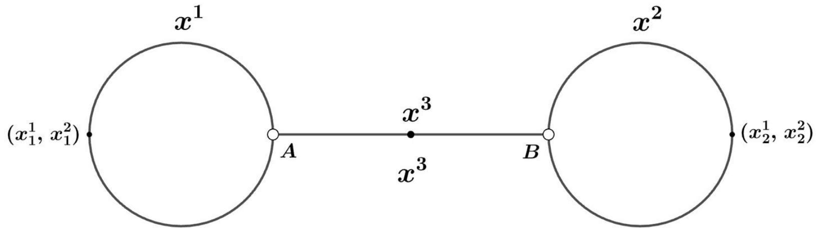

Figure 1. Stratified set Ω on the plane

---
*Corresponding author. *E-mail: kaiyrbek.zhalgas@gmail.com*

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19175972 and No. AP19678089).

Received: 19 December 2024; *Accepted:* 3 June 2025.

The facts given about the stratified set are sufficient for us; more general information about stratified sets can be found in the works [1, 2]. According to the work [3], a measure Ω is introduced on the set μ, as well as the corresponding function spaces. According to the specified work [4, 5] Ω is represented as a union of two non-intersecting parts: $\Omega_0 = C_1 \cup l \cup C_2$ and $\partial\Omega_0 = \{A, B\}$.

2 *Correctly solvable problems for the Laplace-Beltrami operator on a punctured circle* C_1

For convenience, we assume that the punctured circle C_1 is given by equation

$$C_1 = \{x_1 = (x_1^1, x_1^2) \in \mathbb{R}^{\not{k}} \backslash \{(0,0)\} : (x_1^2)^2 + (x_1^1 + 1)^2 = 1\}.$$

It is clear that the punctured circle C_1 can be defined using one card

$$\begin{cases} x_1^1 = \cos t_1 - 1, \\ x_1^2 = \sin t_1. \end{cases}$$

Moreover, the local coordinate t_1 runs through the interval $(0, 2\pi)$. In C_1, one can define classes of functions and the Laplace-Beltrami operator as was done in work [4]. In particular, the Laplace-Beltrami operator in this case represents the operator of twofold differentiation with respect to the variable t_1, if the function on C_1, is represented as a function on the interval $(0, 2\pi)$. If the function on C_1 is represented as a function of $x \in C_1$ then the value of the Laplace-Beltrami operator coincides with a two-fold tangent derivative. Since the Laplace-Beltrami operator is defined invariantly with respect to local coordinates, then when solving the corresponding equations, the equation can be solved in derivative local coordinates. Local coordinates can be chosen at one's discretion, and then the solution found in the chosen coordinates must be able to be written in other arbitrary local coordinates. From the above reasoning, it follows that the statement is true.

Theorem 1. For any numbers a, b and any function $f(x)$, defined on C_1 and belonging to $L_2(C_1)$ the inhomogeneous equation

$$(I - \Delta_{C_1})u(x) = f(x), \quad x \in C_1 \tag{1}$$

with conditions at the point $A(0,0)$

$$U_0(u) = a_1, \quad U_1(u) = b_1 \tag{2}$$

has a unique solution $u(x) \in W_2^2(C_1)$.

Remark 1. If a point P on a circle precedes a point Q on the same circle, we briefly write $P \prec Q$. If points P and Q belong to the same oriented map, then the precedence of one point over another point of the same map is defined according to the orientation. Therefore, the notion of one-sided limit $\lim\limits_{\substack{P \to Q \\ P \prec Q}} f(P) = f(Q - 0)$ is correctly defined.

In Theorem 1, Δ_{C_1} denotes the Laplace-Beltrami operator on C_1. Here, in conditions (1), (2) there are linear functionals $U_0(\cdot)$, $U_1(\cdot)$, which are defined in the following way:

$$U_0(u) = \lim_{\substack{x \to A \\ x \succ A \\ x \in C_1}} u(x) - \lim_{\substack{x \to A \\ A \succ x \\ x \in C_1}} u(x),$$

$$U_1(u) = \lim_{\substack{x \to A \\ x \succ A \\ x \in C_1}} \frac{\partial u(x)}{\partial \tau} - \lim_{\substack{x \to A \\ A \succ x \\ x \in C_1}} \frac{\partial u(x)}{\partial \tau},$$

where $\frac{\partial u}{\partial \tau}$-means the derivative along the tangent to C_1 at point x. The proof of Theorem 1 can be found in the work of [4]. From Theorem 1 and from the results of M. Otelbaev [5–7] the following theorem follows.

Theorem 2. (i) For any function $f(x)$, defined on C_1 and belonging to $L_2(C_1)$ the inhomogeneous equation

$$(I - \Delta_{C_1})u(x) = f(x), \quad x \in C_1,$$

with conditions

$$U_0(u) = \int\limits_{C_1} (I - \Delta_{C_1})u(x)\overline{\sigma_0(x)}dl_x, \quad U_1(u) = \int\limits_{C_1} (I - \Delta_{C_1})u(x)\overline{\sigma_1(x)}dl_x, \tag{3}$$

has a unique solution $u(x) \in W_2^2(C_1)$, if $\sigma_0, \sigma_1 \in L_2(C_1)$.

(ii) Let us assume that we add some conditions to the inhomogeneous operator equation (1) with conditions (2) so that equation (1) for all $f \in L_2(C_1)$ has a unique solution $u(x) \in W_2^2(C_1)$.

Then the added conditions are equivalent to conditions (3) for some $\sigma_0, \sigma_1 \in L_2(C_1)$.

Proof. Proof of Theorem 2. The first part of Theorem 2 follows directly from Theorem 1 if

$$a_1 = \int\limits_{C_1} f(x)\overline{\sigma_0(x)}dl_x, \quad b_1 = \int\limits_{C_1} f(x)\overline{\sigma_1(x)}dl_x.$$

Now let us prove the second part of Theorem 2. By assumption, we add some conditions to equation (1) so that equation (1) for all $f \in L_2(C_1)$ has a unique solution $u(x)$, and

$$\|u(x)\|_{L_2(C_1)} \le M\|f(x)\|_{L_2(C_1)}, \tag{4}$$

where M does not depend on f.

So there is only one solution $u(x)$, subject to inequality (4). It follows from the embedding theorem that there exist values of linear functionals $U_0(u), U_1(u)$. It is easy to understand that linear functionals $U_0(\cdot), U_1(\cdot)$ according to inequality (4), are also functionals bounded in $L_2(C_1)$. Therefore, according to F. Riesz's theorem on the general form of a linear continuous functional in space $L_2(C_1)$ there exist functions $\sigma_0(x), \sigma_1(x) \in L_2(C_1)$ such that

$$U_0(u) = \int\limits_{C_1} f(x)\overline{\sigma_0(x)}dl_x, \quad U_1(u) = \int\limits_{C_1} f(x)\overline{\sigma_1(x)}dl_x.$$

Now it remains to replace $f(x)$ with $(I - \Delta_{C_1})u(x)$, from which the validity of the second part of Theorem 3 follows.

3 Well-solved problems for the Laplace-Beltrami operator on a stratified set Ω

In the previous paragraph we wrote out correctly solvable problems for the Laplace-Beltrami operator on a punctured circle C_1. In the same way, one can write out all possible correctly solvable linear problems for the Laplace-Beltrami operator on a punctured circle C_2. Note that correctly solvable linear problems for the operator of twofold differentiation on the interval $l = (0, 1)$ are well known to [5–7]. Now, using the above results, we write out all possible correctly solvable linear problems for the Laplace-Beltrami operator on a stratified set Ω, consisting of C_1, C_2 and l. In this point, the punctured circle C_1 is defined as follows

$$C_1 = \{x_1 = (x_1^1, x_1^2) \in \mathbf{R^2} \backslash (\{0, 0\}) : (x_1^1 + 1)^2 + (x_1^2)^2 = 1\},$$

where the role of local coordinates is played by the variable $t \in (0, 2\pi)$:

$$\begin{cases} x_1^1 = \cos t - 1, \\ x_1^2 = \sin t. \end{cases}$$

The punctured circle C_2 is defined as the following set

$$C_2 = \{x_2 = (x_2^1, x_2^2) \in \mathbf{R^2} \backslash \{(1,0)\} : (x_2^1 - 2)^2 + (x_2^2)^2 = 1\},$$

where the role of local coordinates is played by the variable τ:

$$x_2^1 = 2 + \cos\tau, \quad x_2^2 = \sin\tau, \quad \tau \in (\pi, 3\pi).$$

Interval l is defined as the horizontal open segment

$$l = \{x_3 = (x_3^1, x_3^2) \in \mathbf{R^2}: \ 0 < x_3^1 = S < 1, \ x_3^2 = 0\}.$$

Here the role of the local coordinate is played by the parameter S, which runs through the interval $(0,1)$. An analogue of Theorem 1 can be formulated for a punctured circle C_2 and interval l. As a result, we have the following statement.

Theorem 3. For any numbers $a_1, b_1, a_2, b_2, a_3, b_3$ and any functions $\vec{F} = \{f_1(x_1), f_2(x_2), f_3(s) \in L_2(\Omega)\}$ non-homogeneous system of equations

$$\begin{cases} (I - \Delta_{C_1})u_1(x_1) = f_1(x_1), \quad x_1 \in C_1, \\ (I - \Delta_{C_2})u_2(x_2) = f_2(x_2), \quad x_2 \in C_2, \\ u_3(s) - u_3''(s) = f_3(s), \quad s \in (0,1), \end{cases} \tag{5}$$

with conditions

$$\begin{gathered} U_0(u_1) = a_1, \quad U_1(u_1) = b_1, \\ V_0(u_2) = a_2, \quad V_1(u_2) = b_2, \\ u_3(0) = a_3, \quad u_3(1) = b_3 \end{gathered} \tag{6}$$

has a unique solution $u = (u_1, u_2, u_3) \in W_2^2(\Omega)$.

In Theorem 3 Δ_{C_2} denotes the Laplace-Beltrami operator on C_2. Also, linear forms determined by limiting ratios are designated by $V_0(\cdot)$ and $V_1(\cdot)$:

$$V_0(u_2) = \lim_{\substack{x \to B \\ x \succ B \\ x \in C_2}} u_2(x) - \lim_{\substack{x \to B \\ B \succ x \\ x \in C_2}} u_2(x),$$

$$V_1(u_2) = \lim_{\substack{x \to B \\ x \succ B \\ x \in C_2}} \frac{\partial u_2(x)}{\partial \tau} - \lim_{\substack{x \to B \\ B \succ x \\ x \in C_2}} \frac{\partial u_2(x)}{\partial \tau},$$

where $B = (1,0)$ and $\frac{\partial u}{\partial \tau}$-means the derivative along the tangent to C_2 at point x.

Similar results for graphs without loops were studied in [8]. This theorem can be interpreted as correctly solvable problems for the Laplace-Beltrami operator on graphs with loops. From Theorem 3 and the results [5–7] of the assertion follows.

Theorem 4. (i) For any function $\vec{F} = \{f_1, f_2, f_3\} \in L_2(\Omega)$ inhomogeneous system of equations (5) with conditions

$$\begin{cases} U_0(u_1) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1)\overline{\sigma_1(x_1)}dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2)\overline{\rho_1(x_2)}dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s))\overline{\varphi_1(s)}ds, \\ U_1(u_1) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1)\overline{\sigma_2(x_1)}dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2)\overline{\rho_2(x_2)}dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s))\overline{\varphi_2(s)}ds, \\ V_0(u_2) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1)\overline{\sigma_3(x_1)}dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2)\overline{\rho_3(x_2)}dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s))\overline{\varphi_3(s)}ds, \\ V_1(u_2) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1)\overline{\sigma_4(x_1)}dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2)\overline{\rho_4(x_2)}dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s))\overline{\varphi_4(s)}ds, \\ u_3(0) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1)\overline{\sigma_5(x_1)}dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2)\overline{\rho_5(x_2)}dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s))\overline{\varphi_5(s)}ds, \\ u_3(1) = \int\limits_{C_1} (I - \Delta_{C_1}) u_1(x_1)\overline{\sigma_6(x_1)}dl_1 + \int\limits_{C_2} (I - \Delta_{C_2}) u_2(x_2)\overline{\rho_6(x_2)}dl_2 + \int\limits_0^1 (u_3(s) - u_3''(s))\overline{\varphi_6(s)}ds, \end{cases} \quad (7)$$

has a unique solution $u = \{u_1, u_2, u_3 \in W_2^2(\Omega)\}$, if

$$\sigma_j \in L_2(C_1), \quad \rho_j \in L_2(C_2), \quad \varphi_j \in L_2(0,1), \quad j = 1,2,3,4,5,6.$$

(ii) Let us assume that we add some conditions to the inhomogeneous system of equations (5) with conditions (6) so that equation (5) for all $\vec{F} = \{f_1, f_2, f_3\} \in L_2(\Omega)$ has a unique solution $u = (u_1, u_2, u_3) \in W_2^2(\Omega))$. Then the added conditions are equivalent to conditions of the form (7) for some

$$\sigma_j \in L_2(C_1), \quad \rho_j \in L_2(C_2), \quad \varphi_j \in L_2(0,1), \quad j = 1,2,3,4,5,6.$$

The proof of Theorem 4 repeats the proof of Theorem 2, only the theorem of F. Riesz is used, which concerns the Hilbert space $L_2(\Omega)$.

The formulation of correct boundary value problems for the Laplace operator in a punctured ball was discussed in the works [9–11]. A description of all possible well-defined problems for the Laplace-Beltrami operator on a punctured sphere can be found [12–14]. Everywhere correctly solvable problems for differential operators in punctured domains or in domains with cuts can be interpreted as singular perturbations of regular differential operators. From this point of view, singular differential operators are studied in the works [15–17], differential operators for the Dirichlet and Neumann problems are studied in the works [18, 19].

4 Examples of well-posed problems on a stratified set

In this section we will give specific examples that follow from the first part of Theorem 4. Let us recall Lemma 1 from work [4].

Lemma 1. [4] For any smooth 2π-periodic function $\hat{F}(t)$ the integral identity is valid

$$\int\limits_0^t \hat{F}(t)dt = \int\limits_{\gamma_x^1} F(\xi^1, \xi^2)(\xi^1 d\xi^2 - \xi^2 d\xi^1),$$

where γ_x^1 positively oriented arc of a punctured circle C_1^1, connecting the dots $(0,0)$ and $x = (x^1, x^2) \in C_1$.

Here the function $F(x)$ for $x \in C_1$ is generated by the function $\hat{F}(t)$ for $t \in (0, 2\pi)$ as follows: first, we expand $\hat{F}(t)$ into a trigonometric series

$$\hat{F}(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt), \quad t \in (0, 2\pi)$$

and then according to the formulas $x^1+1 = \cos t$, $x^2 = \sin t$ we move from t to variables $(x^1, x^2) = x \in C_1$

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k T_k(x^1+1) + b_k x^2 U_{k-1}(x^1+1)),$$

where T_k and U_{k-1} Chebyshev polynomials of the first and second kind, respectively. Similarly, the integral $\int\limits_0^{\tau} \hat{\sigma}(\tau)d\tau$ at $\tau \in (\pi, 3\pi)$ we can rewrite it through the integral

$$\int_{\gamma_x^2} \sigma(\xi^1, \xi^2)(\xi^1 d\xi^2 - \xi^2 d\xi^1),$$

where γ_x^2 positively oriented arc pierced circle C_2, connecting points $(-1, 0)$ and $x = (x^1, x^2) \in C_2$. Here also $\sigma(x)$ for $x \in C_2$ is generated by the function $\hat{\sigma}(\tau)$ for $\tau \in (\pi, 3\pi)$ as follows:

First, we expand $\hat{\sigma}(\tau)$ for $\tau \in (\pi, 3\pi)$ into a trigonometric series

$$\hat{\sigma}(\tau) = \frac{c_0}{2} + \sum_{k=1}^{\infty}(c_k \cos k\tau + d_k \sin k\tau), \quad \tau \in (\pi, 3\pi),$$

and then according to the formulas $x^1 - 2 = \cos\tau$, $x^2 = \sin\tau$ we move from the parameter τ to the variables $(x^1, x^2) = x \in C_2$

$$\sigma(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty}(c_k T_k(x^1-2) + d_k x^2 U_{k-1}(x^1-2)).$$

In conditions (7) the integrals $\int\limits_{C_1} f_1(x_1)\overline{\sigma_1 x_1} dl_1$ and $\int\limits_{C_2} f_2(x_2)\overline{\sigma_2 x_2} dl_2$. These integrals can be rewritten in terms of local coordinates t and τ, respectively:

$$\int\limits_{C_1} f_1(x_1)\overline{\sigma_1(x_1)}dl_1 = \int\limits_0^{2\pi} f_1(\cos t - 1, \sin t)\overline{\sigma_1(\cos t - 1, \sin t)}dt = \int\limits_0^{2\pi} \hat{f}_1(t)\overline{\hat{\sigma}_1(t)}dt,$$

$$\int\limits_{C_2} f_2(x_2)\overline{\sigma_2(x_2)}dl_2 = \int\limits_{\pi}^{3\pi} f_2(2+\cos\tau, \sin\tau)\overline{\sigma_2(2+\cos\tau, \sin\tau)}d\tau = \int\limits_{\pi}^{3\pi} \hat{f}_2(\tau)\overline{\hat{\sigma}_2(\tau)}d\tau.$$

Now we are ready to rewrite the integral $\int\limits_{C_1}(I - \Delta_{C_1})u_1(x_1)\overline{\sigma_1(x_1)}dl_1$ in a form convenient for us

$$\int\limits_{C_1}(I - \Delta_{C_1})u_1(x_1)\overline{\sigma_1(x_1)}dl_1 = \int\limits_{C_1} f_1(x_1)\overline{\sigma_1(x_1)}dl_1 = \int\limits_0^{2\pi} \hat{f}_1(t)\overline{\hat{\sigma}_1(t)}dt = \int\limits_0^{2\pi}\Big(\hat{u}_1(t) - \hat{u}_1''(t)\Big)\overline{\hat{\sigma}_1(t)}dt.$$

We apply the integration by parts to the last integral, assuming that $\hat{\sigma}_1(t)$ is twice continuously differentiable function. As a result, we have

$$\int_0^{2\pi}\Big(\hat{u}_1(t)-\hat{u}_1''(t)\Big)\overline{\hat{\sigma}_1(t)}dt=\int_0^{2\pi}\hat{u}_1(t)\overline{\Big(\hat{\sigma}_1(t)-\hat{\sigma}_1''(t)\Big)}dt-\hat{u}_1'(t)\overline{\hat{\sigma}_1(t)}\Big|_{t=0}^{t=2\pi}+\hat{u}_1(t)\overline{\hat{\sigma}_1'(t)}\Big|_{t=0}^{t=2\pi}=$$

$$=\int_0^{2\pi}\hat{u}_1(t)\overline{\Big(\hat{\sigma}_1(t)-\hat{\sigma}_1''(t)\Big)}dt-\hat{u}'(2\pi-0)\overline{\hat{\sigma}_1(2\pi-0)}+\hat{u}_1(2\pi-0)\overline{\hat{\sigma}'(2\pi-0)}+\hat{u}_1'(+0)\overline{\hat{\sigma}_1(+0)}-\hat{u}_1(+0)\overline{\hat{\sigma}_1'(+0)}.$$

Now, as a result of the change of variables from the local coordinate t to the variables $(x_1^1, x_1^2) = x$, we have

$$\int_{C_1}(I-\Delta_{C_1})u_1(x_1)\overline{\sigma_1(x_1)}dl_1=\int_{C_1}u_1(x_1)\overline{(I-\Delta_{C_1})\sigma_1(x_1)}dl_1--\frac{\partial u_1}{\partial\tau}(\prec(0,0)\overline{\sigma_1(\prec(0,0))}+$$

$$+u_1(\prec(0,0))\overline{\frac{\partial}{\partial\tau}\sigma_1(\prec(0,0))}+\frac{\partial u_1(\prec(0,0))}{\partial\tau}\sigma_1(\prec(0,0))-u_1(\prec(0,0))\frac{\partial\sigma_1}{\partial\tau}(\prec(0,0)),\tag{8}$$

where

$$g(\prec(0,0))=\lim_{\substack{x_1\to(0,0)\\x_1\prec(0,0)\\x_1\in C_1}}g(x_1),\quad g(\succ(0,0))=\lim_{\substack{x_1\to(0,0)\\x_1\succ(0,0)\\x_1\in C_1}}g(x_1),$$

$$\frac{\partial g(\prec(0,0))}{\partial\tau}=\lim_{\substack{x_1\to(0,0)\\x_1\prec(0,0)\\x_1\in C_1}}\frac{\partial g(x_1)}{\partial\tau},\quad\frac{\partial g(\succ(0,0))}{\partial\tau}=\lim_{\substack{x_1\to(0,0)\\x_1\succ(0,0)\\x_1\in C_1}}\frac{\partial g(x_1)}{\partial\tau},$$

where $\frac{\partial}{\partial\tau}$ is derivative along the tangent to C_1 at point x_1. In the same way, for any two sufficiently smooth C_2 functions on $u_2(x_2),\ \rho_2(x_2)$ the following identity holds

$$\int_{C_2}(I-\Delta_{C_2})u_2(x_2)\overline{\rho_2(x_2)}dl_2=\int_{C_2}u_2(x_2)\overline{(I-\Delta_{C_2})\rho_2(x_2)}dl_2-$$

$$-\frac{\partial u_2(\prec(-1,0))}{\partial\tau}\overline{\rho_2\prec(-1,0))}+u_2(\prec(-1,0))\overline{\frac{\partial\rho_2(\prec(-1,0))}{\partial\tau}}+\tag{9}$$

$$+\frac{\partial u_2(\succ(-1,0))}{\partial\tau}\overline{\rho_2(\succ(-1,0))}-u_2(\succ(-1,0))\overline{\frac{\partial\rho_2(\succ(-1,0))}{\partial\tau}},$$

where $\frac{\partial}{\partial\tau}$ is derivative along the tangent to C_2 at the point x_2. The given auxiliary statements allow us to obtain consequences of Theorem 4. Now we will specify the choice of boundary functions $\sigma_j(x_1),\ \rho_j(x_2),\ \varphi_j(x_3)$ for $j=1,2,3,4,5,6$ from Theorem 4. Let for $j=1,2,3,4,5,6$ the functions $\sigma_j(x_1),\ \rho_j(x_2),\ \varphi_j(x_3)$ be chosen so that

$$(I-\Delta_{C_1})\sigma_j(x_1)=0,\quad x_1\in C_1,$$

$$(I-\Delta_{C_2})\rho_j(x_2)=0,\quad x_2\in C_2,$$

$$\varphi_j(s)-\varphi_j''(s)=0,\quad s\in(0,1).$$

Then, from relations (8) and (9) we have

$$\int\limits_{C_1}(I-\Delta_{C_1})u_1(x_1)\overline{\sigma_j(x_1)}dl_1=-\frac{\partial u_1(\prec(0,0))}{\partial\tau}\overline{\sigma_j(\prec(0,0))}+$$

$$+u_1\Big(\prec(0,0)\Big)\overline{\frac{\partial\sigma_j(\prec(0,0))}{\partial\tau}}+\frac{\partial u_1(\succ(0,0))}{\partial\tau}\overline{\sigma_j(\succ(0,0))}-u_1(\succ(0,0))\overline{\frac{\partial\sigma_j(\succ(0,0))}{\partial\tau}},$$

$$\int\limits_{C_2}(I-\Delta_{C_2})u_2(x_2)\overline{\rho_j(x_2)}dl_1=-\frac{\partial u_2(\prec(-1,0))}{\partial\tau}\overline{\rho_j(\prec(-1,0))}+$$

$$+u_2\Big(\prec(-1,0)\Big)\overline{\frac{\partial\rho_j(\prec(-1,0))}{\partial\tau}}+\frac{\partial u_2(\succ(-1,0))}{\partial\tau}\overline{\rho_j(\succ(-1,0))}-u_2(\succ(-1,0))\overline{\frac{\partial\rho_j(\succ(-1,0))}{\partial\tau}},$$

$$\int\limits_0^1(u_3(s)-u_3''(s))\overline{\varphi_j(s)}ds=-\frac{du_3(1-0)}{ds}\overline{\varphi_j(1-0)}+u_3(1-0)\frac{d\varphi_j(1-0)}{ds}+$$

$$\frac{du_3(+0)}{ds}\overline{\varphi_j(+0)}-u_3(+0)\frac{d\varphi_j(+0)}{ds}.$$

Thus, the boundary conditions (7) from Theorem 4 take the form for $j=1,2,3,4,5,6$

$$\begin{gathered}U_j=\overline{\hat{\sigma}_j(+0)}\big[\hat{u}_1'(+0)-\cosh 2\pi\,\hat{u}_1'(2\pi-0)+\sinh 2\pi\,\hat{u}_1'(2\pi-0)\big]+\\
+\overline{\hat{\sigma}_j{}'(+0)}\big[\cosh 2\pi\hat{u}_1'(2\pi-0)-\sinh 2\pi\,\hat{u}_1'(2\pi-0)-\hat{u}_1'(+0)\big]+\\
+\overline{\hat{\rho}_j(\pi+0)}\big[\hat{u}_2'(\pi+0)-\cosh 2\pi\,\hat{u}_2'(3\pi-0)+\sinh 2\pi\,\hat{u}_2'(3\pi-0)\big]+\\
+\overline{\hat{\rho}_j{}'(\pi+0)}\big[\cosh 2\pi\hat{u}_2'(3\pi-0)-\sinh 2\pi\,\hat{u}_1'(3\pi-0)-\hat{u}_2'(\pi+0)\big]+\\
+\overline{\varphi_j(+0)}\big[u_3'(+0)+\frac{\cosh 1}{\sinh 1}u_3(0)-\frac{1}{\sinh 1}u_3(1-0)\big]+\\
+\overline{\varphi_j(1-0)}\big[\frac{\cosh 1}{\sinh 1}u_3(1-0)-u_3'(1-0)-\frac{1}{\sinh 1}u_3(+0)\big],\end{gathered}\tag{10}$$

where

$$U_1(u_1)=\hat{u}_1(+0)-\hat{u}_1(2\pi-0),\quad U_2(u_1)=\hat{u}_1'(+0)-\hat{u}_1'(2\pi-0),$$
$$U_3(u_2)=\hat{u}_2(\pi+0)-\hat{u}_2(3\pi-0),\quad U_4(u_2)=\hat{u}_2'(\pi+0)-\hat{u}_2'(3\pi-0),$$
$$U_5(u_3)=u_3(+0),\quad U_6(u_3)=u_3(1-0).$$

5 *Self-adjoint well-solved problems*

In the previous paragraph, examples of correctly solvable problems that are set using boundary conditions. Now we will select from them those problems that are self-adjoint. Correctly-solvable problems correspond to operators whose resolvent sets contain $\lambda=0$. At the same time, self-adjoint well-solvable problems correspond to operators whose eigenvalues provide nonzero real numbers. Thus, in this section, such well-solvable problems are distinguished whose spectrum is discrete and consists of nonzero real eigenvalues. Recall that for any two sufficiently smooth functions $u_1(x_1),\ u_2(x_2),\ u_3(s)$ and $\vartheta_1(x_1),\ \vartheta_2(x_2),\ \vartheta_3(s)$ the identity holds

$$\int\limits_{C_1}(I-\Delta_{C_1})u_1(x_1)\overline{\vartheta_1(x_1)}dl_1+\int\limits_{C_2}(I-\Delta_{C_2})u_2(x_2)\overline{\vartheta_2(x_2)}dl_2+\int\limits_0^1(u_3(s)-u_3''(s))\overline{\vartheta_3(s)}ds=$$

$$= \int\limits_{C_1} u_1(x_1)\overline{(I-\Delta_{C_1})\vartheta_1(x_1)}dl_1 + \int\limits_{C_2} u_2(x_2)\overline{(I-\Delta_{C_2})\vartheta_2(x_2)}dl_2 + \int\limits_0^1 u_3(s)\overline{(\vartheta_3(s)-\vartheta_3''(s))}ds +$$

$$\begin{gathered} +(\hat{u}_1'(+0)-\hat{u}_1'(2\pi-0))\overline{\hat{\vartheta}_1(2\pi-0)} + \hat{u}_1'(+0)\overline{(\hat{\vartheta}_1(+0)-\hat{\vartheta}_1(2\pi-0))}- \\ -(\hat{u}_1(+0)-\hat{u}_1(2\pi-0))\overline{\hat{\vartheta}_1'(2\pi-0)} + \hat{u}_1(+0)\overline{(\hat{\vartheta}_1'(2\pi-0)-\hat{\vartheta}_1'(+0))}+ \\ +(\hat{u}_2'(\pi+0)-\hat{u}_2'(3\pi-0))\overline{\hat{\vartheta}_2(3\pi-0)} + \hat{u}_2'(\pi+0)\overline{(\hat{\vartheta}_2(\pi+0)-\hat{\vartheta}_2(3\pi-0))}- \\ -(\hat{u}_2(\pi+0)-\hat{u}_2(3\pi-0))\overline{\hat{\vartheta}_2'(3\pi-0)} + \hat{u}_2(\pi+0)\overline{(\hat{\vartheta}_2'(3\pi-0)-\hat{\vartheta}_2'(\pi+0))}- \\ -u_3'(1-0)\overline{\vartheta_3(1-0)} + u_3'(+0)\overline{\vartheta_3(+0)} + u_3(1-0)\overline{\vartheta_3'(1-0)} - u_3(+0)\overline{\vartheta_3'(+0)}, \end{gathered} \tag{11}$$

where $\hat{u}_1(t)=u_1(\cos t-1,\sin t)$ for $t\in(0,2\pi)$, $\hat{u}_2(\tau)=u_2(2+\cos\tau,\sin\tau)$ for $\tau\in(\pi,3\pi)$.

Let D denote the set of functions $u_1(x_1),\ u_2(x_2),\ u_3(s)$ such that
(I) $u_1(x_1)\in W_2^2(C_1),\ u_2(x_2)\in W_2^2(C_2),\ u_3(s)\in W_2^2(0,1)$.

Let us also introduce the set D_0, consisting of functions $u_1(x_1),\ u_2(x_2),\ u_3(s)\in D$ such that
(II) $\hat{u}_1'(+0)=\hat{u}_1'(2\pi-0),\ \hat{u}_1'(+0)=0,\ \hat{u}_1(+0)=\hat{u}_1(2\pi-0), \hat{u}_1(+0)=0,$
$\hat{u}_2'(\pi+0)=\hat{u}_2'(3\pi-0),\ \hat{u}_2'(\pi+0)=0,\ \hat{u}_2(\pi+0)=\hat{u}_2(3\pi-0), \hat{u}_2(\pi+0)=0,$
$u_3(+0)=0,\ u_3'(+0)=0,\ u_3(1-0)=0,\ u_3'(1-0)=0.$

Let us introduce the operator L on D using the formula

$$L=(u_1(x_1),u_2(x_2),u_3(s))=((I-\Delta_{C_1})u_1(x_1),(I-\Delta_{C_2})u_2(x_2),(u_3(s)-u_s''(s))).$$

Let us denote by L_0 the restriction of the operator L on D_0. The operator L_0 is Hermitian and following the scheme from § 17 of the monograph [20] we write all possible self-adjoint extensions of the operator L_0. To do this, we need some properties of the operator L_0.

Lemma 2. Let $(f_1(x_1),f_2(x_2),f_3(s))\in L_2(\Omega)$. Equation

$$L_0=(u_1(x_1),u_2(x_2),u_3(s))=(f_1(x_1),f_2(x_2),f_3(s))$$

has a solution if and only if $(f_1(x_1),f_2(x_2),f_3(s))$ orthogonal to all solutions of the homogeneous system

$$(I-\Delta_{C_1})\omega_1(x_1)=0,\quad (I-\Delta_{C_2})\omega_2(x_2)=0,\quad \omega_3(s)-\omega_3''(s)=0. \tag{12}$$

Proof. Let us denote by $(u_1(x_1),u_2(x_2),u_3(s))$ the solution of the system

$$(I-\Delta_{C_1})u_1(x_1)=f_1(x_1),\ (I-\Delta_{C_2})u_2(x_2)=f_2(x_2),\ u_3(s)-u_3''(s)=f_3(s),$$

satisfying the condition

$$\hat{u}_1'(+0)=\hat{u}_1'(2\pi-0),\ \hat{u}_1(+0)=\hat{u}_1(2\pi-0),$$

$$\hat{u}_2'(\pi+0)=\hat{u}_2'(3\pi-0),\ \hat{u}_2(\pi+0)=\hat{u}_2(3\pi-0),\ u_3(+0)=0,u_3(1-0)=0.$$

From the results of the work [4] it follows that there is a unique solution $(u_1(x_1),u_2(x_2),u_3(s))$ to the indicated problem. In the work [4] the eigenvalues of the given problem are calculated and it is shown that there is no zero among the eigenvalues. For the found solution $(u_1(x_1),u_2(x_2),u_3(s))$ identity (11) will take the form

$$\int\limits_{C_1} f_1(x_1)\overline{\vartheta_1(x_1)}dl_1 + \int\limits_{C_2} f_2(x_2)\overline{\vartheta_2(x_2)}dl_2 + \int\limits_0^1 f_3(s)\overline{\vartheta_3(s)}ds =$$

$$= \int\limits_{C_1} u_1(x_1)\overline{(I-\Delta_{C_1})\vartheta_1(x_1)}dl_1 + \int\limits_{C_2} u_2(x_2)\overline{(I-\Delta_{C_2})\vartheta_2(x_2)}dl_2 + \int\limits_0^1 u_3(s)(\overline{\vartheta_3(s)-\vartheta_3''(s)})ds +$$

$$+\hat{u}_1'(+0)\overline{\hat{\vartheta}_1(+0)-\hat{\vartheta}_1(2\pi-0))} + \hat{u}_1(+0)\overline{(\hat{\vartheta}_1'(2\pi-0)-\hat{\vartheta}_1'(+0))} + \tag{13}$$

$$+\hat{u}_2'(\pi+0)\overline{(\hat{\vartheta}_2(\pi+0)-\hat{\vartheta}_2(3\pi-0))} + +\hat{u}_2(\pi+0)\overline{(\hat{\vartheta}_2'(3\pi-0)-\hat{\vartheta}_2'(\pi-0))} -$$

$$-\hat{u}_3'(1-0)\overline{\vartheta_3(1-0)} + \hat{u}_3'(+0)\overline{(\vartheta_3)(+0)}.$$

Now let's choose $V_1 = (\vartheta_{11}(x_1), \vartheta_{12}(x_2), \vartheta_{13}(s))$ so that the homogeneous equations (12) and additional conditions are satisfied

$$\hat{\vartheta}_{11}(+0) - \hat{\vartheta}_{11}(2\pi-0) = 1, \;\; \hat{\vartheta}_{11}'(2\pi-0) - \hat{\vartheta}_{11}'(+0) = 0,$$

$$\hat{\vartheta}_{12}(\pi+0) - \hat{\vartheta}_{12}(3\pi-0) = 0, \;\; \hat{\vartheta}_{12}'(3\pi-0) - \hat{\vartheta}_{12}'(\pi+0) = 0,$$

$$\vartheta_{13}(1-0) = 0, \;\; \vartheta_{13}(+0) = 0.$$

In fact, $\vartheta_{13}(s) \equiv 0, \; \vartheta_{12}(x_2) \equiv 0, \; \hat{\vartheta}_{11}(t) = \frac{e^{2\pi-t}-e^t}{2(e^{2\pi}-1)}$. In this case, from relation (13) it follows

$$\int\limits_{C_1} f_1(x_1)\overline{\vartheta_{11}(x_1)}dl_1 = \hat{u}_1'(+0). \tag{14}$$

By choosing $V_2 = (V_{21}(x_1), V_{22}(x_2), V_{23}(s))$ in a reasonable way, we can obtain the relation

$$\int\limits_{C_1} f_1(x_1)\overline{\vartheta_{21}(x_1)}dl_1 = \hat{u}_1(+0). \tag{15}$$

Reasoning in the same way as in the monograph [20], we obtain the relations

$$\int\limits_{C_2} f_2(x_2)\overline{\vartheta_{32}(x_2)}dl_2 = \hat{u}_2'(+0), \tag{16}$$

$$\int\limits_{C_2} f_2(x_2)\overline{\vartheta_{42}(x_2)}dl_2 = \hat{u}_2(\pi+0), \tag{17}$$

$$\int\limits_0^1 f_3(s)\overline{\vartheta_{53}(s)}ds = -u_3'(1-0), \tag{18}$$

$$\int\limits_0^1 f_3(s)\overline{\vartheta_{63}(s)}ds = u_3'(+0). \tag{19}$$

From relations (14)–(19) the assertion of Lemma 1 follows.

We will also need the following assertion.

Lemma 3. Whatever the numbers

$$\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4,\, \alpha_5,\, \alpha_6,\, \beta_1,\, \beta_2,\, \beta_3,\, \beta_4,\, \beta_5,\, \beta_6$$

there exists a function $(u_1(x_1), u_2(x_2), u_3(s)) \in D$, satisfying the conditions

$$\hat{u}_1'(+0) - \hat{u}_1(2\pi - 0) = \alpha_1, \;\; \hat{u}_1'(+0) = \beta_1,$$

$$\hat{u}_1'(2\pi - 0) - \hat{u}_1'(+0) = \alpha_2, \;\; \hat{u}_1(+0) = \beta_2,$$

$$\hat{u}_2(\pi + 0) - \hat{u}_2(3\pi - 0) = \alpha_3, \;\; \hat{u}_2'(\pi + 0) = \beta_3,$$

$$\hat{u}_2'(3\pi - 0) - \hat{u}_2'(\pi + 0) = \alpha_4, \;\; \hat{u}_2(\pi + 0) = \beta_4,$$

$$u_3(1 - 0) = \alpha_5, \;\; -u_3'(1 - 0) = \beta_5$$

$$u_3(+0) = \alpha_6, \;\; u_3'(+0) = \beta_6.$$

Proof. The proof of Lemma 2 repeats the reasoning that was used in the proof of Lemma 2 § 17 of the monographs [20]. Now we can formulate the main result of this section, since the construction of § 17 of the monograph [20] in our case is carried out automatically.

Theorem 5. Every self-adjoint correctly solvable extension L_u of the operator L_0 is determined by boundary conditions of the form (10), and

$$\begin{gathered}
(\hat{\sigma}_j'(2\pi - 0) - \hat{\sigma}_j'(+0))\overline{(\hat{\sigma}_k(2\pi - 0) - \delta_{k2})} + (\hat{\sigma}_j'(+0) + \delta_{j1})\overline{(\hat{\sigma}_j(+0) - \hat{\sigma}_k(2\pi - 0))} - \\
-(\hat{\sigma}_j'(+0) - \hat{\sigma}_j'(2\pi - 0))\overline{(\hat{\sigma}_k(2\pi - 0) - \delta_{k1})} + (\hat{\sigma}_j'(+0) + \delta_{j2})\overline{(\hat{\sigma}_j'(2\pi - 0) - \hat{\sigma}_k'(+0))} - \\
- (\hat{\rho}_j'(3\pi - 0) - \hat{\rho}_j'(\pi + 0))\overline{(\hat{\rho}_k(3\pi - 0) - \delta_{k4})} + (\hat{\rho}_j'(+0) + \delta_{j3})\overline{(\hat{\rho}_j(\pi + 0) - \hat{\rho}_k(3\pi - 0))} - \\
-(\hat{\rho}_j'(\pi - 0) - \hat{\rho}_j'(3\pi + 0))(\hat{\rho}_k(3\pi - 0) - \delta_{k3}) + (\hat{\rho}_j'(\pi + 0) - \delta_{j4})(\hat{\rho}_j'(3\pi - 0) - \hat{\rho}_k'(\pi - 0)) - \\
-(\varphi_j'(1 - 0) - \delta_{j6})(\varphi_1(1 - 0)) + (\varphi_j'(+0) + \delta_{j5})(\varphi_k(+0)) + \\
+(\varphi_j(1 - 0))(\varphi_j'(1 - 0) - \delta_{k6}) - \varphi_j(+0)(\varphi_k'(+0) + \delta_{k5}) = 0.
\end{gathered} \tag{20}$$

Proof. Let us consider a well-posed problem defined by conditions (10). For convenience, we rewrite conditions (10) as

$$\begin{gathered}
-\overline{(\hat{\sigma}_j'(2\pi - 0) + \delta_{j1})}(\hat{u}_1(+0) - \hat{u}_1(2\pi - 0)) + \overline{(\hat{\sigma}_j(2\pi - 0) - \delta_{j2})}(\hat{u}_1'(+0) - \hat{u}_1'(2\pi - 0)) - \\
-\overline{(\hat{\rho}_j'(3\pi - 0) + \delta_{j3})}(\hat{u}_2(\pi + 0) - \hat{u}_2(3\pi - 0)) + \overline{(\hat{\rho}_j(3\pi - 0) - \delta_{j4})}(\hat{u}_2'(\pi + 0) - \hat{u}_2'(3\pi - 0)) - \\
-\overline{(\hat{\varphi}_j'(+0) + \delta_{j5})}u_3(+0) + \overline{(\hat{\varphi}_j'(1 - 0) - \delta_{j6})}u_3(1 - 0) + \\
+\overline{(\hat{\sigma}_j(+0) - \hat{\sigma}_j(2\pi - 0))}\hat{u}_1'(+0) + \overline{(\hat{\sigma}_j'(2\pi - 0) - \hat{\sigma}_j'(+0))}\hat{u}_1(+0) + \\
+\overline{(\hat{\rho}_j(\pi + 0) - \hat{\rho}_j(3\pi - 0))}\hat{u}_2'(\pi + 0) + \overline{(\hat{\rho}_j'(3\pi - 0) - \hat{\rho}_j'(\pi + 0))}\hat{u}_2(\pi + 0) + \\
+\overline{\varphi_j(+0)}u_3'(+0) - \overline{\varphi_j(1 - 0)}u_3'(1 - 0) = 0, \;\; j = 1, 2, 3, 4, 5, 6.
\end{gathered}$$

Let us introduce for $j = 1, 2, 3, 4, 5, 6$ a function $\vartheta_{j1}(x_1), \vartheta_{j2}(x_2), \vartheta_{j3}(s)$ such that

$$\hat{\vartheta}_{j1}(2\pi - 0) = \hat{\sigma}_j(2\pi - 0) - \delta_{j2}, \;\; \hat{\vartheta}_{j1}(+0) - \hat{\vartheta}_{j1}(2\pi - 0) = \hat{\sigma}_j(+0) - \hat{\sigma}_j(2\pi - 0),$$

$$\hat{\vartheta}_{j1}'(2\pi - 0) = \hat{\sigma}_j'(2\pi - 0) + \delta_{j1}, \;\; \hat{\vartheta}_{j1}'(2\pi - 0) - \hat{\vartheta}_{j1}'(+0) = \hat{\sigma}_j'(2\pi - 0) - \hat{\sigma}_j'(+0),$$

$$\hat{\vartheta}_{j2}(3\pi-0)=\hat{\rho}_j(3\pi-0)-\delta_{j4},\ \hat{\vartheta}_{j2}(\pi+0)-\hat{\vartheta}_{j2}(3\pi-0)=\hat{\rho}_j(\pi+0)-\hat{\rho}_j(3\pi-0),$$
$$\hat{\vartheta}'_{j2}(3\pi-0)=\hat{\rho}'_j(3\pi-0)+\delta_{j3},\ \hat{\vartheta}'_{j2}(3\pi-0)-\hat{\vartheta}'_{j2}(\pi+0)=\hat{\rho}'_j(3\pi-0)-\hat{\rho}'_j(\pi+0),$$
$$\hat{\vartheta}'_{j3}(+0)=\varphi'_j(+0)+\delta_{j5},\ \vartheta_{j3}(1-0)=\varphi_j(1-0),$$
$$\hat{\vartheta}'_{j3}(1-0)=\varphi'_j(1-0)-\delta_{j6},\ \vartheta_{j3}(+0)=\varphi_j(+0).$$

According to Lemma 2, such functions exist. In order for conditions (10) to be self-adjoint, according to theorem 4 from § 18 of the monographs [20], the following requirements must be met for any $j,k=1,2,3,4,5,6$:

$$\int\limits_{C_1}(I-\Delta_{C_1})\vartheta_{j1}(x_1)\overline{\vartheta_{k1}(x_1)}dl_1+\int\limits_{C_2}(I-\Delta_{C_2})\vartheta_{j2}(x_2)\overline{\vartheta_{k2}(x_2)}dl_2+\int\limits_0^1(\vartheta_{j3}(s)-\vartheta'_{j3}(s))\overline{\vartheta_{k3}(s)}ds=$$

$$=\int\limits_0^1\vartheta_{j3}(s)\overline{(\vartheta_{k3}(s)-\vartheta''_{k3}(s))}ds+\int\limits_{C_1}\vartheta_{j1}(x_1)\overline{(I-\Delta_{C_1})\vartheta_{k1}(x_1)}dl_1+\int\limits_{C_2}\vartheta_{j2}(x_2)\overline{(I-\Delta_{C_2})\vartheta_{k2}(x_2)}dx_2.$$

The above requirements can be written down using the Lagrange identity (11) in the form of the relation (20).

Conclusion

In this paper, the reasoning refers to a special stratified set Ω. The results presented can be extended to more complex stratified sets composed of one-dimensional and zero-dimensional manifolds. In this paper, an important tool is the transition from one-dimensional smooth manifolds defined by a single chart to intervals. In intervals, the theory of the Sturm-Liouville operator is quite advanced. Therefore, a reverse transition from the Sturm-Liouville operators on a system of intervals to the Laplace-Beltrami operators on stratified sets composed of one-dimensional smooth manifolds and zero-dimensional manifolds is possible.

Acknowledgments

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19175972 and No. AP19678089).

Author Contributions

B.E. Kanguzhin statement of the problem and proof of Theorems 1, 2, M.O. Mustafina proof of Lemma 1, 2, O.A. Kaiyrbek proof of Theorems 1, 2. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Pham, F. (1970). *Introduction to the topological study of Landau singularities.* Moscow: Mir.

2 Pokorny, Yu.V., Penkin, O.M., Borovskikh, A.V., Pryadiev, V.L., Lazarev, K.P., & Shabrov, S.A. (2005). *Differentsialnye uravneniia na geometricheskikh grafakh [Differential equations on geometric graphs].* Moscow: Fizmatlit [in Russian].

3 Dairbekov, N.S., Penkin, O.M., & Sarybekova, L.O. (2019). An analog of the Sobolev inequality on a stratified set. *St. Petersburg Math. J., 30*, 869-875. https://doi.org/10.1090/spmj/1573

4 Biyarov, B.N. (2011). About the spectrum of the Laplace operator. *Eurasian Math. J., 2*(2), 129–133.

5 Kokebaev, B.K., Otelbaev, M., & Shynybekov, A.N. (1982). K teorii suzheniia i rasshireniia operatorov, I [On the theory of restriction and extension of operators, I]. *Izvestiia Akademii Nauk Kazakhskoi SSR, Seria Fiziko-Matematicheskaia. — News of the USSR Academy of Sciences. Series Phys-Math., 5*, 24–26 [in Russian].

6 Kokebaev, B.K., Otelbaev, M., & Shynybekov, A.N. (1983). K voprosam rasshireniia i suzheniia operatorov [On questions of extension and restriction of operators]. *Doklady AN SSSR — Reports of the Academy of Sciences of the USSR, 271*(6), 1307–1311 [in Russian].

7 Kokebaev, B.K., Otelbaev, M., & Shynybekov, A.N. (1983). K teorii suzheniia i rasshireniia operatorov, II [On the theory of restriction and extension of operators, II]. *Izvestiia Akademii Nauk Kazkhskoi SSR, Seria Fiziko-Matematicheskaia. — News of the USSR Academy of Sciences. Series Phys-Math., 110*(1) [in Russian].

8 Kanguzhin, B.E., Aimal Rasa, G.H., & Kaiyrbek, Zh.A. (2021). Identification of the Domain of the Sturm-Liouville Operator on a Star Graph. *Symmetry, 13*(7), 1210. https://doi.org/10.3390/sym13071210

9 Pavlov, B.S. (1987). The theory of extensions and explicitly-soluble models. *Russian Math. Surveys, 42*(6), 127–168. https://doi.org/10.1070/RM1987v042n06ABEH001491

10 Abramowitz, M., & Stegun, C. (2016). *Legendre Functions and Orthogonal Polynomials.* 9th printing. New York.

11 Wang, Y., & Ye, L. (2013). Biharmonic Green function and biharmonic Neumann function in a sector. *Complex Variables and Elliptic Equations, 58*(1), 7–22. https://doi.org/10.1080/17476933.2010.551199

12 Begehr, H., & Vaitekhovich, T. (2013). Modified harmonic Robin function. *Complex Variables and Elliptic Equations, 58*(4), 483–496. https://doi.org/10.1080/17476933.2011.625092

13 Biyarov, B.N., Svistunov, D.L., & Abdrasheva, G.K. (2020). Correct singular perturbations of the Laplace operator. *Eurasian Math. J., 11*(4), 25–34. https://doi.org/10.32523/2077-9879-2020-11-4-25-34

14 Biyarov, B.N., & Abdrasheva, G.K. (2017). Relatively bounded perturbations of correct restrictions and extensions of linear operators. *Functional Analysis in Interdisciplinary Applications, 216*, 213–221.

15 Konechnaja, N.N., Mirzoev, K.A., & Shkalikov, A.A. (2018). On the Asymptotic Behavior of Solutions to Two-Term Differential Equations with Singular Coefficients. *Mathematical Notes, 104*(1-2), 244–252. http://dx.doi.org/10.1134/S0001434618070258

16 Savchuk, A.M., & Shkalikov, A.A. (2020). Asymptotic analysis of solutions of ordinary differential equations with distribution coefficients. *Sbornik: Mathematics, 211*(11), 1623–1659. https://doi.org/ 10.1070/SM9340

17 Miloslova, A.A., & Suslina, T.A. (2021). Usrednenie parabolicheskikh uravnenii vysokogo pori-

adka s periodicheskimi koeffitsientami [Averaging of high-order parabolic equations with periodic coefficients]. *Sovremennaia matematika. Fundamentalnye napravleniia — Modern Mathematics. Fundamental Directions, 67*(1), 130–191 [in Russian]. https://doi.org/10.22363/2413-3639-2021-67-1-130-191

18 Nazarov, S.A. (2021). O sobstvennykh chislakh i funktsiiakh zadach Dirikhle i Neimana v oblasti s dyrchatymi peregorodkami [On eigenvalues and functions of Dirichlet and Neumann problems in a domain with perforated partitions]. *Differentsialnye uravneniia — Differential equations, 57*(6), 752–768 [in Russian].

19 Biyarov, B.N. (2025). Similar transformation of one class of well-defined restrictions. *Eurasian Math. J., 16*(1), 8–21. https://doi.org/10.32523/2077-9879-2025-16-1-08-21

20 Naimark, M.A. (1967). *Lineinye differentsialnye operatory [Linear differential operators].* Moscow: Mir [in Russian].

*Author Information**

Baltabek Esmatovich Kanguzhin — Doctor of Physical and Mathematical Sciences, Professor, Institute of Mathematics and Mathematical Modeling, 125 Pushkin street, Almaty, 050010, Kazakhstan; e-mail: *kanguzhin53@gmail.com*; https://orcid.org/0000-0001-5504-6362

Mergul Oralbekovna Mustafina — Doctoral student, East Kazakhstan University named after Sarsen Amanzholov, 30th Guards Division street, Ust-Kamenogorsk, 070000, Kazakhstan; e-mail: *mergulm011@gmail.com*

Olzhas Askaruly Kaiyrbek (*corresponding author*) — Doctoral student, Al-Farabi Kazakh National University, 71 Al-Farabi Avenue, Almaty 050040, Kazakhstan; e-mail: *kaiyrbek.zhalgas@gmail.com*

*Authors' names are presented in the order: first name, middle name, and last name.