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Singularly perturbed problems with rapidly oscillating inhomogeneities in the case of discrete irreversibility of the limit operator

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We consider a linear singularly perturbed differential system, one of the points of the spectrum of the limiting operator of which goes to zero on some discrete subset of the segment of the independent variable. The problem belongs to the class of problems with unstable spectrum. Previously, S.A. Lomov's regularization method was used to construct asymptotic solutions of a similar system. However, it was applied in the case of absence of fast oscillations. The presence of the latter does not allow us to approximate the exact solution by a degenerate one, since the limit transition in the initial system when a small parameter tends to zero in a uniform metric is impossible. Therefore, when constructing the asymptotic solution, it is necessary to take into account the effects introduced into the asymptotics by fast oscillations. In developing the corresponding algorithm, one could use the ideas of the classical Lomov regularization method, but considering that its implementation requires numerous calculations (e.g., to construct the main term of the asymptotics in the simplest case of the second-order zero eigenvalue of the limit operator one has to solve three algebraic systems of order higher than the first), the authors considered it necessary to develop a more economical algorithm based on regularization by means of normal forms.

Keywords: singularly perturbed problem, normal form, discrete irreversibility of the operator, instability of the spectrum, regularized asymptotics, asymptotic solution, solvability of iterative problems, limit transition.

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1 *Problem formulation and its regularization*

Consider the singularly perturbed Cauchy's problem

$$\varepsilon\frac{dy}{dt} = A_0(t)\,y + h_0(t) + h_1(t)\,e^{i\frac{\beta(t)}{\varepsilon}}, \quad y(0,\varepsilon) = y^0, \quad t \in [0,T] \qquad (1_0)$$

where $y = \{y_1(t), ..., y_n(t)\}$ is an unknown vector function, $h_j = \{h_{1j}(t), ..., h_{nj}(t)\}$ are known vector functions, $y^0 = \{y_1^0, ..., y_n^0\}$ is the known constant vector, $\beta'(t) > 0$ is the frequency of rapidly oscillating inhomogeneity, $\varepsilon > 0$ is a small parameter. Let $\{\lambda_j(t),\ j = \overline{1,n}\}$ be the spectrum of the matrix $A_0(t)$. Assuming that the conditions:

1) $A_0(t) \in C^\infty([0,T], \mathbb{C}^{n\times n}),\ h_j(t) \in C^\infty([0,T],\ \mathbb{C}^n),\ j = 0,1,\ \beta(t) \in C^\infty([0,T], \mathbb{C}^1)$;

2) there exists the subset $B \subset [0,T]$ such that

a) $\lambda_1(t) = l_1(t)\prod\limits_{j=1}^{r}(t-t_j)^{s_j},\ l_1(t) < 0,\ l_1(t) \in C^\infty[0,T],\ s_j = 2m_j \in \mathbb{Z}_+,\ t_j \in [0,T],$
$j = \overline{1,r},\ \lambda_k(t) \neq 0\ \forall t \in [0,T],\ k = \overline{2,n};$

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b) $\lambda_i(t) \neq \lambda_j(t)\,,\ \ \lambda_j(t) \neq \beta'(t)\,,\ \ i \neq j,\ i,j = \overline{1,n},\ \ \forall t \in [0,T]\,;$

c) $\beta'(t) > 0,\ \ \mathrm{Re}\lambda_i(t) \leq 0\ \ \forall t \in [0,T]\,,\ \ i = \overline{1,n}$

are satisfied, we develop an algorithm for constructing the asymptotic solution of the problem (1_0). The problem (1_0) belongs to the class of complex problems for the study of singularly perturbed systems with unstable spectrum [1]. In [2], a regularization method is developed for the case when the spectrum of the variable limit operator vanishes at individual points. In [3], the Cauchy problem, is studied in the presence of a "weak" turning point of the limit operator, and estimates are provided that characterize the behavior of singularities at $\varepsilon \to +0$. A generalization of the ideas of the regularization method for problems with a turning point at which the eigenvalues "stick together" at $t = 0$ and initializations are considered in works [4, 5]. An analytical method for solving a Burgers-type equation in a Banach space is investigated in [6]. Namely, after artificially introducing a small parameter into the equation, the existence of an analytical solution with respect to this parameter is proven. The concept of a pseudoanalytic (pseudoholomorphic) solution introduced by S.A. Lomov initiated the development of singularly perturbed analytic theory. In [7, 8], formally singularly perturbed equations are considered in topological algebras, which allows one to formulate the basic concepts of singularly perturbed analytic theory from the standpoint of maximum generality, and conditions for the existence of solutions holomorphic in the parameter are found in the case when the perturbing operator is bilinear. The study of finding conditions for the ordinary convergence of series in powers of a small parameter, representing solutions to perturbation theory problems, is considered in [9]. Their results were generalized to integro-differential equations in [10]. This paper is the first to apply the normal form method to study such problems. The purpose of this paper is to develop this algorithm to construct asymptotic solutions of the problem (1_0) in the presence of a rapidly oscillating inhomogeneity $h_0(t)\,e^{i\frac{\beta(t)}{\varepsilon}}$.

Since the function $e^{\frac{1}{\varepsilon}\beta(t)}$ satisfies the differential equation

$$\varepsilon\frac{dy_{n+1}(t,\varepsilon)}{dt} = i\beta'(t)y_{n+1}(t,\varepsilon),\quad y_{n+1}(0,\varepsilon) = e^{\frac{i}{\varepsilon}\beta(0)},$$

then from the system (1_0) of order n it will be necessary to pass to the system of order $(n+1)$:

$$\varepsilon\frac{d}{dt}\begin{pmatrix} y(t,\varepsilon) \\ y_{n+1}(t,\varepsilon)\end{pmatrix} = \begin{pmatrix} A_0(t) & h_1(t) \\ 0 & i\beta'(t)\end{pmatrix}\begin{pmatrix} y(t,\varepsilon) \\ y_{n+1}(t,\varepsilon)\end{pmatrix} + \begin{pmatrix} h_0(t) \\ 0\end{pmatrix},\quad \begin{pmatrix} y(0,\varepsilon) \\ y_{n+1}(t,\varepsilon)\end{pmatrix} + \begin{pmatrix} y^0 \\ e^{\frac{i}{\varepsilon}\beta(0)}\end{pmatrix}$$

or

$$\varepsilon\frac{dz}{dt} = A(t)\,z + h(t)\,,\quad z(0,\varepsilon) = z^0,\quad t \in [0,T]\,, \tag{1}$$

where notations

$$z = \{y, y_{n+1}\}\,,\ z^0 = \left\{y^0, e^{\frac{i}{\varepsilon}\beta(0)}\right\},\ A(t) = \begin{pmatrix} A_0(t) & h_1(t) \\ 0 & i\beta'(t)\end{pmatrix},\ h(t) = \begin{pmatrix} h_0(t) \\ 0\end{pmatrix}$$

are introduced.

Let's denote by $e_i = \left\{0, ..., 0, \underset{(i)}{1}, 0, ..., 0\right\}$ the i-th ort in $\mathbb{C}^{n+1}$, $\bar{1} = \{1, ..., 1\} \in \mathbb{R}^{n+1}$ is the vector consisting solidly of units, $\lambda_{n+1}(t) = i\beta'(t)$, and through $\Lambda(t) = \mathrm{diag}\{\lambda_1(t), ..., \lambda_{n+1}(t)\}$ is the diagonal matrix with the spectrum of the matrix $A(t)$ on the diagonal. We regularize the problem (1) with the vector $u = \{u_1, ..., u_n, u_{n+1}\}$ of the regularizing variables satisfying the normal form*

$$\varepsilon\frac{du}{dt} = \Lambda(t)\,u + g_0(t)\,e_1 + \sum_{j=1}^{m}\varepsilon^j\sum_{i=1}^{r} g_j(t)\,e_1,\quad u(0,\varepsilon) = \bar{1}, \tag{2}$$

*On regularization by means of normal forms, see, for example, [10].

where the functions $g_j(t) \in C^\infty\left([0,T], \mathbb{C}^1\right)$ are calculated in the process of constructing the asymptotic solution of problem (1). The extended system corresponding to problem (1) will have the form

$$\varepsilon \frac{\partial \tilde{z}}{\partial t} + \frac{\partial \tilde{z}}{\partial u}\left[\Lambda(t)\, u + g_0(t)\, e_1 + \sum_{j=1}^{m} \varepsilon^j \sum_{i=1}^{r} g_j(t)\, e_1\right] - A(t)\, \tilde{z} = h(t), \quad \tilde{z}(t,u,\varepsilon)\,|_{t=0,u=\bar{1}} = y^0, \tag{3}$$

where the function $\tilde{z} = \tilde{z}(t,u,\varepsilon)$ is such that its contraction on the solution $u = u(t,\varepsilon)$ of the normal form (2) coincides with the exact solution $z(t,\varepsilon)$ of problem (1). Since problem (3) is regular in ε at $\varepsilon \to +0$, its solution can be sought in the form of series

$$\tilde{z}(t,u,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k z_k(t,u) \tag{4}$$

by non-negative powers of the parameter ε. Substituting series (4) into (3) and equating the coefficients at the same powers of ε, we obtain the following iterative problems:

$$Lz_0 \equiv \frac{\partial z_0}{\partial u}\Lambda(t)\, u - A(t)\, z_0 = h(t) - \frac{\partial z_0}{\partial u} g_0(t)\, e_1, \quad z_0(0,\bar{1}) = z^0; \tag{4_0}$$

$$Lz_1 = -\frac{\partial z_0}{\partial t} - \frac{\partial z_0}{\partial u} g_1(t)\, e_1 - \frac{\partial z_1}{\partial u} g_0(t)\, e_1, \quad y_1(0,\bar{1}) = 0; \tag{4_1}$$

$$Lz_{k+1} = -\frac{\partial z_k}{\partial t} - \frac{\partial z_0}{\partial u} g_{k+1}(t)\, e_1 - \frac{\partial z_{k+1}}{\partial u} g_0(t)\, e_1 - \sum_{j=1}^{k} \frac{\partial z_j}{\partial u} g_{k+1-j}(t)\, e_1, z_{k+1}(0,\bar{1}) = 0,\ k > 1. \tag{4_{k+1}}$$

Here $g_{kj}(t) \equiv 0$ at $k \geq m+1$.

2 *Solvability of the first iterative problem*

Under the described conditions on the spectrum of operator $A(t)$ there exists a matrix $C(t) \equiv (c_1(t), \ldots, c_{n+1}(t))$ with columns $c_j(t) \in C^\infty([0,T], \mathbb{C}^{n+1})$ such that for all $t \in [0,T]$ the identity

$$C^{-1}(t)A(t)C(t) \equiv \Lambda(t) = \operatorname{diag}(\lambda_1(t), \ldots, \lambda_{n+1}(t)) \tag{5}$$

holds. Let's denote by $d_j(t)$ the j-th column of the matrix $[C^{-1}(t)]^*$, $j = \overline{1, n+1}$. It is clear that for each $t \in [0,T]$ the following equality holds $A^*(t)d_j(t) = \overline{\lambda}_j(t)d_j(t)$ $(c_i(t), d_j(t)) \equiv \delta_{ij}$ $(i,j = \overline{1, n+1})$, where δ_{ij} is Kronecker's symbol (here and below $(*,*)$ denotes the scalar product in $\mathbb{C}^{n+1}$). Note that identity (5) excludes the rotation points in system (1).

The solution of each iterative problem (4_k) we will be defined in the space U of functions $z(t,u) = \{z_1, \ldots z_{n+1}\}$ of the form

$$z(t,u) = \sum_{j=1}^{n+1} z_j(t)\, u_j + z_0(t), \quad z_j(t) \in C^\infty([0,T], \mathbb{C}^{n+1}), \quad j = \overline{0, n+1} \tag{6}$$

in which the scalar product (at each $t \in [0,T]$)

$$< w, z > \equiv \left\langle \sum_{j=1}^{n+1} w_j(t)\, u_j + w_0(t), \sum_{j=1}^{n+1} z_j(t)\, u_j + z_0(t) \right\rangle \triangleq \sum_{j=0}^{n+1} (w_j(t), z_j(t)) \equiv \sum_{j=0}^{n+1} w_j^T(t)\, \bar{z}_j(t)$$

is introduced. Without developing the general theory of solvability of iterative problems (4_k), let us try to solve the problem (4_0). By defining its solution as an element of the space U given by (6):

$$z_0(t,u)=\sum_{j=1}^{n+1} z_j^{(0)}(t)\,u_j+z_0^{(0)}(t), \qquad (7)$$

we obtain the following system of equations for the coefficients $z_j^{(0)}(t)$:

$$-A(t)\,z_0^{(0)}(t)=h(t)-g_0(t)\,z_1^{(0)}(t), \qquad (8)$$

$$[\lambda_j(t)\,I-A(t)]\,z_j^{(0)}(t)=0,\quad j=\overline{1,n+1}. \qquad (9)$$

Solutions of the systems (9) are defined in the form $z_j^{(0)}(t)=\alpha_j(t)\,c_j(t)$, where $\alpha_j(t)\in C^\infty\left([0,T],\mathbb{C}^1\right)$ are arbitrary scalar functions, $j=\overline{1,n+1}$. To compute these functions, we proceed to the iterative system (4_1). Defining its solution in the space U as a function

$$z_1(t,u)=\sum_{j=1}^{n+1} z_j^{(1)}(t)\,u_j+z_0^{(1)}(t),$$

we get similar systems

$$-A(t)\,z_0^{(1)}(t)=-\dot{z}_0^{(0)}(t)-z_1^{(0)}(t)\,g_1(t)-g_0(t)\,z_1^{(1)}(t),$$

$$[\lambda_j(t)\,I-A(t)]\,z_j^{(1)}(t)=-\dot{\alpha}_j(t)\,c_j(t)-\alpha_j(t)\,\dot{c}_j(t),\quad j=\overline{1,n+1}. \qquad (10)$$

For the solvability of systems (10) in the class $C^\infty\left([0,T],\mathbb{C}^{n+1}\right)$ it is necessary and sufficient that

$$\left(-\dot{\alpha}_j(t)\,c_j(t)-\alpha_j(t)\,\dot{c}_j(t),d_j(t)\right)\equiv 0,\quad j=\overline{1,n+1}$$

from where we find the functions

$$\alpha_j(t)=\alpha_j(0)\exp\left\{-\int_0^t\left(\dot{c}_j(\theta),d_j(\theta)\right)d\theta\right\},\quad j=\overline{1,n+1}.$$

The initial values for these functions are found from the condition $z_0(0,\bar{1})=z^0$, which, taking into account (7), is written in the form

$$\sum_{j=1}^{n+1}\alpha_j(0)\,c_j(0)=z^0-z_0^{(0)}(0)\quad\Leftrightarrow\quad \alpha_j(0)=\left(z^0-z_0^{(0)}(0),d_j(0)\right),\ j=\overline{1,n+1}. \qquad (11)$$

However, no function has yet been found in (11) $z_0^{(0)}(t)$. Substituting $z_1^{(0)}(t)=\alpha_1(t)\,c_1(t)$ in (8) and making in the obtained system the transformation $z_0^{(0)}(t)=C(t)\,\xi\equiv\left(c_1(t),...,c_{n+1}(t)\right)\begin{pmatrix}\xi_1\\ ...\\ \xi_{n+1}\end{pmatrix}$, we obtain the following equations for the vector components ξ:

$$-\lambda_1(t)\,\xi_1=\left(h(t),d_1(t)\right)-g_0(t)\,\alpha_1(t),$$

$$-\lambda_j(t)\,\xi_j=\left(h(t),d_j(t)\right),\ j=\overline{2,n+1}.$$

Since $\lambda_j(t) \neq 0$ at $j = \overline{2, n+1}$, then the last equations of this system have unique solutions

$$\xi_j(t) = -\frac{(h(t), d_j(t))}{\lambda_j(t)}, \; j = \overline{2, n+1}.$$

In view of condition 2a), the first equation of the above system is solvable in the class $C^\infty\left([0,T], \mathbb{C}^1\right)$ then and only when

$$D^\nu(\alpha_1 g_0)(t_j) = D^\nu(h, d_1)(t_j), \quad j = \overline{1, r}, \quad \nu = \overline{0, s_j - 1}$$

(here and throughout the following, $D^\nu(f)(t_j)$ denotes the ν-th derivative of a function $f(t)$ at the point t_j).

It follows that the function $\alpha_1(t)\, g_0(t)$ is the Lagrange-Sylvester's polynomial of the function $(h(t), d_1(t))$, i.e.,

$$\alpha_1(t)\, g_0(t) = \sum_{j=1}^{r} \sum_{\nu=0}^{s_j-1} D^\nu(h, d_1)(t_j)\, K_{ji}(t), \tag{12}$$

where $\left\{K_{ji}(t),\, j = \overline{1, r},\, i = \overline{0, s_j - 1}\right\}$ is the basis system of Lagrangian-Sylvester's polynomials constructed by the polynomial $\psi(t) = \prod\limits_{j=1}^{r} (t - t_j)^{s_j}$ [10; §9.2]. Suppose that the number $\alpha_1(0) = \left(z^0 - z_0^{(0)}(0), d_1(0)\right) \neq 0$. Then it follows from (12) that the function $g_0(t)$ is represented as

$$g_0(t) = \frac{1}{\alpha_1(0)\, p_1(t)} \sum_{j=1}^{r} \sum_{\nu=0}^{s_j-1} D^\nu(h, d_1)(t_j)\, K_{j\nu}(t), \tag{13}$$

where $p_1(t) = \exp\left\{-\int\limits_0^t (\dot{c}_1(\theta), d_1(\theta))\, d\theta\right\}$. From (8), taking into account formula (12), we find the function $z_0^{(0)}(t)$:

$$\begin{gathered} z_0^{(0)}(t) = -A^{-1}(t)\left(h(t) - g_0(t)\, z_1^{(0)}(t)\right) = \\ = -C(t)\,\Lambda^{-1}(t)\, C^{-1}(t)\,(h(t) - g_0(t)\,\alpha_1(t)\, c_1(t)) = \\ = -\frac{(h(t), d_1(t)) - g_0(t)\,\alpha_1(t)}{\lambda_1(t)} c_1(t) - \sum_{j=2}^{n+1} \frac{(h(t), d_j(t))}{\lambda_j(t)} c_j(t) \equiv \qquad (14) \\ \equiv -\frac{(h(t), d_1(t)) - \sum\limits_{j=1}^{r} \sum\limits_{\nu=0}^{s_j-1} D^\nu(h, d_1)(t_j)\, K_{j\nu}(t)}{\lambda_1(t)} c_1(t) - \sum_{j=2}^{n+1} \frac{(h(t), d_j(t))}{\lambda_j(t)} c_j(t). \end{gathered}$$

Hence, we can see that the function $z_0^{(0)}(t)$ does not depend on $\alpha_1(0)$. This allows us to find values $\alpha_j(0)$:

$$\begin{gathered} \alpha_1(0) = \left(z^0 - z_0^{(0)}(0), d_1(0)\right) = \left(z^0, d_1(0)\right) + \\ + \frac{(h(0), d_1(0)) - \sum\limits_{j=1}^{r} \sum\limits_{\nu=0}^{s_j-1} D^\nu(h, d_1)(t_j) K_{j\nu}(0)}{\lambda_1(t)}, \end{gathered} \tag{15}$$

$$\alpha_j(0) = \left(z^0, d_j(0)\right) + \sum_{j=2}^{n+1}\left(\tfrac{(h(0),d_j(0))}{\lambda_j(0)}c_j(0), d_j(0)\right) = \\ = \left(z^0, d_j(0)\right) + \tfrac{(h(0),d_j(0))}{\lambda_j(0)}, \quad j = \overline{2, n+1} \tag{16}$$

unambiguously and hence compute the solution (7) to problem (4_0) in the space U in a single-valued way. We come to the following result.

Theorem 1. Let conditions 1), 2a), 2b) be satisfied and the number $\alpha_1(0)$, defined by formula (15), is not equal to zero. Then whatever the functions $z_1(t,u) \in U$ and $g_1(t) \in C^\infty\left([0,T], \mathbb{C}^1\right)$, there exists a single function $g_0(t) \in C^\infty([0,T], \mathbb{C}^1)$, computed by formulas (13), such that the problem (4_0) under the additional condition

$$\left\langle -\frac{\partial z_0}{\partial t} - \frac{\partial z_0}{\partial u} g_1(t) e_1 - \frac{\partial z_1}{\partial u} g_0(t) e_1, d_j(t) u_j \right\rangle \equiv 0 \quad \forall t \in [0,T], \quad j = \overline{1, n+1}$$

has a single solution in the class U. This solution is given by formula (7), where the functions $\alpha_j(t)$ have the form $\alpha_j(t) = \alpha_j(0)\exp\left\{-\int\limits_0^t (\dot{c}_j(\theta), d_j(\theta))\, d\theta\right\}$, $j = \overline{1, n+1}$, and the numbers $\alpha_j(0)$ calculated by the formulas (16).

Remark. If the right part $h(t)$ of system (1) is such that the following equations

$$D^\nu(h)(t_j) = 0 \Leftrightarrow D^\nu(h_1)(t_j) = 0, \quad j = \overline{1, r}, \quad \nu = \overline{0, s_j - 1} \tag{$*$}$$

are satisfied, then, as can be seen from formulas (13) and (14), the function $g_0(t) \equiv 0$, and the function $z_0^{(0)}(t)$ will have the form

$$z_0^{(0)}(t) = -\frac{(h(t), d_1(t))}{\lambda_1(t)} c_1(t) - \sum_{j=2}^{n+1} \frac{(h(t), d_j(t))}{\lambda_j(t)} c_j(t). \tag{$**$}$$

3 Algorithm for constructing solutions to iterative problems (4_k) at $k \geq 1$

Carrying out calculations similar to those used in constructing the solution of the first iterative problem (4_0), we obtain the following algorithm for the sequential solution of the problems (4_k), $k \geq 1$.

1) Each of the iterative systems (4_k), $k \geq 1$, is represented as

$$L\hat{z}_k \equiv \frac{\partial \hat{z}_k}{\partial u}\Lambda(t) u - A(t)\hat{z}_k = -\frac{\partial \hat{z}_{k-1}}{\partial t}, \tag{17}$$

$$-A(t) z_k^{(0)}(t) = -\frac{\partial z_{k-1}^{(0)}}{\partial t} - \frac{\partial \hat{z}_0}{\partial u} g_k(t) e_1 - \frac{\partial \hat{z}_k}{\partial u} g_0(t) e_1 - \sum_{j=1}^{k-1} \frac{\partial \hat{z}_k}{\partial u} g_{k-j}(t) e_1 \tag{18}$$

according to the representation of the solution $z(t,u) \in U$ as $z_k = \hat{z}_k(t,u) + z_k^{(0)}(t)$, where

$$\hat{z}_k(t,u) = \sum_{j=1}^{n+1} \hat{z}_j^{(k)}(t) u_j \in \hat{U}, \quad \hat{z}_k^{(0)}(t) \in U^{(0)} = C^\infty\left([0,T], \mathbb{C}^{n+1}\right).$$

2) We solve the system (17) in the space $\hat{U}$. For its solvability in this space it is necessary and sufficient that the identities $\left\langle -\frac{\partial \hat{z}_{k-1}}{\partial t}, d_j(t) u_j \right\rangle \equiv 0, \quad j = \overline{1, n+1}$ hold [10].

3) Writing the solution of the system (17) in the form $\hat{z}_k(t,u) = \sum_{j=1}^{n+1} \hat{z}_j^{(k)}(t)\, u_j$, substitute it into system (18) and find uniquely the function $g_k(t)$ (using Lagrange-Sylvester's polynomials) and the solution $\hat{z}_k^{(0)}(t) \in U^{(0)}$ of system (18) in the space $U^{(0)}$.

4) Let's compose the function $z_k(t,u) = \hat{z}_k(t,u) + z_k^{(0)}(t)$; it is a solution of the system (4_k), but is found ambiguously so far. To finally compute this function, we proceed to the following iterative problem (4_{k+1}).

The corresponding system $L\hat{z}_{k+1} = -\frac{\partial \hat{z}_k}{\partial t}$ will have a solution in $\hat{U}$ if and only if the following conditions hold

$$< -\frac{\partial \hat{z}_k}{\partial t}, d_j(t)\, u_j > \equiv 0, \quad j = \overline{1, n+1}.$$

These conditions and the initial condition $z_k(0, \bar{1}) = 0$ for the problem (4_k), $k \geq 1$ allow us to find the solution of $z_k(t,u) \in U$ in an unambiguous way.

4 Construction of the asymptotic solution of problem (1)

Let us proceed to the computation of the asymptotic solution of problem (1). Let the solutions $z_0(t,u), ... z_N(t,u) \in U$ of the problems $(4_0), ..., (4_N)$ respectively be constructed by the above algorithm. The functions $g_0(t), ..., g_N(t)$, participating in the formation of the normal form (2) (of order $m = N$) will be uniquely found. This form has the following solution:

$$\begin{aligned} u_j(t,\varepsilon) &= e^{\varepsilon^{-1}\int_0^t \lambda_j(s)ds}, \quad j = \overline{2, n+1}, \\ u_1(t,\varepsilon) &= e^{\varepsilon^{-1}\int_0^t \lambda_1(s)ds} \left[1 + \frac{1}{\varepsilon}\int_0^t e^{\varepsilon^{-1}\int_0^x \lambda_1(s)ds} g_0(x)dx \right] + \sum_{k=0}^{N-1} \varepsilon^k \int_0^t e^{\varepsilon^{-1}\int_0^x \lambda_1(s)ds} g_{k+1}(x)dx. \end{aligned} \tag{19}$$

Let us make a partial sum $S_N(t,u,\varepsilon) = \sum_{j=0}^{N} \varepsilon^j z_j(t,u)$ of the series (4) and form a contraction of this sum on the solution (19) of the normal form (2). We denote the obtained function by $z_{\varepsilon N}(t)$. The following statement holds (which is proved in the same way as the analogous statement in [10; Chap. 3]).

Lemma 1. Let $\alpha_1(0)$, defined by formula (15), is not zero, and conditions 1), 2a) – 2c) are satisfied. Then the function $z_{\varepsilon N}(t)$ satisfies the problem

$$\varepsilon \frac{dz_{\varepsilon N}(t)}{dt} - A(t) z_{\varepsilon N}(t) = h(t) + \varepsilon^N R_N(t,\varepsilon), \quad z_{\varepsilon N}(0) = z^0,$$

where $\|R_N(t,\varepsilon)\|_{C[0,T]} \leq \bar{R}$, $\bar{R} > 0$ is a constant independent of $(t,\varepsilon) \in [0,T] \times (0, \varepsilon_0]$ ($\varepsilon_0 > 0$ is small enough).

Using this lemma, we prove the following result as in [10; Chap. 3, §3.5].

Theorem 2. Let all conditions of the lemma be satisfied. Then the following statements are true:

1. If the right-hand side $h(t)$ of problem (1) does not satisfy the requirement $(*)$, then there is an estimate

$$\|z(t,\varepsilon) - z_{\varepsilon N}(t)\|_{C[0,T]} \leq C_N \varepsilon^N, \tag{20}$$

where $z(t,\varepsilon)$ is the exact solution of problem (1), and $z_{\varepsilon N}(t)$ is the above constructed constriction of the N-th partial sum of the series sum of series (4) on the solution $u = u(t,\varepsilon)$ of the normal form (2) of order $m = N+1$, $C_N > 0$ is a constant independent of $(t,\varepsilon) \in [0,T] \times (0, \varepsilon_0]$, $\varepsilon_0 > 0$ is small enough.

2. If the right-hand side $h(t)$ of problem (1) satisfies the requirement (**), then the estimate

$$\|z(t,\varepsilon)-z_{\varepsilon N}(t)\|_{C[0,T]}\le C_{N+1}\varepsilon^{N+1},$$

where $z(t,\varepsilon)$ and $z_{\varepsilon N}(t)$ are the same functions as in (20) $C_{N+1}>0$ is a constant independent of $(t,\varepsilon)\in[0,T]\times(0,\varepsilon_0]$, $\varepsilon_0>0$ is quite small.

5 *Example*

Consider the differential equation

$$\varepsilon\dot{y}=-t^2 l_0(t)\,y+h_0(t)+h_1(t)\,e^{\frac{i}{\varepsilon}\beta(t)},\quad y(0,\varepsilon)=y^0,\quad t\in[0,T], \tag{21}$$

where $y=y(t,\varepsilon)$ is a scalar function, the coefficient $a(t)=-t^2l_0(t)$ goes to zero only at the point $t=0$ and $l_0(t)<0$, $\forall t\in[0,T]$, $l_0(t)$, $h_0(t)$, $h_1(t)\in C^\infty([0,T],\mathbb{R})$. For this equation we can write out the exact solution, but it will be very difficult to obtain the asymptotics at $\varepsilon\to+0$. Let's attempt to apply the algorithm developed above to extract the leading asymptotic term in this problem's solution. Denoting, as before,

$$z=\{y,y_2\},\quad z^0=\left\{y^0,e^{\frac{i}{\varepsilon}\beta(0)}\right\},\quad A(t)=\begin{pmatrix}a(t)&h_1(t)\\0&i\beta'(t)\end{pmatrix},$$
$$h(t)=\begin{pmatrix}h_0(t)\\0\end{pmatrix},\quad \lambda_1(t)=a(t)=-t^2l_0(t),\quad \lambda_2(t)=i\beta'(t),$$

we obtain the system

$$\varepsilon\frac{dz}{dt}=A(t)\,z+h(t),\quad z(0,\varepsilon)=z^0,\quad t\in[0,T]. \tag{22}$$

Calculating the eigenvalues and eigenvectors of the matrices $A(t)$ and $A^*(t)$, we'll have:

$$c_1(t)=\begin{pmatrix}1\\0\end{pmatrix},\quad c_2(t)=\begin{pmatrix}\frac{h_0(t)}{-a(t)+\lambda_2(t)}\\1\end{pmatrix},$$
$$d_1(t)=\begin{pmatrix}1\\\frac{h_0(t)}{a(t)-\bar{\lambda}_2(t)}\end{pmatrix},\quad d_2(t)=\begin{pmatrix}0\\1\end{pmatrix}.$$

By Theorem 2, in the case of $(h(t),d_1(t))\not\equiv0\ \Leftrightarrow h_1(t)\not\equiv0$ solution of the first-order normal form (2) $(m=1)$:

$$\begin{aligned}&\varepsilon\dot{u}_1=\lambda_1(t)u_1+(g_0(t)+\varepsilon g_1(t)),\quad u_1(0,1)=1,\\&\varepsilon\dot{u}_2=\lambda_2(t)u_2,\quad u_2(0,1)=1\end{aligned} \tag{23}$$

contains a negative degree ε^{-1} since $g_0(t)=\alpha_1^{-1}(t)\,(h(t),d_1(t))\neq0$. Thus the solution of problem (23) tends to infinity at $\varepsilon\to+0$. The physical content of the problem corresponds to bounded solutions, so we will consider problem (22) under the condition $(h(t),d_1(t))=h_1(t)\equiv0$, $\forall t\in[0,T]$. Then $g_0(t)\equiv0$, and the leading asymptotic term in the solution to problem (22) is given by (7)

$$z_{\varepsilon0}(t)=\alpha_1(t)\,c_1(t)\,e^{\frac{1}{\varepsilon}\int_0^t\lambda_1(\theta)d\theta}\left[1+\int_0^t e^{-\frac{1}{\varepsilon}\int_0^s\lambda_1(\theta)d\theta}g_1(s)\,ds\right]+$$
$$+\alpha_2(t)\,c_2(t)\,e^{\frac{1}{\varepsilon}\int_0^t\lambda_2(\theta)d\theta}-\frac{(h(t),d_2(t))}{\lambda_2(t)}c_2(t),$$

where the functions $\alpha_1(t)$ and $\alpha_2(t)$ are calculated from the solvability condition of the problem (4_1) in the space U. Given our notations, we write the main term of the asymptotics of the solution of problem (21) in the following form

$$y_{\varepsilon 0}(t)=\left(y^0+\frac{h_0(0)\,e^{\frac{i}{\varepsilon}\beta(0)}}{a(0)-i\beta'(0)}\right)e^{\frac{1}{\varepsilon}\int\limits_0^t a(\theta)d\theta}\left[1+\int\limits_0^t e^{-\frac{1}{\varepsilon}\int\limits_0^s a(\theta)d\theta}g_1(s)\,ds\right]+\frac{h_0(t)\,e^{\frac{i}{\varepsilon}\beta(t)}}{-a(t)+i\beta'(t)}. \tag{24}$$

Conclusion

From (24) we see that if $h_0(t)\neq 0$ on the segment $[0,T]$, the exact solution $y(t,\varepsilon)$ of problem (24) has no limit at $\varepsilon\to+0$ due to the oscillatory inhomogeneity $e^{\frac{i}{\varepsilon}\beta(t)}$ included in (24). If $h_0(t)=0$, $\forall t\in\in[0,T]$, then the main term of the asymptotics (24) takes the form of

$$y_{\varepsilon 0}(t)=y^0e^{-\frac{1}{\varepsilon}\int\limits_0^t\theta^2 l_0(\theta)d\theta}\left[1+\int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_0^s\theta^2 l_0(\theta)d\theta}g_1(s)\,ds\right].$$

The zero of $t=0$ of the function $a(t)=-t^2l_0(t)$ affects that the summand

$$y^0e^{-\frac{1}{\varepsilon}\int\limits_0^t\theta^2 l_0(\theta)d\theta}\int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_0^s\theta^2 l_0(\theta)d\theta}g_1(s)\,ds$$

outside the boundary zone $[0,\delta(\varepsilon)]$ of length of order $\sqrt[3]{\varepsilon}$ “slows down” the tendency of the exact solution $y(t,\varepsilon)$ of problem (21) to the limit $\bar{\bar{y}}(t)\equiv 0$.

In the case of an exponential boundary layer occurring at $a(t)<0$, $\forall t\in[0,T]$, the exact solution $y(t,\varepsilon)$ differs from the limit outside the boundary layer by an order of magnitude of ε [11]. Thus, the effect of the slowed limit transition (as the small parameter approaches zero) in a singularly perturbed problem is associated with the point wise features of its spectrum.

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Author Contributions

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Conflict of Interest

The authors declare no conflict of interest.

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