Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 199–210

https://doi.org/10.31489/2025M1/199-210 *Research article*

Normal Jonsson theories and their Kaiser classes

A.R. Yeshkeyev, M.T. Kassymetova*, N.M. Mussina

Institute of Applied Mathematics, Karaganda Buketov University, Karaganda, Kazakhstan
(E-mail: aibat.kz@gmail.com, mairushaasd@mail.ru, mussinanazerke@gmail.com)

We present results concerning new notion connected with the study of Jonsson theories. The new notion is a Kaiser class of models for arbitrary Jonsson theories. All results are obtained within the framework of the normality of the considered Jonsson theory. Additionally, we describe the properties of lattices formed by perfect fragments of a fixed Jonsson theory and their relationship with the #-companion of these fragments. The results we obtained are the model-theoretic properties of the #-companion of a normal perfect Jonsson fragment. Furthermore, we establish necessary and sufficient conditions for a normal Jonsson theory to be perfect, expressed in terms of the lattices of existential formulas.

Keywords: Jonsson theory, semantic model, Jonsson set, almost Jonsson set, normality.

2020 Mathematics Subject Classification: 03C35, 03C48, 03C52, 03C65.

Introduction

This article investigates the model-theoretic properties of certain subsets within the semantic model of a fixed normal Jonsson theory. In particular, it considers regular almost Jonsson sets as such subsets. Previously, in [1] introduced the notions of normality for Jonsson theories and the concept of an almost Jonsson set. The notion of regularity is a natural requirement for definable sets, which are Jonsson sets. The formula that defines the Jonsson set for a more convenient study must satisfy certain well-established properties in a model theoretical sense. In this case, the axiomatic formulation of the regularity property is considered as a set defined by a formula that has Morley rank. Moreover, all this is determined within the framework of the study of Jonsson theories regarding the Morley rank, previously it was defined in the work [2]. In addition, it should be noted that the main results of this article are related to previous results from the works [1, 3, 4]. Since the Jonsson theories are not, generally speaking, complete theories, in the general case we do not have the opportunity to consider the Lindenbaum–Tarski Boolean algebra of formulas and its corresponding Stone space of types. Therefore, just as in [3, 4] we consider the lattice of existential formulas and the corresponding existential types for which the results are obtained in the framework of the study of almost Jonsson sets. The most important difference and innovation in this work from the previous studies in [3, 4] is the application of double factorization to the class of cosemanticness under consideration. The main novelty in this article is the transition from fragments obtained by closing Jonsson sets to fragments obtained by closing almost Jonsson sets. In the future, the class of models whose Kaiser hull forms the Jonsson theory will be called the Kaiser class of models of the considered Jonsson theory. Thus, almost Jonsson sets distinguish a special class of models among the class of all models of the considered Jonsson theory.

*Corresponding author. *E-mail: mairushaasd@mail.ru*

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23489523).

Received: 28 August 2024; *Accepted:* 15 December 2024.

1 Necessary information related to the study of the Jonsson theories

In order to understand the content and nuances of our next consideration we have to recall the main definitions and statements which are connected with definitions from [1]. Throughout this article, we will deal with a countable first-order language and all the considered theories will be countable, respectively.

The central concept of this article is the notion of a Jonsson theory.

Definition 1. [5; 80] A theory T is defined as a Jonsson theory if it satisfies the following conditions:
(i) T has infinite models;
(ii) T is an inductive theory (such that, $\forall\exists$-axiomatizable);
(iii) T has the joint embedding property (JEP);
(iv) T has the amalgam property (AP).

The technique reflecting its essence and application in the study of various concepts related to the Jonsson theories is described in the following works: [6–19].

Remark 1. We will always work within the framework of a pregeometry [20], which is defined by the closure operator cl on the set of all subsets of the semantic model of the given Jonsson theory.

The most important type of considered models in the framework of studying Jonsson theories is a class of its existentially closed models. Let us recall this important definition.

Definition 2. [5; 105] Let M be a structure and N be an extension of M. We say that M is existentially closed in N if for any tuple $\bar{a}$ from M and any quantifier-free formula $f(\bar{x}, \bar{y})$ in the language of M, the following holds: if $N \vdash (\exists \bar{y}) f(\bar{a}, \bar{y})$, then $M \vdash (\exists \bar{y}) f(\bar{a}, \bar{y})$.

One of the syntactic invariants of a Jonsson theory is its center.

Definition 3. [20] The center T^* of a Jonsson theory T is the elementary theory of its semantic model C, i.e., $T^* = Th(C)$.

Definition 4. [5; 156] Let T be an arbitrary theory of the language L. We say that T' is a model companion of T, if the following conditions hold:
(i) $T'_\forall = T_\forall$ (i.e. T and T' are mutually model-consistent, meaning any model of T can be embedded into a model of T' and vice versa),
(ii) T' is model complete.

The concept of a model companion is well-known and useful in the study of Jonsson theories. The following notion generalizes this concept.

Definition 5. [5; 158] Let T be an arbitrary theory. A #-companion $T^\#$ of T is a theory in the same signature that satisfies the following conditions:
(i) $(T^\#)_\forall = T_\forall$;
(ii) if $T_\forall = T'_\forall$, then $T^\# = (T')^\#$;
(iii) $T_{\forall\exists} \subseteq T^\#$.

The concept of model completeness is closely connected to the notion of mutual model consistence for the theory under consideration. It is evident that the model companion is a $T^\#$-companion of theory T.

Remark 2. When we say that $T'_\forall = T_\forall$, we mean precisely that:
(i) any model of theory T is a substructure of a model of T';
(ii) any model of theory T' is a substructure of a model of T.

It is clear that T' is a model companion of T if and only if T' is a model companion of $T_\forall$.

Proposition 1. [5; 159] Let T be an arbitrary theory.

(i) T has a model companion if and only if the class of existentially closed models of $T_\forall$ is an elementary class.

(ii) If a model companion of T exists, it is unique and is corresponds to the theory of existentially closed models of $T_\forall$.

In addition to the model companion, other companions are well-known in the study of Jonsson theories, such as the forcing companion, the existentially closed companion, and the Kaiser hull. Moreover, the Kaiser hull is closely connected to the companion that characterizes the class of existentially closed models of the theory being studied, as well as to the companion that defines the class of generic models when studying the forcing companion of the given theory.

It is well-known that the natural interpretations of the companion $T^{\#}$ include T^*, T^f, T^M, and T^l, where:

(i) T^* is the center of the Jonsson theory T;

(ii) T^f is the forcing companion of the Jonsson theory T;

(iii) T^M is the model companion of the theory T;

(iv) $T^e = Th(E_T)$, where E_T denotes the class of existentially closed models of the theory T.

It is important to note that if E_T represents the class of T-existentially closed models of an inductive theory T, then E_T is always non-empty [20].

A. Robinson introduced the concepts of finite forcing and the forcing #-companion in Model Theory [21]. In [21], it is demonstrated that a theory in a countable language that satisfies the Joint Embedding Property (JEP) has a forcing #-companion that is complete.

The following theorem establishes that any Jonsson theory T always has a forcing companion, which is a complete theory.

Theorem 1. [5; 162]

(i) $T^{\#}$ is a #-companion of T.

(ii) $T^{\#}$ is complete if and only if T satisfies JEP.

Consequently, the simultaneous existence of all interpretations of the #-companion is closely related to the existence of a model companion.

In the study of Jonsson theories, the class of perfect Jonsson theories holds a significant role.

Definition 6. [20] A Jonsson theory T is said to be perfect if every its semantic model is an ω^+-saturated model of T^*.

It turns out that the semantic model of the Jonsson theory under consideration is existentially closed.

Lemma 1. [20] The semantic model C_T of a Jonsson theory T is T-existentially closed.

The following theorem provides a criterion for perfectness:

Theorem 2. (Criterion of Perfectness). [20] Let T be an arbitrary Jonsson theory. Then the following conditions are equivalent:

(i) the theory T is perfect;

(ii) the theory T^* is a model companion of theory T.

In the case of a completely Jonsson theory, the concept of a companion coincides with the center of the theory for any type of companion.

Corollary 1. For a perfect Jonsson theory T, the #-companion coincides with the center of T.

Before studying the concepts of fragments of the theory under consideration, it is important to note that the definability of a set in models means that the set is a solution to some formula in the first-order language of the given signature.

Definition 7. [4] A set A is called A-definable if it can be defined by some formula in the language L. A set A is said to be Jonsson in the theory T if it satisfies the following conditions:

(i) A is a definable subset of M;

(ii) $cl(A)$ is a carrier of some existentially closed submodel M, where $cl(A)$ is the set of all A-definable elements $a \in A$ such that, for some formula $\varphi(x) \in L(A)$, it holds that $\varphi(M) = \{a\}$.

Let A be a Jonsson set in the theory T, and let N be an existentially closed submodel of a semantic model M of the considered Jonsson theory T, where $cl(A) = N$.

We denote by $Th_{\forall\exists}(N)$ the sets of all $\forall\exists$-sentences in the language that are true in the model N.

The following concept is simply a Jonsson theory obtained by closing a definable subset that is Jonsson.

Definition 8. [4] The fragment $Fr(X)$ of the Jonsson set X is a Jonsson theory obtained as a $\forall\exists$-sentences true in the model N which is a closure of this set, i.e. $Fr(X) = Th_{\forall\exists}(N)$.

Furthermore, we will use the denotation N^0 for the fragment $Fr(X)$, where N is $cl(X)$ and sometimes we can say instead of N^0 the Kaiser hull of the model N.

The following definition gives us important notion of Jonsson spectrum, the details of this one can extract from [1, 10, 17].

Definition 9. [22] Let K be a class of L-structures. The Jonsson spectrum of K, denoted $JSp(K)$, is defined as the set of all theories satisfying the following conditions:

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } \forall A \in K\ A \models T\}.$$

The following definition is an essential generalization of the concept of a Jonsson set.

Definition 10. [1] A set X is called an almost Jonsson if it satisfies the following conditions:

1) X is a definable subset of C_T, where C_T is the semantic model of theory T;

2) $cl(X) = M \in Mod(T)$.

Furthermore, $Th_{\forall\exists}(M) = M^0 = Fr(X)$ and $M^0 \in JSp(C_T)$, where $cl(X)$ is the definable closure of the set X in the frame of given above pregeometry [20].

This concept can be illustrated by an example of an arbitrary abelian group, which turns out to be the closure of some existential formula defining an almost Jonsson set of this abelian group.

The concept of normality for a Jonsson theory is defined for a class of theories where any fragment of their semantic models belongs to the Jonsson spectrum of the given Jonsson theory's semantic model.

The following definition identifies specific subsets within the semantic model of the given Jonsson theory.

Definition 11. [1] The Jonsson theory T is said to be normal if, for any $X \subseteq C_T$ such that $cl(X) = M \in Mod(T)$, $Fr(X) = M^0 \in JSp(C_T)$ and $C_{M^0} \preceq_{\exists_1} C_T$, where C_{M^0} is a semantic model of M^0.

An example of a normal theory is the universal theory of all unars. This theory is characterized by the fact that it has an empty list of axioms and is Jonsson.

Lemma 2. Let T be a normal Jonsson theory, K be a subset of E_T. Then $Th_{\forall\exists}(K)$ is a normal Jonsson theory.

Proof. Let $K \subseteq E_T$, T be a normal Jonsson theory. Since T is a normal Jonsson theory, it follows that T has JEP, then according to Theorem 1 $\forall A, B \in E_T \Rightarrow T^0(A) = T^0(B)$.

Theorem 3. [23; 353] Let L be a first-order language and T be a theory in L. Suppose that T has JEP, and let A, B be existentially closed models of T. Then each $\forall\exists$-sentence of the language L that is true in A will also be true in B.

Let T be a Jonsson theory, C_T be the semantic model of theory T, E_T be class of existentially closed models of theory T, $E_T \subset ModT$. And now let us define the concept of the Kaiser class of models for the arbitrary of the Jonsson theory. $K_T = \{A \in ModT/Th_{\forall\exists}(A)$ is a Jonsson theory$\}$. It is clear that $E_T \subseteq K_T$.

1) $K_T \neq \emptyset$ (if $T = T^* = Th(C_T)$ and T is perfect, then $K_T = E_T$).

2) $\forall M \subseteq K_T$, $Th_{\forall\exists}(M)$ is a Jonsson theory.

The Jonsson sets and almost Jonsson sets generate fragments in the case of normality of considered Jonsson theory from models of the Kaiser class of this theory, and these fragments' semantic models are existentially closed submodels of semantic model of considered Jonsson theory.

Definition 12. [20] The Jonsson theory T_1 is said to be cosemantic to the Jonsson theory T_2 (denoted by $T_1 \bowtie T_2$) if their semantic models are identical, i.e. $\mathcal{C}_{T_1} = \mathcal{C}_{T_2}$, where $\mathcal{C}_{T_i}$ represents the semantic models of T_i for $i = 1, 2$.

Definition 13. [20] Let T_1 and T_2 be arbitrary Jonsson theories. We say that T_1 and T_2 are Jonsson syntactically similar if there exists a bijection $f : E(T_1) \to E(T_2)$ satisfying the following conditions:

1) Isomorphism of lattices: for each $n < \omega$, the restriction of f to $E_n(T_1)$ is an isomorphism between the lattices $E_n(T_1)$ and $E_n(T_2)$.

2) Preservation of Existential Quantifiers: for every $\varphi \in E_{n+1}(T)$ and $n < \omega$, $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$.

3) Preservation of Equality: the bijection preserves equality formulas, i.e., $f(v_1 = v_2) = (v_1 = v_2)$.

The examples of syntactic similarities of two Jonsson theories are given in [18].

Definition 14. [20] The structure $\langle C, Aut(\mathcal{C}), Sub(C)\rangle$ is said to be the Jonsson semantic triple, where C denotes the domain of the semantic model $\mathcal{C}$ of the theory T, $Aut(\mathcal{C})$ represents the automorphism group of $\mathcal{C}$, and $Sub(C)$ is a class of all subsets of C that serve as domains of the corresponding existentially closed submodels of $\mathcal{C}$.

Definition 15. [20] Two Jonsson theories T_1 and T_2 are said to be Jonsson semantically similar if their Jonsson semantic triples are isomorphic as pure triples.

We introduce the following notation:

(i) the syntactic similarity of complete theories T_1 and T_2 is denoted by $T_1 \overset{S}{\bowtie} T_2$;

(ii) the semantic similarity of complete theories T_1 and T_2 is denoted by $T_1 \underset{S}{\bowtie} T_2$.

For Jonsson theories T_1 and T_2:

(i) the Jonsson syntactic similarity of theories T_1 and T_2 is written as $T_1 \overset{S}{\rtimes} T_2$;

(ii) the Jonsson semantic similarity of theories T_1 and T_2 is written as $T_1 \underset{S}{\rtimes} T_2$.

The following result, obtained by the first author of this paper, highlights the relationship between the Jonsson syntactic similarity of Jonsson theories and the Jonsson syntactic similarity of their centers.

Theorem 4. [20] Let T_1 and T_2 be $\exists$-complete perfect Jonsson theories. The following conditions are equivalent:

1) $T_1 \overset{S}{\rtimes} T_2$;

2) $T_1^* \overset{S}{\bowtie} T_2^*$, where T_1^* and T_2^* are the centers of T_1 and T_2, respectively.

The following results, concerning the case of two Jonsson theories, were obtained by Yeshkeyev A.R.:

Theorem 5. Let T_1 and T_2 be two Jonsson theories. If T_1 and T_2 are Jonsson syntactically similar, then they are also Jonsson semantically similar.

Thus, it follows that

$$T_1 \overset{S}{\rtimes} T_2 \Rightarrow T_1 \underset{S}{\rtimes} T_2$$

for any two Jonsson theories T_1 and T_2. This demonstrates that Jonsson syntactic similarity is a sufficient condition for Jonsson semantic similarity of theories.

There are also cases where this condition is necessary. In this paper, we explore specific classes of Jonsson theories for which these two relations are equivalent. We denote this equivalence by the following notation:

$$T_1 \overset{SS}{\rtimes} T_2.$$

Lemma 3. [18] Any two cosemantic Jonsson theories are also Jonsson semantically similar.

The proof follows from the definition of cosemantic Jonsson theories.

The definitions of Jonsson semantic and syntactic similarity relations were extended to classes of Jonsson theories in [18]:

Definition 16. [18] Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is syntactically similar to the class $[T]_2$, denoted by $[T]_1 \overset{S}{\rtimes} [T]_2$, if for every theory $\Delta \in [T]_1$ there exists a theory $\Delta' \in [T]_2$ such that Δ and Δ' are syntactically similar.

Definition 17. [18] The pure triple $\langle C, Aut(\mathcal{C}), \overline{E}_{[T]} \rangle$ is referred to the Jonsson semantic triple for the class $[T] \in JSp(A)/_{\bowtie}$, where C represents the semantic model of $[T]$, $AutC$ denotes the group of automorphisms of C, and $\overline{E}_{[T]}$ is the class of isomorphically images of all existentially closed models of $[T]$.

Definition 18. [18] Let $\mathcal{A} \in \mathrm{Mod}\sigma_1$, $\mathcal{B} \in \mathrm{Mod}\sigma_2$, $[T]_1 \in JSp(\mathcal{A})/_{\bowtie}$, and $[T]_2 \in JSp(\mathcal{B})/_{\bowtie}$. We say that the class $[T]_1$ is Jonsson semantically similar to class $[T]_2$, denoted by $[T]_1 \underset{S}{\rtimes} [T]_2$, if their semantic triples are isomorphic as pure triples.

2 On fragments and its #-companions for models from K_T

This paragraph is related to the description of the companions of fragments of Jonsson sets or almost Jonsson sets. Actually it is equivalent to the following fact: any considered set of this paragraph belongs to K_T, where T is considered normal Jonsson theory. The properties of the lattice of existential formulas were described quite well in works [20, 24–26]. Let's recall the main definitions.

Let a theory T be fixed as above.

Definition 19. [24; 843] Let $\varphi^T, \psi^T \in E_n(T)$ and assume that $\varphi^T \cap \psi^T = 0$.

(i) Complement: ψ^T is considered a complement of φ^T if their union equals the total element, i.e., $\varphi \cup \psi = 1$.

(ii) Pseudo-Complement: ψ^T is called a pseudo-complement of φ^T if, for every $\mu^T \in E_n(T)$, whenever $\varphi^T \cap \mu^T = 0$, it follows that $\mu^T \leq \psi^T$.

(iii) Weak Complement: ψ^T is weakly complementary to φ^T if, for every $\mu^T \in E_n(T)$, the condition $(\varphi^T \cup \mu^T) \cap \mu^T = 0$ implies $\mu^T = 0$.

Definition 20. [24; 843] Properties of φ^T and $E_n(T)$: φ^T is complemented if there exists another element that serves as its complement; φ^T is weakly complemented if there exists a weak complement for it; φ^T is pseudo-complemented if there exists a pseudo-complement for it; $E_n(T)$ is complemented if every $\varphi^T \in E_n(T)$ has a complement; $E_n(T)$ is weakly complemented if every $\varphi^T \in E_n(T)$ has a weak complement; $E_n(T)$ is pseudo-complemented if every $\varphi^T \in E_n(T)$ has a pseudo-complement.

We now turn our attention to the formulas that are preserved under extensions of models and submodels within the context of the theory T.

Definition 21. [20] A formula $\varphi(x_1, ..., x_n)$ is said to be preserved under extensions in $ModT$ if for any models $A \subset B$ of T, and any $a_1, ..., a_n \in A$, $A \models \varphi[a_1, ..., a_n] \Rightarrow \varphi[a_1, ..., a_n]$.

Definition 22. [20] A formula $\varphi(x_1, ..., x_n)$ is said to be preserved under submodels in $ModT$ if for any models $A \subset B$ of T, and any $a_1, ..., a_n \in A$, $B \models \varphi[a_1, ..., a_n] \Rightarrow A \models \varphi[a_1, ..., a_n]$.

We now define the concept of an invariant formula and examine the connection between the invariance of an existential formula and the complementarity of its class in $E(T)$.

Definition 23. [24; 844]. A formula φ is said to be invariant in $ModT$ if it is preserved both under extensions of models in $ModT$ and submodels in $ModT$.

Definition 24. [24; 843] A theory T is called positively model complete if it satisfies the following conditions:

(i) T is model complete, meaning every formula φ is equivalent to an existential formula within T.

(ii) Every existential formula in the language L of T is equivalent to a positive existential formula within T.

We present the required definitions and summarize established results that clarify the connection between model completeness, quantifier elimination in a theory T, and the structural properties of lattices of existential formulas $E_n([[M^0]])$.

Some results from [20] and classical model theory can be further refined within the context of studying fragments of Jonsson theories.

These refined results will serve as a basis for deriving the key conclusions in this section.

Theorem 6. [24; 846] An existential formula φ is invariant in $Mod(Th_{\forall\exists}(E_T))$, where E_T is the class of all existentially closed models of T, if and only if φ^T is weakly complemented in $E(T)$.

Theorem 7. [20]

(i) A theory T is model complete if and only if all formulas are preserved when passing to submodels in $Mod(T)$.

(ii) A theory T is model complete if and only if all formulas are preserved when extending to larger models in $Mod(T)$.

Theorem 8. [20]

(i) If T' is a model companion of a universal theory T, then T' is a model completion of T if and only if T' allows elimination of quantifiers.

(ii) If T is a model companion of T, then T' is a model completion of T if and only if T satisfies AP.

Theorem 9. [20] A theory T is a submodel complete if and only if T allows elimination of quantifiers.

Theorem 10. [24; 843] A theory T is called positively model complete if and only if every $\varphi^T \in E_n(T)$ has a positive existential complement.

Theorem 11. [24; 843] A theory T has a model companion if and only if $\varphi^T \in E_n(T)$ has a weakly complement.

Theorem 12. [24; 843] A theory T has a model companion if and only if $\varphi^T \in E_n(T)$, the lattice of existential formulas forms a Stone algebra.

Theorem 13. [24; 843] A theory T has a model companion if and only if every $\varphi^T \in E_n(T)$ has a weakly quantifier-free complement.

Theorem 14. [20] Let T be a Jonsson theory. Then the next conditions are equivalent:

1) the theory T is perfect;

2) the theory T has a model companion.

In [26], a connection between the completeness and model completeness of a Jonsson theory was established.

The following result represents a particular case of Lindstrom's theorem, which describes the relationship between model completeness and completeness.

Let T be a fixed, normal Jonsson theory. Within T, we consider N contained in K_T. Next, we examine the Jonsson spectrum $JSp(N)$, and proceed to perform a double factorization of $JSp(N)/_{\underset{\bowtie}{SS}}$.

Theorem 15. Let T be a normal Jonsson theory, $A \in C_T$, where A is an almost Jonsson theory, then $cl(A) = M \in K_T$. And let $K_T \neq \emptyset$, $N \subseteq K_T$. We consider a Jonsson theory $M^0 = Th_{\forall\exists}(M) \in JSp(N)$, and $[M^0] \in JSp(N)/_{\bowtie}$ and $[[M^0]] \in JSp(N)/_{\underset{\bowtie}{SS}}$, where M^0 is a Kaiser hull. Then the following conditions are equivalent:
(i) $[[M^0]]$ is complete;
(ii) $[[M^0]]$ is model complete.

In [27], a connection was demonstrated between the perfectness of a Jonsson theory and the properties of the lattice $E_n(T)$ [3]. Final point of proof of this fact follows from the definition of normality.

Theorem 16. Let T be a normal Jonsson theory. And let $K_T \neq \emptyset$, $N \subseteq K_T$, $[M^0] \in JSp(N)/_{\bowtie}$ and $[[M^0]] \in JSp(N)/_{\underset{\bowtie}{SS}}$. Then the following conditions are equivalent:
(i) $[[M^0]]$ is perfect;
(ii) $[[M^0]]^{\#}$ is normal Jonsson theory;
(iii) $E_n([[M^0]])$ is a Boolean algebra;
(iv) $[[M^0]]^*$ is model-complete.

Proof. It follows from results which connected the properties of lattice of existential formulas $E_n(T)$ and above results from Theorem 15.

Now, we study the model-theoretic properties of the #-companion of a normal perfect Jonsson fragment. Let T be a normal Jonsson theory in a countable language L, and let $A \in C_T$, where A is an almost Jonsson theory, then $cl(A) = M \in K_T$. And let $K_T \neq \emptyset$, $N \subseteq K_T$. We consider a Jonsson theory $M^0 = Th_{\forall\exists}(M) \in JSp(N)$, and $[M^0] \in JSp(N)_{\bowtie}$ and $[[M^0]] \in JSp(N)/_{\underset{\bowtie}{SS}}$. The distributive lattice $E_n([[M^0]])$ consists of equivalence classes of formulas defined as $\varphi^{[[M^0]]} = \{\psi \in E_n(L) | [[M^0]] \vdash \varphi \leftrightarrow \psi, \varphi \in E_n(L), K([[M^0]]) = \cup_{n<\omega} E_n([[M^0]])\}$.

We now focus on a fragment $[[M^0]]$ that is complete for existential sentences and satisfies the conditions described above.

Theorem 17. Let T be a normal Jonsson theory. And let $K_T \neq \emptyset$, $N \subseteq K_T$, $[M^0] \in JSp(N)_{\bowtie}$ and $[[M^0]] \in JSp(N)/_{\underset{\bowtie}{SS}}$. Let $[[M^0]]$ be a perfect fragment of a Jonsson set A, $[[M^0]]^{\#}$ be its #-companion. Then the following hold:
(i) $[[M^0]]^{\#}$ admits elimination of quantifiers if and only if every $\varphi \in E_n([[M^0]])$ has a quantifier-free complement;
(ii) $[[M^0]]^{\#}$ is positively model-complete if and only if every $\varphi \in E_n([[M^0]]$ has an existential complement.

Proof. (i) Given that $[[M^0]]$ is a fragment of a Jonsson set A and M' is a semantic model of $[[M^0]]$, we know $[[M^0]]$ is perfect, we have that $[[M^0]]^{\#} = Th(M')$, where the center of M' admits elimination of quantifiers. By Theorem 9, $[[M^0]]^{\#}$ is submodel complete, and therefore, is model complete, and by Theorem 16, the lattice $E_n([[M^0]])$ is a Boolean algebra, meaning every $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has a complement. Since $[[M^0]]^{\#}$ admits elimination of quantifiers, every class $\varphi \in E_n([[M^0]])$ must have a quantifier-free complement.

Conversely, assume that every class $\varphi \in E_n([[M^0]])$ has a quantifier-free complement. Then $E_n([[M^0]])$ is a Boolean algebra (by definition). By Theorem 16, this implies that $[[M^0]]^{\#}$ is model-complete. By part (ii) of Theorem 7, every formula in $[[M^0]]^{\#}$ is equivalent to some existential formula, i.e., the class of such formulas belongs to $E_n([[M^0]]^{\#})$. Since $[[M^0]]$ is complete existential sentences, we know $E_n([[M^0]]) = En([[M^0]]^{\#})$. Therefore, every $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has a quantifier-free complement, and $E_n([[M^0]]^{\#})$ consists of quantifier-free formulas. Hence, $[[M^0]]^{\#}$ admits elimination of quantifiers.

(ii) Assume that $[[M^0]]^{\#}$ is positively model-complete. By Definition 11, $[[M^0]]^{\#}$ is model-complete and for every existential formula φ there exists a positive existential formula ψ such that $[[M^0]]^{\#} \vdash \varphi \leftrightarrow \psi$. By Theorem 16, $E_n([[M^0]])$ is a Boolean algebra, so every $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has an existential complement. Since $[[M^0]]^{\#}$ ensures the existence of a positive existential formula ψ equivalent to φ, every there is a positive existential formula such that $[[M^0]]^{\#} \vdash \varphi \leftrightarrow \psi$, we obtain that every $\varphi \in E_n([[M^0]])$ has a positive existential complement. The necessary condition of part (ii) is proved.

To prove the sufficiency of part (ii): Assume that every $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has a positive existential complement. By Theorem 10, this implies that $[[M^0]]$ is positive model-complete. By the definition of positive model-completeness, $[[M^0]]$ is also model-complete. By Theorem 15, the fragment $[[M^0]]$ is complete. Since $[[M^0]]^{\#}$ is a central complement of the theory $[[M^0]]$, we have $[[M^0]] = [[M^0]]^{\#}$. Therefore, $[[M^0]]^{\#}$ is positively model-complete. The sufficiency is proven by using Theorem 10 (positive model-completeness of $[[M^0]]$), Theorem 15 (completeness of $[[M^0]]$), and the relationship between $[[M^0]]$ and $[[M^0]]^{\#}$. This completes the proof that $[[M^0]]^{\#}$ is positively model-complete.

In the following theorem, we establish necessary and sufficient conditions for the perfectness of a normal fragment $[[M^0]]$ within the framework of the class $[M^0]$, using the structure of the lattice of existential formulas $E_n([[M^0]])$.

Theorem 18. Let T be a normal Jonsson theory. And let $K_T \neq \emptyset$, $N \subseteq K_T$, $[M^0] \in JSp(N)_{\bowtie}$ and $[[M^0]] \in JSp(N)_{/\overset{SS}{\bowtie}}$, and let $[[M^0]]$ be a perfect fragment with $Fr^{\#}(A)$ as its #-companion. Then the following conditions are equivalent:

(i) $[[M^0]]$ is perfect;

(ii) $E_n([[M^0]])$ is weakly complemented;

(iii) $\varphi \in E_n([[M^0]])$ is a Stone algebra.

Proof. Let $[[M^0]]$ be a perfect fragment of a Jonsson set A, and $[[M^0]]^{\#}$ its #-companion. We prove the equivalence of the three conditions:

(i) $\Rightarrow$ (ii). Assume that a normal Jonsson theory $[[M^0]]$ is perfect. By Theorem 17, $[[M^0]]$ has a #-companion $[[M^0]]^{\#}$. By results in [24], we know $[[M^0]]^f = [[M^0]]^0$, where $[[M^0]]^0 = Th_{\forall\exists}(E_{[[M^0]]})$ is Kaiser's hull of $[[M^0]]$. Since $[[M^0]]^{\#}$ is model-complete (by the definition of a #-companion), part (ii) of Theorem 7 states that every formula of the language is preserved under submodels in $Mod[[M^0]]^{\#}$.

Consequently, every existential formula in the language is preserved under both submodels and extensions in $Mod[[M^0]]^{\#}$. By Definition 23, these formulas are invariant in $Mod[[M^0]]^{\#}$. From Theorem 6, it follows that every existential formula in $E_n([[M^0]])$ is weakly complemented. Thus, $E_n([[M^0]])$ is weakly complemented.

(ii) $\Rightarrow$ (i). Assume $E_n([[M^0]])$ is weakly complemented.

1. By Theorem 11, the fragment $[[M^0]]$ has a model companion.

2. By Theorem 17, if $[[M^0]]$ has a models companion, then $[[M^0]]$ is perfect. Thus, $[[M^0]]$ is perfect. Therefore, (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (iii). Assume $[[M^0]]$ is perfect.

1. By part (ii) of Theorem 10, the model companion of a normal Jonsson theory is its model completion.

2. From the perfectness of $[[M^0]]$, it follows by Theorem 12 that $E_n([[M^0]])$ forms a Stone algebra. Thus, $E_n([[M^0]])$ is a Stone algebra.

(iii) $\Rightarrow$ (i). Assume $E_n([[M^0]])$ is a Stone algebra.

1. By Theorem 12, if $E_n([[M^0]])$ is a Stone algebra, then $[[M^0]]$ has a #-companion $[[M^0]]^{\#}$.

2. By Theorem 16, if $[[M^0]]$ has a #-companion, then $[[M^0]]$ is perfect. Thus, $[[M^0]]$ is perfect.

Theorem 19. Let $[[M^0]]$ be a perfect normal Jonsson theory. And let $K_T \neq \emptyset$, $N \subseteq K_T$, $[M^0] \in JSp(N)_{\bowtie}$ and $[[M^0]] \in JSp(N)/_{\underset{\bowtie}{SS}}$, $[[M^0]]^{\#}$ be its $\#$-companion. Then the following conditions are equivalent:

(i) $[[M^0]]^{\#}$ is a normal Jonsson theory;

(ii) every $\varphi \in E_n([[M^0]])$ has a weakly quantifier-free complement.

To prove the necessity we need the following statement.

If the model companion $[M^0]^m$ is defined, then a model companion $([M^0]_{\forall})^m$ is defined and

$$[M^0]^m = ([M^0]_{\forall})^m \tag{1}$$

(see [5]).

Proof. (i) $\Rightarrow$ (ii). Assume $[[M^0]]^{\#}$ is a normal Jonsson theory. By results from [24], the fragment $[[M^0]]$ is perfect. From Theorem 18, a perfect fragment $[[M^0]]$ has a $\#$-companion, which coincides with the theory $[[M^0]]^{\#}$. By part (ii) of Theorem 7, the $\#$-companion $[[M^0]]^{\#}$ is a model completion of fragment $[[M^0]]$. Due to the mutual model consistency between the fragment $[[M^0]]$ and its universal theory $[[M^0]]_{\forall}$, we have the model completion of $[M^0]$ (the fragment's class) is also the model completion of $[M^0]_{\forall}$ (its universal consequences). By Theorem 13, it follows that every existential formula $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has a weakly quantifier-free complement.

(ii) $\Rightarrow$ (i). Assume that every existential formula $\varphi^{[[M^0]]} \in E_n([[M^0]])$ has a weakly quantifier-free complement. By definition, if every $\varphi^{[[M^0]]}$ has a weakly quantifier-free complement, then every $\varphi^{[[M^0]]} \in E_n([[M^0]])$ is weakly complemented, i.e. $E_n([[M^0]])$ is weakly complemented. By result from [24], it follows that the $\#$-companion $[[M^0]]^{\#}$ is a normal Jonsson theory.

All necessary concepts and statements related to these notions that were not defined and were not noted in the text of this article can be extracted from the following list of sources.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Yeshkeyev, A.R, Ulbrikht, O.I, & Omarova, M.T. (2024). Double factorization of the Jonsson spectrum. *Bulletin of the Karaganda University. Mathematics Series*, 4(116), 185–196. https://doi.org/10.31489/2024M4/185-196

2 Yeshkeyev, A.R. (2016). Some properties of Morly rank over Jonsson sets. *Bulletin of the Karaganda University. Mathematics Series*, 4(84), 57–62.

3 Yeshkeyev, A.R. (2013). The structure of lattices of positive existential formulae of (Δ-PJ)-theories. *Science Asia*, 39(2), 19–24.

4 Yeshkeyev, A.R., Kasymetova, M.T., & Shamatayeva, N.K. (2018). Model-theoretic properties of the #-companion of a Jonsson set. *Eurasian mathematical journal, 9*(2), 68–81.

5 Barwise, J. (1982). *Teoriia modelei: spravochnaia kniga po matematicheskoi logike. Chast 1 [Model theory: Handbook of mathematical logic. Part 1].* Moscow: Izdatelstvo Nauka [in Russian].

6 Yeshkeyev, A.R., & Mussina, N.M. (2021). An algebra of the central types of the mutually model-consistent fragments. *Bulletin of the Karaganda University. Mathematics Series*, 4(101), 111–118. https://www.doi.org/10.31489/2021M1/111-118

7 Yeshkeyev, A.R., & Omarova, M.T. (2021). An essential base of the central types of the convex theory. *Bulletin of the Karaganda University. Mathematics Series*, 1(101), 119–126. https://www.doi.org/10.31489/2021M1/119-126

8 Yeshkeyev, A.R., Kassymetova, M.T., & Ulbrikht, O.I. (2021). Independence and simplicity in Jonsson theories with abstract geometry. *Bulletin of the Karaganda University. Mathematics Series*, 1(18), 433–455. https://www.doi.org/10.33048/semi.2021.18.030

9 Yeshkeyev, A.R., Tungushbayeva, I.O., & Kassymetova, M.T. (2022). Connection between the amalgam and joint embedding properties. *Bulletin of the Karaganda University. Mathematics Series*, 1(105), 127–135. https://www.doi.org/10.31489/2022M1/127-135

10 Yeshkeyev, A.R., Ulbrikht, O.I., & Omarova, M.T. (2023). The Number of Fragments of the Perfect Class of the Jonsson Spectrum. *Lobachevskii Journal of Mathematics, 43*(12), 3658–3673. https://www.doi.org/10.1134/S199508022215029X

11 Yeshkeyev, A.R., Yarullina, A.R., & Kassymetova, M.T. (2023). On Robinson spectrum of the semantic Jonsson quasivariety of unars. *Bulletin of the Karaganda University. Mathematics Series*, 2(110), 169–178. https://www.doi.org/10.31489/2023M2/169-178

12 Yeshkeyev, A.R., Ulbrikht, O.I., & Yarullina, A.R. (2022). Existentially positive Mustafin theories of S-acts over a group. *Bulletin of the Karaganda University. Mathematics Series*, 2(106), 172–185. https://www.doi.org/10.31489/2022M2/172-185

13 Yeshkeyev, A.R., Tungushbayeva, I.O., & Omarova, M.T. (2022). Forcing companions of Jonsson AP-theories. *Bulletin of the Karaganda University. Mathematics Series*, 3(107), 152–163. https://www.doi.org/10.31489/2022M3/152-163

14 Yeshkeyev, A.R., Tungushbayeva, I.O., & Amanbekov, S.M. (2022). Existentially prime Jonsson quasivarieties and their Jonsson spectra. *Bulletin of the Karaganda University. Mathematics Series*, 4(108), 117–124. https://doi.org/10.31489/2022M4/117-124

15 Yeshkeyev, A.R., Ulbrikht, O.I., & Issayeva, A.K. (2023). Algebraically prime and atomic sets. *Turkic World Mathematical Society (TWMS) Journal of Pure and Applied Mathematics, 14*(2), 232–245. https://www.doi.org/10.30546/2219-1259.14.2.2023.232

16 Yeshkeyev, A.R., Tungushbayeva, I.O., & Koshekova, A.K. (2024). The cosemanticness of Kaiser hulls of fixed classes of models. *Bulletin of the Karaganda University. Mathematics Series*, 1(113), 208–217. https://doi.org/10.31489/2024m1/208-217

17 Yeshkeyev, A.R., Ulbrikht, O.I., & Mussina, N.M. (2023). Similarities of Hybrids from Jonsson Spectrum and S-Acts. *Lobachevskii Journal of Mathematics, 44*(12), 5502–5518. https://doi.org/10.1134/s1995080223120399

18 Yeshkeyev, A.R., Ulbrikht, O.I., & Urken, G.A. (2023). Similarities of Jonsson spectra's classes. *Bulletin of the Karaganda University. Mathematics Series*, 4(112), 130–143. https://doi.org/10.31489/2023m4/130-143

19 Yeshkeyev, A.R., Tungushbayeva, I.O., & Ulbrikht, O.I. (2024). On some types of algebras on a Jonsson spectrum. *Siberian Electronic Methematical Reports, 21*(2), 866–881.

20 Yeshkeyev, A.R. (2024). *Theories and their models.* Karaganda, Kazakhstan: izd. KarU [in Russian].

21 Yeshkeyev, A.R. (2015). On Jonsson sets and some of their properties. *Bulletin of the Karaganda University. Mathematics Series*, 1(21), 99–100.

22 Yeshkeyev, A.R., & Ulbrikht, O.I. (2019). JSp-cosemanticness of R-modules. *Siberian Electronic*

Methematical Reports, 16, 1233–1244. https://doi.org/10.33048/semi.2019.16.084

23 Hodges, W. (1993). *Model Theory. Encyclopedia of Mathematics and its Applications.* Cambridge: Cambridge University Press.

24 Weispfenning, V. (1981). The model-theoretic signicance of complemented existential formulas. *The Journal of Symbolic Logic*, 46(4), 843–849.

25 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Mathematics Series*, 3(79), 33–39.

26 Yeshkeyev, A.R., & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Mathematics Series*, 3(79), 25–32.

27 Robinson, A. (1963). Introduction to model theory and to the metamathematics of algebra. *Amsterdam.*

*Author Information**

Aibat Rafhatuly Yeshkeyev — Doctor of physical and mathematical sciences, Professor, Professor Researcher, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *aibat.kz@gmail.com*; https://orcid.org/0000-0003-0149-6143

Maira Tekhnikovna Kassymetova (*corresponding author*) — PhD, Associate Professor of the Department of Algebra, Mathematical Logic and Geometry named after T.G. Mustafin, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *mairushaasd@mail.ru*; https://orcid.org/0000-0002-4659-0689

Nazerke Mukhtaramkyzy Mussina — PhD, Associate Professor of the Department of Algebra, Mathematical Logic and Geometry named after T.G. Mustafin, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *mussinanazerke@gmail.com*; https://orcid.org/0000-0002-4360-8767

*The author's name is presented in the order: First, Middle and Last Names.