https://doi.org/10.31489/2025M1/92-103 *Research article*

BVP for the heat equation with a fractional integro-differentiation operator

M.T. Kosmakova[1,2], N.T. Orumbayeva[1,2,*], D.M. Akhmanova[2], L.Zh. Kasymova[3]

[1]*Institute of Applied Mathematics, Karaganda, Kazakhstan;*
[2]*Karaganda Buketov University, Karaganda, Kazakhstan;*
[3]*Abylkas Saginov Karaganda Technical University, Karaganda, Kazakhstan*
(E-mail: svetlanamir578@gmail.com, orumbayevan@mail.ru, danna.67@mail.ru, l.kasymova2017@mail.ru)

A boundary value problem for a loaded heat conduction equation is considered, when the loaded term has the form of a fractional Riemann-Liouville derivative with respect to a spatial variable, and the loading point moves with a variable velocity. The problem is reduced to a Volterra integral equation of the second kind, the kernel of which contains a special function, namely, a Wright-type function. The kernel of the resulting integral equation is estimated, and it is shown, under certain restrictions on the line along which the load moves, that the kernel of the equation has a weak singularity, which is the basis for the assertion that the loaded term in the equation of the problem is a weak perturbation of its differential part. The study is based on the asymptotic behavior of the Wright function at infinity and at zero.

Keywords: loaded heat equation, fractional derivative, Volterra integral equation, Wright function.

2020 Mathematics Subject Classification: 45D05, 35K20.

Introduction

The heat conduction equation plays a key role in modeling thermal processes in various physical systems. In the classical formulation, it describes the temperature distribution in a medium subject to heat transfer. However, to more accurately account for complex physical effects, such as anomalous diffusion or material memory, generalized models are introduced that include additional terms, for example, containing a fractional integro-differentiation operator.

Fractional derivatives, unlike integer derivatives, make it possible to take into account memory effects and nonlocality of processes. Their application in heat conduction modeling has been actively developing in recent decades. The works [1, 2] consider the fundamentals of the theory of fractional calculus and its applications in mathematical physics. The application of fractional derivatives in heat conduction equations was investigated in [3], where it was shown that such models describe anomalous diffusion processes well. Fractional derivatives can also take into account spatial correlations and coordinate nonlocality in systems where the influence on the state at a given point in space depends not only on neighboring points, but also on more distant ones [4].

Boundary value problems for heat equations with fractional derivatives represent a separate area of research. They require the development of new approaches, since the presence of a fractional term leads to a complication of the mathematical structure of the problem. In [5], the spectral properties of operators with fractional derivatives are analyzed, and in [6, 7] boundary conditions for fractional models are studied.

Problems with loaded terms involving fractional derivatives are of particular interest. These problems arise in the context of modeling processes with heat sources or sinks that depend on time or spatial

*Corresponding author. *E-mail: orumbayevan@mail.ru*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488740, Grant No. AP23488729, 2024–2026.)

Received: 10 September 2024; *Accepted:* 13 December 2024.

coordinates. A loaded differential equation is an equation with a loaded term, which can contain differential or integrodifferential operators. This loaded term can be expressed as a function containing both the variables themselves and their derivatives.

Loaded equations allow you to model more complex physical or mathematical systems that cannot always be described by simple equations. For example, in problems of mathematical physics or control theory, loaded differential equations can be used to take into account the influence of external factors or additional conditions on the dynamics of the system. It is obvious that the presence of a loaded term gives rise to new, still unexplored problems in the theory of boundary value problems, therefore there is a need to develop new methods for solving the evolving theory of loaded differential equations [8].

Loaded differential equations can be considered as weak or strong perturbations of differential equations. In some cases, boundary value problems remain correct in natural classes of functions, where the loaded term is interpreted as a weak perturbation [9]. If the uniqueness of the solution to the boundary value problem is violated, then the load can be considered as a strong perturbation [10]. It turns out that the nature of the load (weak or strong perturbation) depends both on the order of the derivatives included in the loaded (perturbed) part of the operator, and on the manifold on which the trace of the desired function is specified.

The study of boundary value problems with loaded terms, presented in the form of integrals or fractional derivatives, can lead to different results depending on the specifics of the equation and the conditions of the problem. There may also be difficulties associated with the analysis and evaluation of integral operators in the resulting integral equations, since their kernels contain special functions. In [11, 12], the intervals for changing the order of the fractional derivative, that is contained in the loaded term, are determined, for which the theorems of existence and uniqueness of solutions to boundary value problems and arising integral equations are valid. We also note that the boundary value problems of heat conduction and the Volterra integral equations arising in their study with singularities in the kernel, similar to the singularities in this paper, were considered in [13, 14].

Also, integral equations with singularities in the kernel arise when studying boundary value problems in non-cylindrical domains that degenerate into a point at the initial moment of time [15–20].

Fractional derivatives in equations add new aspects and difficulties in the study of boundary value problems, since they take into account not only the previous state of the system, but also its history. The fractional order differentiation operation is a combination of differentiation and integration operations. Recently, work has appeared on the study of inverse boundary value problems with a load of fractional order. In [21], the inverse problem with a nonlinear gluing condition for a loaded equation of parabolic-hyperbolic type is studied for solvability. The problem is reduced to the study of the nonlinear Fredholm integral equation of the second kind. In [22], as an application of the analyticity of the solution, the uniqueness of an inverse problem in determining the fractional orders in the multi-term time-fractional diffusion equations from one interior point observation is established.

This paper examines a boundary value problem (BVP) defined in the open right upper quadrant. The problem is transformed into an integral equation, which, in certain instances, takes the form of a pseudo-Volterra type. The solvability of this equation is influenced by the order of differentiation in the loaded term and the behavior of the load line near the origin. In Section 1, we introduce some necessary definitions and mathematical preliminaries of fractional calculus, special functions and boundary value problems which will be needed in the forthcoming Sections. The problem statement for a heat equation with a loaded term as the Riemann-Liouville fractional derivative in the right upper quadrant (x, t) is given in Section 2. The initial conditions are homogeneous. Process of reducing a boundary value problem to an integral equation is the content of Section 3. In Section 4, we estimate the integral equation's kernel and establish conditions under which it has a weak singularity. Estimating the integral equation's kernel is based on the asymptotic behavior of the Wright function at infinity and at zero. This implies the solvability conditions for the BVP which are provided in Section 5.

In Section 5 the main results is formulated.

1 Preliminaries

Definition 1. [23] Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville integral of the order β is defined as follows

$$ {}_rD_{a,t}^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau, \quad \beta, a \in R, \quad \beta > 0. \tag{1}$$

Definition 2. Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville derivative of the order β is defined as follows

$$ {}_rD_{a,t}^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}} d\tau, \quad \beta, a \in R, \quad n-1 < \beta < n. \tag{2}$$

From formula (2) it follows that

$$ {}_rD_{a,t}^{0} f(t) = f(t), \quad {}_rD_{a,t}^{n} f(t) = f^{(n)}(t), \quad n \in N.$$

Taking into account formula (1), formula (2) can be rewritten as

$$ {}_rD_{a,t}^{\beta} f(t) = \frac{d^n}{dt^n} {}_rD_{a,t}^{\beta-n} f(t), \quad \beta, a \in R, \quad n-1 < \beta < n.$$

Information about the Mittag-Leffler function and the Wright function is taken from [24, 25].

Definition 3. The entire function of the form

$$E_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \mu)}, \quad \lambda > 0, \quad \mu \in C \tag{3}$$

is called the Mittag-Leffler function.

Definition 4. The entire function of the form

$$\phi(\lambda, \mu; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \quad \mu \in C \tag{4}$$

is called the Wright function.

Definition 5. A Wright-type function is a function $e_{\alpha,\beta}^{\mu,\delta}(z)$ defined by the contour integral and the Mittag-Leffler function (3)

$$e_{\alpha,\beta}^{\mu,\delta}(z) = \frac{1}{2\pi i} \int_{\gamma(r,\omega\pi)} e^t t^{-\delta} E_{\alpha,\mu}\left(z t^{\beta}\right) dt,$$

where $\gamma(r, \omega\pi)$ is the Hankel contour, the value of ω is chosen such that

$$1 - \omega\beta > \frac{\alpha}{2}, \quad \frac{1}{2} < \omega \leq 1. \tag{5}$$

Inequalities (5) are always satisfied when

$$0 < \alpha < 2, \quad 0 < \alpha + \beta < 2, \quad \beta < 1, \quad \delta + \beta > 0.$$

For $\alpha>\beta$, $\alpha>0$, for any $z\in\mathbb{C}$ the Wright-type function can be represented as a series

$$e_{\alpha,\beta}^{\mu,\delta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\mu)\Gamma(\delta-\beta n)},\quad \mu\in\mathbb{C},\quad \delta\in\mathbb{C}.$$

When $\alpha=\mu=1$ it coincides with the Wright function:

$$e_{1,\beta}^{1,\delta}(z)=\phi(-\beta,\delta,z). \tag{6}$$

For a Wright-type function, the following autotransformation formula is valid:

$$e_{\alpha,\beta}^{\mu-\alpha,\delta+\beta}(z)=ze_{\alpha,\beta}^{\mu,\delta}(z)+\frac{1}{\Gamma(\mu-\alpha)\Gamma(\delta+\beta)}. \tag{7}$$

If $\pi\geq|\arg z|>\pi(\alpha+\beta)/2+\varepsilon,\ \varepsilon>0,\ k=0,1,2,\ldots$, then the following limit relations are valid for large absolute values of z:

$$\begin{aligned}&\lim_{|z|\to\infty}e_{\alpha,\beta}^{\mu,\delta}(z)=0,\\ &\lim_{|z|\to\infty}ze_{\alpha,\beta}^{\mu,\delta}(z)=-\frac{1}{\Gamma(\mu-\alpha)\Gamma(\delta+\beta)}.\end{aligned} \tag{8}$$

Let $c\in\mathbb{C}$. If $\mu>0$, then

$$D_{0x}^{\nu}x^{\mu-1}e_{\alpha,\beta}^{\mu,\delta}\left(cx^{\alpha}\right)=x^{\mu-\nu-1}e_{\alpha,\beta}^{\mu-\nu,\delta}\left(cx^{\alpha}\right). \tag{9}$$

When $\mu=0$, the following formula is valid

$$D_{0x}^{\nu}\frac{1}{x}e_{\alpha,\beta}^{0,\delta}\left(cx^{\alpha}\right)=x^{-\nu-1}e_{\alpha,\beta}^{-\nu,\delta}\left(cx^{\alpha}\right)-\frac{x^{-\nu-1}}{\Gamma(-\nu)\Gamma(\delta)}.$$

When $\nu=n\in\mathbb{N}$, formula (9) is valid for all $\mu\in\mathbb{R}$

$$\frac{d^n}{dx^n}x^{\mu-1}e_{\alpha,\beta}^{\mu,\delta}\left(cx^{\alpha}\right)=x^{\mu-n-1}e_{\alpha,\beta}^{\mu-n,\delta}\left(cx^{\alpha}\right).$$

The following equalities hold

$$\int_0^{\infty}\frac{1}{t}e_{\alpha,\beta}^{0,\delta}(-\lambda t)dt=-\frac{\alpha}{\Gamma(\delta)'},\quad \int_0^{\infty}\frac{1}{t}e_{\alpha,\beta}^{\mu,0}(-\lambda t)dt=\frac{\beta}{\Gamma(\mu)}.$$

Also the formulas for differentiating a Wright type function are valid

$$\frac{d}{dz}e_{\alpha,\beta}^{\mu,\delta}(z)=\frac{1}{\alpha z}\left[e_{\alpha,\beta}^{\mu-1,\delta}(z)+(1-\mu)e_{\alpha,\beta}^{\mu,\delta}(z)\right].$$

It's known [26; 57] that in the domain $Q=\{(x,t)\mid x>0,\quad t>0\}$ the solution to the boundary value problem of heat conduction

$$u_t=a^2u_{xx}+F\left(x,t\right),$$

$$u\left|_{t=0}\right.=f(x),\quad u\left|_{x=0}\right.=g(x)$$

is described by the formula

$$u\left(x,t\right)=\int_0^{\infty}G\left(x,\xi,t\right)\ f(\xi)\,d\xi+\int_0^{t}H\left(x,t-\tau\right)\ g(\tau)\,d\tau+$$

$$+\int_0^t \int_0^\infty G\left(x, \xi, t-\tau\right) F\left(\xi, \tau\right) d\xi d\tau, \tag{10}$$

where

$$G(x,\, \xi,\, t) = \frac{1}{2\sqrt{\pi\, a\, t}} \left\{ \exp\left(-\frac{(x-\xi)^2}{4\, a\, t}\right) - \exp\left(-\frac{(x+\xi)^2}{4\, a\, t}\right) \right\},$$

$$H(x,\, t) = \frac{1}{2\sqrt{\pi\, a}\, t^{3/2}} \exp\left(-\frac{x^2}{4\, a\, t}\right).$$

The Green function $G\left(x, \xi, t\right)$ satisfies the relation

$$\int_0^\infty G\left(x, \xi, t\right) d\xi = \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right),$$

where $\operatorname{erf}\left(z\right)$ is the error integral.

2 The problem's statement

In the domain $Q = \{(x, t) : x > 0, t > 0\}$, we consider a BVP

$$u_t = u_{xx} - \lambda \left\{ {}_r D^\beta_{0,x} u\left(x, t\right) \right\} \big|_{x=\gamma(t)} + f\left(x, t\right), \tag{11}$$

$$u\left(x, 0\right) = 0, \quad u\left(0, t\right) = 0, \tag{12}$$

where λ is a complex parameter, ${}_r D^\beta_{0,\, t}\, u(x,\, t)$ is the Riemann-Liouville derivative (2) of an order β, $1 < \beta < 2$, $\gamma(t)$ is a continuous increasing function, $\gamma(0) = 0$.

The problem is studied in the class of continuous functions.

For the right side of the equation, we require the following conditions to be satisfied:

$$f(x, t) \in L_\infty\left(A\right) \cap C\left(B\right), \tag{13}$$

where $A = \{(x, t) \,|\, x > 0,\ t \in [0, T]\}$, $B = \{(x, t) \,|\, x > 0,\ t \geq 0\}$, $T = const > 0$,

$$f_1\left(x, t\right) = \int_0^t \int_0^\infty G\left(x, \xi, t-\tau\right) f\left(\xi, \tau\right) d\xi d\tau \in L_1 \ \ (x > 0). \tag{14}$$

Let us introduce the notation

$$D^\nu_{at} g\left(t\right) = \frac{1}{\Gamma\left(-\nu\right)} \int_a^t \frac{g\left(\xi\right) d\xi}{\left(t-\xi\right)^{\nu+1}},\ \ \nu < 0.$$

When $\nu = 0$ $D^0_{at} g\left(t\right) = g\left(t\right)$, then

$$D^\nu_{at} g\left(t\right) = \frac{d^n}{dt^n} D^{\nu-n}_{at} g\left(t\right), \quad n-1 < 0 \leq n, \quad n \in N.$$

We consider the fractional derivative in the Riemann-Liouville sense with respect to the spatial variable. If $a = 0$, $n = 2$, $\nu = \beta \Rightarrow$

$${}_r D^\beta_{0x} u\left(x, t\right) = \frac{d^2}{dx^2} D^{\beta-2}_{0x} u\left(x, t\right) \tag{15}$$

or

$${}_r D^\beta_{0x} u\left(x, t\right) = \frac{d^2}{dx^2} \left(\frac{1}{\Gamma\left(2-\beta\right)} \int_0^x \frac{u\left(x, \xi\right) d\xi}{\left(x-\xi\right)^{\beta-1}} \right). \tag{16}$$

The derivative in the loaded term of equation (11) is determined by the formula (16).

3 Reducing the BVP to an integral equation

According to the formula (10) a solution to BVP (11)-(12) can be represented as

$$u\left(x,t\right)=-\lambda\int_0^t\int_0^\infty G\left(x,\xi,t-\tau\right)\mu\left(\tau\right)d\xi d\tau+f_1\left(x,t\right), \tag{17}$$

where

$$\mu\left(t\right)=\left\{{}_rD_{0,x}^{\beta}u\left(x,t\right)\right\}\big|_{x=\gamma(t)}\ , \tag{18}$$

$$f_1\left(x,t\right)=\int_0^t\int_0^{+\infty}G\left(x,\xi,t-\tau\right)f\left(\xi,\tau\right)d\xi d\tau. \tag{19}$$

In [9] it was proved formulas

$$e^{-\xi^2}=\sqrt{\pi}\phi\left(-\frac{1}{2},\frac{1}{2},-2\xi\right), \tag{20}$$

where $\phi\left(\phi(\lambda,\mu;z)\right)$ is the Wright function (4),

$$\operatorname{erf}\left(z\right)=2\int_0^z\phi\left(-\frac{1}{2},\frac{1}{2},-2\xi\right)d\xi=1-\phi\left(-\frac{1}{2},1,-2z\right). \tag{21}$$

Then, taking into account formulas (20) and (21) representation (17) can be rewritten as:

$$u\left(x,t\right)=-\lambda\int_0^t K\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu\left(\tau\right)d\tau+f_1\left(x,t\right), \tag{22}$$

where

$$K\left(\frac{x}{2\sqrt{t-\tau}}\right)=1-\phi\left(-\frac{1}{2},1,-\frac{x}{\sqrt{t-\tau}}\right) \tag{23}$$

and $\mu(t)$ and $f_1(t)$ are defined by formulas (18) and (19) respectively.

To (22) we apply the fractional integro-differentiation operator by formula (15). Taking into account formulas (23), (6), (7), and (9), we obtain, when $1<\beta<2$:

$${}_rD_{0x}^{\beta}\left(\mathrm{K}\left(\frac{x}{2\sqrt{t-\tau}}\right)\right)=x^{-\beta}\left(\frac{1}{\Gamma(1-\beta)}-e_{1,\frac{1}{2}}^{1-\beta,1}\left(-\frac{x}{\sqrt{t-\tau}}\right)\right)=\frac{x^{1-\beta}}{\sqrt{t-\tau}}e_{1,\frac{1}{2}}^{2-\beta,\frac{1}{2}}\left(-\frac{x}{\sqrt{t-\tau}}\right).$$

Indeed, according to the law of composition, we have

$$D_{0x}^{\beta}(f(x))=D^1D_{0x}^{\beta-1}(f(x))\Rightarrow$$

$${}_rD_{0x}^{\beta-1}\left(K\left(\frac{x}{2\sqrt{t-\tau}}\right)\right)={}_rD_{0x}^{\beta-1}\left(1-\Phi\left(-\frac{1}{2},1;-\frac{x}{\sqrt{t-\tau}}\right)\right)=$$

$$={}_rD_{0x}^{\beta-1}\left(1-e_{1,\frac{1}{2}}^{1,1}\left(-\frac{x}{\sqrt{t-\tau}}\right)\right)=x^{1-\beta}\left(\frac{1}{\Gamma(2-\beta)}-e_{1,\frac{1}{2}}^{2-\beta,1}\left(-\frac{x}{\sqrt{t-\tau}}\right)\right).$$

Then, taking into account the autotransformation formula (7), we get

$$D^1\left(\frac{x^{1-\beta}}{\Gamma(2-\beta)}-x^{1-\beta}e_{1,\frac{1}{2}}^{2-\beta,1}\left(-\frac{x}{\sqrt{t-\tau}}\right)\right)=$$

$$=\frac{1-\beta}{\Gamma(2-\beta)}x^{-\beta}-x^{-\beta}e_{1,\frac{1}{2}}^{1-\beta,1}\left(-\frac{x}{\sqrt{t-\tau}}\right)=\frac{x^{1-\beta}}{\sqrt{t-\tau}}e_{1,\frac{1}{2}}^{2-\beta,\frac{1}{2}}\left(-\frac{x}{\sqrt{t-\tau}}\right).$$

Thus, BVP (11)-(12) is reduced to a Volterra integral equation of the second kind

$$\mu(t) + \lambda \int_0^t K_\beta(t,\tau)\mu(\tau)d\tau = f_2(t), \tag{24}$$

with a kernel

$$K_\beta(t,\tau) = \frac{(\gamma(t))^{1-\beta}}{\sqrt{t-\tau}} e^{2-\beta,\frac{1}{2}}_{1,\frac{1}{2}} \left(-\frac{\gamma(t)}{\sqrt{t-\tau}}\right), \quad 1<\beta<2, \tag{25}$$

and with the right part

$$f_2\left(t\right) = \left\{ {}_r D^\beta_{0,x} f_1\left(x,t\right)\right\} |_{x=\gamma(t)} \,. \tag{26}$$

4 Research of the integral equation

Since in the given problem (11)-(12) the line, along which the load is moving, has the form $x = \gamma\left(t\right)$, and $\gamma\left(t\right)$ increases and $\gamma\left(0\right) = 0$, then there are different cases of behavior for $\frac{x}{\sqrt{t}}\,|_{x=\gamma(t)}$, when $t \to 0$.

Let $0 < x = \gamma(t) \sim t^\omega$ when $t \to 0,\ \omega > 0$.

Let's introduce a change of variable τ:

$$z = \frac{\gamma(t)}{\sqrt{t-\tau}} \Rightarrow \sqrt{t-\tau} = \frac{\gamma(t)}{z}.$$

Then

$$K_\beta(t,z) = (\gamma(t))^{-\beta} z e^{2-\beta,\frac{1}{2}}_{1,\frac{1}{2}}(-z).$$

Let's consider the following cases:

$$a)\ 0<\omega<\frac{1}{2} \quad \Rightarrow \quad |z| \to +\infty, \quad \text{when} \quad t\to 0.$$

Taking into account the limiting ratio (8), we get

$$\lim_{|z|\to+\infty} z e^{\mu,\nu}_{\alpha,\beta}(-z) = -\frac{1}{\Gamma(\mu-\alpha)\cdot\Gamma(\delta+\beta)} \Rightarrow$$

$$\lim_{t\to 0} K_\beta(t,\tau) = \lim_{t\to 0}(\gamma(t))^{-\beta} z e^{2-\beta,\frac{1}{2}}_{1,\frac{1}{2}}(-z) = \lim_{t\to 0} t^{-\beta\omega}\frac{1}{\Gamma(1-\beta)} = +\infty$$

$$b)\ \omega > \frac{1}{2} \quad \Rightarrow \quad |z| \to 0, \quad \text{when} \quad t\to 0.$$

Taking into account the limiting ratio (8), we get

$$\lim_{t\to 0} e^{2-\beta,\frac{1}{2}}_{1,\frac{1}{2}}\left(-\frac{t^\omega}{\sqrt{t-\tau}}\right) = \frac{1}{\Gamma(2-\beta)\sqrt{\pi}} \Rightarrow K_\beta(t,\tau) \sim \frac{t^{\omega(1-\beta)}}{\sqrt{t-\tau}}, \quad \text{when} \quad t\to 0.$$

The kernel (25) of the integral equation (24) has singularities at $t = 0$ and $t = \tau$.

Let us define the conditions under which the integral operator of the equation is compressible in the class of continuous functions. Consider the integral

$$\int_0^t K_\beta(t,\tau)d\tau = t^{\omega(1-\beta)}\sqrt{t} = t^{\omega(1-\beta)+\frac{1}{2}} \underset{t\to 0}{\to} 0$$

if $\omega(1-\beta) + \frac{1}{2} > 0 \Rightarrow \omega < \frac{1}{2(\beta-1)}$.

$$c)\ \omega = \frac{1}{2} \Rightarrow K_\beta(t,\tau) \sim \frac{t^{\frac{1-\beta}{2}}}{\sqrt{t-\tau}} e_{1,\frac{1}{2}}^{2-\beta\frac{1}{2}}\left(-\sqrt{\frac{t}{t-\tau}}\right), \quad \text{when} \quad t \to 0.$$

Since $e_{1,\frac{1}{2}}^{2-\beta\frac{1}{2}}\left(-\sqrt{\frac{t}{t-\tau}}\right) \underset{t\to 0}{\to}$ const, then

$$\int_0^t K_\beta(t,\tau)d\tau \sim t^{\frac{1}{2}-\frac{\beta}{2}}\sqrt{t} = t^{1-\frac{\beta}{2}} \underset{t\to 0+}{\to} 0,$$

as $1 < \beta < 2$.

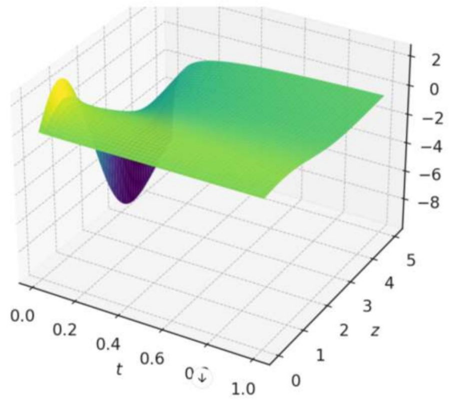

Figure 1. Graph of the kernel

Figure 1 presents the graph of the kernel which shows stability at small time values.

5 *The main results*

So, the following theorem has been proven.

Theorem 1. Integral equation (24) with kernel (25) for $1 < \beta < 2$ and with $\gamma(t) \sim t^\omega$ in the neighborhood of $t = 0$ is $\gamma(t) \sim t^\omega$, $\omega > 0$, $\gamma(0) = 0$ uniquely solvable in the class of continuous functions for any continuous right-hand side $f_2(t)$, if $\omega < \dfrac{1}{2(\beta-1)}$ and $\omega = \dfrac{1}{2}$.

This result coincided with the result obtained in [12].

Let us introduce a class of functions

$$\mathfrak{U} = \Bigg\{ u \,\Big|\, (x\sqrt{t})^{-1}u \in L_\infty(A) \cap C(B); \qquad u_t - u_{xx} \in L_\infty(A) \cap C(B);$$

$$\left\{ {}_rD_{0,x}^{\beta} u(x,t) \right\}\Big|_{x=\gamma(t)} \in C([0;T]), \quad T = const > 0, \quad 1 < \beta < 2 \Bigg\}, \tag{27}$$

where $A = \{(x,t)\,|\,x > 0,\ t \in [0,T]\}$, $B = \{(x,t)\,|\,x > 0,\ t \geq 0\}$, $T = const > 0$.

Since the solution of the integral equation (24) $\mu(t)$ is a continuous and bounded function under the conditions of Theorem (1), it can be shown that for the solution of problem (11)-(12), which has the form (22), where $f\,(x,t)$ belongs to the class (13), the following estimate is valid

$$|u\,(x,t)\,| \leq C(\lambda)\,x\,\sqrt{t},$$

where $C(\lambda) = C_1|\lambda| + C_2$.

Also it can be shown that function (18) satisfies BVP (11)-(12) and belongs to the class (27).

The following main result follows from Theorem 1:

Theorem 2. Let the function $f\,(x,t)$ satisfy conditions (13) and (14), the function $\mu(t) \in C([0;T])$ be a solution of integral equation (24) with the right-hand side $f_2(t) \in C([0;T])$ defined by formulas (19) and (26). Then BVP (11)-(12) with the load motion law $\gamma(t) \sim t^{\omega}$ (in the neighborhood of the point $t=0$) has a unique solution (22) in the class (27), if $\omega < \dfrac{1}{2(\beta-1)}$ and $\omega = \dfrac{1}{2}$.

Conclusion

Under the conditions of the theorem, the kernel (25) of the integral equation (24) has a weak singularity. Therefore, the method of successive approximations can be applied to find a unique solution of the equation (24). Then the corresponding boundary value problems are correct in natural classes of functions, i.e. the loaded term of the posed boundary value problem is a weak perturbation of the differential equation.

Since the problem statement contains a fractional derivative, then the obtained results can be applied in several domains such that:

Thermal processes: the study is particularly relevant to heat conduction problems where the material exhibits memory effects or non-locality. For instance: heat diffusion in heterogeneous materials with varying thermal properties, processes involving spatially moving heat sources or sinks.

Anomalous Diffusion: The fractional derivative approach effectively models systems exhibiting anomalous diffusion, as encountered in porous media, biological tissues with complex transport phenomena.

Engineering Systems: in mechanical and civil engineering, materials with hereditary properties, such as viscoelastic materials, benefit from this approach.

Mathematical Physics: the results are applicable in studying boundary value problems in non-cylindrical domains and domains with degeneracies, enhancing the analysis of complex geometries.

Now we will give a comparison with related studies, incorporating the comparative analysis. References [9, 11] provide foundational insights into the behavior of fractional derivatives in heat equations. Our study extends this by analyzing the effect of weak perturbations caused by the load term. In contrast to [12], which focuses on specific fixed domains, our results address moving load scenarios, offering broader applicability. Prior work, such as [13, 14], emphasizes integral equations with singularities. Our approach diverges by providing a detailed kernel analysis under varying load motion laws, as expressed through. Studies like [21] examine inverse problems for fractional equations but do not address weak perturbations in moving loads. Our results bridge this gap, contributing to a more comprehensive framework.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488740, Grant No. AP23488729, 2024–2026.)

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Podlubny, I. (1999). *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications.* San Diego-Boston-New York-London-Tokyo-Toronto: Academic Press.

2 Mainardi, F. (2012). Fractional calculus: Some basic problems in continuum and statistical mechanics. New York: Springer-Verlag. https://doi.org/10.48550/arXiv.1201.0863

3 Metzler, R., & Klafter, J. (2000). The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Reports, 1*(339), 1–77. https://doi.org/10.1016/S0370-1573(00)00070-3

4 Magin, R. (2004). Fractional Calculus in Bioengineering. *Critical Reviews in Biomedical Engineering, 1*(32), 1–104. https://doi.org/10.1615/critrevbiomedeng.v32.i1.10

5 Kukushkin, M.V. (2018). Spectral properties of fractional differentiation operators. *Electronic Journal of Differential Equations, 2018*(29), 1–24. https://doi.org/10.48550/arXiv.1710.02662

6 Atanackovic, T.M., Pilipovic, S., & Zorica, D. (2011). Distributed-order fractional wave equation on a finite domain. Stress relaxation in a rod. *International Journal of Engineering Science, 2*(49), 175–190. https://doi.org/10.1016/j.ijengsci.2010.11.004

7 Podlubny, I. (2002). Geometric and physical interpretation of fractional integration and fractional differentiation. *Fract. Calculus Appl. Anal., 4*(5), 367–386. https://doi.org/10.48550/arXiv.math/0110241

8 Parasidis, I.N. (2019). Extension method for a class of loaded differential equations with nonlocal integral boundary conditions. *Bulletin of the Karaganda University. Mathematics Series,* 4(96), 58–68. https://doi.org/10.31489/2019M4/58-68

9 Pskhu, A.V., Kosmakova, M.T., Akhmanova, D.M., Kassymova, L.Zh., & Assetov, A.A. (2022). Boundary value problem for the heat equation with a load as the Riemann-Liouville fractional derivative. *Bulletin of the Karaganda University. Mathematics Series,* 1(105), 74–82. https://doi.org/10.31489/2022M1/74-82

10 Ramazanov, M.I., Kosmakova, M.T., Romanovsky, V.G., & Tuleutaeva, Zh.M. (2018). Boundary value problems for essentially-loaded parabolic equation. *Bulletin of the Karaganda University. Mathematics Series,* 4(92), 79–86. https://doi.org/10.31489/2018M4/79-86

11 Kosmakova, M.T., Iskakov, S.A., & Kasymova, L.Zh. (2021). To solving the fractionally loaded heat equation. *Bulletin of the Karaganda University. Mathematics Series,* 1(101), 65–77. https://doi.org/10.31489/2021M1/65-77

12 Kosmakova, M.T., Ramazanov, M.I., & Kasymova, L.Zh. (2021). To Solving the Heat Equation with Fractional Load. *Lobachevskii Journal of Mathematics, 12*(42), 2854–2866. https://doi.org/10.1134/S1995080221120210

13 Kosmakova, M.T. (2016). On an integral equation of the Dirichlet problem for the heat equation in the degenerating domain. *Bulletin of the Karaganda University. Mathematics Series,* 1(81), 62–67. https://mathematics-vestnik.ksu.kz/index.php/mathematics-vestnik/article/view/79/74

14 Kosmakova, M.T., Akhmanova, D.M., Iskakov, S.A., Tuleutaeva, Zh.M., & Kasymova, L.Zh. (2019). Solving one pseudo-Volterra integral equation. *Bulletin of the Karaganda University. Mathematics Series,* 1(93), 72–77. https://doi.org/10.31489/2019m1/72-77

15 Jenaliyev, M.T., Ramazanov, M.I., Kosmakova, M.T., & Tuleutaeva, Zh.M. (2020). On the solution to a two-dimensional heat conduction problem in a degenerate domain. *Eurasian Mathematical Journal, 11*(3), 89–94. https://doi.org/10.32523/2077-9879-2020-11-3-89-94

16 Kosmakova, M.T., Orumbayeva, N.T., Medeubaev, N.K., & Tuleutaeva, Zh.M. (2018). Problems of Heat Conduction with Different Boundary Conditions in Noncylindrical Domains. *AIP Conference Proceedings, 1*(1997). https://doi.org/10.1063/1.5049065

17 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). Uniqueness and non-uniqueness of solutions of the boundary value problems of the heat equation. *AIP Conference Proceedings, 1*(1676). https://doi.org/10.1063/1.4930454

18 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2014). On the spectrum of Volterra integral equation with the incompressible kernel. *AIP Conference Proceedings, 1*(1611), 127–132. https://doi.org/10.1063/1.4893816

19 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2017). On the Solvability of Nonhomogeneous Boundary Value Problem for the Burgers Equation in the Angular Domain and Related Integral Equations. *Springer Proceedings in Mathematics and Statistics, 216,* 123–141. https://doi.org/10.1007/978-3-319-67053-9_12

20 Ramazanov, M.I., Kosmakova, M.T., & Tuleutaeva, Zh.M. (2021). On the Solvability of the Dirichlet Problem for the Heat Equation in a Degenerating Domain. *Lobachevskii Journal of Mathematics, 42*(15), 3715–3725. https://doi.org/10.1134/S1995080222030179

21 Abdullaev, O.Kh., & Yuldashev, T.K. (2023). Inverse Problems for the Loaded Parabolic-Hyperbolic Equation Involves Riemann-Liouville Operator. *Lobachevskii Journal of Mathematics, 3*(44), 1080–1090. https://doi.org/10.1134/S1995080223030034

22 Kubica, A., Ryszewska, K., & Yamamoto, M. (2020). Initial boundary value problems for time-fractional diffusion equations. *Time-Fractional Differential Equations,* 73–108. https://doi.org/10.1007/978-981-15-9066-5_4

23 Samko, S.G., Kilbas, A.A., & Marichev, O.I. (1993). *Fractional Integrals and Derivatives. Theory and Application.* Gordon and Breach: New York.

24 Stanković, B. (1970). On the function of E.M. Wright. *Publ. de l'Institut Mathématique, Beograd, Nouvelle Sér. 10*, 113–124.

25 Gorenflo, R., Luchko Y., & Mainardi, F. (1999). Analytical properties and applications of the Wright function. *Fractional Calculus and Applied Analysis, 2*(4), 383–414. https://doi.org/10.48550/arXiv.math-ph/0701069

26 Polyanin, A.D. (2002). *Handbook of Linear Partial Differential Equations for Engineers and Scientists.* Chapman and Hall/CRC: New York-London.

Author Information[*]

Minzilya Timerbaevna Kosmakova — PhD, Associate professor, Professor, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *svetlanamir578@gmail.com*; https://orcid.org/0000-0003-4070-0215

Nurgul Tumarbekovna Orumbayeva (*corresponding author*) — Candidate of physical and mathematical sciences, Associate professor, Professor, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *orumbayevan@mail.ru*; https://orcid.org/0000-0003-1714-6850

Danna Maratovna Akhmanova — Candidate of physical and mathematical sciences, Associate professor, Professor, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *danna.67@mail.ru*; https://orcid.org/0000-0003-1040-2495

Laila Zhumazhanovna Kasymova — PhD, Senior Lecturer, Abylkas Saginov Karaganda Technical University, 56 Ave. Nursultan Nazarbayev, Karaganda, 100027, Kazakhstan; e-mail: *l.kasymova2017@mail.ru*; https://orcid.org/0000-0002-4696-867X

[*]The author's name is presented in the order: First, Middle and Last Names.