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Solution of the model problem of heat conduction with Bessel operator

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In this work, a model boundary value problem for a parabolic equation with a Bessel operator was investigated. The solution to the problem under consideration is sought as a sum of thermal potentials: the double-layer and volume potentials, which reduces the problem to a Volterra integral equation of the second kind. The questions of existence and uniqueness of the obtained integral equation were investigated. The existence condition for the solution to the given problem was found. It is shown that if this condition is fulfilled, the problem has a single solution. The problem considered in this paper is called a model problem because the region in which the solution of the problem is sought is cylindrical and its results will be used in solving boundary value problems for the parabolic equation in noncylindrical regions having different order of degeneracy of the solution region to a point at the initial moment of time.

Keywords: heat equation, boundary value problem, Bessel operator, cylindrical domain, double layer thermal potential, thermal volume potential, Volterra integral equation, Laplace transform, homogeneous and inhomogeneous integral equation, resolvent.

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Introduction

In modern conditions, the rapid advancement of contact technology and increasing electrical device speeds make precise temperature field measurement in contact systems particularly important. In addition, it is important to study the dynamics of temperature field changes in time. When studying temperature processes in high-current contacts, it is necessary to take into account changes in the dimensions of the contact area, which occur both under the influence of electrodynamic forces and due to melting of the contact material at high temperatures.

During the electrode opening process, the temperature at the contact surface reaches the melting point, resulting in the formation of a liquid metal bridge between the electrodes. As further opening occurs, the bridge separates, causing material transfer from one electrode to the other. This process, known as bridge erosion, can significantly affect the performance of the contact system.

A distinctive characteristic of such problems, from a mathematical perspective, is the presence of a movable boundary in the solution domain, along with the fact that, at the initial moment, the contacts are closed, causing the solution domain to degenerate into a point. The solution of such thermal problems requires the application of generalized thermal potentials and the subsequent transformation of the initial boundary value problem to Volterra-type integral equations. In some cases, for example, when the order of degeneration of the region to a point is high enough, the integral equations will be singular, namely, the classical method of successive approximations is not applicable to them [1–14].

Earlier we considered boundary value problems for parabolic equations with Bessel operator in the domain $Q = \{(r,t)|\, 0 < r < t^{\omega},\ t > 0\}$ at $\omega > \frac{1}{2}$. The problem considered in this paper is called a

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model problem because the domain in which the solution of the problem is sought is cylindrical and its solution will be used in solving the problem in the case when the boundary of the domain will change according to the law $x = t^{\omega},\ 0 < \omega < \frac{1}{2}$.

1 Problem statement

In the region $Q = \{(r,t)|\ 0 < r < 1,\ 0 < t < T\}$, the following boundary value problem is considered:

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{1-2\beta}{r} \cdot \frac{\partial u}{\partial r} + a^2 \cdot \frac{\partial^2 u}{\partial r^2} + f(r,t), \tag{1}$$

$$u(r,t)|_{r=0} = 0,\ \ t > 0, \tag{2}$$

$$u(r,t)|_{r=1} = 0,\ \ t > 0, \tag{3}$$

$$u(r,t)|_{t=0} = 0, \tag{4}$$

where $0 < \beta < 1,\ f(r,t)$ is a given function.

2 Fundamental solution for equation (1)

In the domain $Q^{\infty} = \{(r,t)|\ r > 0,\ t > 0\}$ consider the boundary value problem for the homogeneous equation

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{1-2\beta}{r} \cdot \frac{\partial u}{\partial r} + a^2 \cdot \frac{\partial^2 u}{\partial r^2}, \tag{5}$$

corresponding to the inhomogeneous equation (1) of the basic boundary value problem, at boundary conditions

$$u(r,t)|_{r=0} = 0,\ \ t > 0, \tag{6}$$

$$u(r,t)|_{r=\infty} = 0,\ \ t > 0, \tag{7}$$

and the initial condition

$$u(r,t)|_{t=0} = \frac{\delta(r-\xi)}{r^{1-2\beta}}, \tag{8}$$

where $\delta(z)$ is the Dirac delta function, $\xi > 0$. Applying to the problem (5)–(8) the Laplace transform on the variable t, we obtain the boundary value problem for the ordinary differential equation

$$\frac{\partial^2 \hat{u}}{\partial r^2} + \frac{1-2\beta}{r} \cdot \frac{\partial \hat{u}}{\partial r} - \frac{p}{a^2} \cdot \hat{u} = -\frac{\delta(r-\xi)}{a^2 r^{1-2\beta}} \tag{9}$$

with boundary conditions

$$\hat{u}(r,p)|_{r=0} = 0, \tag{10}$$

$$\hat{u}(r,p)|_{r=\infty} = 0. \tag{11}$$

This boundary value problem (9)–(11) has a single solution $\hat{u}(r,p) = \hat{G}(r,p,\xi)$, where

$$\hat{G}(r,p,\xi) = \begin{cases} \frac{r^{\beta}\cdot\xi^{\beta}}{a^2} \cdot K_{\beta}\left(\frac{\xi\sqrt{p}}{a}\right) \cdot I_{\beta}\left(\frac{r\sqrt{p}}{a}\right),\ 0 < r < \xi, \\ \frac{\xi^{\beta}\cdot r^{\beta}}{a^2} \cdot I_{\beta}\left(\frac{\xi\sqrt{p}}{a}\right) \cdot K_{\beta}\left(\frac{r\sqrt{p}}{a}\right),\ \xi < r < \infty, \end{cases}$$

where $I_{\beta}(z),\ K_{\beta}(z)$ are cylindrical functions of imaginary argument of order β (Infeld and McDonald functions). The function $\hat{G}(r,p,\xi)$ belongs to the class of Laplace transform images. Performing its inversion [15; 350], we obtain

$$G(r,\xi,t) = \frac{1}{2a^2} \cdot \frac{r^{\beta}\cdot\xi^{1-\beta}}{t} \cdot \exp\left[-\frac{r^2+\xi^2}{4a^2 t}\right] \cdot I_{\beta}\left(\frac{r\xi}{2a^2 t}\right).$$

Let us replace the variable t in the function $G(r,\xi,t)$ by $(t-\tau)$, then

$$G(r,\xi,t-\tau)=\frac{1}{2a^2}\cdot\frac{r^{\beta}\cdot\xi^{1-\beta}}{t-\tau}\cdot\exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right]\cdot I_{\beta}\left(\frac{r\xi}{2a^2(t-\tau)}\right),$$

which has the following properties:

$$\lim_{r\to 0} G(r,\xi,t-\tau)\ =0,\quad \tau<t,\quad \xi>0,$$

$$\lim_{r\to \infty} G(r,\xi,t-\tau)\ =0,\quad \tau<t,\quad \xi>0,$$

$$\lim_{\tau\to t} G(r,\xi,t-\tau)\ =0,\quad r\neq\xi,$$

$$\lim_{\tau\to t}\int_0^{\infty} G(r,\xi,t-\tau)\cdot r^{1-2\beta}dr\ =1.$$

This function will be used to construct the thermal potential of the double layer in the domain $Q=\{(r,t)|\ 0<r<\infty,\ 0<t<T\}$:

$$W(r,t)=2a^2\int_0^t \frac{\partial G(r,\xi,t-\tau)}{\partial\xi}\bigg|_{\xi=1}\cdot g(\tau)d\tau,$$

and thermal volume potential in the region $Q=\{(r,t)|\ 0<r<\infty,\ 0<t<T\}$:

$$F(r,t)=\int_0^t d\tau\int_0^1 f(\xi,\tau)\cdot G(r,\xi,t-\tau)\cdot\xi^{1-2\beta}d\xi.$$

Remark 1. The density $f(r,t)$ is defined and continuous in the domain $\{(r,t)|\ 0<r\le 1,\ 0<t<T\}$, and inside the domain there is an estimate:

$$|f(r,t)|\le M\cdot r^{\gamma},\ \ M=\text{const},\ \ \gamma>-2+\beta. \tag{12}$$

The following properties are valid for the function $F(r,t)$.

1. The function $F(r,t)$ is defined and continuous in the domain $Q=\{(r,t)|\ 0<r<\infty,\ 0<t<T\}$ and for any values $t>0$ the equality is true

$$\lim_{r\to 0} F(r,t)\ =0.$$

2. Everywhere in the region $Q=\{(r,t)|\ 0<r<\infty,\ 0<t<T\}$ there exists and is continuous the derivative $\frac{\partial F}{\partial r}$, which is defined as follows:

$$\frac{\partial F}{\partial r}=\int_0^t d\tau\int_0^1 f(\xi,\tau)\cdot\frac{\partial G(r,\xi,t-\tau)}{\partial r}\cdot\xi^{1-2\beta}d\xi.$$

3. In the domain $Q=\{(r,t)|\ 0<r<\infty,\ 0<t<T\}$ there exists and is continuous the derivative $\frac{\partial F}{\partial t}$, which is defined by the equality

$$\frac{\partial F}{\partial t}=\int_0^t d\tau\int_0^1 f(\xi,\tau)\cdot\frac{\partial G(r,\xi,t-\tau)}{\partial t}\cdot\xi^{1-2\beta}d\xi+f(r,t).$$

3 Reduction of the boundary value problem (1)–(4) to the Volterra integral equation

As we found out, the fundamental solution for equation (1) is the function

$$G(r,\xi,t-\tau)=\frac{1}{2a^2}\cdot\frac{r^\beta\cdot\xi^{1-\beta}}{t-\tau}\cdot\exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right]\cdot I_\beta\left(\frac{r\xi}{2a^2(t-\tau)}\right),$$

where ξ is a parameter, $0<\beta<1$, $I_\beta(z)$ is a modified Bessel function of order β. The solution of problem (1)–(4) is found as a sum of the thermal double-layer potential and the volume potential:

$$u(r,t)=\int_0^t\left.\frac{\partial G(r,\xi,t-\tau)}{\partial\xi}\right|_{\xi=1}\mu(\tau)d\tau+\int_0^t\left.\frac{\partial G(r,\xi,t-\tau)}{\partial\xi}\right|_{\xi=0}\nu(\tau)d\tau+F(r,t),$$

where

$$F(r,t)=\int_0^t d\tau\int_0^1 f(\xi,\tau)\cdot G(r,\xi,t-\tau)\cdot\xi^{1-2\beta}d\xi,$$

and the densities $\mu(t)$ and $\nu(t)$ are to be defined. Using the fact that

$$\left.\frac{\partial G(r,\xi,t-\tau)}{\partial\xi}\right|_{\xi=0}=\frac{1}{(2a^2)^{\beta+1}}\cdot\frac{r^{2\beta}}{2^\beta(t-\tau)^{\beta+1}}\cdot\frac{1}{\beta\Gamma(\beta)}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right],$$

and

$$\left.\frac{\partial G(r,\xi,t-\tau)}{\partial\xi}\right|_{\xi=1}=\frac{r^\beta(r-1)}{4a^4(t-\tau)^2}\cdot\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\cdot\exp\left[-\frac{r}{2a^2(t-\tau)}\right]I_\beta\left(\frac{r}{2a^2(t-\tau)}\right)+$$
$$+\frac{r^{\beta+1}}{4a^4(t-\tau)^2}\cdot\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r}{2a^2(t-\tau)}\right]I_{\beta-1,\beta}\left(\frac{r}{2a^2(t-\tau)}\right)+$$
$$+\frac{r^\beta(1-2\beta)}{2a^2(t-\tau)}\cdot\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\cdot\exp\left[-\frac{r}{2a^2(t-\tau)}\right]\cdot I_\beta\left(\frac{r}{2a^2(t-\tau)}\right),$$

where the designation

$$I_{\beta-1,\beta}(z)=I_{\beta-1}(z)-I_\beta(z),$$

we obtain the integral representation of the solution of the equation:

$$\begin{aligned}u(r,t)=\int_0^t\Bigg\{&\frac{r^\beta(r-1)}{4a^4t(t-\tau)^2}\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r}{2a^2(t-\tau)}\right]I_\beta\left(\frac{r}{2a^2(t-\tau)}\right)+\\&+\frac{r^{\beta+1}}{4a^4(t-\tau)^2}\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r}{2a^2(t-\tau)}\right]I_{\beta-1,\beta}\left(\frac{r}{2a^2(t-\tau)}\right)+\\&+\frac{r^\beta(1-2\beta)}{2a^2(t-\tau)}\exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right]\exp\left[-\frac{r}{2a^2(t-\tau)}\right]\cdot I_\beta\left(\frac{r}{2a^2(t-\tau)}\right)\Bigg\}\mu(\tau)d\tau+\\&+\int_0^t\frac{1}{(2a^2)^{\beta+1}}\cdot\frac{r^{2\beta}}{2^\beta(t-\tau)^{\beta+1}}\cdot\frac{1}{\beta\Gamma(\beta)}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\cdot\nu(\tau)d\tau+F(r,t),\end{aligned}\tag{13}$$

where

$$\mu(t)\in L_\infty(0,\infty).\tag{14}$$

Using the boundary condition (2) for (13), we determine that the density

$$\nu(t) = 0.$$

Then

$$u(r,t) = \int_0^t \left\{ \frac{r^\beta(r-1)}{4a^4(t-\tau)^2} \exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r}{2a^2(t-\tau)}\right] I_\beta\left(\frac{r}{2a^2(t-\tau)}\right) + \right.$$
$$+ \frac{r^{\beta+1}}{4a^4(t-\tau)^2} \exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{r}{2a^2(t-\tau)}\right) + \tag{15}$$
$$\left. + \frac{r^\beta(1-2\beta)}{2a^2(t-\tau)} \exp\left[-\frac{(r-1)^2}{4a^2(t-\tau)}\right] \exp\left[-\frac{r}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{r}{2a^2(t-\tau)}\right) \right\} \mu(\tau)d\tau +$$
$$+F(r,t).$$

Using the boundary condition (3), we obtain the integral equation with respect to the unknown density $\mu(t)$:

$$\mu(t) - \int_0^t \sum_{i=1}^{2} N_i(t,\tau) \cdot \mu(\tau)d\tau = f(t), \tag{16}$$

where

$$N_1(t,\tau) = \frac{1-2\beta}{t-\tau} \exp\left[-\frac{1}{2a^2(t-\tau)}\right] \cdot I_\beta\left(\frac{1}{2a^2(t-\tau)}\right),$$

$$N_2(t,\tau) = \frac{1}{2a^2(t-\tau)^2} \exp\left[-\frac{1}{2a^2(t-\tau)}\right] I_{\beta-1,\beta}\left(\frac{1}{2a^2(t-\tau)}\right),$$

$$f(t) = F(t,t).$$

The solution of this integral equation, if it exists in the class of functions (14), is singular and can be found by the method of successive approximations, since the estimates [16] are valid:

$$0 < e^{-z} \cdot I_\beta(z) < \frac{C_1}{\sqrt{z}},\ 0 < e^{-z} \cdot I_{\beta-1,\beta}(z) < \frac{C_1}{\sqrt{z^3}},\ C_1,\ C_2 = \text{const}.$$

To clarify the question of existence of a solution to equation (16), we use the method of integral Laplace transform.

4 Solution of the integral equation (16)

Let us apply the Laplace transform to both parts of the integral equation (16):

$$\widehat{\mu}(p)\left\{1 - \left[\widehat{N_1}(p) + \widehat{N_2}(p)\right]\right\} = \hat{f}(p),\ \text{Re}p > 0,$$

$$\widehat{\mu}(p) = \frac{\hat{f}(p)}{1 - \left[\widehat{N_1}(p) + \widehat{N_2}(p)\right]}. \tag{17}$$

In order to find the image of the function $\widehat{N_1}(p) + \widehat{N_2}(p)$ we will use:
1) formula (29.169) [15; 350];

2) the property: let $f(t) \risingdotseq \hat{f}(p)$, then $\frac{1}{t}f(t) \risingdotseq \int_p^\infty \hat{f}(p)dp$ [17; 506]. Then we have:

$$\widehat{N_1}(p) = 2(1-2\beta)K_\beta\left(\frac{\sqrt{p}}{a}\right)I_\beta\left(\frac{\sqrt{p}}{a}\right), \ \mathrm{Re}p > 0,$$

$$\widehat{N_2}(p) = \frac{1}{a^2}\int_p^\infty \left[K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right)I_{\beta-1}\left(\frac{\sqrt{p}}{a}\right) - K_\beta\left(\frac{\sqrt{p}}{a}\right)I_\beta\left(\frac{\sqrt{p}}{a}\right)\right]dp =$$

$$= 1 - 2\frac{\sqrt{p}}{a}I_\beta\left(\frac{\sqrt{p}}{a}\right)K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right), \ \mathrm{Re}p > 0.$$

Let us show that the homogeneous integral equation

$$\mu(t) - \int_0^t \sum_{i=1}^2 N_i(t,\tau)\cdot\mu(\tau)d\tau = 0 \tag{18}$$

has only zero solution in the class of functions $\mu(t) \in L_\infty(0,\infty)$. For this purpose, let us find the roots of the equation

$$1 - \left[\widehat{N_1}(p) + \widehat{N_2}(p)\right] = 0$$

or

$$2I_\beta\left(\frac{\sqrt{p}}{a}\right)\cdot\left\{\frac{\sqrt{p}}{a}K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right) - (1-2\beta)K_\beta\left(\frac{\sqrt{p}}{a}\right)\right\} = 0. \tag{19}$$

Let $I_\beta\left(\frac{\sqrt{p}}{a}\right) = 0$ in equality (19). According to the definition of Bessel function of imaginary argument $I_\beta\left(\frac{\sqrt{p}}{a}\right) = e^{-\frac{\pi}{2}\beta i}J_\beta\left(\frac{i\sqrt{p}}{a}\right)$, where $J_\beta\left(\frac{i\sqrt{p}}{a}\right)$ is a cylindrical Bessel function of the first kind. The function $J_\beta\left(\frac{i\sqrt{p}}{a}\right)$ has infinitely many valid roots for any valid β; if $\beta > -1$, all its roots are valid and equal to $i\frac{\sqrt{p_k}}{a} = \alpha_k, \ p_k = -a^2\alpha_k^2, \ \alpha_k \in \mathbb{R}, \ k \in \mathbb{Z}\setminus\{0\}$ [18], which contradicts the $\mathrm{Re}p > 0$ condition.

It is clear that the second multiplier at $\frac{1}{2} < \beta < 1$

$$\frac{\sqrt{p}}{a}K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right) - (1-2\beta)K_\beta\left(\frac{\sqrt{p}}{a}\right) \neq 0,$$

and at $0 < \beta < \frac{1}{2}$ it has a single root $p = p_0 > 0$. It follows that in this case the solution of the homogeneous equation (18) is the function $\mu_0(t) = C\cdot e^{p_0 t}$, which does not belong to the class (14). Thus, it is shown that the homogeneous integral equation (18) has only zero solution.

It follows from equality (17) at $0 < \beta < \frac{1}{2}$ that if the function $\hat{f}(p)$ goes to zero at the point p_0, then the expression

$$\frac{\hat{f}(p)}{1 - \left[\widehat{N_1}(p) + \widehat{N_2}(p)\right]}$$

has no poles and in this case equation (16) will have a single solution in the class of functions (14). Thus, for solvability of equation (16) at $0 < \beta < \frac{1}{2}$ it is necessary and sufficient to fulfill the condition

$$\int_0^\infty e^{-p_0 t}f(t)dt = 0.$$

If $\frac{1}{2} < \beta < 1$, then equation (16) is unconditionally solvable.

Let this condition be satisfied. Let us find the solution of the inhomogeneous integral equation. For this purpose, let us represent (17) in the following form:

$$\widehat{\mu}(p) = \hat{f}(p) + \hat{R}(p) \cdot \hat{f}(p),$$

where

$$\hat{R}(p) = \frac{\widehat{N_1}(p) + \widehat{N_2}(p)}{1 - \left[\widehat{N_1}(p) + \widehat{N_2}(p)\right]} = \frac{1 - 2I_\beta\left(\frac{\sqrt{p}}{a}\right)\left[\frac{\sqrt{p}}{a}K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right) - (1-2\beta)K_\beta\left(\frac{\sqrt{p}}{a}\right)\right]}{2I_\beta\left(\frac{\sqrt{p}}{a}\right)\left[\frac{\sqrt{p}}{a}K_{\beta-1}\left(\frac{\sqrt{p}}{a}\right) - (1-2\beta)K_\beta\left(\frac{\sqrt{p}}{a}\right)\right]}.$$

Let us use the properties of [15; 191]:

1. If $\varphi(t) \risingdotseq \widehat{\varphi}(p)$, then

$$\varphi(\alpha t) \risingdotseq \frac{1}{\alpha}\widehat{\varphi}(\frac{p}{\alpha}), \ \alpha > 0. \tag{20}$$

2. If $\widehat{\varphi}(p) \risingdotseq \varphi(t)$, then

$$\widehat{\varphi}(\sqrt{p}) = \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{t^{\frac{3}{2}}} \int_0^\infty \tau \cdot e^{-\frac{\tau^2}{4t}} \varphi(\tau) d\tau. \tag{21}$$

For convenience we introduce the notation $\frac{\sqrt{p}}{a} = z$ and find the original expression

$$\widehat{R^*}(z) = \frac{1 - 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right]}{2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right]}.$$

According to [17; 519]:

$$\widehat{R^*}(z) = \frac{A(z)}{B(z)} \risingdotseq \sum_{-\infty}^{+\infty} \frac{A(z_k)}{B'(z_k)} e^{-z_k y},$$

where z_k are zeros of the function

$$B(z) = 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right].$$

1) Let $y_\beta(z) = zK_{\beta-1}(z) - (1-2\beta)K_\beta(z) = 0$. This equation, as noted earlier, has one root z_0 at $0 < \beta < \frac{1}{2}$.

2) Let $I_\beta(z) = e^{-\frac{\pi}{2}\beta i} J_\beta(iz) = 0$. Therefore, $iz_k = \alpha_k$ or $z_k = -i\alpha_k$, where $\alpha_k \in \mathbb{R}$.

Then

$$\widehat{R^*}(z) = \frac{A(z)}{B(z)} \risingdotseq \sum_{-\infty}^{+\infty} \frac{A(z_k)}{B'(z_k)} e^{-z_k y} = \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{A(z_k)}{B'(z_k)} e^{-z_k y} + \frac{A(z_0)}{B'(z_0)} e^{-z_0 y} = R^*_{-}(y),$$

where

$$B(z) = 2I_\beta(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right],$$

$$B'(z) = 2I_{\beta-1}(z)\left[zK_{\beta-1}(z) - (1-2\beta)K_\beta(z)\right] + 2(1-2\beta)I_\beta(z)K_{\beta-1}(z) + \\ + \left(\frac{4\beta(1-2\beta)}{z} - 2z\right) I_\beta(z)K_\beta(z).$$

Thus, we obtained that at $0 < \beta < \frac{1}{2}$:

$$R^*(y) = \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{e^{-z_k y}}{2I_{\beta-1}(z_k)\left[z_k K_{\beta-1}(z_k) - (1-2\beta)K_\beta(z_k)\right]} + \frac{e^{-z_0 y}}{2I_\beta(z_0)K_{\beta-1}(z_0)\left[1 - \frac{1}{1-2\beta} z_0^2\right]}. \tag{22}$$

Let us introduce the following notations:

$$A_{\beta,k} = \frac{1}{2I_{\beta-1}(z_k)\left[z_k K_{\beta-1}(z_k) - (1-2\beta)K_\beta(z_k)\right]}, \quad A_{\beta,0} = \frac{1}{2I_\beta(z_0)K_{\beta-1}(z_0)\left[1 - \frac{1}{1-2\beta}z_0^2\right]}.$$

From equality (22) and properties of (20) and (21) we have:

$$\hat{R}\left(\frac{\sqrt{p}}{a}\right) \risingdotseq R(t) = \frac{a^2}{2\sqrt{\pi}t^{\frac{3}{2}}} \cdot \sum_{k\in\mathbb{Z}\setminus\{0\}} A_{\beta,k} \cdot \int_0^\infty \tau e^{-\frac{\tau^2}{4t} - ia^2\alpha_k\tau} d\tau + \frac{a^2}{2\sqrt{\pi}t^{\frac{3}{2}}} \cdot A_{\beta,0} \cdot \int_0^\infty \tau e^{-\frac{\tau^2}{4t} - z_0 a^2\tau} d\tau,$$

where α_k are zeros of the function $J_\beta(z)$. For the resolvent $R(t)$ the following estimation is valid

$$R(t) \leq \frac{a^2\pi}{4\sqrt{t}}.$$

Remark 2. At $0 < \beta < \frac{1}{2}$ it follows from the equality (17) that for solvability of the integral equation (15) it is necessary and sufficient to fulfill the condition

$$\int_0^\infty e^{-p_0 t} f(t)dt = 0, \tag{23}$$

where $f(t) = \lim_{r\to t} F(r,t)$.

Theorem 1. For any function $f(t) \in C(0,T)$, equation (16) has a single solution if $\frac{1}{2} < \beta < 1$. When $0 < \beta < \frac{1}{2}$, it is necessary and sufficient for the solvability of the integral equation (16) that condition (23) is satisfied. In this case, for any function $f(t) \in C(0,T)$, the integral equation (16) has a single solution.

Remark 3. If at $0 < \beta < \frac{1}{2}$ condition (23) is not satisfied, then equation (16) has no solutions in the chosen class of functions. However, this result does not contradict the well-known fact that the Volterra equation always has a single solution. Equation (16) belongs to the class of Volterra-type equations of the second kind and, therefore, in case the condition (23) is not satisfied, it will also be solvable, but in a wider space of functions with exponential growth.

5 Solution of the boundary value problem (1)–(4)

Theorem 2. For any function $f(r,t)$ from the class (12), the boundary value problem (1)–(4):

1) for $\frac{1}{2} < \beta < 1$ it has a single solution $u(r,t) \in C(0,T)$;

2) when $0 < \beta < \frac{1}{2}$, it is necessary and sufficient to fulfill condition (23) for the existence of a solution. If this condition is satisfied, the problem has a single solution in the class of functions $u(r,t) \in C(0,T)$.

Conclusion

In this work we study a model boundary value problem for a parabolic equation with a Bessel operator. The existence condition for the solution to this problem at $0 < \beta < \frac{1}{2}$ is found. It is shown that if this condition is fulfilled, the problem has a single solution. If $\frac{1}{2} < \beta < 1$, the problem is unconditionally solvable. The results of this work will be used in solving boundary value problems for parabolic equations in non-cylindrical regions having different order of degeneration of the solution region to a point at the initial moment of time.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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