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Solitary Wave Solutions of the coupled Kawahara Equation

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The field of nonlinear differential equations have made significant contribution in understanding nonlinear dynamics and its complex phenomenon. One such evolution equation is Kawahara equation, which has gained its importance in plasma physics and allied fields. Many researchers are interested to work on their soliton, multi-solitons solutions and to study other properties such as stability, integrability, conservation laws and so on. The aim of the paper is to study the Coupled Kawahara equation and to deduce its soliton solutions. The coupled equation is treated with the ansatz method and the tanh method to compute soliton solutions. The novelty of this work is to demonstrate the fact, that the derived system efficiently gives two governing equations admitting solitary wave solutions. Further, in the coupled equation, one equation has the nonlinear term vv_x addition to the Kawahara equation, while the other is the modified Kawahara equation. Scope for future works is also highlighted.

Keywords: Evolution Equation, Bounded solutions, the Ansatz method, the Tanh method.

2020 Mathematics Subject Classification: 35L55, 35Q51.

Introduction

The study of nonlinear dynamics has significantly advanced our understanding of various physical phenomena through nonlinear partial differential equations. One prominent area within this field is solitary wave theory. The concept of solitary waves was first observed empirically by John Scott Russell in 1844 [1]. Later, in 1965, Korteweg and de Vries formulated the mathematical representation of these waves, now known as the KdV equation. This third-order nonlinear differential equation, involving spatial derivatives, has found extensive applications in areas such as shallow water wave theory, ocean engineering, optics, and related disciplines [2–5].

The Kawahara equation is a significant evolution equation used to model various physical phenomena, including plasma dynamics and gravity waves on viscous liquid surfaces. It also describes magneto-acoustic wave behavior in plasma and the dynamics of long water waves beneath ice-covered surfaces [3–9]. Essentially, the Kawahara equation extends the KdV equation by incorporating a fifth-order term. However, unlike the KdV equation, it is not integrable, as it does not appear in Hietarinta's classification of integrable systems [10].

In [8], the governing model for waves in dispersive media was introduced. In [11], travelling wave solutions for the Kawahara equation and its modified form were derived. A comparison of two numerical approaches for solving the Kawahara equation was presented in [12]. Solitary wave solutions for the modified Kawahara equation were explored in [13], while the soliton solution for the generalized Kawahara equation was provided in [14]. Additionally, solitary wave solutions for the Hirota-Satsuma coupled KdV equation have been studied in [15, 16].

There are many methods to solve the nonlinear evolution equations namely, the Adomian decomposition method, the Homotopy perturbation method, the Hirota's Bilinear method, the Bilinear neural network method, Lie symmetry analysis, the tanh method and so on [17–19].

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In our work, first we obtain the coupled Kawahara equation by transforming it into a function of complex variable. Further, the solitons of the coupled equation are computed using the ansatz method and the tanh method. Computed solutions are simulated using Maple. Further, analysis of solutions is carried out, which conveys that the transformed coupled equation resembles Kawahara type and the modified Kawahara equation.

1 *Soliton Solutions*

The Kawahara equation is the extension of KdV equation with higher order dispersion term p_{xxxxx} which reads as [4, 8, 14] :

$$p_t + 6pp_x + p_{xxx} - p_{xxxxx} = 0\,;\ x, t > 0 \in \mathbb{R}. \tag{1}$$

By considering,

$$p(x,t) = u(x,t) + i\ v(x,t),$$

in equation (1) results in the coupled Kawahara equation:

$$u_t + 6uu_x - 6vv_x + u_{xxx} - u_{xxxxx} = 0, \tag{2}$$

$$v_t + 6uv_x + 6vu_x + v_{xxx} - v_{xxxxx} = 0. \tag{3}$$

If $v = 0$, the above system of equations (2) and (3) will reduce to the well known Kawahara equation.

To study this coupled system and its soliton solution, we use the ansatz method and the tanh method in the following section.

1.1 *The ansatz method*

As noted earlier, $v = 0$ results in the Kawahara equation and its soliton solution is of the form sech^4 we begin with the ansatz,

$$u(x,t) = A\ \operatorname{sech}^M k(x - ct), \tag{4}$$

$$v(x,t) = B\ \operatorname{sech}^N k(x - ct), \tag{5}$$

where $M, N \in \mathbb{N}$; $k, c, A > 0$ and $B > 0$ are scalars from the field $\mathbb{R}$.

The restriction of A and B to be positive is to retain the coupled system.

By balancing the higher nonlinear term and higher linear term, we obtain $M = 4$ and $N = 2$. Substituting this M and N in equation (4) and (5), we get

$$\begin{aligned} u(x,t) &= A\ \operatorname{sech}^4 k(x - ct), \\ v(x,t) &= B\ \operatorname{sech}^2 k(x - ct). \end{aligned} \tag{6}$$

Now, substituting the above expressions u, v of (6) into the equation (2), we obtain,

$$\begin{aligned} &[Ac - 6A^2 + 14Ak^2 + 376Ak^4 + 3B^2] \\ &+ [6A^2 - 30Ak^2 - 120Ak^4]\ \tanh^2 k(x - ct) \\ &+ [6A^2 - 1680Ak^4]\ \tanh^2 k(x - ct)\ \operatorname{sech}^2 k(x - ct) = 0. \end{aligned}$$

As the set $\{1, \tanh^2 k(x - ct), \tanh^2 k(x - ct)\ \operatorname{sech}^2 k(x - ct)\}$ is linearly independent, it leads to,

$$\begin{aligned} Ac - 6A^2 + 14Ak^2 + 376Ak^4 + 3B^2 &= 0, \\ 6A^2 - 30Ak^2 - 120Ak^4 &= 0, \\ 6A^2 - 1680Ak^4 &= 0. \end{aligned}$$

Solving the above system gives the values of A and k as

$$A = 280k^4,\ k = \pm\frac{1}{2\sqrt{13}},\ \text{and } c = \frac{-1014B^2}{35} + \frac{36}{169},\ \text{with } c < \frac{36}{169}.$$

Therefore, solutions are

$$\begin{aligned} u_1(x,t) &= 280k^4 \text{sech}^4\left\{\pm\frac{1}{2\sqrt{13}}\left(x - \left(\frac{-1014B^2}{35} + \frac{36}{169}\right)t\right)\right\} \text{ and} \\ v_1(x,t) &= B\ \text{sech}^2\left\{\pm\frac{1}{2\sqrt{13}}\left(x - \left(\frac{-1014B^2}{35} + \frac{36}{169}\right)t\right)\right\}. \end{aligned} \tag{7}$$

Now, using u and v of (6) in equation (3), we observe that

$$\begin{aligned} &[Bc - 18AB + 8Bk^2 + 136Bk^4] \\ &+ [18AB - 12Bk^2 - 120Bk^4]\ \tanh^2 k(x - ct) \\ &+ [18AB - 360Bk^4] \tanh^2 k(x - ct)\ \text{sech}^2 k(x - ct)) = 0. \end{aligned}$$

This in turn implies the system of equations

$$\begin{aligned} Bc - 18AB + 8Bk^2 + 136Bk^4 &= 0, \\ 18AB - 12Bk^2 - 120Bk^4 &= 0, \\ 18AB - 360Bk^4 &= 0. \end{aligned}$$

Solving the above system, we obtain

$$A = 20k^4,\ B = B,\ c = \frac{4}{25} \text{ and } k = \pm\frac{1}{2\sqrt{5}}.$$

Therefore, the corresponding solutions for above values are given by

$$\begin{aligned} u_2(x,t) &= 20k^4 \text{sech}^4\left(\pm\frac{1}{2\sqrt{5}}\left(x - \frac{4}{25}t\right)\right) \text{ and} \\ v_2(x,t) &= B\ \text{sech}^2\left(\pm\frac{1}{2\sqrt{5}}\left(x - \frac{4}{25}t\right)\right). \end{aligned} \tag{8}$$

1.2 The tanh method

In this subsection, we replicate the soliton solutions that are obtained using the ansatz method using the tanh method. For more details refer [4, 20–23].

By introducing a new variable $z = x - ct$ in (2) and (3), we obtain

$$-cU^{(1)} + 6UU^{(1)} - 6VV^{(1)} + U^{(3)} - U^{(5)} = 0, \tag{9}$$

$$-cV^{(1)} + 6(UV)^{(1)} + V^{(3)} - V^{(5)} = 0. \tag{10}$$

The above equations (9) and (10) can be integrated once to get,

$$-cU + 3U^2 - 3V^2 + U^{(2)} - U^{(4)} = 0, \tag{11}$$

$$-cV + 3UV + V^{(2)} - V^{(4)} = 0. \tag{12}$$

Let $U(Y) = \sum_{j=0}^{M} a_j Y^j$ and $V(Y) = \sum_{j=0}^{N} b_j Y^j$, where $U^{(n)} = \dfrac{d^n U}{dZ^n}$, $Y = \tanh Z$, $M, N \in \mathbb{N}$, a_j and b_j are real coefficients need to be determined.

By balancing the power of highest order of the derivative and nonlinear terms: $U^{(4)}$ and U^2 of (11), we obtain $M = 4$.

Analogously, balancing powers for the equation (12), we obtain $N = 2$. So, we have

$$U(Y) = a_0 + a_1 Y + a_2 Y^2 + a_3 Y^3 + a_4 Y^4, \tag{13}$$

$$V(Y) = b_0 + b_1 Y + b_2 Y^2. \tag{14}$$

Using equation (13) in (9), we obtain the following system of equations:

$$\begin{aligned}
-ca_0 + 3a_0^2 - 3b_0^2 + 2a_2k^2 + 16a_4k^4 &= 0,\\
-ca_1 + 6a_0a_1 + 6a_3k^2 - 2a_1k^2 - 16a_1k^4 + 120a_4k^4 &= 0,\\
-ca_2 + 6a_0a_2 + 3a_1^2 + 12a_4k^2 - 8a_2k^2 - 136a_2k^4 + 480a_4k^4 &= 0,\\
-ca_3 + 6a_0a_3 + 6a_1a_2 + 2a_1k^2 - 18a_3k^2 - 576a_3k^4 + 40a_1k^4 &= 0,\\
-ca_4 + 6a_0a_4 + 6a_1a_3 + 3a_2^2 + 6a_2k^2 - 32a_4k^2 - 1696a_4k^4 + 240a_2k^4 &= 0,\\
6a_1a_4 + 6a_2a_3 + 12a_3k^4 - 24a_1k^4 + 816a_3k^4 &= 0,\\
6a_2a_4 + 3a_3^2 + 20a_4k^2 - 120a_2k^4 + 2080a_4k^4 &= 0,\\
6a_3a_4 - 360a_3k^4 &= 0,\\
3a_4^2 - 840a_4k^4 &= 0.
\end{aligned}$$

Solving the above system of equations, we obtain

$$\begin{aligned}
&c = -\frac{1}{39}\left[\frac{495040k^8 + 31360k^6 + 280k^4 + 117b_0^2 - 117a_0^2}{a_0}\right],\\
&a_1 = 0, a_2 = -\frac{1120}{3}k^4 - \frac{140}{39}k^2, a_3 = 0, a_4 = 280k^4,\\
&b_1 = 0, a_0 \neq 0, b_2, \text{ and } b_0 \text{ are arbitrary constants.}
\end{aligned}$$

By fixing $a_0 = a_4$, $a_2 = -2a_4$ and $b_2 = -b_0$, results in $k = \pm\dfrac{1}{2\sqrt{13}}$ and $c = \dfrac{36}{169} - \dfrac{1014}{35}b_0^2$, which agrees with the ansatz method.

$$\begin{aligned}
u_1(x,t) &= 280k^4 \mathrm{sech}^4\left\{\pm\frac{1}{2\sqrt{13}}\left(x - \Big(\frac{-1014b_0^2}{35} + \frac{36}{169}\Big)t\right)\right\} \text{ and}\\
v_1(x,t) &= b_0 \,\mathrm{sech}^2\left\{\pm\frac{1}{2\sqrt{13}}\left(x - \Big(\frac{-1014b_0^2}{35} + \frac{36}{169}\Big)t\right)\right\}.
\end{aligned}$$

Now, using (14) in (12), we obtain system of equations:

$$\begin{aligned}
-cb_0 + 6a_0b_0 + 2b_2k^2 + 16b_2k^4 &= 0,\\
-cb_1 + 6a_0b_1 + 6a_1b_0 - 2b_1k^2 - 16b_1k^4 &= 0,\\
-cb_2 + 6a_0b_2 + 6a_1b_1 + 6a_2b_0 - 8b_2k^2 - 136b_2k^4 &= 0,\\
6(a_1b_2 + a_2b_1 + b_0a_3) + 2b_1k^2 + 40b_1k^4 &= 0,\\
6(a_4b_0 + a_2b_2 + a_3b_1 + b_2k^2) + 240b_2k^4 &= 0,\\
6(a_3b_2 + a_4b_1) - 24b_1k^4 &= 0,\\
6a_4b_2 - 120b_2k^4 &= 0.
\end{aligned}$$

Solving the above system of equations, we obtain

$$\begin{aligned}
&a_0 = \frac{8}{3}k^4 + \frac{1}{3}k^2 + \frac{1}{6}c, a_1 = 0, a_2 = -(20k^2+1)k^2,\\
&a_3 = 0,\ a_4 = 20k^4,\ b_1 = 0,\\
&b_0,\ b_2, \text{ and } c \text{ are arbitrary constants.}
\end{aligned}$$

By fixing $a_0 = a_4$ and $a_2 = -2a_4$, will result in $k = \pm\dfrac{1}{2\sqrt{5}}$ and $c = \dfrac{4}{25}$. Hence,

$$\begin{aligned}
&u_2(x,t) = 20k^4 \operatorname{sech}^4\left(\pm\frac{1}{2\sqrt{5}}\left(x - \frac{4}{25}t\right)\right) \text{ and}\\
&v_2(x,t) = b_0 \operatorname{sech}^2\left(\pm\frac{1}{2\sqrt{5}}\left(x - \frac{4}{25}t\right)\right).
\end{aligned}$$

2 Plots of the Solutions

In this subsection, we simulate the solutions of the coupled equation using maple package [24].

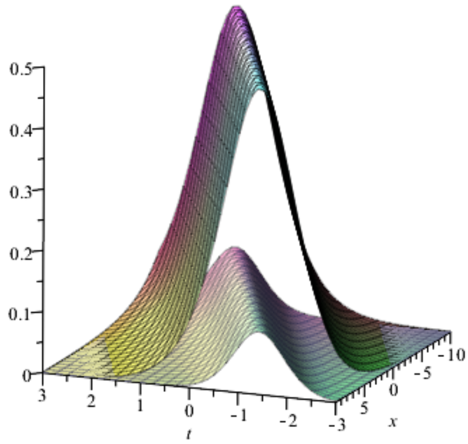

Figure 1. Plots of (7) with $b = \dfrac{1}{2}$

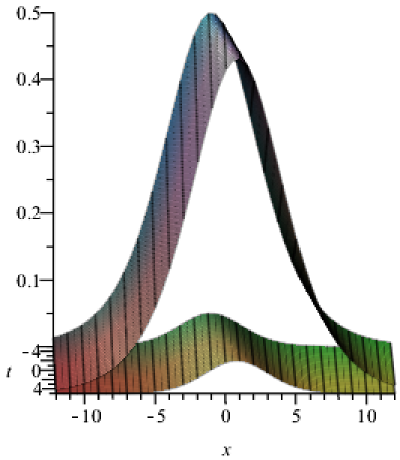

Figure 2. Plots of (8) with $b = \dfrac{1}{2}$

3 Analysis of Solutions

We observe that, the solutions $u(x,t)$ and $v(x,t)$ are related by $u = \frac{A}{B^2}v^2$, then equation (3) will reduce to $v_t + 18\frac{A}{B^2}v^2v_x + v_{xxx} - v_{xxxxx} = 0$.

So, the coupled equation (2) and (3) will be of the form,

$$u_t + 6uu_x - 6vv_x + u_{xxx} - u_{xxxxx} = 0, \tag{15}$$

$$v_t + 18\frac{A}{B^2}v^2v_x + v_{xxx} - v_{xxxxx} = 0. \tag{16}$$

Equation (15) is the Kawahara type equation with the additional term vv_x to the Kawahara equation and equation (16) is the modified Kawahara equation. Further, one can observe that the solutions simulated in Figure 1 and Figure 2, for a particular choice $b = \dfrac{1}{2}$. The solution given in equation (7) indicated by Figure 1 has a slightly high amplitude compared to the solution given by equation (8), which is depicted in Figure 2.

4 Discussion

In conclusion, transforming the Kawahara equation to a coupled system results in two different governing equations in which one is Kawahara type equation and the other is the modified Kawahara equation. As a scope for further work, one can compute the other solutions such as periodic solution, shock solution and singular solution to the discussed system.

Author Contributions

R. Rangarajan suggested the equation and validated the computed solutions. K. Bharatha contributed to the application of the tanh method and established the conditions for obtaining the coupled system. C.J. Neethu developed the ansatz method and performed the simulation of the solutions. All authors participated in revising the manuscript and approved its final submission.

Conflict of Interest

The authors declare no conflict of interest.

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