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On the stability of the third order partial differential equation with time delay

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In this paper, the initial value problem for a third-order partial differential equation with time delay within a Hilbert space was analyzed. We establish a key theorem regarding the stability of this problem. Additionally, we demonstrate how this stability theorem can be applied to the third-order partial differential equation with time delay.

Keywords: stability, third order partial differential equations, time delay.

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Introduction

In physics, various problems give rise to third order partial differential equations (PDEs). In various branches of engineering and science, such as applied mathematics, these problems have become a key research area. Within the last 10 decades, interest towards nonlocal and local boundary value problems (BVPs) for PDEs with space and time variables have increased significantly. Nonlocal and local BVPs for third order PDEs have been investigated widely in a lot literature (for instance, see [1–3]).

One of the most frequently occurring phenomena in various engineering applications is time delay (TD). A typical instance with regards to control theory can be seen in sampled-data control process.

Applications and theory of nonlinear and linear third-order differential and difference equations comprising a delay term were investigated widely (for instance, see [4–11], and the included references).

Lastly, applications and theory of PDEs of the same order having delay operator term with respect to the other operator term were studied for parabolic differential equations with delay term (for example, see [12–18], and the included references).

However, the stability theory of third-order PDEs having a delay term is not well developed. In this paper, our aim is to study the initial value problem (IVP) for the third order PDE having TD

$$\begin{cases} \frac{d^3y(s)}{ds^3} + B\frac{dy(s)}{ds} = cBy(s-z) + h(s), \ 0 < s < \infty, \\ y(s) = k(s), \ -z \le s \le 0 \end{cases} \tag{1}$$

in G, a Hilbert space, having self-adjoint positive definite operator (SAPDO) $B,\ B \ge \lambda I$, where $\lambda > 0$. Here $k(s)$ defined on $[-z, 0]$ is the given abstract continuous function (ACF) with values in $D(B),\ h(s)$ defined on $(0, \infty)$ is the given ACF having values in G, and $c \in R^1$.

The structure of the paper is as follows. In Section 1, we establish the main theorem on the stability of problem (1). Section 2 presents theorems on stability estimates for the solutions of three problems involving third-order PDEs. Finally, Section 3 provides the conclusion.

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1 Main theorem on Stability

If conditions i, ii, iii below are met, then a function $y(s)$ is considered a solution to problem (1):

i. $y(s)$ is twice continuously differentiable over the interval $[0,\infty)$, with the derivative at $s=0$ taken as the unilateral derivative.

ii. The derivative $\frac{dy(s)}{ds}$ lies in $D(B)$ for every $s\in[0,\infty)$, and the function $B\frac{dy(s)}{ds}$ is continuous throughout the interval $[0,\infty)$.

iii. $y(s)$ satisfies the primary equation and the initial conditions described in (1).

Throughout this paper, let $\{E(s), s\geq 0\}$ be an operator function, where $E(s)=\cos(sB^{\frac{1}{2}})$, and is defined by the formula

$$E(s)=\frac{e^{isB^{\frac{1}{2}}}+e^{-isB^{\frac{1}{2}}}}{2}. \tag{2}$$

From the operator function $T(s)=B^{-\frac{1}{2}}\sin(sB^{\frac{1}{2}})$, where $T(s)=\int_0^s E(p)\,dp$, it follows that

$$T(s)=B^{-\frac{1}{2}}\frac{e^{isB^{\frac{1}{2}}}-e^{-isB^{\frac{1}{2}}}}{2i}. \tag{3}$$

We refer to [19] for the theory of cosine operator functions. We now present an important lemma below.

Lemma 1.1. The estimates that follows holds for $s\geq 0$:

$$\left\|\exp\left\{\pm isB^{\frac{1}{2}}\right\}\right\|_{G\to G}\leq 1,\quad ||E(s)||_{G\to G}\leq 1,\quad \left\|B^{\frac{1}{2}}T(s)\right\|_{G\to G}\leq 1. \tag{4}$$

The proof of the lemma above depends on the spectral representation of unit SAPDO B.

Moreover, for all $\frac{dx(s)}{ds}\in D(B)$ we can write

$$\frac{d^3x(s)}{ds^3}+B\frac{dx(s)}{ds}=\left(\frac{d^2}{ds^2}+B\right)\frac{d}{ds}x(s).$$

Therefore, problem (1) be rewritten as the equivalent IVP

$$\begin{cases} \dfrac{dy(s)}{ds}=x(s), \\ \dfrac{d^2x(s)}{ds^2}+Bx(s)=cBy(s-z)+h(s),\ 0<s<\infty, \\ y(s)=k(s), -z\leq s\leq 0 \end{cases}$$

for the system of linear differential equations. Integrating these equations, we can write

$$\begin{cases} y(s)=y(0)+\int_0^s x(r)dr, \\ x(s)=E\left(s\right)x\left(0\right)+T\left(s\right)x'\left(0\right)+\int_0^s T\left(s-r\right)\left[cBk(r-z)+h(r)\right]dr \end{cases}$$

for all $s\in[0,z]$ and

$$\begin{cases} y(s)=y(mz)+\int_{mz}^s x(r)dr, \\ x(s)=E\left(s-mz\right)x\left(mz\right)+T\left(s-mz\right)x'\left(mz\right)+\int_{mz}^s T\left(s-r\right)\left[cBy(r-z)+h(r)\right]dr \end{cases}$$

for all $s\in\left[mz,(m+1)\,z\right],\ m=1,2,\ldots$

Applying (2) and (3), we can write

$$\int_0^s T(r)drx = -B^{-1}\left(E\left(s\right)-I\right)x.$$

From that and equation $\dfrac{dy(s)}{ds}=x(s)$ it follows $x\left(mz\right)=y'\left(mz\right), x'\left(mz\right)=y''\left(mz\right)$ and

$$y(s)=\begin{cases} y\left(0\right)+T\left(s\right)y'\left(0\right)-B^{-1}\left(E\left(s\right)-I\right)y''\left(0\right)+ \\ +\int_0^s B^{-1}\left(I-E\left(s-r\right)\right)\left[cBk(r-z)+h(r)\right]dr,\ s\in\left[0,z\right], \\ y(mz)+T\left(s-mz\right)y'\left(mz\right)-B^{-1}\left(E\left(s-mz\right)-I\right)y''\left(mz\right)+ \\ +\int_{mz}^s B^{-1}\left(I-E\left(s-r\right)\right)\left[cBy(r-z)+h(r)\right]dr,\ s\in\left[mz,\ \left(m+1\right)z\right],\ m=1,\ldots \end{cases} \tag{5}$$

The main theorem is formulated below.

Theorem 1. Assume that $k(s)$ be a twice continuously differentiable function and $k^0(s)\in D(B^{(3)/2})$, $k^1(s)\in D(B^{(2)/2})$, $k^2(s)\in D(B^{(1)/2})$. Then the following estimates hold for the solution of problem (1):

$$\max_{0\le s\le z}\left\|B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2}\right\|_G,\ \max_{0\le s\le z}\left\|B\frac{dy(s)}{ds}\right\|_G,\ \frac{1}{2}\max_{0\le s\le z}\left\|B^{\frac{3}{2}}y(s)\right\|_G \tag{6}$$

$$\le\left(2+|c|\,z\right)a_0+\int_0^z\left\|B^{\frac{1}{2}}h(r)\right\|_G dr,$$

$$a_0=\max\left\{\max_{-z\le s\le 0}\left\|B^{\frac{1}{2}}\frac{d^2k(s)}{ds^2}\right\|_G,\ \max_{-z\le s\le 0}\left\|B\frac{dk(s)}{ds}\right\|_G,\ \max_{-z\le s\le 0}\left\|B^{\frac{3}{2}}k(s)\right\|_G\right\},$$

$$\max_{mz\le s\le(m+1)z}\left\|B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2}\right\|_G,\ \max_{mz\le s\le(m+1)z}\left\|B\frac{dy(s)}{ds}\right\|_G,\ \frac{1}{2}\max_{mz\le s\le(m+1)z}\left\|B^{\frac{3}{2}}y(s)\right\|_G \tag{7}$$

$$\le\left(2+|c|\,z\right)a_m+\sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr,$$

$$a_m=\max\left\{\max_{(m-1)z\le s\le mz}\left\|B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2}\right\|_G,\ \max_{(m-1)z\le s\le mz}\left\|B\frac{dy(s)}{ds}\right\|_G,\ \max_{(m-1)z\le s\le mz}\left\|B^{\frac{3}{2}}y(s)\right\|_G\right\},$$

$$m=1,2,\ldots$$

Proof. Let $s\in[0,z]$. Then, applying (5), we get

$$y(s)=k\left(0\right)+T\left(s\right)k'\left(0\right)-B^{-1}\left(E\left(s\right)-I\right)k''\left(0\right)$$

$$+\int_0^s B^{-1}\left(I-E\left(s-r\right)\right)\left[cBk(r-z)+h(r)\right]dr,$$

$$B\frac{dy(s)}{ds}=E\left(s\right)Bk'\left(0\right)+T\left(s\right)Bk''\left(0\right)$$

$$+\int_0^s BT\left(s-r\right)\left[cBk(r-z)+h(r)\right]dr,$$

$$B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2}=-B^{\frac{1}{2}}T\left(s\right)Bk'\left(0\right)+E\left(s\right)B^{\frac{1}{2}}k''\left(0\right)$$

$$+\int_0^s B^{\frac{1}{2}} E\left(s-r\right)\left[cBk(r-z)+h(r)\right]dr.$$

Using these formulas, estimates (4) and the triangle inequality, we get

$$\left\|B^{\frac{3}{2}} y(s)\right\|_G \le \left\|B^{\frac{3}{2}} k(0)\right\|_G + \left\|Bk'\left(0\right)\right\|_G + 2\left\|B^{\frac{1}{2}} k''\left(0\right)\right\|_G$$

$$+2\left|c\right| z \max_{-z\le s\le 0}\left\|B^{\frac{3}{2}} k(s)\right\|_G + 2\int_0^s \left\|B^{\frac{1}{2}} h(r)\right\|_G dr$$

$$\le 2\left(2+\left|c\right| z\right) a_0 + 2\int_0^z \left\|B^{\frac{1}{2}} h(r)\right\|_G dr,$$

$$\left\|B\frac{dy(s)}{ds}\right\|_G \le \left\|Bk'\left(0\right)\right\|_G + \left\|B^{\frac{1}{2}} k''\left(0\right)\right\|_G$$

$$+\left|c\right| z \max_{-z\le s\le 0}\left\|B^{\frac{3}{2}} k(s)\right\|_G + \int_0^s \left\|B^{\frac{1}{2}} h(r)\right\|_G dr$$

$$\le \left(2+\left|c\right| z\right) a_0 + \int_0^z \left\|B^{\frac{1}{2}} h(r)\right\|_G dr,$$

$$\left\|B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2}\right\|_G \le \left\|B^{\frac{1}{2}} k''\left(0\right)\right\|_G$$

$$+\left|c\right| z \max_{-z\le s\le 0}\left\|B^{\frac{3}{2}} k(s)\right\|_G + \int_0^s \left\|B^{\frac{1}{2}} h(r)\right\|_G dr$$

$$\le \left(2+\left|c\right| z\right) a_0 + \int_0^z \left\|B^{\frac{1}{2}} h(r)\right\|_G dr$$

for $s\in[0,z]$. From that estimate (6) follows. Let $s\in\left[mz,\left(m+1\right)z\right],\ m=1,2,\dots$ Then, applying (5), we get

$$y(s)=y(mz)+T\left(s-mz\right)y'\left(mz\right)-B^{-1}\left(D\left(s-mz\right)-I\right)y''\left(mz\right)$$

$$+\int_{mz}^s B^{-1}\left(I-D\left(s-r\right)\right)\left[cBy(r-z)+h(r)\right]dr,$$

$$B\frac{dy(s)}{ds}=D\left(s-mz\right)By'\left(mz\right)+T\left(s-mz\right)By''\left(mz\right)$$

$$+\int_{mz}^s BT\left(s-r\right)\left[cBy(r-z)+h(r)\right]dr,$$

$$B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2}=-B^{\frac{1}{2}}T\left(s-mz\right)By'\left(mz\right)+D\left(s-mz\right)B^{\frac{1}{2}}y''\left(mz\right)$$

$$+\int_{mz}^s B^{\frac{1}{2}} D\left(s-r\right)\left[cBy(r-z)+h(r)\right]dr.$$

Using these formulas, estimates (4) and the triangle inequality, we get

$$\left\|B^{\frac{3}{2}} y(s)\right\|_G \le \left\|B^{\frac{3}{2}} y(mz)\right\|_G + \left\|By'\left(mz\right)\right\|_G + 2\left\|B^{\frac{1}{2}} y''\left(mz\right)\right\|_G$$

$$+2\left|c\right| z \max_{(m-1)z\le s\le mz}\left\|B^{\frac{3}{2}} y(s)\right\|_G + 2\sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}} h(r)\right\|_G dr$$

$$\leq 2\left(2+|c|\,z\right)a_m + 2\sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\left\|B\frac{dy(s)}{ds}\right\|_G \leq \left\|By'\left(mz\right)\right\|_G + \left\|B^{\frac{1}{2}}y''\left(mz\right)\right\|_G$$

$$+\,|c|\,z \max_{(m-1)z\leq s\leq mz}\left\|B^{\frac{3}{2}}y(s)\right\|_G + \sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\leq \left(2+|c|\,z\right)a_m + \sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\left\|B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2}\right\|_G \leq \left\|By'\left(mz\right)\right\|_G + \left\|B^{\frac{1}{2}}y''\left(mz\right)\right\|_G$$

$$+\,|c|\,z \max_{(m-1)z\leq s\leq mz}\left\|B^{\frac{3}{2}}y(s)\right\|_G + \sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

$$\leq \left(2+|c|\,z\right)a_m + \sum_{j=1}^{m+1}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr$$

for $s\in[mz,(m+1)\,z],\ m=1,2,\ldots$ Estimate (7) follows from it. Theorem 1 is proved.

According to Theorem 1, the following stability estimate holds for the solution of problem (1):

$$\max_{0\leq s\leq(m+1)z}\left\|B^{\frac{1}{2}}\frac{d^2y(s)}{ds^2}\right\|_G,\ \max_{0\leq s\leq(m+1)z}\left\|B\frac{dy(s)}{ds}\right\|_G,\frac{1}{2}\max_{0\leq s\leq(m+1)z}\left\|B^{\frac{3}{2}}y(s)\right\|_G$$

$$\leq \left(2+|c|\,z\right)^m a_0 + \sum_{j=1}^{m}\left(2+|c|\,z\right)^{m-j}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r)\right\|_G dr.$$

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2 Applications

The applications of Theorem 1 are considered in this section.

First, the initial nonlocal BVP for the third order PDE with TD

$$\begin{cases}\frac{\partial^3 x(s,u)}{\partial s^3}-\left(b(u)x_{su}(s,u)\right)_u+\rho x_s(s,u)=c\left(-\left(b(u)x_u(s-z,u)\right)_u+\rho x(s-z,u)\right)+h(s,u),\\ 0<s<\infty,\ 0<u<1,\\ x(s,u)=k(s,u),\ -z\leq s\leq 0,\ 0\leq u\leq 1,\\ x(s,0)=x(s,1),\ x_u(s,0)=x_u(s,1),\ 0\leq s<\infty\end{cases}\qquad(8)$$

is considered. Problem (8) has a unique solution $x(s,u)$, under compatibility conditions, for the smooth functions $b(u)\geq b>0,\ u\in(0,1),\ \rho>0,\ b(1)=b(0),\ k(s,u)\ -z\leq s\leq 0,\ 0\leq u\leq 1,\ h(s,u),$ $0<s<\infty, 0<u<1$, and $c\in R^1$. This allows us to reduce the BVP (8) to the IVP (1) in a Hilbert space $G=L_2[0,1]$ with a SAPDO B^u defined by the formula:

$$B^u x(u) = -(b(u)x_u)_u + \rho x$$

with domain

$$D(B^u) = \{x(u) : x(u),\ x_u(u),\ (b(u)x_u)_u \in L_2[0,1],\ x(1) = x(0),\ x_u(1) = x_u(0)\}.$$

By utilizing the symmetry property of the spatial operator B^u, domain of which is $D(B^u) \subset W_2^2[0,1]$, and incorporating the estimates from Theorem 1, the following theorem concerning the stability of problem (8) is obtained.

Theorem 2. The solutions to problem (8) satisfy the stability estimates that follow:

$$\max_{0\le s\le mz}\left\|\frac{d^2y(s,\cdot)}{ds^2}\right\|_{W_2^1[0,1]},\ \max_{0\le s\le mz}\left\|\frac{dy(s,\cdot)}{ds}\right\|_{W_2^2[0,1]},\ \frac{1}{2}\max_{0\le s\le mz}\|y(s,\cdot)\|_{W_2^3[0,1]}$$

$$\le M_1\left[(2+|c|z)^m a_0 + \sum_{j=1}^{m}(2+|c|z)^{m-j}\int_{(j-1)z}^{jz}\left\|B^{\frac{1}{2}}h(r,\cdot)\right\|_{L_2[0,1]}dr\right],$$

$$a_0 = \max\left\{\max_{-z\le s\le 0}\left\|\frac{d^2k(s,\cdot)}{ds^2}\right\|_{W_2^1[0,1]},\ \max_{-z\le s\le 0}\left\|\frac{dk(s,\cdot)}{ds}\right\|_{W_2^2[0,1]},\ \max_{-z\le s\le 0}\|k(s,\cdot)\|_{W_2^3[0,1]}\right\},$$

where M_1 does not depend on $k(s,u)$ and $h(s,u)$.

In this context, $W_2^1[0,1]$, $W_2^2[0,1]$ and $W_2^3[0,1]$ are Sobolev spaces consisting of all square integrable functions $\phi(u)$ defined on the interval $[0,1]$, endowed with the following norm:

$$\|\phi\|_{W_2^\zeta[0,1]} = \left(\int_0^1\sum_{j=0}^{\zeta}\left(\phi^{(j)}(u)\right)^2du\right)^{\frac{1}{2}},\quad \zeta = 1,2,3.$$

Next, let Ω represent the unit open cube in the n-dimensional Euclidean space R^n, where $u = (u_1,\dots,u_n)$ and $0 < u_\zeta < 1$ for $\zeta = 1,\dots,n$. The boundary of this domain is denoted by P, and we define $\overline{\Omega} = \Omega \cup P$. Within the domain $[0,\infty)\times\Omega$, we consider the initial BVP for a third-order multi-dimensional PDE with a TD, subject to Dirichlet boundary conditions.

$$\begin{cases} \frac{\partial^3 x(s,u)}{\partial s^3} - \sum\limits_{t=1}^{n}(b_t(u)x_{su_t}(s,u))_{u_t} = -c\sum\limits_{t=1}^{n}(b_t(u)x_{u_t}(s-z,u))_{u_t}, \\ 0 < s < \infty,\ u \in \Omega, \\ x(s,u) = k(s,u),\ -z \le s \le 0,\ u \in \overline{\Omega}, \\ x(s,u) = 0,\ u \in P,\ 0 \le s < \infty \end{cases} \tag{9}$$

is considered. Here $b_t(u) \ge b > 0$, $(u \in \Omega)$, $k(s,u)$, $-z \le s \le 0$, $u \in \overline{\Omega}$, $h(s,u)$, $0 < s < \infty$, $u \in \Omega$ are given smooth functions, and $c \in R^1$.

We consider the Hilbert space $L_2(\overline{\Omega})$ of all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\| h \|_{L_2(\overline{\Omega})} = \left(\int_{u\in\overline{\Omega}}|h(u)|^2du_1\cdots du_n\right)^{\frac{1}{2}}.$$

Problem (9) has a unique solution $x(s,u)$, under compatibility conditions, for the smooth functions $b(u) \ge b > 0$, $u \in \Omega$, $\rho > 0$, $k(s,u)$, $-z \le s \le 0$, $u \in \overline{\Omega}$, $h(s,u)$, $0 < s < \infty$, $u \in \Omega$, and $c \in R^1$. With this problem (9) can be reduced to the IVP (1) in the Hilbert space $G = L_2(\overline{\Omega})$ with a SAPDO B^u defined by the formula

$$B^u x(u) = -\sum_{t=1}^{n} (b_t(u) x_{u_t})_{u_t} \tag{10}$$

with domain

$$D(B^u) = \{x(u) : x(u), x_{u_t}(u), (b_t(u)x_{u_t})_{u_t} \in L_2(\Omega), \quad 1 \le t \le n, \quad x(u) = 0, u \in P\}.$$

As a result, we can establish the following theorem concerning the stability of problem (9).

Theorem 3. The following stability estimates are derived for the solutions of problem (9):

$$\max_{0\le s\le mz} \left\| \frac{d^2 y(s,\cdot)}{ds^2} \right\|_{W_2^1(\overline{\Omega})}, \max_{0\le s\le mz} \left\| \frac{dy(s,\cdot)}{ds} \right\|_{W_2^2(\overline{\Omega})}, \frac{1}{2} \max_{0\le s\le mz} \|y(s,\cdot)\|_{W_2^3(\overline{\Omega})}$$

$$\le M_2 \left[(2+|c|\,z)^n b_0 + \sum_{j=1}^{m} (2+|c|\,z)^{m-j} \int_{(j-1)\omega}^{jz} \left\| B^{\frac{1}{2}} h(s,\cdot) \right\|_{L_2(\overline{\Omega})} ds \right],$$

$$b_0 = \max\left\{ \max_{-z\le s\le 0} \left\| \frac{d^2 k(s,\cdot)}{ds^2} \right\|_{W_2^1(\overline{\Omega})}, \max_{-z\le s\le 0} \left\| \frac{dk(s,\cdot)}{ds} \right\|_{W_2^2(\overline{\Omega})}, \max_{-z\le s\le 0} \|k(s,\cdot)\|_{W_2^3(\overline{\Omega})} \right\},$$

where M_2 does not depend on $k(s,u)$ and $h(s,u)$. Here, $W_2^1(\overline{\Omega}), W_2^2(\overline{\Omega})$ and $W_2^3(\overline{\Omega})$ are Sobolev spaces of all square integrable functions $\phi(u)$ defined on $\overline{\Omega}$, equipped with the norm

$$\|\phi\|_{W_2^\zeta(\overline{\Omega})} = \left(\int \cdots \int_{u\in\overline{\Omega}} \sum_{j=0}^{\zeta} \sum_{t=1}^{n} \left(\phi_{\underbrace{u_t \cdots u_t}_{j \text{ times}}}(u) \right)^2 du_1 \cdots du_n \right)^{\frac{1}{2}}.$$

The proof of Theorem 3 is based on Theorem 1 and the symmetry property of the operator B^u defined by formula (10) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 4. For the solution of the elliptic differential problem [10]:

$$\begin{cases} B^u x(u) = \mu(u), u \in \Omega, \\ x(u) = 0, u \in P, \end{cases}$$

the following coercivity inequality holds:

$$\sum_{t=1}^{m} \|x_{u_t u_t}\|_{L_2(\overline{\Omega})} \le M_3 \|\mu\|_{L_2(\overline{\Omega})}.$$

Here, M_3 does not depend on $\mu(u)$.

Third, in $[0,\infty)\times\Omega$, the BVP for the multi-dimensional Schrödinger equation with TD and Neumann boundary condition is considered:

$$\begin{cases} \frac{\partial^3 x(s,u)}{\partial s^3} - \sum\limits_{t=1}^{m} (b_t(u) x_{s u_t}(s,u))_{u_t} + \rho x_s(s,u) = c\left(-\sum\limits_{t=1}^{m} (b_t(u) x_{u_t}(s-z,u))_{u_t} + \rho x(s-z,u) \right), \\ 0 < s < \infty, u \in \Omega, \\ x(s,u) = k(s,u), -z \le s \le 0, u \in \overline{\Omega}, \\ \frac{\partial x(s,u)}{\partial \vec{m}} = 0, u \in P, 0 \le s < \infty. \end{cases} \tag{11}$$

Here, $\vec{m}$ is the normal vector to P, $b_t(u) \geq b > 0, (u \in \Omega)$, $k(s,u)$, $-z \leq s \leq 0, 0 \leq u \leq 1$ and $h(s,u), 0 < s < \infty, 0 < u < 1$ are given smooth functions, and $c \in R^1$.

Problem (11) has a unique solution $x(s,u)$, under compatibility conditions, for the smooth functions $\varphi(u)$ and $b_t(u)$. This enables us to simplify problem (11) into the IVP in the Hilbert space $G = L_2(\overline{\Omega})$ with a SAPDO B^u, defined by the following expression:

$$B^u x(u) = -\sum_{t=1}^{m} (b_t(u) x_{u_t})_{u_t} + \rho x$$

having domain:

$$D(B^u) = \left\{ x(u) : x(u), x_{u_t}(u), (b_t(u) x_{u_t})_{u_t} \in L_2(\overline{\Omega}),\ 1 \leq t \leq m,\ \frac{\partial x(u)}{\partial \vec{m}} = 0,\ u \in P \right\}.$$

Therefore, estimates of Theorem 1 with $G = L_2(\overline{\Omega})$ allow us to state the following theorem on stability of problem (11).

Theorem 5. The following stability estimates hold for the solutions of problem (11):

$$\max_{0 \leq s \leq mz} \left\| \frac{d^2 y(s,\cdot)}{ds^2} \right\|_{W_2^1(\overline{\Omega})}, \max_{0 \leq s \leq mz} \left\| \frac{dy(s,\cdot)}{ds} \right\|_{W_2^2(\overline{\Omega})}, \frac{1}{2} \max_{0 \leq s \leq mz} \| y(s,\cdot) \|_{W_2^3(\overline{\Omega})}$$

$$\leq M_4 \left[(2 + |c|\, z)^m b_0 + \sum_{j=1}^{m} (2 + |c|\, z)^{m-j} \int_{(j-1)z}^{jz} \left\| B^{\frac{1}{2}} h(r,\cdot) \right\|_{L_2(\overline{\Omega})} dr \right],$$

$$b_0 = \max \left\{ \max_{-z \leq s \leq 0} \left\| \frac{d^2 k(s,\cdot)}{ds^2} \right\|_{W_2^1(\overline{\Omega})}, \max_{-z \leq s \leq 0} \left\| \frac{dk(s,\cdot)}{ds} \right\|_{W_2^2(\overline{\Omega})}, \max_{-z \leq s \leq 0} \| k(s,\cdot) \|_{W_2^3(\overline{\Omega})} \right\},$$

where M_4 does not depend on $\varphi(u)$.

The proof of Theorem 5 is based on the stability estimates from Theorem 1, where $G = L_2(\overline{\Omega})$, as well as the symmetry property of the operator B^u defined in formula (11) together with the next theorem regarding the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 6. For the solution of the elliptic differential problem [20],

$$\begin{cases} B^u x(u) = \mu(u), & u \in \Omega, \\ \frac{\partial x(u)}{\partial \vec{m}} = 0, & u \in P, \end{cases}$$

the coercivity inequality that follows holds:

$$\sum_{t=1}^{m} \| x_{u_t u_t} \|_{L_2(\overline{\Omega})} \leq M_5 \| \mu \|_{L_2(\overline{\Omega})}.$$

Here, M_5 is independent of $\mu(u)$.

3 Conclusion

In this paper, we examine the IVP for a third-order PDE with TD in a Hilbert space. We establish a key theorem concerning the stability of this problem and demonstrate its applications. Additionally, some of the results discussed here, albeit without proofs, were previously published in [21].

Using this method, we can investigate the IVP for the nonlinear third order PDE with TD

$$\begin{cases} \frac{d^3y(s)}{ds^3} + B\frac{dy(s)}{ds} = h(s, y(s-z)),\ 0 < s < \infty, \\ y(s) = k(s),\ -z \le s \le 0 \end{cases}$$

in G, a Hilbert space, having SAPDO B, $B \ge \lambda I$, where $\lambda > 0$. Here $k(s)$ defined on $[-z, 0]$ is the given ACF with values in $D(B)$.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Amirov, S., & Kozhanov, A.I. (2013). Mixed boundary value problem for a class of strongly nonlinear Sobolev-type equations of higher order. *Doklady Mathematics, 88*(5), 446–448. https://doi.org/10.1134/S1064562413040236

2 Ashyralyev, A., & Belakroum, K. (2022). A stable difference scheme for a third-order partial differential equation. *Journal of Mathematical Sciences, 260*(4), 399–417. https://doi.org/10.1007/s10958-022-05702-5

3 Niu, J., & Li, P. (2014). Numerical algorithm for the third-order partial differential equation with three-point boundary value problem. *Abstract and Applied Analysis, 2014*(40), 1–7. https://doi.org/10.1155/2014/630671

4 Afuwape, A.U., & Omeike, M.O. (2010). Stability and boundedness of solutions of a kind of third-order delay differential equations. *Journal of Computational and Applied Mathematics, 29*(3), 329–342. https://doi.org/10.1590/S1807-03022010000300001

5 Al Themairi, A., Qaraad, B., Bazighifan, O., & Nonlaopon, K. (2022). Third-order neutral differential equations with damping and distributed delay: new asymptotic properties of solutions. *Symmetry, 14*(10), 1–13. https://doi.org/10.3390/sym14102192

6 Ashyralyev, A., & Ibrahim, S. (2023). A numerical algorithm for the third-order delay partial differential equation with involution and Neumann boundary condition. *AIP Conference Proceedings, 2781*(1), 1–6. https://doi.org/10.1063/5.0145923

7 Baculíková, B., Džurina, J., & Rogovchenko, Y.V. (2012). Oscillation of third-order trinomial delay differential equations. *Applied Mathematics and Computation, 218*(13), 7023–7033. https://doi.org/10.1016/j.amc.2011.12.049

8 Cahlon, B., & Schmidt, D. (2006). Stability criteria for certain third-order delay differential equations. *Journal of Computational and Applied Mathematics, 188*(2), 319–335. https://doi.org/10.1016/j.cam.2005.04.034

9 Grace, S.R. (2015). Oscillation criteria for a third-order nonlinear delay differential equations with time delay. *Opuscula Mathematica, 35*(4), 485–497. http://dx.doi.org/10.7494/OpMath.2015.35.4.485

10 Pikina, P.A. (2017). Predictive time optimal algorithm for a third-order dynamical system with delay. *Journal of Physics: Conference Series, 891*(1), 9–11. https://dx.doi.org/10.1088/1742-6596/891/1/012278

11 Xiang, H. (2016). Oscillation of the third-order nonlinear neutral differential equations with distributed time delay. *Italian Journal of Pure and Applied Mathematics, 36*(2016), 769–782.

12 Agirseven, D. (2012). Approximate solutions of delay parabolic equations with the Dirichlet condition. *Abstract and Applied Analysis, 2012*, 682752. https://doi.org/10.1155/2012/ 682752

13 Ardito, A., & Ricciardi, P. (1980). Existence and regularity for linear delay partial differential equations. *Nonlinear Analysis, 4*(2), 411–414. https://doi.org/10.1016/0362-546X(80)90064-4

14 Ashyralyev, A., & Agirseven, D. (2013). On convergence of difference schemes for delay parabolic equations. *Computers and Mathematics with Applications, 66*(7), 1232–1244. https://doi.org/10.1016/j.camwa.2013.07.018

15 Ashyralyev, A., & Agirseven, D. (2014). Well-posedness of delay parabolic difference equations. *Advances in Difference Equations, 2014*(18), 1–20. https://doi.org/10.1186/1687-1847-2014-18

16 Ashyralyev, A., & Sobolevskii, P.E. (2004). *New difference schemes for partial differential equations. Operator theory advances and applications.* Basel, Boston, Berlin: Birkhäuser Verlag Press.

17 Blasio, G.D. (2003). Delay differential equations with unbounded operators acting on delay terms. *Nonlinear Analysis, 52*(1), 1–18. https://doi.org/10.1016/S0362-546X(01)00868-9

18 Sinestrari, E. (1984). On a class of retarded partial differential equations. *Mathematische Zeitschrift, 186*(1), 223–246. https://doi.org/10.1007/BF01161806

19 Fattorini, H.O. (1985). Second-order linear differential equations in Banach spaces. Amsterdam: Elsevier Science Publishers B.V.

20 Sobolevskii, P.E., & Ashyralyev, A. (1975). *Difference methods for the approximate solution of differential equations.* Voronezh, Russia: Voronezh State University Press.

21 Ashyralyev, A., Hincal, E., & Ibrahim, S. (2018). Stability of the third-order partial differential equations with time delay. *AIP Conference Proceedings, 1997*(1), 020086. https://doi.org/10.1063/1.5049080

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