https://doi.org/10.31489/2025M1/53-62 *Research article*

Gagliardo–Nirenberg type inequalities for smoothness spaces related to Morrey spaces over n-dimensional torus

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In the paper, the Gagliardo–Nirenberg type inequalities for smoothness spaces $B^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ of Nikol'skii–Besov type and spaces $F^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ of Lizorkin–Triebel type both related to Morrey spaces over n-dimensional torus for some range of the parameters s, p, q, τ were proved. These spaces are natural analogues of the spaces $B^{s\tau}_{pq}(\mathbb{R}^n)$ and $F^{s\tau}_{pq}(\mathbb{R}^n)$ in the case of multidimensional torus $\mathbb{T}^n$. The main results of the article are two theorems, each of which proves the Gagliardo–Nirenberg type inequality for the Lizorkin–Triebel type spaces or the Nikol'skii–Besov type spaces respectively.

Keywords: Nikol'skii–Besov/Lizorkin–Triebel smoothness spaces related to Morrey space, multidimensional torus, Gagliardo–Nirenberg type inequalities.

2020 Mathematics Subject Classification: 42A35, 41A63, 41A55.

Introduction

Multiplicative and additive inequalities for (partial) derivatives of functions play crucial role in different areas of Analysis and Applied Mathematics, in particular, in Analysis of Partial Differential Equations.

Multiplicative and additive inequalities for derivatives of functions in single variable (on an axis, a semi-axis, a segment, or a unit circle) with exact constants are an extensive section of modern function theory, originated from the classical works of J. Hadamard and A.N. Kolmogorov. The development of this field can be traced in surveys [1, 2].

In the case of functions in several variables, E. Gagliardo and L. Nirenberg proved important inequality, nowadays known as the Gagliardo–Nirenberg inequality (see [3; ch. III, sect. 15]):

Proposition 1. Let function u belong to $L_q(\mathbb{R}^n)$ and such that all its (distributional) derivatives of order $l(\in \mathbb{N})$ belong to $L_r(\mathbb{R}^n)$, with $1 \le q, r \le \infty$. Then for $0 \le j < l$, the following inequality

$$\sum_{|\alpha|=j} \|\partial^\alpha f \mid L_p(\mathbb{R}^n)\| \le C \|f \mid L_q(\mathbb{R}^n)\|^{1-\theta} \Big(\sum_{|\alpha|=l} \|\partial^\alpha f \mid L_r(\mathbb{R}^n)\| \Big)^{\theta} \qquad (1)$$

holds, where $\frac{1}{p} = \frac{j}{n} + (1-\theta)\frac{1}{q} + \theta\big(\frac{1}{r} - \frac{l}{n}\big)$ for all θ in the interval $[\frac{j}{l}, 1]$ (the positive constant C depending only on n, l, j, q, r, θ), with the following exceptional cases:

1. If $j = 0$, $rl < n$, $q = \infty$, then we make the additional assumption that either u tends to zero at infinity or $u \in L_{q^*}(\mathbb{R}^n)$ for some finite $q^* > 0$.

2. If $1 < r < \infty$ and $l - j - \frac{n}{r}$ is a nonnegative integer then inequality (1) holds only for θ satisfying $\frac{j}{l} \le \theta < 1$.

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This research was supported by Ministry of Science and Higher Education, Republic of Kazakhstan, under grant AP14869246.

Received: 30 August 2024; *Accepted:* 31 October 2024.

The multiplicative inequality (1) is equivalent to the corresponding additive inequality with an arbitrary parameter $\varepsilon > 0$:

$$\sum_{|\alpha|=j} \|\partial^\alpha f \mid L_p(\mathbb{R}^n)\| \le C\Big(\varepsilon^{-\frac{1}{1-\theta}} \|f \mid L_q(\mathbb{R}^n)\| + \varepsilon^{\frac{1}{\theta}} \sum_{|\alpha|=l} \|\partial^\alpha f \mid L_r(\mathbb{R}^n)\|\Big), \quad \forall \varepsilon > 0.$$

Note that under some particular assumptions, the inequality (1) and its additive analogue for some special cases of mixed L_p-, L_q- and L_r-norms were established by V.P. Il'in, L. Nirenberg and others; further, M. Troisi, V.A. Solonnikov and others obtained analogues of inequality (1) for the anisotropic case of specifying differential properties of functions in L_r (see details and general results in [3; ch. III, sect. 15]).

The classical Gagliardo–Nirenberg inequalities and their generalizations mentioned above are a very useful tool in connection with partial differential equations (see, for example, the monograph [3]). For this reason, there is also some interest in their analogues in various non-classical situations.

In 2001 H. Brezis and P. Mironescu [4] proved the following Gagliardo–Nirenberg type inequalities for the (isotropic) Lizorkin–Triebel spaces.

Proposition 2. (i) Let a tempered distribution f belongs to both Lizorkin–Triebel spaces $F^{s_0}_{p_0q_0}(\mathbb{R}^n)$ and $F^{s_1}_{p_1q_1}(\mathbb{R}^n)$, with $0 < p_0,\, p_1 < \infty$, $0 < q_0,\, q_1 \le \infty$, $-\infty < s_0 < s_1 < \infty$. Then for any $\theta : 0 < \theta < 1$ and $q : 0 < q \le \infty$, the following inequality

$$\|f \mid F^s_{pq}(\mathbb{R}^n)\| \le C\|f \mid F^{s_0}_{p_0q_0}(\mathbb{R}^n)\|^{1-\theta}\|f \mid F^{s_1}_{p_1q_1}(\mathbb{R}^n)\|^\theta \tag{2}$$

holds, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\quad s = (1-\theta)s_0 + \theta s_1$ (the positive constant C depending only on n, s_0, s_1, p_0, p_1, q_0, q_1, q, θ).

(ii) Let a tempered distribution f belongs to both the Lizorkin–Triebel spaces $F^{s_0}_{p_0q_0}(\mathbb{R}^n)$ and $F^{s_1}_{\infty\infty}(\mathbb{R}^n)$, with $0 < p_0 < \infty$, $0 < q_0 \le \infty, -\infty < s_0 \ne s_1 < \infty$. Then for any $\theta : 0 < \theta < 1$ and $q : 0 < q \le \infty$, the following inequality

$$\|f \mid F^s_{pq}(\mathbb{R}^n)\| \le C\|f \mid F^{s_0}_{p_0q_0}(\mathbb{R}^n)\|^{1-\theta}\|f \mid F^{s_1}_{\infty\infty}(\mathbb{R}^n)\|^\theta \tag{3}$$

holds, where $\frac{1}{p} = \frac{1-\theta}{p_0}$, $s = (1-\theta)s_0 + \theta s_1$ (the positive constant C depending only on $n, s_0, s_1, p_0, q_0, q, \theta$).

The analogues of the inequalities (2) and (3) for the (isotropic) Besov–Nikol'skii spaces are as follows.

Proposition 3. Let a tempered distribution f belong to both Nikol'skii–Besov spaces $B^{s_0}_{p_0q_0}(\mathbb{R}^n)$ and $B^{s_1}_{p_1q_1}(\mathbb{R}^n)$, with $0 < p_0,\, p_1 < \infty$, $0 < q_0,\, q_1 \le \infty$, $-\infty < s_0 < s_1 < \infty$. Then for any $\theta : 0 < \theta < 1$, the following inequality

$$\|f \mid B^s_{pq}(\mathbb{R}^n)\| \le C\|f \mid B^{s_0}_{p_0q_0}(\mathbb{R}^n)\|^{1-\theta}\|f \mid B^{s_1}_{p_1q_1}(\mathbb{R}^n)\|^\theta \tag{4}$$

holds, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $s = (1-\theta)s_0 + \theta s_1$ (the positive constant C depending only on $n, s_0, s_1, p_0, p_1, q_0, q_1, \theta$).

The inequality (4) is a classical result of J. Peetre, proved in middle of 1960s.

Note that for the Nikol'skii–Besov spaces the Gagliardo–Nirenberg type inequality (4) is established for full natural range of parameters $n, s_0, s_1, p_0, p_1,\, q_0, q_1, \theta$, in contrast to that inequality for the Lizorkin–Triebel spaces: here, there is a gap for the case where $0 < p_0 < \infty, \quad 0 < q_0 \le \infty$, $p_1 = \infty, \quad 0 < q_1 < \infty$.

Moreover, as can be seen from the inequalities (2) and (3), the result (Gagliardo–Nirenberg type inequality for the Lizorkin–Triebel spaces) is completely independent of the values of the "microscopic" parameters q, q_0, q_1. Unlike Lizorkin–Triebel type spaces, in the inequality (4) the parameter q is strictly connected with q_0 and q_1 like the other parameters.

The gap mentioned above was fulfilled by W. Sickel [5]:

Proposition 4. Let a tempered distribution f belongs to $F^{s_0}_{p_0q_0}(\mathbb{R}^n)\cap F^{s_1}_{\infty q_1}(\mathbb{R}^n)$, with $0<p_0<\infty$, $0<q_0\leq\infty$, $0<q_1<\infty$, $-\infty<s_0\neq s_1<\infty$. Then for any $\theta:0<\theta<1$ and $q:0<q\leq\infty$, the following inequality

$$\|f \mid F^{s}_{pq}(\mathbb{R}^n)\|\leq C\|f\mid F^{s_0}_{p_0q_0}(\mathbb{R}^n)\|^{1-\theta}\|f\mid F^{s_1}_{\infty\infty}(\mathbb{R}^n)\|^{\theta} \tag{5}$$

holds, where $\frac{1}{p}=\frac{1-\theta}{p_0}$, $s=(1-\theta)s_0+\theta s_1$ (the positive constant C depending only on $n,s_0,s_1,p_0,q_0,q,\theta$).

In fact, W. Sickel established the Gagliardo–Nirenberg type inequalities for two scales (of Nikol'skii–Besov) $B^{s\tau}_{pq}(\mathbb{R}^n)$ and (Lizorkin–Triebel) $F^{s\tau}_{pq}(\mathbb{R}^n)$ (with additional real parameter τ) of smoothness spaces related to Morrey spaces over whole Euclidean space $\mathbb{R}^n$ in full range of parameters involved. The inequalities from [5] contain the inequalities (2)–(5) as special cases because $B^{s0}_{pq}(\mathbb{R}^n)\equiv B^{s}_{pq}(\mathbb{R}^n)$ and $F^{s0}_{pq}(\mathbb{R}^n)\equiv B^{s}_{pq}(\mathbb{R}^n)$ in sense of equivalent (quasi)norms.

Goal of the paper is to prove the Gagliardo–Nirenberg type inequalities for the spaces $B^{s\tau}_{pq}(\mathbb{T}^n)$ and $F^{s\tau}_{pq}(\mathbb{T}^n)$, which are natural analogues of the spaces $B^{s\tau}_{pq}(\mathbb{R}^n)$ and $F^{s\tau}_{pq}(\mathbb{R}^n)$ in the case of multidimensional torus $\mathbb{T}^n$.

The rest of the paper is organized as follows. In Section 2 we introduce some notation, define the spaces of distributions $B^{s\tau}_{pq}(\mathbb{R}^n)$, $F^{s\tau}_{pq}(\mathbb{R}^n)$, $B^{s\tau}_{pq}(\mathbb{T}^n)$ and $F^{s\tau}_{pq}(\mathbb{T}^n)$ and formulate main results of the paper (Theorems 1 and 2). Section 3 contains the proof of crucial Lemma. Finally, in Section 4, we give proofs of Theorems 1 and 2.

1 *The Gagliardo–Nirenberg type inequalities for the smoothness spaces related to Morrey spaces*

First we introduce some notation and give definitions of (the two scales of) spaces of distributions under consideration.

Let $n\in\mathbb{N}$, $n\geq 2$, $\mathrm{z}_n=\{1,\ldots,n\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $x=(x_1,\ldots,x_n), y=(y_1,\ldots,y_n)\in\mathbb{R}^n$, we put $xy=x_1y_1+\ldots+x_ny_n$, $|x|=|x_1|+\ldots+|x_n|$, $|x|_\infty=\max(|x_\nu|:\nu\in\mathrm{z}_n)$; $x\leq y$ $(x<y)$ $\Leftrightarrow x_\nu\leq y_\nu$ $(x_\nu<y_\nu)$ for all $\nu\in\mathrm{z}_n$.

Let $\mathcal{S}:=\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'=\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of test functions and tempered distributions respectively; $\widehat{f}$ is Fourier transform for $f\in\mathcal{S}'(\mathbb{R}^n)$; in particular, for $\varphi\in\mathcal{S}$,

$$\widehat{\varphi}(\xi)=\int_{\mathbb{R}^n}\varphi(x)e^{-2\pi i\,\xi x}dx.$$

For $0<p\leq\infty$ and a measurable set $G\subset\mathbb{R}^n$, as usual, let $L_p(G)$ be the space of functions $f:G\to\mathbb{C}$ integrable in sense of Lebesgue to the power p (essentially bounded if $p=\infty$) over G, endowed with standard (quasi)norm (norm if $p\geq 1$)

$$\|\,f\,|\,L_p(G)\,\|=\Big(\int_G|\,f(x)\,|^p dx\Big)^{\frac{1}{p}}\ (p<\infty),$$

$$\|\,f\,|\,L_\infty(G)\,\|=\text{ess sup}(|\,f(x)\,|:x\in G).$$

For $0<q\leq\infty$ let $\ell_q:=\ell_q(\mathbb{N}_0)$ be the space of (complex–valued) sequences $(c_j)=(c_j:j\in\mathbb{N}_0)$ with finite standard (quasi)norm (norm if $q\geq 1$) $\|(c_j)\,|\,\ell_q\|$.

Further, let $\ell_q(L_p(G))$ ($L_p(G;\ell_q)$ respectively) be the space of function sequences $(g_j(x))=(g_j(x):k\in\mathbb{N}_0)$ $(x\in G)$ with finite (quasi)norm (norm if $p,q\geq 1$)

$$\|\,(g_j(x))\,|\,\ell_q(L_p(G))\,\|=\|\,(\,\|\,g_j\,|\,L_p(G)\|)\,|\,\ell_q\,\|,$$

$$(\|\,(g_j(x))\,|\,L_p(G;\ell_q)\,\|=\|\,\|\,(g_j(\cdot))\,|\,\ell_q\,\|\,|\,L_p(G)\|$$

respectively).

We choose a test function $\eta_0 \in \mathcal{S}$ such that

$$0 \le \widehat{\eta}_0(\xi) \le 1,\ \ \xi \in \mathbb{R}^n;\ \ \widehat{\eta}_0(\xi) = 1 \ \text{ if } |\xi|_\infty \le 1;\ \operatorname{supp} \widehat{\eta}_0 = \{\xi \in \mathbb{R}^n \,|\, |\xi|_\infty \le 2\}.$$

Put $\widehat{\eta}(\xi) = \widehat{\eta}_0(2^{-1}\xi) - \widehat{\eta}_0(\xi)$, $\widehat{\eta}_j(\xi) := \widehat{\eta}_j(\xi) = \widehat{\eta}(2^{1-j}\xi)$, $j \in \mathbb{N}$. Then

$$\sum_{j=0}^{\infty} \widehat{\eta}_j(\xi) \equiv 1,\ \ \xi \in \mathbb{R}^n,$$

i.e. $\{\widehat{\eta}_j(\xi) \,|\, j \in \mathbb{N}_0\}$ is the resolution of unity over $\mathbb{R}^n$. It is clear that

$$\eta(x) = 2^n \eta_0(2x) - \eta_0(x),\ \eta_j(x) := 2^{(j-1)n}\eta(2^{j-1}x),\ \ j \in \mathbb{N}.$$

We define the operator Δ_j^η on $\mathcal{S}'$ as follows: for $f \in \mathcal{S}'$ put

$$\Delta_j^\eta(f,x) = f * \eta_j(x) = \langle f, \eta_j(x-\cdot)\rangle.$$

Let $\mathcal{Q}$ be the set of all dyadic cubes in $\mathbb{R}^n$ of the form

$$Q = Q_{j\xi} = \{\, x \in \mathbb{R}^n \,:\, 2^j x - \xi \in [0,1)^n \,\}\ (j \in \mathbb{Z}, \xi \in \mathbb{Z}^n).$$

Denote by $j(Q)(=j)$ and $|Q|(=2^{-jm})$ "level" and the volume of the cube $Q = Q_{j\xi}$ respectively.

Now we recall important definition of the Lizorkin–Triebel space $F^s_{\infty\, q}(\mathbb{R}^n)$ $(0 < q < \infty)$, invented by M. Frazier and B. Jawerth [6].

Definition 1. Let $s \in \mathbb{R}$, $0 < q < \infty$. The space $F^s_{\infty\, q} := F^s_{\infty\, q}(\mathbb{R}^n)$ consists of all distributions $f \in \mathcal{S}'$ for which (quasi)norm

$$\| f \,|\, F^s_{\infty\, q} \| = \Big(\sup_{Q \in \mathcal{Q}: j(Q) \ge 0} \frac{1}{|Q|} \int_Q \sum_{j=j(Q)}^{\infty} |2^{sj}\Delta_j^\eta(f,x)|^q dx \Big)^{1/q}$$

is finite.

Further, denote by $\widetilde{\mathcal{S}}' := \mathcal{S}'(\mathbb{T}^n)$ the space of all distributions $f \in \mathcal{S}'$, 1–periodic in each variable (i.e. such that $\langle f, \varphi(\cdot + \xi)\rangle = \langle f, \varphi\rangle$ for all $\varphi \in \mathcal{S}$ and any $\xi \in \mathbb{Z}^n$), and $\tilde{\mathcal{S}} := \mathcal{S}(\mathbb{T}^n)$ the space of all infinitely differentiable functions over $\mathbb{T}^n$ endowed with the topology of uniform convergence of all partial derivatives over $\mathbb{T}^n$. Then $\mathcal{S}'(\mathbb{T}^n)$ is identified naturally with the space topologically dual to $\mathcal{S}(\mathbb{T}^n)$. It is known that $f \in \tilde{\mathcal{S}}'$ if and only if $\operatorname{supp} \widehat{f} \subset \mathbb{Z}^n$, i.e. $\widehat{f} = 0$ on open set $\mathbb{R}^n \backslash \mathbb{Z}^n$. Here $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ is n-dimensional torus.

Let $g : \mathbb{R}^n \to \mathbb{C}$ be an arbitrary function, then its periodization $\widetilde{g} : \mathbb{T}^n \to \mathbb{C}$ is defined as (at least formal) sum of series $\sum\limits_{\xi \in \mathbb{Z}^n} g(x+\xi)$.

By the Poisson summation formula it is easy to verify that for $\varphi \in \mathcal{S}$, $\widetilde{\varphi} \in \tilde{\mathcal{S}}$, and, moreover, $\widetilde{\varphi}(x) = \sum_{\xi \in \mathbb{Z}^n} \widehat{\varphi}(\xi) e^{2\pi i \xi x}$.

Now we define operators $\widetilde{\Delta}_j^\eta$ on $\widetilde{\mathcal{S}}'$ $(j \in \mathbb{N}_0)$ as follows: for $f \in \tilde{\mathcal{S}}'$, put

$$\widetilde{\Delta}_j^\eta(f,x) = f * \widetilde{\eta}_j(x) = \langle f, \widetilde{\eta}_j(x-\cdot)\rangle = \sum_{\xi \in \mathbb{Z}^n} \widehat{\eta}_j(\xi)\widehat{f}(\xi) e^{2\pi i\, \xi x}.$$

Let $(\mathbf{0} = (0,\dots,0), \mathbf{1} = (1,\dots,1) \in \mathbb{R}^n)$

$$\widetilde{\mathcal{Q}} = \{Q \in \mathcal{Q} \,|\, Q \subset Q_0 := [0,1)^n\} = \{Q_{j\xi} \,|\, j \in \mathbb{N}_0,\, \xi \in \mathbb{Z}^n : \mathbf{0} \le \xi < 2^j \mathbf{1}\}.$$

In analogy with 1, we give

Definition 2. Let $s \in \mathbb{R}$, $0 < q < \infty$. The Lizorkin–Triebel space $\widetilde{F}^s_{\infty\, q} := F^s_{\infty\, q}(\mathbb{T}^n)$ consists of all distributions $f \in \widetilde{\mathcal{S}}'$, for which (quasi)norm

$$\| f \,|\, \widetilde{F}^s_{\infty\, q} \| = \Big(\sup_{Q \in \widetilde{\mathcal{Q}}} \frac{1}{|Q|} \int_Q \sum_{j=j(Q)}^{\infty} |2^{sj} \widetilde{\Delta}^{\eta}_j(f,x)|^q dx \Big)^{1/q}$$

is finite.

Now we recall definitions of two scales (of Nikol'skii–Besov type) $B^{s\,\tau}_{p\,q}(\mathbb{R}^n)$ and (Lizorkin–Triebel type) $F^{s\,\tau}_{p\,q}(\mathbb{R}^n)$ of (inhomogeneous) smoothness spaces related to Morrey spaces and their periodic analogues $B^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ and $F^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ (below $t_+ := \max\{0, t\}$ if $t \in \mathbb{R}$).

Definition 3. Let $s, \tau \in \mathbb{R}$, $0 < p, q \le \infty$. Then

I. the Nikol'skii–Besov type space $B^{s\,\tau}_{p\,q} := B^{s\,\tau}_{p\,q}(\mathbb{R}^n)$ consists of all distributions $f \in \mathcal{S}'$, for which (quasi)norm

$$\| f \,|\, B^{s\,\tau}_{p\,q} \| = \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^{\tau}} \|(2^{sj} \Delta^{\eta}_j(f,x)(j+1-j(Q))^0_+) \,|\, \ell_q(L_p(Q))\|$$

is finite;

II. the Lizorkin–Triebel type space $F^{s\,\tau}_{p\,q} := F^{s\,\tau}_{p\,q}(\mathbb{R}^n)$ $(p < \infty)$ consists of all distributions $f \in \mathcal{S}'$, for which (quasi)norm

$$\| f \,|\, F^{s\,\tau}_{p\,q} \| = \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^{\tau}} \|(2^{sj} \Delta^{\eta}_j(f,x)(j+1-j(Q))^0_+) \,|\, L_p(Q; \ell_q)\|$$

is finite.

Remark 1. Inhomogeneous spaces $B^{s\,\tau}_{p\,q}$ and $F^{s\,\tau}_{p\,q}$ were introduced in [7] and have been studied thoroughly (see, in particular, [5, 7–10]). We also noted that (local) Morrey spaces and Nikol'skii–Besov–Morrey and Lizorkin–Triebel–Morrey spaces have been attracting a lot of attention, see, for instance, [5, 7–14].

Definition 4. Let $s, \tau \in \mathbb{R}$, $0 < p, q \le \infty$. Then

I. the Nikol'skii–Besov type space $\widetilde{B}^{s\,\tau}_{p\,q} := B^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ consists of all distributions $f \in \tilde{\mathcal{S}}'$, for which (quasi)norm

$$\| f \,|\, B^{s\,\tau}_{p\,q}(\mathbb{T}^n) \| = \sup_{Q \in \widetilde{\mathcal{Q}}} \frac{1}{|Q|^{\tau}} \|(2^{sj} \widetilde{\Delta}^{\eta}_j(f,x)(j+1-j(Q))^0_+) \,|\, \ell_q(L_p(Q))\|$$

is finite;

II. the Lizorkin–Triebel type space $\widetilde{F}^{s\,\tau}_{p\,q} := F^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ $(p < \infty)$ consists of all distributions $f \in \tilde{\mathcal{S}}'$, for which (quasi)norm

$$\| f \,|\, F^{s\,\tau}_{p\,q}(\mathbb{T}^n) \| = \sup_{Q \in \widetilde{\mathcal{Q}}} \frac{1}{|Q|^{\tau}} \|(2^{sj} \widetilde{\Delta}^{\eta}_j(f,x)(j+1-j(Q))^0_+) \,|\, L_p(Q; \ell_q)\|$$

is finite.

Remark 2. Obviously, the spaces $\widetilde{B}^{s\,0}_{p\,q}$ and $\widetilde{F}^{s\,0}_{p\,q}$ coincide with the isotropic periodic Nikol'skii–Besov spaces $\widetilde{B}^s_{p\,q}$ and Lizorkin–Triebel spaces $\widetilde{F}^s_{p\,q}$ respectively. Furthermore, it is not hard to see that for any $\tau \le 0$, we have coincidence $\widetilde{B}^{s\,\tau}_{p\,q} = \widetilde{B}^s_{p\,q}$ and $F^{s\,\tau}_{p\,q} = \widetilde{F}^s_{p\,q}$ in sense of equivalent (quasi)norms, in contrast to the spaces $B^{s\,\tau}_{p\,q}$ and $F^{s\,\tau}_{p\,q}$: as known, $B^{s\,\tau}_{p\,q} = \{0\}$ and $F^{s\,\tau}_{p\,q} = \{0\}$ when $\tau < 0$ (see [7]).

We noted that periodic Morrey spaces and Nikol'skii–Besov–Morrey and Lizorkin–Triebel–Morrey spaces have been attracting increasing attention as well, see, for instance, [15–18].

First, we consider the Gagliardo–Nirenberg type inequalities for the Lizorkin–Triebel type spaces $F^{s\,\tau}_{p\,q}(\mathbb{T}^n)$.

Theorem 1. Let $0 < q_0, q_1 \le \infty,\ -\infty < s_0 < s_1 < \infty,\ \tau_0, \tau_1 \ge 0$.

(i) Let $0 < p_0, p_1 < \infty$. Then for any $0 < \theta < 1$ and $0 < q \le \infty$, there exists constant $C > 0$ such that the inequality

$$\|f \mid F^{s\,\tau}_{p\,q}(\mathbb{T}^n)\| \le C\|f \mid F^{s_0\,\tau_0}_{p_0\,q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid F^{s_1\,\tau_1}_{p_1\,q_1}(\mathbb{T}^n)\|^{\theta}$$

is satisfied for all $f \in \mathcal{S}'(\mathbb{T}^n)$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\tau = (1-\theta)\tau_0 + \theta\tau_1$, $s = (1-\theta)s_0 + \theta s_1$.

(ii) Let $0 < p_0 < \infty$. Then for any $0 < \theta < 1$ and $0 < q \le \infty$, there exists constant $C > 0$ such that the inequality

$$\|f \mid F^{s\tau}_{pq}(\mathbb{T}^n)\| \le C\|f \mid F^{s_0\tau_0}_{p_0q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid B^{s_1\tau_1}_{\infty\infty}(\mathbb{T}^n)\|^{\theta}$$

holds for all $f \in \mathcal{S}'(\mathbb{T}^n)$, where $\frac{1}{p} = \frac{1-\theta}{p_0}$, $\tau = (1-\theta)\tau_0 + \theta\tau_1$, $s = (1-\theta)s_0 + \theta s_1$.

Remark 3. The proof given below is due to H. Brezis and P. Mironescu [4] for $\tau = 0$ and W. Sickel [5] for $\tau > 0$ in the non-periodic case of $\mathbb{R}^n$.

The Gagliardo–Nirenberg type inequalities for the Nikol'skii–Besov type spaces $B^{s\,\tau}_{p\,q}(\mathbb{T}^n)$ are as follows.

Theorem 2. Let $0 < p_0, p_1 \le \infty$, $0 < q_0, q_1 \le \infty$, $-\infty < s_0 < s_1 < \infty$, τ_0, $\tau_1 \ge 0$. Then for any $0 < \theta < 1$, there exists constant $C > 0$ such that the inequality

$$\|f \mid B^{s\,\tau}_{p\,q}(\mathbb{T}^n)\| \le C\|f \mid B^{s_0\,\tau_0}_{p_0\,q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid B^{s_1\,\tau_1}_{p_1\,q_1}(\mathbb{T}^n)\|^{\theta}$$

is valid for all $f \in \mathcal{S}'(\mathbb{T}^n)$ where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\tau = (1-\theta)\tau_0 + \theta\tau_1$, $s = (1-\theta)s_0 + \theta s_1$.

Remark 4. If we replace $\mathbb{T}^n$ by $\mathbb{R}^n$ in Theorems 1 and 2, we obtain an exact formulation of above–mentioned W. Sickel's results for $B^{s\,\tau}_{p\,q}(\mathbb{R}^n)$ and $F^{s\,\tau}_{p\,q}(\mathbb{R}^n)$.

2 Crucial Lemma

Key ingredient in what follows is the following inequality of F. Oru (see Lemma 3.7 in [4]).

Lemma 1. Let $0 < \theta < 1$, $-\infty < s_0, s_1 < \infty$, $s = (1-\theta)s_0 + \theta s_1$, $0 < q \le \infty$. Then there exists $C = C(s_0, s_1, \theta, q) > 0$ such that for any sequence $(a_j)_j$ of complex numbers the inequality

$$\|(2^{sj}a_j)_j \mid \ell_q\| \le C\|(2^{s_0j}a_j)_j \mid \ell_\infty\|^{1-\theta}\|(2^{s_1j}a_j)_j \mid \ell_\infty\|^{\theta} \tag{6}$$

holds true.

For completeness, we present the proof of Lemma 1 from [4].

Proof. Let $C_1 = \sup 2^{s_1j} \mid a_j \mid$, $C_2 = \sup 2^{s_2j} \mid a_j \mid$, so that $C_1 \le C_2$. We will assume that $C_1 > 0$, otherwise there is nothing to prove. Since $s_1 < s_2$, there exists some $j_0 > 0$ such that

$$\min\left\{\frac{C_1}{2^{s_1j}}, \frac{C_2}{2^{s_2j}}\right\} = \begin{cases} \frac{C_1}{2^{s_1j}}, & j \le j_0, \\ \frac{C_2}{2^{s_2j}}, & j > j_0. \end{cases}$$

Since $\frac{C_1}{2^{s_1j}} \le \frac{C_2}{2^{s_2j}}$ and $\frac{C_2}{2^{s_2(j_0+1)}} \le \frac{C_1}{2^{s_1(j_0+1)}}$, we find

$$C_2 \sim C_1 2^{(s_2-s_1)j_0}.$$

Therefore,

$$\|(2^{s_1j}a_j) \mid \ell_\infty\|^{\theta}\|(2^{s_2j}a_j) \mid \ell_\infty\|^{1-\theta} \sim C_1 2^{(s_2-s_1)j_0(1-\theta)}. \tag{7}$$

On the other hand, we have $a_j \le \min\left\{\frac{C_1}{2^{s_1 j}}, \frac{C_2}{2^{s_2 j}}\right\}$, so that

$$a_j \le \frac{C_1}{2^{s_1 j}} \quad \text{for } 0 \le j \le j_0, \quad a_j \le \frac{C_2}{2^{s_2 j}} \text{ for } j > j_0.$$

It follows that

$$\|(2^s_j a_j) \,|\, \ell_q\| \le \left(\sum_{j \le j_0} C_1^q 2^{(s-s_1)jq} + \sum_{j > j_0} C_2^q 2^{(s-s_2)jq} \right)^{1/q} \le$$

$$\le C \left(\sum_{j \le j_0} C_1^q 2^{(s-s_1)jq} + \sum_{j > j_0} C_1^q 2^{-\theta(s_2-s_1)jq+(s_2-s_1)j_0 q} \right)^{1/q}.$$

Therefore,

$$\|(2^s_j a_j) \,|\, \ell_q\| \le C C_1 2^{(s_2-s_1)j_0(1-\theta)}.$$

Finally, we find that the inequality

$$\|(2^{sj} a_j) \,|\, \ell_q\| \le C C_1 2^{(s_2-s_1)j_0(1-\theta)}. \tag{8}$$

Now (6) follows from (7) and (8). Thus, Lemma 1 is completely proved.

3 *Proofs of Theorems 1 and 2*

Proof of Theorem 1.

Proof. As mentioned above, the line of argument follows [4]. First, we prove (i). Successively applying Lemma 1 with $a_j = |\widetilde{\Delta}^\eta_j(f,x)|$, Holder's integral inequality (with exponents $P_0 = \frac{p_0}{(1-\theta)p}$ and $P_1 = \frac{p_1}{\theta p}$) and Jensen's inequalities ($\|\cdot \,|\, \ell_{q_0}\| \ge \|\cdot \,|\, \ell_\infty\|$ and $\|\cdot \,|\, \ell_{q_1}\| \ge \|\cdot \,|\, \ell_\infty\|$), we find

$$\|f \mid F^{s\tau}_{pq}(\mathbb{T}^n)\| \le c\|f \mid F^{s_0\tau_0}_{p_0\infty}(\mathbb{T}^n)\|^{1-\theta}\|f \mid F^{s_1\tau_1}_{p_1\infty}(\mathbb{T}^n)\|^{\theta} \le$$

$$\le C\|f \mid F^{s_0\tau_0}_{p_0q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid F^{s_1\tau_1}_{p_1q_1}(\mathbb{T}^n)\|^{\theta},$$

thus part (i) is established.

Now we turn to proof of part (ii). It follows from the condition that $p_1 = \infty$, $p_0 < p < \infty$. Therefore, successively applying Lemma 1 with $a_j = |\widetilde{\Delta}^\eta_j(f,x)|$, the inequality $\| g \,|\, L_p(Q)\| \le (\| g \,|\, L_{p_0}(Q)\|)^{1-\theta}(\| g \,|\, L_\infty(Q)\|)^{\theta}$ and further arguing as in case (i), we obtain

$$\|f \mid F^{s\tau}_{pq}(\mathbb{T}^n)\| \le c\|f \mid F^{s_0\tau_0}_{p_0\infty}(\mathbb{T}^n)\|^{1-\theta}\|f \mid F^{s_1\tau_1}_{\infty\infty}(\mathbb{T}^n)\|^{\theta} \le$$

$$\le C\|f \mid F^{s_0\tau_0}_{p_0q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid F^{s_1\tau_1}_{\infty\infty}(\mathbb{T}^n)\|^{\theta} \equiv$$

$$\equiv C\|f \mid F^{s_0\tau_0}_{p_0q_0}(\mathbb{T}^n)\|^{1-\theta}\|f \mid B^{s_1\tau_1}_{\infty\infty}(\mathbb{T}^n)\|^{\theta},$$

Thus, part (ii) is also obtained.

Proof of Theorem 2.

Proof. Here, successively applying the Holder inequality for integrals (with exponents $P_0 = \frac{p_0}{(1-\theta)p}$ and $P_1 = \frac{p_1}{\theta p}$), the Holder inequality for series $\left(\text{with exponents } Q_0 = \frac{q_0}{(1-\theta)q} \text{ and } Q_1 = \frac{q_1}{\theta q}\right)$ and using elementary properties of suprema, we obtain

$$\|f \mid B^{s\tau}_{pq}(\mathbb{T}^n)\| \equiv \sup_{Q \in \widetilde{Q}} \frac{1}{|Q|^\tau} \left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{jsp} |\widetilde{\Delta}^\eta_j(f,x)|^p dx \right]^{q/p} \right\}^{1/q} \le$$

$$\leq \sup_{Q\in\widetilde{Q}} \frac{1}{|Q|^{\tau}} \left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{js_0p_0} |\widetilde{\Delta}_j^{\eta}(f,x)|^{p_0} dx \right]^{(q(1-\theta))/p_0} \left[\int_Q 2^{js_1p_1} |\widetilde{\Delta}_j^{\eta}(f,x)|^{p_1} dx \right]^{q\theta/p_1} \right\}^{1/q} \leq$$

$$\leq \sup_{Q\in\widetilde{Q}} \frac{1}{|Q|^{\tau}} \left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{js_0p_0} |\widetilde{\Delta}_j^{\eta}(f,x)|^{p_0} dx \right]^{q_0/p_0} \right\}^{(1-\theta)/q_0} \times$$

$$\times \sup_{Q\in\widetilde{Q}} \frac{1}{|Q|^{\tau_1\theta}} \left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{js_1p_1} |\widetilde{\Delta}_j^{\eta}(f,x)|^{p_1} dx \right]^{q_1/p_1} \right\}^{\theta/q_1} \leq$$

$$\leq \sup_{Q\in\widetilde{Q}} \frac{1}{|Q|^{\tau_0(1-\theta)}} \left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{js_0p_0} |\widetilde{\Delta}_j^{\eta}(f,x)|^{p_0} dx \right]^{q_0/p_0} \right\}^{(1-\theta)/q_0} \times$$

$$\times \sup_{Q\in\widetilde{Q}} \frac{1}{|Q|^{\tau_1\theta}} \left\{ \sum_{j=j(Q)}^{\infty} \left[\int_Q 2^{js_1p_1} |\widetilde{\Delta}_j^{\eta}(f,x)|^{p_1} dx \right]^{q_1/p_1} \right\}^{\theta/q_1} =$$

$$= \|f \mid B_{p_0,q_0}^{s_0,\tau_0}(\mathbb{T}^n)\|^{1-\theta} \|f \mid B_{p_1,q_1}^{s_1,\tau_1}(\mathbb{T}^n)\|^{\theta}.$$

Thus, Theorem 2 is completely proved as well.

Acknowledgments

The authors are grateful to the reviewer for his attentive attitude and comments, thanks to which the article has changed for the better.

Author Contributions

All authors contributed equally to this work. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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