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Solutions of boundary value problems for loaded hyperbolic type equations

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This paper investigates a class of second-order partial differential equations describing wave processes with nonlocal effects, including cases involving fractional derivatives. Such equations often arise in the theory of elasticity, aerodynamics, acoustics, and electrodynamics. The presented equations include both integral and differential terms, evaluated either at a fixed point $x = x_0$ or $x = \alpha(t)$. An equation with a fractional derivative of order $0 \leq \beta < 1$ is considered, making it possible to model memory effects and other nonlocal properties. For each equation, supplemented by initial conditions, either a closed-form analytical solution is obtained or the main steps of its derivation are outlined. The article employs the Laplace transform to solve the resulting integral equation, enabling the solution to be presented in an explicit form.

Keywords: differential equations, partial derivatives, loaded equations, boundary value problem, Laplace transform, convolution, wave equations, fractional derivative, integral equations, Cauchy problem.

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Introduction

The study of optimal control problems and long-term forecasting has led to the emergence of a new class of mathematical equations known as "loaded equations".

Initially, these equations were studied by N.N. Nazarov and N.N. Kochin, though the term "loaded equation" was first introduced by A.M. Nakhushev [1–4], who developed a general classification and applications for this type of equations. Loaded equations can be differential, integral, integro-differential, or functional equations, in which the differentiation, integration, or functional transformation operator is applied not to the entire function, but only to its value at a specific point or over a certain set.

Loaded second-order partial differential equations are of particular interest and have been examined in a number of studies [5–9]. These equations find applications in various fields of science and technology, for instance, in modeling heat propagation, wave processes, population dynamics, and other phenomena, and they remain highly relevant today [10–15].

It should be noted that the authors of the present article have experience in solving boundary value problems for hyperbolic equations [16–18]. This expertise has enabled them to analyze loaded equations, taking into account the specifics of methods developed for classical partial differential equations [19–21].

Thus, this study contributes to the theory of loaded equations by providing new methods and results that can be used to model and analyze a variety of physical, biological, and technical processes.

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1 Solution of the boundary value problem for a loaded wave equation, where the load depends on the value of the function $u(x,t)$ at a fixed point $x = x_0$

Let us consider the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \mu u(x,t)\Big|_{x=x_0} + f(x,t), \quad -\infty < x < \infty, \quad t > 0 \tag{1}$$

with initial conditions

$$u(x,t)\Big|_{t=0} = g_1(x), \tag{2}$$

$$\frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = g_2(x), \tag{3}$$

where a is const, the function $f(x,t)$ is continuous on $-\infty < x < \infty,\ t > 0$, functions $g_1(x), g_2(x)$ are continuous on $-\infty < x < \infty$, μ is parameter (load coefficient). The parameter μ equation (1) can take values depending on the physical context of the problem, with $\mu \in \mathbb{R}$ (an arbitrary real number). For example:

$\mu > 0$: models an increase in the load (an additional force);

$\mu = 0$: in this case, the equation reduces to the classical wave equation;

$\mu < 0$: may describe dissipative effects.

Here, the load in equation (1) signifies that the system's state at the point x_0 directly impacts wave propagation throughout the entire space.

The solution to problem (1)–(3) can be expressed as follows

$$u(x,t) = \frac{\mu}{2a} \int\limits_0^t \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} u(x,\tau)\Big|_{x=x_0} d\xi d\tau + \frac{1}{2} u_1(x,t) =$$

$$= \mu \int\limits_0^t (t-\tau) u(x,\tau)\Big|_{x=x_0} d\tau + \frac{1}{2} u_1(x,t), \tag{4}$$

where

$$u_1(x,t) = [g_1(x-at) + g_1(x+at)] + \frac{1}{a} \int\limits_{x-at}^{x+at} g_2(\xi) d\xi + \frac{1}{a} \int\limits_0^t \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau) d\xi d\tau. \tag{5}$$

Now, from (4) we obtain

$$u(x,t)\Big|_{x=x_0} = \mu \int\limits_0^t (t-\tau) u(x,\tau)\Big|_{x=x_0} d\tau + \frac{1}{2} u_1(x,t)\Big|_{x=x_0}. \tag{6}$$

We introduce the notation $\psi(t) = u(x,t)\Big|_{x=x_0}$. Then equation (6) takes the form

$$\psi(t) = \mu \int\limits_0^t (t-\tau) \psi(\tau) d\tau + \frac{1}{2} u_1(x,t)\Big|_{x=x_0}. \tag{7}$$

To solve equation (7), we use the Laplace transform. Let $\Psi(p) = L[\psi(t)]$ be the transform of $\psi(t)$. Let $U(p) = L\left[\frac{1}{2}u_1(x,t)\Big|_{x=x_0}\right]$ be the transform of $\frac{1}{2}u_1(x,t)\Big|_{x=x_0}$. Since $L[t] = \frac{1}{p^2}$, $\int\limits_0^t (t-\tau)\psi(\tau)d\tau = t * \psi(t)$ is convolution. Applying the convolution theorem for the Laplace transform:

$$L[f(t) * g(t)] = L[f(t)]L[g(t)], \tag{8}$$

where $f(t) * g(t) = \int\limits_0^t f(\tau)g(t-\tau)d\tau$, we obtain from (7):

$$\Psi(p) = \mu L[t * \psi(t)] + U(p).$$

Taking (8) into account, we get

$$\Psi(p) = \mu L[t]L[\psi(t)] + U(p).$$

Since $L[t] = \frac{1}{p^2}$, $\Psi(p) = L[\psi(t)]$, we have:

$$\Psi(p) = \frac{\mu}{p^2}\Psi(p) + U(p).$$

Hence,

$$\Psi(p) = \frac{p^2 U(p)}{p^2 - \mu}.$$

Thus, we have found the transform $\Psi(p)$ of the function $\psi(t)$ in the image space. Applying the inverse Laplace transform to $\Psi(p)$:

$$\psi(t) = L^{-1}\left[\frac{p^2 U(p)}{p^2 - \mu}\right] = L^{-1}\left[U(p) + \frac{\mu U(p)}{p^2 - \mu}\right],$$

$$\psi(t) = L^{-1}[U(p)] + L^{-1}\left[\frac{\mu U(p)}{p^2 - \mu}\right].$$

For the second term, we use a Laplace transform table:

$$L^{-1}\left[\frac{\mu}{p^2 - \mu}\right] = \sqrt{\mu}\sinh(\sqrt{\mu}t).$$

Therefore, the second term can be written as:

$$L^{-1}\left[\frac{\mu U(p)}{p^2 - \mu}\right] = \sqrt{\mu}\sinh(\sqrt{\mu}t) * \frac{1}{2}u_1(x,t)\Big|_{x=x_0}.$$

Here "*" denotes convolution. Consequently,

$$\psi(t) = \frac{1}{2}u_1(x,t)\Big|_{x=x_0} + \sqrt{\mu}\sinh(\sqrt{\mu}t) * \frac{1}{2}u_1(x,t)\Big|_{x=x_0}.$$

This is the solution to the original integral equation (7). In our particular case, we need the convolution of $\sqrt{\mu}\sinh(\sqrt{\mu}t)$ и $\frac{1}{2}u_1(x_0,t)$. Thus,

$$\psi(t) = \frac{1}{2}u_1(x,\tau)\Big|_{x=x_0} + \frac{\sqrt{\mu}}{2}\int\limits_0^t u_1(x_0,\tau)\sinh(\sqrt{\mu}(t-\tau))d\tau.$$

Hence, the solution to problem (1)–(3) is given by

$$u(x,t)=\frac{\mu}{2}\int\limits_0^t (t-\tau)\Big(u_1(x,\tau)\Big|_{x=x_0}+\sqrt{\mu}\int\limits_0^\tau u_1(x_0,\tau_1)\sinh(\sqrt{\mu}(\tau-\tau_1))d\tau_1\Big)d\tau+\frac{1}{2}u_1(x,t).$$

Influence of the parameter μ on the class of solutions. For $\mu\neq 0$ the solution (6) includes integral terms with a load. In this case, we require smoothness: $u(x,t)\in C^2$.

For $\mu>0$, the solution contains hyperbolic functions. For $\mu<0$, trigonometric functions appear. In the case $\mu=0$, the equation becomes the classical wave equation with the d'Alembert solution. In this case, we require smoothness $g_1(x)\in C^2$ and $g_2(x)\in C^1$.

2 *Solution of the boundary value problem for a loaded wave equation, where the load depends on the derivative of the function $u(x,t)$ at the fixed point $x=x_0$*

Consider the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}=a^2\frac{\partial^2 u(x,t)}{\partial x^2}+\mu\frac{\partial u(x,t)}{\partial x}\Big|_{x=x_0}+f(x,t), \tag{9}$$

subject to the initial conditions (2) and (3).

The nonlocal term in equation (9) indicates that the rate of change of the function at the point x_0 affects wave propagation.

The solution to problem (8), (2), (3) can be written in the form

$$u(x,t)=\frac{\mu}{2a}\int\limits_0^t\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)}\frac{\partial u(x,\tau)}{\partial x}\Big|_{x=x_0}d\xi d\tau+\frac{1}{2}u_1(x,t)=$$

$$=\mu\int\limits_0^t (t-\tau)\frac{\partial u(x,\tau)}{\partial x}\Big|_{x=x_0}d\tau+\frac{1}{2}u_1(x,t).$$

Introducing the notation

$$\psi(t)=\frac{\partial u(x,\tau)}{\partial x}\Big|_{x=x_0},$$

we obtain:

$$u(x,t)=\frac{\mu}{2a}\int\limits_0^t\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)}\psi(\tau)d\xi d\tau+\frac{1}{2}u_1(x,t)=\mu\int\limits_0^t (t-\tau)\psi(\tau)d\tau+\frac{1}{2}u_1(x,t). \tag{10}$$

From (10), take the derivative with respect to x and then substitute $x=x_0$. This yields an explicit representation for the unknown function $\psi(t)$:

$$\psi(t)=\frac{1}{2}\frac{\partial u_1(x,t)}{\partial x}\Big|_{x=x_0}.$$

Hence, the solution to the boundary value problem (9), (2), (3) is

$$u(x,t)=\frac{\mu}{2}\int\limits_0^t (t-\tau)\frac{\partial u_1(x,\tau)}{\partial x}\Big|_{x=x_0}d\tau+\frac{1}{2}u_1(x,t). \tag{11}$$

Thus, the following theorem holds:

Theorem 1. Let the following conditions be satisfied:
a) the functions $g_1(x) \in C^2(\mathbb{R})$, $g_2(x) \in C^1(\mathbb{R})$;
b) the function $f(x,t) \in C(\mathbb{R} \times [0,T])$;
c) the parameters $a > 0$, $\mu \in \mathbb{R}$, $x_0 \in \mathbb{R}$ are fixed.

Then, for equation (9) with a nonlocal term, under the initial conditions (2) and (3), there exists a unique solution $u(x,t) \in C^2(\mathbb{R} \times [0,T])$, representable in the form (11), where $u_1(x,t)$ is the solution of the corresponding problem without the nonlocal term ($\mu = 0$), computed by formula (5).

Next, consider the boundary value problem for a loaded equation in which the loaded term involves the time derivative:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \mu \left.\frac{\partial u(x,t)}{\partial t}\right|_{x=x_0} + f(x,t). \tag{12}$$

A solution to problem (12), (2), (3) can be written as

$$u(x,t) = \frac{\mu}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} \left.\frac{\partial u(x,\tau)}{\partial \tau}\right|_{x=x_0} d\xi d\tau + \frac{1}{2} u_1(x,t) =$$

$$= \mu \int_0^t (t-\tau) \left.\frac{\partial u(x,\tau)}{\partial \tau}\right|_{x=x_0} d\tau + \frac{1}{2} u_1(x,t). \tag{13}$$

Since

$$\frac{d}{dt}\left[\mu \int_0^t (t-\tau) \left.\frac{\partial u(x,\tau)}{\partial \tau}\right|_{x=x_0} d\tau\right] = \mu \int_0^t \frac{\partial u(x_0,\tau)}{\partial \tau} d\tau,$$

it follows that

$$\frac{\partial u(x,t)}{\partial t} = \mu \int_0^t \left.\frac{\partial u(x,\tau)}{\partial \tau}\right|_{x=x_0} d\tau + \frac{1}{2}\frac{\partial u_1(x,t)}{\partial t}.$$

Substituting $x = x_0$ gives

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{x=x_0} = \mu \int_0^t \left.\frac{\partial u(x,\tau)}{\partial t}\right|_{x=x_0} d\tau + \frac{1}{2} \left.\frac{\partial u_1(x,t)}{\partial t}\right|_{x=x_0}.$$

Introduce $\psi(t) = \left.\frac{\partial u(x,t)}{\partial t}\right|_{x=x_0}$, then

$$\psi(t) = \mu \int_0^t \psi(\tau) d\tau + \frac{1}{2} \left.\frac{\partial u_1(x,t)}{\partial t}\right|_{x=x_0}.$$

Taking the derivative with respect to t:

$$\psi'(t) - \mu\psi(t) = \frac{1}{2} \left.\frac{\partial^2 u_1(x,t)}{\partial t^2}\right|_{x=x_0}. \tag{14}$$

From condition (3), it follows that

$$\psi(t)|_{t=0} = g_2(x_0). \tag{15}$$

Let us solve the problem (14), (15). We introduce the notation $\widetilde{f}(t) = \frac{1}{2}\frac{\partial^2 u_1(x_0,t)}{\partial t^2}$ and find the integrating factor:

$$M(t) = e^{-\mu t}.$$

Multiply both sides of the equation by $M(t)$:

$$e^{-\mu t}\psi'(t) - \mu e^{-\mu t}\psi(t) = e^{-\mu t}\widetilde{f}(t).$$

The left-hand side is the derivative of a product:

$$\frac{d}{dt}\left(e^{-\mu t}\psi(t)\right) = e^{-\mu t}\widetilde{f}(t).$$

Integrate both sides with respect to t from 0 to t:

$$e^{-\mu t}\psi(t) - \psi(0) = \int_0^t e^{-\mu\tau}\widetilde{f}(\tau)d\tau.$$

Taking into account the initial condition $\psi(0) = g_2(x_0)$, we obtain

$$e^{-\mu t}\psi(t) = g_2(x_0) + \int_0^t e^{-\mu\tau}\widetilde{f}(\tau)d\tau,$$

hence,

$$\psi(t) = e^{\mu t}g_2(x_0) + e^{\mu t}\int_0^t e^{-\mu\tau}\widetilde{f}(\tau)d\tau.$$

Substituting $\widetilde{f}(t)$, we get the solution of problem (14), (15):

$$\psi(t) = e^{\mu t}g_2(x_0) + \frac{1}{2}e^{\mu t}\int_0^t e^{-\mu\tau}\frac{\partial^2 u_1(x_0,\tau)}{\partial\tau^2}d\tau.$$

Let us analyze the effect of the parameter μ on the solution to problem (13), (14). When $\mu = 0$, the solution simplifies to:

$$\psi(t) = g_2(x_0) + \frac{1}{2}\int_0^t \frac{\partial^2 u_1(x_0,\tau)}{\partial\tau^2}d\tau = g_2(x_0) + \frac{1}{2}\left[\frac{\partial u_1(x_0,t)}{\partial t} - \frac{\partial u_1(x_0,0)}{\partial t}\right].$$

For $\mu > 0$ the solution contains a growing exponential term, which requires additional conditions on $\widetilde{f}(t)$ to ensure boundedness.

For $\mu < 0$ the solution decays exponentially as $t \to \infty$.

Substituting $\psi(t)$ into (13) yields the solution of problem (12), (2), (3).

Theorem 2. Let the following conditions be satisfied:

a) the initial functions satisfy the smoothness requirements: $g_1(x) \in C^2(\mathbb{R})$, $g_2(x) \in C^1(\mathbb{R})$;

b) the inhomogeneous term $f(x,t) \in C(\mathbb{R} \times [0,T])$;

c) the problem parameters satisfy $a > 0$, $\mu \in \mathbb{R}$, $x_0 \in \mathbb{R}$.

Then, for the equation with a nonlocal term in time (12) under the initial conditions (2), (3), the following statements hold:

if $\mu < 0$, there exists a unique solution $u(x,t) \in C^2(\mathbb{R} \times [0,T])$;

if $\mu = 0$, a solution exists and is unique in the class $C^2(\mathbb{R} \times [0,T])$;

if $\mu > 0$, a solution may exist only locally on some interval $[0,t_0)$.

Finally, consider the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \mu \frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0} + f(x,t), \tag{16}$$

in which the nonlocal term depends on the second derivative of $u(x,t)$ with respect to t at the fixed point $x = x_0$. This implies that the acceleration of the wave process at x_0 influences the wave propagation.

Then the solution to problem (16) under conditions (2) and (3) can be represented as

$$u(x,t) = \frac{\mu}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} \frac{\partial^2 u(x,\tau)}{\partial \tau^2}\bigg|_{x=x_0} d\xi d\tau + \frac{1}{2} u_1(x,t) =$$

$$= \mu \int_0^t (t-\tau) \frac{\partial^2 u(x,\tau)}{\partial \tau^2}\bigg|_{x=x_0} d\tau + \frac{1}{2} u_1(x,t). \tag{17}$$

Since

$$\frac{d}{dt}\left[\mu \int_0^t (t-\tau) \frac{\partial^2 u(x,\tau)}{\partial \tau^2}\bigg|_{x=x_0} d\tau\right] = \mu \frac{\partial u(x_0,t)}{\partial t}\bigg|_0^t = \mu \frac{\partial u(x,t)}{\partial t}\bigg|_{x=x_0} - \mu \frac{\partial u(x,0)}{\partial t}\bigg|_{x=x_0},$$

$$\frac{d}{dt}\left[\mu \frac{\partial u(x,t)}{\partial t}\bigg|_{x=x_0} - \mu \frac{\partial u(x,0)}{\partial t}\bigg|_{x=x_0}\right] = \mu \frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0},$$

then by setting $x = x_0$ in (17), we obtain

$$\frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0} = \mu \frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0} + \frac{1}{2} \frac{\partial^2 u_1(x,t)}{\partial t^2}\bigg|_{x=x_0}$$

or

$$\frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0} = \frac{1}{2(1-\mu)} \frac{\partial^2 u_1(x,t)}{\partial t^2}\bigg|_{x=x_0}.$$

From this, we arrive at the following theorem.

Theorem 3. Suppose the following conditions hold:

1) $f(x,t) \in C((-\infty, +\infty) \times [0, +\infty))$, $g_1(x), g_2(x) \in C(-\infty, +\infty)$, a is a const,

2) $\mu \neq 1$, μ is a parameter,

then the solution to problem (16), (2), (3) exists, is unique, and can be expressed in terms of the solution of the classical wave problem together with an integral representation:

$$u(x,t) = \mu \int_0^t (t-\tau) \frac{\partial^2 u(x,\tau)}{\partial \tau^2}\bigg|_{x=x_0} d\tau + \frac{1}{2} u_1(x,t),$$

where

$$\frac{\partial^2 u(x,t)}{\partial t^2}\bigg|_{x=x_0} = \frac{1}{2(1-\mu)} \frac{\partial^2 u_1(x,t)}{\partial t^2}\bigg|_{x=x_0},$$

and the function $u_1(x,t)$ is given by formula (5).

3 Solution of the boundary value problem for a hyperbolic equation with a fractional derivative

Now let us consider the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \mu \left[{}_0D_t^{\beta} u(x,t) \right]_{x=\alpha(t)} + f(x,t), \tag{18}$$

subject to conditions (2), (3), where $0 \le \beta < 1$. Here, the loading term is the fractional derivative with respect to time of order β of $u(x,t)$, evaluated at the point $x = \alpha(t)$, which moves through space as a function of time. The presence of a fractional time derivative in the nonlocal term implies that the state of the system at any given time depends on all of its past states, affecting how the wave propagates.

Assume the functions

$$f(x,t) \in C((-\infty, +\infty) \times [0, +\infty)), \quad g_1(x), g_2(x) \in C(-\infty, +\infty), \tag{19}$$

$$\left[{}_0D_t^{\beta} u(x,t) \right] \Big|_{x=\alpha(t)} \in C(-\infty, +\infty). \tag{20}$$

Then the solution of the problem takes the form

$$u(x,t) = \frac{\mu}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} \left[{}_0D_\tau^{\beta} u(x,\tau) \right] \Big|_{x=\alpha(\tau)} d\xi d\tau + \frac{1}{2} u_1(x,t) =$$

$$= \mu \int_0^t (t-\tau) \left[{}_0D_\tau^{\beta} u(x,\tau) \right] \Big|_{x=\alpha(\tau)} d\tau + \frac{1}{2} u_1(x,t). \tag{21}$$

Here,

$$\left[{}_0D_t^{\beta} u(x,t) \right] \Big|_{x=\alpha(t)} = {}_0D_t^{\beta} \left\{ \mu \int_0^t (t-\tau) \left[{}_0D_\tau^{\beta} u(x,\tau) \right] d\tau \right\} \Big|_{x=\alpha(\tau)} + \widehat{u}_1(x,t) \Big|_{x=\alpha(t)}, \tag{22}$$

where

$$\widehat{u}_1(x,t) \Big|_{x=\alpha(t)} = {}_0D_t^{\beta} \left\{ \frac{1}{2} u_1(x,t) \right\} \Big|_{x=\alpha(t)} = \frac{1}{2\Gamma(1-\beta)} \int_0^t \frac{\frac{\partial u_1(x,\tau)}{\partial \tau}}{(t-\tau)^{\beta}} d\tau \Big|_{x=\alpha(t)}. \tag{23}$$

Introduce the notation

$$\psi(t) = {}_0D_t^{\beta} u(x,t) \Big|_{x=\alpha(t)}. \tag{24}$$

Then equation (22) becomes:

$$\psi(t) = \mu {}_0D_t^{\beta} \left\{ \int_0^t (t-\tau) \psi(\tau) d\tau \right\} + \widehat{u}_1(x,t) \Big|_{x=\alpha(t)} =$$

$$= \frac{\mu}{\Gamma(1-\beta)} \int_0^t \left[\frac{\int_0^{\tau} (\tau - \tau_1) \psi(\tau_1) d\tau_1}{(t-\tau)^{\beta}} \right]_{\tau}' d\tau + \widehat{u}_1(x,t) \Big|_{x=\alpha(t)} =$$

$$= \frac{\mu}{\Gamma(1-\beta)} \int\limits_0^t \frac{\int\limits_0^\tau \psi(\tau_1) d\tau_1}{(t-\tau)^\beta} d\tau + \widehat{u}_1(x,t)\Big|_{x=\alpha(t)}. \tag{25}$$

Let us make a change of variables for the outer integral: $t - \tau = \eta$. Then

$$\psi(t) = \frac{\mu}{\Gamma(1-\beta)} \int\limits_0^t \frac{\int\limits_0^{t-\eta} \psi(\tau_1) d\tau_1}{\eta^\beta} d\eta + \widehat{u}_1(x,t)\Big|_{x=\alpha(t)}.$$

Hence,

$$\psi'(t) = \frac{\mu}{\Gamma(1-\beta)} \int\limits_0^t \frac{\psi(t-\eta)}{\eta^\beta} d\eta + \frac{\partial \widehat{u}_1(x,t)}{\partial t}\Big|_{x=\alpha(t)}.$$

Reversing the substitution, we get

$$\psi'(t) = \frac{\mu}{\Gamma(1-\beta)} \int\limits_0^t \frac{\psi(\tau)}{(t-\tau)^\beta} d\tau + \frac{\partial \widehat{u}_1(x,t)}{\partial t}\Big|_{x=\alpha(t)}$$

or

$$\psi'(t) = \frac{\mu}{\Gamma(1-\beta)} \psi(t) * t^{-\beta} + \frac{\partial \widehat{u}_1(x,t)}{\partial t}\Big|_{x=\alpha(t)}. \tag{26}$$

From conditions (2) and (25) it follows that $\psi(0) = 0$. We shall solve this equation using the Laplace transform. Let $\Psi(p) = \mathcal{L}[\psi(t)]$ denote the Laplace transform of $\psi(t)$ and let $U_1(p) = \mathcal{L}\left[\frac{\partial \widehat{u}_1(x,t)}{\partial t}\Big|_{x=\alpha(t)}\right]$ be the transform of $\frac{\partial \widehat{u}_1(x,t)}{\partial t}\Big|_{x=\alpha(t)}$, then $\mathcal{L}[\psi'(t)] = p\Psi(p) - \psi(0) = p\Psi(p)$, $\mathcal{L}[t^{-\beta}] = \frac{\Gamma(1-\beta)}{p^{1-\beta}}$. Substituting into (26), we obtain

$$p\Psi(p) = \frac{\mu}{\Gamma(1-\beta)} \mathcal{L}[\psi(t) * t^{-\beta}] + U_1(p).$$

By the convolution theorem for the Laplace transform,

$$p\Psi(p) = \frac{\mu}{\Gamma(1-\beta)} \Psi(p) \frac{\Gamma(1-\beta)}{p^{1-\beta}} + U_1(p).$$

From this, it follows that

$$\Psi(p) = \frac{p^{1-\beta}}{p^{2-\beta} - \mu} U_1(p). \tag{27}$$

From formula (1.80) [8, p. 21] with $k = 0$ we have

$$\mathcal{L}[t^{\widetilde{\beta}-1} E_{\widetilde{\alpha}\widetilde{\beta}}(\pm s t^{\widetilde{\alpha}})] = \frac{p^{\widetilde{\alpha}-\widetilde{\beta}}}{p^{\widetilde{\alpha}} \mp s},$$

where $s = \mu$, $\widetilde{\alpha} = 2 - \beta$, $\widetilde{\alpha} - \widetilde{\beta} = 1 - \beta$, and which implies $\widetilde{\beta} = 1$. Therefore,

$$\mathcal{L}^{-1}\left[\frac{p^{1-\beta}}{p^{2-\beta} - \mu}\right] = E_{2-\beta;1}(\mu t^{2-\beta}),$$

where $E_{2-\beta;1}(\mu t^{2-\beta})$ is the Mittag-Leffler function [8, p. 16-17] . Applying the inverse Laplace transform to (27) yields:

$$\psi(t) = E_{2-\beta;1}(\mu t^{2-\beta}) * \left.\frac{\partial \widehat{u}_1(x,t)}{\partial t}\right|_{x=\alpha(t)} \tag{28}$$

or

$$\psi(t) = \int_0^t E_{2-\beta,1}(\mu(t-s)^{2-\beta}) \cdot \left[\left.\frac{\partial \hat{u}_1(x,s)}{\partial s}\right|_{x=\alpha(s)}\right] ds.$$

Thus, the theorem is proved.

Theorem 4. The integro-differential equation (26) with the homogeneous initial condition $\psi(0) = 0$ has a solution in the class $C([0,T])$ for any continuous right-hand side $\left[{}_0D_t^{\beta} u(x,t)\right]_{x=\alpha(t)}$, where $0 \le \beta < 1$, and its solution is given by formula (28).

By substituting (28) into (21) and taking into account (24), we obtain the solution of problem (18), (2), (3).

Theorem 5. Let the functions $f(x,t)$, $g_1(x)$, $g_2(x)$ satisfy conditions (19). Then the boundary-value problem (18), (2), (3) has, in the class (20), a unique solution given by

$$u(x,t) = \mu \int_0^t (t-\tau)\psi(\tau)d\tau + \frac{1}{2}u_1(x,t),$$

where the functions $\psi(t)$, $u_1(x,t)$ и $\left.\widehat{u}_1(x,t)\right|_{x=\alpha(t)}$ are determined by formulas (28), (23) and (5).

Conclusion

Thus, in this work, various forms of the wave equation with nonlocal terms and a fractional derivative were investigated. Solutions were presented for equations whose nonlocal terms depend on the values of the function and its derivatives at fixed points, and the case of an equation with a fractional derivative was examined, which makes it possible to model memory effects. The solution method, based on applying the Laplace transform, enabled us to obtain analytical or semi-analytical solutions. These solutions illustrate the influence of nonlocal effects and hereditary properties on the dynamics of wave processes. The results obtained are of great significance for understanding the behavior of complex systems, where interaction is not restricted to local interactions and where the system's past states affect its current behavior.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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