https://doi.org/10.31489/2025M3/176-183 *Research article*

Approximations of Theories of Unars

N.D. Markhabatov

L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
(E-mail: markhabatov@gmail.com)

Łoś's theorem states that a first-order formula holds in an ultraproduct of structures if and only if it holds in "almost all" factors, where "almost all" is understood in terms of a given ultrafilter. This fundamental result plays a key role in understanding the behavior of first-order properties under ultraproduct constructions. Pseudofinite structures – those that are elementarily equivalent to ultraproducts of finite models–serve as an important bridge between the finite and the infinite, allowing the transfer of finite combinatorial intuition to the study of infinite models. In the context of unary algebras (unars), a classification of unar theories provides a foundation for analyzing pseudofiniteness within this framework. Based on this classification, a characterization of pseudofinite unar theories is obtained, along with several necessary and sufficient conditions for a unar theory to be pseudofinite. Furthermore, various forms of approximation to unar theories are investigated. These include approximations not only for arbitrary unar theories but also for the strongly minimal unar theory. Different types of approximating sequences of finite structures are examined, shedding light on the model-theoretic and algebraic properties of unars and enhancing our understanding of their finite counterparts.

Keywords: pseudofinite theory, pseudofinite structure, strongly minimal unar, smoothly approximated structure, unar, Collatz Hypotesis, connected unar, bounded unar, ω-categorical unar.

2020 Mathematics Subject Classification: 03C13, 03C45, 03C52, 03C60.

Introduction

We are dealing with a structure called mono-unary algebra, or unar. Unars have often been studied in connection with various algebraic structures and branches of mathematics, such as universal algebra and model theory. Model-theoretic properties of theories formulated in the language of a single unary function have been studied in a number of works, including [1].

For additional properties, see [2–4]. Besides, unars can be applied in other fields such as computer science and sometimes engineer, physics and life sciences etc. [5–7]

The paper [8] considers surjective quadratic Jordan algebras, which has connections with problems of decomposition of algebraic structures, as in [9] which studies an algebraic approaches to binary formulas and compositions of theories. In both cases, the issues of decomposition and model construction are important.

Pseudofinite structures [10] are a fascinating area of mathematical logic that bridge the gap between finite and infinite structures. They allow for the study of infinite structures in ways that resemble finite structures, and they provide a connection to various other concepts in model theory. One of the most important examples of a pseudofinite structure is the ultraproduct of a sequence of finite structures. Given a sequence of finite structures $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, ...)$, their ultraproduct is an infinite structure that "approximates" each finite structure in the sequence. In fact, any first-order sentence that is true in almost all of the finite structures in the sequence (meaning all but a finite number) is true in the ultraproduct. This ultraproduct is a pseudofinite structure. Sergei Vladimirovich Sudoplatov raised

The research was supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (AP26194247).

Received: 14 January 2025; *Accepted:* 6 June 2025.

a natural question [11, 12] about the types and powers of approximation of the theory. In paper [13], approximations of acyclic graphs are studied. It is proved that any theory of an acyclic graph (tree) of finite diameter is pseudofinite with respect to acyclic graphs (trees), that is, any such theory is approximated by theories of finite structures (acyclic graphs, trees). The works [14, 15] are devoted to the study of ranks, topologies and closures of families of theories, as well as algebras associated with definable families of theories.

The paper also investigates the smooth approximability of unars. Smoothly approximable structures were first studied in detail in the works [16, 17]. The model theory of smoothly approximable structures was significantly advanced by G. Cherlin and E. Hrushovskii. Automorphisms and their properties are an important aspect of the study of smoothly approximable unary algebras, as shown in [18, 19], which considers the features of automorphisms in more complex algebraic structures.

1 *Definitions and Basic Concepts*

As usual, we will use the standard terminology. A unar is a structure $\mathcal{U} = \langle U; f^{(1)}\rangle$, whose language consists of one single operation f. For any $u \in U$, let $f^0(u) = u$, $f^{n+1}(u) = f(f^n(u))$ for all $n \in \omega$, $f^{-1}(u) = \{w \in U | f(w) = u\}$. A unar $\mathcal{U}$ is called a *cycle* of length $n \in \mathbb{N}$, if there exists $u \in U$ such as $U = \{f^i(u) | 0 \le i < n\}$, $f^n(u) = u$, $f^i(u) \ne f^j(u)$ for all different $i, j \in \{0, \dots, n-1\}$. The set $\{u_i | i \in \omega\} \subseteq U$ is called a *semichain*, if $f(u_i) = u_{i+1}$ and $u_i \ne u_j$ for all distinct $i, j \in \omega$. The set $\{u_i | i \in \omega\} \subseteq U$ is called an infinite *antichain*, if $f(u_{i+1}) = u_i$ and $u_i \ne u_j$ for all distinct $i, j \in \omega$. If $|f^{-1}(u)| = k$, we say that u is a *k-branching point*, or *k-valence point*.

Definition 1. Let $X \subseteq U$ and $u, v \in U$. We say that u, v are *connected*, if there is $n, m \in \mathbb{N}$ such as $f^n(u) = f^m(v)$. The set $X \subseteq U$ is *connected* if any two elements of X are connected. A maximal connected set is called a *connected component* of U.

Definition 2. A theory T is said to be *bounded* if there exists a natural number N such that the following formula is true in T:

$$(\forall u)[\bigvee_{n,m=1}^{N} (f^n(u) = f^{n+m}(u))].$$

Fact. [20] The T is ω-categorical iff

i) T is bounded;

ii) if $\mathcal{U} \models T$, then there are only a finite number of non-isomorphic sets of the form $\bigcup_{n<\omega} f^{-n}(u)$ in $\mathcal{U}$ or equivalently, $\mathcal{U}$ realizes a finite number of 1-types.

The *root of depth* n of an element u is the set $K_n(u) = \{w \in \mathcal{U} | \exists i \le n \text{ such that } f^i(w) = u\}$. The *root* of u is

$$K(u) = \bigcup_{i\in\omega} K_n(u).$$

A connected subset of the root $K_n(u)$ that contains u is called a subroot of depth n of the element u.

A set of N-neighborhood of $V \subseteq U$ is the set

$$\{u \in U : \exists v \in V \text{such that} \bigvee_{n,m}^{N} f^n(v) = f^m(u)\}.$$

The concept of pseudofiniteness was first introduced by J. Ax. A structure $\mathcal{M}$ in a fixed language L is called pseudofinite if it is infinite but satisfies the following property: for every sentence φ in L, if $\mathcal{M}$ satisfies φ, then there exists a finite structure $\mathcal{M}_0$ that also satisfies φ. The theory $T = Th(\mathcal{M})$ of a pseudofinite structure $\mathcal{M}$ is called a pseudofinite theory.

Ultraproducts have been a powerful tool in model theory since the 1950s and 1960s. They are also important in set theory because they are used to construct elementary embeddings, which are key to studying large cardinals. J. Ax linked the idea of pseudofiniteness to ultraproducts, showing how these constructions can help understand pseudofinite structures.

In classical logic, pseudofinite structures have an interesting property related to definable functions.

Proposition 1. Let $\mathcal{M}$ be a pseudofinite structure, and let $f : M^k \to M^k$ be a definable function. Then: f is injective (one-to-one) if and only if f is surjective (onto).

This property is a direct consequence of pseudofiniteness and highlights the "finite-like" behavior of pseudofinite structures, even though they are infinite.

Definition 3. [12] Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin \mathcal{T}$. The theory T is said to be *$\mathcal{T}$-approximated,* or *approximated by the family* $\mathcal{T}$, or a *pseudo-$\mathcal{T}$-theory,* if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ for which $\varphi \in T'$.

If the theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and theories $T' \in \mathcal{T}$ are said to be *approximations* for T.

Definition 4. [21] *A disjoint union* $\bigsqcup_{n\in\omega} \mathcal{M}_n$ of pairwise disjoint systems $\mathcal{M}_n$ of pairwise disjoint predicate signatures $\Sigma_n, n \in \omega$, is a system of signature $\bigcup_{n\in\omega} \Sigma_n \cup \{P_n^{(1)} | n \in \omega\}$ with support $\bigsqcup_{n\in\omega} M_n$, $P_n = M_n$, and interpretations of predicate symbols from Σ_n that coincide with their interpretations in systems $\mathcal{M}_n$, $n \in \omega$.

A disjoint union of theories T_n, pairwise disjoint predicate signatures Σ_n, respectively, $n \in \omega$, is the theory

$$\bigsqcup_{n\in\omega} T_n \rightleftharpoons Th(\bigsqcup_{n\in\omega} \mathcal{M}_n),$$

where $\mathcal{M}_n \models T_n,\ n \in \omega$.

Obviously, the $T_1 \sqcup T_2$ theory does not depend on the choice of the disjunctive union $\mathcal{M}_1 \sqcup \mathcal{M}_2$ of the models $\mathcal{M}_1 \models T_1$ and $\mathcal{M}_2 \models T_2$.

2 Smoothly Approximability of Unars

The study of countably infinite and countably categorical smoothly approximable structures is relevant in many areas of mathematics, including topology, analysis, and algebra.

A. Lachlan introduced the concept of smoothly approximable structures to shift the focus from analyzing finite structures to analyzing infinite ones. The idea is to classify large finite structures that behave as if they are "approximations" to an infinite limit structure. This approach provides a bridge between finite and infinite model theory.

Definition 5. [16] Let L be a countable signature and let $\mathcal{M}$ be a countable and ω-categorical L-structure. L-structure $\mathcal{M}$ (or $Th(\mathcal{M})$) is said to be *smoothly approximable* if there is an ascending chain of finite substructures $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \ldots \subseteq \mathcal{M}$ such that $\bigcup_{i\in\omega} \mathcal{M}_i = \mathcal{M}$ and for every i, and for every $\bar{a}, \bar{b} \in \mathcal{M}_i$ if $tp_{\mathcal{M}}(\bar{a}) = tp_{\mathcal{M}}(\bar{b})$, then there is an automorphism σ of M such that $\sigma(\bar{a}) = \bar{b}$ and $\sigma(\mathcal{M}_i) = \mathcal{M}_i$, or equivalently, if it is the union of an ω-chain of finite homogeneous substructures; or equivalently, if any sentence in $Th(\mathcal{M})$ is true of some finite homogeneous substructure of $\mathcal{M}$.

This means that $\mathcal{M}$ can be "approximated" by a sequence of finite substructures that are homogeneous in a certain sense.

It is important to note that a finitely homogeneous substructure does not necessarily mean that the substructure is homogeneous in the usual sense. Instead, it refers to a weaker property related to the existence of automorphisms preserving the substructure.

Theorem 1. Any infinite ω-categorical unar $\mathcal{U} = \langle U, f \rangle$ is smoothly approximable.

Proof. Let U be a countably categorical unar that does not have a ∞-branching point. Since by [20] in U the set of degrees of points is finite and U is bounded and realizes a finite number of 1-types, i.e., either all connected components are isomorphic, or U consists of a countable number of copies of non-isomorphic connected components. Each connected component can be considered as a finite homogeneous substructure. Thus, U can be represented as a union of finite homogeneous substructures, $U = \bigcup_{i\in\omega} U_i = \bigsqcup_{i\in\omega} U_i$, where U_i are finite homogeneous connected components.

Now U have the connected components with ∞-branching points. Let $U \setminus V$ is the connected components with ∞-branching points and V is the union of finite connected components. Then there are $W_0, W_1, \ldots, : |W_i| < \omega$, $W_0 \subset W_1 \subset \ldots$, and $U \setminus V = \bigsqcup_{i\in\omega} W_i$.

3 Pseudofiniteness of Unars

Theorem 2. A theory T of an infinite unar is pseudofinite if and only if any sentence $\varphi \in T$ is consistent with a theory of bounded unar.

Proof. Let T be the theory of pseudofinite unar, and let T' be the theory of bounded unar. By the definition of pseudofiniteness, any sentence $\varphi \in T$ has a finite model. Since $\varphi \cup T'$ is finitely consistent and, by the compactness theorem, T is consistent with T'.

To the opposite side. Since any sentence φ of a theory T of infinite unar is consistent with a theory T' of bounded unar, any sentence φ belongs to T'. Take $\varphi \in T \cap T'$. Again, by compactness, φ has a model that is either finite or infinite. Hence any sentence φ of the theory T has a finite model.

The following corollary is a direct consequence of Theorem 2 and summarizes Theorem 1.

Proposition 2. Any theory T of a bounded infinite unar is pseudofinite.

If in $\mathcal{U} = \langle U, f \rangle$ the unary function f is injective (surjective) then $\mathcal{U}$ is an *injective (surjective) unar.*

Proposition 3. Any surjective infinite unar is pseudofinite if and only if it is bijective.

Proof. Follows directly from Proposition 1.

Proposition 4. Any injective non-surjective infinite unar is not pseudofinite.

Proof. Let $\mathcal{U}$ be an infinite injective unar. The connected components in $\mathcal{U}$ can be classified to be either a copy of $\langle \mathbb{N}, succ \rangle$, $\langle \mathbb{Z}, succ \rangle$, or a cycle of period p, where $p \in \mathbb{N}^+$. We exclude the last two cases from consideration due to surjectivity. It remains to consider unary $\mathcal{U}$ components that isomorphic to $\langle \mathbb{N}, succ \rangle$. By Proposition 1, $\mathcal{U}$ is not pseudofinite, since there exists an element that does not have a preimage.

Remark 1. There are:

1) surjective pseudofinite and non-pseudofinite infinite unars, e.g., an infinite permutation or $\langle \mathbb{Z}, succ \rangle$ and, respectively, a function with at least two preimages for every element, or a cycle with its preimages out of this cycle;

2) injective non-surjective non-pseudofinite unars, e.g., a Peano successor function;

3) non-injective non-surjective pseudofinite and non-pseudofinite unars, e.g., a unar consisting of an element and its infinitely many preimages, and, respectively, this unar united with a connected component forming a Peano successor function.

These will be described in more detail in the following sections.

Remark 2. Consider the unary function

$$f(x) = \begin{cases} x/2, & \text{if } x \text{ is even,} \\ 3x+1, & \text{if } x \text{ is odd.} \end{cases}$$

Let's call the structure $\langle \mathbb{Z}^+, f \rangle$ as $3x+1$-*unar* or *Collatz unar*. It is easy to see that any point in this model is 1-branching or 2-branching. Therefore, the $3x+1$-unar is not strongly minimal and has an infinite number of antichains. Moreover, the $3x+1$-unar is a surjective unar. By Remark 1 is not pseudofinite.

3.1 Types of Approximations for Families of Theories of Unars

Definition 6. $\mathcal{T}$-approximated theory T is said to be *CYCLE-approximated,* if $\mathcal{T}$ is a family of theories of finite unars with cycles. Also, the $\mathcal{T}$-approximated theory T is said to be *FOREST-approximated,* if $\mathcal{T}$ is a family of theories of finite unars without cycles. In particular, if $\mathcal{T}$ is a family of the theory of connected unars, then T is said to be *TREE-approximated.*

Proposition 5. The theory T of unbounded unar is CYCLE-approximated if and only if each connected component contains a semichain and only one antichain.

Proof. Let $\mathcal{U}_n$ be a cycle of length $n < \omega$. Increasing the length of the cycle in the limit we obtain an acyclic unar $\mathcal{U} = \bigsqcup_{n \leq i \leq \omega} \mathcal{U}_i$, which is a copy of $\langle \mathbb{Z}, succ \rangle$ with a semichain and an antichain. The proof from right to left is similar to [13; Theorem 2].

Proposition 6. The theory T of unar is FOREST-approximated if and only if T is the theory of a non-injective and non-surjective bounded unar, each component containing an infinitely branching point.

Proof. By the definition of a FOREST-approximated theory T, the family $\mathcal{T}$=FOREST consists of finite acyclic unars. If all connected components are finite, then T is approximated by increasing the number of connected components. And if there is an ∞-branching point in the components, then T is approximated by increasing the valency of the root points. It is easy to see that T is a theory of neither injective nor surjective bounded unar. The proof from right to left is similar to [13; Theorem 4].

3.2 Approximations of Strongly Minimal Unars

The study of uncountable categoricity and ω-stability in certain types of structures is of principal importance.

Definition 7. A structure $\mathcal{M}$ is said to be *minimal*, if any subset definable in the structure using parameters is either finite or co-finite (a complement to a finite set). $\mathcal{M}$ is said to be *strongly minimal*, if any model of $Th(\mathcal{M})$ is minimal.

The notion of strong minimality is important in model theory because it provides a way to classify theories based on the complexity of their definable sets. Strongly minimal theories have many interesting and useful properties, including simplicity and stability, which make them amenable to study and applications in other areas of mathematics.

Proposition 7. The theory $T = Th(\mathfrak{A})$ of bounded strongly minimal unar $\mathfrak{A}$ is pseudofinite.

Proof. A bounded strongly minimal unar $\mathfrak{A}$ can have either one or no ∞-branching point. If bounded and has ∞-branching point, then $\mathfrak{A}$ is connected. By Proposition 2. $\mathfrak{A}$ is pseudofinite and by Proposition 6 the theory $Th(\mathfrak{A})$ is TREE-approximated.

If bounded and has no ∞-branching point, Then every connected component of $\mathfrak{A}$ is finite and all but finitely many connected components are cycles of the same length m. By the classical results of Zilber and Cherlin, Harrington, Lachlan [22, 23] say that strongly minimal (in fact ω-stable) ω-categorical theories are pseudofinite.

Proposition 8. There is a theory T of unbounded strongly minimal non-injective non-surjective pseudofinite unar.

Proposition 9. The theory $T = Th(\mathcal{U})$ of unbounded strongly minimal injective unar $\mathcal{U}$ is pseudofinite if and only if $\mathcal{U}$ is bijective.

Model-theoretic properties such as definable minimality of unars were studied in [2].

Proposition 10. Let T be the theory of a strongly minimal unar such that each vertex has n preimages for some natural n. Then the theory T is pseudofinite if and only if $n = 1$.

3.3 Connected Unars

Theorem 3. [24, 25] Let $\mathcal{U}$ be a connected unary without cycles, containing no infinite antichains, and there exist $m \in \omega$ and a semichain $S \subseteq \mathcal{U}$ such that

1) $|f^{-1}(u)| \leq m$ for all $u \in \mathcal{U}$;

2) for any $n \in \omega$ there are $u, v, u_0 \in S, v_0 \in \mathcal{U}$, satisfying the following conditions:

a) $a = f^n(u_0),\ b = f^n(v_0)$,

b) $b = f^{2n+k}(u)$ for some $k \in \omega$,

c) $f^{n-1}(v_0) \notin S$,

d) there is a finite partial isomorphism $\alpha : \mathcal{U} \to \mathcal{U}$ such that $dom\alpha = O^n(u)$, $rang\alpha \subseteq O^n(v)$ and $\alpha(u_0) = v_0$.

Then $\mathcal{U}$ is a pseudofinite unar.

In the work [26, 27] a study of pseudofinite polygons was started.

The following statements are easily derived from the above results.

Proposition 11. The theory T of connected unar is CYCLE-approximated if and only if it contains a semichain and only one antichain.

Proposition 12. The theory T of unar is TREE-approximated if and only if T is the theory of a non-injective and non-surjective bounded unar, containing an infinitely branching point.

Proposition 13. There is an pseudofinite unar that is not CYCLE-approximated and TREE-approximated.

4 Concluding remarks

On a base of classification of unar theories, a characterization of pseudofiniteness of unar theories is found, as well as some necessary and sufficient conditions of pseudofiniteness. Approximations of the theory of unars are shown, as well as for the strongly minimal theory of unars. Various types of approximation of the unar theory are considered. Unars are special cases of polygons. In the future, it is planned to study pseudofinite polygons and their approximations.

5 Acknowledgments

The author thanks his scientific supervisor Sergey Vladimirovich Sudoplatov for comprehensive support. As well as the participants of the A.D. Taimanov seminar on Model Theory (Astana).

Conflict of Interest

The author declare no conflict of interest

References

1 Ivanov, A.A. (1984). Complete theories of unars. *Algebra and Logic, 23*(1), 36–55. https://doi.org/10.1007/BF01979698

2 Baisalov, E.R., Meirembekov, K.A., & Nurtazin, A.T. (2006). Opredelimo minimalnye modeli [Definably-minimal models]. *Teoriia modelei v Kazakhstane: sbornik nauchnykh rabot, posviashchennyi pamiati A.D. Taimanova — Model Theory in Kazakhstan: collection of scientific works, dedicated to the memory of A.D. Taimanov,* 140–157 [in Russian].

3 Kulpeshov, B.Sh., & Sudoplatov, S.V. (2022). Properties of ranks for families of strongly minimal theories. *Siberian Electronic Mathematical Reports, 19*(1), 120–124. https://doi.org/10.33048/semi.2022.19.011

4 Kulpeshov, B.S. (2016). On almost binarity in weakly circularly minimal structures. *Eurasian Mathematical Journal, 7*(2), 38–49.

5 Akhmetbaev, D.S., Dzhandigulov, A.R., & Akhmetbaev, A.D. (2019). Topological algorithm for forming nodal stresses of complex networks energy systems. *E3S Web of Conferences, 139*, 01066. https://doi.org/10.1051/e3sconf/201913901066

6 Akhmetbayev, D.S., Dzhandigulov, A.R., & Bystrova, S.V. (2020). Topological system method of formation of transformer transformation coefficients. *In E3S Web of Conferences, 216*, 01087. https://doi.org/10.1051/e3sconf/202021601087

7 Akhmetbaev, D.S., Jandigulov, A.R., & Akhmetbaev, A.D. (2024). Improving algorithms for searching for trees of directed graphs of electrical power systems. *Power Technology and Engineering, 58*, 349–355. https://doi.org/10.1007/s10749-024-01819-6

8 Baisalov, E.R., & Aljouiee, A. (2020). Surjective quadratic Jordan algebras. *Eurasian Mathematical Journal, 11*(2), 19–29. https://doi.org/10.32523/2077-9879-2020-11-2-19-29

9 Emel'yanov, D.Y., Kulpeshov, B.S., & Sudoplatov, S.V. (2020). Algebras of binary formulas for compositions of theories. *Algebra and Logic, 59*, 295–312. https://doi.org/10.1007/s10469-020-09602-y

10 Ax, J. (1968). The elementary theory of finite fields. *Annals of Mathematics, 88*(2), 239–271. https://doi.org/10.2307/1970573

11 Sudoplatov, S.V. (2017). Combinations related to classes of finite and countably categorical structures and their theories. *Siberian Electronic Mathematical Reports, 14*, 135–150. https://doi.org/10.17377/semi.2017.14.014

12 Sudoplatov, S.V. (2020). Approximations of theories. *Siberian Electronic Mathematical Reports, 17*, 715–725. https://doi.org/10.33048/semi.2020.17.049

13 Markhabatov, N.D. (2022). Approximations of acyclic graphs. *The Bulletin of Irkutsk State University. Series Mathematics, 40*, 104–111. https://doi.org/10.26516/1997-7670.2022.40.104

14 Markhabatov, N.D., & Sudoplatov, S.V. (2019). Algebras for definable families of theories. *Siberian Electronic Mathematical News, 16*, 600–608. https://doi.org/10.33048/semi.2019.16.037

15 Markhabatov, N.D., & Sudoplatov, S.V. (2021). Topologies, ranks, and closures for families of theories. II. *Algebra and Logic, 60*(1), 38–52. https://doi.org/10.1007/s10469-021-09626-y

16 Kantor, W.M., Liebeck, M.W., & Macpherson, H.D. (1989). $\aleph_0$-categorical structures smoothly approximated by finite substructures. *Proceedings of the London Mathematical Society, 59*, 439–463, https://doi.org/10.1112/plms/s3-59.3.439

17 Wolf, D. (2020). Multidimensional exact classes, smooth approximation and bounded 4-types. *The Journal of Symbolic Logic, 85*(4), 1305–1341. https://doi.org/10.1017/jsl.2020.37

18 Alimbaev, A.A., Nauryzbaev, R.Z., & Umirbaev, U.U. (2020). On the automorphisms of a free Lie algebra of rank 3 over an integral domain. *Siberian Mathematical Journal, 61*(1), 1–10. https://doi.org/10.1134/S0037446620010012

19 Nauryzbaev, R. (2015). On Generators of the Tame Automorphism Group of Free Metabelian Lie Algebras. *Communications in Algebra, 43*(5), 1791–1801. https://doi.org/10.1080/00927872.2013.876038

20 Shishmarev, Yu.E. (1972). Categorical theories of a function. *Matematicheskie Zametki, 11*, 58–63. https://doi.org/10.1007/BF01366918

21 Woodrow, R.E., (1976). *Theories with a finite number of countable models and a small language.* PhD Thesis. Ottawa.

22 Cherlin, G., Harrington L., & Lachlan, A.H. (1985). $\aleph_0$-categorical. $\aleph_0$-stable structures, *Ann. Pure Appl. Logic, 28*(2), 103–135, https://doi.org/10.1016/0168-0072(85)90023-5

23 Zil'ber, B. (1981). K probleme konechnoi aksiomatiziruemosti teorii, kategorichnykh vo vsekh beskonechnykh moshchnostyakh [On the problem of finite axiomatizability for theories categorical in all infinite powers]. In Baizhanov, B. (Ed.), *Issledovaniia po teoreticheskomu programmirovaniiu — Investigations in Theoretical Programming*, 69–75. Alma-Ata: Nauka [in Russian].

24 Stepanova, A.A., Efremov, E.L., & Chekanov, S.G. (2022). On the pseudofiniteness of connected unars. In Efremov, E.L. (Ed.), *Syntax and semantics of logical systems: Materials of the 7th International School-Seminar.* Vladivostok: Publishing House of the Far Eastern Federal University, 48–49.

25 Efremov, E.L., Stepanova, A.A., & Chekanov, S.G. (2025). Connected pseudofinite unars. *Algebra and Logic, 63*, 189–194. https://doi.org/10.1007/s10469-025-09782-5

26 Stepanova, A.A., Efremov, E.L., & Chekanov, S.G. (2024). Pseudofinite S-acts. *Siberian Electronic Mathematical Reports, 21*(1), 271–276.

27 Stepanova, A.A., Efremov, E.L., & Chekanov, S.G. (2025). On T-pseudofinite models of universal theories T. *Kazakh Mathematical Journal, 25*(2), 19–24. https://doi.org/10.70474/fxyhwj58

Author Information[*]

Nurlan Darkhanuly Markhabatov — PhD in Mathematics, Department of Cryptology, L.N. Gumilyov Eurasian National University, Qajymuqan 13, Astana, 010000, Kazakhstan; e-mail: *markhabatov@gmail.com*; https://orcid.org/0000-0002-5088-0208

[*]The author's name is presented in the order: first name, middle name, and last name.