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Properties of semigroups of elementary types of model classes

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The study of classes of first-order countable language models and their properties is an important direction in model theory. Of particular interest are axiomatizable classes of models (varieties, quasivarieties, finitely axiomatizable classes, Jonssonian classes, etc.). In this paper we present the results obtained on the properties of formula-definable classes of models and formula-definable semigroups of elementary types, namely, we study the properties of semigroups of elementary types of models in a first-order language. We consider products of elementary types which form a commutative semigroup with unit. A two-place relation of absorption of one elementary type by another is introduced, which allows us to distinguish formula-definable semigroups of elementary types and corresponding classes of models. On the basis of the axiomatizability property of formula-definite semigroups of elementary types, their connection with ultraproducts and infinite products is established. Examples of idempotently formula-definite and non-idempotently formula-definite semigroups are given, and their peculiarities are discussed. The paper demonstrates both the study of semigroups of elementary types and the study of properties of formula-definite classes of models.

Keywords: idempotent, axiomatizable class, formula-definable semigroups, properties of semigroups, model companion, formula-definable model, elementary types of model classes, non-formula-definable model classes, countable signature.

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Introduction

On the set of elementary types of a countable signature σ of the first-order language L, the product of elementary types is considered. This forms a commutative semigroup with an identity element. Certain properties of subsemigroups of the semigroup of elementary types are established. Within this semigroup, a binary relation of absorption of one elementary type by another is studied. This allows for the identification of formula-definable semigroups of elementary types and formula-definable classes of models. Several properties of formula-definable semigroups of elementary types and formula-definable classes of models are proven.

1 Definitions and preliminary results

Let L be a language of countable signature σ of first order. For any model A of the language L, let $Th(A)$ denote the set of all sentences (closed formulas) of the language L that are true in the model A. The theory $Th(A)$ is called the elementary type of the model A.

An arbitrary (abstract) class of all models of the counting signature σ of the first-order language L is divided into classes by the relation of elementary equivalence of models (classification of A. Tarski [1,2]). This results in a set of classes (elementary types).

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The cardinality of the set Th_L, consisting of all elementary types of the countable signature σ in L, does not exceed 2^ω. In what follows, T denotes the elementary type of a model.

Historically, within the class of all models of the language L, greatest research interest has been focused on axiomatizable subclasses of models defined by certain theories: varieties, quasivarieties, finitely axiomatizable classes, Jonsson classes, etc.

The symbol $*$ indicates a known result, with references provided.

Theorem 1. * [1] Filtered and direct products of models preserve elementary equivalence.

Definition 1. The *product* of the elementary types T_1 of model A and T_2 of model B is defined as $T_1 \cdot\ T_2 = Th(A \times B)$, where $A \times B$ is the Cartesian product of models A and B. Analogously, infinite products of theories $\prod_{i \in I} T_i$, ultrapowers $\prod_{i \in I} T_i/D$, ultraproducts T^I/D with ultrafilters D, and filtered products of theories are defined.

Proposition 1. Definition 1 is well-defined.

Proof. It follows from Theorem 1*.

The algebra $\langle Th_L, \cdot\rangle$ is a commutative semigroup with an identity element.

If K is some class of models of the language L, then the set of elementary types of all models in the class K is denoted by Th_K and is called the set of elementary types of the class K. If H is a set of elementary types of theories in L, then K_H is the class of all models of all elementary types in H.

Examples of subsemigroups of $\langle Th_L, \cdot\rangle$ with specific properties are studied in various articles, books, and monographs: semigroups of elementary types of models of Horn classes, varieties, and quasivarieties [1, 2]. J. Wierzejewski [3] proved that the set of stable (superstable, ω-stable) elementary types of models forms a semigroup of stable (superstable, ω-stable) types under the product. M.V. Shvidefski [4] explored the complexity of the lattice of subsemigroups of the semigroup of elementary types. D.E. Palchunov [5] studied the semigroup of elementary types of Boolean algebras.

Later in the text, in the class of all models of the language L, a binary relation of one model *absorbing* another is defined.

Definition 2. A model A *absorbs* a model B, if $Th(A \times B) = Th(A)$, where $A \times B$ is the direct product of models.

Then the definitions are given and the following statements are obtained, which immediately follow from the corresponding theorems with the sign *.

Definition 3. We say that an elementary type T_2 *absorbs* an elementary type T_1, denoted $T_1 \leq T_2$, if $T_1 \cdot T_2 = T_2$. An elementary type T is called *idempotent*, if $T \cdot T = T$. A model A is called an *idempotent model*, if $Th(A \times A) = Th(A)$.

Model B *absorbs* model A, if $Th(A) \leq Th(B)$.

The absorption relation on the set Th_L is antisymmetric and transitive.

Definition 4. A set H of elementary types of models in the language L is called *an axiomatizable set* of elementary types if the class K_H, consisting of all models of all elementary types in H, forms an axiomatizable class.

Not every set of elementary types is axiomatizable.

The problem of axiomatizability of model classes is one of the central questions in model theory [1, 2, 5].

Theorem 2. * (J. Keisler [1]) A class of models is axiomatizable, if and only if it is closed under ultraproducts and elementary equivalence.

Proposition 2. A set H of elementary types is axiomatizable, if and only if H is closed under ultraproducts.

Proof. Follows from Theorem 2*.

Theorem 3. * [1] For any two sets of models $\{A_i \mid i \in I\}, \{B_i \mid i \in I\}$, and any ultrafilter D on I, the following holds:

$$\prod_{i\in I}(A_i \times B_i)/D \cong \prod_{i\in I} A_i/D \times \prod_{i\in I} B_i/D.$$

Proposition 3. For any two sets of elementary types $\{T_i \mid i \in I\}, \{T'_i \mid i \in I\}$, and any ultrafilter D on I, the following holds:

$$\prod_{i\in I}(T_i \cdot T'_i)/D = \prod_{i\in I} T_i/D \times \prod_{i\in I} T'_i/D.$$

Proof. Follows from Definition 1 of the ultraproduct of elementary types and Theorem 3*.

Theorem 4. [1] For any model A and any ultrafilter D, the following holds: $A \equiv A^I/D$.

Proposition 4. For any elementary type T and any ultrafilter D, the following holds: $T = T^I/D$.

Proof. Follows from Definition 1 of the ultrapower of elementary types and Theorem 4*.

Theorem 5. * Let A, B, and C be models of the language L. If $A \times B \times C \equiv A$, then $A \times B \equiv A$ ($\equiv$ denotes elementary equivalence of models).

Proposition 5. Let T_1 , T_2 , T_3 be elementary types. If $T_1 \cdot T_2 \cdot T_3 = T3$, then $T_1 \cdot T_3 = T_3$.

Proof. Follows from Definition 1 of the product of elementary types and Theorem 5*.

The main focus of the above results is the transition from studying the properties of model classes to examining the properties of sets of elementary types of these classes. This enables consideration of the semigroup $\langle Th_L, \cdot\rangle$ and the properties of its subsemigroups. That is, it allows us to discover new properties of model classes using the direct product operation for models.

Studies on axiomatizable classes of models closed with respect to direct products are available in textbooks and articles of many authors. However, the problem of characterizations of axiomatizable classes closed with respect to direct products is still open [6–8].

2 Formula-definable semigroups of elementary types

This section presents results related to formula-definable semigroups of elementary types and formula-definable model classes.

Definition 5. [8, 9] A set of elementary types H of a signature is called *a formula-definable set* of elementary types if there exists an elementary type T such that for any elementary type T_1, holds $T_1 \in H$ if and only if $T_1 \cdot T = T$. In this case, the elementary type T is called the *determinant* of the set H. If the determinant of H is idempotent, H is called an *idempotent formula-definable set* of elementary types.

Definition 6. A class of models K is called *a formula-definable model* class if Th_K is a formula-definable set of elementary types. If Th_K is an idempotent formula-definable set of elementary types, K is called *an idempotent formula-definable model class.* The model of the determinant of the set Th_K is called the determinant of the class K [10].

Examples:

1. The class of models with a single equivalence relation is formula-definable. The determinant of this class of models is a model with an infinite number of equivalence classes, each of which is infinite [11].

2. The set of all ω-stable, the set of all superstable and the set of all stable elementary types, these sets are not formula-definite sets of elementary types.

Example 1 is fairly self-explanatory.

Explanation of Example 2:

From an example provided in [3], there exists an unstable elementary type T such that $T \cdot T$ is ω-stable. If the set of all ω-stable types were formula-definable, i.e., defined by some elementary type T_1, it would follow that $T \cdot T \cdot T_1 = T_1$. Then, by Proposition 5, $T \cdot T_1 = T_1$. Hence, the set of all ω-stable elementary types is not formula-definable. The same reasoning applies to the sets of superstable and stable types.

By analogy, this is true for the set of all superstable and the set of all stable elementary types.

Theorem 6. If a set of elementary types H is closed under direct products, then there exists an idempotent T such that for any $T_1 \in H$, holds $T_1 \cdot T = T$.

Proof. Since H is closed under infinite products and the cardinality of elementary types is at most 2^ω, there exists an elementary type T in H such that the product of all types in H equals elementary type T. Applying Proposition 5, we conclude that for any $T_1 \in H$, holds $T_1 \cdot T = T$. The type T is idempotent.

However, the idempotent T obtained in Theorem 6 may not necessarily serve as the determinant of H.

Thus, a set of elementary types closed under infinite products may not be an idempotent formula-definable set, even if it is an axiomatizable set. Examples of such sets of elementary types can be found among quasivarieties. We will provide such an example later.

Theorem 7. * (J. Keisler [1]) By any proposition φ one can efficiently find a number n such that for any index set I and any models A_i , $i \in I$, there exists a subset J in I that contains at most n elements, and for any V, $J \subseteq V \subseteq I$, $\prod_{i\in V} A_i \models \varphi$ if and only if $\prod_{i\in I} A_i \models \varphi$.

Theorem 8. A formula-definable set of elementary types H is closed under ultraproducts, finite, and infinite direct products of elementary types. That is, H is an axiomatizable set of elementary types, forms a commutative semigroup with an identity, and the formula-definable class K_H of models is an axiomatizable class.

Proof. Let $T_1, ..., T_n \in H$. By definition, H is a formula-definable set, so there exists a type T such that $T_i \cdot T = T$, $i \leq n$. Since the operation $\cdot$ is commutative and associative, $T_1 \cdot ... \cdot T_n \cdot T = T$, which implies $T_1 \cdot ... \cdot T_n \in H$. Thus, H is closed under finite products.

Let $\{T_i \mid i \in I, T_i \in H\}$ The equality $\prod_{i\in I} T_i \cdot T = T$ follows from the closedness with respect to finite products and Theorem 7 *. Therefore, H is closed under infinite products.

Let $\prod_{i\in I} T_i/D$ be an ultraproduct of elementary types with ultrafilter D, where $T_i \in H$ for $i \in I$. Using Propositions 3 and 4,

$$\prod_{i\in I} T_i/D \cdot T = \prod_{i\in I} T_i/D \cdot T^I/D = \prod_{i\in I} (T_i \cdot T)/D = T.$$

Hence, H is closed under ultraproducts, meaning H is an axiomatizable set of elementary types. Consequently, the formula-definable class of models is axiomatizable class of models.

Not every axiomatizable class of models is a formula-definable class. For instance, the axiomatizable class of fields is not a formula-definable class. If it were, the product of fields would have to be a field, which is not generally true.

Therefore, the set of formula-definable sets of elementary types is a proper subset of the set of all axiomatizable sets of elementary types.

A formula-definable set of elementary types forms a commutative semigroup with an identity, referred to as *a formula-definable semigroup of elementary types* [9]. Each elementary type T defines a formula-definable set of elementary types $GT = \{T_1 \mid T_1 \cdot T = T, T_1 \in Th_L\}$. This set GT is axiomatizable, and the class of models H_{GT} is formula-definable class of models.

Definition 7. If the determinant of a formula-definable semigroup of elementary types is idempotent, then such a semigroup is called *an idempotent formula-definable semigroup* of elementary types. The class of all models of all elementary types in this semigroup is called *an idempotent formula-definable model class.*

Not every determinant of a formula-definable semigroup is idempotent. For example, the elementary theory of a dense order without endpoints defines a formula-definable semigroup of theories but is not itself idempotent.

Theorem 9. A formula-definable semigroup G of elementary types is an idempotent formula-definable semigroup, and the class H_G of models of this semigroup is an idempotent formula-definable model class.

Proof. Since G is a formula-definable semigroup, by Theorem 8 it is closed under infinite products. By Theorem 6, there exists an idempotent $T \in G$ such that for any $T_1 \in G$, holds $T_1 \cdot T = T$. We now show that the idempotent T is the determinant of G. Since G is formula-definable, there exists a determinant T^G such that for any elementary type $T_1 \in Th_L$ holds $T_1 \in G$ if and only if $T_1 \cdot T^G = T^G$. If for some of elementary type $T' \in Th_L$ holds $T' \cdot T = T$, then $T' \cdot T \cdot T^G = T^G$.

By Proposition 5, $T' \cdot T^G = T^G$. Therefore, $T' \in G$, meaning G is an idempotent formula-definable semigroup, and H_G is an idempotent formula-definable model class.

Examples of formula-definable and non-formula-definable model classes.

An example of minimal quasivarieties from A.I. Maltsev's work [2]:

"Consider the signature with two predicate symbols P and Q. The quasivariety K, defined by the formulas $x = y$ and $P(x) \to Q(x)$, consists of three single-element models U_1, U_2, U_3, having respective diagrams:

$$D(U_1) = \{P(a), Q(a)\},\ D(U_2) = \{\neg P(a), \neg Q(a)\},\ D(U_3) = \{\neg P(a), Q(a)\}.$$

The model U_1 is unitary, the model U_2 is absolutely free. The pair U_1, U_2 forms a minimal quasivariety defined by the formulas

$$x = y,\ P(x) \to Q(x),\ Q(x) \to P(x),$$

while the pair U_1, U_3 forms a minimal quasivariety defined by the formulas $x = y, Q(x)$, and the quasivariety K itself is not minima".

In this example, we can see that the subquasivariety $\{U_1, U_2\}$ is not an idempotently formula-definable class, but the subquasivariety $\{U_1, U_3\}$ is an idempotently formula-definable class like the quasivariety K itself.

That is, we have examples of idempotently formula-definable semigroups of elementary types and not idempotently formula-definable semigroups of elementary types.

Each idempotent defines a unique idempotent formula-definable semigroup of theories. And to each idempotent formula-definable semigroup of elementary types corresponds a unique idempotent determinant of this semigroup. This semigroup is an axiomatizable set of theories by Theorem 8.

By analogy, this can be said of idempotently formula-definable model classes.

From the previous considerations we see that idempotent formula-definable semigroups of elementary types differ from semigroups in the classical sense in that they consider infinite products and Proposition 5 and the idempotent determinant for each idempotent formula-definable semigroup of elementary types plays the role of a zero element.

Among quasivarieties there are quasivarieties V which are not idempotently formula-definable classes of models. But:

Theorem 10. If K is a variety of models, then Th_K is an idempotent formula-definable semigroup of elementary types of class K. In other words, any variety of models is an idempotent formula-definable class.

Proof. This follows from Theorem 6 and the fact that a variety is defined by identities that are stable under direct products of models.

For example, the set of all elementary types of Boolean algebras, under the product operation, forms an idempotent formula-definable semigroup. We give examples of formula-definable model classes that are quasivarieties but are not varieties.

Theorem 11. * [2] The class of semigroups embeddable in groups forms a quasivariety.

Let V be the class of semigroups embeddable in groups. Then, the corresponding set Th_V, consisting of all elementary types of semigroups in V, forms a semigroup under the product of elementary types. Moreover, it is closed under infinite products.

Question: Is the quasivariety of semigroups embeddable in groups an idempotent formula-definable class?

It is known [2] that for commutative semigroups, the validity of the quasidentity of contraction in the semigroup

$$xy = xz \to y = z \qquad (*)$$

is sufficient for embedding the semigroup into a group.

Thus, the class of commutative semigroups satisfying the quasidentity forms a quasivariety of semigroups embeddable in groups. Consequently, the set of all elementary types of such semigroups forms a semigroup of elementary types.

Theorem 12. Let K be the class of commutative semigroups (with reduction) satisfying the quasidentity (*), and let Th_K denote the set of all elementary types of semigroups in K. Then Th_K is an idempotent formula-definable semigroup, and K is an idempotent formula-definable class of semigroups embeddable in groups.

Proof. Since K, the class of commutative semigroups embeddable in groups, is a quasivariety, Th_K is closed under infinite products. By Theorem 6, there exists an idempotent T such that for any $T_1 \in Th_K$ holds $T_1 \cdot T = T$. But, since K is defined by the quasidentity (*), this identity is present in T due to the multiplicative stability of quasidentities under products. Additionally, in any semigroup true $xx = xx \to x = x$.

If T' is the elementary type of a commutative semigroup that does not satisfy (*), then $T' \cdot T \neq T$. Thus, elementary type T serves as the determinant of Th_K, making Th_K an idempotent formula-definable semigroup.

It is clear that in this case, the class K is a quasivariety that is not a variety. Moreover, since in a semigroup embeddable in a group there can exist only one idempotent, which is the identity, the semigroup Th_K itself is not embeddable in a group.

The following theorem gives a sufficient condition when the formula-definite class will be an inductive class, that is, closed with respect to the union of chains.

Theorem 13. A formula-definite class K of models will be an inductive class, when the determinant T of the semigroup of elementary types of this class is a $\forall\exists$-elementary type.

Proof. Let $M_1 \subseteq M_2 \subseteq ...$ be a chain of models in K. By Theorem 8, K is an axiomatizable class. Take any model A of the elementary type T and consider the chain $M_1 \times A \subseteq M_2 \times A \subseteq ....$ Take the union of this chain. Since T is $\forall\exists$-elementary type, the union of this chain is the model of T. Since K is a formula-definable, axiomatizable class of models, the union $M_1 \subseteq M_2 \subseteq ...$ is a model in K.

Theorem 14. If G_1, G_2 are formula-definable semigroups of elementary types, their intersection $G1 \cap G_2$ is also a formula-definable semigroup of elementary types.

Proof. The intersection $G_1 \cap G_2 \neq \oslash$, as G_1 and G_2 both contain the identity element. $G_1 \cap G_2$ is closed under infinite direct products of theories. By Theorem 8, there exists an idempotent T such that for any $T' \in G_1 \cap G_2$ holds $T' \cdot T = T$. It remains to show that the elementary type T is the determinant of the semigroup $G_1 \cap G_2$. Let T_1 and T_2 be determinants of the semigroups G_1 and G_2, respectively. If for some elementary type $T^C \in Th_L$, $T^C \cdot T = T$, then $T_1 \cdot T \cdot T^C = T_1$ and $T_2 \cdot T \cdot T^C = T_2$. By Proposition 5, $T_1 \cdot T^C = T_1$ and $T_2 \cdot T^C = T_2$. Thus, $T^C \in G_1 \cap G_2$, that is, $T \cdot T^C = T$, meaning T is the determinant of the semigroup $G_1 \cap G_2$.

This theorem allows us to construct, for any set of elementary types M, a minimal formula-definable semigroup G such that $M \subseteq G$, where the model class K_G is the minimal formula-definable class satisfying $K_M \subseteq K_G$. The class K_G is an axiomatizable class of models.

Conclusion

In this paper we investigated properties of semigroups of elementary types of models in a first-order language. The formula-definite semigroups of elementary types, their relation to axiomatizable classes of models and the role of idempotent elements in their structure are considered. The presented results emphasize the importance of studying semigroups of elementary types for analyzing properties of classes of models and reveal new approaches to their classification.

The revealed properties of formula-definite and idempotently formula-definite semigroups demonstrate the potential of using these structures to solve open questions in model theory, such as the problem of axiomatizability of classes closed with respect to products. The examples given in the paper illustrate the variety and complexity of such structures.

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Author Contributions

All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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