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Generalizing Semi-n-Potent Rings

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The present article deals with the problem of characterizing a widely large class of associative and possibly non-commutative rings. So, we define and explore the class of rings R for which each element in R is a sum of a tripotent element from R and an element from the subring $\Delta(R)$ of R which commute with each other, calling them *strongly* Δ*-tripotent* rings, or shortly just *SDT* rings. Succeeding in obtaining a complete description of these rings R modulo their Jacobson radical $J(R)$ as the direct product of a Boolean ring and a Yaqub ring, our results somewhat generalize those established by Koşan-Yildirim-Zhou in Can. Math. Bull. (2019). Specifically, it is proved that if a ring R is SDT, then the factor ring $R/J(R)$ is always reduced and 6 lies in $J(R)$. Even something more, as already noticed before, it is shown that the quotient $R/J(R)$ is a tripotent ring, which means that each of its elements satisfies the cubic equation $x^3 = x$. Furthermore, examining triangular matrix rings $\mathrm{T}_n(R)$, we succeeded to classify its structure rather completely in the case where R is a local ring and $n \geq 3$ by establishing a satisfactory necessary and sufficient condition in terms of the ring R and its sections, resp., divisions.

Keywords: idempotent, tripotent, strongly nil-clean ring, Boolean ring, semi-tripotent ring, Δ subring, Jacobson radical, triangular matrix ring.

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Introduction and Motivation

Throughout this paper, all rings are assumed to be unital and associative. Almost all symbols, notation and concepts are standard being consistent with the classical book [1]. The Jacobson radical, the lower nil-radical, the set of nilpotent elements, the set of idempotent elements, and the set of units of R are denoted, respectively, by $J(R)$, $\mathrm{Nil}_*(R)$, $\mathrm{Nil}(R)$, $\mathrm{Id}(R)$, and $U(R)$. Additionally, we write $\mathrm{M}_n(R)$, $\mathrm{T}_n(R)$ and $R[x]$ for the $n \times n$ full matrix ring, the $n \times n$ upper triangular matrix ring, and the polynomial ring over R, respectively.

The core focus of this exploration is the set

$$\begin{aligned} J(R) \subseteq \Delta(R) &= \{x \in R : x + u \in U(R) \text{ for all } u \in U(R)\} \\ &= \{x \in R : 1 - xu \text{ is invertible for all } u \in U(R)\} \\ &= \{x \in R : 1 - ux \text{ is invertible for all } u \in U(R)\}, \end{aligned}$$

which was examined by Lam in [2; Exercise 4.24] and recently explored in detail by Leroy-Matczuk in [3]. It was indicated in [3; Theorems 3 and 6] that $\Delta(R)$ represents the (proper) largest Jacobson radical subring of R that remains closed under multiplication by all units (resp., quasi-invertible elements) of R, and it is an ideal of R exactly when $\Delta(R) = J(R)$.

In the contemporary ring theory, the class of strongly nil-clean rings possesses significant importance. A ring R is called *strongly nil-clean* if every element of R can be expressed as the sum of an

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idempotent in R and a nilpotent element in R that commute with each other (see [4–6]). Later on, Chen and Sheibani generalized in [7] this concept and introduced the so-called strongly 2-nil-clean rings: a ring R is defined as *strongly 2-nil-clean* if every element of R can be written as the sum of a tripotent element of R (i.e., an element $x \in R$ such that $x^3 = x$) and a nilpotent element of R that commute.

On the other hand, in a way of similarity, *strongly J-clean* rings are those rings in which every element can be written as the sum of an idempotent and an element from the Jacobson radical that commute [8, 9]. In this vein, Koşan et al. introduced in [10] the so-termed *semi-tripotent* rings R in which each element is the sum of a tripotent element from R and an element from $J(R)$.

Considering and analyzing these definitions, as well as the fact that $\Delta(R)$ is a (possibly proper) subset of $J(R)$, that is *not* necessarily an ideal, and which also does *not* have useful properties like the set $\mathrm{Nil}(R)$, a question naturally arises about the properties of those rings R for which each element is the sum of a tripotent element from R and an element from $\Delta(R)$ that commute with each other. The main objective of the current article is namely to investigate these types of rings and to conduct a comprehensive study of their structure.

Thereby, we come to the following key notion, motivated by the discussion alluded to above.

Definition 1. We say that R is a *strongly Δ-tripotent* ring, or just an *SDT ring* for short, if every element of R is the sum of a tripotent from R and an element from $\Delta(R)$ that commute with each other. Such a sum's presentation is also said to be an *SDT representation.*

Our further plan in the organization of our study is the following: In the next section, we obtain some crucial examples and principal properties of such rings establishing their connection with many standard properties – e.g., such as uniquely clean (see Corollary 2). In the subsequent section, we achieve the major result describing the algebraic structure of the SDT rings in an appropriate form showing that these rings modulo their Jacobson radical are the direct product of a Boolean ring and a Yaqub ring (see Theorem 1). Some other closely related statements are also proved such as Propositions 2 and 6. In the fourth section, we study the behavior of the given SDT concept under various ring extensions and, specifically, we characterize when $\mathrm{T}_n(R)$ is an SDT ring by finding a necessary and sufficient condition, provided that R is local and $n \geq 3$ (see Theorem 2). In the final fifth section, we conclude with some commentaries and two challenging open problems (see, e.g., Problems 1 and 2) which, hopefully, will stimulate a future intensive examination of the present subject.

1 Examples and Basic Properties

The following claim can easily be proven, so we omit the details leaving them to the interested reader for check.

Lemma 1. (1) Suppose $R = \prod_{i \in I} R_i$. Then, R is an SDT ring if, and only if, for each $i \in I$, R_i is an SDT ring.

(2) Suppose R is a ring and I is an ideal of R such that $I \subseteq J(R)$. Then, R/I is an SDT ring.

We proceed by proving the following three technical assertions.

Lemma 2. For every $e = e^3 \in R$ and $d \in \Delta(R)$, we have $(e \pm e^2)d, d(e \pm e^2), 2e^2d$, and $2ed \in \Delta(R)$.

Proof. For every $e = e^3 \in R$, we have

$$((1 - e^2) - e)((1 - e^2) - e) = 1 = ((1 - e^2) + e)((1 - e^2) + e).$$

Therefore, $(1 - e^2 \pm e) \in U(R)$, so it follows from [3; Lemma 1(2)] that, for every $d \in \Delta(R)$, both $((1 - e^2 \pm e)d$ and $d((1 - e^2 \pm e) \in \Delta(R)$. Since $\Delta(R)$ is a subring of R, we have $(e \pm e^2)d$ and $d(e \pm e^2) \in \Delta(R)$. This implies that $2ed$ and $2e^2d \in \Delta(R)$, as required. □

Lemma 3. Let R be an SDT ring, and $a \in R$. If $a^2 \in \Delta(R)$, then $a \in \Delta(R)$.

Proof. Assume that $a = e + d$ is an SDT representation. We have $a^2 = e^2 + 2ed + d^2$. By Lemma 2, it must be that

$$e^2 = a^2 - 2ed - d^2 \in \Delta(R) \cap \mathrm{Id}(R) = \{0\},$$

which implies $e = 0$. Thus, $a = d \in \Delta(R)$, as expected. □

Lemma 4. Let R be an SDT ring. Then, for every $a \in R$, $a - a^3 \in \Delta(R)$.

Proof. Assume $a = e + d$ is an SDT representation. We calculate that

$$a - a^3 = (d - d^3) - (2e^2d + 2ed) - (e^2d + ed^2).$$

Furthermore, according to Lemma 2, it suffices to show that $e^2d + ed^2 \in \Delta(R)$. But, Lemma 2 tells us that $e^2d + ed^2 \in \Delta(R)$ precisely when $ed + e^2d^2 \in \Delta(R)$. Consequently, we show that $ed + e^2d^2 \in \Delta(R)$.

To this target, assume $ed = f + b$ is an SDT representation. Then,

$$e^2d^2 = f^2 + 2fb + b^2.$$

Thus,

$$ed + e^2d^2 = (f + f^2) + (b + 2fb + b^2).$$

Now, multiplying by d and d^2 both sides of the previous relation, we have

$$ed^2 + e^2d^3 = (f + f^2)d + (b + 2fb + b^2)d \in \Delta(R),$$

$$ed^3 + e^2d^4 = (f + f^2)d^2 + (b + 2fb + b^2)d^2.$$

Owing to Lemma 2, we infer that $ed^2 + e^2d^3, ed^3 + e^2d^4 \in \Delta(R)$. Also,

$$ed^2 + e^2d^3 = ed^2 + e^2d^3 - ed^3 + ed^3 - e^2d^2 + e^2d^2 = e^2d^2 + ed^3 + (e^2 - e)d^3 + (e - e^2)d^2.$$

Thus, in virtue of Lemma 2, it follows that $e^2d^2 + ed^3 \in \Delta(R)$. Therefore, we get

$$\begin{cases} e^2d^2 + ed^3 \in \Delta(R), \\ ed^3 + e^2d^4 \in \Delta(R), \end{cases} \implies \quad e^2d^2 + e^2d^4 \in \Delta(R).$$

We now have that

$$(ed + e^2d^2)^2 = e^2d^2 + 2ed^3 + e^2d^4 \in \Delta(R).$$

So, Lemma 3 enables us that $ed + e^2d^2 \in \Delta(R)$, as pursued. □

We now arrive at the following concrete application of the last lemma.

Example 1. Let R be an arbitrary ring. Then, $R[x]$ is *not* an SDT ring.

Proof. Assume the contrary. Then, applying Lemma 4, we derive that $x - x^3 \in \Delta(R[x])$, and thus $1 - x + x^3 \in U(R[x])$, which is the wanted contradiction. □

With the previous example in mind, the ring $R[x]$ is surely not SDT. However, a logical question arises about the form of elements with an SDT representation in the polynomial ring $R[x]$. We will attempt to answer this question below.

Recall that a ring R is said to be *2-primal* if $\mathrm{Nil}_*(R) = \mathrm{Nil}(R)$. For instance, it is well known that any commutative ring and any reduced ring are definitely 2-primal.

Likewise, for an endomorphism σ of R, the ring R is called *σ-compatible* if, for every $a, b \in R$, the equality $ab = 0$ if, and only if, $a\sigma(b) = 0$ [11]. In this case, it is clear that σ is always injective.

We now manage to prove the following two pivotal statements.

Proposition 1. Let R be a 2-primal and α-compatible ring. Then,

$$\Delta(R[x,\alpha]) = \Delta(R) + Nil_*(R[x,\alpha])x.$$

Proof. Assuming $f = \sum_{i=0}^n a_i x^i \in \Delta(R[x,\alpha])$, then, for each $u \in U(R)$, we have that $1 - uf \in U(R[x,\alpha])$. Thus, taking into account [12; Corollary 2.14], $1 - ua_0 \in U(R)$ holds and, for every $1 \le i \le n$, it holds $ua_i \in \mathrm{Nil}_*(R)$. Since $\mathrm{Nil}_*(R)$ is an ideal, we deduce $a_0 \in \Delta(R)$ and hence, for each $1 \le i \le n$, we obtain $a_i \in Nil_*(R)$. Since R is a 2-primal ring, [12; Lemma 2.2] applies to get that $Nil_*(R)[x,\alpha] = Nil_*(R[x,\alpha])$, as desired.

Conversely, assume $f \in \Delta(R) + Nil_*(R[x,\alpha])x$ and $u \in U(R[x,\alpha])$. Then, employing [12; Corollary 2.14], we have $u \in U(R) + Nil_*(R[x,\alpha])x$. But, since R is a 2-primal ring, we receive $1 - uf \in U(R) + Nil_*(R[x,\alpha])x \subseteq U(R[x,\alpha])$, whence $f \in \Delta(R[x,\alpha])$, as promised. □

Proposition 2. Let R be a 2-primal and α-compatible ring, and let $e^3 = e = \sum_{i=0}^n e_i x^i \in R[x,\alpha]$. Then, $e_0^3 = e_0$ and, for every $1 \le i \le n$, the inclusion $e_i \in Nil(R)$ is true.

Proof. It is easy to see that $e_0^3 = e_0$, so it suffices to show that, for every $1 \le i \le n$, the relation $e_i \in \mathrm{Nil}(R)$ is valid. Since $e^3 = e$, we inspect that $e_n\alpha^n(e_n)\alpha^{2n}(e_n) = 0$. And because R is α-compatible, [13; Lemma 2.1] is applicable to get that $e_n^3 = 0$.

Now, set $g := f - e_n x^n$. Since $f^3 = f$ and $e_n \in \mathrm{Nil}_*(R)$, we have $g - g^3 \in \mathrm{Nil}_*(R)[x,\alpha]$, so $\bar{g} = \bar{g}^3 \in R/\mathrm{Nil}_*(R)[x,\alpha]$. Thus, one verifies that

$$e_{n-1}\alpha^{n-1}(e_{n-1})\alpha^{2n-2}(e_{n-1}) \in Nil_*(R).$$

But, since R is an α-compatible ring, [13; Lemma 2.1] works to obtain that $e_{n-1} \in \mathrm{Nil}(R)$. Continuing in this aspect, it can be shown that, for each $1 \le i \le n$, the condition $e_i \in Nil(R)$ is fulfilled, as asked for. □

To specify the elements with an SDT representation of the ring $R[x,\alpha]$, we need new notation. For convenience of the exposition, we just put the set of elements with an SDT representation in the ring R to be abbreviated as $SDT(R)$.

So, we have the validity of the following.

Lemma 5. Let R be a 2-primal and α-compatible ring. Then,

$$SDT(R[x,\alpha]) \subseteq SDT(R) + Nil_*(R)[x,\alpha]x.$$

Proof. Assume $f = \sum_{i=0}^n f_i x^i \in SDT(R[x,\alpha])$ and $f = \sum_{i=0}^n e_i x^i + \sum_{i=0}^n d_i x^i$ is an SDT representation. In accordance with Propositions 1 and 2, we have $e_0 = e_0^3$ and $d_0 \in \Delta(R)$, and hence clearly $e_0 d_0 = d_0 e_0$, so that $f_0 \in SDT(R)$.

Moreover, with the aid of Proposition 2, for every $1 \le i \le n$, it must be that $e_i, d_i \in \mathrm{Nil}_*(R)$, whence $f_i = e_i + d_i \in \mathrm{Nil}_*(R)$, as required. □

The next affirmation is crucial.

Lemma 6. Let R be an SDT ring. Then, $R/J(R)$ is reduced.

Proof. Assume $x^2 \in J(R) \subseteq \Delta(R)$. Thus, by Lemma 3, we have $x \in \Delta(R)$. Let $r \in R$. Since $1 - r^2x^2 \in U(R)$, we may set $u := 1 - rx^2 r \in U(R)$. Therefore,

$$(1 - rx)(1 + rx) = 1 - rx + xr - rx^2 r = xr - rx + u.$$

It suffices to show that $xr - rx \in \Delta(R)$. To this goal, assume $r = e + d$ is an SDT representation. Then,

$$xr - rx = x(e + d) - (e + d)x = xe - ex + (xd - dx),$$

and as $x, d \in \Delta(R)$, it is just sufficient to prove that $xe - ex \in \Delta(R)$.

Since
$$\left[e^2x(1-e^2)\right]^2 = 0 = \left[(1-e^2)xe^2\right]^2.$$
Lemma 3 assures that
$$\begin{cases} e^2x(1-e^2) \in \Delta(R) \Longrightarrow e^2x - e^2xe^2 \in \Delta(R), \\ (1-e^2)xe^2 \in \Delta(R) \Longrightarrow xe^2 - e^2xe^2 \in \Delta(R). \end{cases}$$
However, because $\Delta(R)$ is closed under addition, we arrive at $e^2x - xe^2 \in \Delta(R)$. Consequently,
$$xe - ex = xe + xe^2 - xe^2 - ex - e^2x + e^2x = e^2x - xe^2 + x(e+e^2) - (e+e^2)x \in \Delta(R).$$
Hence,
$$(1-rx)(1+xr) \in U(R).$$
But R was arbitrary, and so $x \in J(R)$, as needed. □

Given the truthfulness of Lemma 4, we have that, for every SDT ring R, $6 = 2^3 - 2 \in \Delta(R)$. This raises a logical question: if R is an SDT ring, is $6 \in J(R)$? We will answer this query in the following lemma.

Lemma 7. Let R be an SDT ring. Then, $6 \in J(R)$.

Proof. Invoking Lemma 4, we know that $6 \in \Delta(R)$, which implies $12 = 6 + 6 \in \Delta(R)$. Letting $r \in R$ be arbitrary, and letting $r = e + d$ be an SDT representation, Lemma 2 ensures that
$$1 - 12r = 1 - 12e - 12d = 1 - 2(6e) - 12d \in 1 + \Delta(R) \subseteq U(R).$$
Thus, $12 \in J(R)$.

Furthermore, since $6^2 = 36 = 3 \times 12 \in J(R)$, Lemma 6 helps us to conclude that $6 \in J(R)$, as stated. □

As a useful consequence, we deduce the following.

Corollary 1. Let R be an SDT ring. Then, the following two points hold:
(1) $2 \in U(R)$ if, and only if, $3 \in J(R)$.
(2) $3 \in U(R)$ if, and only if, $2 \in J(R)$.

Proof. The proof is pretty straightforward being based on Lemma 7, so we leave it voluntarily. □

The next two assertions are worthy of documentation.

Proposition 3. Let R be an SDT ring such that $2 \in U(R)$. Then, $\Delta(R)$ is an ideal. In particular, under these conditions, $\Delta(R) = J(R)$.

Proof. Since $\Delta(R)$ is closed under addition, it is sufficient to show that, for any $d \in \Delta(R)$ and $r \in R$, the relations $rd, dr \in \Delta(R)$ are valid. Assume, for this aim, that $rd = e+b$ and $r = f+b'$ are two SDT representations. Exploiting Lemma 2, we know $2fd \in \Delta(R)$. Since $2 \in U(R)$, [3; Lemma 1(2)] teaches us that $fd \in \Delta(R)$. So, we have
$$rd = e + b = fd + b'd \Longrightarrow e - fd = b'd - b \in \Delta(R).$$
But, since $fd \in \Delta(R)$, it follows that $e \in \Delta(R)$, so $e^2 \in \Delta(R) \cap Id(R) = \{0\}$, which forces $e = 0$. Therefore, $rd = b \in \Delta(R)$. Similarly, it can be shown that $dr \in \Delta(R)$, guaranteeing the claim. □

Proposition 4. Let R be an SDT ring with $3 \in U(R)$. Then, for any $a \in R$, we have $a = f + b$, where $f = f^2 \in R$, $b \in \Delta(R)$ and $fb = bf$.

Proof. Suppose $a = f + d$ is an SDT representation. Then,

$$a - a^2 = (f - f^2) + (d - 2fd - d^2).$$

Since $3 \in U(R)$ by Corollary 1, we get $2 \in J(R)$. Thus, $(f - f^2)^2 = -2(f - f^2) \in J(R)$ and, with Lemma 7 at hand, we observe that $f - f^2 \in J(R)$. This gives $a - a^2 \in \Delta(R)$.

On the other hand, since

$$a - f^2 = (a - a^2) + 2(a^2 - f^2 - fd) - d^2 \in \Delta(R),$$

by setting $e := f^2$, we finish the proof after all. □

A ring R is called an *SDI ring* if, for every $r \in R$, there exist $e = e^2 \in R$ and $b \in \Delta(R)$ such that $r = e + b$ and $eb = be$. Recall also that a ring is called a ΔU *ring*, provided $1 + \Delta(R) = U(R)$ [14].

The following closely related results are of some interest as well.

Lemma 8. Every SDI ring is a ΔU ring.

Proof. Suppose $u \in U(R)$ and $u = e + d$ is an SDI representation. Then, we have

$$e = u - d \in U(R) + \Delta(R) \subseteq U(R) \cap Id(R) = \{1\},$$

as required. □

Lemma 9. ([14; Proposition 2.3]) The ring R is a ΔU ring if and only if $U(R) + U(R) \subseteq \Delta(R)$; and then, $U(R) + U(R) = \Delta(R)$.

Recall that a ring R is said to be *uniquely clean*, provided that each element in R has a unique representation as the sum of an idempotent and a unit [15].

The next valuable consequence gives some transversal between the notions of SDI rings and unique cleanness.

Corollary 2. Let R be a ring. Then the following are equivalent:

(1) R is uniquely clean.

(2) R is SDI and all idempotents are central.

Proof. (1) $\Rightarrow$ (2). Assume R is a uniquely clean ring. Consulting with [15; Lemma 4], every idempotent in R is central. Besides, by virtue of [15; Theorem 20], for every $a \in R$, there exists a unique idempotent e such that $a - e \in J(R) \subseteq \Delta(R)$. Thus, there exists $d \in \Delta(R)$ such that $a = e + d$. Since all idempotents are central, we have $ed = de$.

(2) $\Rightarrow$ (1). Assume R is an SDI ring, and let $a \in R$ be arbitrary. Suppose $a + 1 = e + d$ is an SDI representation. Then, $a = e + (d - 1)$, which is a clean representation. Assume now that $e + u = f + v$ are two clean representations. So, Lemma 9 informs us that $e - f = v - u \in \Delta(R)$. Since all idempotents are central, we find $e - f = (e - f)^3$, and so $(e - f)^2 \in \Delta(R) \cap Id(R) = \{0\}$. Therefore, $e - f = (e - f)^3 = (e - f)(e - f)^2 = 0$. Hence, $e = f$, as it must be. □

2 The Main Characterizations

We start our considerations here with some relationships between certain classes of rings.

Proposition 5. Suppose R is an SDT ring and a domain. Then, R is a local ring.

Proof. Let $a \in R$. We want to show that either $a \in U(R)$ or $a \in \Delta(R)$. To that end, suppose $a = e + d$ is an SDT representation. If $e = 0$, then $a = d \in \Delta(R)$. If $e \neq 0$, then as $e^3 = e$ it must be $e(1 - e^2) = 0$. But, since R is a domain, $(1 - e)(1 + e) = 1 - e^2 = 0$, so either $e = 1$ or $e = -1$. Therefore, either $a = 1 + d \in U(R)$ or $a = -1 + d \in U(R)$. It can next easily be shown that R is a local ring if, and only if, $R = U(R) \cup \Delta(R)$, as required. □

As an immediate consequence, we yield:

Corollary 3. Suppose R is a strongly 2-nil clean and local ring. Then, R is an SDT ring.

Proof. It is pretty easy, because in a local ring the containment $\mathrm{Nil}(R) \subseteq J(R)$ always holds. □

The next assertion is of some importance by giving some close relevance between the notion of a semi-tripotent ring as stated in [10] and the new concept of an SDT ring given above.

Proposition 6. Suppose R is a semi-tripotent and local ring. Then, R is an SDT ring.

Proof. Since R is a local ring, either $2 \in J(R)$ or $2 \in U(R)$. If $2 \in J(R)$, then in virtue of [10; Theorem 3.5] the factor-ring $R/J(R)$ is Boolean. On the other hand, as R is local, it has to be that $R/J(R) \cong \mathbb{Z}_2$, and so $R = J(R) \cup (1 + J(R))$, yielding R is an SDT ring. If, however, $2 \in U(R)$, then again [10; Theorem 3.5] works to get that the quotient-ring $R/J(R)$ is a Yaqub ring. However, because R is local, it must be that $R/J(R) \cong \mathbb{Z}_3$, and thus $R = J(R) \cup (1 + J(R)) \cup (-1 + J(R))$ implying R is an SDT ring, as asserted. □

It is well known that a ring is Boolean if and only if it is a subdirect product of copies of $\mathbb{Z}_2$. Analogously, in [7], Chen and Sheibani called a non-zero ring R a *Yaqub ring* if it is a subdirect product of copies of $\mathbb{Z}_3$. They proved that R is a Yaqub ring if, and only if, 3 is nilpotent and R is a tripotent ring (that is, each of its element is tripotent).

We are now ready to attack the chief characterizing result, thereby completely describing the structure of the SDT rings.

Theorem 1. Assume R is an SDT ring. Then, $R/J(R)$ is a tripotent ring, i.e., $R/J(R) \cong R_1 \times R_2$, where R_1 is a Boolean ring and R_2 is a Yaqub ring.

Proof. Referring to Lemma 7, we have $6 \in J(R)$. Set $\bar{R} := R/J(R)$. Thanks to the famous Chinese Remainder Theorem, we write $\bar{R} \cong R_1 \times R_2$, where $R_1 := \bar{R}/2\bar{R}$ and $R_2 := \bar{R}/3\bar{R}$. Since R is an SDT ring, Lemma 1(2) guarantees that $\bar{R}$ is an SDT ring too. Therefore, again in view of Lemma 1(1), R_1 is an SDT ring. Since $2 = 0$ in R_1, we have $3 \in U(R_1)$. Thankfully, Proposition 4 yields R_1 is an SDI ring. Also, Lemma 6 implies that R_1 is reduced, and thus all idempotents in R_1 are central. Therefore, Corollary 2 shows that R_1 is a uniquely clean ring. Note that, as $J(R) = 0$, it must be that $J(R_1) = 0$. Using now [15; Theorem 19], we conclude that R_1 is a Boolean ring, as formulated.

On the other hand, since $3 = 0$ in $R_2 \neq \{0\}$, we have $2 \in U(R_2)$. Knowing Proposition 3, we obtain $J(R_2) = \Delta(R_2)$. This means, with the help of Lemma 4, that, for any $a \in R_2$, the relations $a - a^3 \in \Delta(R_2) = J(R_2) = 0$ are true. Thus, for any $a \in R_2$, we get that $a = a^3$. Furthermore, using [7; Lemma 4.4], we infer that R_2 is a Yaqub ring, as given. □

It is worthwhile noticing that the extra requirement on the first direct component R_1 and the second direct component R_2 to be not simultaneously $\{0\}$ can be freely ignored here, as opposed to what was shown in [16], where an analogous shortcoming was unambiguously detected for the main result of the paper [17].

Let R be a ring, and let $a \in R$. Suppose $ann_l a := \{r \in R : ra = 0\}$ and $ann_r a := \{r \in R : ar = 0\}$.

We continue by verifying the following two needed technicalities.

Lemma 10. Let R be a ring and $a = e + d$ an SDT representation in R. Then, $\mathrm{ann}_l(a) \subseteq \mathrm{ann}_l(e)$ and $\mathrm{ann}_r(a) \subseteq \mathrm{ann}_r(e)$.

Proof. Assume $ra = 0$. Now, Lemma 2 applies to ensure that there exists $d' \in \Delta(R)$ such that $a^2 = e^2 + d'$. Since $ra = 0$, we have $re^2 + rd' = 0$. Now, multiplying by e from the right, we get $re + red' = 0$, and so $re(1 + d') = 0$. Since $d' \in \Delta(R)$, it follows that $1 + d' \in U(R)$ which forces $re = 0$. Thus, $r \in \mathrm{ann}_l(e)$. Similarly, it can be shown that the inclusion $\mathrm{ann}_r(a) \subseteq \mathrm{ann}_r(e)$ is too valid, as required. □

Lemma 11. Let R be a ring and $e \in R$ an idempotent. If $a \in eRe$ is an SDT element in R, then a is an SDT element in the corresponding corner subring eRe.

Proof. Write $a = f + d$, where $f = f^3$, $d \in \Delta(R)$ and $fd = df$. Since $1 - e \in \mathrm{ann}_l(a) \cap \mathrm{ann}_r(a)$, Lemma 10 is a guarantor that $1 - e \in \mathrm{ann}_l(f) \cap \mathrm{ann}_r(f)$ implying $(1 - e)f = f(1 - e) = 0$. Thus, $f = ef = fe$. Likewise, since $a \in eRe$, we receive $a = ea = ae = eae$. But, subsequently multiplying $a = f + d$ by e from the left and right, we obtain that $a = efe + ede$. Note that, since $f = ef = fe$ and f is a tripotent, efe is also a tripotent. So, it suffices to show that $ede \in \Delta(eRe)$.

On the other hand, since $f = ef = fe = efe$ and $a = ea = ae = eae$, it is evident that

$$d = ed = de = ede \in \Delta(R) \cap eRe.$$

Now, we show that $eRe \cap \Delta(R) \subseteq \Delta(eRe)$ always holds. To this purpose, assume $r \in eRe \cap \Delta(R)$ and $u \in U(eRe)$. Then, $(u + (1-e))(u^{-1} + (1-e)) = 1$, so $u + (1-e) \in U(R)$. Since $r \in \Delta(R)$, there exists $v \in R$ such that $(1-(u+(1-e))r)v = 1$. But $r \in eRe$, so that $(1-ur)v = 1$. Furthermore, multiplying subsequently by e from the left and right, we extract that $(e - ur)eve = e$ forcing $r \in \Delta(eRe)$. Finally, $d \in \Delta(eRe)$, and we are done. □

As an automatic consequence, we yield the following.

Corollary 4. Let R be a ring, and let $e \in R$ be an idempotent. If R is an SDT ring, then so is the corresponding corner subring eRe.

Furthermore, in regard to the last corollary, a logically arising question is whether or not the converse in its formulation holds, that is, if both eRe and $(1 - e)R(1 - e)$ are SDT rings, is it true that so does R? However, the next construction, suggested to us by Dr. Omer Cantor to whom we express our sincere gratitude, illustrates that this question has a negative solution. In fact, let $R := \mathrm{M}_2(\mathbb{Z}_2)$ and set $e := E_{11}$. An easy check shows that both eRe and $(1 - e)R(1 - e)$ are isomorphic to $\mathbb{Z}_2$, so they are obviously SDT rings. However, it is readily to verify that $\Delta(R) = (0)$ by direct computation and, of course, some elements of R, such as E_{12}, are *not* tripotent or even *not* n-potent for any natural number $n \geq 3$. Therefore, R is *not* an SDT ring, as suspected.

3 Triangular Matrix Rings

As usual, a ring R is termed *local*, provided $R/J(R)$ is a division ring, that is, each element in $R \setminus J(R)$ is a unit, which set-theoretically means that $R = J(R) \cup U(R)$.

We begin here with the following technicality.

Lemma 12. Let R be a local ring with $2 \in U(R)$. Then, R has only trivial tripotent elements.

Proof. Suppose that $e = e^3 \in R$. If $e \in J(R)$, then $e(1-e^2) = 0$, whence $e = 0$. If now $e \in U(R)$, then $e^2 = 1$, and so $(1-e)(1+e) = 0$. Since $(1-e)+(1+e) = 2 \in U(R)$ and R is a local ring, we have either $1-e \in U(R)$ or $1+e \in U(R)$. This, in turn, means that either $e = 1$ or $e = -1$, as required. □

Based on the above claim, we now considerably extend the well-known Workhorse Lemma (see [18; Lemma 6]) as follows.

Lemma 13. (Generalized Workhorse Lemma) Let R be a local ring such that $2 \in U(R)$, $n \geq 2$ and $A, E \in \mathrm{T}_n(R)$. Suppose that, for all $(i,j) \neq (1,n)$, $(E^3)_{ij} = E_{ij}$ and $(AE - EA)_{ij} = 0$. Suppose also that

$$A = \begin{pmatrix} a & \alpha & c \\ & B & \beta \\ & & b \end{pmatrix} \text{ and } E = \begin{pmatrix} e & \gamma & z \\ & F & \delta \\ & & f \end{pmatrix},$$

where $B, F \in \mathrm{T}_{n-2}(R)$, $a, b, c, e, f, z \in R$, $\alpha, \gamma \in \mathrm{M}_{1,n-2}(R)$ and $\beta, \delta \in \mathrm{M}_{n-2,1}(R)$. Then, the following items are fulfilled:

(i) Given $e = f = 1$, then $E^3 = E$ if and only if $z = -1/2(\gamma F\delta + 2\gamma\delta)$, and in this case, $AE = EA$.

(ii) Given $e = f = -1$, then $E^3 = E$ if and only if $z = -1/2(\gamma F\delta - 2\gamma\delta)$, and in this case, $AE = EA$.

(iii) Given $e = f = 0$, then $E^3 = E$ if and only if $z = \gamma F\delta$, and in this case, $AE = EA$.

(iv) Given $e = 1$ and $f = -1$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta + 2c$.

(v) If $e = -1$ and $f = 1$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta - 2c$.

(vi) If $e = 1$ and $f = 0$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta + c$.

(vii) If $e = 0$ and $f = 1$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta - c$.

(viii) If $e = -1$ and $f = 0$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta - c$.

(w) If $e = 0$ and $f = -1$, then $E^3 = E$. Further, $AE = EA$ if and only if z satisfies the equation $az - zb = \gamma\beta - \alpha\delta + c$.

Proof. (i) It is apparent that $E^3 = E$ if and only if $z = -1/2(\gamma F\delta + 2\gamma\delta)$. We show that $AE = EA$. Given the assumptions, we have

$$z = -1/2(\gamma F\delta + 2\gamma\delta), \tag{1}$$
$$\gamma F = -\gamma F^2, \tag{2}$$
$$F\delta = -F^2\delta, \tag{3}$$
$$\alpha + \gamma B = a\gamma + \alpha F, \tag{4}$$
$$F\beta + \delta b = B\delta + \beta. \tag{5}$$

In virtue of the above equations, we compute that

$$\begin{aligned}(EA)_{1n} &= c + \gamma\beta + zb \overset{(1)}{=} c + \gamma\beta - 1/2\gamma F\delta b - \gamma\delta b \\ &\overset{(5)}{=} c + \gamma\beta + \gamma(F\beta - B\delta - \beta) + 1/2\gamma F(F\beta - B\delta - \beta) \\ &= c + \gamma\beta + \gamma F\beta - \gamma B\delta - \gamma\beta + 1/2\gamma F^2\beta - 1/2\gamma FB\delta - 1/2\gamma F\beta \\ &\overset{(2)}{=} c - \gamma B\delta - 1/2\gamma FB\delta.\end{aligned}$$

$$\begin{aligned}(AE)_{1n} &= az + \alpha\delta + c \overset{(1)}{=} -a\gamma\delta - 1/2a\gamma F\delta + \alpha\delta + c \\ &\overset{(4)}{=} (\alpha F - \alpha - \gamma B)\delta + 1/2(\alpha F - \alpha - \gamma B)F\delta + \alpha\delta + c \\ &= \alpha F\delta - \alpha\delta - \gamma B\delta + 1/2\alpha F^2\delta - 1/2\alpha F\delta - 1/2\gamma BF\delta + \alpha\delta + c \\ &\overset{(3)}{=} c - \gamma B\delta - 1/2\gamma BF\delta = c - \gamma B\delta - 1/2\gamma FB\delta.\end{aligned}$$

Note that, since $(AE - EA)_{ij} = 0$, we establish $FB = BF$.

(ii) The proof is similar to part (i).

(iii) It is obvious that $E^3 = E$ if and only if $z = \gamma F\delta$. We show that $AE = EA$. Given the assumptions, we have

$$z = \gamma F\delta, \tag{6}$$
$$\gamma = \gamma F^2, \tag{7}$$
$$\delta = F^2\delta, \tag{8}$$
$$\gamma B = a\gamma + \alpha F, \tag{9}$$
$$B\delta = F\beta + \delta b. \tag{10}$$

From the above equations, we calculate that

$$\begin{aligned}(EA)_{1n} &= \gamma\beta + zb \overset{(6)}{=} \gamma\beta + \gamma F\delta b \\ &\overset{(10)}{=} \gamma\beta + \gamma F(B\delta - F\beta) \\ &= \gamma\beta + \gamma FB\delta - \gamma F^2\beta \\ &\overset{(7)}{=} \gamma FB\delta.\end{aligned}$$

$$\begin{aligned}(AE)_{1n} &= az + \alpha\delta \overset{(6)}{=} a\gamma F\delta + \alpha\delta \\ &\overset{(9)}{=} (\gamma\beta - \alpha F)F\delta + \alpha\delta \\ &= \gamma BF\delta - \alpha F^2\delta + \alpha\delta \\ &\overset{(8)}{=} \gamma BF\delta = \gamma FB\delta.\end{aligned}$$

(iv) Assume $e = 1$ and $f = -1$. Then, under the assumptions, we deduce that

$$\gamma F = -\gamma F^2, \quad F^2\delta = F\delta \implies \gamma F\delta = -\gamma F^2\delta = -\gamma F\delta \implies 2\,\gamma F\delta = 0.$$

But, since $2 \in U(R)$, we have $\gamma F\delta = 0$. Thus, we get $(E^3)_{1n} = \gamma F\delta + z = z = E_{1n}$ and, therefore, $E^3 = E$. Moreover, it is clear that $EA = AE$ if and only if

$$az + \alpha\delta - c = c + \gamma\beta + zb,$$

which is equivalent to

$$az - zb = \gamma\beta - \alpha\delta + 2c.$$

(v) The proof is similar to part (iv).

(vi) Assume $e = 1$ and $f = 0$. So, under the given assumptions, we have

$$\gamma F = -\gamma F^2, \quad \delta = F^2\delta \implies \gamma F\delta = -\gamma F^2\delta = -\gamma\delta.$$

Consequently, we derive $(E^3)_{1n} = z + \gamma\delta + \gamma F\delta = z = E_{1n}$, and hence $E^3 = E$. It is also readily checked that $EA = AE$ if and only if

$$az - zb = \gamma\beta - \alpha\delta + c.$$

Finally, one sees that points (vii), (viii) and (w) possess proofs which are similar to that of (vi). □

The next preliminary facts are worthy of discussion: let $a \in R$. The mappings $l_a : R \to R$ and $r_a : R \to R$ represent the (additive) abelian group endomorphisms defined respectively by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Consequently, the expression $l_a - r_b$ defines an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$. According to [5], a local ring R is classified as *bleached* if, for any $a \in U(R)$ and $b \in J(R)$, both $l_a - r_b$ and $l_b - r_a$ are surjective. The category of bleached local rings includes many well-established examples, such as commutative local rings, local rings with nil Jacobson radicals, and local rings in which some power of each element of their Jacobson radicals is central [18; Example 13].

Now, we need the following.

Lemma 14. Let R be a local ring such that $2 \in U(R)$, and suppose that $A \in \mathrm{T}_n(R)$. Write A as (a_{ij}). Then, for any set $\{e_{ii}\}_{i=1}^n$ of tripotents in R such that $e_{ii} = e_{jj}$ whenever $l_{a_{ii}} - r_{a_{jj}}$ is not a surjective abelian group endomorphism of R, there exists a tripotent $E \in \mathrm{T}_n(R)$ such that $AE = EA$ and $E_{ii} = e_{ii}$ for every $i \in \{1, \dots, n\}$.

Proof. Leveraging Lemma 13, the proof process mirrors that of [18; Lemma 7]. To avoid redundancy, we omit the detailed proof. □

We are now in a position to attack the main result in this section, in which the proof we shall apply the established above Theorem 1.

Theorem 2. Let R be a local ring and $n > 2$. Then, the following conditions are equivalent:
(1) $\mathrm{T}_n(R)$ is an SDT ring;
(2) either
(2.1) R is a bleached ring and $R/J(R) \cong \mathbb{Z}_2$;
or
(2.2) R is a bleached ring, $R/J(R) \cong \mathbb{Z}_3$ and, if $a, b \in R$ such that $a - 1 \in \Delta(R)$ and $b + 1 \in \Delta(R)$, then $l_a - r_b : R \to R$ is surjective.

Proof. Since R is a local ring, we have either $2 \in J(R)$ or $2 \in U(R)$. We prove the theorem for both cases independently.

Case 1: If $2 \in J(R)$.

(1) ⇒ (2.1). Since 2 belongs to $J(R)$, Theorem 1 discovers that $R/J(R)$ is a Boolean ring. But, since R is local, we must have $R/J(R) \cong \mathbb{Z}_2$. Because $\mathrm{T}_n(R)$ is an SDT ring, Corollary 4 gives that $\mathrm{T}_2(R)$ is an SDT ring too. Moreover, Proposition 4 allows us to detect that $T_2(R)$ is an SDI ring.

Suppose now $a \in U(R)$ and $b \in J(R)$. We intend to show that $l_a - r_b : R \to R$ is surjective. Thereby, it suffices to prove that, for every $v \in R$, there exists $x \in R$ such that $ax - xb = v$. Put $r := \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$. Assume $r = g + j$ is an SDI representation, where $g = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix}$ and $j = \begin{pmatrix} d & y \\ 0 & d' \end{pmatrix}$. Since e is an idempotent and $a \in U(R)$, we deduce $e = 1$. However, since f is an idempotent and

$b \in J(R)$, we derive $f = 0$. Thus, $g = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$. Since $rg = gr$, we now have $ax - xb = v$. Therefore, $l_a - r_b : R \to R$ is surjective. Similarly, we can show that $l_b - r_a : R \to R$ is surjective, as desired.

(2.1) $\Rightarrow$ (1). Since $2 \in J(R)$, we only have the case $R/J(R) \cong \mathbb{Z}_2$. Thus, by [8; Theorem 4.4], there is nothing left to prove.

Case 2: If $2 \in U(R)$.

(1) $\Rightarrow$ (2.2). Since 2 belongs to $U(R)$, Theorem 1 demonstrates that $R/J(R)$ is a Yaqub ring. But, since R is local, we must have $R/J(R) \cong \mathbb{Z}_3$. Because $\mathrm{T}_n(R)$ is an SDT ring, Corollary 4 gives that $\mathrm{T}_2(R)$ is an SDT ring too.

Suppose now $a \in U(R)$ and $b \in J(R)$. We intend to show that $l_a - r_b : R \to R$ is surjective. Thereby, it suffices to establish that, for each $v \in R$, there is $x \in R$ such that $ax - xb = v$. Set $r := \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$. Assume $r = g + j$ is an SDT representation, where $g = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix}$ and $j = \begin{pmatrix} d & y \\ 0 & d' \end{pmatrix}$. Since $b \in J(R)$ and f is a tripotent, we detect $f = 0$. On the other hand, Lemma 12 allows us to conclude that R has no non-trivial tripotents. Hence, since $a \in U(R)$, e is simultaneously a unit and a tripotent element, and thus either $e = 1$ or $e = -1$. If $g = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$, then since $rg = gr$, we have $ax - xb = v$. If, however, $g = \begin{pmatrix} -1 & x \\ 0 & 0 \end{pmatrix}$, then again since $rg = gr$, we have $a(-x) - (-x)b = v$. Consequently, $l_a - r_b : R \to R$ is surjective. Similarly, we can establish that $l_b - r_a : R \to R$ is surjective.

We now show that under the given assumptions, the SDT representation of elements is unique. In this light, suppose $e + d = f + b$ are two SDT representations in R. Note that, Lemma 12 manifestly yields $e, f \in \{-1, 0, 1\}$, so that one easily sees that either $e = f$ or $e = -f$. If $e = -f$, then $2e = b - d \in \Delta(R)$. Since $2 \in U(R)$, we have $e \in \Delta(R)$. Thus, $e^2 \in \Delta(R) \cap \mathrm{Id}(R) = \{0\}$, which leads to $e = 0$. Therefore, $e = f = 0$.

Suppose now that $a = 1 + d$ and $b = -1 + d'$ are two SDT representations. Assume that

$$r = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$$

is an element of $\mathrm{T}_2(R)$. Also, suppose that $r = g + w$ is an SDT representation, where

$$g = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} d & y \\ 0 & d' \end{pmatrix}.$$

Bearing in mind the above note, we can assume without loss of generality that $e = 1$ and $f = -1$. Since $gw = wg$ and $2 \in U(R)$, we deduce $a(1/2)x - (1/2)xb = v$. This obviously implies that the map $l_a - r_b : R \to R$ is surjective.

(2.1) $\Rightarrow$ (1). Suppose $A \in \mathrm{T}_n(R)$. We show that A has an SDT representation such that $A = E + D$ in $\mathrm{T}_n(R)$. Since $R/J(R) \cong \mathbb{Z}_3$, we see with no any technical difficulty that $R = J(R) \cup (1 + J(R)) \cup (-1 + J(R))$. First, we construct the elements on the main diagonal E. Suppose

$$e_{ii} := \begin{cases} 0 & \text{if } a_{ii} \in J(R), \\ 1 & \text{if } a_{ii} \in 1 + J(R), \\ -1 & \text{if } a_{ii} \in -1 + J(R). \end{cases}$$

Therefore, one inspects that $a_{ii} - e_{ii} \in J(R)$ for each i. Notice that, since $2 \in U(R)$, it must be that $(1 + J(R)) \cap (-1 + J(R)) = \emptyset$. If $e_{ii} \neq e_{jj}$, then we come to

$$\begin{cases} (1)\ e_{ii} \in U(R) \text{ and } e_{jj} \in J(R), \\ (2)\ e_{ii} \in J(R) \text{ and } e_{jj} \in U(R), \\ (3)\ e_{ii} \text{ and } e_{jj} \in U(R). \end{cases}$$

We prove that, in all three cases, $l_{a_{ii}} - r_{a_{jj}} : R \to R$ is necessarily surjective.

In fact, for case (1), $a_{ii} \in U(R)$ and $a_{jj} \in J(R)$ and, because R is bleached, $l_{a_{ii}} - r_{a_{jj}} : R \to R$ is indeed surjective.

The case (2) is observed to be similar to case (1).

In case (3), with no harm of generality, assuming $e_{ii} = 1$ and $e_{jj} = -1$, we obtain that $a_{ii} - 1$, $a_{jj} + 1 \in \Delta(R)$. Therefore, by the requested assumption, $l_{a_{ii}} - r_{a_{jj}} : R \to R$ is surjective. Hence, with Lemma 14 in hand, there is a tripotent $E \in T_n(R)$ such that $AE = EA$ and $E_{ii} = e_{ii}$ for each $i \in \{1, \dots, n\}$. In addition,

$$A - E \in J(T_n(R)) \subseteq \Delta(T_n(R)),$$

thus completing the proof. □

The case when $n = 2$ can be considered separately in the following manner.

Example 2. Suppose R is an integral domain and an SDT ring. Then, $\mathrm{T}_2(R)$ is an SDT ring.

Proof. Utilizing Proposition 2, R is a local ring. In the other vein, since R is a domain, arguing as in the proof of Proposition 2, we can assume that R has no non-trivial tripotents.

Since R is local, we have either $2 \in U(R)$ or $2 \in J(R)$. First, we assume that $2 \in J(R)$, and let $A = \begin{pmatrix} a & \beta \\ 0 & b \end{pmatrix} \in \mathrm{T}_2(R)$. Note that an SDT ring with $2 \in J(R)$ is always an SDI ring. We show that $\mathrm{T}_2(R)$ is also SDI. Precisely, we consider the following four cases:

1. If $a, b \in J(R)$, then $A \in J(R)$, so $A = 0 + A$ is an SDI representation.

2. If $a, b \in U(R)$, then since R is both SDI and local, we have $a - 1 \in J(R)$ and $b - 1 \in J(R)$. Therefore,

$$A = I_2 + \begin{pmatrix} a-1 & \beta \\ 0 & b-1 \end{pmatrix}$$

is an SDI representation for A.

3. $a \in U(R), b \in J(R)$. Since R is an SDI ring, we obtain $a - 1 \in J(R)$. Thus,

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a-1 & \beta - \alpha \\ 0 & b \end{pmatrix}$$

is an SDT representation, where $\alpha = \beta((a-1) + (1-b))^{-1}$.

4. $b \in U(R), a \in J(R)$. Since R is an SDI ring, we receive $b - 1 \in J(R)$. So,

$$A = \begin{pmatrix} 0 & \alpha \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & \beta - \alpha \\ 0 & b-1 \end{pmatrix}$$

is an SDT representation, where $\alpha = \beta((b-1) + (1-a))^{-1}$.

Now, suppose $2 \in U(R)$.

1. If $a, b \in J(R)$, then $A \in J(R)$, so $A = 0 + A$ is an SDT representation.

2. Given $a, b \in U(R)$. If the SDT representations of a and b are of the form $a = 1 + (a-1)$ and $b = 1 + (b-1)$, then

$$A = I_2 + \begin{pmatrix} a-1 & \beta \\ 0 & b-1 \end{pmatrix}$$

is an SDT representation for A.

If the SDT representations of a and b are of the form $a = -1 + (a+1)$ and $b = -1 + (b+1)$, then

$$A = -I_2 + \begin{pmatrix} a+1 & \beta \\ 0 & b+1 \end{pmatrix}$$

is an SDT representation for A.

If the SDT representations of a and b are of the form $a = -1 + (a + 1)$ and $b = 1 + (b - 1)$, then

$$A = \begin{pmatrix} -1 & \alpha \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a+1 & \beta - \alpha \\ 0 & b-1 \end{pmatrix}$$

is an SDT representation, where $\alpha = 2\beta(2 + (b - 1) - (a + 1))^{-1}$.

If the SDT representations of a and b are of the form $a = 1 + (a - 1)$ and $b = -1 + (b + 1)$, then

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} a-1 & \beta - \alpha \\ 0 & b+1 \end{pmatrix}$$

is an SDT representation, where $\alpha = 2\beta(2 + (a - 1) - (b + 1))^{-1}$. Note that $2 \in U(R)$ is assumed.

3. Given $a \in U(R)$ and $b \in J(R)$. If the SDT representation of a is of the form $a = 1 + (a - 1)$, then

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a-1 & \beta - \alpha \\ 0 & b \end{pmatrix}$$

is an SDT representation for A, where $\alpha = \beta((1 - b) - (1 - a))^{-1}$.

If the SDT representation of a is of the form $a = -1 + (a + 1)$, then

$$A = \begin{pmatrix} -1 & \alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a+1 & \beta - \alpha \\ 0 & b \end{pmatrix}$$

is an SDT representation for A, where $\alpha = \beta((1 + b) - (1 + a))^{-1}$.

4. Given $a \in J(R)$ and $b \in U(R)$. If the SDT representation of b is of the form $b = 1 + (b - 1)$, then

$$A = \begin{pmatrix} 0 & \alpha \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & \beta - \alpha \\ 0 & b-1 \end{pmatrix}$$

is an SDT representation for A, where $\alpha = \beta((b - 1) + (1 - a))^{-1}$.

If the SDT representation of b is of the form $b = -1 + (b + 1)$, then

$$A = \begin{pmatrix} 0 & \alpha \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} a & \beta - \alpha \\ 0 & b+1 \end{pmatrix}$$

is an SDT representation for A, where $\alpha = \beta((1 + a) - (1 + b))^{-1}$, as claimed. □

Now, we manage to examine the above stated example in a more general situation like the following one.

Proposition 7. Let R be a ring that has no non-trivial tripotent elements. Then, the following conditions are equivalent:

(1) $T(R, V)$ is an SDT ring.

(2) Either $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$.

Proof. (1) $\Rightarrow$ (2). If $T(R, V)$ is an SDT ring, it is easily verified that R is also an SDT ring. Moreover, since R has no non-trivial tripotent elements, as shown in Proposition 2, we can prove that R is a local ring. Therefore, according to a combination of the locality of R and Theorem 1, we conclude $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$.

(2) $\Rightarrow$ (1). If $R/J(R) \cong \mathbb{Z}_2$, then from [15; Theorem 15] we deduce that $T(R, V)$ is a uniquely clean ring. Thus, it is an SDI ring and, consequently, an SDT ring.

If, however, $R/J(R) \cong \mathbb{Z}_3$, we so derive

$$R = J(R) \cup (1 + J(R)) \cup (-1 + J(R)).$$

Assume now that $\begin{pmatrix} a & v \\ 0 & a \end{pmatrix} \in T(R, V)$ is fulfilled. So, we have:

(a) If $a \in J(R)$, then $\begin{pmatrix} a & v \\ 0 & a \end{pmatrix} \in J(T(R, V))$.

(b) If $a \in 1 + J(R)$, then

$$\begin{pmatrix} a & v \\ 0 & a \end{pmatrix} = I_2 + \begin{pmatrix} a-1 & v \\ 0 & a-1 \end{pmatrix},$$

which is an SDT representation.

(c) If $a \in -1 + J(R)$, then

$$\begin{pmatrix} a & v \\ 0 & a \end{pmatrix} = -I_2 + \begin{pmatrix} a+1 & v \\ 0 & a+1 \end{pmatrix},$$

which is an SDT representation, as claimed. □

We finish our examinations with the following exhibitions which we leave to the interested reader for a direct check.

Example 3. Let R be a ring in which all tripotent elements are central. Then, the following issues hold:

(1) R is an SDT ring if and only if $R[[x]]$ is an SDT ring.
(2) R is an SDT ring if and only if $R[x]/(x^n)$ is an SDT ring.
(3) R is an SDT ring if and only if $T(R, R)$ is an SDT ring.

Concluding Discussion and Questions

As above noticed, in [10] the authors defined and investigated those rings R, calling them *semi-tripotent*, whose elements are a sum of a tripotent element from R and an element from the Jacobson radical of R which, generally, need *not* commute each other.

Now, regarding Proposition 6, one may ask whether the classes of semi-tripotent rings and SDT rings are independent of each other; that is, does there exist an SDT ring what is *not* semi-tripotent as well as a semi-tripotent ring that is *not* SDT? However, it was proved in [10; Theorem 3.5 (6)] that $R/J(R)$ has the same presentation as in our Theorem 1 plus the requirement that all idempotents of R lift modulo $J(R)$. That is why, it quite surprisingly follows that *every SDI ring whose idempotent lift modulo the Jacobson radical is always semi-tripotent.* However, as the opposite claim of Theorem 1 is not at all guaranteed in order to be a satisfactory criterion, we do *not* know yet if any semi-tripotent ring is SDT. Likewise, due to the lifting restriction of the idempotents, the reciprocal implication *cannot* happen in all generality or, in other words, there is an SDT ring that is *not* semi-tripotent.

Our first intriguing query is related to the study in-depth of a generalized version of the SDT rings like this, which presents a more general setting of the *semi-n-potent rings* as defined in [10].

Problem 1. Describe those rings R, naming them *strongly* Δ *n-potent*, whose elements are a sum of a n-potent element in R (i.e., an element $a \in R$ such that $a^n = a$ for some $n \in \mathbb{N}$) and an element from $\Delta(R)$ that commute with each other.

On the other side, in conjunction with [19], we close our work with the following interesting question.

Problem 2. Characterize those rings R, calling them *C*Δ *rings*, whose elements are a sum of an element from the center $\mathrm{Z}(R)$ and from $\Delta(R)$.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Lam, T.Y. (2001). *A First Course in Noncommutative Rings* (2nd ed., Graduate Texts in Mathematics, Vol. 131). New York: Springer. https://doi.org/10.1007/978-1-4419-8616-0

2 Lam, T.Y. (2003). *Exercises in Classical Ring Theory 2nd ed., Problem Books in Mathematics.* New York: Springer. https://doi.org/10.1007/b97448

3 Leroy, A., & Matczuk, J. (2019). Remarks on the Jacobson radical. *Contemporary Mathematics. Rings, Modules and Codes, 727*, 269–276. https://doi.org/10.1090/conm/727

4 Chen, H. (2011). *Rings Related Stable Range Conditions.* (Series in Algebra, 11). Hackensack: World Scientific. https://doi.org/10.1142/8006

5 Diesl, A.J. (2013). Nil clean rings. *Journal of Algebra*, *383*, 197–211. https://doi.org/10.1016/j.jalgebra.2013.02.020

6 Koşan, M.T., & Zhou, Y. (2016). On weakly nil-clean rings. *Front. Math. China*, *11*, 949–955. https://doi.org/10.1007/s11464-016-0555-6

7 Chen, H., & Sheibani, M. (2017). Strongly 2-nil-clean rings. *Journal of Algebra and Its Applications, 16*(9), 1750178. https://doi.org/10.1142/S021949881750178X

8 Chen, H. (2010). On strongly J-clean rings. *Communications in Algebra, 38*(10), 3790–3804. https://doi.org/10.1080/00927870903286835

9 Chen, H. (2012). Strongly J-clean matrices over local rings. *Communications in Algebra, 40*(4), 1352–1362. https://doi.org/10.1080/00927872.2010.551529

10 Koşan, M.T., Yildirim, T., & Zhou, Y. (2019). Rings whose elements are the sum of a tripotent and an element from the Jacobson radical. *Canadian Mathematical Bulletin, 62*(4), 810–821. https://doi.org/10.4153/S0008439519000092

11 Annin, S. (2002). Associated primes over skew polynomials rings. *Communications in Algebra, 30*(5), 2511–2528. https://doi.org/10.1081/AGB-120003481

12 Chen, W. (2015). On constant products of elements in skew polynomial rings. *Bulletin of the Iranian Mathematical Society, 41*(2), 453–462.

13 Hashemi, E., & Moussavi, A. (2005). Polynomial extensions of quasi-Baer rings. *Acta Mathematica Hungarica, 107*(3), 207–224. https://doi.org/10.1007/s10474-005-0191-1

14 Karabacak, F., Koşan, M.T., Quynh, T., & Tai, D. (2021). A generalization of UJ-rings. *Journal of Algebra and Its Applications, 20*(12), 2150217. https://doi.org/10.1142/S0219498821502170

15 Nicholson, W.K., & Zhou, Y. (2004). Rings in which elements are uniquely the sum of an idempotent and a unit. *Glasg. Math. J., 46*(2), 227–236. https://doi.org/10.1017/S0017089504001727

16 Pandey, S.K. (2024). A note on rings in which each element is a sum of two idempotents. *Elemente der Mathematik, 79*(3), 126–127. https://doi.org/10.4171/EM/507

17 Ying, Z., Koşan, T., & Zhou, Y. (2016). Rings in which every element is a sum of two tripotents. *Canadian Mathematical Bulletin, 59*(3), 661–672. https://doi.org/10.4153/CMB-2016-009-0

18 Borooah, G., Diesl, A.J., & Dorsey, T.J. (2007). Strongly clean triangular matrix rings over local rings. *Journal of Algebra, 312*(2), 773–797. https://doi.org/10.1016/j.jalgebra.2006.10.029

19 Ma, G., Wand, Y., & Leroy, A. (2024). Rings in which elements are sum of a central element and an element in the Jacobson radical. *Czechoslovak Mathematical Journal, 74*(2), 515–533. https://doi.org/10.21136/CMJ.2024.0433-23

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