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On two four-dimensional curl operators and their applications

M.T. Jenaliyev, A.S. Kassymbekova, M.G. Yergaliyev*

Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
(E-mail: muvasharkhan@gmail.com, kasar1337@gmail.com, ergaliev@math.kz)

Academician O.A. Ladyzhenskaya emphasized the importance of constructing a fundamental system in the space of solenoidal functions for simple domains such as squares, cubes, and similar regions. This article examines the problem of constructing such fundamental systems for a four-dimensional parallelepiped and cube. As is well known, applying the stream functions known from the two- and three-dimensional cases, the spectral problem for the Stokes operator reduces to the so-called clamped plate problem, which, in turn, has no solution in domains such as the square, cube, or parallelepiped. Thus, in higher-dimensional cases, the necessity of an analogous stream function becomes evident. In this work, the authors propose two curl operators that satisfy the above-mentioned requirements. Using the introduced curl operators, the spectral problem for the biharmonic operator in a four-dimensional parallelepiped and cube is formulated. Alternative approaches to constructing a fundamental system are presented, given the unsolvability of the spectral problem. Furthermore, the growth orders of the obtained eigenvalues are established.

Keywords: spectral problem, fundamental system, curl operator.

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Introduction

As is well known, the theoretical foundation of classical electromagnetic field (EMF) theory is based on Maxwell's equations, which generalize the experimental results obtained by the end of the 18th century. The development of classical EMF theory led to its description as an antisymmetric second-rank tensor, from which Maxwell's equations follow. These equations played a key role in the development of theoretical physics and had a profound influence on the creation of the special theory of relativity and other theories. By the early 20th century, classical electrodynamics was considered a completed science, and the EMF theory received its further development in the form of quantum electrodynamics.

In this work, we consider two four-dimensional curl operators. While the first curl operator is closely related to electromagnetic field theory and Maxwell's equations, the second curl operator is introduced artificially. The first curl operator is introduced (theoretically well-founded) using an antisymmetric second-rank tensor [1; 146, 149]. In fact, the four-dimensional curl operator is introduced on a six-dimensional vector field. In contrast to the first, the artificially chosen four-dimensional curl operator is introduced on a four-dimensional vector field. These operators are used by us to construct fundamental systems in the space of solenoidal functions. These systems are not only important theoretically but also computationally efficient for the approximate solution of boundary value problems for the Stokes and Navier-Stokes equation systems.

It should be noted that spectral problems for the Stokes operator (with periodicity conditions) in a cubic domain were also considered in the works [2–4]. In the work [2], the spectra of the curl and Stokes operators in a cube for functions satisfying the periodicity condition are studied. The Cauchy

*Corresponding author. *E-mail: ergaliev@math.kz*

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problem for the 3-D Navier-Stokes equations with periodic boundary conditions in the spatial variable was studied in [4].

First of all, let us formulate the spectral problem for the Stokes operator. Let $x = (x_1, ..., x_d) \in \Omega \subset \mathbb{R}^d,\ d \geq 2$, be an open bounded (simply connected) domain with boundary $\partial\Omega$. We seek nontrivial solutions $\{\vec{w}_k(x),\ p_k(x),\ x \in \Omega,\ \ k \in \mathbb{N}\}$ and corresponding values of the parameter $\{\mu_k^2,\ k \in \mathbb{N}\}$ for the following boundary value problem [5; 38]:

$$\begin{cases} -\Delta\vec{w}(x) + \nabla p(x) &= \mu^2 w(x), \quad x \in \Omega, \\ \operatorname{div}\{\vec{w}(x)\} &= 0, \qquad\quad x \in \Omega, \\ \vec{w}(x) &= 0, \qquad\quad x \in \partial\Omega. \end{cases} \tag{A}$$

In the terminology of inverse problem theory for differential equations [6], problem (A) can be interpreted as a coefficient inverse problem, where the condition of overdetermination is represented by the requirement of the nontriviality of the solution $\{\vec{w}(x), \nabla p(x)\}$, corresponding to the sought coefficient μ^2.

Let us introduce the main spaces that will be used. Let $x = (x_1, ..., x_d) \in \Omega \subset \mathbb{R}^d,\ d \geq 2$, be an open bounded (simply connected) domain with a sufficiently smooth boundary $\partial\Omega$, and $m \geq 0$ be an integer,

$$W_2^m(\Omega) = \left\{v |\ \partial_x^{|\alpha|} v \in L^2(\Omega),\ |\alpha| \leq m\right\}, \quad \text{where} \ \ \partial_x^{|\alpha|} = \partial_{x_1}^{\alpha_1} ... \partial_{x_d}^{\alpha_d},\ \ |\alpha| = \sum_{j=1}^{d} \alpha_j,\ \ \partial_{x_j} = \frac{\partial}{\partial_{x_j}},$$

$$\overset{\circ}{W}{}_2^m(\Omega) = \left\{v |\ v \in W_2^m(\Omega),\ \partial_{\vec{n}}^j v = 0,\ j = 0, 1, 2..., m-1,\ \vec{n} \text{ is the outward normal to } \partial\Omega\right\}.$$

Otherwise, in the notation of the spaces, we will follow the monograph [7].

1 The first four-dimensional curl operator

Let us consider the four-dimensional case of the spectral problem (A). We start from the case of the four-dimensional rectangular parallelepiped.

Let $\Omega_4 = \{x_0 < x < x_1,\ y_0 < y < y_1,\ z_0 < z < z_1,\ \zeta_0 < \zeta < \zeta_1\}$ be a rectangular parallelepiped, where $x_0, x_1, y_0, y_1, z_0, z_1, \zeta_0, \zeta_1$ are given.

Problem 1.1. Find the vector function $\vec{U}(x, y, z, \zeta)$ for the given solenoidal vector function $\vec{w}(x, y, z, \zeta)$, i.e.

$$\operatorname{curl}\vec{U}(x, y, z, \zeta) = \vec{w}(x, y, z, \zeta),\ \ \operatorname{div}\vec{w}(x, y, z, \zeta) = 0,\ \ (x, y, z, \zeta) \in \Omega_4, \tag{1}$$

where $\vec{U} = \{U_1, U_2, U_3, U_4, U_5, U_6\},\ \vec{w} = \{w_1, w_2, w_3, w_4\}$,

$$U_k \in W_2^2(\Omega_4),\ \ k = 1, 2, 3, 4, 5, 6; \quad w_j \in W_2^1(\Omega_4),\ \ j = 1, 2, 3, 4. \tag{2}$$

We introduce the first four-dimensional curl operator in the following way

$$\vec{w} = \operatorname{curl}\vec{U} = \begin{pmatrix} -\partial_y U_1 - \partial_z U_2 - \partial_\zeta U_3 \\ \partial_x U_1 + \partial_\zeta U_5 - \partial_z U_6 \\ \partial_x U_2 + \partial_y U_6 - \partial_\zeta U_4 \\ \partial_x U_3 + \partial_z U_4 - \partial_y U_5 \end{pmatrix},\ \operatorname{div}\operatorname{curl}\vec{U} = 0. \tag{3}$$

Remark 1. The curl operator in equation (3) acts on a six-dimensional vector $\vec{U}$, which, in particular, corresponds to the following vector composed of the intensity vectors of the electric field $\vec{E}$ and the magnetic field $\vec{H}$: $\vec{E} = \{E^1, E^2, E^3\},\ \vec{H} = \{H^1, H^2, H^3\}$ [1; 149, 274] namely, $\vec{U} = \{E^1, E^2, E^3, H^1, H^2, H^3\}$.

We introduce the following notations

$$\begin{gathered} y = x_1, \quad z = x_2, \quad \zeta = x_3, \quad x = x_4, \\ U_k = E^k = ic\varepsilon_0 E_k, \quad U_{k+3} = H^k = \tfrac{1}{\mu_0} B_k, \quad k = 1,2,3, \\ w_4 = \tfrac{\varrho}{\varepsilon_0}, \quad w_k = j_k, \quad k = 1,2,3, \end{gathered} \tag{4}$$

where $E_k,\ k = 1,2,3$ are the components of the electric field intensity vector, $B_k,\ k = 1,2,3$ are the components of the magnetic field intensity vector, c is the speed of light in a vacuum, ε_0 is the dielectric constant in a vacuum, μ_0 is the magnetic permeability in a vacuum, ϱ is the charge density, $\vec{j} = \{j_1, j_2, j_3\}$ is the electric current density vector, and $i = \sqrt{-1}$.

Proposition 1. According to (3)–(4) and [1; 149] we will have Maxwell's equations for the electromagnetic field in a vacuum:

$$\begin{cases} -ic\varepsilon_0 \partial_{x_1} E_1 + \tfrac{1}{\mu_0} \partial_{x_3} B_3 - \tfrac{1}{\mu_0} \partial_{x_4} B_2 = j_1, \\ -ic\varepsilon_0 \partial_{x_1} E_2 + \tfrac{1}{\mu_0} \partial_{x_4} B_1 - \tfrac{1}{\mu_0} \partial_{x_2} B_3 = j_2, \\ -ic\varepsilon_0 \partial_{x_1} E_3 + \tfrac{1}{\mu_0} \partial_{x_2} B_2 - \tfrac{1}{\mu_0} \partial_{x_3} B_1 = j_3, \\ ic\varepsilon_0 \partial_{x_1} E_1 + ic\varepsilon_0 \partial_{x_2} E_2 + ic\varepsilon_0 \partial_{x_3} E_3 = \tfrac{\varrho}{\varepsilon_0}, \end{cases} \tag{5}$$

$$\begin{cases} \partial_{x_2} B_1 + \partial_{x_3} B_2 + \partial_{x_4} B_3 = 0, \\ \partial_{x_4} B_1 + \tfrac{i}{c} \partial_{x_3} E_2 - \tfrac{i}{c} \partial_{x_2} E_3 = 0, \\ \partial_{x_4} B_2 + \tfrac{i}{c} \partial_{x_1} E_3 - \tfrac{i}{c} \partial_{x_3} E_1 = 0, \\ \partial_{x_4} B_3 + \tfrac{i}{c} \partial_{x_2} E_1 - \tfrac{i}{c} \partial_{x_1} E_2 = 0. \end{cases} \tag{6}$$

Proof of Proposition 1. Indeed, if according to (3)–(4) and [1; 149] we introduce new independent variables instead of the spacetime coordinates (x, y, z, t) as

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict, \tag{7}$$

where c is the speed of light in a vacuum, one can observe a remarkable symmetry in Maxwell's equations describing the electromagnetic field.

With the notation (7), the equations satisfied by the electrodynamic potentials (V, A_1, A_2, A_3) can be written as:

$$\begin{cases} \partial^2_{x_1} A_k + \partial^2_{x_2} A_k + \partial^2_{x_3} A_k + \partial^2_{x_4} A_k & = & -\mu_0 j_k, \\ \partial^2_{x_1} V + \partial^2_{x_2} V + \partial^2_{x_3} V + \partial^2_{x_4} V & = & -\tfrac{\varrho}{\varepsilon_0}, \end{cases}$$

where $k = 1,2,3$, and the Lorentz condition can be written as:

$$\partial_{x_1} A_1 + \partial_{x_2} A_2 + \partial_{x_3} A_3 + \frac{i}{c} \partial_{x_4} V = 0.$$

Introducing the notation

$$\Phi_k = A_k, \quad k = 1,2,3; \quad \Phi_4 = \frac{i}{c} V, \tag{8}$$

the relations between the electromagnetic field vectors $\vec{E} = \{E_1, E_2, E_3\}$, $\vec{B} = \{B_1, B_2, B_3\}$, and the potentials V, $\vec{A} = \{A_1, A_2, A_3\}$:

$$\vec{E} = -\nabla V - \vec{A}, \ \ \vec{B} = \nabla \times \vec{A},$$

can be written as follows:

$$\begin{cases} E_k = -\partial_{x_k} V + \frac{c}{i}\partial_{x_4} A_k, & k = 1,2,3, \\ B_j = \partial_{x_k} A_l - \partial_{x_l} A_k, & j,k,l = 1,2,3, \end{cases}$$

or, taking into account (8), we will have

$$\begin{cases} -\frac{i}{c}E_k &=& \partial_{x_k}\Phi_4 - \partial_{x_4}\Phi_k, \quad k = 1,2,3, \\ B_j &=& \partial_{x_k}\Phi_l - \partial_{x_l}\Phi_k, \quad j,k,l = 1,2,3. \end{cases} \tag{9}$$

Considering the right-hand side of the relations (9), we will define the elements of the matrix $F_{\mu\nu}$ using Table 1.

Table 1

Matrix $F_{\mu\nu}$

$\mu \parallel \nu$	1	2	3	4
1	0	B_3	$-B_2$	$-(i/c)E_1$
2	$-B_3$	0	B_1	$-(i/c)E_2$
3	B_2	$-B_1$	0	$-(i/c)E_2$
4	$(i/c)E_1$	$(i/c)E_2$	$(i/c)E_3$	0

Therefore, we have

$$\begin{array}{lll} F_{12} = B_3, & F_{13} = -B_2, & F_{23} = B_1, \\ F_{14} = -\frac{i}{c}E_1, & F_{24} = -\frac{i}{c}E_2, & F_{34} = -\frac{i}{c}E_3, \end{array}$$

and we confirm the validity of the conditions

$$F_{\mu\nu} = -F_{\nu\mu}.$$

In these notations, the relations (9) will be written as:

$$F_{\mu\nu} = \partial_{x_\mu}\Phi_\nu - \partial_{x_\nu}\Phi_\mu, \quad \mu,\nu = 1,2,3,4. \tag{10}$$

From here, according to (3)–(4), we obtain the equations (5).

It remains to establish the equations (6). From (10), we obtain

$$\partial_{x_\lambda} F_{\mu\nu} + \partial_{x_\nu} F_{\lambda\mu} + \partial_{x_\mu} F_{\nu\lambda} = 0, \quad \lambda,\mu,\nu = 1,2,3,4. \tag{11}$$

It can be verified that, according to Table 1, the relations (11) are equivalent to the relations (6).

On the other hand, taking into account the notations (4) we can now write the spectral problem for the Stokes operator corresponding to the equations (5). We have

$$\begin{cases} -\Delta\vec{w}(x) + \nabla p(x) &=& \mu^2 w(x), & x \in \Omega_4, \\ \operatorname{div}\{\vec{w}(x)\} &=& 0, & x \in \Omega_4, \\ \vec{w}(x) &=& 0, & x \in \partial\Omega_4, \end{cases} \tag{12}$$

where, when returning from the notation of independent variables (x_1, x_2, x_3, x_4) to the notation (x, y, z, ζ):

$$x = x_1, \quad y = x_2, \quad z = x_3, \quad \zeta = x_4,$$

we get

$$w_4(x,y,z,\zeta)=\frac{\varrho(x,y,z,\zeta)}{\varepsilon_0},\quad w_k(x,y,z,\zeta)=j_k(x,y,z,\zeta),\quad k=1,2,3,$$

$p(x,y,z,\zeta)$ is the scalar function of the artificial pressure.

Let us introduce the notation for the spaces

$$\mathbf{W}_2^2(\Omega_4)=\left(W_2^2(\Omega_4)\right)^6,\ \ \mathbf{W}_2^1(\Omega_4)=\left(W_2^1(\Omega_4)\right)^4. \tag{13}$$

Proposition 2. In the notation (4), the following equality of the sets holds [5; 470]:

$$\operatorname{curl}\left\{\mathbf{W}_2^2(\Omega_4)\right\}=\mathbf{W}(\Omega_4)=\left\{\vec{w}\in\mathbf{W}_2^1(\Omega_4),\ \operatorname{div}\vec{w}=0\right\}. \tag{14}$$

Proposition 3. If $U_1=U_2=U_3=U_4=U_5=U_6=U(x,y,z,\zeta)\in W_2^2(\Omega_4)$, then instead of (14) we obtain:

$$\vec{w}\in\mathbf{W}_0(\Omega_4)=\operatorname{curl}\{\vec{U}\}_{|U_1=U_2=U_3=U_4=U_5=U_6=U\in W_2^2(\Omega_4)}\subset\mathbf{W}(\Omega_4).$$

The following statement has been proven.

Proposition 4. For each four-dimensional vector function $\vec{w}(x,y,z,\zeta)\in\mathbf{W}(\Omega_4)$ (14), there exists a six-dimensional vector function $\vec{U}(x,y,z,\zeta)\in\mathbf{W}_2^2(\Omega_4)$ (13) that satisfies the relations (1)–(3). The converse statement is also true: for each six-dimensional vector function $\vec{U}(x,y,z,\zeta)\in\mathbf{W}_2^2(\Omega_4)$ (13), there exists a four-dimensional vector function $\vec{w}(x,y,z,\zeta)\in\mathbf{W}(\Omega_4)$ (14) that satisfies the relations (1)–(3).

Now we turn to the case of the four-dimensional cube $\Omega=\{0<x,y,z,\zeta<l\}$.
Let $U_1=U_2=U_3=U_4=U_5=U_6=U$,

$$\vec{U}=\{U,U,U,U,U,U\},\ \ (x,y,z,\zeta)\in\Omega, \tag{15}$$

then we have:

$$\vec{U}(x,y,z,\zeta)=\{U,U,U,U,U,U\},$$

$$\vec{w}(x,y,z,\zeta)=\{w_1,w_2,w_3,w_4\}.$$

We introduce the curl operator (3) for the "four-dimensional cube" Ω under the condition (15) in the following way

$$\vec{w}=\operatorname{curl}\vec{U}=\begin{pmatrix}(-\partial_y-\partial_z-\partial_\zeta)U\\(\partial_x+\partial_\zeta-\partial_z)U\\(\partial_x+\partial_y-\partial_\zeta)U\\(\partial_x+\partial_z-\partial_y)U\end{pmatrix},\ \ \operatorname{div}\operatorname{curl}\vec{U}=0.$$

In this case, the spectral problem for the Stokes operator (12) takes the form:

$$(-\Delta)^2U=\lambda^2(-\Delta)U,\quad (x,y,z,\zeta)\in\Omega,$$

$$U=\partial_{\vec{n}}U=0,\quad (x,y,z,\zeta)\in\partial\Omega,$$

where $\boxed{\lambda^2=3\mu^2}$ and $\vec{n}$ is the outward unit normal to $\partial\Omega$.

The spectral problem (A1), along with its extensions to polyharmonic operators, has been the subject of extensive research [8–11]. It has been shown that an explicit solution for a square domain is unattainable due to the failure of the separation of variables method in this case. The only known exceptions are circular and spherical domains [12–14]. In [15], lower bounds for the first eigenvalue of problem (A1) were derived for various manifolds. Numerical approximations of problem (A1) are provided in [16, 17]. Various issues of biharmonic operators were also studied in [18–20].

Let us replace the biharmonic operator in the problem (12) with a fourth-order differential operator.

Problem 1.1.

$$(\partial_x^4+\partial_y^4+\partial_z^4+\partial_\zeta^4)U=\lambda^2(-\Delta)U,\ \ (x,y,z,\zeta)\in\Omega, \tag{16}$$

$$U_{|\partial\Omega}=\partial_{\vec{n}}U_{|\partial\Omega}=0. \tag{17}$$

The spaces $V_1(\Omega)$ and $V_2(\Omega)$, with dim $=4$. Let $V_1(\Omega)$ and $V_2(\Omega)$ denote the spaces equipped with the scalar products: $(\nabla u,\nabla v)_{L^2(\Omega)}\ \forall\, u,v\in\overset{\circ}{W}{}_2^1(\Omega)$ and $((u,v))\overset{\text{def}}{=}\left(\partial_x^2u,\partial_x^2v\right)_{L^2(\Omega)}+\left(\partial_y^2u,\partial_y^2v\right)_{L^2(\Omega)}+$ $+\left(\partial_z^2u,\partial_z^2v\right)_{L^2(\Omega)}+\left(\partial_\zeta^2u,\partial_\zeta^2v\right)_{L^2(\Omega)}\ \forall\, u,v\in\overset{\circ}{W}{}_2^2(\Omega)$.

We will show that the set of "generalized eigenfunctions" of the inverse operator T^{-1} to the operator from (20), belonging to the space $V_2(\Omega)$, forms an orthonormal basis in the space $V_1(\Omega)$.

For this purpose, we will consider the following auxiliary boundary value problem.

$$\left(\partial_x^4+\partial_y^4+\partial_z^4+\partial_\zeta^4\right)u(x,y,z,\zeta)=(-\Delta)\,h(x,y,z,\zeta)\quad\text{in}\quad\Omega, \tag{18}$$

$$u(x,y,z,\zeta)=\partial_{\vec{n}}u(x,y,z,\zeta)=0\quad\text{on}\quad\partial\Omega, \tag{19}$$

which, in operator form, is expressed as:

$$Tu=B_1h, \tag{20}$$

$$T\in\mathscr{L}(\overset{\circ}{W}{}_2^2(\Omega);W_2^{-2}(\Omega)),\ \ B_1\in\mathscr{L}(L^2(\Omega);W_2^{-2}(\Omega)).$$

Let $h\in L^2(\Omega)$, i.e. $B_1h\in W_2^{-2}(\Omega)$. Then the boundary value problem (18)–(19) (or (20)) can be written as the following (highlighted) identity:

$$\langle Tu,v\rangle\overset{\text{def}}{=}\underline{((u,v))}=\langle B_1h,v\rangle\overset{\text{def}}{=}\underline{(h,-\Delta v)}_{L^2(\Omega)}\ \ \forall\ v(x,y,z,\zeta)\in\ V_2(\Omega), \tag{21}$$

where

$$((u,v))\overset{\text{def}}{=}\left(\partial_x^2u,\partial_x^2v\right)_{L^2(\Omega)}+\left(\partial_y^2u,\partial_y^2v\right)_{L^2(\Omega)}+\left(\partial_z^2u,\partial_z^2v\right)_{L^2(\Omega)}+\left(\partial_\zeta^2u,\partial_\zeta^2v\right)_{L^2(\Omega)}, \tag{22}$$

$$((u,u))\ge\frac{1}{2}\|\Delta u\|^2_{L^2(\Omega)}. \tag{23}$$

Remark 2. According to (24) from [7; 117, 125] for convex bounded domains $x=(x_1,...,x_d)\in\Omega\subset\mathbb{R}^d$, $d\ge 2$, with a piecewise smooth boundary $\partial\Omega$, the following inequality holds:

$$\|u_{xx}\|_{L^2(\Omega)}\le C\|\Delta u\|_{L^2(\Omega)},\ \ \forall\, u\in W^2_{2,0}(\Omega)\equiv W_2^2(\Omega)\cap\overset{\circ}{W}{}_2^1(\Omega); \tag{24}$$

$$(u,v)^{(2)}_{2,\Omega}=\int\limits_{\Omega}(uv+u_xv_x+u_{xx}v_{xx})\,dx,\ \ u_{xx}=\left(u_{x_jx_k}\right),\ j,k=1,...,d,$$

therefore, due to the inequalities (24) and (23), the equivalent norm (22) is defined in the space $W_2^2(\Omega)\cap\overset{\circ}{W}{}_2^1(\Omega)$. This norm induces an equivalent norm on the subspace $\overset{\circ}{W}{}_2^2(\Omega)\subset W^2_{2,0}(\Omega)$.

Theorem 1. Let $h\in L^2(\Omega))$, i.e. $B_1h\in W_2^{-2}(\Omega)$. Then the boundary value problem (16)–(17) has a unique solution $u\in\ V_2(\Omega)$.

Remark 3. The statement of Theorem 1 remains valid if $h\in W^2_{2,0}(\Omega)$, i.e., $Bh\in L^2(\Omega)$, where $B\in\mathscr{L}(W^2_{2,0}(\Omega);L^2(\Omega))$.

Remark 4. According to (23) and (24) for the continuous bilinear functional $((u,v))$ (22), the condition of positive definiteness holds:

$$((u,u)) \geq K\|u\|^2_{V_2(\Omega)} \;\; \forall\, u \in V_2(\Omega). \tag{25}$$

Theorem 2. Let the operator B in the spectral problem (16)–(17) be defined as in Remark 3: $B \in \mathscr{L}(W^2_{2,0}(\Omega); L^2(\Omega))$. Then the operator T^{-1}:

$$T^{-1}:\; L^2(\Omega) \to V_2(\Omega) \subset L^2(\Omega)$$

is self-adjoint and compact, as it acts in the space $L^2(\Omega)$, i.e. the set of "generalized eigenfunctions" of the operator T^{-1}, belonging to the space $V_2(\Omega)$, forms an orthonormal basis in the space $V_1(\Omega)$.

Proof of Theorem 1. The statement of Theorem 1 follows from (21), (23), (25), and the continuity of the bilinear functional (22) in the space $\overset{\circ}{W}{}^2_2(\Omega)$ [21; 629, 653]. Definition of condition (E) [22; 169]: positive definiteness of the principal self-adjoint part of the system — the ellipticity condition for one equation.

Proof of Theorem 2. To prove Theorem 2, let us consider the mapping $T^{-1}: Bh \to u$, defined from the statement of Remark 3 to Theorem 1. It is linear and acts continuously from $L^2(\Omega)$ to $V_2(\Omega)$. Due to the compact embedding $V_2(\Omega) \hookrightarrow L^2(\Omega)$, the linear operator T^{-1}, considered as a linear operator on $L^2(\Omega)$, is compact. This operator is also self-adjoint ("relative to the operator B") because

$$(T^{-1}Bh_1, Bh_2)_{L^2(\Omega)} = ((u_1,u_2)) = \left(Bh_1, T^{-1}Bh_2\right)_{L^2(\Omega)},$$

where

$$T^{-1}Bh_i = u_i,\;\; Bh_i \in L^2(\Omega),\;\; T^{-1}Bh_i \in V_2(\Omega),\;\; i=1,2.$$

Consequently, the operator T^{-1} possesses a complete orthonormal sequence of "generalized eigenfunction" (see the formulas below in (27)): $v_j \in V_2(\Omega)$,

$$T^{-1}Bv_j = \lambda_j^{-2}v_j,\;\; j \geq 1,\;\; \lambda_j^{-2} > 0,\;\; \lambda_j^{-2} \to +0,\;\; j \to +\infty. \tag{26}$$

By multiplying equatio (26) scalarly by Tv, we obtain

$$v_j \in V_2(\Omega),\;\; ((v_j,v)) = \lambda_j^2(\nabla v_j, \nabla v)_{L^2(\Omega)},\;\; \forall\, v \in V_2(\Omega), \tag{27}$$

where $j \geq 1$, $\lambda_j^2 > 0$, $\lambda_j^2 \to +\infty$, $j \to +\infty$, we indeed have

$$(v_j, Tv)_{L^2(\Omega)} = \underline{((v_j,v))} = \lambda_j^2(Bv_j,v)_{L^2(\Omega)} = \underline{\lambda_j^2(\nabla v_j, \nabla v)_{L^2(\Omega)}}.$$

From the underlined identity (which coincides with (27)), we obtain, as usual:

$$(\nabla v_j, \nabla v_k)_{L^2(\Omega)} = \delta_{jk},\quad ((v_j,v_k)) = \lambda_j^2\delta_{jk},\quad \forall\, j,k.$$

This completes the proof of Theorem 2.

Theorem 3. The spectral problem (16)–(17) has the following solution:

$$U_n(x,y,z,\zeta) = X_n(x)Y_n(y)Z_n(z)\Upsilon_n(\zeta),\quad \lambda_n^2,\quad n \in \mathbb{N}, \tag{28}$$

where $X_n(x) = \Phi_n(\sigma)_{|\sigma=x}$, $Y_n(y) = \Phi_n(\sigma)_{|\sigma=y}$, $Z_n(z) = \Phi_n(\sigma)_{|\sigma=z}$, $\Upsilon_n(\zeta) = \Phi_n(\sigma)_{|\sigma=\zeta}$ are defined as follows

$$\begin{cases} \Phi_{2n-1}(\zeta) = \sin^2\frac{\lambda_{2n-1}\zeta}{2},\; \lambda_{2n-1}^2 = \left(\frac{2(2n-1)\pi}{l}\right)^2,\;\; n \in \mathbb{N}, \\ \Phi_{2n}(\zeta) = [\lambda_{2n}l - \sin\lambda_{2n}l]\sin^2\frac{\lambda_{2n}\zeta}{2} - \sin^2\frac{\lambda_{2n}l}{2}\left[\lambda_{2n}\zeta - \sin\lambda_{2n}\zeta\right], \\ \qquad \lambda_{2n}^2 = \left(\frac{2\nu_n}{l}\right)^2,\;\; n \in \mathbb{N}, \end{cases} \tag{29}$$

and $\{\nu_n,\; n \in \mathbb{N}\}$ are the positive roots of the equation $\tan\nu = \nu$, and $\mu_n^2 = \lambda_n^2/3$, $n \in \mathbb{N}$.

The arrangement of eigenvalues on the positive half-axis is shown in Figure 1 (here $l = 2$).

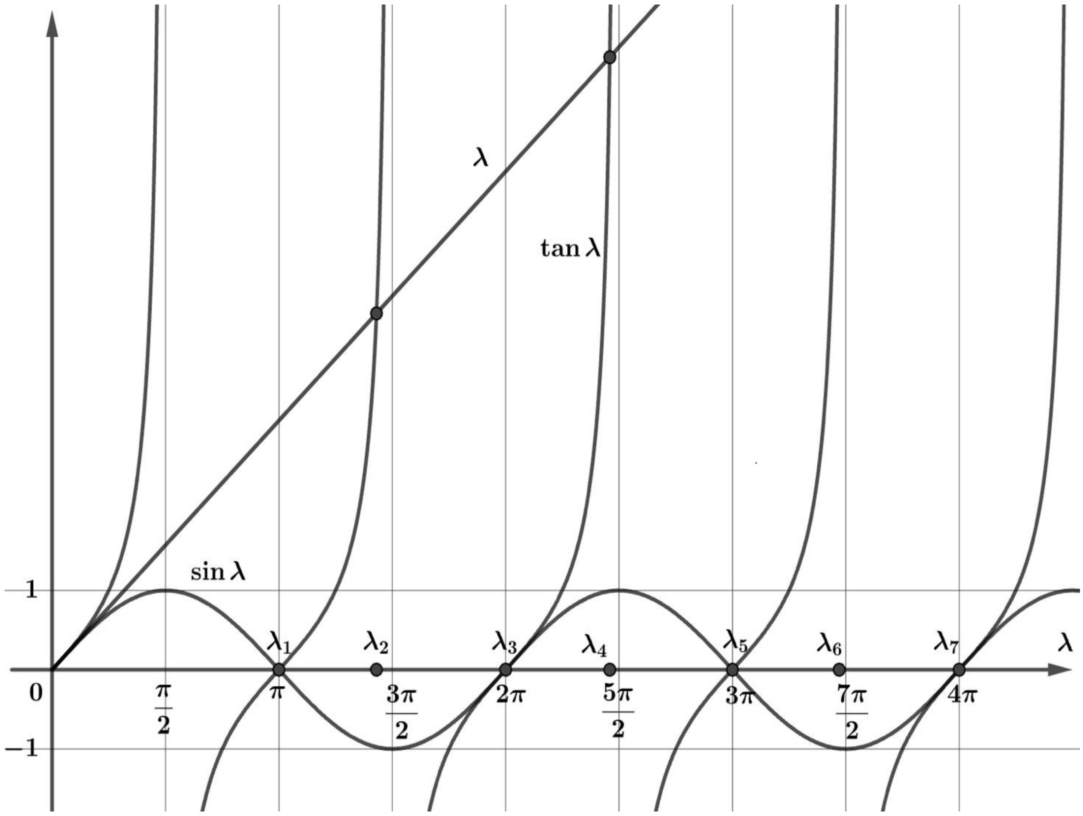

Figure 1. The positive roots of the equations (for $l = 2$):
$\tan \nu_n = \nu_n$, $\nu_n = \frac{\lambda_n l}{2} = \lambda_n$; $\sin \lambda_n = 0$, $n \in \mathbb{N}$.

From Figure 1 we get:

$$0 < \lambda_1 = \pi < \lambda_2 = \frac{3\pi}{2} - \varepsilon_1 < \lambda_3 = 2\pi < \lambda_4 = \frac{5\pi}{2} - \varepsilon_2 <$$

$$< \lambda_5 = 3\pi < \lambda_6 = \frac{7\pi}{2} - \varepsilon_3 < \lambda_7 = 4\pi < ...$$

Next, from Theorem 1, we obtain:

Corollary 1. The eigenvalues $\{\lambda_{2n},\ n \in \mathbb{N}\}$ are ordered as follows:

$$0 < \lambda_{2n} = \frac{2\nu_n}{l} < \frac{(2n+1)\pi}{2}, \ \forall\, n \in \mathbb{N},$$

$$\lambda_{2n} = \frac{2\nu_n}{l} \to \frac{(2n+1)\pi}{2}, \ n \to \infty,$$

where $\{\nu_n,\ n \in \mathbb{N}\}$ are the positive roots of the equation $\tan \nu = \nu$.

Theorem 4. The system of functions $\{U_n(x, y, z, \zeta)\}_{n=1}^{\infty} \subset V_2(\Omega)$, defined by the relations (28)–(29), forms a complete orthogonal sequence of "generalized eigenfunctions" in the space $V_1(\Omega)$.

Let us introduce the notations:

$$\overset{\circ}{\mathbf{W}}{}_2^2(\Omega) = (\overset{\circ}{W}{}_2^2(\Omega))^6\,, \quad \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) = (\overset{\circ}{W}{}_2^1(\Omega))^4\,, \tag{30}$$

$$\overset{\circ}{\mathbf{W}}(\Omega) = \{\vec{w} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega),\ \operatorname{div} \vec{w} = 0\}\,. \tag{31}$$

Proposition 5. For each four-dimensional vector-function $\vec{w}(x,y,z,\zeta) \in (V_1(\Omega))^4$, satisfying the condition $\operatorname{div}\vec{w}(x,y,z,\zeta) = 0$, there exists a unique six-dimensional vector-function $\vec{U}(x,y,z,\zeta) \in (V_2(\Omega))^6$. The converse statement is also true.

The validity of Proposition 5 directly follows from the proof of Proposition 4.

Proposition 6. The equality of sets holds [5; 470]:

$$\operatorname{curl}\{\overset{\circ}{\mathbf{W}}{}_2^2(\Omega)\} = \overset{\circ}{\mathbf{W}}(\Omega).$$

Let $\mathbf{H}(\Omega)$ denote the space of solenoidal functions, defined as follows:

$$\mathbf{H}(\Omega) = \{\vec{w} |\, \vec{w} \in \mathbf{L}^2(\Omega),\ \operatorname{div}\vec{w} = 0,\ \vec{w}\cdot\vec{n}|_{\partial\Omega} = 0\},\ \ \mathbf{L}^2(\Omega) = (L^2(\Omega))^4, \tag{32}$$

where $\vec{w}\cdot\vec{n}$ is the normal component of the vector $\vec{w}$.

Proposition 7. If $U_1 = U_2 = U_3 = U_4 = U_5 = U_6 = U(x,y,z,\zeta) \in \overset{\circ}{W}{}_2^2(\Omega)$, then we get

$$\vec{w} \in \overset{\circ}{\mathbf{W}}_0(\Omega) = \operatorname{curl}\{\vec{U}\}_{|U_1=U_2=U_3=U_4=U_5=U_6=U\in \overset{\circ}{W}{}_2^2(\Omega)} \subset \overset{\circ}{\mathbf{W}}(\Omega).$$

Next, we introduce the extended system of functions $\left\{U_m^1(x,y,z,\zeta)\right\}_{m=0}^{\infty}$, where

$$U_0^1(x,y,z,\zeta) \equiv 0,\ \ U_m^1(x,y,z,\zeta) = U_m(x,y,z,\zeta),\ \ m \in \mathbb{N},$$

and construct vector-functions

$$\vec{U}^1_{mjkqrs}(x,y,z,\zeta) = \left\{U_m^1, U_j^1, U_k^1, U_q^1, U_r^1, U_s^1\right\} \in \overset{\circ}{\mathbf{W}}{}_2^2(\Omega), \tag{33}$$

where $m,j,k,q,r,s \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, which, as is evident, will form a fundamental system in the space $\mathbf{V}_1(\Omega) = (V_1(\Omega))^6$.

Theorem 5. By applying the curl operator from (3) to the extended system of vector-functions (33), we obtain the desired fundamental system $(m,j,k,q \in \mathbb{N}_0)$:

$$\vec{w}_{mjkqrs}(x,y,z,\zeta) = \{w_{1,mjk}, w_{2,mrs}, w_{3,jsq}, w_{4,kqr}\} \in \overset{\circ}{\mathbf{W}}(\Omega) \tag{34}$$

in the space of solenoidal functions $\mathbf{H}(\Omega)$ (32), where

$$w_{1,mrs}(x,y,z,\zeta) = \partial_x U_m^1 + \partial_\zeta U_r^1 - \partial_z U_s^1,\ \ (x,y,z,\zeta)\in\Omega,\ \ m,r,s\in\mathbb{N}_0,$$

$$w_{2,jsq}(x,y,z,\zeta) = \partial_x U_j^1 + \partial_y U_s^1 - \partial_\zeta U_q^1,\ \ (x,y,z,\zeta)\in\Omega,\ \ j,s,q\in\mathbb{N}_0,$$

$$w_{3,kqr}(x,y,z,\zeta) = \partial_x U_k^1 + \partial_z U_q^1 - \partial_y U_r^1,\ \ (x,y,z,\zeta)\in\Omega,\ \ k,q,r\in\mathbb{N}_0,$$

$$w_{4,mjk}(x,y,z,\zeta) = -\partial_y U_m^1 - \partial_z U_j^1 - \partial_\zeta U_k^1,\ \ (x,y,z,\zeta)\in\Omega,\ \ m,j,k\in\mathbb{N}_0,$$

$$\operatorname{div}\vec{w}_{mjkqrs}(x,y,z,\zeta) = 0,\ \ (x,y,z,\zeta)\in\Omega,\ \ m,j,k,q,r,s\in\mathbb{N}_0, \tag{35}$$

$$\vec{w}_{mjkqrs}(x,y,z,\zeta) = 0,\ \ (x,y,z,\zeta)\in\partial\Omega,\ \ m,j,k,q,r,s\in\mathbb{N}_0. \tag{36}$$

Proof of Theorem 5. From the formulas defining the vector-function (34), we sequentially obtain:

$$|w_1 - w_1^\varepsilon|^2 \le 3\left[|\partial_x(U_1 - U_1^\varepsilon)|^2 + |\partial_\zeta(U_5 - U_5^\varepsilon)|^2 + |\partial_z(U_6 - U_6^\varepsilon)|^2\right],$$

$$|w_2 - w_2^\varepsilon|^2 \le 3\left[|\partial_x(U_2 - U_2^\varepsilon)|^2 + |\partial_y(U_6 - U_6^\varepsilon)|^2 + |\partial_\zeta(U_4 - U_4^\varepsilon)|^2\right],$$

$$|w_3 - w_3^\varepsilon|^2 \le 3\left[|\partial_x(U_3 - U_3^\varepsilon)|^2 + |\partial_z(U_4 - U_4^\varepsilon)|^2 + |\partial_y(U_5 - U_5^\varepsilon)|^2\right],$$

$$|w_4 - w_4^\varepsilon|^2 \leq 3\left[|\partial_y(U_1 - U_1^\varepsilon)|^2 + |\partial_z(U_2 - U_2^\varepsilon)|^2 + |\partial_\zeta(U_3 - U_3^\varepsilon)|^2\right],$$

i.e.,

$$\|\vec{w} - \vec{w}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \leq 2\sqrt{3}\|\vec{U} - \vec{U}^\varepsilon\|_{\mathbf{V}_1(\Omega)} \leq 2\sqrt{3}\,\varepsilon,$$

where, first, according to Proposition 4, for each vector-function $\vec{w}(z, y, z, \zeta) \in \overset{\circ}{\mathbf{W}}(\Omega)$ from (31), there corresponds a unique vector-function $\vec{U}(z, y, z, \zeta) \in \overset{\circ}{\mathbf{W}}{}_2^2(\Omega)$ (30).

Secondly, the finite sum

$$\vec{U}^\varepsilon(x, y, z, \zeta) = \sum_{n=0}^{N_\varepsilon} a_n \vec{U}_n^1(x, y, z, \zeta), \quad N_\varepsilon < \infty,$$

ensuring the fulfillment of inequality

$$\|\vec{U}(x, y, z, \zeta) - \vec{U}^\varepsilon(x, y, z, \zeta)\|_{\mathbf{V}_1(\Omega)} \leq \varepsilon$$

corresponds to the sum defined by the formulas:

$$\vec{w}^\varepsilon(x, y, z, \zeta) = \sum_{n=0}^{N_\varepsilon} a_n \vec{w}_n(x, y, z, \zeta), \quad N_\varepsilon < \infty,$$

and satisfying inequality:

$$\frac{1}{2\sqrt{3}}\|\vec{w}(x, y, z, \zeta) - \vec{w}^\varepsilon(x, y, z, \zeta)\|_{\mathbf{H}(\Omega)} \leq \|\vec{U}(x, y, z, \zeta) - \vec{U}^\varepsilon(x, y, z, \zeta)\|_{(V_1(\Omega))^6} \leq \varepsilon,$$

due to the fact that equality

$$\|\vec{w}(x, y, z, \zeta) - \vec{w}^\varepsilon(x, y, z, \zeta)\|_{\mathbf{L}^2(\Omega)} = \|\vec{w}(x, y, z, \zeta) - \vec{w}^\varepsilon(x, y, z, \zeta)\|_{\mathbf{H}(\Omega)}$$

and identity

$$\|\vec{w} - \vec{w}^\varepsilon\|_{\mathbf{H}^\perp(\Omega)} = 0, \;\; \mathbf{L}^2(\Omega) = \mathbf{H}(\Omega) \oplus \mathbf{H}^\perp(\Omega)$$

hold. The relations (28)–(31) show that the system of four-dimensional vector-functions

$$\{\vec{w}_n(x, y, z, \zeta)\}_{n=0}^\infty \subset (V_1(\Omega))^4,$$

forms a fundamental system in the space of solenoidal vector fields $\mathbf{H}(\Omega)$, satisfying the condition

$$\operatorname{div} \vec{w}_n(x, y, z, \zeta) = 0.$$

This concludes the proof of Theorem 5.

Proposition 8. Any vector-functions of the form

$$\vec{w}(x, y, z, \zeta) = \{0, 0, 0, w_4(x, y, z, \zeta)\},$$
$$\vec{w}(x, y, z, \zeta) = \{0, 0, w_3(x, y, z, \zeta), 0\},$$
$$\vec{w}(x, y, z, \zeta) = \{0, w_2(x, y, z, \zeta), 0, 0\},$$
$$\vec{w}(x, y, z, \zeta) = \{w_1(x, y, z, \zeta), 0, 0, 0\},$$

from the space $(\overset{\circ}{W}{}_2^1(\Omega))^4$ (where the functions $w_1(x, y, z, \zeta)$, $w_2(x, y, z, \zeta)$, $w_3(x, y, z, \zeta)$ and $w_4(x, y, z)$ are identically nonzero) cannot be solenoidal, i.e., they will not satisfy both the equation (incompressibility condition of the incompressible fluid) $\operatorname{div} \vec{w}(x, y, z, \zeta) = 0$ (35) and the boundary condition (36).

Thus, we have constructed a fundamental system in the space of solenoidal functions for a four-dimensional "cubic" domain.

2 *The second four-dimensional* curl *operator for a 4-D domain*

Let us formulate Problem 1.1 for the second four-dimensional curl operator. We start from the case of the four-dimensional rectangular parallelepiped, $\dim \Omega_4 = 4$.

Problem 2.1. Find the vector-function $\vec{U}(x,y,z,\zeta)$ for a given solenoidal vector-function $\vec{w}(x,y,z,\zeta)$, i.e.

$$\operatorname{curl} \vec{U}(x,y,z,\zeta) = \vec{w}(x,y,z,\zeta), \ \operatorname{div} \vec{w}(x,y,z,\zeta) = 0, \ (x,y,z,\zeta) \in \Omega_4, \tag{37}$$

where $\vec{U} = \{U_1, U_2, U_3, U_4\}, \ \vec{w} = \{w_1, w_2, w_3, w_4\}$,

$$U_j \in W_2^2(\Omega_4), \ w_j \in W_2^1(\Omega_4), \ j = 1,2,3,4. \tag{38}$$

We introduce curl operator [5; 141] in the following way

$$\vec{w} = \operatorname{curl} \vec{U} = \begin{pmatrix} \partial_y U_3 - \partial_z U_2 - \partial_\zeta U_2 \\ \partial_z U_4 - \partial_\zeta U_3 - \partial_x U_3 \\ \partial_\zeta U_1 + \partial_x U_2 - \partial_y U_4 \\ \partial_x U_2 + \partial_y U_3 - \partial_z U_1 \end{pmatrix}, \ \operatorname{div} \operatorname{curl} \vec{U} = 0. \tag{39}$$

We recall the notation for the spaces

$$\mathbf{W}_2^2(\Omega_4) = \left(W_2^2(\Omega_4)\right)^4, \ \mathbf{W}_2^1(\Omega_4) = \left(W_2^1(\Omega_4)\right)^4. \tag{40}$$

Proposition 9. If $U_1 = U_2 = U_3 = U_4 = U(x,y,z,\zeta) \in W_2^2(\Omega_4)$, then instead of (14), we obtain:

$$\vec{w} \in \mathbf{W}_0(\Omega_4) = \operatorname{curl}\{\vec{U}\}_{|U_1=U_2=U_3=U_4=U \in W_2^2(\Omega_4)} \subset \mathbf{W}(\Omega_4).$$

The following statement is true.

Proposition 10. For each four-dimensional vector-function $\vec{w}(x,y,z,\zeta) \in \mathbf{W}(\Omega_4)$ (14), there exists a four-dimensional vector-function $\vec{U}(x,y,z,\zeta) \in \mathbf{W}_2^2(\Omega_4)$ (40) that satisfies the relations (37)–(39). The converse statement is also true: for each four-dimensional vector-function $\vec{U}(x,y,z,\zeta) \in \mathbf{W}_2^2(\Omega_4)$ (40), there exists a four-dimensional vector-function $\vec{w}(x,y,z,\zeta) \in \mathbf{W}(\Omega_4)$ (14) that satisfies the relations (37)–(39).

Now we turn to the case of the four-dimensional cube $\Omega = \{0 < x,y,z,\zeta < l\}$.

Let

$$U_1 = U_2 = U_3 = U_4 = U, \ \vec{U} = \{U,U,U,U\}, \ (x,y,z,\zeta) \in \Omega, \tag{41}$$

then we have

$$\vec{U}(x,y,z,\zeta) = \{U,U,U,U\},$$
$$\vec{w}(x,y,z,\zeta) = \{w_1, w_2, w_3, w_4\}.$$

The curl operator (39) for the "four-dimensional cube" Ω under the condition (41).

$$\vec{w} = \operatorname{curl} \vec{U} = \begin{pmatrix} (\partial_y - \partial_z - \partial_\zeta) U \\ (\partial_z - \partial_\zeta - \partial_x) U \\ (\partial_\zeta + \partial_x - \partial_y) U \\ (\partial_x + \partial_y - \partial_z) U \end{pmatrix}, \ \operatorname{div} \operatorname{curl} \vec{U} = 0.$$

In this case, the spectral problem for the Stokes operator (A) takes the form:

$$\Delta(\Delta - S)U = \mu^2(-\Delta + S)U, \ (x,y,z,\zeta) \in \Omega, \tag{42}$$

$$U = \partial_{\vec{n}} U = 0, \;\; (x,y,z,\zeta) \in \partial\Omega, \tag{43}$$

$$S = \frac{4}{3}\left(\partial^2_{xz} + \partial^2_{yz} + \partial^2_{y\zeta} - \partial^2_{x\zeta}\right), \tag{44}$$

where $\vec{n}$ is the outward unit normal to $\partial\Omega$.

In equation (42), let the operator $S \equiv 0$, and then replace the resulting biharmonic operator $(-\Delta)^2$ with a fourth-order differential operator. As a result, instead of the spectral problem (42)–(44), we obtain the spectral problem (16)–(17) with $\lambda^2 = \mu^2$, for which Theorems 3 and 4 remain valid.

Let us introduce the notations:

$$\overset{\circ}{\mathbf{W}}{}^2_2(\Omega) = (\overset{\circ}{W}{}^2_2(\Omega))^4\,, \quad \overset{\circ}{\mathbf{W}}{}^1_2(\Omega) = (\overset{\circ}{W}{}^1_2(\Omega))^4\,, \tag{45}$$

$$\overset{\circ}{\mathbf{W}}(\Omega) = \{\vec{w} \in \overset{\circ}{\mathbf{W}}{}^1_2(\Omega), \; \operatorname{div}\vec{w} = 0\}\,. \tag{46}$$

Proposition 11. For each four-dimensional vector-function $\vec{w}(x,y,z,\zeta) \in (V_1(\Omega))^4$, satisfying the condition $\operatorname{div}\vec{w}(x,y,z,\zeta) = 0$, there exists a unique four-dimensional vector-function $\vec{U}(x,y,z,\zeta) \in (V_2(\Omega))^6$. The converse statement is also true.

The validity of Proposition 11 directly follows from the proof of Proposition 10.

Proposition 12. If $U_1 = U_2 = U_3 = U_4 = U(x,y,z,\zeta) \in \overset{\circ}{W}{}^2_2(\Omega)$, then we get:

$$\vec{w} \in \overset{\circ}{\mathbf{W}}_0(\Omega) = \operatorname{curl}\{\vec{U}\}_{|U_1=U_2=U_3=U_4=U\in \overset{\circ}{W}{}^2_2(\Omega)} \subset \overset{\circ}{\mathbf{W}}(\Omega).$$

Next, we introduce the extended system of functions $\left\{U^1_m(x,y,z,\zeta)\right\}_{m=0}^{\infty}$, where

$$U^1_0(x,y,z,\zeta) \equiv 0, \;\; U^1_m(x,y,z,\zeta) = U_m(x,y,z,\zeta), \;\; m \in \mathbb{N},$$

and construct vector-functions

$$\vec{U}^1_{mjkq}(x,y,z,\zeta) = \left\{U^1_m, U^1_j, U^1_k, U^1_q\right\} \in \overset{\circ}{\mathbf{W}}{}^2_2(\Omega), \tag{47}$$

$m,j,k,q \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, which, as evident, will form a fundamental system in the space $\mathbf{V}_1(\Omega) = (V_1(\Omega))^4$.

Theorem 6. By applying the curl operator from (39) to the extended system of vector-functions (47), we obtain the desired fundamental system ($m,j,k,q \in \mathbb{N}_0$):

$$\vec{w}_{mjkq}(x,y,z,\zeta) = \{w_{1,kjj}, w_{2,qkk}, w_{3,mjq}, w_{4,jkm}\} \in \overset{\circ}{\mathbf{W}}(\Omega) \tag{48}$$

in the space of solenoidal functions $\mathbf{H}(\Omega)$ (32), where

$$w_{1,kjj}(x,y,z,\zeta) = \partial_y U^1_k - \partial_z U^1_j - \partial_\zeta U^1_j, \;\; (x,y,z,\zeta) \in \Omega, \quad j,k \in \mathbb{N}_0,$$

$$w_{2,qkk}(x,y,z,\zeta) = \partial_z U^1_q - \partial_\zeta U^1_k - \partial_\zeta U^1_k, \;\; (x,y,z,\zeta) \in \Omega, \quad k,q \in \mathbb{N}_0,$$

$$w_{3,mjq}(x,y,z,\zeta) = \partial_\zeta U^1_m + \partial_x U^1_j - \partial_y U^1_q, \;\; (x,y,z,\zeta) \in \Omega, \quad m,j,q \in \mathbb{N}_0,$$

$$w_{4,jkm}(x,y,z,\zeta) = \partial_x U^1_j + \partial_y U^1_k - \partial_z U^1_m, \;\; (x,y,z,\zeta) \in \Omega, \quad m,j,k \in \mathbb{N}_0,$$

$$\operatorname{div}\vec{w}_{mjkq}(x,y,z,\zeta) = 0, \;\; (x,y,z,\zeta) \in \Omega, \;\; m,j,k,q \in \mathbb{N}_0, \tag{49}$$

$$\vec{w}_{mjkq}(x,y,z,\zeta) = 0, \;\; (x,y,z,\zeta) \in \partial\Omega, \;\; m,j,k,q \in \mathbb{N}_0. \tag{50}$$

Proof of Theorem 6. From the formulas defining the vector-function (48), we sequentially obtain

$$|w_1 - w_1^\varepsilon|^2 \leq 3\left[|\partial_y(U_3 - U_3^\varepsilon)|^2 + |\partial_z(U_2 - U_2^\varepsilon)|^2 + |\partial_\zeta(U_2 - U_2^\varepsilon)|^2\right],$$

$$|w_2 - w_2^\varepsilon|^2 \leq 3\left[|\partial_x(U_3 - U_3^\varepsilon)|^2 + |\partial_z(U_4 - U_4^\varepsilon)|^2 + |\partial_\zeta(U_3 - U_3^\varepsilon)|^2\right],$$

$$|w_3 - w_3^\varepsilon|^2 \leq 3\left[|\partial_x(U_2 - U_2^\varepsilon)|^2 + |\partial_y(U_4 - U_4^\varepsilon)|^2 + |\partial_\zeta(U_1 - U_1^\varepsilon)|^2\right],$$

$$|w_4 - w_4^\varepsilon|^2 \leq 3\left[|\partial_x(U_2 - U_2^\varepsilon)|^2 + |\partial_y(U_3 - U_3^\varepsilon)|^2 + |\partial_z(U_1 - U_1^\varepsilon)|^2\right],$$

i.e.,

$$\|\vec{w} - \vec{w}^\varepsilon\|_{\mathbf{L}^2(\Omega)} \leq 2\sqrt{3}\|\vec{U} - \vec{U}^\varepsilon\|_{\mathbf{V}_1(\Omega)} \leq 2\sqrt{3}\,\varepsilon,$$

where, first, according to Proposition 11, for each vector-function $\vec{w}(z, y, z, \zeta) \in \overset{\circ}{\mathbf{W}}(\Omega)$ from (46), there corresponds a unique vector-function $\vec{U}(z, y, z, \zeta) \in \overset{\circ}{\mathbf{W}}{}_2^2(\Omega)$ (45).

Secondly, the finite sum

$$\vec{U}^{\,1\varepsilon}(x, y, z, \zeta) = \sum_{n=0}^{N_\varepsilon} a_n \vec{U}_n^1(x, y, z, \zeta), \quad N_\varepsilon < \infty, \tag{51}$$

ensuring the fulfillment of inequality

$$\|\vec{U}^1(x, y, z, \zeta) - \vec{U}^{\,1\varepsilon}(x, y, z, \zeta)\|_{\mathbf{V}_1(\Omega)} \leq \varepsilon, \tag{52}$$

corresponds to the sum defined by the formulas:

$$\vec{w}^{\,\varepsilon}(x, y, z, \zeta) = \sum_{n=0}^{N_\varepsilon} a_n \vec{w}_n(x, y, z, \zeta), \quad N_\varepsilon < \infty, \tag{53}$$

and satisfying inequality:

$$\frac{1}{2\sqrt{3}}\|\vec{w}(x, y, z, \zeta) - \vec{w}^{\,\varepsilon}(x, y, z, \zeta)\|_{\mathbf{H}(\Omega)} \leq \|\vec{U}^1(x, y, z, \zeta) - \vec{U}^{\,1\varepsilon}(x, y, z, \zeta)\|_{(V_1(\Omega))^6} \leq \varepsilon, \tag{54}$$

due to the fact that equality

$$\|\vec{w}(x, y, z, \zeta) - \vec{w}^{\,\varepsilon}(x, y, z, \zeta)\|_{\mathbf{L}^2(\Omega)} = \|\vec{w}(x, y, z, \zeta) - \vec{w}^{\,\varepsilon}(x, y, z, \zeta)\|_{\mathbf{H}(\Omega)},$$

and identity

$$\|\vec{w} - \vec{w}^{\,\varepsilon}\|_{\mathbf{H}^\perp(\Omega)} = 0, \ \ \mathbf{L}^2(\Omega) = \mathbf{H}(\Omega) \oplus \mathbf{H}^\perp(\Omega)$$

hold.

The relations (51)–(54) show that the system of four-dimensional vector-functions

$$\{\vec{w}_n(x, y, z, \zeta)\}_{n=0}^\infty \subset (V_1(\Omega))^4,$$

forms a fundamental system in the space of solenoidal vector fields $\mathbf{H}(\Omega)$, satisfying the condition

$$\operatorname{div} \vec{w}_n(x, y, z, \zeta) = 0.$$

This concludes the proof of Theorem 6.

Proposition 13. Any vector-functions of the form

$$\vec{w}(x,y,z,\zeta) = \{0,0,0,w_4(x,y,z,\zeta)\}\,,$$

$$\vec{w}(x,y,z,\zeta) = \{0,0,w_3(x,y,z,\zeta),0\}\,,$$
$$\vec{w}(x,y,z,\zeta) = \{0,w_2(x,y,z,\zeta),0,0\}\,,$$
$$\vec{w}(x,y,z,\zeta) = \{w_1(x,y,z,\zeta),0,0,0\}\,,$$

from the space $(\overset{\circ}{W}{}_2^1(\Omega))^4$ (where the functions $w_1(x,y,z,\zeta)$, $w_2(x,y,z,\zeta)$, $w_3(x,y,z,\zeta)$ and $w_4(x,y,z)$ are identically nonzero) cannot be solenoidal, i.e. they will not satisfy both the equation (incompressibility condition of the incompressible fluid) $\operatorname{div}\vec{w}(x,y,z,\zeta) = 0$ (49), and the boundary condition (50).

Thus, we have constructed a fundamental system in the space of solenoidal functions for a "four-dimensional cubic" domain.

Conclusion

The paper considers two four-dimensional curl operators. The first is the classical one, which is used in the description of Maxwell's equations for electromagnetic fields. The second curl operator is new and has not been known before. Based on these operators, an explicit construction of a fundamental system in the space of solenoidal functions for the "four-dimensional cube" is obtained. This fundamental system of functions can be used for the approximate solution of boundary value problems for stationary and evolutionary Stokes and Navier-Stokes equations. It is worth noting that in the works [23] and [24], the solution to the spectral problem (A1) for the biharmonic operator in the domain Ω, represented by a 3-D sphere, was found.

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Author Contributions

M.T. Jenaliyev served as the principal investigator of the research grant and supervised the research process. A.S. Kassymbekova assisted in data collection and analysis. M.G. Yergaliyev collected and analyzed data, and led manuscript preparation. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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*Author Information**

Muvasharkhan Tanabayevich Jenaliyev — Doctor of Physical and Mathematical Sciences, Professor, Chief Researcher, Institute of Mathematics and Mathematical Modeling, Almaty, 050010, Kazakhstan; e-mail: muvasharkhan@gmail.com; https://orcid.org/0000-0001-8743-7026

Arnai Sarsembekovna Kassymbekova — Candidate of Physical and Mathematical Sciences, Senior Researcher, Institute of Mathematics and Mathematical Modeling, Almaty, 050010, Kazakhstan; e-mail: kasar1337@gmail.com; https://orcid.org/0000-0002-4105-625X

Madi Gabidenovich Yergaliyev (*corresponding author*) — PhD, Senior Researcher, Institute of Mathematics and Mathematical Modeling, Almaty, 050010, Kazakhstan; e-mail: ergaliev@math.kz; https://orcid.org/0000-0001-8638-4647

*The author's name is presented in the order: First, Middle, and Last Names.