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Uniform asymptotic expansion of the solution for the initial value problem with a piecewise constant argument

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The article is devoted to the study of a singularly perturbed initial problem for a linear differential equation with a piecewise constant argument second-order for a small parameter. This paper is considered the asymptotic expansion of the solution to the Cauchy problem for singularly perturbed differential equations with piecewise-constant argument. The initial value problem for first order linear differential equations with piecewise-constant argument was obtained that determined the regular members. The Cauchy problems for linear nonhomogeneous differential equations with a constant coefficient were obtained, which determined the boundary layer terms. An asymptotic estimate for the remainder term of the solution of the Cauchy problem was obtained. Using the remainder term, we construct a uniform asymptotic solution with accuracy $O(\varepsilon^{N+1})$ on the $\theta_i \le t \le \theta_{i+1}, i = \overline{0,p}$ segment of the singularly perturbed Cauchy problem with a piecewise constant argument.

Keywords: singular perturbation, asymptotics, small parameter, boundary layer part, piecewise constant argument.

2020 Mathematics Subject Classification: 34D15, 34E10, 34K26.

Introduction

The singularly perturbed differential equations arise in various fields of chemical kinetics, mathematical biology fluid dynamics and in a variety models for control theory. These problems depend on a small positive parameter such that the solution varies rapidly in some domains and varies slowly in other domains. Asymptotics of the solution of singularly perturbed initial and boundary value problems with the phenomenon of an initial jump for ordinary differential equations with smooth coefficients in the general formulation were studied by K.A. Kasymov [1] and others. Asymptotics of the solution of the first boundary value problem for the linear ordinary differential equation of the second order with piecewise smooth coefficients and a small parameter at the highest derivative was first studied in the work of V. G. Sushko [2]. In particular, in the work of Kasymov [3] the case is considered when the highest coefficient of a degenerate equation has discontinuities of the first kind at points $t = t_i$, $i = \overline{1,n}$ and it is proved that the desired solution of the boundary value problem has initial jumps at these points. However, the case when the coefficients of the of a linear differential equation depend on a piecewise constant variable has not been investigated by them and others. The initial and boundary value problems considered in the studies [4–11] are equivalent to the Cauchy problem with the initial jump for differential and integro-differential equations in the stable case. Methods of solving nonlocal problems for hyperbolic equations with piecewise constant argument of the generalized type are given in papers [12–14]. A mathematical model including a piecewise constant argument was first considered by Busenberg and Cooke in 1982. Systematic studies of theoretical and practical problems involving

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piecewise constant arguments were initiated in the early 1980s. Since then, differential equations with piecewise constant arguments have attracted great attention from researchers in mathematics, biology, engineering, and other fields. They constructed a first-order linear equation to investigate vertically transmitted diseases. Following this work, using the method of reduction to discrete equations, many authors have analyzed various types of differential equations with piecewise constant arguments. A system of differential equations with piecewise constant argument of generalized type was introduced in [15, 16]. Asymptotic estimations of the solution to a singularly perturbed equation with piecewise constant argument were published [17, 18].

1 *Statement of the problem*

We consider the initial value problem for linear differential equations with a piecewise constant argument of a small parameter

$$\varepsilon y''(t) + A(t)y'(t) + B(t)y(t) + C(t)y(\beta(t)) = F(t), \tag{1}$$

$$y(0,\varepsilon) = d_0,\ y'(0,\varepsilon) = d_1, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, d_0, d_1 are known constants. The piecewise constant argument is determined with the function $\beta(t) = \theta_i$, if $t \in [\theta_i, \theta_{i+1}),\ i = \overline{1,p},\ 0 < \theta_1 < \theta_2 < ... < \theta p < T$.

Let us assume that the following conditions are satisfied:

C1) $A(t), B(t), C(t), F(t) \in C[0,T]$;

C2) $A(t) > 0,\ 0 \le t \le T$.

Theorem 1. Suppose that conditions (C1)-(C2) are fulfilled. Then, for the solution of the initial problem (1), (2) and its derivatives in the interval $0 \le t \le T$ for $\varepsilon > 0$, the following limit transitions are valid

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = \overline{y}(t), \quad 0 \le t \le T, \tag{3}$$

$$\lim_{\varepsilon\to 0} y'(t,\varepsilon) = \overline{y'}(t), \quad 0 < t \le T,$$

where $\overline{y}^{(q)}(t),\ q = 0,1$ is the solution to the following initial problem:
if $t \in [0;\theta_1)$

$$\begin{cases} A(t)\overline{y}'(t) + B(t)\overline{y}(t) = F(t) - C(t) - d_0, \\ \overline{y}(0) = d_0, \end{cases}$$

and if $t \in [\theta_i;\theta_{i+1}),\ i = \overline{1,p}$

$$\begin{cases} A(t)\overline{y}'(t) + B(t)\overline{y}(t) = F(t) - C(t)\overline{y}(\theta_i), \\ \overline{y}(\theta_i) = \overline{y}(\theta_i). \end{cases}$$

The convergence (3) can be nonuniform near several points, that is to say, that multi-layers emerge. These layers occur on the neighborhoods of $t = 0$ and $t = \theta_i,\ i = \overline{1,p}$.

For example, we take $A(t) = 1, B(t) = 0, C(t) = -3, \beta(t) = \left[\frac{t}{2}\right], F(t) = 1$ and $t \in [2n, 2n+2)$, $n = 0,1,2,\ d_0 = 1,\ d_1 = 3$. Then the graph of the solution is shown in Figure.

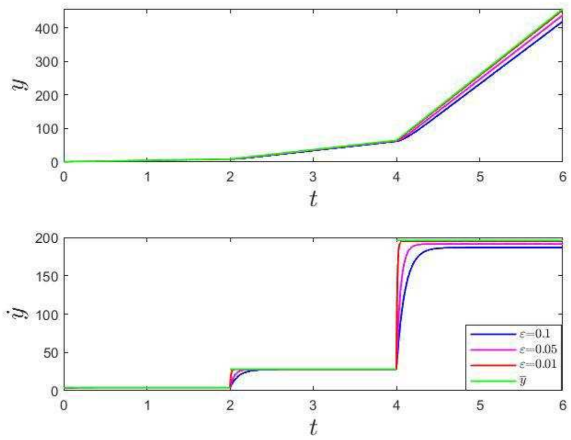

Figure. The blue, pink and red lines are graphs of solutions of example with initial values $d_0 = 1, d_1 = 3$ with values of $\varepsilon : 0.1, 0.05, 0.01$ respectively. The green line is the solution of unperturbed problem.

The derivative of solution of unperturbed problem is a discontinuity of the first kind.

2 *Uniform asymptotic expansion of the solution for the initial problem*

In the interval $\theta_i \leq t \leq \theta_{i+1},\ i = \overline{0,p}$, we look for a uniform asymptotic expansion of the solution to the initial problem (1), (2) in the following form

$$y(t,\varepsilon) = y_\varepsilon(t) + \varepsilon w_\varepsilon^{(i)}(\tau_i), \quad \tau_i = \frac{t-\theta_i}{\varepsilon}, \tag{4}$$

where

$$y_\varepsilon(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + ... + \varepsilon^k y_k(t) + ... \tag{5}$$

$$w_\varepsilon^{(i)}(\tau_i) = w_0^{(i)}(\tau_i) + \varepsilon w_1^{(i)}(\tau_i) + \varepsilon^2 w_2^{(i)}(\tau_i) + ... + \varepsilon^k w_k^{(i)}(\tau_i). \tag{6}$$

(5) is called the regular part of the asymptotics, and (6) is called the boundary-layer part of the asymptotics.

If we substitute expression (4) into equation (1), we obtain the following equality

$$\varepsilon \left(y''_\varepsilon(t) + \frac{\varepsilon}{\varepsilon^2} \ddot{w}_\varepsilon^{(i)}(\tau_i) \right) + A(t) \left(y'_\varepsilon(t) + \frac{\varepsilon}{\varepsilon} \dot{w}\varepsilon^{(i)}(\tau_i) \right) + \tag{7}$$

$$+B(t) \left(y_\varepsilon(t) + \varepsilon \dot{w}_\varepsilon^{(i)}(\tau_i) \right) + C(t)(y_\varepsilon(\theta_i) + \varepsilon w_\varepsilon^{(i)}(0)) = F(t).$$

From equation (7) we select equations that depend on the variables t and τ separately:

$$\varepsilon y''_\varepsilon(t) + A(t) y'_\varepsilon(t) + B(t) y_\varepsilon(t) + C(t) y_\varepsilon(\theta_i) = F(t), \tag{8}$$

$$\ddot{w}_\varepsilon^{(i)}(\tau_i) + A(\theta_i + \varepsilon\tau_i)\dot{w}_\varepsilon^{(i)}(\tau_i) + \varepsilon B(\theta_i + \varepsilon\tau_i) w_\varepsilon^{(i)}(\tau_i) + \varepsilon C(\theta_i + \varepsilon\tau_i) w_\varepsilon^{(i)}(0) = 0. \tag{9}$$

We substitute expression (5) into equation (8)

$$\varepsilon\left(y_0'(t)+\varepsilon y_1'(t)+...+\varepsilon^k y_k'(t)+...\right)+A(t)\left(y_0'(t)+\varepsilon y_1''(t)+...+\varepsilon^k y_k''(t)+...\right)+ \tag{10}$$

$$+B(t)\left(y_0(t)+\varepsilon y_1(t)+...+\varepsilon^k y_k(t)+...\right)+$$

$$+C(t)\left(y_0(\theta_i)+\varepsilon y_1(\theta_i)+...+\varepsilon^k y_k(\theta_i)+...\right)=F(t).$$

Equating the coefficients for a small parameter of the same degree in both sides of equation (10), we obtain a sequence of equations defining the $y_k(t),\ k=0,1,...$ functions

$$\varepsilon^0: A(t)y_0'(t)+B(t)y_0(t)+C(t)y_0(\theta_i)=F(t),$$

$$\varepsilon^1: A(t)y_1'(t)+B(t)y_1(t)+C(t)y_1(\theta_i)=-y_0''(t),$$

$$\varepsilon^k: A(t)y_k'(t)+B(t)y_k(t)+C(t)y_k(\theta_i)=-y_{k-1}''(t).$$

We classify the functions $A(\theta_i+\varepsilon\tau_i)$, $B(\theta_i+\varepsilon\tau_i)$, $i=\overline{0,p}$ into a Taylor series in the neighborhood of the point θ_i by degree ε

$$A(\theta_i+\varepsilon\tau_i)=A(\theta_i)+\frac{\varepsilon\tau_i}{1!}A'(\theta_i)+\frac{(\varepsilon\tau_i)^2}{2!}A''(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}A^{(k)}(\theta_i)+... \tag{11}$$

$$B(\theta_i+\varepsilon\tau_i)=B(\theta_i)+\frac{\varepsilon\tau_i}{1!}B'(\theta_i)+\frac{(\varepsilon\tau_i)^2}{2!}B''(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}B^{(k)}(\theta_i)+...$$

$$C(\theta_i+\varepsilon\tau_i)=C(\theta_i)+\frac{\varepsilon\tau_i}{1!}C'(\theta_i)+\frac{(\varepsilon\tau_i)^2}{2!}C''(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}C^{(k)}(\theta_i)+...$$

Substituting formulas (6), (11) into equation (9), we obtain the following expression

$$\ddot{w}_0^{(i)}(\tau_i)+\varepsilon\ddot{w}_1^{(i)}(\tau_i)+...+\varepsilon^k\ddot{w}_k^{(i)}(\tau_i)+...+\left(A(\theta_i)+\frac{\varepsilon\tau_i}{1!}A'(\theta_i)+\right. \tag{12}$$

$$\left.+\frac{(\varepsilon\tau_i)^2}{2!}A''(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}A^{(k)}(\theta_i)+...\right)(\ddot{w}_0^{(i)}(\tau_i)+\varepsilon\ddot{w}_1^{(i)}(\tau_i)+...+\varepsilon^k\ddot{w}_k^{(i)}(\tau_i)+...)+$$

$$+\varepsilon\left(B(\theta_i)+\frac{\varepsilon\tau_i}{1!}B'(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}B^{(k)}(\theta_i)+...\right)(\ddot{w}_0^{(i)}(\tau_i)+\varepsilon\ddot{w}_1^{(i)}(\tau_i)+...+\varepsilon^k\ddot{w}_k^{(i)}(\tau_i)+...)+$$

$$+\varepsilon\left(C(\theta_i)+\frac{\varepsilon\tau_i}{1!}C'(\theta_i)+...+\frac{(\varepsilon\tau_i)^k}{k!}C^{(k)}(\theta_i)+...\right)(\ddot{w}_0^{(i)}(\tau_i)+\varepsilon\ddot{w}_1^{(i)}(\tau_i)+...+\varepsilon^k\ddot{w}_k^{(i)}(\tau_i)+...)=0.$$

By equalizing the coefficients for a small parameter of the same degree in both sides of the equation (12), we obtain a sequence of equations defining the $w_k^{(i)}(\tau_i)$, $k=0,1,...$ functions

$$\varepsilon^0:\quad \ddot{w}_0^{(i)}(\tau_i)+A(\theta_i)\dot{w}_0^{(i)}(\tau_i)=0, \tag{13}$$

$$\varepsilon^1: \quad \ddot{w}_1^{(i)}(\tau_i) + A(\theta_i)\dot{w}_1^{(i)}(\tau_i) = \Phi_1(\tau_i), \tag{14}$$

where

$$\Phi_1(\tau_i) = -\tau_i A'(\theta_i)\dot{w}_0^{(i)}(\tau_i) - B(\theta_i)w_0^{(i)}(\theta_i) - C(\theta_i)w_0^{(i)}(0),$$

$$\varepsilon^k: \quad \ddot{w}_k^{(i)}(\tau_i) + A(\theta_i)\dot{w}_k^{(i)}(\tau_i) = \Phi_k(\tau_i), \tag{15}$$

where

$$\Phi_k(\tau_i) = -\sum_{m=1}^{k} \frac{(\tau_i)^m}{m!} A^{(m)}(\theta_i)\dot{w}_{k-m}^{(i)}(\tau_i) - \tag{16}$$

$$-\sum_{l=1}^{k} \frac{(\tau_i)^{l-1}}{(l-1)!} \left(B^{(l-1)}(\theta_i)w_{k-l}^{(i)}(\tau_i) + C^{(l-1)}(\theta_i)w_{k-l}^{(i)}(0) \right).$$

Consider the interval $t \in [0, \theta_1)$. Applying conditions (2) to the uniform asymptotic expansion of solution (4), equating the coefficients in front of the small parameter ε of the same degree, we determine the following conditions:

$$y_0(0) + \varepsilon y_1(0) + ... + \varepsilon^k y_k(0) + ...+$$

$$+\varepsilon \left(w_0^{(0)}(0) + \varepsilon w_1^{(0)}(0) + ... + \varepsilon^k w_k^{(0)}(0) + ... \right) = d_0,$$

$$y_0'(0) + \varepsilon y_1'(0) + ... + \varepsilon^k y_k'(0) + ...+$$

$$+\varepsilon \left(\dot{w}_0^{(0)}(0) + \varepsilon \dot{w}_1^{(0)}(0) + ... + \varepsilon^k \dot{w}_k^{(0)}(0) + ... \right) = d_1.$$

$$\varepsilon^0: y_0(0) = d_0, \;\; \dot{w}_0^{(0)}(0) = d_1 - y_0'(0),$$

$$\varepsilon^1: y_1(0) = -w_0^{(0)}(0), \;\; \dot{w}_1^{(0)}(0) = -y_1'(0),$$

$$\varepsilon^k: y_k(0) = -w_{k-1}^{(0)}0, \;\; \dot{w}_k^{(0)}(0) = -y_k'(0).$$

To determine the $w_k^{(i)}(\tau_i)$, $k = 0, 1, ...$ functions of the boundary layer, one more condition is necessary, since the order of equations (13)-(15) is equal to two. If we integrate equation (13) over $[\tau_i, \infty)$ and take into account condition $w_0^{(i)}(\infty) = 0$, $\dot{w}_0^{(i)}(\infty) = 0$, we determine the following expression

$$w_0^{(i)}(\tau_i) = -\frac{\dot{w}_0^{(i)}(\tau_i)}{A(\theta_i)}. \tag{17}$$

We substitute $\tau_0 = 0$ into equation (17):

$$w_0^{(0)}(0) = -\frac{d_1 - y_0'(0)}{A(0)}.$$

We continue this process and find the following conditions

$$w_k^{(0)}(0) = -\frac{1}{A(0)}\left(-y_k'(0) + \int_0^{\infty}\Phi_k(s)ds\right). \tag{18}$$

Let us determine the initial conditions for the interval $\theta_i \le t \le \theta_{i+1}$, $i = \overline{1,p}$:

$$y_0(\theta_i) + \varepsilon y_1(\theta_i) + ... + \varepsilon^k y_k(\theta_i) + ... + \varepsilon\left(w_0^{(i)}(0) + \varepsilon w_1^{(i)}(0) + ... + \varepsilon^k w_k^{(i)}(0) + ...\right) = y(\theta_0),$$

$$y_0'(\theta_i) + \varepsilon y_1'(\theta_i) + ... + \varepsilon^k y_k'(\theta_i) + ... + \varepsilon\left(\dot{w}_0^{(i)}(0) + \varepsilon \dot{w}_1^{(i)}(0) + ... + \varepsilon^k \dot{w}_k^{(i)}(0) + ...\right) = y'(\theta_i).$$

$$\varepsilon^0 : y_0(\theta_i) = y(\theta_i), \;\; \dot{w}_0^{(0)}(0) = y'(\theta_i) - y_0'(\theta_i),$$

$$\varepsilon^1 : y_1(\theta_i) = -w_0^{(i)}(0), \;\; \dot{w}_1^{(i)}(0) = -y_1'(\theta_i),$$

$$\varepsilon^k : y_k(\theta_i) = -w_{k-1}^{(i)}(0), \;\; \dot{w}_k^{(0)}(0) = -y_k'(\theta_i).$$

To determine the functions $w_k^{(i)}(\tau_0)$, $k = 0, 1, ...$ of the boundary layer, one more condition is necessary, since the order of equations (13)–(15) is equal to two. If we integrate equation (13) over $[\tau_i, \infty)$, $i = 1, 2, 3, ...$ and take into account the conditions $w_0^{(i)}(\infty) = 0$, $\dot{w}_0^{(i)}(\infty) = 0$, we determine the following expression:

$$w_0^{(i)}(\tau_i) = -\frac{1}{A(\theta_i)}\dot{w}_0^{(i)}(\tau_i). \tag{19}$$

If we substitute $\tau_i = \theta_i$, $i = 1, 2, 3, ...$ into equation (19),

$$w_0^{(i)}(0) = -\frac{1}{A(\theta_i)}(y'(\theta_i) - y_0'(\theta_i)).$$

Continuing this process, we obtain the following conditions

$$w_k^{(i)}(0) = -\frac{1}{A(\theta_i)}\left(-y_k'(\theta_i) + \int_{\theta_i}^{\infty}\Phi_k(s)ds\right), \;\; i = 1, 2, 3, ...$$

Problems defining regular terms for the interval $t \in [0, \theta_1)$

$$\begin{cases} A(t)y_0'(t) + B(t)y_0(t) + C(t)y_0(0) = F(t), \\ y_0(0) = d_0. \end{cases} \tag{20}$$

From the initial calculation (20), the $y_0(t)$ term of the regular part of the asymptotics is determined uniquely:

$$\begin{cases} A(t)y_1'(t) + B(t)y_1(t) + C(t)y_1(0) = -y_0''(t), \\ y_1(0) = -w_0^{(0)}(0). \end{cases} \tag{21}$$

From the initial calculation (21), the $y_1(t)$ term of the regular part of the asymptotics is determined uniquely:

$$\begin{cases} A(t)y_k'(t) + B(t)y_k(t) + C(t)y_k(0) = -y_{k-1}''(t), \\ y_k(0) = -w_{k-1}^{(0)}(0). \end{cases} \tag{22}$$

From the original description (22) the term $y_k(t)$ of the regular part of the asymptotics is determined uniquely.

Problems defining boundary-layer members for the interval $t \in [0, \theta_1)$

$$\ddot{w}_0^{(0)}(\tau_0) + A(0)\dot{w}_0^{(0)}(\tau_0) = 0, \tag{23}$$

$$w_0^{(0)}(0) = -\frac{d_1 - y_0'(0)}{A(0)},$$

$$\dot{w}_0^{(0)}(0) = d_1 - y_0'(0).$$

From the initial calculation (18) and (23), the zeroth approximation $w_0^{(0)}(\tau_0)$ of the boundary-layer part of the asymptotics is uniquely determined:

$$\ddot{w}_k^{(0)}(\tau_0) + A(0)\dot{w}_k^{(0)}(\tau_0) = \Phi_k(\tau_0), \tag{24}$$
$$w_k^{(0)}(0) = -\frac{1}{A(0)}\left(-y_k'(0) + \int_0^\infty \Phi_k(s)ds\right),$$
$$\dot{w}_k^{(0)}(0) = -y_k'(0),$$

where the function $\Phi_k(\tau_0)$ is determined by formula (16). From the initial calculation (24) the $w_k^{(0)}(\tau_0)$ $k-$th approximation of the boundary layer part of the asymptotics is uniquely determined.

Problems defining regular terms for the interval $t \in [\theta_i, \theta_{i+1})$, $i = 1, 2, ...$

$$\begin{cases} A(t)y_0'(t) + B(t)y_0(t) + C(t)y_0(\theta_i) = F(t), \\ y_0(\theta_i) = y(\theta_i), \end{cases}$$

$$\begin{cases} A(t)y_k'(t) + B(t)y_k(t) + C(t)y_k(\theta_i) = -y_{k-1}''(t), \\ y_k(0) = -w_{k-1}^{(0)}(\theta_i). \end{cases}$$

Problems of determining boundary-layer elements for the interval $t \in [\theta_i, \theta_{i+1})$, $i = 1, 2, ...$

$$\ddot{w}_0^{(i)}(\tau_i) + A(0)\dot{w}_0^{(i)}(\tau_i) = 0, \;\; i = \overline{1, p}$$

$$w_0^{(i)}(0) = -\frac{1}{A(\theta_i)}\left(y'(\theta_i) - y_0'(\theta_i)\right),$$

$$\dot{w}_0^{(i)}(0) = y'(\theta_i) - y_0'(\theta_i).$$

$$\ddot{w}_k^{(i)}(\tau_i) + A(\theta_i)\dot{w}_k^{(i)}(\tau_i) = \Phi_k(\tau_i), \;\; i = \overline{1, p}$$
$$w_k^{(i)}(0) = -\frac{1}{A(\theta_i)}\left(-y_k'(\theta_i) + \int_{\theta_i}^\infty \Phi_k(s)ds\right),$$
$$\dot{w}_k^{(i)}(0) = -y_k'(\theta_i).$$

3 Justification of the asymptotic behavior of the solution to the initial problem

Theorem 2. Let conditions (C1), (C2) be satisfied. Then, for a sufficiently small value of the small parameter ε (1), the initial problem (2) has a solution $y(t, \varepsilon)$ on the interval $\theta \leq t \leq \theta_{i+1}$, $i = \overline{0, p}$, which is unique and is expressed as

$$y(t, \varepsilon) = y_N(t, \varepsilon) + R_N(t, \varepsilon),$$

where the function $y_N(t, \varepsilon)$ is defined by the formula

$$y_N(t, \varepsilon) = \sum_{k=0}^{N} \varepsilon^k y_k(t) + \varepsilon \sum_{k=0}^{N} \varepsilon^k w_k^{(i)}(\tau_i), \;\; \tau_i = \frac{t - \theta_i}{\varepsilon}, \quad \theta_i \leq t \leq \theta_{i+1}, \; i = \overline{0, p} \tag{25}$$

and the following estimates are suitable for the remainder term $R_N(t,\varepsilon)$

$$|R_N^{(q)}(t,\varepsilon)| \le C\varepsilon^{N+1}, \quad q=0,1, \;\; \theta_i \le t \le \theta_{i+1}, \quad i=\overline{0,p}, \tag{26}$$

where $C>0$ is a quantity independent of ε.

Proof. We obtain the independent sum of series (25) from (4). If we substitute function (25) into equation (1), we obtain an equation

$$\varepsilon y_N''(t,\varepsilon) + A(t)y_N'(t,\varepsilon) + B(t)y_N(t,\varepsilon) + C(t)y_N'(\beta(t)) + D(t)y_N(\beta(t)) = F(t) + O\left(\varepsilon^{N+1}\right). \tag{27}$$

That is, the function $y_N(t,\varepsilon)$ satisfies the equation with accuracy $O\left(\varepsilon^{N+1}\right)$.

Satisfying the function (25) with conditions (2), we define the following conditions

$$y_N(0,\varepsilon) = d_0 + O\left(\varepsilon^{N+1}\right), \quad y_N'(0,\varepsilon) = d_1. \tag{28}$$

Let us introduce the difference between the exact solution and the approximate solution in the following form

$$R_N(t,\varepsilon) = y(t,\varepsilon) - y_N(t,\varepsilon) \Rightarrow y(t,\varepsilon) = y_N(t,\varepsilon) - R_N(t,\varepsilon). \tag{29}$$

The function $R_N(t,\varepsilon)$ is called the remainder term of the asymptotics.

Substituting formula (29) into equation (1), taking into account that the function $y_N(t,\varepsilon)$ satisfies equation (27) and conditions (28), we obtain the following equation defining the remainder term $R_N(t,\varepsilon)$

$$\varepsilon R_N''(t,\varepsilon) + A(t)R_N'(t,\varepsilon) + B(t)R_N(t,\varepsilon) + C(t)R_N'(\beta(t)) + D(t)R_N(\beta(t)) = O\left(\varepsilon^{N+1}\right), \tag{30}$$

$$R_N(0,\varepsilon) = O\left(\varepsilon^{N+1}\right), \;\; R_N'(0,\varepsilon) = 0.$$

Since the type of problem (30) is the same as the type of problem (1), (2), to solve problem (26) we use the solution estimate (1), (2)

$$|y^{(q)}(t,\varepsilon)| \le C\left(|d_0|(1+\max_{\theta_i\le t\le\theta_{i+1}}|C(t)|) + \varepsilon|d_1| + \max_{\theta_i\le t\le\theta_{i+1}}|F(t)|\right) +$$

$$+C\varepsilon^{1-q}\exp^{-\gamma\frac{t-\theta_i}{\varepsilon}}\left(|d_0|(1+\max_{\theta_i\le t\le\theta_{i+1}}|C(t)|) + |d_1| + \max_{\theta_i\le t\le\theta_{i+1}}|F(t)|\right), \;\; q=0,1, \;\; i=\overline{0,p+1}.$$

Then we obtain the following estimate for the solution

$$|R_N(t,\varepsilon)| \le C\varepsilon^{N+1} + C\varepsilon^{N+2}\exp^{-\gamma\frac{t-\theta_i}{\varepsilon}} \le C\varepsilon^{N+1}, \;\; \theta_i \le t \le \theta_{i+1}, \;\; i=\overline{0,p+1}$$

$$|R_N'(t,\varepsilon)| \le C\varepsilon^{N+1} + C\varepsilon^{N+1}\exp^{-\gamma\frac{t-\theta_i}{\varepsilon}} \le C\varepsilon^{N+1}, \;\; \theta_i \le t \le \theta_{i+1}, \;\; i=\overline{0,p+1}.$$

The following conclusion follows: the function $y_N(t,\varepsilon)$ is called an asymptotic solution obtained with an accuracy of $O\left(\varepsilon^{N+1}\right)$. From Theorem 2 it is clear that for the solution of the perturbed problem there is a uniform limit transition and it has a discontinuity of the 1st kind. Theorem 2 is proved.

Conclusion

In this paper, we considered the asymptotic expansion of the solution to the Cauchy problem for a singularly perturbed initial value problem for a linear differential equation with a piecewise constant second-order argument in a small parameter. We have obtained the initial problem for first-order linear differential equations with piecewise constant argument that determine the regular terms. Cauchy problems were also obtained for linear nonhomogeneous differential equations with a constant coefficient, which determine the terms of the boundary layer. Using an estimate for the solution to the initial problem, we obtained an asymptotic estimate for the remainder term for the solution to the Cauchy problem. And using the remainder term, we constructed a uniform asymptotic solution with an accuracy of $O(\varepsilon^{N+1})$ on $\theta_i \leq t \leq \theta_{i+1}$, $i = \overline{0,p}$ segment of a singularly perturbed Cauchy problem with piecewise constant argument.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The author declares no conflict of interest.

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