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The compact eighth-order of approximation difference schemes for fourth-order differential equation

A. Ashyralyev[1,2,3], I.M. Ibrahim[4,5,*]

[1] Bahcesehir University, Istanbul, Turkey;
[2] Peoples Friendship University Russia, Moscow, Russia;
[3] Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;
[4] Akre University for Applied Science, Akre, Iraq;
[5] Near East University, Nicosia, Turkey
(E-mail: aallaberen@gmail.com; ibrahimkalash81@gmail.com)

Local and nonlocal boundary value problems (LNBVPs) related to fourth-order differential equations (FODEs) were explored. To tackle these problems numerically, we introduce novel compact four-step difference schemes (DSs) that achieve eighth-order of approximation. These DSs are derived from a novel Taylor series expansion involving five points. The theoretical foundations of these DSs are validated through extensive numerical experiments, demonstrating their effectiveness and precision.

Keywords: Taylor's decomposition on five points (TDFP), LNBVPs, DSs, approximation, numerical experiment.

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Introduction

In applied sciences, achieving high precision in numerical algorithms is crucial, particularly when exact solutions are not feasible. Currently, a key focus is on developing and analyzing highly accurate DSs for ordinary and partial DEs with variable coefficients. Previous research has extensively explored the use of Taylor series expansions for constructing high-order compact finite DSs. For example, on two and three points Taylor's decomposition (TDs) has been used for approximate solutions of linear ordinary and partial DSs, as detailed in sources [1], [2], [3]. Further advancements include the use of three-step schemes with fourth-order of accuracy, derived from TDs on four points, for the numerical solution of several LNBVPs related to third-order DEs, as discussed in [4], [5], and [6]. These techniques have also been applied to third-order time-varying linear dynamical systems, as evidenced by the numerical analysis conducted on an up-converter in communication systems.

Recent studies [7] and [8] have expanded this work to include four-step DSs with fourth- and sixth-order accuracy, generated from TDFPs, specifically for linear ordinary DEs with boundary value problems (BVPs).

BVPs for ordinary DEs are fundamental in both theoretical and applied contexts, modeling a wide array of physical, biological, and chemical processes. Notable applications include Timoshenko's work on elasticity [9], Soedel's analysis of structural deformation [10], and Dulacska's research on soil settlement effects [11].

The literature on BVPs for higher-order DEs is extensive, including recent contributions [12], [13], and [14]. For a comprehensive overview of known results and additional references, see the monographs [15], [16], and paper [17].

*Corresponding author. *E-mail: ibrahimkalash81@gmail.com*
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Nonlinear FODEs, often termed beam equations, have also been studied under various boundary conditions. Zill and Cullen [18] provide a clear discussion and physical interpretation of boundary conditions for linear beam equations, contrasting with other conditions like conjugate [19], focal [12], [20], and [21].

In this paper, we introduce new compact eighth-order finite DSs, derived from an innovative TDFPs, for solving FODEs with variable coefficients.

We consider FSDSs of eighth-order approximation for the numerical solutions of three types of BVPs

$$\begin{cases} u^{(4)}(s)+a(s)u(s)=F(s), 0<s<\Upsilon, \\ u(0)=\varphi, u^{(1)}(0)=\eta, u(\Upsilon)=\omega, u^{(1)}(\Upsilon)=\varrho, \end{cases} \tag{1}$$

$$\begin{cases} u^{(4)}(s)+a(s)u(s)=F(s), 0<s<\Upsilon, \\ u(0)=\varphi, u(0)=\eta, u(\Upsilon)=\omega, u^{(2)}(\Upsilon)=\varrho, \end{cases} \tag{2}$$

$$\begin{cases} u^{(4)}(s)+a(s)u(s)=F(s), 0<s<\Upsilon, \\ u(0)=\varphi, u^{(3)}(0)=\eta, u(\Upsilon)=\omega, u^{(3)}(\Upsilon)=\varrho, \end{cases} \tag{3}$$

and of the nonlocal BVP

$$\begin{cases} u^{(4)}(s)+a(s)u(s)=F(s), 0<s<\Upsilon, u(0)=u(\Upsilon)+\varphi, \\ u^{(1)}(0)=u^{(1)}(\Upsilon)+\eta, u^{(2)}(0)=u^{(2)}(\Upsilon)+\omega, u^{(3)}(0)=u^{(3)}(\Upsilon)+\varrho \end{cases} \tag{4}$$

for the FODEs. We introduce the uniform grid space

$$[0,\Upsilon]_h=\{y_k=kh, k=0,1,\cdots,N, Nh=\Upsilon\}.$$

The primary objective of this paper is to develop highly accurate four-step DSs for solving local and nonlocal FODEs. We introduce eighth-order accurate DSs generated by a new technique based on a five-point stencil: $y_{k\pm2}$, $y_{k\pm1}$, and y_k within the interval $[0,\Upsilon]_h$. The theoretical underpinnings of these schemes are corroborated by numerical experiments. The structure of the paper is as follows: Section 1 details the construction of the new technique using five points. Sections 2 through 5 explore local BVPs (1), (2), (3) and a nonlocal BVP (4).

1 *A new TDFPs*

The design of eighth order of approximation DSs for the numerical solutions of the LNBVPs (1), (2), (3), and (4) is based on the subsequent theorem on new TDFPs.

Theorem 1.1. Let $W(y)$ be a function defined on the interval $[0,\Upsilon]$ with a continuous twelfth derivative. Then the subsequent relation is satisfied:

$$\begin{aligned} &h^{-4}(W(y_{k+2})-4W(y_{k+1})+6W(y_k)-4W(y_{k-1})+W(y_{k-2})) \\ &=\frac{76}{105}W^{(4)}(y_k)+\frac{9}{70}(W^{(4)}(y_{k+1})+W^{(4)}(y_{k-1}))+\frac{1}{105}(W^{(4)}(y_{k+2})+W^{(4)}(y_{k-2})) \\ &\quad -\frac{97}{1680}h^4W^{(8)}(y_k)+o(h^8). \end{aligned} \tag{5}$$

Proof. By applying Taylor's formula, we obtain

$$\begin{aligned} &h^{-4}(W(y_{k+2})-4W(y_{k+1})+6W(y_k)-4W(y_{k-1})+W(y_{k-2})) \\ &=W^{(4)}(y_k)+W^{(6)}(y_k)\frac{1}{6}h^2+W^{(8)}(y_k)\frac{1}{80}h^4+W^{(10)}(y_k)\frac{17}{7!6}h^6+o(h^8). \end{aligned} \tag{6}$$

Applying the method of undetermined coefficients(MUCs), we will aim to find

$$h^{-4}(W(y_{k+2})-4W(y_{k+1})+6W(y_k)-4W(y_{k-1})+W(y_{k-2}))$$
$$-\alpha W(y_k)-\beta(W^{(4)}(y_{k+1})+W^{(4)}(y_{k-1}))-\gamma(W^{(4)}(y_{k+2})+W^{(4)}(y_{k-2}))-dh^4W^{(8)}(y_k)=o(h^8).$$

Utilizing Taylor's formula, we derive

$$\alpha W^{(4)}(y_k)+\beta(W^{(4)}(y_{k+1})+W^{(4)}(y_{k-1}))+\gamma(W^{(4)}(y_{k+2})+W^{(4)}(y_{k-2}))$$
$$=(\alpha+2\beta+2\gamma)W^{(4)}(y_k)+(\beta+4\gamma)W^{(6)}(y_k)h^2+(\frac{1}{12}\beta+\frac{4}{3}\gamma)W^{(8)}(y_k)h^4$$
$$+(\frac{1}{5!3}\beta+\frac{2^7}{6!}\gamma)W^{(10)}(y_k)h^6+o(h^8).$$

Using formula (6) and above formula , we get

$$h^{-4}(W(y_{k+2})-4W(y_{k+1})+6W(y_k)-4W(y_{k-1})+W(y_{k-2}))$$
$$-\alpha W^{(4)}(y_k)-\beta(W^{(4)}(y_{k+1})+W^{(4)}(y_{k-1}))-\gamma(W^{(4)}(y_{k+2})+W^{(4)}(y_{k-2}))-dh^4W^{(8)}(y_k)$$
$$=(1-\alpha-2\beta-2\gamma)W^{(4)}(y_k)+(\frac{1}{6}-\beta-4\gamma)W^{(6)}(y_k)h^2+(\frac{1}{80}-\frac{1}{12}\beta-\frac{4}{3}\gamma-d)W^{(8)}(y_k)h^4$$
$$+(\frac{17}{7!6}-\frac{1}{5!3}\beta-\frac{2^7}{6!}\gamma)W^{(10)}(y_k)h^6+o(h^8).$$

By setting the coefficient of the lowest power of h to zero, we derive the following system of algebraic equations(SAEs).

$$\begin{cases} \alpha+2\beta+2\gamma=1, \\ \beta+4\gamma=\frac{1}{6}, \\ \frac{1}{12}\beta+\frac{4}{3}\gamma+d=\frac{1}{80}, \\ \frac{1}{5!3}\beta+\frac{2^7}{6!}\gamma=\frac{17}{7!6}. \end{cases}$$

Upon resolving this SAEs, we find $\alpha=\dfrac{76}{105}$, $\beta=\dfrac{9}{70}$, $\gamma=\dfrac{1}{105}$, $d=-\dfrac{97}{1680}$. The relation (5) is obtained. Theorem 1.1 is established.

Theorem 1.2. Let $W(y)$ be a function defined on the interval $[0,\Upsilon]$ with a continuous fifth derivative. Then the subsequent relation holds:

$$W^{(1)}(y_k)=\frac{2}{3h}(W(y_{k+1})-W(y_{k-1}))-\frac{1}{12h}(W(y_{k+2})-W(y_{k-2}))+o\left(h^4\right). \tag{7}$$

Proof. By applying Taylor's formula, we obtain

$$W^{(1)}(y_k)=\beta(W(y_{k+1})-W(y_{k-1}))+\gamma(W(y_{k+2})-W(y_{k-2}))+o\left(h^4\right).$$

Utilizing Taylor's formula, we derive

$$(h^{-1}-(2\beta+4\gamma))W^{(1)}(y_k)h+(\frac{2}{3!}\beta+\frac{16}{3!}\gamma)W^{(3)}(y_k)h^3+(\beta+\gamma)o(h^5).$$

By setting the coefficient of the lowest power of h to zero, we derive the following SAEs.

$$\begin{cases} 2\beta+4\gamma=h^{-1}, \\ \frac{2}{3!}\beta+\frac{16}{3!}\gamma=0. \end{cases}$$

Upon resolving this SAEs, we find $\beta=\frac{2}{3}h^{-1}$, $\gamma=-\frac{1}{12}h^{-1}$. So, relation (7) is proved. Theorem 1.2 is established.

Theorem 1.3. Let $W(y)$ be a function defined on the interval $[0, \Upsilon]$ with a continuous sixth derivative. Then the subsequent relation holds:

$$W^{(2)}(y_k) = \frac{4}{3h^2}\left(W(y_{k+1}) + W(y_{k-1}) - 2W(y_k)\right) \tag{8}$$

$$-\frac{1}{12h^2}\left(W(y_{k+2}) + W(y_{k-2}) - 2W(y_k)\right) + o\left(h^4\right).$$

Proof. By applying Taylor's formula, we obtain

$$W^{(2)}(y_k) = \beta\left(W(y_{k+1}) + W(y_{k-1}) - 2W(y_k)\right) + \gamma\left(W(y_{k+2}) + W(y_{k-2}) - W(y_k)\right) + o\left(h^4\right).$$

Utilizing Taylor's formula, we derive

$$\left(h^{-2} - \left(\frac{2}{2!}\beta + \frac{8}{2!}\gamma\right)\right) W^{(2)}(y_k)\, h^2 + \left(\frac{2}{4!}\beta + \frac{32}{4!}\gamma\right) W^{(4)}(y_k)\, h^4 + (\beta+\gamma)o(h^6).$$

To obtain the SAEs, equate the coefficients of the smallest power of h in the above identity to 0.

$$\begin{cases} \frac{2}{2!}\beta + \frac{8}{2!}\gamma = h^{-2}, \\ \frac{2}{4!}\beta + \frac{32}{4!}\gamma = 0. \end{cases}$$

Upon resolving this SAEs, we find $\beta = \frac{4}{3}h^{-2}$, $\gamma = -\frac{1}{12}h^{-2}$. So, relation (8) is proved. Theorem 1.3 is established.

Theorem 1.4. Let $W(y)$ be a function defined on the interval $[0, \Upsilon]$ with a continuous seventh derivative. Then the subsequent relation holds:

$$W^{(3)}(y_k) = \frac{896}{159h^3}\left(W(y_{k+1}) - W(y_{k-1}) - 2W^{(1)}(y_k)\, h\right) \tag{9}$$

$$-\frac{419}{1272h^3}\left(W(y_{k+2}) - W(y_{k-2}) - 4W^{(1)}(y_k)\, h\right) + o\left(h^4\right).$$

Proof. Applying the MUCs, we will aim to find

$$W^{(3)}(y_k) = \beta\left(W(y_{k+1}) - W(y_{k-1}) - 2W^{(1)}(y_k)\, h\right) + \gamma\left(W(y_{k+2}) - W(y_{k-2}) - 4W^{(1)}(y_k)\, h\right)$$

$$+h^4\left(p\left(W^{(4)}(y_{k+1}) - W^{(4)}(y_{k-1})\right) + q\left(W^{(4)}(y_{k+2}) - W^{(4)}(y_{k-2})\right)\right) + o\left(h^8\right).$$

By applying Taylor's formula, we obtain

$$\left(h^{-3} - \left(\frac{2}{3!}\beta + \frac{16}{3!}\gamma\right)\right) W^{(3)}(y_k)\, h^3 + \left(\frac{2}{5!}\beta + \frac{64}{5!}\gamma\right) W^{(5)}(y_k)\, h^5 + (\beta+\gamma)\, o\left(h^7\right).$$

To obtain the SAEs, equate the coefficients of the smallest power of h in the above identity to 0.

$$\begin{cases} \frac{2}{3!}\beta + \frac{16}{3!}\gamma = h^{-3}, \\ \frac{2}{5!}\beta + \frac{64}{5!}\gamma = 0. \end{cases}$$

Upon resolving this SAEs, we find $\beta = \dfrac{896}{159}h^{-3}$, $\gamma = -\dfrac{419}{1272}h^{-3}$. So, relation (9) is proved. Theorem 1.4 is established.

2 Local BVP (1)

Let us consider BVP (1). For the application of TDFPs (5), we have to give the eighth order of approximation formulas for $W^{(1)}(0)$ and $W^{(1)}(\Upsilon)$.

Theorem 2.1. Let $W(y)$ be a function defined on the interval $[0, \Upsilon]$ with a continuous fifth derivative. Then the subsequent relations hold:

$$\begin{aligned} W^{(1)}(0) = h^{-1} \Bigg\{ & \frac{2223}{518}\left(W(h) - W(0)\right) - \frac{3735}{1036}\left(W(2h) - W(0)\right) + \frac{1535}{777}\left(W(3h) - W(0)\right) \\ & -\frac{45}{74}\left(W(4h) - W(0)\right) + \frac{243}{2590}\left(W(5h) - W(0)\right) - \frac{23}{3108}\left(W(6h) - W(0)\right)\Bigg\} \\ & - \frac{69h^3}{518}\left(W^{(4)}(h) - W^{(4)}(0)\right) + \frac{21h^3}{148}\left(W^{(4)}(2h) - W^{(4)}(0)\right) + o\left(h^8\right), \end{aligned} \tag{10}$$

$$\begin{aligned} W^{(1)}(\Upsilon) = h^{-1} \Bigg\{ & -\frac{2223}{518}\left(W(\Upsilon - h) - W(\Upsilon)\right) + \frac{3735}{1036}\left(W(\Upsilon - 2h) - W(\Upsilon)\right) \\ & -\frac{243}{2590}\left(W(\Upsilon - 5h) - W(\Upsilon)\right) + \frac{23}{3108}\left(W(\Upsilon - 6h) - W(\Upsilon)\right) \\ & - \frac{1535}{777}\left(W(\Upsilon - 3h) - W(\Upsilon)\right) + \frac{45}{74}\left(W(\Upsilon - 4h) - W(\Upsilon)\right) + o\left(h^8\right). \end{aligned} \tag{11}$$

Proof. Applying the MUCs, we will aim to find

$$\begin{aligned} W^{(1)}(0) = \beta\left(W(h) - W(0)\right) + \gamma\left(W(2h) - W(0)\right) + d\left(W(3h) - W(0)\right) \\ +p\left(W(4h) - W(0)\right) + q\left(W(5h) - W(0)\right) + w\left(W(6h) - W(0)\right) \\ +h^4 m\left(W^{(4)}(h) - W^{(4)}(0)\right) + h^4 n\left(W^{(4)}(2h) - W^{(4)}(0)\right) + o\left(h^8\right). \end{aligned}$$

By applying Taylor's formula, we obtain

$$\begin{aligned} W^{(1)}(0) = \beta\sum_{l=1}^{8}\frac{h^l}{l!}W^{(l)}(0) + \gamma\sum_{l=1}^{8}\frac{(2h)^l}{l!}W^{(l)}(0) + d\sum_{l=1}^{8}\frac{(3h)^l}{l!}W^{(l)}(0) \\ +p\sum_{l=1}^{8}\frac{(4h)^l}{l!}W^{(l)}(0) + q\sum_{l=1}^{8}\frac{(5h)^l}{l!}W^{(l)}(0) + w\sum_{n=1}^{8}\frac{(6h)^l}{l!}W^{(l)}(0) \\ +h^4 m\sum_{l=1}^{4}\frac{h^l}{l!}W^{(l+4)}(0) + h^4 n\sum_{l=1}^{4}\frac{(2h)^l}{l!}W^{(l+4)}(0) + o\left(h^8\right). \end{aligned}$$

To obtain the SAEs, equate the coefficients of the smallest power of h in the above identity to 0.

$$\begin{cases} 3d + 4p + \beta + 2\gamma + 5q + 6w = h^{-1}, \\ \frac{9}{2!}d + \frac{16}{2!}p + \frac{1}{2!}\beta + \frac{4}{2!}\gamma + \frac{25}{2!}q + \frac{36}{2!}w = 0, \\ \frac{27}{3!}d + \frac{64}{3!}p + \frac{1}{3!}\beta + \frac{8}{3!}\gamma + \frac{125}{3!}q + \frac{216}{3!}w = 0, \\ \frac{81}{4!}d + \frac{256}{4!}p + \frac{1}{4!}\beta + \frac{16}{4!}\gamma + \frac{625}{4!}q + \frac{1296}{4!}w = 0, \\ \frac{243}{5!}d + \frac{1024}{5!}p + \frac{1}{5!}\beta + \frac{32}{5!}\gamma + \frac{3125}{5!}q + \frac{7776}{5!}w + m + 2n = 0, \\ \frac{729}{6!}d + \frac{4096}{6!}p + \frac{1}{6!}\beta + \frac{64}{6!}\gamma + \frac{15\,625}{6!}q + \frac{46\,656}{6!}w + \frac{1}{2!}m + \frac{4}{2!}n = 0, \\ \frac{2187}{7!}d + \frac{16\,384}{7!}p + \frac{1}{7!}\beta + \frac{128}{7!}\gamma + \frac{78\,125}{7!}q + \frac{279\,936}{7!}w + \frac{1}{3!}m + \frac{8}{3!}n = 0, \\ \frac{6561}{8!}d + \frac{65\,536}{8!}p + \frac{1}{8!}\beta + \frac{256}{8!}\gamma + \frac{390\,625}{8!}q + \frac{1679\,616}{8!}w + \frac{1}{4!}m + \frac{16}{4!}n = 0, \\ \frac{19\,683}{9!}d + \frac{262\,144}{9!}p\frac{1}{9!}\beta + \frac{512}{9!}\gamma + \frac{1953\,125}{9!}q + \frac{10\,077\,696}{9!}w + \frac{1}{5!}m + \frac{32}{5!}n = 0, \\ \frac{59\,049}{10!}d + \frac{1048\,576}{10!}p + \frac{1}{10!}\beta + \frac{1024}{10!}\gamma + \frac{9765\,625}{10!}q + \frac{60\,466\,176}{10!}w + \frac{1}{6!}m + \frac{64}{6!}n = 0. \end{cases}$$

Upon resolving this SAEs, we find $\beta = \dfrac{2223}{518h}$, $\gamma = -\dfrac{3735}{1036h}$, $d = \dfrac{1535}{777h}$, $p = -\dfrac{45}{74h}$, $q = \dfrac{243}{2590h}$, $w = -\dfrac{23}{3108h}$, $m = -\dfrac{69}{518h}$, $n = \dfrac{21}{148h}$. So, relation (10) is established. In a similar fashion, one can derive the relationship (11). Theorem 2.1 is established.

Now, we consider the application of Theorems 1.1–1.4 and Theorem 2.1 for the numerical solution of the BVP (1). Using the equation (1) and formulas (5), (7), (8), (9), (10), (11), and disregarding minor terms, we can present the eighth order of approximation DS

$$\begin{cases} h^{-4}(u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}) \\ + \left(\frac{76}{105} - \frac{97}{1680}h^4 a(x_k)\right) a(t_k)u_k + \frac{9}{70}a(x_{k+1})u_{k+1} \\ + a(x_{k-1})u_{k-1} + \frac{1}{105}\left(a(x_{k+2})u_{k+2} + a(x_{k-2})u_{k-2}\right) \\ = \left(\frac{76}{105} + \frac{97}{1680}h^4 a(x_k)\right) F(x_k) + \frac{9}{70}\left(F(x_{k+1}) + F(x_{k-1})\right) \\ + \frac{1}{105}\left(F(x_{k+2}) + F(x_{k-2})\right) - \frac{97}{1680}h^4 F^{(4)}(x_k), \\ \\ \left(-\frac{5543}{2590} + \frac{9}{1036}h^4 a(0)\right) u_0 + \left(\frac{2223}{518} + \frac{69}{518}h^4 a(h)\right) u_1 \\ - \left(\frac{3735}{1036} + \frac{21}{148}h^4 a(2h)\right) u_2 + \frac{1535}{777}u_3 - \frac{45}{74}u_4 + \frac{243}{2590}u_5 - \frac{23}{3108}u_6 \\ = h\eta + h^4\left[\frac{69}{518}\left(F(x_1) - F(x_0)\right) - \frac{21}{148}\left(F(x_2) - F(x_0)\right)\right], u_0 = \varphi, \\ \\ \left(\frac{5543}{2590} - \frac{9}{1036}h^4 a(0)\right) u_N - \left(\frac{2223}{518} + \frac{69}{518}h^4 a(h)\right) u_{N-1} \\ + \left(\frac{3735}{1036} + \frac{21}{148}h^4 a(2h)\right) u_{N-2} - \frac{1535}{777}u_{N-3} + \frac{45}{74}u_{N-4} - \frac{243}{2590}u_{N-5} \\ + \frac{23}{3108}u_{N-6} = h\rho - h^4\frac{69}{518}\left(F(x_{N-1}) - F(x_N)\right) \\ - h^4\frac{21}{148}\left(F(x_{N-2}) - F(x_N)\right), u_N = \omega \end{cases} \tag{12}$$

for the numerical solution of the BVP (1).

3 Local BVP (2)

Consider the BVP (2). For the application of TD's on five points (5), we have to give the eighth order of approximation formulas for $W^{(2)}(0)$ and $W^{(2)}(\Upsilon)$.

Theorem 3.1. Let $W(y)$ be a function defined on the interval $[0, \Upsilon]$ with a continuous tenth derivative. Then the subsequent relations hold:

$$W^{(2)}(0) - h^{-2}\left\{\frac{6937\,573}{3439\,828}W(0) - \frac{26\,121\,217}{5159\,742}W(h) + \frac{21\,060\,241}{5159\,742}W(2h)\right. \tag{13}$$

$$\left. -\frac{892\,879}{859\,957}W(3h) - \frac{26\,209}{10\,319\,484}W(4h) + \frac{24\,995}{5159\,742}W(5h)\right\} - h^2\frac{23\,426\,639}{206\,389\,680}W^{(4)}(0)$$

$$-h^2\left(\frac{12\,741\,989}{20\,638\,968}W^{(4)}(h) + \frac{5216\,939}{29\,484\,240}W^{(4)}(2h) - \frac{1324\,691}{103\,194\,840}W^{(4)}(3h)\right) = o(h^8),$$

$$W^{(2)}(\Upsilon) - h^{-2}\left\{\frac{6937\,573}{3439\,828}W(\Upsilon) - \frac{26\,121\,217}{5159\,742}W(\Upsilon - h)\right. \tag{14}$$

$$\left. +\frac{21\,060\,241}{5159\,742}W(\Upsilon - 2h) - \frac{892\,879}{859\,957}W(\Upsilon - 3h) - \frac{26\,209}{10\,319\,484}W(\Upsilon - 4h)\right\}$$

$$-h^{-2}\frac{24\,995}{5159\,742}W(\Upsilon - 5h) - h^2\left(\frac{23\,426\,639}{206\,389\,680}W^{(4)}(\Upsilon) + \frac{12\,741\,989}{20\,638\,968}W^{(4)}(\Upsilon - h)\right.$$

$$\left. +\frac{5216\,939}{29\,484\,240}W^{(4)}(\Upsilon - 2h) - \frac{1324\,691}{103\,194\,840}W^{(4)}(\Upsilon - 3h)\right) = o(h^8).$$

Proof. Applying the MUCs, we will aim to find

$$W^{(2)}(0)=\alpha W(0)+\beta W(h)+\gamma W(2h)+dW(3h)+pW(4h)+qW(5h)$$

$$+h^4mW^{(4)}(0)+h^4nW^{(4)}(h)+h^4fW^{(4)}(2h)+h^4wW^{(4)}(3h)+o\left(h^8\right).$$

By applying Taylor's formula, we obtain

$$W^{(2)}(0)=\alpha W(0)+\beta\sum_{l=0}^{9}\frac{h^l}{l!}W^{(l)}(0)+\gamma\sum_{l=0}^{9}\frac{(2h)^l}{l!}W^{(l)}(0)+d\sum_{l=0}^{9}\frac{(3h)^l}{l!}W^{(l)}(0)$$

$$+p\sum_{l=0}^{9}\frac{(4h)^l}{l!}W^{(l)}(0)+q\sum_{l=0}^{9}\frac{(5h)^l}{l!}W^{(l)}(0)+h^4mW^{(4)}(0)$$

$$+h^4n\sum_{l=0}^{5}\frac{h^l}{l!}W^{(l+4)}(0)+h^4f\sum_{l=0}^{5}\frac{(2h)^l}{l!}W^{(l+4)}(0)+h^4w\sum_{l=0}^{5}\frac{(3h)^l}{l!}W^{(l+4)}(0)+o\left(h^8\right).$$

To obtain the SAEs, equate the coefficients of the smallest power of h in the above identity to 0.

$$\begin{cases}
d+p+\alpha+\beta+\gamma+q=0,\\
3d+4p+\beta+2\gamma+5q=0,\\
\frac{9}{2!}d+\frac{16}{2!}p+\frac{1}{2!}\beta+\frac{4}{2!}\gamma+\frac{25}{2!}q=h^{-2},\\
\frac{27}{3!}d+\frac{64}{3!}p+\frac{1}{3!}\beta+\frac{8}{3!}\gamma+\frac{125}{3!}q=0,\\
\frac{81}{4!}d+\frac{256}{4!}p+\frac{1}{4!}\beta+\frac{16}{4!}\gamma+\frac{625}{4!}q+m+n+f+w=0,\\
\frac{243}{5!}d+\frac{1024}{5!}p+\frac{1}{5!}\beta+\frac{32}{5!}\gamma+\frac{3125}{5!}q+n+2f+3w=0,\\
\frac{729}{6!}d+\frac{4096}{6!}p+\frac{1}{6!}\beta+\frac{64}{6!}\gamma+\frac{15\,625}{6!}q+\frac{1}{2!}n+\frac{4}{2!}f+\frac{9}{2!}w=0,\\
\frac{2187}{7!}d+\frac{16\,384}{7!}p+\frac{1}{7!}\beta+\frac{128}{7!}\gamma+\frac{78\,125}{7!}q+\frac{1}{3!}n+\frac{8}{3!}f+\frac{27}{3!}w=0,\\
\frac{3^8}{8!}d+\frac{4^8}{8!}p+\frac{1}{8!}\beta+\frac{2^8}{8!}\gamma+\frac{5^8}{8!}q+\frac{1}{4!}n+\frac{16}{4!}f+\frac{81}{4!}w=0,\\
\frac{3^9}{9!}d+\frac{4^9}{9!}p+\frac{1}{9!}\beta+\frac{2^9}{9!}\gamma+\frac{5^9}{9!}q+\frac{1}{5!}n+\frac{32}{5!}f+\frac{243}{5!}w=0.
\end{cases}$$

Upon resolving this SAEs, we find $\alpha=\frac{6937\,573}{3439\,828}$, $\beta=-\frac{26\,121\,217}{5159\,742}$, $\gamma=\frac{21\,060\,241}{5159\,742}$, $d=-\frac{892\,879}{859\,957}$, $p=-\frac{26\,209}{10\,319\,484}$, $q=\frac{24\,995}{5159\,742}$, $m=\frac{23\,426\,639}{206\,389\,680}$, $n=\frac{12\,741\,989}{20\,638\,968}$, $f=\frac{5216\,939}{29\,484\,240}$, $w=-\frac{1324\,691}{103\,194\,840}$. So, relation (13) is proved. In a similar fashion, one can derive the relationship (14). Theorem 3.1 is established.

Now, we consider the application of Theorems 1.1–1.4 and Theorem 3.1 for the numerical solution of the BVP (2). Using the equation (2) and formulas (5), (7), (8), (9), (13), (14), and disregarding minor terms, we can present the eighth order of approximation DS

$$\begin{cases} \left(1+\frac{1}{105}h^4a(x_{k-2})\right)u_{k-2}+\left(-4+\frac{9}{70}h^4a(x_{k-1})\right)u_{k-1} \\ +\left(6+\left(\frac{76}{105}-\frac{97}{1680}h^4a(x_k)\right)a(x_k)h^4\right)u_k \\ +\left(-4+\frac{9}{70}h^4a(x_{k+1})\right)u_{k+1}+\left(1+\frac{1}{105}h^4a(x_{k+2})\right)u_{k+2} \\ =h^4\left[\left(\frac{76}{105}+\frac{97}{1680}h^4a(x_k)\right)F(x_k)+\frac{9}{70}\left(F(x_{k+1})+F(x_{k-1})\right)\right. \\ \left.+\frac{1}{105}\left(F(x_{k+2})+F(x_{k-2})\right)-\frac{97}{1680}h^4F^{(4)}(x_k)\right],\ 2\le k\le N-2, \\ \left(\frac{6937\,573}{3439\,828}-\frac{23\,426\,639}{206\,389\,680}h^4a(0)\right)u_0-\left(\frac{26\,121\,217}{5159\,742}+\frac{12\,741\,989}{20\,638\,968}h^4a(h)\right)u_1 \\ +\left(\frac{21\,060\,241}{5159\,742}-\frac{5216\,939}{29\,484\,240}h^4a(2h)\right)u_2 \\ -\left(\frac{892\,879}{859\,957}-\frac{1324\,691}{103\,194\,840}h^4a(3h)\right)u_3 \\ -\frac{26\,209}{10\,319\,484}u_4+\frac{24\,995}{5159\,742}u_5=h^2\eta-\frac{23\,426\,639}{206\,389\,680}f(0) \\ -\left[\frac{12\,741\,989}{20\,638\,968}f(h)+\frac{5216\,939}{29\,484\,240}f(2h)-\frac{1324\,691}{103\,194\,840}f(3h)\right],u_0=\varphi, \\ \left(\frac{6937\,573}{3439\,828}-\frac{23\,426\,639}{206\,389\,680}h^4a(\Upsilon)\right)u_N \\ -\left(\frac{26\,121\,217}{5159\,742}+\frac{12\,741\,989}{20\,638\,968}h^4a(\Upsilon-h)\right)u_{N-1} \\ +\left(\frac{21\,060\,241}{5159\,742}-\frac{5216\,939}{29\,484\,240}h^4a(\Upsilon-2h)\right)u_{N-2} \\ \left(-\frac{892\,879}{859\,957}+\frac{1324\,691}{103\,194\,840}h^4a(\Upsilon-3h)\right)u_{N-3}-\frac{26\,209}{10\,319\,484}u_{N-4} \\ +\frac{24\,995}{5159\,742}u_{N-5}=h^2\rho-h^4\left[\frac{23\,426\,639}{206\,389\,680}F(\Upsilon)+\frac{12\,741\,989}{20\,638\,968}F(\Upsilon-h)\right. \\ \left.+\frac{5216\,939}{29\,484\,240}F(\Upsilon-2h)-\frac{1324\,691}{103\,194\,840}F(\Upsilon-3h)\right],u_N=\omega, \end{cases} \tag{15}$$

for the numerical solution of the BVP (2).

4 Local BVP (3)

Let us consider BVP (3). For the application of TD's on five points (5), we have to give the eighth order of approximation formulas for $W^{(3)}(0)$ and $W^{(3)}(\Upsilon)$.

Theorem 4.1. Let $W(y)$ be a function defined on the interval $[0,\Upsilon]$ with a continuous eleventh derivative. Then the subsequent relations hold:

$$W^{(3)}(0)-h^{-3}\left\{\frac{126\,630\,131}{4505\,760}(W(h)-W(0))-\frac{78\,574\,591}{1501\,920}(W(2h)-W(0))\right. \tag{16}$$

$$+\frac{45\,949\,355}{901\,152}(W(3h)-W(0))-\frac{24\,699\,239}{901\,152}(W(4h)-W(0))+\frac{1667\,173}{214\,560}(W(5h)-W(0))$$

$$\left.-\frac{4609\,391}{4505\,760}(W(6h)-W(0))+\frac{309\,293}{4505\,760}(W(7h)-W(0))\right\}$$

$$+h\left\{-\frac{597\,497}{1501\,920}\left(W^{(4)}(h)-W^{(4)}(0)\right)+\frac{1528\,979}{500\,640}\left(W^{(4)}(2h)-W^{(4)}(0)\right)\right.$$

$$\left.-\frac{1173\,833}{500\,640}\left(W^{(4)}(3h)-W^{(4)}(0)\right)\right\}=o\left(h^8\right),$$

$$W^{(3)}(\Upsilon)-h^{-3}\left\{-\frac{126\,630\,131}{4505\,760}\left(W(\Upsilon-h)-W(\Upsilon)\right)\right. \tag{17}$$
$$+\frac{78\,574\,591}{1501\,920}\left(W(\Upsilon-2h)-W(\Upsilon)\right)-\frac{45\,949\,355}{901\,152}\left(W(\Upsilon-3h)-W(\Upsilon)\right)$$
$$+\frac{24\,699\,239}{901\,152}\left(W(\Upsilon-4h)-W(\Upsilon)\right)-\frac{1667\,173}{214\,560}\left(W(\Upsilon-5h)-W(\Upsilon)\right)$$
$$\left.+\frac{4609\,391}{4505\,760}\left(W(\Upsilon-6h)-W(\Upsilon)\right)-\frac{309\,293}{4505\,760}\left(W(\Upsilon-7h)-W(\Upsilon)\right)\right\}$$
$$+h\left\{\frac{597\,497}{1501\,920}\left(W^{(4)}(\Upsilon-h)-W^{(4)}(\Upsilon)\right)-\frac{1528\,979}{500\,640}\left(W^{(4)}(\Upsilon-2h)-W^{(4)}(\Upsilon)\right)\right.$$
$$\left.+\frac{1173\,833}{500\,640}\left(W^{(4)}(\Upsilon-3h)-W^{(4)}(\Upsilon)\right)\right\}=o\left(h^{8}\right).$$

Proof. Applying the MUCs, we will aim to find

$$W^{(3)}(0)-\beta\left(W(h)-W(0)\right)+\gamma\left(W(2h)-W(0)\right)+d\left(W(3h)-W(0)\right)$$
$$+p\left(W(4h)-W(0)\right)+q\left(W(5h)-W(0)\right)+w(W(6h)-W(0))+f\left(W(7h)-W(0)\right)$$
$$+h^{4}m\left(W^{(4)}(h)-W^{(4)}(0)\right)+h^{4}n(W^{(4)}(2h)-W^{(4)}(0))+h^{4}s\left(W^{(4)}(3h)-W^{(4)}(0)\right)=o\left(h^{8}\right).$$

By applying Taylor's formula, we obtain

$$W^{(3)}(0)=\beta\sum_{l=1}^{10}\frac{h^{l}}{l!}W^{(l)}(0)+\gamma\sum_{l=1}^{10}\frac{(2h)^{l}}{l!}W^{(l)}(0)+d\sum_{l=1}^{10}\frac{(3h)^{l}}{l!}W^{(l)}(0)$$
$$+p\sum_{l=1}^{10}\frac{(4h)^{l}}{l!}W^{(l)}(0)+q\sum_{l=1}^{10}\frac{(5h)^{l}}{l!}W^{(l)}(0)+w\sum_{l=1}^{10}\frac{(6h)^{l}}{l!}W^{(l)}(0)$$
$$+f\sum_{l=1}^{10}\frac{(7h)^{l}}{l!}W^{(l)}(0)+h^{4}m\sum_{l=1}^{5}\frac{h^{l}}{l!}W^{(4+l)}(0)$$
$$+h^{4}n\sum_{l=1}^{5}\frac{(2h)^{l}}{l!}W^{(4+l)}(0)+h^{4}s\sum_{l=1}^{5}\frac{(3h)^{l}}{l!}W^{(4+l)}(0)+o\left(h^{8}\right).$$

To obtain the SAEs, equate the coefficients of the smallest power of h in the above identity to 0.

$$\begin{cases}
3d+4p+5q+6w+\beta+2\gamma+7f=0,\\
\frac{9}{2!}d+\frac{4^2}{2!}p+\frac{5^2}{2!}q+\frac{6^2}{2!}w+\frac{1}{2!}\beta+\frac{4}{2!}\gamma+\frac{7^2}{2!}f=0,\\
\frac{3^3}{3!}d+\frac{4^3}{3!}p+\frac{5^3}{3!}q+\frac{6^3}{3!}w+\frac{1}{3!}\beta+\frac{8}{3!}\gamma+\frac{7^3}{3!}f=h^{-3},\\
\frac{3^4}{4!}d+\frac{4^4}{4!}p+\frac{5^4}{4!}q+\frac{6^4}{4!}w+\frac{1}{4!}\beta+\frac{2^4}{4!}\gamma+\frac{7^4}{4!}f=0,\\
\frac{3^5}{5!}d+\frac{4^5}{5!}p+\frac{5^5}{5!}q+\frac{6^5}{5!}w+\frac{1}{5!}\beta+\frac{2^5}{5!}\gamma+\frac{7^5}{5!}f+m+2n+3s=0,\\
\frac{3^6}{6!}d+\frac{4^6}{6!}p+\frac{5^6}{6!}q+\frac{6^6}{6!}w+\frac{1}{6!}\beta+\frac{2^6}{6!}\gamma+\frac{7^6}{6!}f+\frac{1}{2!}m+\frac{4}{2!}n+\frac{3^2}{2!}s=0,\\
\frac{3^7}{7!}d+\frac{4^7}{7!}p+\frac{5^7}{7!}q+\frac{6^7}{7!}w+\frac{1}{7!}\beta+\frac{2^7}{7!}\gamma+\frac{7^7}{7!}f+\frac{1}{3!}m+\frac{8}{3!}n+\frac{3^3}{3!}s=0,\\
\frac{3^8}{8!}d+\frac{4^86}{8!}p+\frac{5^8}{8!}q+\frac{6^8}{8!}w+\frac{1}{8!}\beta+\frac{2^8}{8!}\gamma+\frac{7^8}{8!}f+\frac{1}{4!}m+\frac{16}{4!}n+\frac{3^4}{4!}s=0,\\
\frac{3^9}{9!}d+\frac{4^9}{9!}p+\frac{5^9}{9!}q+\frac{6^9}{9!}w+\frac{1}{9!}\beta+\frac{2^9}{9!}\gamma+\frac{7^9}{9!}f+\frac{1}{5!}m+\frac{32}{5!}n+\frac{3^5}{5!}s=0,\\
\frac{3^{10}}{10!}d+\frac{4^{10}}{10!}p+\frac{5^{10}}{10!}q+\frac{6^{10}}{10!}w+\frac{1}{10!}\beta+\frac{2^{10}}{10!}\gamma+\frac{7^{10}}{10!}f+\frac{1}{6!}m+\frac{64}{6!}n+\frac{3^6}{6!}s=0.
\end{cases}$$

Upon resolving this SAEs, we find $\beta = \frac{126\,630\,131}{4505\,760h^3}$, $\gamma = -\frac{78\,574\,591}{1501\,920h^3}$, $d = \frac{45\,949\,355}{901\,152h^3}$, $p = -\frac{24\,699\,239}{901\,152h^3}$, $q = \frac{1667\,173}{214\,560h^3}$, $w = -\frac{4609\,391}{4505\,760h^3}$, $f = \frac{309\,293}{4505\,760h^3}$, $m = -\frac{597\,497}{1501\,920h^3}$, $n = \frac{1528\,979}{500\,640h^3}$, $s = -\frac{1173\,833}{500\,640h^3}$. So, relation (16) is proved. In a similar fashion, one can derive the relationship (17). Theorem 4.1 is established.

Now, we consider the application of Theorems 1.1–1.4 and Theorem 4.1 for the numerical solution of the BVP (3). Using the equation (3) and formulas (5), (7), (8), (9), (16), (17), and disregarding minor terms, we can present the eighth order of approximation DS

$$h^{-4}(u_{k+2} - 4u_{k+1} + 6u_k - 4u_{k-1} + u_{k-2}) + b_k u_k \tag{18}$$

$$+c_k u_{k+1} + d_k u_{k-1} + h_k u_{k+2} + g_k u_{k-2} = \varphi_k, 2 \le k \le N-2,$$

$$a_{1,0}u_0 + a_{1,1}u_1 + a_{1,2}u_2 + a_{1,3}u_3 + a_{1,4}u_4 + a_{1,5}u_5 + a_{1,6}u_6 = -a_{1,7} + \eta,$$

$$u_0 = \varphi, u_N = \omega,$$

$$a_{1,N}u_N + a_{1,N-1}u_{N-1} + a_{1,N-2}u_{N-2} + a_{1,N-3}u_{N-3}$$

$$+a_{1,N-4}u_{N-4} + a_{1,N-5}u_{N-5} + a_{1,N-6}u_{N-6} = -a_{1,N-7} + \rho,$$

where

$$b_k = \left(\frac{76}{105} - \frac{97}{1680}h^4 a(y_k)\right) a(y_k) - \frac{97}{1680}h^4 a^{(4)}(y_k) - \frac{697}{1680}h^4 a''(y_k)6A_k^{20},$$

$$c_k = \frac{9}{70}a(y_{k+1}) - \frac{97}{1680}h^4\left[4a'(y_k)B_k^{30} + 6a''(y_k)B_k^{20} + 4a'''(y_k)B_k^{10}\right],$$

$$d_k = \frac{9}{70}a(y_{k-1}) - \frac{97}{1680}h^4\left[4a'(y_k)C_k^{30} + 6a''(y_k)C_k^{20} + 4a'''(y_k)C_k^{10}\right],$$

$$h_k = \frac{1}{105}a(y_{k+2}) - \frac{97}{1680}h^4\left[4a'(y_k)D_k^{30} + 6a''(y_k)D_k^{20} + 4a'''(y_k)D_k^{10}\right],$$

$$g_k = \frac{1}{105}a(y_{k-2}) - \frac{97}{1680}h^4\left[4a'(y_k)E_k^{30} + 6a''(y_k)E_k^{20} + 4a'''(y_k)E_k^{10}\right],$$

$$\varphi_k = \left(\frac{76}{105} + \frac{97}{1680}h^4 a(y_k)\right) F(y_k) + \frac{9}{70}\left(F(y_{k+1}) + F(y_{k-1})\right)$$

$$+\frac{1}{105}\left(F(y_{k+2}) + F(y_{k-2})\right) - \frac{97}{1680}h^4 F^{(4)}(y_k)$$

for the numerical solution of the BVP (3).

5 The nonlocal BVP (4)

Now, we consider the application of Theorems 1.1–1.4 and Theorems 2.1, 3.1, and 4.1 for the numerical solution of the nonlocal BVP (4). Using the equation (4) and formulas (5), (7), (8), (9), (10), (11), (13), (14), (16), (17), and disregarding minor terms, we can present the eighth order of approximation DS

$$
\left\{
\begin{array}{l}
h^{-4}(u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2})+\big(\frac{76}{105} \\
-\frac{97}{1680}h^4a(y_k)\big)\,a(y_k)u_k+\frac{9}{70}a(y_{k+1})u_{k+1}+a(y_{k-1})u_{k-1} \\
+\frac{1}{105}\left(a(y_{k+2})u_{k+2}+a(y_{k-2})u_{k-2}\right)=\big(\frac{76}{105}+\frac{97}{1680}h^4a(y_k)\big)\,\digamma(y_k) \\
+\frac{9}{70}\left(\digamma(y_{k+1})+\digamma(y_{k-1})\right)+\frac{1}{105}\left(\digamma(y_{k+2})+\digamma(y_{k-2})\right) \\
-\frac{97}{1680}h^4\digamma^{(4)}(y_k), \\
u_0=u_N+\varphi, \\
h^{-1}\big\{\big(-\frac{5543}{2590}+\frac{9}{1036}h^4a\left(0\right)\big)\,u_0+\big(\frac{2223}{518}+\frac{69}{518}h^4a(h)\big)\,u_1 \\
-\frac{45}{74}u_4-\big(\frac{3735}{1036}+\frac{21}{148}h^4a(2h)\big)\,u_2 \\
+\frac{1535}{777}u_3+\frac{243}{2590}u_5-\frac{23}{3108}u_6\big\} \\
=h^{-1}\big\{\big(\frac{5543}{2590}-\frac{9}{1036}h^4a\left(0\right)\big)\,u_N-\big(\frac{2223}{518}+\frac{69}{518}h^4a(h)\big)\,u_{N-1} \\
+\big(\frac{3735}{1036}+\frac{21}{148}h^4a(2h)\big)\,u_{N-2}-\frac{1535}{777}u_{N-3} \\
+\frac{45}{74}u_{N-4}-\frac{243}{2590}u_{N-5}+\frac{23}{3108}u_{N-6}\big\}+\eta, \\
\\
h^{-2}\big\{\big(\frac{6\,937\,573}{3\,439\,828}-\frac{23\,426\,639}{206\,389\,680}h^4a\left(0\right)\big)\,u_0-\big(\frac{26\,121\,217}{5\,159\,742} \\
-+\frac{12\,741\,989}{20\,638\,968}h^4a(h)\big)\,u_1+\big(\frac{21\,060\,241}{5\,159\,742}-\frac{5\,216\,939}{29\,484\,240}h^4a(2h)\big)\,u_2 \\
-\big(\frac{892\,879}{859\,957}-\frac{1\,324\,691}{103\,194\,840}h^4a(3h)\big)\,u_3-\frac{26\,209}{10\,319\,484}u_4 \\
+\frac{24\,995}{5\,159\,742}u_5\big\}=h^{-2}\big\{\big(\frac{6\,937\,573}{3\,439\,828}-\frac{23\,426\,639}{206\,389\,680}h^4a\left(\Upsilon\right)\big)\,u_N \\
+\big(-\frac{26\,121\,217}{5\,159\,742}-\frac{12\,741\,989}{20\,638\,968}h^4a(\Upsilon-h)\big)\,u_{N-1} \\
+\big(\frac{21\,060\,241}{5\,159\,742}-\frac{5\,216\,939}{29\,484\,240}h^4a(\Upsilon-2h)\big)\,u_{N-2} \\
+\big(-\frac{892\,879}{859\,957}+\frac{1\,324\,691}{103\,194\,840}h^4a(\Upsilon-3h)\big)\,u_{N-3} \\
-\frac{26\,209}{10\,319\,484}u_{N-4}+\frac{24\,995}{5\,159\,742}u_{N-5}\big\}+\omega, \\
\\
h^{-3}\big\{\big(-\frac{17\,265\,457}{300\,384}+\frac{467\,941}{1\,501\,920}h^4a\left(0\right)\big)\,u_0 \\
+\big(\frac{126\,630\,131}{4\,505\,760}+\frac{597\,497}{1\,501\,920}h^4a(h)\big)\,u_1-\big(\frac{4\,609\,391}{4\,505\,760} \\
+\frac{1\,528\,979}{500\,640}h^4a(2h)\big)\,u_2+\big(\frac{45\,949\,355}{901\,152}+\frac{1\,173\,833}{500\,640}h^4a(3h)\big)\,u_3 \\
-\frac{24\,699\,239}{901\,152}u_4+\frac{1\,667\,173}{214\,560}u_5-\frac{4\,609\,391}{4\,505\,760}u_6+\frac{309\,293}{4\,505\,760}u_7\big\} \\
=\varrho+h^{-3}\big\{\big(\frac{27\,867\,473}{4\,505\,760}-\frac{467\,941}{1\,501\,920}h^4a\left(0\right)\big)\,u_N \\
-\big(\frac{126\,630\,131}{4\,505\,760}+\frac{597\,497}{1\,501\,920}h^4a(h)\big)\,u_{N-1}+\big(\frac{78\,574\,591}{1\,501\,920}+ \\
\frac{1\,528\,979}{500\,640}h^4a(2h)\big)\,u_{N-2}+\big(-\frac{45\,949\,355}{901\,152}-\frac{1\,173\,833}{500\,640}h^4a(3h)\big)\,u_{N-3} \\
+\frac{24\,699\,239}{901\,152}u_{N-4}-\frac{1\,667\,173}{214\,560}u_{N-5}+\frac{4\,609\,391}{4\,505\,760}u_{N-6}-\frac{309\,293}{4\,505\,760}u_{N-7}\big\}
\end{array}
\right. \tag{19}
$$

for the numerical solution of the nonlocal BVP (4).

Now, for numerical analysis we consider the BVPs (1)–(4), for the simple case when $\Upsilon = 1$, $a\left(y\right)=1, \varphi=\eta=\omega=\chi=0$, and

$$\digamma(y)=\frac{y^8(1-y)^8}{8!}+\frac{1}{120}y^4\ (y-1)^4\left(130y^4-260y^3+182y^2-52y+5\right).$$

Then,

$$\mathbb{U}(y)=\frac{y^8(1-y)^8}{8!}$$

is the exact solution of these BVPs. For solving these problems, we use the eighth order of approximation DSs (12), (15), (18), and (19), respectively, with different values of h. The error is computed by

$$E_N=\max_{0\le k\le N}|u(y_k)-u_k|.$$

The error analysis shown in Table indicates that all DSs have correct convergence rates.

Table

Numerical Results

$h = \frac{1}{N}$	$N = 40$	$N = 80$	$N = 160$
DS (12)	1.2225e-13	5.5447e-15	3.0095e-16
DS (15)	2.0641e-12	3.7546e-14	6.4559e-16
DS (18)	5.2435e-13	1.2208e-14	3.1736e-16
DS (19)	3.6094e-09	1.7743e-10	6.4050e-12

Conclusion

1. In this work, we examine LNBVPs for FODEs with variable coefficients. We develop and analyze finite DSs of eighth-order accuracy using a novel method based on five-point grids for addressing these problems. Our findings are validated through extensive numerical experiments.

2. Highly accurate four-step finite DSs for solving LNBVPs of the general FODE

$$u^{(4)}(s) + d(s)u^{(3)}(s) + c(s)u^{(2)}(s) + b(s)u^{(1)}(s) + a(s)u(s) = \Psi(s), 0 < s < \Upsilon$$

will be presented and investigated.

3. Highly accurate four-step finite DSs for solving LNBVPs for elliptic FODEs

$$u^{(4)}(s) + Au(s) = \Psi(s), 0 < s < \Upsilon$$

will be constructed and studied. Here A is a self-adjoint positive definite operator in a Hilbert space H. The stability of these DSs is ensured by the operator method discussed in reference [1].

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Author Information[*]

Allaberen Ashyralyev — Doctor of physical and mathematical sciences, Professor, Department of Mathematics, Bahcesehir University, 34353, Istanbul, Turkey; e-mail: aallaberen@gmail.com; https://orcid.org/0000-0002-4153-6624

Ibrahim Mohammed Ibrahim (*corresponding author*) — Doctor of mathematical sciences, Department of Mathematics, Akre University for Applied Sciences, Akre, Iraq; e-mail: ibrahimkalash81@gmail.com; https://orcid.org/0009-0006-3901-1526

[*]The author's name is presented in the order: First, Middle and Last Names.