https://doi.org/10.31489/2025M3/34-45 *Research article*

On solving the second boundary value problem for the Viscous Transonic Equation

Yu.P. Apakov[1,2], Kh.K. Ibrokhimov[2,*]

[1] *V.I. Romanovskiy Institute of Mathematics, Tashkent, Uzbekistan;*
[2] *Namangan State Technical University, Namangan, Uzbekistan*
(E-mail: yusupjonapakov@gmail.com, xusniddin571@mail.ru)

In a rectangular domain, the second boundary value problem for the Viscous Transonic Equation is considered. The uniqueness of the solution to the problem is proved using the energy integral method. The existence of a solution is proved by the method of separation of variables, i.e. it is sought in the form of a product of two functions $X(x)$ and $Y(y)$. For definition $Y(y)$, an ordinary differential equation of the second order with two boundary conditions on the boundaries of segment $[0, q]$ is obtained. For this problem, the eigenvalues and the corresponding eigenfunctions are found at $n \in N$. For definition $X(x)$, an ordinary differential equation of the third order with three boundary conditions on the boundaries of segment $[0, q]$ is obtained. The solution to this problem is found in the form of an infinite series, uniform convergence, and the possibility of term-by-term differentiation under certain conditions on the given functions is proven. The convergence of the second-order derivative of the solution with respect to variable y is proved using the Cauchy-Bunyakovsky and Bessel inequalities. When substantiating the uniform convergence of the solution, the absence of a "small denominator" is proved.

Keywords: equations with multiple characteristics, boundary value problem, uniqueness, existence, method of separated variables, eigenvalue, eigenfunction, functional series, absolute and uniform convergence.

2020 Mathematics Subject Classification: 35G15.

Introduction

Third-order partial differential equations are considered when solving problems in the theory of nonlinear acoustics and in the hydrodynamic theory of space plasma and fluid filtration in porous media [1, 2]. Quite often, sharp changes in flow parameters occur in narrow regions adjacent to shock waves. The gradients of flow parameters in them can be so significant that, along with the nonlinear nature of the movement, it becomes necessary to take into account the influence of viscosity and thermal conductivity. Such currents are called short waves. The theory of transonic flows refers to the theory of short waves. It should be noted that recently in the literature this equation is increasingly called the viscous transonic equation, or simply the VT equation.

In [3], taking into account the properties of viscosity and thermal conductivity of the gas, a thirdorder equation with multiple characteristics was obtained from the Navier-Stokes system, containing the second derivative with respect to time

$$u_{xxx} + u_{yy} - \frac{\nu}{y} u_y = u_x u_{xx}, \quad \nu = const.$$

This equation, at $\nu = 1$, describes an axisymmetric flow, while at $\nu = 0$, it describes a plane-parallel flow [4].

[*]Corresponding author. *E-mail: yusupjonapakov@gmail.com*
Received: 27 December 2024; *Accepted:* 6 June 2025.

L. Cattabriga in [5] for equation $D_x^{2n+1}u - D_y^2 u = 0$ constructed a fundamental solution in the form of a double improper integral and studied the properties of the potential and solved boundary value problems.

In [6, 7], fundamental solutions for a third-order equation with multiple characteristics were constructed, containing second derivatives with respect to time, expressed through degenerate hypergeometric functions, their properties were studied, and estimates were found for $|t| \to \infty$.

In [8], the Dirichlet problem for third-order hyperbolic equations was investigated, while in [9], an analogue of the Goursat problem for a third-order equation with singular coefficients was studied.

In works [10, 11], nonlocal problems for third-order differential equations were examined, while in works [12–14], the stability of boundary value problems for third-order partial differential equations is studied.

In works [15–17], boundary value problems for third-order partial differential equations were investigated.

1 *Formulation of the problem*

In the domain $D = \{(x, y) : \ 0 < x < p, \ 0 < y < q\}$, consider the equation:

$$L[u] \equiv \frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{1}$$

where $p > 0, \ q > 0$ are given sufficiently smooth functions.

Problem A. Find a function $u(x, y)$ from class $C_{x,y}^{3,2}(D) \cap C_{x,y}^{2,1}(\overline{D})$, that satisfies equation (1) in the domain D and the following boundary conditions:

$$u_y(x, 0) = u_y(x, q) = 0, \tag{2}$$

$$\begin{cases} au(0, y) + bu_{xx}(0, y) = \psi_1(y), \\ cu(p, y) + du_{xx}(p, y) = \psi_2(y), \\ u_x(0, y) = \psi_3(y), \end{cases} \tag{3}$$

where a, b, c and d are given constants, and also $a^2+b^2 \neq 0$, $a^2+c^2 \neq 0$, $c^2+d^2 \neq 0$, and $\psi_i(y)$, $i = \overline{1,3}$, are given sufficiently smooth functions, and

$$\psi'_i(0) = \psi'_i(q) = 0, \quad i = \overline{1,3}. \tag{4}$$

Note that a similar problem for the adjoint equation was studied in [18–19]. The boundary value problem close to the topic of this work was studied in [20–22]. The case $a = 1, \ b = 0, \ c = 1, \ d = 0$ was considered in work [23].

2 *The uniqueness of solution*

Theorem 1. If Problem A has a solution, then if conditions $ab \leq 0, \ cd \geq 0$ are met, it is unique.

Proof. Assume the opposite, let Problem A have two solutions $u_1(x, y)$ and $u_2(x, y)$. Then the function $u(x, y) = u_1(x, y) - u_2(x, y)$ satisfies equation (1) with homogeneous boundary conditions. Let us prove that $u(x, y) \equiv 0$ is in $\overline{D}$.

In the domain D the identity

$$uL[u] = uu_{xxx} + uu_{yy} = 0,$$

or

$$u\,L[u] \equiv \frac{\partial}{\partial x}\left(u\,u_{xx} - \frac{1}{2}\,u_x^2\right) + \frac{\partial}{\partial y}\left(u\,u_y\right) - u_y^2 = 0 \tag{5}$$

holds. Integrating identity (5) over the domain D and taking into account homogeneous boundary conditions, we obtain

$$\int_0^p\int_0^q \tfrac{\partial}{\partial x}\left[u\left(x,y\right)\,u_{xx}\left(x,y\right) - \tfrac{1}{2}\,u_x^2\left(x,y\right)\right]dxdy +$$
$$+\int_0^p\int_0^q \tfrac{\partial}{\partial y}\left[u\left(x,y\right)\,u_y\left(x,y\right)\right]dxdy - \int_0^p\int_0^q u_y^2\left(x,y\right)dxdy = 0.$$

Taking into account homogeneous boundary conditions and requiring $a \neq 0,\ c \neq 0$, we obtain

$$\frac{d}{c}\int_0^q u_{xx}^2\left(p,y\right)dy - \frac{b}{a}\int_0^q u_{xx}^2\left(0,y\right)dy + \frac{1}{2}\int_0^q u_x^2\left(p,y\right)dy + \int_0^p\int_0^q u_y^2\left(x,y\right)dxdy = 0.$$

Taking into account the condition $ab \leq 0,\ cd \geq 0$, we obtain $u_y\left(x,y\right) = 0$. From this, it follows that $u\left(x,y\right) = f\left(x\right)$. Substituting $u\left(x,y\right)$ into equation (1), we get $f'''\left(x\right) = 0$. The solution to this equation is $f\left(x\right) = C_1x^2 + C_2x + C_3$. To satisfy the boundary conditions in equation (3), the constants $C_1,\ C_2,\ C_3$ are determined,

$$\begin{cases} 2bC_1 + aC_3 = \psi_{10}, \\ C_1\left(cp^2 + 2d\right) + cC_3 = \psi_{20}, \\ C_2 = \psi_{30}. \end{cases}$$

The value of the main determinant of this system is as follows:

$$\Delta = \begin{vmatrix} 2b & 0 & a \\ cp^2 + 2d & cp & c \\ 0 & 1 & 0 \end{vmatrix} = acp^2 + 2ad - 2bc.$$

Assume that $\Delta = 0$. In this case, $acp^2 + 2ad - 2bc = 0$, and from this, we derive $p^2 = 2\left(\frac{b}{a} - \frac{d}{c}\right)$ which represents the uniqueness condition. According to the theorem and the condition $a^2 + c^2 \neq 0$, if we consider $p^2 < 0$, it leads to a contradiction because $p > 0$. Thus, $\Delta \neq 0$. Consequently, this implies $C_1 = C_2 = C_3 = 0$ then $f\left(x\right) = 0$. Hence, the function $u\left(x,y\right) \equiv 0$ for all $\left(x,y\right) \in \overline{D}$. Finally, from the last equation, it follows that $u_1\left(x,y\right) = u_2\left(x,y\right)$.

In cases $b \neq 0,\ d \neq 0;\ a \neq 0,\ d \neq 0;\ c \neq 0,\ b \neq 0$ similarly we obtain the equality of $u\left(x,y\right) \equiv 0$ in $\overline{D}$.

The proof of Theorem 1 is complete.

3 Existence of a solution

Theorem 2. If the functions are $\psi_i\left(y\right) \in C^2\left[0 < y < q\right],\ i = \overline{1,3}$ and conditions (4) are satisfied, then a solution to Problem A exists.

Proof. To prove the existence of a solution to Problem A, we search in the form

$$u\left(x,y\right) = X\left(x\right)Y\left(y\right). \tag{6}$$

Substituting (6) into equation (1) and separating the variables, and taking into account the boundary condition (2), we obtain a Sturm-Liouville type problem with respect to the function $Y(y)$ [24]:

$$\begin{cases} Y'' + \lambda Y = 0, \\ Y'(0) = Y'(q) = 0, \end{cases} \tag{7}$$

where λ is the separation parameter.

We know that the solution to problem (7) is expressed as follows:

$$Y_n(y) = B_n \cos \frac{n\pi}{q} y.$$

It is known that a nontrivial solution to the problem (7) exists only when

$$\lambda_0 = 0, \quad \lambda_n = \left(\frac{n\pi}{q}\right)^2, \quad n \in N.$$

λ_n, with $n \in N \cup \{0\}$, are the eigenvalues, and their corresponding eigenfunctions are as follows:

$$Y_n(y) = \begin{cases} \frac{1}{\sqrt{q}}, & n = 0, \\ \sqrt{\frac{2}{q}} \cos \frac{n\pi}{q} y, & n \in N. \end{cases} \tag{8}$$

Note that the system of eigenfunctions (8) of problem (7) is complete and orthonormal in the $L_2(0, q)$ space [25].

1) When $\lambda_0 = 0$, we get the following problem for the function $X(x)$:

$$\begin{cases} X'''_0 = 0, \\ aX_0(0) + bX''_0(0) = \psi_{10}, \\ cX_0(p) + dX''_0(p) = \psi_{20}, \\ X'_0(0) = \psi_{30}. \end{cases} \tag{9}$$

The solution to the boundary value problem equation (9) is as follows:

$$X_0(x) = C_1 x^2 + C_2 x + C_3,$$

then, taking into account (6) and (9), from equality (7) we search the solution to Problem A in the form:

$$u_0(x) = \frac{1}{\sqrt{q}} \left(C_1 x^2 + C_2 x + C_3\right). \tag{10}$$

Taking into account condition (3), we obtain a system of algebraic equations:

$$\begin{cases} 2bC_1 + aC_3 = \psi_{10}, \\ C_1 \left(cp^2 + 2d\right) + cC_3 = \psi_{20}, \\ C_2 = \psi_{30}, \end{cases} \tag{11}$$

where $\psi_{i0},\ i = \overline{1,3}$, are the Fourier coefficients of the function $\psi_i(y),\ i = \overline{1,3}$, i.e.,

$$\psi_{i0} = \sqrt{\frac{1}{q}} \int_0^q \psi_i(y)\, dy, \quad i = \overline{1,3}.$$

Now we find the solution of system (11). To do this, we first calculate the main determinant of system (11), which has the following form

$$\Delta = \begin{vmatrix} 2b & 0 & a \\ cp^2+2d & cp & c \\ 0 & 1 & 0 \end{vmatrix} = acp^2 + 2ad - 2bc.$$

Since $p > 0$, then $\Delta \neq 0$ and system (11) has a solution:

$$C_1 = \frac{-c\psi_{10} + a\psi_{20} - acp\psi_{30}}{acp^2 + 2ad - 2bc},$$

$$C_2 = \psi_{30},$$

$$C_3 = \frac{\psi_{10}\left(cp^2 + 2d\right) - 2b\psi_{20} + 2bcp\psi_{30}}{acp^2 + 2ad - 2bc}.$$

Substituting $C_i,\ i=\overline{1,3}$ into (10), we obtain

$$u_0(x) = \sqrt{\frac{1}{q}}\frac{1}{acp^2+2ad-2bc}\left[\psi_{10}\left(-cx^2+cp^2+2d\right)+\right.$$
$$\left.+\,\psi_{20}\left(ax^2-2b\right)+\psi_{30}\left(acp^2x+2adx-2bcx+2bcp\right)\right]. \tag{12}$$

In what follows, the maximum value among all positive known numbers found in estimates will be denoted by M. Now we find estimates (12) and $u_0(x)$ in the domain D. From (12) we have

$$|u_0(x)| \leq M\left[|\psi_{10}| + |\psi_{20}| + |\psi_{30}|\right] \leq M,$$

$$\left|{u_0}'''(x)\right| \leq M.$$

2) Now, when $\lambda_n = \left(\frac{n\pi}{q}\right)^2$, $n \in N$, we get the following problem for the function $X(x)$:

$$\begin{cases} X'''_n - \lambda_n X_n = 0, \\ aX_n(0) + bX''_n(0) = \psi_{1n}, \\ cX_n(p) + dX''_n(p) = \psi_{2n}, \\ X'_n(0) = \psi_{3n}; \end{cases} \tag{13}$$

here

$$\psi_{i\,n} = \sqrt{\frac{2}{q}}\int_0^q \psi_i(y)\cos\left(\frac{n\pi}{q}y\right)dy, \quad i=\overline{1,3}, \quad n \in N.$$

The general solution to the equation in problem (13) has the form:

$$X_n(x) = C_{1n}e^{k_n x} + e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\frac{\sqrt{3}}{2}k_n x + C_{3n}\sin\frac{\sqrt{3}}{2}k_n x\right), \tag{14}$$

where

$$k_n = \sqrt[3]{\lambda_n} = \left(\frac{n\pi}{q}\right)^{\frac{2}{3}}, \quad n \in N.$$

Taking into account the boundary conditions of problem (13) for the solution in the form of (14), we obtain the following:

$$\begin{cases} C_{1n}\left(a+bk_n^2\right)+C_{2n}\left(a-\frac{bk_n^2}{2}\right)-C_{3n}\frac{\sqrt{3}bk_n^2}{2}=\psi_{1n}, \\ C_{1n}e^{k_np}\left(c+dk_n^2\right)+C_{2n}e^{-\frac{1}{2}k_np}\left(c\cos\left(\frac{\sqrt{3}}{2}k_np\right)+dk_n^2\cos\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right)\right)+ \\ +C_{3n}e^{-\frac{1}{2}k_np}\left(c\sin\left(\frac{\sqrt{3}}{2}k_np\right)+dk_n^2\sin\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right)\right)=\psi_{2n}, \\ k_nC_{1n}-\frac{1}{2}k_nC_{2n}+\frac{\sqrt{3}}{2}k_nC_{3n}=\psi_{3n}. \end{cases} \tag{15}$$

So, to determine the coefficients $C_{in},\ i=\overline{1,3}$, we received a system of algebraic equations (15).

Let us introduce the notation:

$$\alpha_n=\cos\left(\frac{\sqrt{3}}{2}k_np\right),\ \beta_n=\sin\left(\frac{\sqrt{3}}{2}k_np\right),\ \gamma_n=\cos\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right),\ \delta_n=\sin\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right).$$

Then (15) has the following form:

$$\begin{cases} C_{1n}\left(a+bk_n^2\right)+C_{2n}\left(a-\frac{bk_n^2}{2}\right)-C_{3n}\frac{\sqrt{3}bk_n^2}{2}=\psi_{1n}, \\ C_{1n}e^{k_np}\left(c+dk_n^2\right)+C_{2n}e^{-\frac{1}{2}k_np}\left(c\alpha_n+dk_n^2\gamma_n\right)+C_{3n}e^{-\frac{1}{2}k_np}\left(c\beta_n+dk_n^2\delta_n\right)=\psi_{2n}, \\ k_nC_{1n}-\frac{1}{2}k_nC_{2n}+C_{3n}\frac{\sqrt{3}}{2}k_n=\psi_{3n}. \end{cases} \tag{16}$$

The main determinant of system (16) has

$$\Delta=\begin{vmatrix} a+bk_n^2 & a-\frac{1}{2}bk_n^2 & -\frac{\sqrt{3}}{2}bk_n^2 \\ e^{k_np}\left(c+dk_n^2\right) & e^{-\frac{k_n}{2}p}\left(c\alpha_n+dk_n^2\gamma_n\right) & e^{-\frac{k_n}{2}p}\left(c\beta_n+dk_n^2\delta_n\right) \\ k_n & -\frac{1}{2}k_n & \frac{\sqrt{3}}{2}k_n \end{vmatrix}=\frac{\sqrt{3}}{2}k_n^5e^{k_np}\overline{\Delta},$$

where

$$\overline{\Delta}=\left(\frac{c}{k_n^2}+d\right)\left(b-\frac{a}{k_n^2}\right)+e^{-\frac{3k_n}{2}p}\left\{\left(\frac{ac}{k_n^4}-\frac{2ad}{k_n^2}+\frac{2bc}{k_n^2}-bd\right)\cos\frac{\sqrt{3}}{2}k_np+\sqrt{3}\left(\frac{ac}{k_n^4}+bd\right)\sin\frac{\sqrt{3}}{2}k_np\right\}.$$

We show that $\Delta\neq 0$. To do this, we prove the following lemma:

Lemma 1. The boundary value problem

$$\begin{cases} X'''_n-\lambda_nX_n=0, \\ aX_n(0)+bX''_n(0)=0, \\ cX_n(p)+dX''_n(p)=0, \\ X'_n(0)=0, \end{cases} \tag{17}$$

has only a trivial solution.

Proof. Let's assume the opposite, let $X_n(x)\neq 0$. Consider the identity

$$X_n\left(X'''_n-\lambda_nX_n\right)=0,$$

or

$$\left(X_nX''_n-\frac{1}{2}\left(X'_n\right)^2\right)'-\lambda_nX_n^2=0,$$

integrating over interval $(0 < x < p)$, and taking into account the boundary conditions, we obtain

$$\frac{d}{c}{X''_n}^2(p) - \frac{b}{a}{X''_n}^2(0) + \frac{1}{2}{X_n'}^2(p) + \lambda_n \int\limits_0^p X_n^2 dx = 0.$$

Since $ab \leq 0,\ cd \geq 0,\ \lambda_n > 0$, then $X_n \equiv 0$.

Lemma 1 has been proved.

If there is a number n^* such that $\Delta(n^*) = 0$, then there are constants $C_1^*,\ C_2^*,\ C_3^*$ that are not all equal to zero at the same time, satisfying the system

$$\begin{cases} C_{1n^*}^*\left(a + bk_{n^*}^2\right) + C_{2n^*}^*\left(a - \dfrac{bk_{n^*}^2}{2}\right) - C_{3n^*}^*\dfrac{\sqrt{3}bk_{n^*}^2}{2} = 0, \\ C_{1n^*}^{*\ k_{n^*}p}\left(c + dk_{n^*}^2\right) + C_{2n^*}^* e^{-\frac{1}{2}k_{n^*}p}\left(c\cos\dfrac{\sqrt{3}}{2}k_{n^*}p + dk_{n^*}^2\cos\left(\dfrac{\sqrt{3}}{2}k_{n^*}p - \dfrac{2\pi}{3}\right)\right) + \\ + C_{3n^*}^* e^{-\frac{1}{2}k_{n^*}p}\left(c\sin\dfrac{\sqrt{3}}{2}k_{n^*}p + dk_{n^*}^2\sin\left(\dfrac{\sqrt{3}}{2}k_{n^*}p - \dfrac{2\pi}{3}\right)\right) = 0, \\ k_{n^*}C_{1n^*}^* - \dfrac{1}{2}k_{n^*}C_{2n^*}^* + \dfrac{\sqrt{3}}{2}k_{n^*}C_{3n^*}^* = 0. \end{cases}$$

From this we have that the function

$$X_{n^*}(x) = C_{1n^*}^* e^{k_{n^*}x} + e^{-\frac{1}{2}k_{n^*}x}\left(C_{2n^*}^*\cos\frac{\sqrt{3}}{2}k_{n^*}x + C_{3n^*}^*\sin\frac{\sqrt{3}}{2}k_{n^*}x\right)$$

is a solution to boundary value problem (17), but according to the proven lemma it should be

$$C_{1n^*}^* e^{k_{n^*}x} + e^{-\frac{1}{2}k_{n^*}x}\left(C_{2n^*}^*\cos\frac{\sqrt{3}}{2}k_{n^*}x + C_{3n^*}^*\sin\frac{\sqrt{3}}{2}k_{n^*}x\right) \equiv 0,$$

but this is impossible due to the linear independence of the functions

$$e^{k_{n^*}x}, \quad e^{-\frac{1}{2}k_{n^*}x}\cos\frac{\sqrt{3}}{2}k_{n^*}x, \quad e^{-\frac{1}{2}k_{n^*}x}\sin\frac{\sqrt{3}}{2}k_{n^*}x.$$

Hence the function in the form:

$$u^*(x,y) = u_0(x) + \sqrt{\frac{2}{q}}\sum_{n^*=1}^{+\infty} X_{n^*}(x)\cos\frac{n^*\pi}{q}y$$

are nontrivial solutions to Problem A, and this contradicts the uniqueness theorem. So $\Delta(n) \neq 0$, $n \in N$.

Note that in case $a = 1,\ b = 0,\ c = 1,\ d = 0$, we obtain the result from [23], as a special case.

$$C_{1n} = \tfrac{2e^{-k_n p}}{\sqrt{3}\Delta}\left[\tfrac{\psi_{1n}}{k_n^2}e^{-\frac{k_n}{2}p}\left(\tfrac{c}{k_n^2}\cos\left(\tfrac{\sqrt{3}}{2}k_n p - \tfrac{\pi}{6}\right) + d\cos\left(\tfrac{\sqrt{3}}{2}k_n p + \tfrac{\pi}{6}\right)\right) - \tfrac{\sqrt{3}}{2}\tfrac{\psi_{2n}}{k_n^2}\left(\tfrac{a}{k_n^2} - b\right) + \right.$$
$$\left. + \tfrac{\psi_{3n}e^{-\frac{k_n}{2}p}}{k_n}\left(\left(\tfrac{a}{k_n^2} - \tfrac{1}{2}b\right)\left(\tfrac{c}{k_n^2} - \tfrac{1}{2}d\right)\sin\tfrac{\sqrt{3}}{2}k_n p + \tfrac{\sqrt{3}}{2}\left(\tfrac{ad}{k_n^2} - bd + \tfrac{bc}{k_n^2}\right)\cos\tfrac{\sqrt{3}}{2}k_n p\right)\right],$$

$$C_{2n} = \tfrac{2}{\sqrt{3}\Delta}\left[\tfrac{\psi_{1n}}{k_n^2}\left(e^{-\frac{3k_n}{2}p}\left(\tfrac{c}{k_n^2}\sin\tfrac{\sqrt{3}}{2}k_n p + d\sin\left(\tfrac{\sqrt{3}}{2}k_n p - \tfrac{2\pi}{3}\right)\right) - \tfrac{\sqrt{3}}{2}\left(\tfrac{c}{k_n^2} + d\right)\right) + \tfrac{\sqrt{3}}{2}\tfrac{\psi_{2n}}{k_n^2}e^{-k_n p}\left(\tfrac{a}{k_n^2} + 2b\right) - \right.$$
$$\left. - \tfrac{\psi_{3n}}{k_n}\left(e^{-\frac{3k_n}{2}p}\left(\tfrac{a}{k_n^2} + b\right)\left(\tfrac{c}{k_n^2}\sin\tfrac{\sqrt{3}}{2}k_n p + d\sin\left(\tfrac{\sqrt{3}}{2}k_n p - \tfrac{2\pi}{3}\right)\right) + \tfrac{\sqrt{3}}{2}b\left(\tfrac{c}{k_n^2} + d\right)\right)\right],$$

$$C_{3n}=\frac{2}{\sqrt{3}\Delta}\left[-\frac{\psi_{1n}}{k_n^2}\left(\frac{c}{2k_n^2}+\frac{d}{2}-e^{-\frac{3k_n}{2}p}\left(\frac{c}{k_n^2}\cos\frac{\sqrt{3}}{2}k_np+d\cos\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right)\right)\right)+\frac{3a\psi_{2n}}{2k_n^4}e^{-k_np}+\right.$$
$$\left.+\frac{\psi_{3n}}{k_n}\left(e^{-\frac{3k_n}{2}p}\left(\frac{a}{k_n^2}+\frac{b}{2}\right)\left(\frac{c}{k_n^2}\cos\frac{\sqrt{3}}{2}k_np+d\cos\left(\frac{\sqrt{3}}{2}k_np-\frac{2\pi}{3}\right)\right)-\left(\frac{a}{k_n^2}-\frac{b}{2}\right)\left(\frac{c}{k_n^2}+d\right)\right)\right].$$

Then the solution to Problem A is written in the following form:

$$u(x,y)=u_0(x)+\sqrt{\frac{2}{q}}\sum_{n=1}^{+\infty}\left[C_{1n}e^{k_n x}+e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\frac{\sqrt{3}}{2}k_nx+C_{3n}\sin\frac{\sqrt{3}}{2}k_nx\right)\right]\cos\frac{n\pi}{q}y. \quad (18)$$

Now we prove the absolute and uniform convergence of series (18) in the domain $\overline{D}$. From (18) we have

$$|u(x,y)|\le|u_0(x)|+\sqrt{\frac{2}{q}}\sum_{n=1}^{+\infty}\left|\left[C_{1n}e^{k_n x}+e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\frac{\sqrt{3}}{2}k_nx+C_{3n}\sin\frac{\sqrt{3}}{2}k_nx\right)\right]\cos\frac{n\pi}{q}y\right|\le$$
$$\le M\sum_{n=1}^{+\infty}\left|\left[|C_{1n}|\,e^{k_n x}+|C_{2n}|+|C_{3n}|\right]\right|. \quad (19)$$

Estimating $C_{in},\ i=\overline{1,3}$, we get

$$|C_{1n}|\le Me^{-k_np}\left[\frac{|\psi_{1n}|}{k_n^2}+\frac{|\psi_{2n}|}{k_n^2}+\frac{|\psi_{3n}|}{k_n}\right],$$

$$|C_{2n}|\le M\left[\frac{|\psi_{1n}|}{k_n^2}+\frac{|\psi_{2n}|}{k_n^2}+\frac{|\psi_{3n}|}{k_n}\right],$$

$$|C_{3n}|\le M\left[\frac{|\psi_{1n}|}{k_n^2}+\frac{|\psi_{2n}|}{k_n^2}+\frac{|\psi_{3n}|}{k_n}\right].$$

Substituting the found estimates for $C_{in},\ i=\overline{1,3}$ into (19), we have

$$|u(x,y)|\le M\sum_{n=1}^{+\infty}e^{k_n(x-p)}\left[\frac{|\psi_{1n}|}{k_n^2}+\frac{|\psi_{2n}|}{k_n^2}+\frac{|\psi_{3n}|}{k_n}\right]\le M\sum_{n=1}^{+\infty}\left[\frac{|\psi_{1n}|}{k_n^2}+\frac{|\psi_{2n}|}{k_n^2}+\frac{|\psi_{3n}|}{k_n}\right]. \quad (20)$$

Integrating by parts ψ_{1n}, ψ_{2n}, ψ_{3n} and taking into account condition (4), we obtain

$$\psi_{in}=\left(\frac{q}{\pi}\right)^2\frac{\Psi_{in}}{n^2},\quad i=\overline{1,3},$$

where

$$\Psi_{in}=-\sqrt{\frac{2}{q}}\int_0^q\psi''_i(y)\cos\frac{n\pi y}{q}dy.$$

Then

$$|\psi_{in}|\le M\frac{|\Psi_{in}|}{n^2},\quad i=1,3,\quad M=const>0.$$

Taking these estimates into account, from (20) we find

$$|u(x,y)|\le M\sum_{n=1}^{+\infty}\left[\frac{|\Psi_{1n}|}{n^{\frac{10}{3}}}+\frac{|\Psi_{2n}|}{n^{\frac{10}{3}}}+\frac{|\Psi_{3n}|}{n^{\frac{8}{3}}}\right]<\infty.$$

It follows that series (18) converges absolutely and uniformly.

Now we prove that the derivatives of series (18) included in equation (1) also converge absolutely and uniformly in the domain $\overline{D}$. To do this, we calculate the derivatives with respect to y, from (18) we obtain

$$\frac{\partial^2 u}{\partial y^2} = -\sqrt{\frac{2}{q}}\left(\frac{\pi}{q}\right)^2 \sum_{n=1}^{+\infty} n^2 \left[C_{1n}e^{k_n x} + e^{-\frac{1}{2}k_n x}\left(C_{2n}\cos\frac{\sqrt{3}}{2}k_n x + C_{3n}\sin\frac{\sqrt{3}}{2}k_n x\right)\right]\cos\frac{n\pi}{q}y,$$

taking into account the estimate $u(x, y)$, we have

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \leq M\sum_{n=1}^{+\infty}\left[\frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \frac{|\Psi_{2n}|}{n^{\frac{4}{3}}} + \frac{|\Psi_{3n}|}{n^{\frac{2}{3}}}\right].$$

Using the Cauchy-Bunyakovsky and Bessel inequalities, we obtain

$$\left|\frac{\partial^2 u}{\partial y^2}\right| \leq M\left[\sum_{n=1}^{+\infty}\frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \sum_{n=1}^{+\infty}\frac{|\Psi_{2n}|}{n^{\frac{4}{3}}} + \sqrt{\sum_{n=1}^{+\infty}|\Psi_{3n}|^2}\sqrt{\sum_{n=1}^{+\infty}\frac{1}{n^{\frac{4}{3}}}}\right] \leq$$
$$\leq M\left[\sum_{n=1}^{+\infty}\frac{|\Psi_{1n}|}{n^{\frac{4}{3}}} + \sum_{n=1}^{+\infty}\frac{|\Psi_{2n}|}{n^{\frac{4}{3}}} + \|\psi''_{3n}(y)\|\sqrt{\sum_{n=1}^{+\infty}\frac{1}{n^{\frac{4}{3}}}}\right] < \infty;$$

here

$$\sum_{n=1}^{+\infty}|\Psi_{3n}|^2 \leq \left\|\psi''_{3n}(y)\right\|^2_{L_2[0,q]}.$$

Consequently, the series corresponding to the function $\frac{\partial^2 u}{\partial y^2}$ converges absolutely and uniformly. The absolute and uniform convergence of the third derivative with respect to x of series (18) follows from $\left|\frac{\partial^3 u}{\partial x^3}\right| \leq \left|\frac{\partial^2 u}{\partial y^2}\right|$ and what was proven above.

Theorem 2 is proven.

Conclusion

In this paper, the second boundary value problem for the Viscous Transonic Equation in a rectangular domain is investigated. The uniqueness of the solution to the problem is proved using the energy integral method. The existence of a solution is proved using the method of separation of variables. The solution to the problem is found in the form of an infinite series, uniform convergence, and the possibility of term-by-term differentiation under certain conditions on the given functions is proved. The convergence of the second-order derivative of the solution with respect to the variable is proved using the Cauchy-Bunyakovsky and Bessel inequalities. When substantiating the uniform convergence of the solution, the absence of a "small denominator" is proved.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Yuldashev, T.K. (2014). Obratnaia zadacha dlia odnogo integro-differentsialnogo uravneniia Fredgolma v chastnykh proizvodnykh tretego poriadka [Inverse problem for one Fredholm integro-differential equation in third order partial derivatives]. *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Seriia Fiziko-matematicheskie nauki — Bulletin of Samara State Technical University, Series Physical and mathematical sciences, 1*(34), 56–65 [in Russian]. https://doi.org/10.14498/vsgtu1299

2 Yuldashev, T.K. (2021). *Smeshannye i nelokalnye kraevye zadachi dlia differentsialnykh i integro-differentsialnykh uravnenii [Mixed and Nonlocal Boundary Value Problems for Differential and Integro-Differential Equations].* Tashkent: Universitet [in Russian].

3 Ryjov, O.S. (1965). Asimptoticheskaia kartina obtekaniia tel vrashcheniia so zvukovym potokom viazkogo i teploprovodiashchego gaza [Asymptotic picture of flow around bodies of revolution with a sonic flow of viscous and heat-conducting gas]. *Prikladnaia Matematika i Mekhanika — Applied Mathematics and Mechanics, 29*(6), 1004–1014 [in Russian].

4 Diesperov, V.N. (1972). On Green's function of the linearized viscous transonic equation. *USSR Computational Mathematics and Mathematical Physics, 12*(5), 225–241. https://doi.org/10.1016/0041-5553(72)90013-4

5 Cattabriga, L. (1961). Potenziali di linea e di dominio per equazioni non paraboliche in due variabilia caratteristiche multiple [Line and domain potentials for non-parabolic equations in two variables with multiple characteristics]. *Rendiconti del seminario matimatico della univ. di Padava — Proceedings of the Mathematical Seminar of the University of Padua, 31*, 1–45 [in Italian].

6 Dzhuraev, T.D, & Apakov, Yu.P. (2007). Ob avtomodelnom reshenii odnogo uravneniia tretego poriadka s kratnymi kharakteristikami [On the self-similar solution of a third-order equation with multiple characteristics]. *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Seriia «Fiziko-matematicheskie nauki» — Journal of Samara State Technical University, Series Physical and Mathematical Sciences, 2*(15), 18–26 [in Russian].

7 Dzhuraev, T.D., & Apakov, Yu.P. (2010). On the theory of the third-order equation with multiple characteristics containing the second time derivative. *Ukrainian Mathematical Journal, 62*(1), 43—55. https://doi.org/10.1007/s11253-010-0332-8

8 Zikirov, O.S. (2014). Dirichlet problem for third-order hyperbolic equations. *Russian Mathematics, 58*, 53–60. https://doi.org/10.3103/S1066369X14070068

9 Karimov, Sh.T., & Khojiakbarova, G. (2017). Analog zadachi Gursa dlia odnogo neklassicheskogo uravneniia tretego poriadka s singuliarnym koeffitsientom [An analog of the Goursat problem for a third-order non-classical equation with singular coefficient]. *Modern Problems of Dynamical Systems and Their Applications: Abstracts of the Republic Scientific Conference with Participation of Foreign Scientists.* Turin Polytechnic University in Tashkent (p. 121–122) [in Russian].

10 Kozhanov, I., & Potapova, S.V. (2018). Boundary value problems for odd order forward-backward-type differential equations with two time variables. *Siberian Mathematics Journal, 59*(5), 870–884. https://doi.org/10.1134/S0037446618050117

11 Kozhanov, A.I., & Diuzheva, A.V.(2020). Nelokalnye zadachi s integralnym usloviem dlia differentsialnykh uravnenii tretego poriadka [Nonlocal problems with integral conditions for third-order differential equations]. *Vestnik Samarskogo gosudarstvennogo tekhnicheskogo universiteta. Seriia «Fiziko-matematicheskie nauki» — Journal of Samara State Technical University, Series. Physical and Mathematical Sciences, 24*(4), 607–620 [in Russian].

12 Ashyralyev, A., Belakroum, K., & Guezane-Lakoud, A. (2017). Stability of boundary-value

problems for third-order partial differential equations. *Electronic Journal of Differential Equations, 2017*(53), 1–11.

13 Ashyralyev, A., Ibrahim, S., & Hincal, E. (2021). On stability of the third order partial delay differential equation with involution and Dirichlet condition. *Bulletin of the Karaganda University. Mathematics Series, 2*(102), 25–34. https://doi.org/10.31489/2021M2/25-34

14 Ashyralyev, A., & Belakroum, K. (2022). A stable difference scheme for a third-order partial differential equation *Journal of Mathematical Sciences, 260*(4), 399–417. https://doi.org/10.1007/s10958-022-05702-5

15 Sabitov, K.B. (2021). Initial–Boundary Value Problems for Equation of Oscillations of a Rectangular Plate. *Russian Mathematics, 65*, 52–62. https://doi.org/10.3103/S1066369X 21100054

16 Umarov, R.A. (2025). *Kraevye zadachi dlia uravneniia tretego poriadka s kratnymi kharakteristikami s mladshimi chlenami [Boundary value problems for a third-order equation with multiple characteristics and lower terms]*. Fergana [in Russian].

17 Khamitov, A.A. (2024). O korrektnosti kraevoi zadachi dlia uravneniia tretego poriadka s kratnymi kharakteristikami v trekhmernom prostranstve [Well-posedness of a boundary value problem for a third-order PDE with multiple characteristics in three-dimensional space]. *Doklady Akademii nauk Respubliki Uzbekistan — Reports of the Academy of Sciences of the Republic of Uzbekistan, 3*, 32–38 [in Russian].

18 Apakov, Yu.P., & Umarov, R.A. (2022). Solution of the Boundary Value Problem for a Third Order Equation with Little Terms. Construction of the Green's Function. *Lobachevskii Journal of Mathematics, 43*(3), 738–748. https://doi.org/10.1134/S199508022206004X

19 Umarov, R.A. (2024). Ob odnoi kraevoi zadache dlia neodnorodnogo uravneniia tretego poriadka s peremennymi koeffitsientami [On a Boundary Value Problem for an Inhomogeneous Third-Order Equation with Variable Coefficients]. *Nauchnyi vestnik Namanganskogo Gosudarstvennogo Universiteta — Scientific Bulletin of Namangan State University, 2*, 16–27 [in Russian].

20 Irgashev, B.Yu. (2023). Construction of Fundamental Solution for an Odd-Order Equation. *Vestnik St. Petersburg University, Mathematics, 56*, 512–520. https://doi.org/10.1134/S106345 4123040180

21 Irgashev, B.Yu. (2022). A Boundary Value Problem with Conjugation Conditions for a Degenerate Equation with the Caputo Fractional Derivative. *Russian mathematics, 66*(4), 24–31. https://doi.org/10.3103/S1066369X2204003X

22 Irgashev, B.Yu. (2023). A nonlocal problem for a mixed equation of high even order with a fractional Caputo derivative. *Journal of elliptic and parabolic equations, 9*, 389–399. https://doi.org/10.1007/s41808-023-00205-z

23 Apakov, Yu.P. (2019). *K teorii uravnenii tretego poriadka s kratnymi kharakteristikami [On the theory of third-order equations with multiple characteristics]*. Tashkent: Fan va texnologiya [in Russian].

24 Khasanov, A.B. (2016). *Shturm-Liuvil chegaravii masalalari nazariyasiga kirish [Introduction to the Theory of Sturm-Liouville Boundary Value Problems*. Tashkent: TURON-IQBOL [in Uzbek].

25 Nikolskii, S.M. (1983). *Kurs matematicheskogo analiza [Course of Mathematical Analysis]*. Novosibirsk: Nauka [in Russian].

*Author Information**

Yusupjon Pulatovich Apakov — Doctor of Physical and Mathematical Sciences, Professor, Leading Researcher, V.I. Romanovskiy Institute of Mathematics of the Academy of Sciences of Uzbekistan, 4-B University str., Tashkent, 100174, Uzbekistan; Professor, Namangan State Technical University, 12 Islam Karimov street, Namangan, 160103, Uzbekistan; e-mail: *yusupjonapakov@gmail.com*; https://orcid.org/0000-0001-8805-8917

Khusniddin Komiljon ugli Ibrokhimov (*corresponding author*) — Teacher, Namangan State Technical University, 12 Islam Karimov street, Namangan, 160103, Uzbekistan; e-mail: *xusniddin571@mail.ru*; https://orcid.org/0009-0001-7142-0956

*Authors' names are presented in the order: first name, middle name, and last name.