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Dirichlet type boundary value problem for an elliptic equation with three singular coefficients in the first octant

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The paper investigates a Dirichlet-type boundary value problem for a three-dimensional elliptic equation with three singular coefficients in the first octant. The uniqueness of the solution within the class of regular solutions is established using the energy integral method. To prove the existence of a solution, the Hankel integral transform method is employed. The use of the Hankel transform is particularly appropriate when the variables in the equation range from zero to infinity. This transform is an effective method for obtaining solutions to such problems. In three-dimensional space, to derive the image equation, the Hankel integral transform is applied to the original equation with respect to the variables x and y. As a result, a boundary value problem for an ordinary differential equation in the variable z arises. By solving this problem, a solution to the original boundary value problem is constructed in the form of a double improper integral involving Bessel functions of the first kind and Macdonald functions. To justify the uniform convergence of the improper integrals, asymptotic estimates of the Bessel functions of the first kind and Macdonald functions are utilized. Based on these estimates, bounds for the integrands are obtained, which ensure the convergence of the resulting double improper integral, that is, the solution to the original boundary value problem and its derivatives up to second order, inclusively, as well as the theorem of existence within the class of regular solutions.

Keywords: Hankel's integral transform, Bessel function, modified Bessel function, Macdonald function, singular coefficient, equation of elliptic type, Bessel operator, first octant.

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Introduction. Formulation of the problem

In recent years, interest in degenerate and singular equations has grown significantly, including equations containing the Bessel differential operator. These equations are often encountered in applications, for example, in problems with axial symmetry in continuum mechanics. Interest in problems related to the Bessel operator is also known from fundamental physics. This is due to its numerous applications in gas dynamics, shell theory, magnetohydrodynamics, and other fields of science and technology. A special place in the theory of degenerate and singular equations is occupied by equations containing the Bessel differential operator

$$B_q^z \equiv \frac{d^2}{dz^2} + \frac{2q+1}{z}\frac{d}{dz}, \quad q > -1/2.$$

According to the terminology by the Voronezh mathematician Ivan Aleksandrovich Kipriyanov, equations of three main classes containing the Bessel operator are called B-elliptic, B-hyperbolic, and B-parabolic, respectively. The monograph [1] studies boundary value problems for B-elliptic equations, in addition to this, the account of multi-dimension integral Fourier-Bessel-Hankel transformation theory is given in the monograph. The theory of boundary value problems for the equations with peculiarity

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has been reflected there, while the study of B-hyperbolic equations is presented in the monograph by A.K. Urinov, S.M. Sitnik and Sh.T. Karimov [2]. A wide range of questions for equations with Bessel operators was studied by I.A. Kipriyanov [1] and his students [3–5] and others.

In this paper, we study a Dirichlet-type problem for an elliptic equation with Bessel operators. The solution to the problem under consideration is solved by the Hankel transform method [6–8].

In the domain $\Omega=\{(x,y,z):\, x\in(0,+\infty)\,,\, y\in(0,+\infty)\,,\, z=(0,+\infty)\}\,,$ we consider the following three-dimensional equation with Bessel operators

$$Lu\equiv\left(B^{x}_{\alpha-1/2}+B^{y}_{\beta-1/2}+B^{z}_{\gamma-1/2}\right)u\,(x,y,z)=0, \tag{1}$$

where $u\,(x,y,z)$ is an unknown function, and $0<\alpha,\beta,\gamma<1/2$.

In the domain Ω, equation (1) is of elliptic type. The planes $x=0$, $y=0$ and $z=0$ are the planes of the singularity of the coefficients of the equation.

In the domain Ω, we consider the following problem for equation (1):

Problem D_∞. Find a solution to the equation (1) in the domain Ω, satisfying the conditions

$$u\,(x,y,z)\in C\left(\bar\Omega\right)\cap C^{2,2,2}_{x,y,z}\left(\Omega\right),\quad x^{2\alpha}u_x,\ y^{2\beta}u_y,\ \ z^{2\gamma}u_z\in C\left(\bar\Omega\right), \tag{2}$$

$$u\,(0,y,z)=0,\ \ \lim_{x\to+\infty}u\,(x,y,z)=0,\ \ y,z\in[0,+\infty), \tag{3}$$

$$u\,(x,0,z)=0,\ \ \lim_{y\to+\infty}u\,(x,y,z)=0,\ \ x,z\in[0,+\infty), \tag{4}$$

$$u\,(x,y,0)=f\,(x,y)\,,\ \ \lim_{z\to+\infty}u\,(x,y,z)=0,\ \ x,y\in[0,+\infty), \tag{5}$$

where $\bar\Omega=\{(x,y,z):\ x\in[0,+\infty),\ y\in[0,+\infty),\ z=[0,+\infty)\}\,,$ $f\,(x,y)$ is a given continuous function, such that $f\,(0,y)=0,\ f\,(x,0)=0,\ \lim\limits_{x\to+\infty}f\,(x,y)=0,\ \lim\limits_{y\to+\infty}f\,(x,y)=0.$

In recent years, there has been a steady increase in interest in studying boundary value problems for elliptic equations that involve singularities. Examples of such studies can be found in works [9, 10], among others.

In this paper, we study the stated Problem D_∞ using the Hankel transform method. Many problems in physics, applied mathematics, and mathematical modeling reduce to solving differential, integral, and integro-differential equations. One of the effective methods for obtaining an analytical solution is the method of integral transforms. Among all Bessel-type transforms, the Hankel integral transform is the most thoroughly studied and widely used.

The integral Hankel transform of the order ν of a function is called the integral [6–8]

$$\bar f\,(p)=\int\limits_0^{+\infty}f\,(t)\,tJ_\nu\,(pt)\,dt,\ \ \nu\geq-1/2,\ \ 0<p<+\infty,$$

where $J_\nu\,(z)$ is the Bessel function of the first kind of order ν [6].

The Hankel transform of a function $f\,(t)$ is true for any points on the interval $(0,+\infty)$ in which the function $f\,(t)$ is continuous or piecewise continuous with a finite number of discontinuity points of the first kind, and

$$\int\limits_0^{+\infty}|f\,(t)|\,t^{1/2}dt<+\infty.$$

The inversion formula of the Hankel transform is determined by the integral

$$f\,(t)=\int\limits_0^{+\infty}\bar f\,(p)\,pJ_\nu\,(pt)\,dp,\ \ 0<t<+\infty.$$

The function $\bar{f}(p)$ is often called the Fourier-Bessel-Hankel image [11], and the function $f(t)$ is the original.

The Hankel transform is advisable to apply, obviously, in the case when the variables in the equation change from 0 to $+\infty$.

1 *Uniqueness of the solution to the problem* D_∞

Theorem 1. If there exists solution to Problem D_∞, then it is unique.

Proof. Let Problem D_∞ have two solutions $u_1(x,y,z)$ and $u_2(x,y,z)$. Then $u(x,y,z)=u_1(x,y,z)-$ $-u_2(x,y,z)$ satisfies equation (1) and the homogeneous boundary conditions. We will prove that $u(x,y,z)\equiv 0$ in $\bar{\Omega}$. In the domain Ω the identity is valid

$$x^{2\alpha}y^{2\beta}z^{2\gamma}uLu=\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_x\right)_x+\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_y\right)_y+\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_z\right)_z-$$

$$-x^{2\alpha}y^{2\beta}z^{2\gamma}\left(u_x^2+u_y^2+u_z^2\right)=0.$$

Integrating this identity over the domain

$$\Omega_{\delta_1\delta_3\delta_5}^{\delta_2\delta_4\delta_6}=\left\{(x,y,z):\ \delta_1<x<\delta_2,\ \delta_3<y<\delta_4,\ \delta_5<z<\delta_6\right\},$$

where δ_j, $j=\overline{1,6}$ are positive numbers, we have

$$\iiint\limits_{\Omega_{\delta_1\delta_3\delta_5}^{\delta_2\delta_4\delta_6}}\left[\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_x\right)_x+\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_y\right)_y+\left(x^{2\alpha}y^{2\beta}z^{2\gamma}uu_z\right)_z\right]dxdydz=$$

$$=\iiint\limits_{\Omega_{\delta_1\delta_3\delta_5}^{\delta_2\delta_4\delta_6}}\left[x^{2\alpha}y^{2\beta}z^{2\gamma}\left(u_x^2+u_y^2+u_z^2\right)\right]dxdydz. \tag{6}$$

It is obvious that if $\delta_1,\ \delta_3,\ \delta_5\to 0,\ \ \delta_2,\delta_4,\delta_6\to+\infty$, then $\Omega_{\delta_1\delta_3\delta_5}^{\delta_2\delta_4\delta_6}\to\Omega$.

Applying the Gauss-Ostrogradsky formula [12] to the left side of equality (6), we have

$$\int\limits_{\delta_5}^{\delta_6}\int\limits_{\delta_3}^{\delta_4}y^{2\beta}z^{2\gamma}\left[\delta_2^{2\alpha}u(\delta_2,y,z)\,u_x(\delta_2,y,z)-\delta_1^{2\alpha}u(\delta_1,y,z)\,u_x(\delta_1,y,z)\right]dydz+$$

$$+\int\limits_{\delta_5}^{\delta_6}\int\limits_{\delta_1}^{\delta_2}x^{2\alpha}z^{2\gamma}\left[\delta_4^{2\beta}u(x,\delta_4,z)\,u_y(x,\delta_4,z)-\delta_3^{2\beta}u(x,\delta_3,z)\,u_y(x,\delta_3,z)\right]dxdz+$$

$$+\int\limits_{\delta_3}^{\delta_4}\int\limits_{\delta_1}^{\delta_2}x^{2\alpha}y^{2\beta}\left[{\delta_6}^{2\gamma}u(x,y,\delta_6)\,u_z(x,y,\delta_6)-\ \delta_5^{2\gamma}u(x,y,\delta_5)\,u_z(x,y,\delta_5)\right]dxdy=$$

$$=\iiint\limits_{\Omega_{\delta_1\delta_3\delta_5}^{\delta_2\delta_4\delta_6}}\left[x^{2\alpha}y^{2\beta}z^{2\gamma}\left(u_x^2+u_y^2+u_z^2\right)\right]dxdydz.$$

Hence, passing to the limit at $\delta_1,\ \delta_3,\ \delta_5 \to 0,\ \ \delta_2, \delta_4, \delta_6 \to +\infty$ and taking into account conditions (2), (3), (4) and (5) (for $f(x,y) \equiv 0$), from the last equality, we obtain

$$\iiint\limits_{\Omega} \left[x^{2\alpha} y^{2\beta} z^{2\gamma} \left(u_x^2 + u_y^2 + u_z^2\right)\right] dxdydz = 0.$$

From the last, we have

$$u_x(x,y,z) \equiv u_y(x,y,z) \equiv u_z(x,y,z) \equiv 0,\ \ (x,y,z) \in \Omega.$$

Then, $u(x,y,z) \equiv const$, $(x,y,z) \in \Omega$. Since $u \in C\left(\bar{\Omega}\right)$ and $u(0,y,z) \equiv 0$, then $u(x,y,z) \equiv 0$, $(x,y,z) \in \bar{\Omega}$. From this follows the statement of Theorem 1.

2 Existence of the solution to the problem D_∞

Let $\tilde{u}(\lambda,\mu,z)$ be the Hankel transformation of the unknown function $u(x,y,z)$ with respect to the variables x and y. Then, by the definition, we have

$$\tilde{u}(\lambda,\mu,z) = \int\limits_0^{+\infty}\int\limits_0^{+\infty} xy\left[x^{\alpha-1/2} y^{\beta-1/2} u(x,y,z)\right] J_{1/2-\alpha}(\lambda x)\, J_{1/2-\beta}(\mu y)\, dxdy. \tag{7}$$

Considering inverse thansform, we also have

$$u(x,y,z) = x^{1/2-\alpha} y^{1/2-\beta} \int\limits_0^{+\infty}\int\limits_0^{+\infty} \lambda\mu\tilde{u}(\lambda,\mu,z)\, J_{1/2-\alpha}(\lambda x)\, J_{1/2-\beta}(\mu y)\, d\lambda d\mu.$$

Based on (7), we introduce the functions

$$\tilde{u}_{\varepsilon_1\varepsilon_3}^{\varepsilon_2\varepsilon_4}(\lambda,\mu,z) = \int\limits_{\varepsilon_3}^{\varepsilon_4}\int\limits_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} y^{1/2+\beta} u(x,y,z)\, J_{1/2-\alpha}(\lambda x)\, J_{1/2-\beta}(\mu y)\, dxdy, \tag{8}$$

where ε_j, $j = \overline{1,4}$ are positive numbers.

It's obvious that $\lim\limits_{\substack{\varepsilon_1,\varepsilon_3 \to 0\\ \varepsilon_2,\varepsilon_4 \to +\infty}} \tilde{u}_{\varepsilon_1\varepsilon_3}^{\varepsilon_2\varepsilon_4}(\lambda,\mu,z) = \tilde{u}(\lambda,\mu,z)$.

Using the function (8) and the equation (1), we simplify the expression of $B_{\gamma-1/2}^z \tilde{u}_{\varepsilon_1\varepsilon_3}^{\varepsilon_2\varepsilon_4}(\lambda,\mu,z)$:

$$B_{\gamma-1/2}^z \tilde{u}_{\varepsilon_1\varepsilon_3}^{\varepsilon_2\varepsilon_4} = \int\limits_{\varepsilon_3}^{\varepsilon_4}\int\limits_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} y^{1/2+\beta} J_{1/2-\alpha}(\lambda x)\, J_{1/2-\beta}(\mu y)\, B_{\gamma-1/2}^z u(x,y,z)\, dxdy =$$

$$= -\int\limits_{\varepsilon_3}^{\varepsilon_4}\int\limits_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} y^{1/2+\beta} J_{1/2-\alpha}(\lambda x)\, J_{1/2-\beta}(\mu y) \left(B_{\alpha-1/2}^x + B_{\beta-1/2}^y\right) u(x,y,z)\, dxdy =$$

$$= -\int\limits_{\varepsilon_3}^{\varepsilon_4} \left[\int\limits_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x)\, B_{\alpha-1/2}^x u(x,y,z)\, dx\right] y^{1/2+\beta} J_{1/2-\beta}(\mu y)\, dy -$$

$$- \int\limits_{\varepsilon_1}^{\varepsilon_2} \left[\int\limits_{\varepsilon_3}^{\varepsilon_4} y^{1/2+\beta} J_{1/2-\beta}(\mu y)\, B_{\beta-1/2}^y u(x,y,z)\, dy\right] x^{1/2+\alpha} J_{1/2-\alpha}(\mu x)\, dx. \tag{9}$$

Applying the rule of integration by parts, from (9), we obtain

$$B_{\gamma-1/2}^{z}\tilde{u}_{\varepsilon_1\varepsilon_3}^{\varepsilon_2\varepsilon_4}(\lambda,\mu,z)=-\int\limits_{\varepsilon_3}^{\varepsilon_4}\left\{\left[J_{1/2-\alpha}(\lambda x)\,u_x-\lambda J_{-1/2-\alpha}(\lambda x)\,u\right]x^{1/2+\alpha}\Big|_{x=\varepsilon_1}^{x=\varepsilon_2}-\right.$$

$$\left.-\lambda^2\int\limits_{\varepsilon_1}^{\varepsilon_2}u(x,y,z)x^{1/2+\alpha}J_{1/2-\alpha}(\lambda x)\,dx\right\}y^{1/2+\beta}J_{1/2-\beta}(\mu y)\,dy-$$

$$-\int\limits_{\varepsilon_1}^{\varepsilon_2}\left\{\left[J_{1/2-\beta}(\mu y)\,u_y-\mu J_{-1/2-\beta}(\mu y)\,u\right]y^{1/2+\beta}\Big|_{y=\varepsilon_3}^{y=\varepsilon_4}-\right.$$

$$\left.-\mu^2\int\limits_{\varepsilon_3}^{\varepsilon_4}u(x,y,z)y^{1/2+\beta}J_{1/2-\beta}(\mu y)\,dy\right\}x^{1/2+\alpha}J_{1/2-\alpha}(\lambda x)\,dx. \tag{10}$$

By direct calculation, one can easily verify that the following limits for fixed $\lambda\in(0,+\infty)$ and $\mu\in(0,+\infty)$, exist and are finite:

$$\lim_{x\to 0}x^{1/2+\alpha}J_{-1/2-\alpha}(\lambda x)=2^{1/2+\alpha}\lambda^{-1/2-\alpha}/\Gamma(1/2-\alpha), \tag{11}$$

$$\lim_{y\to 0}y^{1/2+\beta}J_{-1/2-\beta}(\mu y)=2^{1/2+\beta}\mu^{-1/2-\beta}/\Gamma(1/2-\beta). \tag{12}$$

The behavior of the function $J_v(x)$ for sufficiently small and large values of x is described by the formulas given in [13], respectively:

$$J_\nu(x)\underset{x\to 0}{\approx}\frac{x^\nu}{2^\nu\Gamma(1+\nu)},\quad J_\nu(x)\underset{x\to+\infty}{\approx}\left(\frac{2}{\pi x}\right)^{1/2}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right). \tag{13}$$

From the equality (10), passing to the limit at $\varepsilon_1\to 0$, $\varepsilon_3\to 0$, $\varepsilon_2\to+\infty$, $\varepsilon_4\to+\infty$, and taking the conditions (2), (3), (4) and equalities (11), (12), (13) into account, as well as the notation (7), we obtain the following equation

$$\tilde{u}_{zz}(\lambda,\mu,z)+\frac{2\gamma}{z}\tilde{u}_z(\lambda,\mu,z)-\chi^2\tilde{u}(\lambda,\mu,z)=0,\ \ 0<\lambda,\mu,z<+\infty, \tag{14}$$

where $\chi^2=\lambda^2+\mu^2$.

Moreover, due to the boundary conditions (5), from (7) it follows that the function $\tilde{u}(\lambda,\mu,z)$ satisfies the following boundary conditions:

$$\tilde{u}(\lambda,\mu,0)=f_{\lambda\mu},\quad \lim_{z\to+\infty}\tilde{u}(\lambda,\mu,z)=0, \tag{15}$$

where

$$f_{\lambda\mu}=\int\limits_0^{+\infty}\int\limits_0^{+\infty}x^{1/2+\alpha}y^{1/2+\beta}f(x,y)\,J_{1/2-\alpha}(\lambda x)\,J_{1/2-\beta}(\mu y)\,dxdy. \tag{16}$$

We solve the problem (14), (15). It knows that the general solution of the equation (14) has the form [9]

$$\tilde{u}(\lambda,\mu,z)=c_1z^{1/2-\gamma}I_{1/2-\gamma}(\chi z)+c_2z^{1/2-\gamma}K_{1/2-\gamma}(\chi z),\quad z\in[0,c], \tag{17}$$

where c_1 and c_2 are arbitrary constants, $I_l(x)$ and $K_l(x)$ are the Bessel function of the imaginary argument and the Macdonald function of order l [6], respectively.

From the equality (17), based on the asymptotic behavior of the functions $I_\nu(x)$ and $K_\nu(x)$ for sufficiently large x [13], we have

$$I_\nu(x) \approx \frac{e^x}{(2\pi x)^{1/2}}, \quad K_\nu(x) \approx \left(\frac{\pi}{2x}\right)^{1/2} e^{-x},$$

from which follows that the solution of equation (14) satisfying the second condition (15) is determined by the equality

$$\tilde{u}(\lambda, \mu, z) = c_2 z^{1/2-\gamma} K_{1/2-\gamma}(\chi z). \tag{18}$$

By the first condition of (15) from (17), we obtain the equality

$$\tilde{u}(\lambda, \mu, 0) = c_2 2^{-1/2-\gamma} \chi^{-1/2+\gamma} \Gamma(1/2-\gamma) = f_{\lambda\mu},$$

from which we uniquely find c_2 as follows:

$$c_2 = 2^{1/2+\gamma} \chi^{1/2-\gamma} f_{\lambda\mu} / \Gamma(1/2-\gamma).$$

Substituting the value of c_2 into the equality (18), we uniquely find a solution to the problem (14), (15) in the form

$$\tilde{u}(\lambda, \mu, z) = \bar{K}_{1/2-\gamma}(\chi z) f_{\lambda\mu}, \tag{19}$$

where $\bar{K}_\nu(x) = 2^{1-\nu} x^\nu K_\nu(x)/\Gamma(\nu)$, $\nu > 0$.

The solution of the original problem will be obtained by using the inverse Hankel transform as follows:

$$u(x, y, z) = \int_0^{+\infty}\int_0^{+\infty} \lambda\mu X_\lambda(x) Q_\mu(y) \tilde{u}(\lambda, \mu, z)\, d\lambda d\mu, \tag{20}$$

where $X_\lambda(x) = x^{1/2-\alpha} J_{1/2-\alpha}(\lambda x), Q_\mu(y) = y^{1/2-\beta} J_{1/2-\beta}(\mu y)$, and $\tilde{u}(\lambda, \mu, z)$ is determined by the formula (19) and they are respectively solutions of the following equations:

$$B^x_{\alpha-1/2} X_\lambda(x) = -\lambda^2 X_\lambda(x), \ \ 0 < x < +\infty, \tag{21}$$

$$B^y_{\beta-1/2} Q_\mu(y) = -\mu^2 Q_\mu(y), \ \ 0 < y < +\infty, \tag{22}$$

$$B^z_{\gamma-1/2} \tilde{u}(\lambda, \mu, z) = \chi^2 \tilde{u}(\lambda, \mu, z), \ \ \chi^2 = \lambda^2 + \mu^2, \ \ 0 < \lambda, \mu, z < +\infty. \tag{23}$$

If differentiation under the integral sign is possible in (20), then the function $u(x, y, z)$ is a solution to equation (1). Indeed,

$$B^x_{\alpha-1/2} u(x,y,z) + B^y_{\beta-1/2} u(x,y,z) + B^z_{\gamma-1/2} u(x,y,z) =$$

$$= \int_0^{+\infty}\int_0^{+\infty} \lambda\mu \left[B^x_{\alpha-1/2} X_\lambda(x)\right] Q_\mu(y) \tilde{u}(\lambda, \mu, z)\, d\lambda d\mu +$$

$$+ \int_0^{+\infty}\int_0^{+\infty} \lambda\mu X_\lambda(x) \left[B^y_{\beta-1/2} Q_\mu(y)\right] \tilde{u}(\lambda, \mu, z)\, d\lambda d\mu +$$

$$+\int\limits_{0}^{+\infty}\int\limits_{0}^{+\infty}\lambda\mu X_{\lambda}(x)\,Q_{\mu}(y)\left[B^{z}_{\gamma-1/2}\tilde{u}(\lambda,\mu,z)\right]d\lambda d\mu.$$

Hence, by virtue of (21), (22) and (23), we have

$$B^{x}_{\alpha-1/2}u(x,y,z)+B^{y}_{\beta-1/2}u(x,y,z)+B^{z}_{\gamma-1/2}u(x,y,z)=0.$$

Let us demonstrate that the function (20) satisfies conditions (3) and (4). Using formulas (13), the functions $X_{\lambda}(x)$ and $Q_{\mu}(y)$ for small and large argument values, respectively, can be rewritten in the form [12]

$$X_{\lambda}(x)\approx\frac{\lambda^{1/2-\alpha}x^{1-2\alpha}}{2^{1/2-\alpha}\Gamma(3/2-\alpha)},\ \ 0<x,\lambda<1; \tag{24}$$

$$X_{\lambda}(x)\approx x^{-\alpha}\left(\frac{2}{\pi\lambda}\right)^{1/2}\sin\left(\lambda x+\frac{\alpha\pi}{2}\right),\ \ 1<x,\lambda<+\infty; \tag{25}$$

$$Q_{\mu}(y)\approx\frac{\mu^{1/2-\beta}y^{1-2\beta}}{2^{1/2-\beta}\Gamma(3/2-\beta)},\ \ 0<y,\mu<1;$$

$$Q_{\mu}(y)\approx y^{-\beta}\left(\frac{2}{\pi\mu}\right)^{1/2}\sin\left(\mu y+\frac{\beta\pi}{2}\right),\ \ 0<y,\mu<+\infty.$$

From these equalities, it follows that the function (20) satisfies the conditions (3) and (4).

Now, we prove several lemmas used in establishing the uniform convergence of the double integral (23).

Lemma 1. If $\alpha\in(0,1/2)$, then, with respect to the functions at $X_{\lambda}(x)=x^{1/2-\alpha}J_{1/2-\alpha}(\lambda x)$, as $x\in[0,+\infty)$, the following estimates hold:

$$|X_{\lambda}(x)|\leq\begin{cases}c_3x^{1-2\alpha}\lambda^{1/2-\alpha}, & 0<x,\lambda<1,\\ c_4x^{-\alpha}\lambda^{-1/2}, & 1<x,\lambda<+\infty,\end{cases} \tag{26}$$

$$\left|x^{2\alpha}X'_{n}(x)\right|\leq\begin{cases}c_5\lambda^{1/2-\alpha}, & 0<x,\ \lambda<1,\\ c_6\lambda^{1/2}x^{\alpha}, & 1<x,\lambda<+\infty,\end{cases} \tag{27}$$

$$\left|B^{x}_{\alpha-1/2}X_{\lambda}(x)\right|\leq\begin{cases}c_7x^{1-2\alpha}\lambda^{5/2-\alpha}, & 0<x,\lambda<1,\\ c_8x^{-\alpha}\lambda^{3/2}, & 1<x,\lambda<+\infty,\end{cases} \tag{28}$$

where $c_j,\ j=\overline{3,8}$ are positive constants.

Proof. From the equalities (24) and (25), we obtain estimate (26). Next, consider the functions $x^{2\alpha}X'_{\lambda}(x)=\lambda x^{1/2+\alpha}J_{-1/2-\alpha}(\lambda x)$ and (23). By virtue of the asymptotic formula (13), it is straightforward to show that these functions satisfy the estimates (27) and (28), respectively. Lemma 1 has been proved.

Similarly, the following lemma can be proved.

Lemma 2. If $\beta\in(0,1/2)$, then with respect to the functions $Q_{\mu}(y)=y^{1/2-\beta}J_{1/2-\beta}(\mu y)$, at $y\in[0,+\infty)$ the following estimates hold:

$$|Q_{\mu}(y)|\leq\begin{cases}c_9y^{1-2\beta}\mu^{1/2-\beta}, & 0<y,\mu<1,\\ c_{10}y^{-\beta}\mu^{-1/2}, & 1<y,\mu<+\infty,\end{cases} \tag{29}$$

$$\left|y^{2\beta}Q'_{\mu}(y)\right|\leq\begin{cases}c_{11}\mu^{1/2-\beta}, & 0<y,\mu<1,\\ c_{12}\mu^{1/2}y^{\beta}, & 1<y,\mu<+\infty,\end{cases}$$

$$\left|B^{y}_{\beta-1/2}Q_{\mu}\left(y\right)\right| \leq \left\{ \begin{array}{l} c_{13}y^{1-2\beta}\mu^{5/2-\beta}, \;\; 0<y,\mu<1, \\ c_{14}y^{-\beta}\mu^{3/2}, \;\; 1<y,\mu<+\infty, \end{array}\right.$$

where $c_j,\;\; j=\overline{9,14}$ are positive constants.

Lemma 3. For any $\lambda,\mu,z\in(0,+\infty)$, the functions $\tilde{u}(\lambda,\mu,z)$, defined by equality (19) satisfy the estimates

$$|\tilde{u}(\lambda,\mu,z)|\leq|f_{\lambda\mu}|\,, \quad \left|B^{z}_{\gamma-1/2}\tilde{u}(\lambda,\mu,z)\right|\leq\chi^{2}\left|f_{\lambda\mu}\right|. \tag{30}$$

Proof. It is known [9] that if $\nu = const > 0$, then

$$\bar{K}_{\nu}\left(t\right)\leq 1, \quad \bar{K}_{\nu}\left(0\right)=1. \tag{31}$$

From equality (19), according to (31) the first estimate in (30) follows.

As demonstrated earlier, the function $\tilde{u}\left(\lambda,\mu,z\right)$ satisfies the equation (23). Therefore, by virtue of the first estimate in (30), the validity of the second estimate in (30) immediately follows. Lemma 3 has been proved.

Lemma 4. Let $\alpha,\beta,\gamma\in(0,1/2)$ and the function $f\left(x,y\right)$ satisfy the following conditions:

I. $f\left(x,y\right)\in C^{4,4}_{x,y}\left(\bar{\Pi}\right)$, where $\Pi=\left\{(x,y):\; 0<x<+\infty,\, 0<y<+\infty\right\};$

II. $\lim\limits_{x\to 0}\left(\partial^{j}/\partial x^{j}\right)f\left(x,y\right)=0,\;\; \lim\limits_{x\to+\infty}x^{\alpha}\left(\partial^{j}/\partial x^{j}\right)f\left(x,y\right)=0,\;\lim\limits_{y\to 0}\left(\partial^{j}/\partial y^{j}\right)f\left(x,y\right)=0,$ $\lim\limits_{y\to+\infty}y^{\beta}\left(\partial^{j}/\partial y^{j}\right)f\left(x,y\right)=0, \quad j=\overline{0,3}.$

Then, for the coefficients (16), the following estimate holds:

$$|f_{\lambda\mu}|\leq c_{15}(\lambda\mu)^{-4}, \tag{32}$$

where c_{15} is some positive constant.

Proof. The coefficients $f_{\lambda\mu}$, according to formula (16), can be rewritten as

$$f_{\lambda\mu}=\int\limits_{0}^{+\infty}y^{1/2+\beta}J_{1/2-\beta}\left(\mu y\right)F_{j\lambda}\left(y\right)dy, \tag{33}$$

where

$$F_{\lambda}\left(y\right)=\int\limits_{0}^{+\infty}x^{1/2+\alpha}J_{1/2-\alpha}\left(\lambda x\right)f\left(x,y\right)dx.$$

First, consider the function $F_{\lambda}\left(y\right)$. Using the equalities

$$x^{1/2+\alpha}J_{1/2-\alpha}\left(\lambda x\right)=-\frac{1}{\lambda}\frac{d}{dx}\left[x^{1/2+\alpha}J_{-1/2-\alpha}\left(\lambda x\right)\right],$$

the function $F_{\lambda}\left(y\right)$ can be represented as

$$F_{\lambda}\left(y\right)=-\frac{1}{\lambda}\int\limits_{0}^{+\infty}\frac{d}{dx}\left[x^{1/2+\alpha}J_{-1/2-\alpha}\left(\lambda x\right)\right]f\left(x,y\right)dx.$$

Applying integration by parts four times to the above integral, we obtain

$$F_{\lambda}\left(y\right)=-\frac{1}{\lambda}x^{1/2+\alpha}J_{-1/2-\alpha}\left(\lambda x\right)f\left(x,y\right)\Big|_{x=0}^{x=+\infty}+\frac{1}{\lambda^{2}}x^{1/2+\alpha}J_{1/2-\alpha}\left(\lambda x\right)f_{x}\left(x,y\right)\Big|_{x=0}^{x=+\infty}+$$

$$+\frac{1}{\lambda^{3}}x^{1/2+\alpha}J_{-1/2-\alpha}\left(\lambda x\right)B^{x}_{\alpha-1/2}f\left(x,y\right)\Big|_{x=0}^{x=+\infty}-\frac{1}{\lambda^{4}}x^{1/2+\alpha}J_{1/2-\alpha}\left(\lambda x\right)\frac{\partial}{\partial x}B^{x}_{\alpha-1/2}f\left(x,y\right)\Big|_{x=0}^{x=+\infty}+$$

$$+\frac{1}{\lambda^4}\int\limits_0^{+\infty} X_\lambda(x)\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\,dx. \tag{34}$$

By the conditions of Lemma 4, the boundary terms in (34) vanish. Consequently,

$$F_\lambda(y)=\frac{1}{\lambda^4}\int\limits_0^{+\infty} X_\lambda(x)\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\,dx. \tag{35}$$

Using the decomposition of the operator $B^x_{\alpha-1/2}$, it is easy to verify that the functions $\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)$, based on the conditions of Lemma 4, satisfy $\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\in C\left(\bar{\Pi}\right)$. Taking this into account and the fact that $X_n(x)\in C[0,+\infty)$, we conclude that the integral in (35) exists and that $F_\lambda(y)\in C[0,+\infty)$.

Now, consider the coefficient $f_{\lambda\mu}$, defined by equality (33). Similarly to the previous case, applying integration by parts four times to the integral in (33), we obtain

$$f_{\lambda\mu}=-\frac{1}{\mu}y^{1/2+\beta}J_{-1/2-\beta}(\mu y)F_\lambda(y)\Big|_{y=0}^{y=+\infty}+\frac{1}{\mu^2}y^{1/2+\beta}J_{1/2-\beta}(\mu y)F'_\lambda(y)\Big|_{y=0}^{y=+\infty}+$$

$$+\frac{1}{\mu^3}y^{1/2+\beta}J_{-1/2-\beta}(\mu y)B^y_{\beta-1/2}F_\lambda(y)\Big|_{y=0}^{y=+\infty}-\frac{1}{\mu^4}y^{1/2+\beta}J_{1/2-\beta}(\mu y)\frac{\partial}{\partial y}B^y_{\beta-1/2}F_\lambda(y)\Big|_{y=0}^{y=+\infty}+$$

$$+\frac{1}{\mu^4}\int\limits_0^{+\infty} Q_\mu(y)\frac{\partial}{\partial y}y^{2\beta}\frac{\partial}{\partial y}B^y_{\beta-1/2}F_\lambda(y)dy. \tag{36}$$

Since the integral in (36) converges uniformly with respect to y, all derivatives and operators with respect to y acting on the functions $F_\lambda(y)$ can be transferred to the functions $f(x,y)$. Then, by the conditions of Lemma 4, the boundary terms in (36) vanish, and therefore

$$f_{\lambda\mu}=\frac{1}{\mu^4}\int\limits_0^{+\infty} Q_\mu(y)\frac{\partial}{\partial y}y^{2\beta}\frac{\partial}{\partial y}B^y_{\beta-1/2}F_\lambda(y)\,dy.$$

Hence, taking (36) into account, we have

$$f_{\lambda\mu}=\frac{1}{\lambda^4\mu^4}\int\limits_0^{+\infty}\int\limits_0^{+\infty} X_\lambda(x)Q_\mu(y)\frac{\partial}{\partial y}y^{2\beta}\frac{\partial}{\partial y}B^y_{\beta-1/2}\left[\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\right]dxdy. \tag{37}$$

By virtue of the conditions of Lemma 4, the following hold:

$$\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\in C\left(\bar{\Pi}\right),\quad \frac{\partial}{\partial y}y^{2\beta}\frac{\partial}{\partial y}B^y_{\beta-1/2}f(x,y)\in C\left(\bar{\Pi}\right),$$

therefore

$$\frac{\partial}{\partial y}y^{2\beta}\frac{\partial}{\partial y}B^y_{\beta-1/2}\left[\frac{\partial}{\partial x}x^{2\alpha}\frac{\partial}{\partial x}B^x_{\alpha-1/2}f(x,y)\right]\in C\left(\bar{\Pi}\right).$$

Taking this into account, along with $X_\lambda(x)Q_\mu(y)\in C\left(\bar{\Pi}\right)$, we conclude that the integrand is continuous on $\bar{\Pi}$, and the multiple integral in (37) exists. These considerations complete the proof of Lemma 4.

Based on (32), estimate (30) can be rewritten as

$$|\tilde{u}\,(\lambda,\mu,z)| \le c_{16}(\lambda\mu)^{-4}, \quad \left|B^z_{\gamma-1/2}\tilde{u}\,(\lambda,\mu,z)\right| \le c_{17}(\lambda\mu)^{-2}, \tag{38}$$

where c_{16} and c_{17} are positive constants.

Lemma 5. Let $\alpha,\beta,\gamma \in (0,1/2)$, and let $f(x,y)$ be a function such that for $\lambda,\mu \in (0,1)$, the following condition holds:

$$\int\limits_0^1\int\limits_0^1 xyf(x,y)dxdy < +\infty,$$

then the following estimate is valid:

$$|f_{\lambda\mu}| \le c_{18}\lambda^{1/2-\alpha}\mu^{1/2-\beta},\ \ c_{18} = const > 0. \tag{39}$$

Proof. We estimate the coefficient $f_{\lambda\mu}$ defined by equality (16). Taking into account that $0 < x,y,\lambda,\mu < 1$, and using the asymptotic formulas for Bessel functions for small values of arguments (13), as well as the condition of Lemma 5, we obtain inequality (39).

Taking into account (39) and (31), the function in (19) is estimated in the following form

$$|\tilde{u}(\lambda,\mu,z)| \le c_{19}\lambda^{1/2-\alpha}\mu^{1/2-\beta}, \quad c_{19} = const > 0. \tag{40}$$

Now, let us analyze the function (20), i.e., we find an estimate for the function (20). By virtue of the estimates (26), (29), (38) and (40), the integral (23) is bounded, respectively, for $0 < x,y,z < 1$ and for $1 < x,y,z < +\infty$ by the following absolutely convergent improper double integrals:

$$|u\,(x,y,z)| \le \int\limits_0^{+\infty}\int\limits_0^{+\infty} |\lambda\mu X_\lambda\,(x)\,Q_\mu\,(y)\,\tilde{u}\,(\lambda,\mu,z)|\,d\lambda d\mu \le$$

$$\le c_{20}x^{1-2\beta}y^{1-2\beta}\int\limits_0^1\int\limits_0^1 \lambda^{1,5-\alpha}\mu^{1,5-\beta}d\lambda d\mu, \quad 0 < x,y < 1,$$

$$|u\,(x,y,z)| \le \int\limits_0^{+\infty}\int\limits_0^{+\infty} |\lambda\mu X_\lambda\,(x)\,Q_\mu\,(y)\,\tilde{u}\,(\lambda,\mu,z)|\,d\lambda d\mu \le$$

$$\le c_{21}x^{-\alpha}y^{-\beta}\int\limits_1^{+\infty}\int\limits_1^{+\infty} \lambda^{-3,5}\mu^{-3,5}d\lambda d\mu, \quad 1 < x,y < +\infty.$$

Similarly, it can be shown that the integrals $x^{2\alpha}u_x,\ y^{2\beta}u_y,\ z^{2\gamma}u_z,\ B^x_{\alpha-1/2}u,\ B^y_{\beta-1/2}u$ and $B^z_{\gamma-1/2}u$ are bounded by absolutely convergent improper double integrals.

According to Theorem 4 from [14; 233], the double integral in (20) converges uniformly.

Due to the uniform convergence of the double series (20), it can be integrated term by term, and for each term, the order of integration can be interchanged.

Consequently, the integrand in (20) is continuous, and the double integral in (20) converges uniformly for $0 < x,y,z < +\infty$. Therefore, by Theorem 1 from [14; 231], this integral represents a continuous function of x,y and z. Hence, $u(x,y,z)$ is a continuous function in its domain of definition.

Based on these statements, the following theorem holds:

Theorem 2. Let $\alpha,\beta,\gamma \in (0,1/2)$ and the function $f(x,y)$ satisfy the conditions of Lemma 4 and Lemma 5. Then the solution of Problem D_∞ exists and is given by formula (20).

Conclusion

In this work, a Dirichlet type boundary value problem for a three-dimensional elliptic equation with three singular coefficients is formulated and studied. The uniqueness of the solution to the problem has been proved by the method of energy integrals. The Hankel transform method was used to prove the existence of solutions. The solution of the original problem was obtained using the inverse Hankel transform in the form of a two-fold improper integral. Asymptotic methods were used to substantiate the uniform convergence of improper integrals. The obtained estimate made it possible to prove the convergence of these improper integrals and its derivatives up to and including the second order.

Author Contributions

M.R. Murodova collected and analyzed data, and led manuscript preparation, assisted in data collection and analysis. K.T. Karimov served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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