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On solvability of the initial-boundary value problems for a nonlocal hyperbolic equation with periodic boundary conditions

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In this paper, the solvability of initial-boundary value problems for a nonlocal analogue of a hyperbolic equation in a cylindrical domain is studied. The elliptic part of the considered equation involves a nonlocal Laplace operator, which is introduced using involution-type mappings. Two types of boundary conditions are considered. These conditions are specified as a relationship between the values of the unknown function at points in one half of the lateral part of the cylinder and the values at points in the other part of the cylinder boundary. The boundary conditions specified in this form generalize known periodic and antiperiodic boundary conditions for circular domains. The unknown function is represented in the form $u(x) = v(x) + w(x)$, where $v(x)$ is the even part of the function and $w(x)$ is the odd part of the function with respect to the mapping. Using the properties of these functions, we obtain auxiliary initial-boundary value problems with classical hyperbolic equations. In this case, the boundary conditions of these problems are specified in the form of the Dirichlet and Neumann conditions. Further, using the known assertions for the auxiliary problems, theorems on the existence and uniqueness of the solution to the main problems are proved. The solutions to the problems are constructed as a series in systems of eigenfunctions of the Dirichlet and Neumann problems for the classical Laplace operator.

Keywords: antiperiodic condition, Dirichlet problem, eigenfunctions, eigenvalues, Fourier series, hyperbolic equation, initial-boundary value problem, involution, Neumann problem, nonlocal operator, periodic condition.

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Introduction

This paper considers differential equations that belong to the class of equations containing shifts of arguments. Such equations are widely used in describing various scientific models, for example, in modeling immune processes [1, 2], in various population models [3, 4], in modeling the dynamics of nonlinear optical systems [5, 6], and other systems.

Among equations with shifts of arguments, equations with involution occupy a special place. Boundary value and initial-boundary value problems for analogues of elliptic and parabolic equations with involution have been studied by Al-Salti et al. [7, 8], Ashyralyev and Sarsenbi [9, 10], Baranetskij et al. [11], Borikhanov and Mambetov [12], Kozhanov and Bzheumikhova [13], Mussirepova et al. [14, 15], and Yarka et al. [16].

The analogues of hyperbolic equations with involution were considered in [17–19]. In [17], a nonlocal analogue of a hyperbolic equation with involution with respect to the time variable was examined. In the paper, the initial problem was solved by reducing it to an equivalent initial problem

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for a fourth-order equation without involution. The estimates of stability of the solution and its first- and second-order derivatives of the above problem were established. Similar studies were conducted in [18, 19]. In these works, hyperbolic equations with involution with respect to the spatial variable are considered in the one-dimensional case.

In this paper we investigate the solvability of initial-boundary value problems with periodic and antiperiodic boundary conditions in the multidimensional case. Moreover, periodic and antiperiodic boundary conditions are specified on the boundary of a circular cylinder. Boundary value problems with periodic and antiperiodic boundary conditions in circular domains for the Poisson equation were first studied in [20, 21], and for the nonlocal Poisson equation they were investigated in [22]. Note also that boundary value problems with periodic conditions for a hyperbolic equation in rectangular domains were studied in [23].

Let us turn to the formulation of the problems that are considered in this paper. Let Ω be a unit ball, $\partial\Omega$ be a unit sphere, $Q_T = \Omega \times (0, T)$ be an open cylinder. For any $x = (x_1, x_2, ..., x_n)$ we assign a point $Sx = (-x_1, \alpha_2 x_2, ..., \alpha_n x_n)$, where $\alpha_j, j = 2, 3, ..., n$ takes one of the values ± 1.

Let us introduce the operator

$$L_x v(x) \equiv a_0 \triangle v(x) + a_1 \triangle v(Sx),$$

where a_0, a_1 are real numbers, $\triangle = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator.

Denote

$$\partial\Omega_+ = \{x \in \partial\Omega : x_1 \geq 0\}, \quad \partial\Omega_- = \{x \in \partial\Omega : x_1 \leq 0\}, \quad I = \{x \in \partial\Omega : x_1 = 0\}.$$

In the domain Q_T we consider a following problem:

$$\frac{\partial^2 u(t,x)}{\partial t^2} - L_x u(t,x) = f(t,x), \quad (t,x) \in Q_T, \tag{1}$$

$$u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \quad x \in \overline{\Omega}, \tag{2}$$

$$u(t,x) + (-1)^k u(t,Sx) = 0, \quad 0 \leq t \leq T, \quad x \in \partial\Omega_+, \tag{3}$$

$$\partial_\nu u(t,x) - (-1)^k \partial_\nu u(t,Sx) = 0, \quad 0 \leq t \leq T, \quad x \in \partial\Omega_+, \tag{4}$$

where k takes one of the values ± 1 , $\partial_\nu = \frac{\partial}{\partial r}$ is the normal vector, $r = |x|$, $\varphi(x)$ and $\psi(x)$ are the given functions.

A classical solution to problem (1)–(4) is a function $u(t,x)$ from the class $C^{2,2}_{t,x}(Q_T) \cap C^{1,1}_{t,x}(\overline{Q}_T)$ satisfying conditions (1)–(4) in the usual sense.

1 Initial-boundary value problem with Dirichlet boundary condition

In this section we present the well-known statements from V.A. Ilyin's paper [24] regarding the initial-boundary value problem for the classical wave equation

$$\triangle z(t,x) - \frac{1}{a^2}\frac{\partial^2 z(t,x)}{\partial t^2} = -f(t,x), \quad f(t,x) \in Q_T. \tag{5}$$

For equation (5), problems with initial conditions

$$z(0,x) = \tau(x), \quad z_t(0,x) = \rho(x), \quad x \in \overline{\Omega}, \tag{6}$$

and with the Dirichlet boundary condition

$$z(t,x) = 0, \quad [0,T] \times \partial\Omega, \tag{7}$$

or with the Neumann boundary condition

$$\partial_\nu z(t,x)=0,\quad [0,T]\times\partial\Omega, \tag{8}$$

were studied.

A classical solution to the problem with conditions (5)–(7) (or with conditions (5), (6) and (8)) is a function $z(t,x)$ from the class $C^{2,2}_{t,x}(Q_T)\cap C^{1,1}_{t,x}(\overline{Q}_T)$ satisfying conditions (5)–(7) (or with conditions (5), (6) and (8)) in the usual sense. The following assertions are proved.

Lemma 1. [24] Let the functions $\tau(x),\rho(x)$ and $f(t,x)$ satisfy the following conditions:

1) the function $\tau(x)$ is continuous in the domain $\overline{\Omega}$ and has continuous derivatives up to order $[n/2]+2$ and square-integrable derivatives of order $[n/2]+3$ in this domain. In addition,

$$\tau(x)=\triangle\tau(x)=...=\triangle^k\tau(x)=0,\quad k=[(n+4)/4];$$

2) the function $\rho(x)$ is continuous in the domain and has continuous derivatives up to order $[n/2]+1$ and square-integrable derivatives of order $[n/2]+2$ in this domain. In addition,

$$\rho(x)=\triangle\rho(x)=...=\triangle^k\rho(x)=0,\quad k=[(n+2)/4];$$

3) the function $f(t,x)$ is continuous in a closed cylinder $\overline{Q}_T=\overline{\Omega}\times[0,T]$ and has continuous derivatives up to order $[n/2]+1$ and square-integrable derivatives of order $[n/2]+2$ in this cylinder. In addition,

$$f(t,x)=\triangle f(t,x)=...=\triangle^k f(t,x)=0,\quad k=[(n+2)/4].$$

Then, a classical solution to problem (5)–(7) exists, is unique, and can be represented as

$$z(t,x)=\sum_{m=1}^{\infty}\left\{\tau_m\cos a\sqrt{\mu_m}t+\frac{\rho_m}{a\sqrt{\mu}_m}\sin a\sqrt{\mu_m}t\right\}z_{m,D}(x)+$$

$$+\sum_{m=1}^{\infty}\left\{\frac{a}{\sqrt{\mu}_m}\int_0^t f_m(s)\sin a\sqrt{\mu_m}t-s\right\}z_{m,D}(x).$$

Here $z_{m,D}(x)$ are normalized eigenfunctions of the Dirichlet problem

$$\triangle z(x)+\mu z(x)=0,\quad x\in\Omega,\quad z(x)=0,\quad x\in\partial\Omega, \tag{9}$$

and τ_m, ρ_m, and $f_m(t)$ are Fourier coefficients in the expansion of functions $\tau(x),\rho(x)$ and $f(t,x)$ in the system $z_{m,D}(x)$, i.e., $\tau_m=(\tau_m,z_{m,D}(x))$, $\rho_m=(\rho_m,z_{m,D}(x))$ and $f_m=(f_m,z_{m,D}(x))$.

Lemma 2. [24] Let the functions $\tau(x),\rho(x)$ and $f(t,x)$ satisfy the following conditions:

1) the function $\tau(x)$ is continuous in the domain $\overline{\Omega}$ and has continuous derivatives up to order $[n/2]+2$ and square-integrable derivatives of order $[n/2]+3$ in this domain. In addition,

$$\tau(x)=\triangle\tau(x)=...=\triangle^k\tau(x)=0,\quad k=[(n+2)/4];$$

2) the function $\rho(x)$ is continuous in the domain and has continuous derivatives up to order $[n/2]+1$ and square-integrable derivatives of order $[n/2]+2$ in this domain. In addition,

$$\rho(x)=\triangle\rho(x)=...=\triangle^k\rho(x)=0,\quad k=[n/4];$$

3) the function $f(t,x)$ is continuous in a closed cylinder $\overline{Q}_T = \overline{\Omega} \times [0,T]$ and has continuous derivatives up to order $[n/2]+1$ and square-integrable derivatives of order $[n/2]+2$ in this cylinder. In addition,

$$f(t,x) = \triangle f(t,x) = ... = \triangle^k f(t,x) = 0, \quad k = [n/4].$$

Then, a classical solution to the initial boundary value problem for equation (5) with conditions (6), (8) exists, is unique and can be represented as

$$z(t,x) = \sum_{m=1}^{\infty} \left\{ \tau_m \cos a\sqrt{\mu_m} t + \frac{\rho_m}{a\sqrt{\mu}_m} \sin a\sqrt{\mu_m} t \right\} z_{m,N}(x) +$$

$$+ \sum_{m=1}^{\infty} \left\{ \frac{a}{\sqrt{\mu}_m} \int_0^t f_m(s) \sin a\sqrt{\mu_m}(t-s) ds \right\} z_{m,N}(x).$$

Here $z_{m,N}(x)$ are normalized eigenfunctions of the Neumann problem

$$\triangle z(x) + \mu z(x) = 0, x \in \Omega, z(x) = 0, x \in \partial\Omega, \tag{10}$$

and τ_m, ρ_m and $f_m(t)$ are Fourier coefficients in the expansion of functions $\tau(x), \rho(x)$ and $f(t,x)$ in the system $z_{m,N}(x)$.

Further, we present some properties of eigenfunctions $z_{m,D}(x)$ and $z_{m,N}(x)$. In [21], the following statement is proved.

Lemma 3. All eigenfunctions of the Dirichlet problem (9) and the Neumann problem (10) can be chosen so that they have one of the symmetry properties:

$$z(x) - z(Sx) = 0, \tag{11}$$

or

$$z(x) + z(Sx) = 0. \tag{12}$$

2 The main problem

Let $u(t,x)$ be a solution to problem (1)–(4) in the case $k=1$. From equation (1) we obtain the system

$$\begin{cases} u_t(t,x) - a_0 \triangle u(t,x) - a_1 \triangle u(t,Sx) = f(t,x), \\ u_t(t,Sx) - a_1 \triangle u(t,x) - a_0 \triangle u(t,Sx) = f(t,Sx). \end{cases} \tag{13}$$

We denote the operator of the type $I_S u(t,x) = u(t,Sx)$ as I_S. In [25] it was proved that if S is an orthogonal matrix, then the operator I_S commutes with the operators $\triangle$ and $\Lambda \equiv r\frac{\partial}{\partial r}$, where $r=|x|$. In our case, the mapping matrix S is orthogonal and therefore from (13) it follows that

$$f(t,x) + f(t,Sx) = u_t(t,x) - a_0 \triangle u(t,x) - a_1 \triangle u(t,Sx) + u_t(t,Sx) - a_1 \triangle u(t,x) - a_0 \triangle u(t,Sx) =$$
$$= \partial_t[u(t,x) + u(t,Sx)] - a_o \triangle [u(t,x) + u(t,Sx)] - a_1 \triangle [u(t,x) + u(t,Sx)] =$$
$$= \partial_t[u(t,x) + u(t,Sx)] - (a_0 + a_1) \triangle [u(t,x) + u(t,Sx)],$$

$$f(t,x) - f(t,Sx) = u_t(t,x) - a_0 \triangle u(t,x) - a_1 \triangle u(t,Sx) - [u_t(t,Sx) - a_1 \triangle u(t,x) - a_0 \triangle u(t,Sx)] =$$
$$= \partial_t[u(t,x) - u(t,Sx)] - a_o \triangle [u(t,x) - u(t,Sx)] - a_1 \triangle [u(t,x) - u(t,Sx)] =$$
$$= \partial_t[u(t,x) - u(t,Sx)] - (a_0 - a_1) \triangle [u(t,x) - u(t,Sx)].$$

Let us introduce the notations

$$v(t,x)=\frac{1}{2}[u(t,x)+u(t,Sx)],\quad w(t,x)=\frac{1}{2}[u(t,x)-u(t,Sx)].$$

It is obvious that $u(t,x)=v(t,x)+w(t,x)$ and for all $x\in\Omega$ the symmetry properties

$$v(t,Sx)=v(t,x),\quad w(t,Sx)=-w(t,x)$$

are satisfied.

Then, for the functions $v(t,x)$ and $w(t,x)$, we obtain the following equations

$$v_{tt}(t,x)-(a_0+a_1)\triangle v(t,x)=f^{+}(t,x),\quad w_{tt}(t,x)-(a_0-a_1)\triangle w(t,x)=f^{-}(t,x),$$

where $2f^{\pm}(t,x)=f(t,x)\pm f(t,Sx)$.

From initial conditions (2) for the functions $v(t,x)$ and $w(t,x)$, we obtain

$$v(0,x)=\frac{1}{2}[u(0,x)+u(0,Sx)]=\frac{1}{2}[\varphi(x)+\varphi(Sx)]\equiv\varphi^{+}(x),$$

$$v_t(0,x)=\frac{1}{2}[u_t(0,x)+u_t(0,Sx)]=\frac{1}{2}[\psi(x)+\psi(Sx)]\equiv\psi^{+}(x),$$

$$w(0,t)=\frac{1}{2}[u(0,x)-u(0,Sx)]=\frac{1}{2}[\varphi(x)-\varphi(Sx)]\equiv\varphi^{-}(x),$$

$$w_t(0,x)=\frac{1}{2}[u_t(0,x)-u_t(0,Sx)]=\frac{1}{2}[\psi(x)-\psi(Sx)]\equiv\psi^{-}(x).$$

Further, from boundary condition (3) it follows that if $0\le t\le T,\ x\in\partial\Omega_{+}$, then

$$v(t,x)\mid_{t\in[0,T],x\in\Omega_{+}}=u(t,x)+u(t,Sx)\mid_{t\in[0,T],x\in\partial\Omega_{+}}=0,$$

and if $x\in\partial\Omega_{-}$, then $Sx\in\partial\Omega_{+}$, hence

$$v(t,x)\mid_{t\in[0,T],x\in\partial\Omega_{-}}=u(t,x)+u(t,Sx)\mid_{t\in[0,T],x\in\partial\Omega_{-}}=u(t,Sx)+u(t,x)\mid_{t\in[0,T],Sx\in\partial\Omega_{+}}=0.$$

Thus, for the function $v(t,x)$ for all $t\in[0,T]$ and $x\in\partial\Omega$, we have $v(t,x)=0$.

From the symmetry properties of functions $v(t,x)$ and $w(t,x)$, we get the following equalities:

$$\partial_\nu v(t,Sx)\mid_{\partial\Omega}=\Lambda v(t,Sx)\mid_{\partial\Omega}=\Lambda v(t,x)\mid_{\partial\Omega}=\partial_\nu v(t,x)\mid_{\partial\Omega},$$

$$\partial_\nu w(t,Sx)\mid_{\partial\Omega}=\Lambda w(t,Sx)\mid_{\partial\Omega}=-\Lambda w(t,x)\mid_{\partial\Omega}=-\partial_\nu w(t,x)\mid_{\partial\Omega}.$$

Then from boundary condition (4) for the function $w(t,x)$ for all $t\in[0,T]$ and $x\in\partial\Omega$, we obtain the following edge condition

$$\partial_\nu w(t,x)=0.$$

Hence, if $u(t,x)$ is a solution to problem (1)–(4) for $k=1$, then the function $v(t,x)$ satisfies the conditions of the problem

$$v_{tt}(t,x)-(a_0+a_1)\triangle v(t,x)=f^{+}(t,x),(t,x)\in Q_T, \tag{14}$$

$$v(0,x)=\varphi^{+}(x),\quad v_t(0,x)=\psi^{+}(x),\quad x\in\overline{\Omega}, \tag{15}$$

$$v(t,x)=0,\quad [0,T]\times\partial\Omega. \tag{16}$$

Therefore, the function $w(t,x)$ satisfies the conditions of the problem

$$w_{tt}(t,x) - (a_0 - a_1)\triangle v(t,x) = f^-(t,x), \quad (t,x) \in Q_T, \tag{17}$$

$$w(0,x) = \psi^-(x), \quad w_t(0,x) = \psi^-(x), \quad x \in \overline{\Omega}, \tag{18}$$

$$\partial_\nu w(t,x) = 0, \quad [0,T] \times \partial\Omega. \tag{19}$$

Thus, we have obtained two auxiliary initial-boundary value problems for the classical wave equation. In the first problem, the boundary condition is specified in the form of the Dirichlet condition, and in the second problem, in the form of the Neumann condition.

Further, we assume that $a_0 \pm a_1 > 0$ and rewrite equations (14) and (17) as

$$\triangle v(t,x) - \frac{1}{\varepsilon_0^2} v_{tt}(t,x) = -\frac{1}{\varepsilon_0^2} f^+(t,x), \quad (t,x) \in Q_T,$$

$$\triangle w(t,x) - \frac{1}{\varepsilon_1^2} w_{tt}(t,x) = -\frac{1}{\varepsilon_1^2} f^-(t,x), \quad (t,x) \in Q_T,$$

where $\varepsilon_0 = \sqrt{a_0 + a_1}$, $\varepsilon_1 = \sqrt{a_0 - a_1}$.

To study the solvability of problem (14)–(16), we can use the assertion of Lemma 1. If the functions $f^+(t,x)$, $\varphi^+(x)$ and $\psi^+(x)$ satisfy the conditions of this lemma, then the classical solution to problem (14)–(16) exists, is unique, and can be represented as

$$v(t,x) = \sum_{m=1}^{\infty} \left\{ \varphi_m^+ \cos \varepsilon_0 \sqrt{\mu_{m,D}} t + \frac{\psi_m^+}{\varepsilon_0 \sqrt{\mu_{m,D}}} \sin \varepsilon_0 \sqrt{\mu_{m,D}} t \right\} z_{m,D}(x) +$$
$$+ \sum_{m=1}^{\infty} \left\{ \frac{1}{\varepsilon_0 \sqrt{\mu_{m,D}}} \int_0^t f_m^+(s) \sin \varepsilon_0 \sqrt{\mu_{m,D}} (t-s) ds \right\} z_{m,D}(x), \tag{20}$$

where $\varphi_m^+ = (\varphi^+, z_{m,D})$, $\psi_m^+ = (\psi^+, z_{m,D})$ and $f_m^+(t) = (f^+, z_{m,D})$.

Similarly, if the functions $f^-(t,x)$, φ^- and ψ^- satisfy the conditions of Lemma 2, then the classical solution to problem (17)-(18) exists, is unique, and is represented as

$$w(t,x) = \sum_{m=1}^{\infty} \left\{ \varphi_m^- \cos \varepsilon_1 \sqrt{\mu_{m,N}} t + \frac{\psi_m^-}{\varepsilon_1 \sqrt{\mu_{m,N}}} \sin \varepsilon_1 \sqrt{\mu_{m,N}} t \right\} z_{m,N}(x) +$$
$$+ \sum_{m=1}^{\infty} \left\{ \frac{1}{\varepsilon_1 \sqrt{\mu_{m,N}}} \int_0^t f_m^-(s) \sin \varepsilon_1 \sqrt{\mu_{m,N}} (t-s) ds \right\} z_{m,N}(x), \tag{21}$$

where $\varphi_m^- = (\varphi^-, z_{m,D})$, $\psi_m^+ = (\psi^+, z_{m,D})$ and $f_m^-(t) = (f^-, z_{m,D})$.

Let us transform the functions $v(t,x)$ and $w(t,x)$ from equalities (20) and (21). To do this, we use the properties of the eigenfunctions $z_{m,D}(x)$ and $z_{m,N}(x)$ formulated in Lemma 3. In this case, we renumber the eigenfunctions $z_{m,D}(x)$ as follows: we denote the eigenfunctions with property (11) as $z_{2m,D}(x)$, and the eigenfunctions with property (12) as $z_{2m-1,D}(x)$. We will use a similar notation for the eigenfunctions $z_{m,D}(x)$ and $z_{m,N}(x)$.

Then, for the coefficients of the function $\varphi(x)$, we have

$$\varphi_m^+ = \frac{1}{2} \int\limits_{\Omega} [\varphi(x) + \varphi(Sx)] z_{m,D}(x) dx = \frac{1}{2} \int\limits_{\Omega} \varphi(x) [z_{m,D}(x) + z_{m,D}(Sx)] dx.$$

Further, if $m = 2j-1,\ j = 1,2,...$, then $z_{2j-1,D}(x) + z_{2j-1,D}(Sx) = 0$ and if $m = 2j,\ j = 1,2,...$, then $z_{2j,D}(x) + z_{2j,D}(Sx) = 2z_{2j,D}(x)$, thus

$$\varphi_{2m}^{+} = \int\limits_{\Omega} \varphi(x) z_{m,D}(x) dx = \varphi_{2m}, \quad m \geq 1.$$

Similarly, for the coefficients φ_{2m-1}^{-}, we obtain the equalities

$$\varphi_{2m-1}^{-} = \int\limits_{\Omega} \varphi(x) z_{2m-1,N}(x) dx = \varphi_{2m-1}, \quad m \geq 1.$$

Similar equalities can be obtained for the coefficients $f_m^{\pm}(t,x)$ and $\psi^{-}(x)$:

$$\psi_{2m}^{+} = \psi_{2m} \equiv (\psi, z_{2m,D}), \quad \psi_{2m-1}^{-} = \psi_{2m-1} \equiv (\psi, z_{2m-1,N}),$$

$$f_{2m}^{+}(t) = f_{2m} \equiv (f, z_{2m,D}), \quad f_{2m-1}^{-}(t) = f_{2m-1} \equiv (f, z_{2m-1,N}).$$

Then, formula (20), or more precisely the solution to problem (14)–(16) can be rewritten as

$$v(t,x) = \sum_{m=1}^{\infty} \left\{ \varphi_{2m} \cos \varepsilon_0 \sqrt{\mu_{2m,D}} t + \frac{\psi_{2m}}{\varepsilon_0 \sqrt{\mu_{2m,D}}} \sin \varepsilon_0 \sqrt{\mu_{2m,D}} t \right\} z_{2m,D}(x) +$$
$$+ \sum_{m=1}^{\infty} \left\{ \frac{1}{\varepsilon_0 \sqrt{\mu_{2m,D}}} \int_0^t f_{2m}(s) \sin \varepsilon_0 \sqrt{\mu_{2m,D}} (t-s) ds \right\} z_{2m,D}(x), \quad (22)$$

and formula (21) as

$$w(t,x) = \sum_{m=1}^{\infty} \left\{ \varphi_{2m-1} \cos \varepsilon_1 \sqrt{\mu_{2m-1,N}} t + \frac{\psi_{2m-1}}{\varepsilon_1 \sqrt{\mu_{2m-1,N}}} \sin \varepsilon_1 \sqrt{\mu_{2m-1,N}} t \right\} z_{2m-1,N}(x) +$$
$$+ \sum_{m=1}^{\infty} \left\{ \frac{1}{\varepsilon_1 \sqrt{\mu_{2m-1,N}}} \int_0^t f_{2m-1}(s) \sin \varepsilon_1 \sqrt{\mu_{2m-1,N}} (t-s) ds \right\} z_{2m-1,N}(x). \quad (23)$$

Now we present the main assertion regarding problem (1)–(4).

Theorem 1. Let $k = 1, a_0 \pm a_1 > 0$, functions $f(t,x)$, $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Lemma 1. Then, the classical solution to problem (1)–(4), exists, is unique, and can be represented as

$$u(t,x) = \sum_{m=1}^{\infty} \left\{ \varphi_{2m} \cos \sqrt{(a_0+a_1)\,\mu_{2m,D}} t + \frac{\psi_{2m}}{\sqrt{(a_0+a_1)\,\mu_{2m,D}}} \sin (a_0+a_1) \sqrt{\mu_{2m}} t \right\} z_{2m,D}(x) +$$
$$+ \sum_{m=1}^{\infty} \left\{ \varphi_{2m-1} \cos \sqrt{(a_0-a_1)\,\mu_{2m-1,N}} t + \frac{\psi_{2m-1}}{\sqrt{(a_0-a_1)\,\mu_{2m-1,N}}} \sin \sqrt{(a_0-a_1)\,\mu_{2m-1,N}} t \right\} z_{2m-1,N}(x) +$$
$$+ \sum_{m=1}^{\infty} \left\{ \frac{1}{\sqrt{(a_0+a_1)\,\mu_{2m,D}}} \int_0^t f_{2m}(s) \sin \sqrt{(a_0+a_1)\,\mu_{2m,D}} (t-s) ds \right\} z_{2m,D}(x) +$$
$$+ \sum_{m=1}^{\infty} \left\{ \frac{1}{\sqrt{(a_0-a_1)\,\mu_{2m-1,N}}} \int_0^t f_{2m-1}(s) \sin \sqrt{(a_0-a_1)\,\mu_{2m-1,N}} (t-s) ds \right\} z_{2m-1,N}(x). \quad (24)$$

Proof. If the functions $f(t,x)$, $\varphi(x)$ and $\psi(x)$ satisfy the conditions of Lemma 1, then the functions $f^+(t,x)$, $\varphi^+(x)$ and $\psi^+(x)$ satisfy the same conditions. Then, by the assertion of Lemma 1, the solution to problem (14)–(16) exists, is unique, and can be represented in the form (20). If the functions $f^-(t,x)$, $\varphi^-(x)$ and $\psi^-(x)$ satisfy the conditions of Lemma 1, they also satisfy the conditions of Lemma 2. Then, by the assertion of Lemma 2, the solution to problem (17)–(19) with functions exists, is unique, and can be represented in the form (21). Note that the functions $v(t,x)$ and $w(t,x)$ from equalities (22) and (23) have the symmetry properties $v(t,Sx) = v(t,x)$ and $w(t,Sx) = -w(t,x)$. We will show that the function $u(t,x) = v(t,x) + w(t,x)$ will be a classical solution to problem (1)–(4).

Indeed, the following equalities hold for this function

$$u_{tt}(t,x) - L_x u(t,x) =$$

$$= v_{tt}(t,x) - a_0 \triangle v(t,x) - a_1 \triangle v(t,Sx) + w_{tt}(t,x) - a_0 \triangle w(t,x) - a_1 \triangle w(t,Sx) =$$

$$= v_{tt}(t,x) - (a_0 + a_1) \triangle v(t,x) + w_{tt}(t,x) - (a_0 - a_1) \triangle w(t,x) =$$

$$= f^+(t,x) + f^-(t,x) = f(t,x),$$

$$u(0,x) = v(0,x) + w(0,x) = \varphi^+(x) + \varphi^-(x) = \varphi(x), \quad x \in \overline{\Omega},$$

$$u_t(0,x) = v_t(0,x) + w_t(0,x) = \psi^+(x) + \psi^-(x) = \psi(x), \quad x \in \overline{\Omega}.$$

From the symmetry conditions, as well as from boundary conditions (16) and (19) for $x \in \partial\Omega_+$ for $k = 1$, we obtain

$$u(t,x) + u(t,Sx) = v(t,x) + w(t,x) + v(t,Sx) + w(t,Sx) =$$

$$= [v(t,x) + v(t,Sx)] + [w(t,x) + w(t,Sx)] = 2v(t,x) + [w(t,x) - w(t,Sx)] = 0$$

and

$$\partial_\nu u(t,x) - \partial_\nu u(t,Sx) = \partial_\nu \left[v(t,x) - v(t,Sx)\right] + \partial_\nu \left[w(t,x) + w(t,Sx)\right] =$$

$$= \partial_\nu [0] + \partial_\nu w(t,x) = 0.$$

Thus, boundary conditions (3) and (4) are also satisfied. Then, substituting the values of the functions $v(t,x)$ and $w(t,x)$ from equalities (22) and (23) into the left-hand side of the equality $u(t,x) = v(t,x) + w(t,x)$, we obtain representation (24). The theorem is proved.

We conduct similar studies in the case $k = 2$. In this case, if we choose functions $v(t,x)$ and $w(t,x)$ in the form (13), then we obtain a problem with conditions (14), (15) and the Neumann boundary condition $\partial_\nu v(t,x) = 0$, $[0,T] \times \partial\Omega$.

Hence, for the function $w(t,x)$, we obtain a problem with conditions (17), (18) and the Dirichlet boundary condition $w(t,x) = 0, [0,T] \times \partial\Omega$. The main assertion regarding problem (1)–(4) in the case $k = 2$ is the following theorem.

Theorem 2. Let $k = 2, a_0 \pm a_1 > 0$, functions $f(t,x), \varphi(x)$ and $\psi(x)$ the functions and satisfy the conditions of Lemma 1. Then the classical solution to problem (1)–(4) exists, is unique and can be represented in the form

$$u(t,x)=\sum_{m=1}^{\infty}\left\{\varphi_{2m-1}\cos\sqrt{(a_0+a_1)\mu_{2m-1,D}}t+\frac{\psi_{2m-1}}{\sqrt{(a_0+a_1)\mu_{2m-1,D}}}\sin(a_0+a_1)\sqrt{\mu_{2m}}t\right\}z_{2m-1,D}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\varphi_{2m}\cos\sqrt{(a_0-a_1)\mu_{2m,N}}t+\frac{\psi_{2m}}{\sqrt{(a_0-a_1)\mu_{2m,N}}}\sin(a_0-a_1)\sqrt{\mu_{2m,N}}t\right\}z_{2m,N}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\frac{1}{\sqrt{(a_0+a_1)\mu_{2m-1,D}}}\int_0^t f_{2m-1}(s)\sin\sqrt{(a_0+a_1)\mu_{2m-1,D}}(t-s)ds\right\}z_{2m-1,D}(x)+$$
$$+\sum_{m=1}^{\infty}\left\{\frac{1}{\sqrt{(a_0-a_1)\mu_{2m,N}}}\int_0^t f_{2m}(s)\sin\sqrt{(a_0-a_1)\mu_{2m,N}}(t-s)ds\right\}z_{2m,N}(x).$$

Conclusion

In this paper, the initial-boundary value problem for an analogue of a hyperbolic equation with involution is studied in a multidimensional circular cylinder. Periodic and antiperiodic conditions are considered as boundary conditions. The unknown function is represented as the sum of an even and odd part with respect to the involution transformation. For auxiliary functions, initial-boundary functions for the classical hyperbolic equation are obtained. Using known assertions for the obtained problems, theorems on the existence and uniqueness of the main problems are proved.

It is planned to study similar problems for analogues of hyperbolic equations with multiple involution.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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