MATHEMATICS

https://doi.org/10.31489/2025M2/4-15 *Research article*

Boundary Value Problems on a Star Thermal Graph and their Solutions

L.A. Alexeyeva[1,2], A.N. Dadayeva[1,3,*], D.A. Prikazchikov[4], N.Zh. Ainakeyeva[1,5]

[1] *Joldasbekov Institute of Mechanics and Engineering, Almaty, Kazakhstan;*
[2] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[3] *Kazakh National Research Technical University named after K.I. Satpayev, Almaty, Kazakhstan;*
[4] *Keele University, Staffordshire, United Kingdom;*
[5] *Narxoz University, Almaty, Kazakhstan*
(E-mail: alexeeva@math.kz, dady1262@mail.ru, d.prikazchikov@keele.ac.uk, nursaule_math@mail.ru)

In this study, heat conductivity boundary value problems on a star graph are considered, inspired by engineering applications, e.g., heat conduction phenomena in mesh-like structures. Based on the generalized function method, a unified technique for solving boundary value problems on such graphs is developed. Generalized solutions to transient and stationary boundary value problems are constructed for different conditions at the end edges, with the Kirchhoff conditions at the common node. Regular integral representations of solutions to boundary value problems are obtained using the properties and symmetry of the fundamental solution's Fourier transform. The derived results allow the action of various heat sources to be simulated, including concentrated ones by using singular generalized functions. The generalized function method enables a wide variety of boundary value problems to be tackled, including those with local boundary conditions at the ends of the graph, and various transmission conditions at the common node. Based on the research, the authors propose an analytical solution method under the action of various heat sources to solve various boundary value problems on a star thermal graph.

Keywords: star graph, temperature, heat flow, transmission conditions, generalized functions method, generalized solution, Fourier transform, boundary equations.

Mathematics Subject Classification: 35M10, 35K05, 35L05, 94C15.

Introduction

As a branch of applied mathematics, graph theory has wide applications in subjects such as economics, logistics, sociology, optimal control, and navigation. The properties of graphs are also actively used to solve boundary value problems (BVPs) on network-like structures, e.g., oil pipelines, gas pipelines, and electrical networks. The concept of graphs was first introduced by Leonhard Euler in 1736. In his work [1], he pioneered the approach of graph theory to solve the famous problem of the Königsberg bridges. At the beginning of the 20th century, the Hungarian mathematician Dénes König

*Corresponding author. *E-mail: dady1262@mail.ru*

This research was funded by the Ministry of Science and Higher Education of Kazakhstan (grants: AP19674789, 2023–2025; AP23488145, 2024–2026).

Received: 25 November 2024; *Accepted:* 27 March 2025.

was the first to propose the term “graph” in reference to diagrams containing a discrete set of points connected by straight lines and began to study their general properties. In 1936, he published the first book on graph theory [2] that reviewed all related works from 1736 onwards (see also [3]).

Graph Theory (GT) has potential applications in the field of Computer Science (CS) for various purposes. Unique applications of GT in the field of CS, such as web document clustering, cryptography, and algorithm execution analysis, among others, are promising applications. Further, the concepts of GT can be used to simplify and analyze electronic circuits. Recently, graphs have been widely used in social networks (SN) for many purposes related to modeling and analyzing SN structures, modeling SN operation, analyzing SN users, and many other related aspects [4–9]. Determining a distributed parameter of conductance for a neuronal cable model on a tree graph was researched in [10, 11].

The development of analytical methods for the mathematical modeling of network-like structures is based on the solution and analysis of the corresponding direct and inverse problems for equation systems with parameters that are distributed on a graph. These methods primarily analyze the spectral completeness and basis properties of the corresponding BVPs’ eigenfunctions in the space of square integrable functions, in addition to finding conditions for their unique solutions. The field of theory differential equations on networks is relatively new, with most articles on the subject focused on the direct problems of spectral theory.

The theory of ordinary differential equations on networks gave rise to research on the theory of inverse problems on geometric graphs, which can be traced to the articles of V. Yurko [12, 13]. The spectral problems for Sturm-Liouville operators on star graphs were also considered in [14–16]. A number of other contributions are devoted to different types of differential equations on graphs. In [17], discrete Yamabe equations on star graphs were considered, in [18, 19] the fractional boundary value problem on a graph was studied.

A new class of partial differential equations on graphs of different structures has only begun to be studied in the last decade. Boundary value problems for the d’Alembert wave equation, their spectral properties, and the issues of existence and uniqueness of solutions, as well as a number of inverse problems, were considered in the works [20–24].

The studies of boundary value problems for heat conduction equations on graphs appeared relatively recently. They are largely related to the solution of thermoelasticity problems on graphs for studying the strength properties of rod structures, widely used in mechanical engineering, robotics, and construction. Their solutions are based on the solution of systems of equations of spatially one-dimensional thermoelasticity problems [25–27].

Mathematical modeling of the process of heat propagation in a system of rods on a “tree-like” graph in the form of a bundle of linear differential operators was carried out by Yu. Martynova [28], who posed the inverse problem of finding the parameters of boundary conditions for given eigenvalues. Here we consider boundary value problems for the heat equation on an undirected star graph with an arbitrary number of links of different lengths, which may have different thermal characteristics. An analytical solution to the posed boundary value problems is constructed for given boundary conditions at the ends of the graph and a known total heat flow in its node. The constructed solution allows us to determine the temperature and heat flows in any link of the graph under periodic and non-stationary thermal effects. Other authors have not yet constructed such solutions to boundary value problems on thermal graphs.

We used the generalized function method to solve boundary value problems, leading to a differential equation solution with a singular right-hand side containing simple and double layers, the densities of which are determined by the boundary and initial conditions of the solution. The solution is constructed as the convolution of the Green’s function of the equation with the appropriate right-hand side in every edge. To determine the unknown boundary values of the solution and its derivatives, connection equations are constructed boundary functions at the edge ends, employing the asymptotic properties of

Green's functions and their derivatives at zero. To construct a closed system of equations, the obtained algebraic equations for each edge of the graph are supplemented with transmission conditions at the node and linear boundary conditions at its ends. These conditions can be either locally or not locally connected. Thus, the proposed method is applicable to a wide range of BVPs, including those on mesh structures.

1 *Statement of the boundary value problem on a heat star graph*

We consider a heat star graph which contains N edges (A_0, A_j) of the length L_j $(j = 1, 2, ..., N)$ with a common node A_0 (Fig. 1). On each edge $S_j = \{x \in R^1 : 0 \le x \le L_j\}$, there is a coordinate system (x, t) whose origin can be found at the point $A_0 : x = 0$. At S_j, the temperature $\theta_j(x, t)$ satisfies the following heat conduction equation:

$$\frac{\partial \theta_j}{\partial t} - \kappa_j \frac{\partial^2 \theta_j}{\partial x^2} = F_j(x, t), \quad 0 \le x \le L_j, \quad t \ge 0. \tag{1}$$

Here, κ_j is the heat diffusivity coefficient, $F_j(x, t)$ describes the power of acting heat sources on the j-edge.

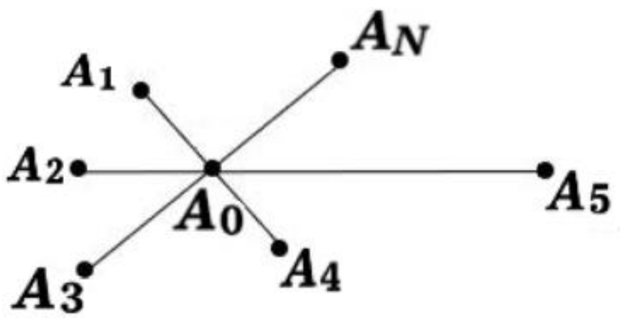

Figure 1. Star graph

The initial conditions at t = 0 for the temperature of the graph are as follows:
(Cauchy conditions)

$$\theta_j(x, 0) = \theta_0^j(x), \quad 0 \le x \le L_j, \quad \forall j, \tag{2}$$

where $\theta_0^j(x) \in C^2(R^1)$. Next, we consider the following boundary value problem (BVP).

Dirichlet problem. Temperature values are known at the ends of the graph:

$$\theta_j(L_j, 0) = \theta_2^j(t), \quad 0 \le x \le L_j, \quad t \ge 0. \tag{3}$$

The following continuity conditions and the Kirchhoff condition are specified in the common node A_0 of the graph:

$$\theta_1^1(t) = \theta_1^2(t) = \ldots = \theta_1^N(t), \quad x = 0, \quad t \ge 0, \tag{4}$$

$$\kappa_1 q_1^1(t) + \kappa_2 q_1^2(t) + \ldots + \kappa_N q_1^N(t) = 0. \tag{5}$$

Here $q_1^j(t) = \frac{\partial \theta_j}{\partial x}|_{x=0}$, $q_2^j(t) = \frac{\partial \theta_j}{\partial x}|_{x=L}$, the superscript indicates the number of the graph edge and $\theta_1^j(t) = \theta^j(0, t)$, $\theta_2^j(t) = \theta^j = (L_j, t)$ designate the temperatures at the ends of the j-th edge, $1 \le j \le N$.

We need to find the solution to the Dirichlet problem on this star graph.

2 Statement of the boundary value problem on the edge of a graph

To define the connection between the boundary values of temperature and heat flows on each element of a graph, at first we construct the solution to the boundary value problem (BVP) on the segment $[0, L]$.

Let's define $\theta(x,t)$, which is the solution of the heat equation:

$$\frac{\partial \theta}{\partial t} - \kappa \frac{\partial \theta}{\partial x^2} = F(x,t),\ 0 \leq x \leq L,\ t \geq 0. \tag{6}$$

The *initial conditions* are as follows, where the temperature is known at $t = 0$:

$$\theta(x,0) = \theta_0(x),\ \theta_0(x) \in C^2(R^1)\left\{0 \leq x \leq L\right\}. \tag{7}$$

We denote

$$\begin{aligned} \theta(0,t) = \theta_1(t),\ x = 0,\ t \geq 0, \\ \theta(L,t) = \theta_2(t),\ x = L,\ t \geq 0. \end{aligned} \tag{8}$$

The following matching conditions apply to the initial and boundary conditions:

$$\theta_1(t) = \theta_0(0), \quad \theta_2(t) = \theta_0(L).$$

We can construct the solutions of this BVP using the generalized function method [29].

Boundary conditions can be written for different BVP in a generalized form:

$$\begin{cases} (\alpha_1\theta_1 + \beta_1\Pi_1(t))|_{x=0} = G_1(t), \\ (\alpha_2\theta_2 + \beta_2\Pi_2(t))|_{x=L} = G_2(t), \end{cases} \tag{9}$$

where $\theta_j(t),\ \Pi_j(t) = -k\frac{\partial \theta}{\partial x}|_{x=x_j}$ $(j = 1, 2)$ are the temperature and heat flows at the ends of segment, known functions $G_j(t) \in L_1(R_+),\ R_+ = [0, \infty)$; coefficients (α_j, β_j) are known arbitrary constants. By choosing them we can solve different BVPs. For example,

1. *Dirichlet problem:* $\alpha_1 = 1,\ \beta_1 = 0,\ \alpha_2 = 1,\ \beta_2 = 0$;
2. *Neumann problem:* $\alpha_1 = 0,\ \beta_1 = 1,\ \alpha_2 = 0,\ \beta_2 = 1$.

3 Generalized solution to boundary value problems on a graph segment using a generalized function method

To determine the solution, we consider a BVP in the space of generalized functions of slow growth $S'(R^2) = \left\{\hat{f}(x,t),\, (x,t) \in R^2\right\}$ [29, 30]. To achieve this, we introduce a regular generalized function (marked with a cap):

$$\hat{\theta}(x,t) = \begin{cases} \theta(x,t),\ (x,t) \in D^-, \\ 0,\ x \notin D^-. \end{cases}$$

Here $D^- = [0, L] \times [0, \infty)$, $\theta(x,t)$ is the classical solution to the BVP in D^-. It can be represented as $\hat{\theta}(x,t) = \theta(x,t)H(L-x)H(x)H(t)$, where $H(x)$ is the Heaviside step function:

$$H(x) = \begin{cases} 1,\ x > 0, \\ 0,\ x < 0. \end{cases}$$

To construct the equation for $\hat{\theta}(x,t)$ in $S'(R^2)$, we find the generalized derivatives of $\hat{\theta}(x,t)$:

$$\frac{\partial \hat{\theta}}{\partial x} = \frac{\partial \theta}{\partial x} H_D^-(x,t) - \theta_2(t)\delta(L-x)H(t) + \theta_1(t)\delta(x)H(t),$$

$$\frac{\partial^2\hat{\theta}}{\partial x^2} = \frac{\partial^2\theta}{\partial x^2} H_D^-(x,t) - q_2(t)\delta(L-x)H(t) + q_1(t)\delta(x)H(t) + \theta_2(t)\delta'(L-x)H(t) + \theta_1(t)H(t)\delta'(x),$$

$$\frac{\partial\hat{\theta}}{\partial t} = \frac{\partial\theta}{\partial t} H_D^-(x,t) - \theta_0(t)H(L-x)\delta(t).$$

Here $H_D^-(x,t) = H_L^-(x)H(t)$ is characteristic function of the set D^-, $H_L^-(x) = H(L-x)H(x)$ is characteristic function of $[0,L]$, $\delta(x)$ is the singular generalized δ-function, $q_1(t) = \frac{\partial\theta}{\partial x}\big|_{x=0}$, $q_2(t) = \frac{\partial\theta}{\partial x}\big|_{x=L}$. Then Eq. (6) has the following form for $\hat{\theta}(x,t)$ in $S'(R^2)$:

$$\begin{gathered}\frac{\partial\hat{\theta}}{\partial t} - \kappa\frac{\partial^2\hat{\theta}}{\partial x^2} = \hat{F}(x,t) + \kappa q_2(t)\delta(L-x)H(x)H(t) - \kappa q_1(t)H(L-x)\delta(x)H(t) - \\ -\kappa\theta_2(t)\delta'(L-x)H(x)H(t) - \kappa\theta_1(t)\delta'(x)H(L-x)H(t) + \theta_0(x)H_L^-(x)\delta(t), \\ \hat{F}(x,t) = F(x,t)H_D^-(x,t).\end{gathered} \tag{10}$$

Note that the right side of this equation includes all initial (7), (8) and boundary temperatures (9) and heat flows $\theta_j(t)\Pi_j(t) = \kappa q_j(t)$.

To shorten the formulas, we will use the following notation for partial derivatives: $u_{,x} = \frac{\partial u}{\partial x}$, $u_{,t} = \frac{\partial u}{\partial t}$.

According to the theory of generalized functions [30], the solution of Eq. (10) can be represented as a convolution of the fundamental solution to the heat equation (6) with the right-hand side of this equation:

$$\begin{gathered}\hat{\theta}(x,t) = \hat{F}_2(x,t) * U(x,t) + \kappa q_2(t)H(x)H(t) \underset{t}{*} U(L-x,t) - \\ -\kappa q_1(t)H(L-x)H(t) \underset{t}{*} U(x,t) - \kappa\theta_2(t)H(t)H(x) \underset{t}{*} U_{,x}(L-x,t) - \\ -\kappa\theta_1(t)H(L-x)H(t) \underset{t}{*} U_{,x}(x,t) + \theta_0(x)H(L-x)H(x) \underset{x}{*} U(x,t),\end{gathered} \tag{11}$$

where $U(x,t)$ is the Green's function of Eq. (6) — fundamental solution to the heat equation (6) by $F(x,t) = \delta(x)\delta(t)$ which decays at ∞ [30]:

$$U(x,t) = \frac{1}{\sqrt{2\pi\kappa t}} exp(-x^2/4\kappa t)H(t). \tag{12}$$

If $\hat{F}(x,t)$ is a regular function, then the formulae shown in (11) can be represented in the following integral form:

$$\begin{gathered}\hat{\theta}(x,t) = \theta(x,t)H(L-x)H(x)H(t) = \\ = H(x)\int_0^t d\tau \int_{-\infty}^{+\infty} U(x-y,t-\tau)F(y,\tau)dy + \kappa H(x)H(t)\int_0^t q_2(t-\tau)U(L-x,\tau)d\tau - \\ -\kappa H(L-x)H(t)\int_0^t U(x-y,t-\tau)q_1(\tau)d\tau - \kappa H(x)H(t)\int_0^t \theta_2(t-\tau)U_{,x}(L-x,\tau)d\tau - \\ -\kappa H(L-x)H(t)\int_0^t U_{,x}(x,t-\tau)\theta_1(\tau)d\tau + \int_0^L U(x-y,t)\theta_0(y)H(L-y)H(y)dy.\end{gathered} \tag{13}$$

Comment. This formula determines the temperature inside a segment using known temperatures and heat flows at its ends. For this reason, it is very useful for engineering applications, as it allows one to study the influence of boundary temperature, heat flows, and initial conditions on the temperature of a segment. In practical problems, these can be measured, and this formula gives the heat state of various rod structures under arbitrary thermal conditions.

To determine unknown boundary functions (heat flows), the resolving boundary equations should be constructed using the boundary conditions at the ends of the segment.

4 Solution of BVP in Fourier transformation space in time. Resolving system of equations

To find four boundary functions in (13) and construct the resolving system of equations, we use the Fourier transform in time:

$$\begin{gathered}\bar{\theta}(x,\omega) = F\left[\hat{\theta}(x,t)\right] = H(x)H(L-x)\int_0^{\infty} \theta(x,t)e^{i\omega t}dt, \\ \hat{\theta}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\theta}(x,\omega)e^{-i\omega t}d\omega. \end{gathered} \tag{14}$$

To define the Fourier transform of the generalized solution (13), we use the property of the Fourier transform of convolution [30]:

$$\begin{gathered}\hat{\theta}(x,\omega)\bar{F}_2(x,\omega) \underset{x}{*} \bar{U}(x,\omega) + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega) + \\ +\kappa\bar{q}_2(\omega)H(x)\bar{U}(L-x,\omega) - \kappa\bar{q}_1(\omega)H(L-x)\bar{U}(x,\omega) - \\ -\kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L-x,\omega) - \kappa\bar{\theta}_1 H(L-x)\bar{U}_{,x}(x,\omega). \end{gathered} \tag{15}$$

Here, a variable under the convolution sign $\left(\underset{x}{*}\right)$ shows convolution over the variable x. The integral representation of (15) has the following form:

$$\begin{gathered}\bar{\theta}(x,\omega)H(L-x)H(x)H(\omega) = H(x)\int_0^L \bar{U}(x-y,\omega)F_2(y,\omega)dy + \kappa H(x)\int_0^L \bar{U}(x-y,\omega)\theta_0(y)dy + \\ +\kappa\bar{q}_2(\omega)H(x)\bar{U}(L-x,\omega) - \kappa\bar{q}_1(x)H(L-x)\bar{U}(x,\omega) - \\ -\kappa\bar{\theta}_2(\omega)H(x)\bar{U}_{,x}(L-x,\omega) - \kappa\bar{\theta}_1 H(L-x)\bar{U}_{,x}(x,\omega), \end{gathered} \tag{16}$$

where the Fourier transform of the Green's function of the heat equation is equal to

$$\bar{U}(x,\omega) = -0,5\frac{sin(k|x|)}{\kappa(k+i0)}, \tag{17}$$

where $k = \sqrt{i\omega\kappa^{-1}} = e^{j\pi/4}\sqrt{\omega\kappa^{-1}} = (1+i)\sqrt{\frac{\omega}{2\kappa}}$. It satisfies the following equation:

$$\frac{d^2\bar{U}}{dx^2} + i\omega\kappa^{-1}\bar{U} = \delta(x).$$

Its derivative has a gap at the point $x = 0$, and is equal to

$$\frac{\partial\bar{U}(x,\omega)}{\partial x} = \bar{U}_{,x}(x,\omega) = -\frac{sgnx}{2\kappa}cos(\kappa|x|), \quad sgnx = \begin{cases} 1, \ x > 0, \\ -1, \ x < 0. \end{cases}$$

There are the next symmetry conditions:

$$\bar{U}(x,\omega) = \bar{U}(-x,\omega), \quad \bar{U}_{,x}(\pm 0,\omega) = \mp\frac{1}{2\kappa}. \tag{18}$$

We use these properties (17), (18) to construct the solving system of equations.

To find the unknown boundary functions, we pass in relation (16) to the limits at the ends of a segment:

$$\lim_{\varepsilon\to 0}\bar{\theta}(0+\varepsilon,\omega) = \bar{\theta}_1(x) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)|_{x=0} +$$
$$+\kappa\bar{q}_2(\omega)\bar{U}(L,\omega) - \kappa\bar{q}_1(\omega)\bar{U}(0,\omega) - \kappa\bar{\theta}_2(x)\bar{U}_x(L,\omega) - \kappa\bar{\theta}_1\bar{U}_{,x}(+0,\omega).$$

From (18) to follow: $-\kappa\bar{\theta}_1(\omega)\bar{U}_{,x}(+0,\omega) = 0,5\bar{\theta}_1(\omega)$. Next, we factor out the last term on the left side and obtain the following equation at the left end of the segment:

$$\frac{1}{2}\bar{\theta}_1(\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)|_{x=0} +$$
$$+\kappa\bar{q}_2(x)\bar{U}(L,\omega) - \kappa\bar{q}_1(\omega)\bar{U}(0,\omega) - \kappa\bar{\theta}_2(\omega)\bar{U}_{,x}(L,\omega).$$

Similarly, we consider the limit at $x = L-\varepsilon,\ \varepsilon \to +0$, and obtain the second equation on the right end of the segment. Let us formulate the obtained results.

Theorem 1. The Fourier time transformants of the boundary functions of the boundary value problems (6)–(9) satisfy the following system of linear algebraic equations:

$$\begin{bmatrix} 0,5 & 0 \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(\omega) \\ \bar{q}_1(\omega) \end{bmatrix} + \begin{bmatrix} \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ 0,5 & 0 \end{bmatrix} \begin{bmatrix} \bar{\theta}_2(\omega) \\ \bar{q}_2(\omega) \end{bmatrix} = \begin{bmatrix} \bar{Q}_1(0,\omega) \\ \bar{Q}_2(L,\omega) \end{bmatrix}, \tag{19}$$

where

$$\bar{Q}_1(0,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)|_{x=0} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)|_{x=0},$$
$$\bar{Q}_2(L,\omega) = \bar{F}(x,\omega) \underset{x}{*} \bar{U}(x,\omega)|_{x=L} + \theta_0(x)H(L-x)H(x) \underset{x}{*} \bar{U}(x,\omega)|_{x=L}.$$

The system (19) gives the possibility to determine only two boundary functions at the ends of the segment if two functions from $\bar{\theta}_1(\omega)\bar{q}_1(\omega)\bar{\theta}_2(\omega)\bar{q}_2(\omega)$ are known. If to add two boundary conditions (9) we have the full system of four linear algebraic equations to determine these boundary functions:

$$\mathbf{A}(\omega)\cdot B(\omega) = C(\omega), \tag{20}$$

where

$$\mathbf{A}(\omega) = \begin{pmatrix} 0,5 & 0 & \kappa\bar{U}_{,x}(L,\omega) & -\kappa\bar{U}(L,\omega) \\ \kappa\bar{U}_{,x}(L,\omega) & \kappa\bar{U}(L,\omega) & 0,5 & 0 \\ \alpha_1 & \beta_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 \end{pmatrix},$$
$$B(\omega) = (\bar{\theta}_1(\omega), \bar{q}_1(\omega)\bar{\theta}_2(\omega), \bar{q}_2(\omega))^T,$$
$$C(\omega) = (\bar{Q}_1(0,\omega), \bar{Q}_2(L,\omega)\bar{b}_3(\omega), \bar{b}_4(\omega))^T.$$

The solution of resolving system (20) is

$$B(\omega) = \mathbf{A}^{-1}(\omega)\cdot C(\omega), \tag{21}$$

where $\mathbf{A}^{-1}$ is the inverse matrix of $\mathbf{A}(\omega)$.

Substituting (21) into (16) we obtain the temperature transformant at any point x. Performing the inverse Fourier transform (14), we obtain $\hat{\theta}(x,t)$. Thus, the temperature $\theta(x,t)$ on the interval $[0,L]$ has been determined at any time t. We have solved BVPs.

5 *Resolving system of equations on a star thermal graph*

Let's return to the consideration of Dirichlet problem for a star heat graph (Fig. 1) by using the system (12). On each edge L_j, $j = \overline{1, N}$, we have the following system of linear algebraic equations of connection of four boundary functions $(\bar{\theta}_1^j(\omega)\bar{q}_1^j(\omega), \bar{\theta}_2^j(\omega)\bar{q}_2^j(\omega))$:

$$\begin{pmatrix} 1 & 0 & -\cos(k_j L_j) & k_j^{-1}\sin(k_j L_j) \\ -\cos(k_j L_j) & -k_j^{-1}\sin(k_j L_j) & 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\theta}_1^j(\omega) \\ \bar{q}_1^j(\omega) \\ \bar{\theta}_2^j(\omega) \\ \bar{q}_2^j(\omega) \end{pmatrix} = \begin{pmatrix} \bar{F}_1^j(\omega) \\ \bar{F}_2^j(\omega) \end{pmatrix},$$

where j is the number of the corresponding graph edge, and $\bar{F}_1^j(\omega) = 2\bar{Q}_{1j}(0,\omega)$, $\bar{F}_2^j(\omega) = 2\bar{Q}_{2j}(L,\omega)$.

$$\bar{Q}_{1j}(0,\omega) = \bar{F}_j(x,\omega) \underset{x}{*} \bar{U}^j(x,\omega)|_{x=0} + \theta_0^j(x) H_L^-(x) \underset{x}{*} \bar{U}^j(x,\omega)|_{x=0},$$

$$\bar{Q}_{2j}(L,\omega) = \bar{F}_j(x,\omega) \underset{x}{*} \bar{U}^j(x,\omega)|_{x=L} + \theta_0^j(x) H_L^-(x) \underset{x}{*} \bar{U}^j(x,\omega)|_{x=L},\ \tilde{U}^j(x,\omega) = -0,5\frac{\sin(k_j|x|)}{k_j(k_j + i0)}.$$

Consequently, we have $2N$ equations for defining $4N$ boundary functions:

$$B(\omega) = (\bar{\theta}_1^1, \bar{q}_1^1, \bar{\theta}_2^1, \bar{q}_2^1, ..., \bar{\theta}_1^N, \bar{q}_1^N \bar{\theta}_2^N, \bar{q}_2^N)^T.$$

This graph has N edges with one boundary condition at the end of every edge. Consequently, we add N boundary conditions at the ends of this graph.

The next N equations contain the condition of continuity (4) and Kirchhoff condition (5) for N edges in common boundary points $x_1 = 0$ at the node A_0. Therefore, the complete resolving system of $4N$ equations has been written in the next form.

Theorem 2. The resolving system of boundary value problem equations (1)–(5) on a star heat graph with N different edges has the following form:

$$\mathbf{A}(\omega) \cdot B(\omega) = C(\omega), \tag{22}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \kappa_1\bar{U}^1_{,x}(L_1) & -\kappa_1\bar{U}^1(L_1) & .. & 0 & 0 & 0 & 0 \\ \kappa_1\bar{U}^1_{L_1} & \kappa_1\bar{U}^1(L_1) & 1 & 0 & .. & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & .. & ... & ... & ... & ... \\ ... & ... & ... & ... & .. & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & .. & 1 & 0 & \kappa_N\bar{U}^N_{,x}(L_N) & -\kappa_N\bar{U}^N(L_N) \\ 0 & 0 & 0 & 0 & .. & \kappa_N\bar{U}^N_{,x}(L_n,\omega) & \kappa_N\bar{U}^N(L_N) & 1 & 0 \\ 0 & 0 & 1 & 0 & .. & 0 & 0 & 0 & 0 \\ 0 & ... & ... & ... & .. & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & .. & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & .. & 0 & 0 & 0 & 0 \\ 1 & ... & ... & ... & .. & ... & ... & ... & ... \\ 1 & ... & ... & ... & .. & -1 & ... & 0 & 0 \\ 0 & \kappa_1 & 0 & 0 & .. & 0 & \kappa_n & 0 & 0 \end{bmatrix}$$

and dimension of the matrix $\mathbf{A}$ is $4N \times 4N$;

$$C(\omega) = \left(\bar{F}_1^1(0,\omega)\bar{F}_2^{-1}(L_1,\omega), ..., \bar{F}_1^N(L_N,\omega)\bar{F}_2^N(L_N,\omega); \bar{\theta}_1(L_1,\omega), ..., \bar{\theta}_N(L_N,\omega); 0, \underset{N-1}{...}, 0, \bar{G}(\omega)\right)^T.$$

Its solution is equal to

$$B(\omega) = \mathbf{A}^{-1}(\omega) \cdot C(\omega),$$

$\mathbf{A}^{-1}(\omega)$ is inverse matrix to matrix $\mathbf{A}(\omega)$.

The first $2N$ rows of matrix A contain at "diagonal" the resulting system (22) for each edge of this graph. The other elements of these lines are null (see $A(\omega)$).

The next N lines contain the boundary conditions on the ends of graph (3). Here in the line $2N+j$ only the column $4j-1$ contains 1, others are null.

The line $(3N+j)$, $j = 1, T \ldots, N-1$ in the first column stands 1, and -1 in column $(4j+1)$, $j = 1, T \ldots, N-1$ (condition of temperature continuity (4)). In the last line (Kirchhoff condition (5)), the value κ_j stands in the column $2+4j$, $j = 0, T \ldots, N-1$.

Substituting $(\bar{\theta}_1^j, \bar{q}_1^j, \bar{\theta}_2^j, \bar{q}_2^j)$ — the corresponding components of $B(\omega)$ for j-edge into (16), we obtain $\bar{\theta}_j(x,\omega)$ at any point x of this edge:

$$\bar{\theta}(x,\omega)H_D^-(x,t) = H(x)\int_0^{L^-} \bar{U}^j(x-y,\omega)\bar{F}^j(y,\omega)dy + \kappa_j H(x)\int_0^{L} \bar{U}^j(x-y,\omega)\theta_0^j(y)dy +$$
$$+\kappa_j \bar{q}_2^j(\omega)\bar{U}^j(L_j - x,\omega) - \kappa_j \bar{q}_1^j(\omega)H(L_j - x)\bar{U}^j(x,\omega) -$$
$$-\kappa_j \bar{\theta}_2^j(\omega)\bar{U}^j_{,x}(L_j - x,\omega) - \kappa_j \bar{\theta}_1^j(\omega)H(L_j - x)\bar{U}^j_{,x}(x,\omega), \quad 0 \le x \le L_j.$$

Performing the inverse Fourier transform (14), we obtain $\hat{\theta}(x,t) = \theta_j(x,t)H_D^-(x,t)$. Thus, the temperature on the star graph is determined at any time t at any edge.

So BVP for the heat star graph has been solved.

Conclusion

Using the generalized function method, the boundary value problems of thermal conductivity on a thermal star graph have been solved, which can be used to study various network-like structures under conditions of thermal heating (cooling). A unified technique has been developed for solving various boundary value problems typical for practical applications.

The action of heat sources can be modeled by both regular and singular generalized functions under various boundary conditions at the ends of the graph edge. The obtained regular integral representations of generalized solutions make it possible to determine the temperature and heat flows on each element and at any point of a graph for stationary oscillations with a constant frequency and in the case of periodic oscillations.

At first, a boundary value problem was solved on one edge of the graph. Using the generalized function method, a heat equation with a singular right-hand side was obtained. The solution to the Dirichlet problem was determined through the convolution of the fundamental solution with the singular right-hand side of the heat equation. Thus, the solution found on the edge was determined by the initial functions, boundary functions, and their derivatives (the unknown boundary functions). A resolving system of two linear algebraic equations in the space of the Fourier transform in time was constructed to determine the unknown boundary functions. After determining all the solutions on all graph edges and taking the continuity condition and Kirchhoff joint condition into account, we obtained the solution to the heat equation on the star graph.

The generalized function method presented here makes it possible to solve a wide range of boundary value problems at the ends of the graph edges and various transmission conditions at its common node and can be extended to network structures of very different types. This distinguishes this method from all others that are used to solve similar problems. This algorithm can be recommended and used for engineering calculations of heating networks and it will find wide application in the design and calculation of rod structures in mechanical engineering and construction.

The method of generalized functions presented here allows not only to solve a wide range of problems with different conditions at the ends of the edges of the graph and the conditions of transmission in its common node, but can also be extended to network structures of various types. This distinguishes this method from all others that are used to solve similar problems.

Acknowledgments

The work was performed with the financial support of the Science Committee of the Ministry of Science and Higher Education of Kazakhstan (grants: AP19674789, 2023–2025; AP23488145, 2024–2026).

Author contributions

L.A: Project administration, Conceptualization, Methodology, Formal Analysis, Investigation; D.P: Investigation; A.D: Investigation, Formal Analysis, Validation, Software; N.A: Investigation, Formal Analysis, Validation, Software.

Conflict of interest

The authors declare no conflict of interest.

References

1 Euler, L. (1741). Solutio problematis ad geometriam situs pertinentis. *Commentarii academiae scientiarum Petropolitanae, 8,* 128–140.

2 König, D. (2001). *Theorie der endlichen und unendlichen graphen.* AMS Chelsea Publishing.

3 König, D. (1990). *Theory of finite and infinite graphs.* Boston: Birkhäuser.

4 Majeed, A., & Raulf, I. (2020). Graph theory: Comprehensive survey about graph theory, applications in computer science and social networks. *Journal Inventions, 5*(1), 10. https://doi.org/10.3390/inventions5010010

5 Priyadarsini, P.L.K. (2015). A survey on some applications of graph theory in cryptography. *J. Discret. Math. Sci. Cryptogr., 18*(3), 209–217. https://doi.org/10.1080/09720529.2013.878819

6 Polak, M., Romańczuk, U., Ustimenko, V., & Wróblewska, A. (2013). On the applications of extremal graph theory to coding theory and cryptography. *Electronic Notes in Discrete Mathematics, 43*, 329–342. https://doi.org/10.1016/ j.endm.2013.07.051

7 Can, U., & Alatas, B. (2019). A new direction in social network analysis: Online social network analysis problems and applications. *Physica A: Statistical Mechanics and its Applications, 535*, 122372. https://doi.org/10.1016/j.physa.2019.122372

8 Sahu, S., Mhedhbi, A., Salihoglu, S., Lin, J., & Özsu, M.T. (2019). The ubiquity of large graphs and surprising challenges of graph processing: extended survey. *VLDB Journal, 29*, 420–431.

9 Bliss, N.T, & Schmidt, M.C. (2013). Confronting the challenges of graphs and networks. *Lincoln Laboratory Journal, 20*(1), 4–9.

10 Avdonin, S., & Bell, J. (2015). Determining a distributed conductance parameter for a neuronal cable model defined on a tree graph. *Inverse Problems and Imaging, 9*(3), 645–659. https://doi.org/10.3934/ipi.2015.9.645

11 Avdonin, S., Belland, J., & Nurtazina, K. (2017). Determining distributed parameters in a neuronal cable model on a tree graph. *Mathematical Methods in the Applied Sciences, 40*(11), 3973–3981. https://doi.org/10.1002/mma.4277

12 Freiling, G., Ignatiev, M., & Yurko, V. (2008). An inverse spectral problem for Sturm-Liouville operators with singular potentials on star-type graphs. *Analysis on Graphs. Proceedings of Symposia in Pure Mathematics. Amer. Math. Soc. Providence, 77*, 397–408.

13 Yurko, V. (2019). Inverse spectral problems for differential pencils on arbitrary compact graphs. *Differential Equations, 55*, 24–33. https://doi.org/10.1134/S0012266119010038

14 Kanguzhin, B.E., & Seitova, A.A. (2020). On degenerate Sturm-Liouville boundary value problems on geometric graphs. *Journal of Mathematics Mechanics and Computer Science, 105*(1), 79–86. https://doi.org/10.26577/JMMCS.2020.v105.i1.07

15 Zhabko, A.P., Nurtazina, K.B., & Provotorov, V.V. (2020). Uniqueness solution to the inverse spectral problem with distributed parameters on the graph-star. *Bulletin of Saint Petersburg University. Appl. Math. Computer Science. Control Processes, 16*(2), 129–143. https://doi.org/10.21638/11701/spbu10.2020.205

16 Kanguzhin, B.E., Kaiyrbek, Zh.A., & Mustafina, M.A. (2024). Recovering of the stiffness coefficients of the Sturm–Liouville operator on a star graph from a finite set of its eigenvalues. *Lobachevski Journal of Mathematics, 45*, 2711–2716. https://doi.org/10.1134/s1995080224602881

17 Grigir'yan, A., Lin, Y., & Yang, Y. (2016). Yamabe type equations on graph. *J. Differential Equations, 216*, 4924–4943. https://doi.org/10.48550/arXiv.1607.04521

18 Graef, J.R., Kong, L., & Wang, M. (2014). Existence and uniqueness of solutions for a fractional boundary value problem on a graph. *Fract. Calc. Appl. Anal., 17*, 499–510. https://doi.org/10.2478/s13540-014-0182-4

19 Mehandiratta, V., Mehra, M., & Leugering, G. (2019). Existence and uniqueness results for a nonlinear Caputo fractional boundary value problem on a star graph. *J. Math. Anal. Appl., 477*(2), 1243–1264. https://doi.org/10.1016/j.jmaa.2019.05.011

20 Alam, G.M., Avdonin, S.A., Choque-Rivero, A.E., & Nurtazina, K. (2025). A new method for solving the exact control problem for the one-dimensional wave equation. *Boletin de la Sociedad Matematica Mexicana, 31*(27). https://doi.org/10.1007/s40590-024-00705-5

21 Avdonin, S.A., & Nicaise, S. (2015). Source and coefficient identification problems for the wave equation on graphs. *Inverse Problems, 31*(9), 095007. https://doi.org/10.1088/0266-5611/31/9/095007

22 Rasa, G.H.A., Kanguzhin, B.E., & Kaiyrbek, Zh.A. (2021). Propagation of waves along the star-graph. *Kazakh Mathematical Journal, 21*(2), 6–14.

23 Arepova, G., Alexeyeva, L.A., & Arepova, D. (2023). Solution to the Dirichiet problem of the wave equation on a star graph. *Mathematics, 11*(20), 4234. https://doi.org/10.3390/math11204234

24 Han, Zh.J., & Zuazua, E. (2017). Decay rates elastics-shaped networks. *J. Netw. Heterog. Media, 12*(3), 461–488. https://doi.org/10.3934/nhm.2017020

25 Alexeyeva, L.A., Dadayeva, A.N., & Ainakeyeva, N.Zh. (2021). Green tensor and regular solutions of equations of rods thermodynamics and their properties. *Journal of Theoretical and Applied Mechanics, 59*(2), 227–238. http://dx.doi.org/10.15632/jtam-pl/133233

26 Alexeyeva, L.A., & Akhmetzhanova, M.M. (2022). Spatially one-dimensional boundary value problems of coupled thermoelasticity: generalized functions method. *Mechanics of Solids, 57*(8), 2151–2165. https://doi.org/10.3103/S0025654422080325

27 Alexeyeva, L.A., Prikazchikov, D.A., Dadayeva, A.N., & Ainakeyeva, N.Zh. (2023). Kraevye

zadachi dinamiki termouprugikh sterzhnei i ikh resheniia [Boundary value problems of the dynamics of thermoelastic rods and their solutions.] *Vestnik Natsionalnoi inzhenernoi akademii Respubliki Kazakhstan — Bulletin of the National Engineering Academy of the Republic of Kazakhstan, 4*(90), 156–168 [in Russian]. https://doi.org/10.47533/2023.1606-146X.43

28 Martynova, Yu.V. (2021). Ob odnoi zadache raspredeleniia tepla v sisteme sterzhnei na grafe tipa "drevo" [On a heat propagation problem in a system of rods on a tree graph]. *Itogi nauki i tekhniki. Sovremennaia matematika i ee prilozheniia. Tematicheskie obzory — Results of Science and Technology. Modern Mathematics and its Applications. Subject Reviews, 198*, 89–95 [in Russian]. https://doi.org/10.36535/0233-6723-2021-198-89-95

29 Alexeyeva, L.A. (2006). Metod obobshchennykh funktsii v nestatsionarnykh kraevykh zadachakh dlia uravneniia volny [Method of generalized functions in non-stationary boundary value problems for the wave equation]. *Matematicheskii zhurnal — Mathematical journal, 1*(6), 16–32 [in Russian].

30 Vladimirov, V.S. (1988). *Obobshchennye funktsii v matematicheskoi fizike [Generalized functions in mathematical physics]*. Moscow: Nauka [in Russian].

*Author Information**

Lyudmila Alexeyevna Alexeyeva — Doctor of Physical and Mathematical Sciences, Professor, Chief researcher of Institute of Mathematics and Mathematical Modeling, Head of the project AP19674789 (2023–2025), Head of the project AP23488145 (2024–2026) at Joldasbekov Institute of Mechanics and Engineering, Almaty, Kazakhstan; e-mail: *alexeeva@math.kz*; https://orcid.org/0000-0002-7131-4635

Assiyat Nesipbayevna Dadayeva (*corresponding author*) — Doctor of Philosophy (PhD), Associate Professor of the Department of Higher Mathematics and Modeling of Kazakh National Research Technical University, Leading researcher of the project AP19674789 (2023–2025) at Institute of Mathematics and Mathematical Modeling, Leading researcher of the project AP23488145 (2024–2026) at Joldasbekov Institute of Mechanics and Engineering, Almaty, Kazakhstan; e-mail: *dady1262@mail.ru*; https://orcid.org/0000-0001-8954-6007

Danila Aleksandrovich Prikazchikov — Program Director in Mathematics, School of Computing and Mathematics, Keele University, UK; e-mail: *d.prikazchikov@keele.ac.uk*; https://orcid.org/0000-0001-7682-3079

Nursaule Zhumatkyzy Ainakeyeva — Researcher in Joldasbekov Institute of Mechanics and Engineering, Senior lecturer at Narxoz University, Researcher of the project AP19674789 (2023–2025) at Institute of Mathematics and Mathematical Modeling, Researcher of the project AP23488145 (2024–2026) at Joldasbekov Institute of Mechanics and Engineering, Almaty, Kazakhstan; e-mail: *nursaule_math@mail.ru*, http://orcid.org/0000-0003-4643-6072

*The author's name is presented in the order: First, Middle, and Last Names.