https://doi.org/10.31489/2025M2/28-43 *Research article*

Homogenization of Attractors to Reaction–Diffusion Equations in Domains with Rapidly Oscillating Boundary: Subcritical Case

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We consider the reaction–diffusion system of equations with rapidly oscillating terms in the equation and in boundary conditions in a domain with locally periodic oscillating boundary. In the subcritical case (the Fourier boundary condition is changed to the Neumann boundary condition in the limit) we proved that the trajectory attractors of this system converge in a weak sense to the trajectory attractors of the limit (homogenized) reaction–diffusion systems in domain independent of the small parameter, characterizing the oscillation rate. To obtain the results we use the approach of homogenization theory, asymptotic analysis and methods of the theory concerning trajectory attractors of evolution equations. Defining the appropriate functional and topological spaces with weak topology, we prove the existence of trajectory attractors and global attractors for these systems. Then we formulate the main Theorem and prove it with the help of auxiliary Lemmata. Applying the homogenization method and asymptotic analysis we derive the homogenized (limit) system of equations, prove the existence of trajectory attractors and global attractors and show the convergence of trajectory and global attractors.

Keywords: attractors, homogenization, reaction–diffusion equations, nonlinear equations, weak convergence, rapidly oscillating boundary.

2020 Mathematics Subject Classification: 35B27, 35B40, 35B41, 35D30.

Introduction

Recently, the attention of scientists has been attracted by various problems for evolution equations with dissipation, in which small parameters are present. To study such problems it is important to use asymptotic methods and homogenization theory. In the present paper we study homogenization problem for reaction–diffusion system of equations in domains with very rapidly oscillating boundary (see for detailed geometric settings [1]). We derive the homogenized (limit) system of equations in domain without oscillation of the boundary, then we prove the existence of trajectory and global attractors for the given and homogenized systems and also prove the convergence of attractors of the given system to the attractors of the homogenized system as the small parameter characterizing the oscillations, tends to zero, i.e. we prove the Hausdorff convergence of attractors as the small parameter

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The research G.F. Azhmoldaev and K.A. Bekmaganbetov in Sections 1 and 4 was funded by the Committee of Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant AP26199535). The work of G.A. Chechkin in Section 3 was financially supported by the state budget programs. The work of V.V. Chepyzhov in Section 2 is partially supported by the Russian Science Foundation (project 23-71-30008).

Received: 24 November 2024; *Accepted:* 28 March 2025.

tends to zero. In many pure mathematical papers, one can find the asymptotic analysis of problems in domains with rapidly oscillating boundaries (see, for example, [1–12]). We want to mention here the basic frameworks [13, 14], where one can find these approaches and methods as well as the detail bibliography.

The basic theory of attractors one can find, for instance, [15–17] and see also the references in these monographs (see also [18]). Homogenization of attractors were studied in [17,19–21] (see also [22–30]).

In this present paper we give the proofs of weak convergence of the trajectory attractor $\mathfrak{A}_\varepsilon$ of the reaction–diffusion systems in a domain with oscillating boundary, as $\varepsilon \to 0$, to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized systems in some natural functional space. We also proved the convergence of the global attractor $\mathcal{A}_\varepsilon$ to $\overline{\mathcal{A}}$ as $\varepsilon \to 0$. Here, the small parameter ε characterizes the period and the amplitude of the oscillations. The parameter ε is included also in Fourier condition on a part of the boundary, and we consider the case in which the Fourier condition transforms to the Neumann one (subcritical case) as the small parameter tends to zero.

The first section is devoted to basic settings, in the second section we describe the limiting (homogenized) reaction–diffusion system and its trajectory attractor. The third section contains auxiliary results and in the fourth section the proof of the main Theorem is presented.

1 *Statement of the problem*

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Ω lies in a half-space $x_d > 0$ and $\Gamma_1 \subset \{x : x_d = 0\}$. Given smooth nonpositive 1-periodic in the $\hat{\xi}$ function $F(\hat{x}, \hat{\xi})$, $\hat{x} = (x_1, ..., x_{d-1}), \hat{\xi} = (\xi_1, ..., \xi_{d-1})$, define the domain Ω_ε as follows: $\partial\Omega_\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2$, where we set $\Gamma_1^\varepsilon = \{x = (\hat{x}, x_d) : (\hat{x}, 0) \in \Gamma_1, x_d = \varepsilon^\alpha F(\hat{x}, \hat{x}/\varepsilon)\}$, $\alpha < 1$, i.e. we add thin oscillating layer $\Pi_\varepsilon = \{x = (\hat{x}, x_d) : (\hat{x}, 0) \in \Gamma_1, x_d \in [0, \varepsilon^\alpha F(\hat{x}, \hat{x}/\varepsilon))\}$ to the domain Ω. Usually, we assume $F(\hat{x}, \hat{\xi})$ to be compactly supported on Γ_1 uniformly in $\hat{\xi}$. Consider the following boundary-value problem:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \lambda \Delta u_\varepsilon - a\left(x, \frac{x}{\varepsilon}\right) f(u_\varepsilon) + h\left(x, \frac{x}{\varepsilon}\right), & x \in \Omega_\varepsilon, t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} + \varepsilon^\beta p(\hat{x}, \frac{\hat{x}}{\varepsilon}) u_\varepsilon = \varepsilon^{1-\alpha} g(\hat{x}, \frac{\hat{x}}{\varepsilon}), & x = (\hat{x}, x_d) \in \Gamma_1^\varepsilon, t > 0, \\ u_\varepsilon = 0, & x \in \Gamma_2, t > 0, \\ u_\varepsilon = U(x), & x \in \Omega_\varepsilon, t = 0, \end{cases} \tag{1}$$

where $u_\varepsilon = u_\varepsilon(x,t) = (u^1, \ldots, u^n)^\top$ is an unknown vector function, the nonlinear function $f = (f^1, \ldots, f^n)^\top$ is given, $h = (h^1, \ldots, h^n)^\top$ is the known right-hand side function, and λ is an $n \times n$-matrix with constant coefficients, having a positive symmetrical part: $\frac{1}{2}(\lambda + \lambda^\top) \geq \varpi I$ (where I is the unit matrix with dimension n). We assume that $p\left(\hat{x}, \hat{\xi}\right) = \operatorname{diag}\{p^1, \ldots, p^n\}$, $g\left(\hat{x}, \hat{\xi}\right) = (g^1, \ldots, g^n)^\top$ are continuous, 1-periodic in $\hat{\xi}$ and $p^i\left(\hat{x}, \hat{\xi}\right)$, $i = 1, \ldots n$, are positive. Here $\frac{\partial}{\partial \nu}$ is the normal derivative of the function multiplied by a matrix, i.e. $\frac{\partial}{\partial \nu} := \sum\limits_{j=1}^{n} \sum\limits_{k=1}^{d} \lambda_{ij} \frac{\partial}{\partial x_k} N_k$, $i = 1, \ldots, n$, and $N = (N_1, \ldots, N_d)$ is the unit outer normal to the boundary of the domain Ω_ε. Let us denote by $p_{\max}$ the maximum of p on Γ_1.

In this paper we investigate evolution equations and their trajectory attractors depending on a small parameter $\varepsilon > 0$ (see for details [26]). We consider the subcritical case, i.e. $\beta > 1 - \alpha$.

Function $a(x, \xi) \in C(\overline{\Omega}_\varepsilon \times \mathbb{R}^d)$ such that $0 < a_0 \leq a(x, \xi) \leq A_0$ with some coefficient a_0, A_0. Assuming that function $a_\varepsilon(x) = a\left(x, \frac{x}{\varepsilon}\right)$ has average $\overline{a}(x)$ when $\varepsilon \to 0+$ in space $L_{\infty,*w}(\Omega)$, that is

$$\int\limits_\Omega a\left(x, \frac{x}{\varepsilon}\right) \varphi(x) dx \to \int\limits_\Omega \overline{a}(x) \varphi(x) dx \quad (\varepsilon \to 0+) \tag{2}$$

for any function $\varphi \in L_1(\Omega)$.

Denote by V (respectively V_ε) the Sobolev space $H^1(\Omega,\Gamma_2)$ (respectively $H^1(\Omega_\varepsilon,\Gamma_2)$), i.e. the space of functions from the Sobolev space $H^1(\Omega)$ (respectively $H^1(\Omega_\varepsilon)$) with zero trace on Γ_2. We also denote by V' (respectively V'_ε) the dual space for V (respectively V_ε), i.e. the space of linear bounded functionals on V (respectively V_ε). Denote by Ω^+ such a domain that $\Omega_\varepsilon \subset \Omega^+$ for any ε. For the vector function $h\,(x,\xi)$, assume that for any $\varepsilon>0$ the function $h^i_\varepsilon(x) = h^i\left(x,\frac{x}{\varepsilon}\right) \in L_2(\Omega^+)$ and has the average $\overline{h^i}(x)$ in the space $L_2(\Omega^+)$ for $\varepsilon \to 0+$, that is

$$h^i\left(x,\frac{x}{\varepsilon}\right) \rightharpoonup \overline{h^i}(x) \quad (\varepsilon\to 0+) \text{ weakly in } L_2(\Omega^+),$$

or

$$\int\limits_{\Omega^+} h^i\left(x,\frac{x}{\varepsilon}\right)\varphi(x)dx \to \int\limits_{\Omega^+} \overline{h^i}(x)\varphi(x)dx \quad (\varepsilon\to 0+) \tag{3}$$

for any function $\varphi \in L_2(\Omega^+)$ and for all $i=1,\dots,n$.

From the condition (3), it follows that the norm of the function $h^i_\varepsilon(x)$ is bounded uniformly in ε, in the space $L_2(\Omega_\varepsilon)$, i.e.

$$\|h^i_\varepsilon(x)\|_{L_2(\Omega_\varepsilon)} \le M_0, \quad \forall\varepsilon\in(0,1]. \tag{4}$$

It is assumed that the vector function $f(v)\in C(\mathbb{R}^n;\mathbb{R}^n)$ satisfies the following inequalities

$$\sum_{i=1}^n |f^i(v)|^{p_i/(p_i-1)} \le C_0\left(\sum_{i=1}^n |v^i|^{p_i}+1\right), \quad 2\le p_1\le\dots\le p_{n-1}\le p_n, \tag{5}$$

$$\sum_{i=1}^n \gamma_i|v^i|^{p_i} - C \le \sum_{i=1}^n f^i(v)v^i, \quad \forall v\in\mathbb{R}^n, \tag{6}$$

for $\gamma_i>0$ for any $i=1,\dots,n$. The inequality (5) is due to the fact that in real reaction-diffusion systems, the functions $f^i(u)$ are polynomials with possibly different degrees. Inequality (6) calls *dissipativity condition* for the reaction-diffusion system (1). In a simple model case $p_i\equiv p$ for any $i=1,\dots,n$, condition (5) and (6) reduce to the following inequalities

$$|f(v)| \le C_0\left(|v|^{p-1}+1\right), \quad \gamma|v|^p - C \le f(v)v, \quad \forall v\in\mathbb{R}^n.$$

Note that the fulfillment of the Lipschitz condition for the function $f(v)$ with respect to the variable v is *not expected.*

Remark 1. Using the methods presented, it is also possible to study systems in which nonlinear terms have the form $\sum\limits_{j=1}^m a_j\left(x,\frac{x}{\varepsilon}\right)f_j(u)$, where a_j are matrices whose elements allow averaging and $f_j(u)$ polynomial vectors of u, which satisfy conditions of the form (5)–(6). For brevity, we study the case $m=1$ and $a_1\left(x,\frac{x}{\varepsilon}\right) = a\left(x,\frac{x}{\varepsilon}\right)I$, where I is the identity matrix.

Denote

$$G(\hat{x}) = \int\limits_{[0,1)^{d-1}} \sqrt{|\nabla_{\hat{\xi}}F(\hat{x},\hat{\xi})|^2}\, g(\hat{x},\hat{\xi})\,d\hat{\xi}, \tag{7}$$

and we have the following convergence (see [1]):

$$\varepsilon^{1-\alpha}\int\limits_{\Gamma_1^\varepsilon} g^i\left(\hat{x},\frac{\hat{x}}{\varepsilon}\right)\upsilon\left(\hat{x},\varepsilon^\alpha F\left(\hat{x},\frac{\hat{x}}{\varepsilon}\right)\right)\,ds \to \int\limits_{\Gamma_1} G^i\left(\hat{x}\right)\upsilon\left(x\right)\,ds$$

for any $\upsilon\in H^1(\Omega_\varepsilon)$ by $\varepsilon\to 0$. Here ds is the element of $(d-1)$-dimensional measure on the hypersurface.

Let us introduce the following notation for the spaces $\mathbf{H} := [L_2(\Omega)]^n$, $\mathbf{H}_\varepsilon := [L_2(\Omega_\varepsilon)]^n$, $\mathbf{V} := [H^1(\Omega, \Gamma_2)]^n$, $\mathbf{V}_\varepsilon := [H^1(\Omega_\varepsilon; \Gamma_2)]^n$. The norms in these spaces are determined as follows

$$\|v\|^2 := \int\limits_{\Omega} \sum_{i=1}^{n} |v^i(x)|^2 dx, \ \|v\|_\varepsilon^2 := \int\limits_{\Omega_\varepsilon} \sum_{i=1}^{n} |v^i(x)|^2 dx,$$

$$\|v\|_1^2 := \int\limits_{\Omega} \sum_{i=1}^{n} |\nabla v^i(x)|^2 dx, \ \|v\|_{1,\varepsilon}^2 := \int\limits_{\Omega_\varepsilon} \sum_{i=1}^{n} |\nabla v^i(x)|^2 dx.$$

We denote by $\mathbf{V}'$ the dual space to the space $\mathbf{V}$, and by $\mathbf{V}'_\varepsilon$ the dual space to the space $\mathbf{V}_\varepsilon$.

Let $q_i = p_i/(p_i - 1)$ for any $i = 1, \dots, n$. We will use the following vector notation $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$, and also define spaces

$$\mathbf{L_p} := L_{p_1}(\Omega) \times \ldots \times L_{p_n}(\Omega), \ \mathbf{L}_{\mathbf{p},\varepsilon} := L_{p_1}(\Omega_\varepsilon) \times \ldots \times L_{p_n}(\Omega_\varepsilon),$$

$$\mathbf{L_p}(\mathbb{R}_+; \mathbf{L_p}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(\Omega)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(\Omega)),$$

$$\mathbf{L_p}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(\Omega_\varepsilon)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(\Omega_\varepsilon)).$$

As in [17, 31] we study weak solutions of the initial boundary value problem (1), that is, functions

$$u_\varepsilon(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}_\varepsilon) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}_{\mathbf{p}}^{loc}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon})$$

which satisfy the equation (1) in the distributional sense (the sense of generalized functions), that is, the integral identity holds

$$-\int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} u_\varepsilon \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} \lambda \nabla u_\varepsilon \cdot \nabla \psi\, dxdt + \int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} a_\varepsilon(x) f(u_\varepsilon) \cdot \psi\, dxdt +$$

$$+\varepsilon^\beta \int\limits_{\Gamma_1^\varepsilon \times \mathbb{R}_+} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) u_\varepsilon \cdot \psi\, dsdt = \int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} h_\varepsilon(x) \cdot \psi\, dxdt + \varepsilon^{1-\alpha} \int\limits_{\Gamma_1^\varepsilon \times \mathbb{R}_+} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) \cdot \psi\, dsdt$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{V}_\varepsilon \cap \mathbf{L}_{\mathbf{p},\varepsilon})$. Here $y_1 \cdot y_2$ means scalar product of vectors $y_1, y_2 \in \mathbb{R}^n$.

If $u_\varepsilon(x,t) \in \mathbf{L_p}(0, M; \mathbf{L}_{\mathbf{p},\varepsilon})$, then from the condition (5) it follows that $f(u(x,t)) \in \mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon})$. At the same time, if $u_\varepsilon(x,t) \in \mathbf{L}_2(0, M; \mathbf{V}_\varepsilon)$, then $\lambda \Delta u_\varepsilon(x,t) + h_\varepsilon(x) \in \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon)$. Therefore, for an arbitrary weak solution $u_\varepsilon(x,s)$ to problem (1), satisfies

$$\frac{\partial u_\varepsilon(x,t)}{\partial t} \in \mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon).$$

From the Sobolev embedding theorem it follows that

$$\mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon) \subset \mathbf{L_q}\left(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}}\right),$$

where space $\mathbf{H}_\varepsilon^{-\mathbf{r}} := H^{-r_1}(\Omega_\varepsilon) \times \ldots \times H^{-r_n}(\Omega_\varepsilon)$, $\mathbf{r} = (r_1, \dots, r_n)$ and indexes $r_i = \max\{1, d(1/q_i - 1/2)\}$ by $i = 1, \dots, n$. Here $H^{-r}(\Omega_\varepsilon)$ denotes the space conjugate to the Sobolev space $\overset{\circ}{W}{}_2^r(\Omega_\varepsilon)$ with index $r > 0$ in the domain Ω_ε.

Therefore, for any weak solution $u_\varepsilon(x,t)$ to problem (1), its time derivative $\frac{\partial u_\varepsilon(x,t)}{\partial t}$ belongs to $\mathbf{L_q}(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}})$.

Remark 2. Existence of a weak solution $u(x,t)$ to problem (1) for any initial data $U \in \mathbf{H}_\varepsilon$ and fixed ε, can be proved in the standard way (see, for example, [16, 31]). This solution may not be unique, since the function $f(v)$ satisfies only the conditions (5), (6) and it is not assumed that the Lipschitz condition is satisfied with respect to v.

The following Lemma is proved in a similar way to the proposition XV. 3.1 from [17].

Lemma 1. Let $u_\varepsilon(x,t) \in \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon})$ be the weak solution of problem (1). Then

(i) $u_\varepsilon \in \mathbf{C}(\mathbb{R}_+;\mathbf{H}_\varepsilon)$;

(ii) function $\|u_\varepsilon(\cdot,t)\|^2$ is absolutely continuous on $\mathbb{R}_+$, and moreover

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(\cdot,t)\|^2 + \int\limits_{\Omega_\varepsilon} \lambda\nabla u_\varepsilon(x,t)\cdot\nabla u_\varepsilon(x,t)dx + \int\limits_{\Omega_\varepsilon} a_\varepsilon(x) f(u_\varepsilon(x,t))\cdot u_\varepsilon(x,t)dx + \tag{8}$$

$$+\varepsilon^\beta \int\limits_{\Gamma_1^\varepsilon} p\left(\hat{x},\frac{\hat{x}}{\varepsilon}\right) u_\varepsilon(x,t)\cdot u_\varepsilon(x,t)\,ds = \int\limits_{\Omega_\varepsilon} h_\varepsilon(x)\cdot u_\varepsilon(x,t)dx + \varepsilon^{1-\alpha}\int\limits_{\Gamma_1^\varepsilon} g\left(\hat{x},\frac{\hat{x}}{\varepsilon}\right)\cdot u_\varepsilon(x,t)\,ds,$$

for almost all $t \in \mathbb{R}_+$.

To define the trajectory space $\mathcal{K}_\varepsilon^+$ for (1), we use the general approaches of Section 2 from [26] and for every $[t_1,t_2] \in \mathbb{R}$, we have the Banach spaces

$$\mathcal{F}_{t_1,t_2} := \mathbf{L}_\mathbf{p}(t_1,t_2;\mathbf{L}_\mathbf{p}) \cap \mathbf{L}_2(t_1,t_2;\mathbf{V}) \cap \mathbf{L}_\infty(t_1,t_2;\mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}\left(t_1,t_2;\mathbf{H}^{-r}\right)\right\}$$

(sometimes we omit the parameter ε for brevity) with the following norm:

$$\|v\|_{\mathcal{F}_{t_1,t_2}} := \|v\|_{\mathbf{L}_\mathbf{p}(t_1,t_2;\mathbf{L}_\mathbf{p})} + \|v\|_{\mathbf{L}_2(t_1,t_2;\mathbf{V})} + \|v\|_{\mathbf{L}_\infty(0,M;\mathbf{H})} + \left\|\frac{\partial v}{\partial t}\right\|_{\mathbf{L}_\mathbf{q}(t_1,t_2;\mathbf{H}^{-r})}.$$

Setting $\mathcal{D}_{t_1,t_2} = \mathbf{L}_\mathbf{q}\left(t_1,t_2;\mathbf{H}^{-r}\right)$, we obtain $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$ and for $u(t) \in \mathcal{F}_{t_1,t_2}$, we have $A(u(t)) \in \mathcal{D}_{t_1,t_2}$. One considers now weak solutions to (1) as solutions of an equation in the general scheme of Section 2 from [26].

Consider the spaces

$$\mathcal{F}_+^{loc} = \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+;\mathbf{L}_\mathbf{p}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}^{loc}(\mathbb{R}_+;\mathbf{H}^{-r})\right\},$$

$$\mathcal{F}_{\varepsilon,+}^{loc} = \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}^{loc}(\mathbb{R}_+;{\mathbf{H}_\varepsilon}^{-r})\right\}.$$

We introduce the following notation. Let $\mathcal{K}_\varepsilon^+$ be the set of all weak solutions to (1). For any $U \in \mathbf{H}$, there exists at least one trajectory $u(\cdot) \in \mathcal{K}_\varepsilon^+$ such that $u(0) = U(x)$. Consequently, the space $\mathcal{K}_\varepsilon^+$ to (1) is not empty and is sufficiently large.

We define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ in the spaces $\mathcal{F}_{t_1,t_2}$ by means of the norms from $\mathbf{L}_2(t_1,t_2;\mathbf{H})$. We get

$$\rho_{t_1,t_2}(u,v) = \left(\int\limits_{t_1}^{t_2} \|u(t)-v(t)\|_\mathbf{H}^2 dt\right)^{1/2} \qquad \forall u(\cdot), v(\cdot) \in \mathcal{F}_{t_1,t_2}.$$

The topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$ is generated by these metrics. Let us recall that $\{v_k\} \subset \mathcal{F}_+^{loc}$ converges to $v \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\|v_k(\cdot) - v(\cdot)\|_{\mathbf{L}_2(t_1,t_2;\mathbf{H})} \to 0$ $(k \to \infty)$ for all $[t_1,t_2] \subset \mathbb{R}_+$. The topology Θ_+^{loc} is metrizable. We consider this topology in the trajectory space $\mathcal{K}_\varepsilon^+$ of (1). Similarly, we define the topology $\Theta_{\varepsilon,+}^{loc}$ in $\mathcal{F}_{\varepsilon,+}^{loc}$.

Denote by $S(\tau)$ the translation semigroup, i.e. $S(\tau)\,u(t) = u(t+\tau)$. The translation semigroup $S(\tau)$ acting on $\mathcal{K}_\varepsilon^+$, is continuous in the topology $\Theta_{\varepsilon,+}^{loc}$. It is easy to see that $\mathcal{K}_\varepsilon^+ \subset \mathcal{F}_{\varepsilon,+}^{loc}$ and the space $\mathcal{K}_\varepsilon^+$ is translation invariant, i.e. $S(\tau)\mathcal{K}_\varepsilon^+ \subseteq \mathcal{K}_\varepsilon^+$ for all $\tau \geq 0$.

Using the scheme of Section 3 from [18] and Section 2 from [26], one can define bounded sets in the space $\mathcal{K}_\varepsilon^+$ by means of the Banach space $\mathcal{F}_{\varepsilon,+}^b$. We naturally get

$$\mathcal{F}_{\varepsilon,+}^b = \mathbf{L}_{\mathbf{p}}^b(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}_2^b(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_{\mathbf{q}}^b(\mathbb{R}_+;\mathbf{H}_\varepsilon^{-r}) \right\}$$

and the space $\mathcal{F}_{\varepsilon,+}^b$ is a subspace of $\mathcal{F}_{\varepsilon,+}^{loc}$.

Definition 1. [17] A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called the *trajectory attractor* of the translation semigroup $\{S(\tau)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, if

(i) $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc};

(ii) the set $\mathfrak{A}$ is strictly invariant with respect to the semigroup: $S(\tau)\mathfrak{A} = \mathfrak{A}$ for all $\tau \geq 0$;

(iii) $\mathfrak{A}$ is an attracting set for $\{S(\tau)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, that is, for each $M > 0$, we have

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\mathfrak{A}) \to 0 \quad (\tau \to +\infty),$$

where $\mathrm{dist}_{\mathcal{M}}(X,Y) := \sup_{x\in X} \mathrm{dist}_{\mathcal{M}}(x,Y) = \sup_{x\in X} \inf_{y\in Y} \rho_{\mathcal{M}}(x,y)$ is the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$. We remember that the Hausdorff semidistance is not symmetric, for any $\mathcal{B} \subseteq \mathcal{K}^+$ bounded in $\mathcal{F}_+^b$ and for each $M > 0$.

Suppose that $\mathcal{K}_\varepsilon$ is the kernel to (1), that consists of all weak complete solutions $u(t), t \in \mathbb{R}$, to our system, bounded in

$$\mathcal{F}_\varepsilon^b = \mathbf{L}_{\mathbf{p}}^b(\mathbb{R};\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}_2^b(\mathbb{R};\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty(\mathbb{R};\mathbf{H}_\varepsilon) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_{\mathbf{q}}^b(\mathbb{R};\mathbf{H}_\varepsilon^{-r}) \right\}.$$

In analogous way we define the topology Θ_ε^{loc} in $\mathcal{F}_\varepsilon^b$.

Proposition 1. Problem (1) has the trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space $\Theta_{\varepsilon,+}^{loc}$. The set $\mathfrak{A}_\varepsilon$ is bounded in $\mathcal{F}_{\varepsilon,+}^b$ and compact in $\Theta_{\varepsilon,+}^{loc}$. Moreover, $\mathfrak{A}_\varepsilon = \Pi_+\mathcal{K}_\varepsilon$, the kernel $\mathcal{K}_\varepsilon$ is non-empty and bounded in $\mathcal{F}_\varepsilon^b$ and compact in Θ_ε^{loc}.

To prove this proposition we use the approach of the proof from [17]. To prove the existence of an absorbing set (bounded in $\mathcal{F}_{\varepsilon,+}^b$ and compact in $\Theta_{\varepsilon,+}^{loc}$) one can use Lemma 1 similar to [17].

It is easy to verify, that $\mathfrak{A}_\varepsilon \subset \mathcal{B}_0(R)$ for all $\varepsilon \in (0,1)$. Here $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_{\varepsilon,+}^b$ with a sufficiently large radius R. Due to the Aubin-Lions-Simon Lemma (see Lemma from [32]), we have

$$\mathcal{B}_0(R) \Subset \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon^{1-\delta}), \tag{9}$$

$$\mathcal{B}_0(R) \Subset \mathbf{C}^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon^{-\delta}), \quad 0 < \delta \leq 1. \tag{10}$$

Bearing in mind (9) and (10), the attraction to the constructed trajectory attractor can be strengthen.

Corollary 1. For any bounded in $\mathcal{F}_{\varepsilon,+}^b$ set $\mathcal{B} \subset \mathcal{K}_\varepsilon^+$, we get

$$\mathrm{dist}_{\mathbf{L}_2(0,M;\mathbf{H}_\varepsilon^{1-\delta})}(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\varepsilon) \to 0,$$
$$\mathrm{dist}_{\mathbf{C}([0,M];\mathbf{H}_\varepsilon^{-\delta})}(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\varepsilon) \to 0 \ (\tau \to \infty),$$

where M is a positive constant.

Recall that $\Omega \subset \Omega_\varepsilon$ and Ω lies in the positive half-space $\{x_d > 0\}$. Therefore, any function $u(x,t)$ with $x \in \Omega_\varepsilon$ that belongs to the space $\mathcal{F}_{\varepsilon,+}^b$ and is restricted to the domain Ω, belongs to the space $\mathcal{F}_+^b$ and, moreover,

$$\|u\|_{\mathcal{F}_+^b} \leq \|u\|_{\mathcal{F}_{\varepsilon,+}^b}.$$

Using this observation, we have

Corollary 2. The trajectory attractors $\mathfrak{A}_\varepsilon$ are uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$. The kernels $\mathcal{K}_\varepsilon$ are uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}^b$.

Definition 2. We say that the trajectory attractors $\mathfrak{A}_\varepsilon$ converge to the trajectory attractor $\overline{\mathfrak{A}}$ as $\varepsilon \to 0$ in the topological space Θ_+^{loc} if for any neighbourhood $\mathcal{O}(\overline{\mathfrak{A}})$ in Θ_+^{loc}, there is an $\varepsilon_1 \geq 0$ such that $\mathfrak{A}_\varepsilon \subseteq \mathcal{O}(\overline{\mathfrak{A}})$ for any $\varepsilon < \varepsilon_1$, that is, for each $M > 0$, we have

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0 \ (\varepsilon \to 0).$$

2 Homogenized reaction-diffusion system and its trajectory attractor (the case $\beta > 1 - \alpha$)

In the next sections, we study the behaviour of the problem (1) as $\varepsilon \to 0$ in the subcritical case $\beta > 1 - \alpha$. We have the following "formal" limit problem with inhomogeneous Fourier boundary condition

$$\begin{cases} \frac{\partial u_0}{\partial t} = \lambda \Delta u_0 - \overline{a}\,(x)\, f(u_0) + \overline{h}\,(x)\,, & x \in \Omega,\ t > 0, \\ \frac{\partial u_0}{\partial \nu} = G(\hat{x}), & x = (\hat{x}, 0) \in \Gamma_1,\ t > 0, \\ u_0 = 0, & x \in \Gamma_2,\ t > 0, \\ u_0 = U(x), & x \in \Omega,\ t = 0. \end{cases} \tag{11}$$

Here $\overline{a}(x)$ and $\overline{h}(x)$ are defined in (2) and (3), respectively, $G(\hat{x})$ was defined in (7).

As before, we consider weak solutions of the problem (11), that is, functions

$$u(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L_p})\,,$$

which satisfy the following integral identity:

$$-\int\limits_{\Omega\times\mathbb{R}_+} u \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{\Omega\times\mathbb{R}_+} \lambda \nabla u \cdot \nabla \psi \, dxdt + \int\limits_{\Omega\times\mathbb{R}_+} \overline{a}(x) f(u) \cdot \psi \, dxdt = \int\limits_{\Omega\times\mathbb{R}_+} \bar{h}(x) \cdot \psi \, dxdt + \int\limits_{\Gamma_1\times\mathbb{R}_+} G\,(\hat{x}) \cdot \psi \, dsdt \tag{12}$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{V} \cap \mathbf{L_p})$. For any weak solution $u(x,t)$ to problem (11), we have that $\frac{\partial u(x,t)}{\partial t} \in \mathbf{L_q}\,(0, M; \mathbf{H^{-r}})$ (see Section 1). Recall, that the "limit" domain Ω in (11) and (12) is independent of ε and its boundary contains the plain part Γ_1.

Similar to (1), for any initial data $U \in \mathbf{H}$, the problem (11) has at least one weak solution (see Remark 2). Lemma 1 also holds true for the problem (11) with replacing the ε-depending coefficients a, h, p and g by the corresponding averaged coefficients $\overline{a}(x), \overline{h}(x), P(\hat{x})$, and $G(\hat{x})$.

As usual, let $\overline{\mathcal{K}}^+$ be the the trajectory space for (11) (the set of all weak solutions), that belong to the corresponding spaces $\mathcal{F}_+^{loc}$ and $\mathcal{F}_+^b$ (see Section 2 from [26]). Recall that $\overline{\mathcal{K}}^+ \subset \mathcal{F}_+^{loc}$ and the space $\overline{\mathcal{K}}^+$ is translation invariant with respect to translation semigroup $\{S(\tau)\}$, that is, $S(\tau)\overline{\mathcal{K}}^+ \subseteq \overline{\mathcal{K}}^+$ for all $\tau \geq 0$. We now construct the trajectory attractor in the topology Θ_+^{loc} for the problem (11) (see Section 1 and Section 2 from [26]).

Similar to Proposition 1, we have

Proposition 2. Problem (11) has the trajectory attractor $\overline{\mathfrak{A}}$ in the topological space Θ_+^{loc}. The set $\overline{\mathfrak{A}}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover,

$$\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}},$$

the kernel $\overline{\mathcal{K}}$ of the problem (11) is non-empty and bounded in $\mathcal{F}^b$.

We also have that $\overline{\mathfrak{A}} \subset \mathcal{B}_0(R)$, where $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_+^b$ with a sufficiently large radius R. Finally, the analog of Corollary 1 holds for the trajectory attractor $\overline{\mathfrak{A}}$.

Corollary 3. For any bounded in $\mathcal{F}^b_+$ set $\mathcal{B} \subset \overline{\mathcal{K}}^+$, we have

$$\operatorname{dist}_{\mathbf{L}_2(0,M;\mathbf{H}^{1-\delta})}\left(\Pi_{0,M} S(\tau)\mathcal{B}, \Pi_{0,M}\overline{\mathcal{K}}\right) \to 0,$$
$$\operatorname{dist}_{\mathbf{C}([0,M];\mathbf{H}_\varepsilon^{-\delta})}\left(\Pi_{0,M} S(\tau)\mathcal{B}, \Pi_{0,M}\overline{\mathcal{K}}\right) \to 0 \ (\tau \to \infty), \ \forall M > 0.$$

3 Preliminary Lemmata (The case $\beta > 1 - \alpha$)

Next Lemmata are proved in [1].

Lemma 2. The convergence

$$v\left(\hat{x}, \varepsilon^\alpha F\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right)\right) \to v(\hat{x}, 0) \quad \text{as} \quad \varepsilon \to 0$$

strongly in $[L_2(\Gamma_1)]^n$ and the inequality

$$\|v\|_{[L_2(\Pi_\varepsilon)]^n} \le C_1 \sqrt{\varepsilon^\alpha} \|v\|_{\mathbf{V}_\varepsilon} \tag{13}$$

take place for any $v \in \mathbf{V}_\varepsilon$.

Lemma 3. Let (ds) be an element of the $(n-1)$-dimensional volume of Γ_1^ε. Then

$$ds = \left(\sqrt{1 + \varepsilon^{2-2\alpha}\left|\nabla_{\hat{\xi}} F\left(\hat{x}, \hat{\xi}\right)\right|^2}\,\Big|_{\hat{\xi}=\frac{\hat{x}}{\varepsilon}}\right) d\hat{x}(1 + O(\varepsilon)) = \varepsilon^{\alpha-1}\left(\sqrt{\left|\nabla_{\hat{\xi}} F\left(\hat{x}, \hat{\xi}\right)\right|^2}\,\Big|_{\hat{\xi}=\frac{\hat{x}}{\varepsilon}} + O(\varepsilon^{1-\alpha})\right) d\hat{x}.$$

Proposition 3. Uniformly in $u, v \in [H^{1/2}(\Gamma_1)]^n$

$$\left|\int_{\Omega_\varepsilon} u \cdot v \, d\hat{x}\right| \le C_2 \|u\|_{[H^{1/2}(\Gamma_1)]^n} \|v\|_{[H^{1/2}(\Gamma_1)]^n}.$$

Lemma 4. There exists such a positive constant C_3, independent of ε, that

$$\int_{\Omega_\varepsilon} |\nabla v|^2 dx + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) v \cdot v \, ds \ge C_3 \|v\|_{\mathbf{V}_\varepsilon}$$

for any $v \in \mathbf{V}_\varepsilon$.

Let us consider auxiliary elliptic problems

$$\begin{cases} \lambda \Delta v_\varepsilon + h\left(x, \frac{x}{\varepsilon}\right) = 0, & x \in \Omega_\varepsilon, \\ \frac{\partial v_\varepsilon}{\partial \nu} + \varepsilon^\beta p(\hat{x}, \frac{\hat{x}}{\varepsilon}) v_\varepsilon = \varepsilon^{1-\alpha} g(\hat{x}, \frac{\hat{x}}{\varepsilon}), & x = (\hat{x}, x_d) \in \Gamma_1^\varepsilon, \\ v_\varepsilon = 0, & x \in \Gamma_2, \end{cases} \tag{14}$$

and

$$\begin{cases} \lambda \Delta v_0 + \overline{h}(x) = 0, & x \in \Omega, \\ \frac{\partial v_0}{\partial \nu} = G(\hat{x}), & x = (\hat{x}, 0) \in \Gamma_1, \\ v_0 = 0, & x \in \Gamma_2, \end{cases}$$

and $\overline{h}(x)$ is defined in (3), $G(\hat{x})$ was defined in (7).

Lemma 5. Let $\beta > 1 - \alpha$. For all $v \in \mathbf{V}_\varepsilon$ the following convergences:

$$\left|\varepsilon^\beta \int_{\Gamma_1^\varepsilon} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) v\left(\hat{x}, \varepsilon^\alpha F\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right)\right) \cdot v\left(\hat{x}, \varepsilon^\alpha F\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right)\right) ds\right| \to 0,$$

$$\left| \varepsilon^{1-\alpha} \int\limits_{\Gamma_1^\varepsilon} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) \cdot v\left(\hat{x}, \varepsilon^\alpha F\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right)\right) ds - \int\limits_{\Gamma_1} G(\hat{x}) \cdot v(x)\, ds \right| \to 0$$

are valid as $\varepsilon \to 0$.

Remark 3. Due to the smoothness of the boundary $\partial\Omega$, the solution v_0 belongs to $H^2(\Omega)$ [33], and, hence, can be continued on Π_ε to belong to $H^2(\Omega_\varepsilon)$ [34].

Lemma 6. Let $\beta > 1-\alpha$ and $F(\hat{x},\hat{\xi})$, $g(\hat{x},\hat{\xi})$, $p(\hat{x},\hat{\xi})$ be periodic in ξ smooth functions. λ is a given matrix, $h(x,\frac{x}{\varepsilon})$ is right-hand function which satisfies conditions (3) and (4). Suppose that $F(\hat{x},\hat{\xi})$ compactly supported in $x \in \Gamma_1$ uniformly in ξ. Then, for all $\varepsilon > 0$ the existence and uniqueness of solution to problem (14) follow, and the strong convergence

$$v_\varepsilon \to v_0 \tag{15}$$

in $\mathbf{V}$ as $\varepsilon \to 0$ is valid.

Proof. Due to Lemma 4 the existence and the uniqueness of solution to problem (1) can be obtained on the base of the Lax-Milgram Lemma ([35]). We extend the function v_0 to the oscillating layer keeping the norm. Then, after simple transformations, we find

$$\int_{\Omega_\varepsilon} \nabla(v_0-v_\varepsilon)\cdot\nabla w dx+\varepsilon^\beta\int_{\Gamma_1^\varepsilon} p(v_0-v_\varepsilon)\cdot w ds = \int_{\Omega_\varepsilon}\nabla v_0\cdot\nabla w dx-\int_{\Omega_\varepsilon} h\cdot w dx-\varepsilon^{1-\alpha}\int_{\Gamma_1^\varepsilon} g\cdot w ds+\varepsilon^\beta\int_{\Gamma_1^\varepsilon} pv_0\cdot w ds =$$

$$= \int_\Omega \nabla v_0 \cdot \nabla w dx - \int_{\Omega_\varepsilon} h \cdot w dx - \varepsilon^{1-\alpha}\int_{\Gamma_1^\varepsilon} g\cdot w ds + \int_{\Pi_\varepsilon} \nabla v_0 \nabla \cdot w dx + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} pv_0 \cdot w ds =$$

$$= \int_{\Pi_\varepsilon} \nabla v_0 \cdot \nabla w dx - \varepsilon^{1-\alpha}\int_{\Gamma_1^\varepsilon} g\cdot w ds + \int_{\Gamma_1} G(\hat{x})\cdot w ds - \int_{\Pi_\varepsilon} h\cdot w dx + \varepsilon^\beta\int_{\Gamma_1^\varepsilon} pv_0\cdot w ds. \tag{16}$$

According to Lemma 3 and Proposition 3 the last integral in the right-hand side of (16) is estimated as follows

$$\varepsilon^\beta \left| \int_{\Gamma_1^\varepsilon} pv_0 \cdot w ds \right| = \varepsilon^\beta \left| \int_{\Gamma_1^\varepsilon} pv_0 \cdot w \left[\varepsilon^{\alpha-1} \left(\sqrt{\left|\nabla_{\hat{\xi}} F\left(\hat{x},\hat{\xi}\right)\right|^2} \Bigg|_{\hat{\xi}=\hat{x}/\varepsilon} + O(\varepsilon^{1-\alpha}) \right) d\hat{x} \right] d\hat{x} \right|$$

$$\le \varepsilon^{\beta-1+\alpha} C_4 \left| \int_{\Gamma_1} pv_0 \cdot w d\hat{x} \right| \le \varepsilon^{\beta-1+\alpha} C_4 \|w\|_{H^{1/2}(\Gamma_1)} \le \varepsilon^{\beta-1+\alpha} C_5 \|w\|_{H^1(\Omega_\varepsilon)}.$$

Recall that in the subcritical case $\beta - 1 + \alpha > 0$ and, therefore, this term is vanishes as $\varepsilon \to 0$.

By (13) considering the uniform boundedness of $\|v_0\|_{H^2(\Omega_\varepsilon)}$, we have

$$\left| \int_{\Pi_\varepsilon} \nabla v_0 \cdot \nabla w dx \right| \le \|\nabla v_0\|_{L_2(\Pi_\varepsilon)} \|w\|_{H^1(\Omega_\varepsilon)} \le C_6 \sqrt{\varepsilon^\alpha} \|v_0\|_{H^2(\Omega_\varepsilon)} \|w\|_{H^1(\Omega_\varepsilon)}$$

and

$$\left| \int_{\Pi_\varepsilon} h \cdot w dx \right| \le \|h\|_{L_2(\Pi_\varepsilon)} \|w\|_{L_2(\Pi_\varepsilon)} \le C_6 \sqrt{\varepsilon^\alpha} \|h\|_{L_2(\Omega_\varepsilon)} \|w\|_{H^1(\Omega_\varepsilon)}.$$

Then, Lemma 5 implies

$$\left| \varepsilon^{1-\alpha} \int_{\Gamma_1^\varepsilon} g \cdot w ds - \int_{\Gamma_1} G(\hat{x}) \cdot w ds \right| \to 0 \text{ as } \varepsilon \to 0. \tag{17}$$

Combining these inequalities and convergence (17) with (16), we deduce

$$\left|\int_{\Omega_\varepsilon} \nabla(v_0 - v_\varepsilon)\cdot\nabla w dx\right| + \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p(v_0 - v_\varepsilon)\cdot w ds \to 0 \ \text{ as } \ \varepsilon \to 0.$$

It remains to substitute $w = v_0 - v_\varepsilon$. Then, (15) follows from Lemma 4 and the Friedrichs type inequality (see, for example, [34], [36] and [37]). Lemma is proved.

Lemma 7. 1) All solutions $u_\varepsilon(t)$ to (1) satisfy

$$\|u_\varepsilon(t)\|_\varepsilon^2 \le \|u_\varepsilon(0)\|_\varepsilon^2 e^{-\varkappa_1 t} + R_1^2, \tag{18}$$

$$\varpi\int_t^{t+1}\|u_\varepsilon(s)\|_{\varepsilon,1}^2 ds + 2a_0\sum_{i=1}^n \gamma_i \int_t^{t+1}\|u_\varepsilon^i(s)\|_{L_{p_i}(\Omega_\varepsilon)}^{p_i} ds + 2p_{\max}\varepsilon^{1-\alpha}\int_t^{t+1}\|u_\varepsilon(s)\|_{\mathbf{L}_2(\Gamma_1^\varepsilon)}^2 ds \le \|u_\varepsilon(t)\|_\varepsilon^2 + R_2^2, \tag{19}$$

where $\varkappa_1 > 0$ is a constant independent of ε. Positive values R_1 and R_2 depend on M_0 (see (4)) and do not depend on $u_\varepsilon(0)$ and ε.

2) All solutions $u(t)$ to (11) satisfy the same inequalities (18) and (19) with the norms in the function spaces over the domain Ω instead Ω_ε.

Proof. We give a brief outline of the proof (see the details in [17]).

In the right hand side of (8) the integral over the part of the boundary Γ_1^ε is nonnegative, because of the positiveness of the matrix p. We integrate (8) with respect to t. Then, to estimate the terms

$$\varepsilon^{1-\alpha}\int_{\Gamma_1^\varepsilon} g\cdot w ds \qquad \text{and} \qquad \varepsilon^\beta \int_{\Gamma_1^\varepsilon} p u_\varepsilon \cdot w ds,$$

we use the Cauchy inequality and the compactness of embedding $\mathbf{L}_2(\Gamma_1^\varepsilon) \Subset \mathbf{V}_\varepsilon$. For other terms we use a standard procedure (see [17]). Lemma is proved.

4 Main assertion

Here we formulate the main result concerning the limit behaviour of the trajectory attractors $\mathfrak{A}_\varepsilon$ of the reaction-diffusion systems (1) as $\varepsilon \to 0$ in the subcritical case $\beta > 1 - \alpha$.

Theorem 1. The following limit holds in the topological space Θ_+^{loc}

$$\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}} \ \text{ as } \varepsilon \to 0+. \tag{20}$$

Moreover,

$$\mathcal{K}_\varepsilon \to \overline{\mathcal{K}} \text{ as } \varepsilon \to 0+ \ \text{ in } \ \Theta^{loc}. \tag{21}$$

Proof. It is easy to see that (21) implies (20). Hence, it is sufficient to prove (21), i.e., for every neighbourhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc}, there exists $\varepsilon_1 = \varepsilon_1(\mathcal{O}) > 0$, such that

$$\mathcal{K}_\varepsilon \subset \mathcal{O}(\overline{\mathcal{K}}) \quad \text{for} \quad \varepsilon < \varepsilon_1. \tag{22}$$

Assume that (22) is not true. Then there exists a neighbourhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\varepsilon_k \to 0+ \ (k \to \infty)$, and a sequence $u_{\varepsilon_k}(\cdot) = u_{\varepsilon_k}(t) \in \mathcal{K}_{\varepsilon_k}$, such that

$$u_{\varepsilon_k} \notin \mathcal{O}'(\overline{\mathcal{K}}) \quad \text{for all} \quad k \in \mathbb{N}.$$

The function $u_{\varepsilon_k}(x,t), t \in \mathbb{R}$ is a solution to

$$\begin{cases} \frac{\partial u_{\varepsilon_k}}{\partial t} = \lambda\Delta u_{\varepsilon_k} - a\left(x, \frac{x}{\varepsilon_k}\right) f(u_{\varepsilon_k}) + h\left(x, \frac{x}{\varepsilon_k}\right), & x \in \Omega_{\varepsilon_k}, \\ \frac{\partial u_{\varepsilon_k}}{\partial \nu} + \varepsilon_k^{\beta} p(\hat{x}, \frac{\hat{x}}{\varepsilon_k}) u_{\varepsilon_k} = \varepsilon_k^{1-\alpha} g(\hat{x}, \frac{\hat{x}}{\varepsilon_k}), & x \in \Gamma_1^{\varepsilon_k}, \\ u_{\varepsilon_k} = 0, & x \in \Gamma_2, \end{cases} \tag{23}$$

where $\beta > 1-\alpha$. To obtain a uniform estimate of the solution in ε, we use Lemma 7. By means of (18) and (19), we obtain that the sequence $\{u_{\varepsilon_k}(x,t)\}$ is bounded in $\mathcal{F}^b$, i.e.,

$$\|u_{\varepsilon_k}\|_{\mathcal{F}^b} = \sup_{t\in\mathbb{R}} \|u_{\varepsilon_k}(t)\| + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\varepsilon_k}(\vartheta)\|_1^2 \, d\vartheta \right)^{1/2} + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\varepsilon_k}(\vartheta)\|_{\mathbf{L_p}}^p \, d\vartheta \right)^{1/p} +$$
$$+ \varepsilon^{\beta} \sup_{t\in\mathbb{R}} \int_t^{t+1} \int_{\Gamma_1^{\varepsilon}} p\Big(\hat{x}, \frac{\hat{x}}{\varepsilon}\Big) u_\varepsilon(x,\vartheta) \cdot u_\varepsilon(x,\vartheta)\, ds\, d\vartheta + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \frac{\partial u_{\varepsilon_k}}{\partial t}(\vartheta) \right\|_{\mathbf{H}^{-r}}^q d\vartheta \right)^{1/q} \le C \ \forall\, k \in \mathbb{N}. \tag{24}$$

Remind that here $\beta > 1-\alpha$. The constant C is independent of ε. Consequently, there exists a subsequence $\{u_{\varepsilon'_k}(x,t)\} \subset \{u_{\varepsilon_k}(x,t)\}$, such that $u_{\varepsilon'_k}(x,t) \to \overline{u}(x,t)$ as $k \to \infty$ in Θ^{loc}. Here $\overline{u}(x,t) \in \mathcal{F}^b$ and $\overline{u}(t)$ satisfy (24) with the same constant C. Because of (24), we get $u_{\varepsilon'_k}(x,t) \rightharpoonup \overline{u}(x,t)$ $(k \to \infty)$ weakly in $\mathbf{L}_2^{loc}(\mathbb{R}; \mathbf{V})$, weakly in $\mathbf{L}_{\mathbf{p}}^{loc}(\mathbb{R}; \mathbf{L_p})$, $*$-weakly in $\mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H})$ and $\frac{\partial u_{\varepsilon'_k}(x,t)}{\partial t} \rightharpoonup \frac{\partial \overline{u}(x,t)}{\partial t}$ $(k \to \infty)$ weakly in $\mathbf{L}_{\mathbf{q},w}^{loc}(\mathbb{R}; \mathbf{H}^{-r})$. We claim that $\overline{u}(x,t) \in \overline{\mathcal{K}}$. We have $\|\overline{u}\|_{\mathcal{F}^b} \le C$. Hence, we have to verify that $\overline{u}(x,t) = u_0(x,t)$, i.e. is a weak solution to (11).

Using (24) and (3), we find that

$$\frac{\partial u_{\varepsilon_k}}{\partial t} - \lambda\Delta u_{\varepsilon_k} - h_{\varepsilon_k}(x) \longrightarrow \frac{\partial \bar{u}}{\partial t} - \lambda\Delta\bar{u} - \overline{h}(x) \quad \text{as} \quad k \to \infty$$

in the space $D'(\mathbb{R}; \mathbf{H}_\varepsilon^{-\mathbf{r}})$, since the derivative operators are continuous in the space of distributions.

Let us prove that

$$a\left(x, \frac{x}{\varepsilon_k}\right) f(u_{\varepsilon_k}) \rightharpoonup \bar{a}(x) f(\bar{u}) \ \text{ as } k \to \infty \tag{25}$$

weakly in $\mathbf{L}_{\mathbf{q},w}^{loc}(\mathbb{R}; \mathbf{L_q})$. We fix an arbitrary number $M > 0$. The sequence $\{u_{\varepsilon_k}(x,t)\}$ is bounded in $\mathbf{L_p}(-M, M; \mathbf{L_p})$ (see (24)). Then, due to (5) the sequence $\{f(u_{\varepsilon_k}(t))\}$ is bounded in $\mathbf{L_q}(-M, M; \mathbf{L_q})$. Since $\{u_{\varepsilon_k}(x,t)\}$ is bounded in $\mathbf{L}_2(-M, M; \mathbf{V})$ and $\left\{\frac{\partial u_{\varepsilon_k}}{\partial t}(t)\right\}$ is bounded in $\mathbf{L_q}(-M, M; \mathbf{H^{-r}})$, we can assume that $u_{\varepsilon_k}(x,t) \to \bar{u}(x,t)$ as $k \to \infty$ strongly in $\mathbf{L}_2(-M, M; \mathbf{L}_2) = \mathbf{L}_2(\Omega \times]-M, M[)$ and, therefore,

$$u_{\varepsilon_k}(x,t) \to \bar{u}(x,t) \ \text{ as } k \to \infty \ \text{ for almost all } (x,t) \in \Omega \times]-M, M[.$$

Since the function $f(v)$ is continuous in $v \in \mathbb{R}$, we conclude that

$$f(u_{\varepsilon_k}(x,t)) \to f(\bar{u}(x,t)) \ \text{ as } k \to \infty \ \text{ for almost all } (x,t) \in \Omega \times]-M, M[. \tag{26}$$

We have

$$a\left(x, \frac{x}{\varepsilon_k}\right) f(u_{\varepsilon_k}) - \bar{a}(x) f(\bar{u}) = a\left(x, \frac{x}{\varepsilon_k}\right) (f(u_{\varepsilon_k}) - f(\bar{u})) + \left(a\left(x, \frac{x}{\varepsilon_k}\right) - \bar{a}(x)\right) f(\bar{u}). \tag{27}$$

Let us show that both terms in the right-hand side of (27) tend to zero as $k \to \infty$ weakly in $\mathbf{L_q}(-M, M; \mathbf{L_q}) = \mathbf{L_q}(\Omega \times]-M, M[)$. First, the sequence $a\left(x, \frac{x}{\varepsilon_k}\right)(f(u_{\varepsilon_k}) - f(\bar{u}))$ tends to zero as $k \to \infty$ for almost all $(x,t) \in \Omega \times]-M,M[$ (see (26)). Applying Lemma 1.3 from [38; Ch. 1, Sec. 1], we conclude that

$$a\left(x, \frac{x}{\varepsilon_k}\right)\Big(f(u_{\varepsilon_k}) - f(\bar{u})\Big) \rightharpoonup 0 \text{ as } k \to \infty$$

weakly in $\mathbf{L_q}(\Omega \times]-M,M[)$. Second, the sequence $\left(a\left(x, \frac{x}{\varepsilon_k}\right) - \bar{a}(x)\right) f(\bar{u})$ also tends to zero a $k \to \infty$ weakly in $\mathbf{L_q}(\Omega \times]-M,M[)$, since $a\left(x, \frac{x}{\varepsilon_k}\right) \rightharpoonup \bar{a}(x)$ as $k \to \infty$ $*$-weakly in $\mathbf{L}_{\infty,*w}(-M,M;\mathbf{L}_2)$ and $f(\bar{u}) \in \mathbf{L_q}(\Omega \times]-M,M[)$. Thus, (25) is proved.

The convergences of $\varepsilon_k^{\beta} p\Big(\hat{x}, \frac{\hat{x}}{\varepsilon_k}\Big) u_{\varepsilon_k}$ to zero and $\varepsilon_k^{1-\alpha} g\Big(\hat{x}, \frac{\hat{x}}{\varepsilon_k}\Big)$ to $G(\hat{x})$ are obvious due to Lemma 5.

Hence, for $\overline{u}(x,t) = u_0(x,t)$, we have

$$-\int_{-M}^{M}\int_{\Omega_{\varepsilon_k}} u_{\varepsilon_k} \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int_{-M}^{M}\int_{\Omega_{\varepsilon_k}} \lambda \nabla u_{\varepsilon_k} \cdot \nabla \psi\, dxdt + \int_{-M}^{M}\int_{\Omega_{\varepsilon_k}} a_{\varepsilon_k}(x) f(u_{\varepsilon_k}) \cdot \psi\, dxdt +$$

$$+{\varepsilon_k}^{\beta} \int_{-M}^{M}\int_{\Gamma_1^{\varepsilon_k}} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon_k}\right) u_{\varepsilon_k} \cdot \psi\, dsdt + {\varepsilon_k}^{1-\alpha} \int_{-M}^{M}\int_{\Gamma_1^{\varepsilon_k}} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon_k}\right) \cdot \psi\, dsdt \longrightarrow -\int_{-M}^{M}\int_{\Omega} u_0 \cdot \frac{\partial \psi}{\partial t}\, dxdt +$$

$$+\int_{-M}^{M}\int_{\Omega} \lambda \nabla u_0 \cdot \nabla \psi\, dxdt + \int_{-M}^{M}\int_{\Omega} \overline{a}(x) f(u_0) \cdot \psi\, dxdt + \int_{-M}^{M}\int_{\Gamma_1} G(\hat{x}) \cdot \psi\, dxdt$$

as $k \to \infty$.

Using (26), we pass to the limit in the equation (23) as $k \to \infty$ in the space $D'(\mathbb{R}; \mathbf{H}^{-r})$ and obtain that the function $u_0(x,t)$ satisfies the integral identity (12) and, therefore, it is a complete trajectory of the equation (11).

Consequently, $u_0 \in \overline{\mathcal{K}}$. We have proved above that $u_{\varepsilon_k} \to u_0$ as $k \to \infty$ in Θ^{loc}. Assumption $u_{\varepsilon_k} \notin \mathcal{O}'(\overline{\mathcal{K}})$ (see [39]) implies $u_0 \notin \mathcal{O}'(\overline{\mathcal{K}})$, and, hence, $u_0 \notin \overline{\mathcal{K}}$. We arrive to the contradiction that complete the proof of the theorem.

Using the compact inclusions (9) and (10), we can improve the convergence (20).

Corollary 4. For any $0 < \delta \le 1$ and for all $M > 0$

$$\mathrm{dist}_{\mathbf{L_2}([0,M];\mathbf{H}^{1-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0, \tag{28}$$

$$\mathrm{dist}_{\mathbf{C}([0,M];\mathbf{H}^{-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0 \quad (\varepsilon \to 0+). \tag{29}$$

To prove (28) and (29), we repeat the proof of Theorem 1 changing the topology Θ^{loc} on $\mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta})$ or $\mathbf{C}^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta})$.

Finally, we consider the reaction–diffusion systems for which the uniqueness theorem is true for the Cauchy problem. It suffices to assume that the nonlinear term $f(u)$ in (1) satisfies the condition

$$(f(v_1) - f(v_2), v_1 - v_2) \ge -C|v_1 - v_2|^2 \text{ for any } v_1, v_2 \in \mathbb{R}^n. \tag{30}$$

(see [17,31]). In [31] it was proved that if (30) is true, then (1) and (11) generate dynamical semigroups in $\mathbf{H}$, possessing global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ are bounded in $\mathbf{V}$ (see also [16], [15]). Moreover

$$\mathcal{A}_\varepsilon = \{u(0) \mid u \in \mathfrak{A}_\varepsilon\},\ \overline{\mathcal{A}} = \{u(0) \mid u \in \overline{\mathfrak{A}}\}.$$

The convergence (29) gives

Corollary 5. Under the assumption of Theorem 1 the limit formula takes place

$$\mathrm{dist}_{\mathbf{H}^{-\delta}}\left(\mathcal{A}_\varepsilon, \overline{\mathcal{A}}\right) \to 0 \ (\varepsilon \to 0+).$$

Conclusion

In the paper we study the reaction–diffusion systems of equations with rapidly oscillating terms in domains with locally periodic rapidly oscillating boundary depending on a small parameter. We construct the homogenized system of equations, define and proved the existence of the trajectory and global attractors to these systems and prove that they converge in a weak sense to the trajectory and global attractors of the limit (homogenized) reaction–diffusion systems in domain independent of the small parameter. In this paper we consider the subcritical case in which the Fourier type boundary condition transforms to the Neumann boundary condition under the limit passage.

Acknowledgments

The authors express their thanks to the anonymous reviewer for careful reading of the text and for useful comments that allowed us to improve the presentation of the text and results.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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