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Mixed problem for a third order parabolic-hyperbolic model equation

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In 1978, the journal *Differential Equations* published an article by A.M. Nakhushev, that presented a method for correctly formulating a boundary value problem for a class of second-order parabolic-hyperbolic equations in an arbitrarily bounded domain with a smooth or piecewise smooth boundary. In that work, a boundary value problem was formulated and investigated using the method of a priori estimates, which is currently called the first boundary value problem for a second-order mixed parabolic-hyperbolic equation. In this work, a boundary value problem for a third-order model parabolic-hyperbolic equation is formulated and investigated in a mixed domain, following the approach of A.M. Nakhushev for second-order mixed parabolic-hyperbolic equations. In one part of the mixed domain, the equation under consideration is a degenerate hyperbolic equation of the first kind of the second order, and in the other part, it is a nonhomogeneous equation of the third order with multiple characteristics and reverse-time parabolic type. For various values of the parameter, existence and uniqueness theorems for a regular solution are proved. The uniqueness theorem is proved using the method of energy integrals combined with A.M. Nakhushev's method. The existence theorem is proved by the method of integral equations. In terms of the Mittag-Leffler function, the solution of the problem is found and written out explicitly. Sufficient smoothness conditions for the given functions are found, which ensure the regularity of the obtained solution.

Keywords: second order degenerate hyperbolic equation of the first kind, third-order equation with multiple characteristics, third-order parabolic-hyperbolic equation, Volterra integral equation, Fredholm integral equation, Tricomi method, method of integral equations, integral equation method, Mittag-Leffler functions.

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Introduction

Boundary value problems for model second order parabolic-hyperbolic equations were first studied in [1,2]. The classification of parabolic-hyperbolic equations into equations with characteristic and non-characteristic lines of type change was carried out by [3]. Moreover, in the work of [1], the problem was studied for a model equation of parabolic-hyperbolic type with a characteristic line of type change, and in the work of [2], the problem was studied for a model equation with a non-characteristic line of type change. In 1978, the journal Differential Equations published an article by A.M. Nakhushev, which provided a method for correctly formulating a boundary value problem for a general second-order parabolic-hyperbolic equation in an arbitrary bounded domain with a smooth or piecewise smooth boundary. The boundary value problem investigated in the aforementioned work by A.M. Nakhushev is currently called the first boundary value problem for a mixed parabolic-hyperbolic equation. The most complete review on boundary value problems for parabolic-hyperbolic equations one can find in monographs [4, 5].

The paper considers one mixed problem for a third-order parabolic-hyperbolic model equation. One part of the mixed domain, involves a third order nonhomogeneous parabolic type equation with multiple characteristics, while the other part involves a second order degenerate hyperbolic type equation of the first kind. The paper presents proofs of the existence and uniqueness theorems for a regular solution.

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The uniqueness proof is based on the method of energy integrals combined with A.M. Nakhushev's method. The existence proof is based on the method of integral equations. In solving the problem, we also used the Mittag-Leffler function and wrote down the solution explicitly.

1 *Formulation of the problem*

On the Euclidean plane with independent variables x and y consider the equation

$$0 = \begin{cases} (-y)^m u_{xx} - u_{yy} + \lambda (-y)^{(m-2)/2} u_x, & y < 0, \\ u_{xxx} + u_y - f, & y > 0, \end{cases} \tag{1}$$

where λ, m are the given numbers, and $m > 0$, $|\lambda| \leq \frac{m}{2}$; $f = f(x,y)$ is the given function; $u = u(x,y)$ is the desired function.

Equation (1) is a model third order equation of the parabolic-hyperbolic type. For $y < 0$, it is equivalent to the degenerate hyperbolic equation of the first kind

$$(-y)^m u_{xx} - u_{yy} + \lambda (-y)^{\frac{m-2}{2}} u_x = 0, \tag{2}$$

and for $y > 0$ with the third-order inhomogeneous parabolic type equation with multiple characteristics

$$u_{xxx} + u_y = f(x,y). \tag{3}$$

The paper [6] is devoted to the study of the problem with a shift for a degenerate hyperbolic equation of the first kind of the form (2), and the local first and second boundary value problems for a degenerate hyperbolic equation of the second kind are investigated in the papers [7, 8]. The papers [9, 10] are devoted to the study of nonlocal problems of degenerate hyperbolic equations with singular coefficients. In [11], a nonlocal problem of the Frankl type for a second-order mixed parabolic-hyperbolic equation with a characteristic line of type change is investigated. The papers [12, 13] are devoted to the problems of conjugation of the generalized diffusion equation and degenerate hyperbolic equations. The problem with a shift for one second-order mixed parabolic-hyperbolic equation with two perpendicular lines of type change is studied in the paper [14]. Nonlocal problems with a shift in the conjugation of a third-order equation with multiple characteristics and a degenerate hyperbolic equation of the first kind of the second order are formulated and investigated in [15, 16]. A nonlocal problem for a third-order mixed parabolic-hyperbolic equation is investigated in [17].

Equation (1), in this paper, is considered in the mixed domain $\Omega = \Omega_1 \cup \Omega_2 \cup I$, where Ω_1 is the domain limited by the characteristics $\sigma_1 = AC : x - \frac{2}{m+2}(-y)^{(m+2)/2} = 0$ and $\sigma_2 = CB : x + \frac{2}{m+2}(-y)^{(m+2)/2} = r$ of equation (2) for $y < 0$, coming out from the point $C = (r/2, y_c)$, $y_c = -\left[\frac{(m+2)\,r}{4}\right]^{\frac{2}{m+2}}$, passing through the points $A = (0,0)$, $B = (r,\,0)$ and the segment $J = AB$ of the straight line $y = 0$; and Ω_2 is the rectangular domain with vertices at $A = (0,\,0)$, $A_0 = (0,\,h)$, $B_0 = (r,\,h)$ and $B = (r,\,0)$, $h = const > 0$; $J = \{(x,0) :\ 0 < x < r\}$ is the interval of AB of the straight line $y = 0$.

A regular, in the domain Ω, solution to equation (1) we call the function $u = u(x,y)$ by the class $C\left(\bar{\Omega}\right) \cap C^1\left(\Omega\right) \cap C^2\left(\Omega_1\right) \cap C^{x,y}_{3,1}\left(\Omega_2\right)$; $u_x(x,0)$, $u_y(x,0) \in L_1\left(J\right)$, when substituted, equation (1) becomes an identity.

Problem 1. Find a solution to equation (1) regular in the domain and satisfies the conditions

$$u(0,y) = \varphi_1(y), \quad u(r,y) = \varphi_2(y), \quad u_x(r,y) = \varphi_3(y), \quad 0 < y < h, \tag{4}$$

$$u\left[\theta_0(x)\right] = \psi(x), \quad 0 \leq x \leq r, \tag{5}$$

where $\theta_0(x) = \left(\frac{x}{2}, -\left(\frac{m+2}{4}\right)^{2/(m+2)} x^{2/(m+2)}\right)$ is the affix of the intersection point of a characteristic emanating from the point $(x, 0) \in J$ with the characteristic AC; $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$ are the functions defined on the segment $0 \le y < h$; $\psi(x)$ is the function given on the segment $0 \le x \le r$ with the matching condition $\varphi_1(0) = \psi(0)$ satisfied.

2 Uniqueness theorem

Let there be a regular solution $u = u(x, y)$ of equation (1) in the domain Ω by the class $C\left(\bar{\Omega}\right) \cap C^1(\Omega)$ and let

$$u(x, 0) = \tau(x), \quad 0 \le x \le r, \tag{6}$$

$$u_y(x, 0) = \nu(x), \quad 0 < x < r. \tag{7}$$

Then, passing in equation (1) to the limit at $y \to +0$, taking into account the notations (6), (7) and conditions (4), we immediately obtain the first fundamental relationship between the functions $\tau(x)$ and $\nu(x)$, transferred from the parabolic part Ω_2 of the domain Ω to the line of type change J:

$$\tau'''(x) + \nu(x) = f(x, 0), \quad 0 < x < r, \tag{8}$$

$$\tau(0) = \varphi_1(0), \quad \tau(r) = \varphi_2(0), \quad \tau'(r) = \varphi_3(0). \tag{9}$$

Next, find the relationship between the functions $\tau(x)$ and $\nu(x)$, brought from the hyperbolic domain Ω_1 of equation (1) to the segment AB of the straight line $y = 0$. To do this, we first note that in the characteristic coordinates $\xi = x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}}$, $\eta = x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}}$, equation (2) becomes the Euler–Darboux–Poisson equation

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta} - \frac{\beta_1}{\eta - \xi}\frac{\partial u}{\partial\eta} + \frac{\beta_2}{\eta - \xi}\frac{\partial u}{\partial\xi} = 0,$$

where $\beta_1 = \frac{m-2\lambda}{2(m+2)}$, $\beta_2 = \frac{m+2\lambda}{2(m+2)}$. Designate additionally: $\beta = \beta_1 + \beta_2 = \frac{m}{m+2}$.

First assume $|\lambda| < \frac{m}{2}$ and then $\tau(x) \in C[0, r] \cap C^2(0, r)$, $\nu(x) \in C^1(0, r) \cap L_1(0, r)$. Hence, the regular solution to problem (6), (7) for equation (2) in Ω_1 is written out by the formula in [18; p. 14]:

$$u(x, y) = \frac{\Gamma(\beta)}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^1 \tau\left[x + (1-\beta)(-y)^{1/(1-\beta)}(2t-1)\right] t^{\beta_2 - 1}(1-t)^{\beta_1 - 1}\,dt +$$

$$+\frac{\Gamma(2-\beta)\,y}{\Gamma(1-\beta_1)\Gamma(1-\beta_2)}\int_0^1 \nu\left[x + (1-\beta)(-y)^{1/(1-\beta)}(2t-1)\right] t^{-\beta_1}(1-t)^{-\beta_2}\,dt, \tag{10}$$

where $\Gamma(p) = \int\limits_0^\infty \exp(-t)\, t^{p-1}\,dt$ is the Euler integral of the second kind (Gamma function).

Satisfying in (10) condition (5), we get:

$$u[\theta_0(x)] = u\left[\frac{x}{2}, -(2-2\beta)^{\beta-1}x^{1-\beta}\right] = \frac{\Gamma(\beta)}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^1 \tau(xt)\, t^{\beta_2-1}(1-t)^{\beta_1-1}\,dt -$$

$$-\frac{(2-2\beta)^{\beta-1}x^{1-\beta}\,\Gamma(2-\beta)}{\Gamma(1-\beta_1)\Gamma(1-\beta_2)}\int_0^1 \nu(xt)\, t^{-\beta_1}(1-t)^{-\beta_2}\,dt = \psi(x).$$

Introducing the new integration variable $z = xt$, rewrite the last equality as

$$\frac{\Gamma(\beta)\, x^{1-\beta}}{\Gamma(\beta_1)\,\Gamma(\beta_2)} \int\limits_0^x \frac{\tau(z)\, z^{\beta_2-1}}{(x-z)^{1-\beta_1}}\, dz - \frac{(2-2\beta)^{\beta-1}\ \Gamma(2-\beta)}{\Gamma(1-\beta_1)\,\Gamma(1-\beta_2)} \int\limits_0^1 \frac{\nu(z)\, z^{-\beta_1}}{(x-z)^{\beta_2}}\, dz = \psi(x).$$

Employing the fractional integro-differentiation operator D_{cx}^{α} (in the Riemann–Liouville sense) [19], rewrite the last equality as follows

$$\frac{\Gamma(\beta)\ x^{1-\beta}}{\Gamma(\beta_2)} D_{0x}^{-\beta_1} \left\{\tau(t)\ t^{\beta_2-1}\right\} - \frac{(2-2\beta)^{\beta-1}\ \Gamma(2-\beta)}{\Gamma(1-\beta_1)} D_{0x}^{\beta_2-1} \left\{\nu(t)\ t^{-\beta_1}\right\} = \psi(x). \tag{11}$$

Inverting equation (11) relative to the function $\nu(x)$, and using the well-known weighted Riemann–Liouville fractional integral and differential operators with the same origins [19; p. 18], find

$$\nu(x) = \gamma_1\, D_{0x}^{1-\beta}\tau(t) - \gamma_2\, x^{\beta_1}\, D_{0x}^{1-\beta_2}\psi(t), \tag{12}$$

where $\gamma_1 = \frac{\Gamma(1-\beta_1)\,\Gamma(\beta)\,(2-2\beta)^{1-\beta}}{\Gamma(\beta_2)\,\Gamma(2-\beta)}$, $\gamma_2 = \frac{\Gamma(1-\beta_1)(2-2\beta)^{1-\beta}}{\Gamma(2-\beta)}$.

Indeed relation (12) is the main fundamental relation between the sought functions $\tau(x)$ and $\nu(x)$ transferred from the domain Ω_1 to the line of type change J when $|\lambda| < \frac{m}{2}$.

In the case if $\lambda = \frac{m}{2}$, the coefficients $\beta_1 = 0$, $\beta_2 = \beta = \frac{m}{m+2}$, $\gamma_1 = \gamma_2 = \frac{(2-2\beta)^{1-\beta}}{\Gamma(2-\beta)}$, and the solution to problem (6), (7) for equation (2) can be written by the formula [18; p. 15]:

$$u(x,y) = \tau\left[x + \frac{2}{m+2}(-y)^{(m+2)/2}\right] + \frac{2y}{m+2}\int\limits_0^1 \nu\left[x + \frac{2}{m+2}(-y)^{(m+2)/2}(2t-1)\right](1-t)^{-\beta}\, dt. \tag{13}$$

Satisfying condition (5) in representation (13), we arrive at the fundamental relationship between the functions $\tau(x)$ and $\nu(x)$ as bellow

$$\nu(x) = \gamma_1\left[D_{0x}^{1-\beta}\tau(t) - D_{0x}^{1-\beta}\psi(t)\right]. \tag{14}$$

In the case if $\lambda = -\frac{m}{2}$, then $\beta_1 = \beta = \frac{m}{m+2}$, $\beta_2 = 0$, $\gamma_1 = 0$, $\gamma_2 = 2^{1-\beta}(1-\beta)^{-\beta}$. The solution to problem (6), (7) for equation (2) here has the form [18; p. 15]:

$$u(x,y) = \tau\left[x - \frac{2}{m+2}(-y)^{(m+2)/2}\right] + \frac{2y}{m+2}\int\limits_0^1 \nu\left[x - \frac{2}{m+2}(-y)^{(m+2)/2}(2t-1)\right](1-t)^{-\beta}\, dt. \tag{15}$$

By (15) under condition (5), we immediately get:

$$\nu(x) = -2^{1-\beta}(1-\beta)^{-\beta}\, x^{\beta}\, \psi'(x). \tag{16}$$

The following theorem on the unique solution to *Problem 1* is true.

Theorem 1. There cannot be more than one regular solution for *Problem 1* in the domain Ω.

Proof. Let's take a homogeneous problem equivalent to *Problem 1.* For instance, assume that $f(x,y) \equiv 0\ \forall\,(x,y) \in \bar{\Omega}_2$, $\varphi_1(y) = \varphi_2(y) = \varphi_3(y) \equiv 0\ \forall\, y \in [0,h]$ and $\psi(x) \equiv 0\ \forall\, x \in [0,r]$. Moreover, taking into account that $\tau(0) = \psi(0) = 0$ by relations (12), (14), (16) for different λ, obtain the bellow equalities:

$$\nu(x) = \gamma_1\, D_{0x}^{1-\beta}\tau(t) = \gamma_1\, D_{0x}^{-\beta}\tau'(t) = \gamma_1\, \partial_{0x}^{1-\beta}\tau(t), \quad -\frac{m}{2} < \lambda \le \frac{m}{2}, \tag{17}$$

$$\nu(x) \equiv 0, \quad \lambda = -\frac{m}{2}, \tag{18}$$

where $\partial_{0x}^{\alpha}\varphi\,(t)$ is the fractional differential operator (in the sense of Caputo).

To further discuss, make use of the operator $\partial_{0x}^{\alpha}\varphi\,(t)$ following property [20]: for any absolutely continuous function $\varphi = \varphi(x)$ on the segment $[0, r]$ that satisfies the condition $\varphi(0) = 0$, the inequality

$$\varphi(x)\,\partial_{0x}^{\alpha}\varphi\,(t) \geq \frac{1}{2}\,\partial_{0x}^{\alpha}\varphi^2\,(t)\,, \quad 0 < \alpha \leq 1 \tag{19}$$

holds.

Let us consider the integral

$$I = \int\limits_0^r \tau(x)\,\nu(x)\,dx. \tag{20}$$

When $-\frac{m}{2} < \lambda \leq \frac{m}{2}$ by (17) and (20), taking into account inequality (19), we arrive at

$$I = \int\limits_0^r \tau(x)\,\nu(x)\,dx = \gamma_1 \int\limits_0^r \tau(x)\,\partial_{0x}^{1-\beta}\tau\,(t)\;dx \geq$$

$$\geq \frac{\gamma_1}{2}\int\limits_0^r \partial_{0x}^{1-\beta}\tau^2\,(t)\;dx = \frac{\gamma_1}{2\Gamma\,(\beta)}\int\limits_0^r (r-x)^{\beta-1}\,\tau^2(x)\,dx \geq 0. \tag{21}$$

On the other hand, for a homogeneous problem equivalent to *Problem 1* write, bearing in mind (8), (9), the integral (20) as follows

$$I = \int\limits_0^r \tau(x)\,\nu(x)\,dx = -\int\limits_0^r \tau(x)\,\tau'''(x)\,dx = -\frac{1}{2}\,\left[\tau'(0)\right]^2 \leq 0. \tag{22}$$

By inequalities (21) and (22) it follows that the integral $I = 0$, which as follows from the equality,

$$I = \frac{\gamma_1}{2\Gamma\,(\beta)}\int\limits_0^r (r-x)^{\beta-1}\,\tau^2(x)\,dx = 0$$

may occur if and only if $\tau(x) \equiv 0\;\forall\, x \in [0,\, r]$. Then basing on relations (8) and (17) find out that $\nu(x) \equiv 0$ for all $x \in [0,\, r]$ and any $\lambda \in \left(-\frac{m}{2};\, \frac{m}{2}\right]$.

However, if $\lambda = -\frac{m}{2}$, then by (8), (9), and (18) we come to the homogeneous problem

$$\tau(0) = 0, \quad \tau(r) = 0, \quad \tau'(r) = 0 \tag{23}$$

for equation

$$\tau'''(x) = 0, \quad 0 < x < r. \tag{24}$$

Just like in the case $\lambda \in \left(-\frac{m}{2};\, \frac{m}{2}\right]$, the solution to problem (23) for equation (24) cannot be anything but trivial: $\tau(x) \equiv 0$ and $\nu(x) \equiv 0$ for all $x \in [0,\, r]$.

Consequently, as per formula (10), (13) and (15), the solution $u(x, y) \equiv 0$ in Ω_1 to be considered as the solution to homogeneous Cauchy problem (6), (7) for equation (2) for all $\lambda \in \left[-\frac{m}{2};\, \frac{m}{2}\right]$.

Let's show now that even for the homogenous problem

$$Lu = u_{xxx} + u_y = 0, \quad (x, y) \in \Omega_2, \tag{25}$$

$$u(0,y)=0, \quad u(r,y)=0, \quad u_x(r,y)=0, \quad 0<y<h, \tag{26}$$

$$u(x,0)=0, \quad 0\le x\le r \tag{27}$$

in the domain Ω_2 regular solutions are not possible except for trivial ones.

Indeed, let's assume that problems (25)–(27) have a nontrivial solution $u=u(x,y)\neq 0$. Following the work [4; p. 237], in equation (26) put

$$u(x,y)=\upsilon(x,y)\,\exp\left(\mu\, y\right), \tag{28}$$

where $\mu = const$ is some real number.

In this case, by (25) relative to the function $\upsilon=\upsilon(x,y)$, we arrive at the equation

$$L_\mu \upsilon = \upsilon_{xxx}+\upsilon_y+\mu\,\upsilon=0, \quad (x,y)\in\Omega_2 \tag{29}$$

with initial boundary conditions

$$\upsilon(0,y)=0, \quad \upsilon(r,y)=0, \quad \upsilon_x(r,y)=0, \quad 0<y<h, \tag{30}$$

$$\upsilon(x,0)=0, \quad 0\le x\le r. \tag{31}$$

Since, by assumption $u=u(x,y)\neq 0$, then, as follows from (28), the solutions to problems (29)–(31) will also be non-trivial $\upsilon=\upsilon(x,y)\neq 0$.

Introduce an auxiliary domain $\Omega_{2\varepsilon}=\{(x,y):\ \varepsilon<x<r-\varepsilon,\ \varepsilon<y<h-\varepsilon,\ \varepsilon>0\}$, where the identity

$$2\left(\upsilon,\, L_\mu \upsilon\right)_0 = 2\int\limits_{\Omega_{2\varepsilon}} \upsilon\, L_\mu\upsilon\, d\Omega_{2\varepsilon} = 2\int\limits_{\Omega_{2\varepsilon}} \upsilon\,\left[\upsilon_{xxx}+\upsilon_y+\mu\,\upsilon\right]\, d\Omega_{2\varepsilon} =$$

$$=\int\limits_{\Omega_{2\varepsilon}}\left[\frac{\partial}{\partial x}\left(2\,\upsilon\,\upsilon_{xx}-\upsilon_x^2\right)+\frac{\partial}{\partial y}\left(\upsilon^2\right)+2\mu\,\upsilon^2\right]\,d\Omega_{2\varepsilon}=0$$

is valid.

Applying Green's formula to the latter equality, obtain

$$2\left(\upsilon,\, L_\mu \upsilon\right)_0 = \int\limits_{\Gamma_{2\varepsilon}}\left(2\,\upsilon\,\upsilon_{xx}-\upsilon_x^2\right)\,dy-\upsilon^2\,dx+2\,\mu\int\limits_{\Omega_{2\varepsilon}}\upsilon^2(x,y)\,d\Omega_{2\varepsilon}=0, \tag{32}$$

where $\Gamma_{2\varepsilon}$ is the auxiliary boundary for $\Omega_{2\varepsilon}$. Let us pass to the limit in the last equality at $\varepsilon\to 0$. It is easy to see that in this case the auxiliary domain $\Omega_{2\varepsilon}$ goes into the domain Ω_2, and the boundary $\Gamma_{2\varepsilon}$ of the auxiliary domain $\Omega_{2\varepsilon}$ goes into the boundary Γ_2 of the domain Ω_2. Taking into account the homogeneous initial-boundary conditions (26)-(27) and the above circumstances, by (32) we arrive at the equality

$$2\left(\upsilon,\, L_\mu \upsilon\right)_0 = \int\limits_0^h \upsilon_x^2(0,y)dy+\int\limits_0^r \upsilon^2\left(x,\,h\right)dx+2\,\mu\int\limits_{\Omega_2}\upsilon^2(x,y)\,d\Omega_2=0. \tag{33}$$

By choosing a positive value for the parameter $\mu>0$, we note that (33) can occur if and only if $\upsilon(x,y)\equiv 0$ in the closure of the domain $\bar{\Omega}_2$, which contradicts the initial assumption that $\upsilon=\upsilon(x,y)\neq 0$. However then $u(x,y)\equiv 0$ in $\bar{\Omega}_2$ as follows by (28). Thus, $u(x,y)\equiv 0$ in $\bar{\Omega}$, that is, the solution to problem (1), (4), (5) is unique in the class of regular functions. The theorem is proved.

3 Existence theorem

Let us move on to the existence of a regular solution in Ω to *Problem 1.*

Theorem 2. Let the given functions $f(x,y)$, $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$, $\psi(x)$ be such that they have the properties

$$\varphi_1(y) \in C\,[0,h] \cap C^2\,(0,h)\,,\ \varphi_2(y) \in C\,[0,h] \cap C^2\,(0,h)\,,\ \varphi_3(y) \in C\,[0,h] \cap C^1\,(0,h)\,; \tag{34}$$

$$\psi(x) \in C^1\,[0,r] \cap C^2\,(0,r)\,; \tag{35}$$

$$f(x,y) \in C^1\left(\bar{\Omega}_2\right). \tag{36}$$

Then there is a regular solution to problem (1), (4), (5) in the domain Ω.

Proof. In fact, following the fundamental relationships (8), (12) and (14) obtained above, with respect to the sought functions $\tau(x)$ and $\nu(x)$ at $\lambda \in \left(-\frac{m}{2};\ \frac{m}{2}\right]$ we arrive at the system of equations

$$\begin{cases} \nu(x) = \gamma_1\, D_{0x}^{1-\beta}\tau\,(t) - \gamma_2\, x^{\beta_1}\, D_{0x}^{1-\beta_2}\psi\,(t)\ , \\ \tau'''(x) + \nu(x) = f(x,0). \end{cases} \tag{37}$$

From system (37) we arrive at the problem of finding a regular solution $\tau = \tau(x)$ of an ordinary differential equation of the third order of the form

$$\tau'''(x) + \gamma_1\, D_{0x}^{1-\beta}\tau\,(t) = f(x,0) + \gamma_2\, x^{\beta_1}\, D_{0x}^{1-\beta_2}\psi\,(t)\,, \quad 0 < x < r, \tag{38}$$

satisfying conditions (9).

Repeating integration of (38) three times from 0 to x, arrive at an integral equation equivalent to the given differential equation:

$$\tau(x) = -\frac{\gamma_1}{\Gamma\,(\beta+2)} \int\limits_0^x (x-t)^{\beta+1}\ \tau\,(t)\ dt + \frac{1}{2}\int\limits_0^x (x-t)^2\ F\,(t)\ dt + c_1 + c_2\, x + \frac{1}{2}\, c_3\, x^2, \tag{39}$$

where $F(x) = f(x,0) + \gamma_2\, x^{\beta_1}\, D_{0x}^{1-\beta_2}\psi\,(t)$, and c_1, c_2, c_3 are still arbitrary constants.

Equation (39) is the Volterra integral equation of the second kind with convolution kernel $K\,(x,\,t) = \frac{(x-t)^{\beta+1}}{\Gamma(\beta+2)}$. The functions $K_n\,(x,\,t) = \frac{(x-t)^{n(\beta+2)+\beta+1}}{\Gamma[n(\beta+2)+\beta+2]}$, $n = 0,\,1,\,2,\,\ldots$ are considered iterated kernels of $K\,(x,\,t)$, and the function

$$R\,(x,\,t;\,\beta) = \sum_{n=0}^{\infty} (-\gamma_1)^n\ K_n\,(x,\,t) = (x-t)^{\beta+1} \sum_{n=0}^{\infty} \frac{\left[(-\gamma_1)\ (x-t)^{(\beta+2)}\right]^n}{\Gamma\,[n\,(\beta+2)+\beta+2]}$$

is the resolving kernel $K\,(x,\,t)$ of equation (39).

With the Mittag-Leffler function, the resolving $R\,(x,\,t;\,\beta)$ of equation (39) kernel $K\,(x,\,t)$ of equation (39) takes the following form

$$R\,(x,\,t;\,\beta) = (x-t)^{\beta+1}\ E_{1/(\beta+2)}\left[-\gamma_1\ (x-t)^{\beta+2}\,;\,\beta+2\right],$$

where $E_\rho\,(z,\,\mu) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho^{-1}\,n+\mu)}$ is the Mittag-Leffler function.

The solution of (39) can be written with the resolving $R\,(x,\,t;\,\beta)$ of $K\,(x,\,t)$ as follows

$$\tau(x) = c_1 + c_2\, x + \frac{1}{2}\, c_3\, x^2 + \frac{1}{2}\int\limits_0^x (x-t)^2\ F\,(t)\ dt - c_1\,\gamma_1 \int\limits_0^x R\,(x,\,t;\,\beta)\ dt-$$

$$-c_2\,\gamma_1\int\limits_0^x t\,R\,(x,\,t;\,\beta)\,dt-\frac{\gamma_1\,c_3}{2}\int\limits_0^x t^2R\,(x,\,t;\,\beta)\,dt-\frac{\gamma_1}{2}\int\limits_0^x R\,(x,\,t;\,\beta)\left(\int\limits_0^t (t-s)^2\;F\,(s)\,ds\right)dt. \quad (40)$$

By direct calculation find out that

$$\int\limits_0^x R\,(x,\,t;\,\beta)\;dt=\int\limits_0^x (x-t)^{\beta+1}\;E_{1/(\beta+2)}\left[-\gamma_1\;(x-t)^{\beta+2};\,\beta+2\right]\,dt=$$

$$=x^{\beta+2}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\;\beta+3\right);$$

$$\int\limits_0^x t\,R\,(x,\,t;\,\beta)\;dt=x^{\beta+3}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\;\beta+4\right);$$

$$\int\limits_0^x t^2\,R\,(x,\,t;\,\beta)\;dt=2\,x^{\beta+4}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\;\beta+5\right);$$

$$\int\limits_0^x R\,(x,\,t;\,\beta)\left(\int\limits_0^t (t-s)^2\,F\,(s)\,ds\right)dt=\int\limits_0^x\left(\int\limits_s^x (t-s)^2\,R\,(x,\,t;\,\beta)\,dt\right)F\,(s)\,ds=$$

$$=2\int\limits_0^x (x-t)^{\beta+4}\,E_{1/(\beta+2)}\left[-\gamma_1\;(x-t)^{\beta+2};\;\beta+5\right]F\,(t)\,dt.$$

Considering the above calculations, rewrite representation (40) as follows

$$\tau(x)=\left[1-\;\gamma_1\,x^{\beta+2}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\;\beta+3\right)\right]c_1+$$

$$+\left[x-\gamma_1 x^{\beta+3}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\;\beta+4\right)\right]c_2+$$

$$+\frac{1}{2}\left[x^2-2\gamma_1\,x^{\beta+4}\,E_{1/(\beta+2)}\left(-\gamma_1\,x^{\beta+2};\;\beta+5\right)\right]c_3+$$

$$+\frac{1}{2}\int\limits_0^x\left\{(x-t)^2-2\,\gamma_1\,(x-t)^{\beta+4}\;E_{1/(\beta+2)}\left[-\gamma_1\;(x-t)^{\beta+2};\;\beta+5\right]\right\}F\,(t)\,dt. \quad (41)$$

Satisfying conditions (9) for (41), get to the next system of equation with respect to c_2, c_3:

$$\tau(0)=c_1=\varphi_1(0),$$

$$\begin{cases} \left[r-\gamma_1\,r^{\beta+3}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\;\beta+4\right)\right]c_2+\frac{1}{2}\left[r^2-2\gamma_1\,r^{\beta+4}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\;\beta+5\right)\right]c_3= \\ =\varphi_2(0)-\left[1-\;\gamma_1\,r^{\beta+2}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\;\beta+3\right)\right]\;\varphi_1(0)- \\ -\frac{1}{2}\int\limits_0^r\left\{(r-t)^2-2\,\gamma_1\,(r-t)^{\beta+4}\;E_{1/(\beta+2)}\left[-\gamma_1\;(r-t)^{\beta+2};\;\beta+5\right]\right\}F\,(t)\;dt; \\ \\ \left[1-\gamma_1 r^{\beta+2}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\;\beta+3\right)\right]c_2+\left[r-\gamma_1\,r^{\beta+3}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\;\beta+4\right)\right]c_3= \\ =\varphi_3(0)+\gamma_1\,r^{\beta+1}\,E_{1/(\beta+2)}\left(-\gamma_1\,r^{\beta+2};\;\beta+2\right)\;\varphi_1(0)- \\ -\int\limits_0^r\left\{(r-t)-\gamma_1\,(r-t)^{\beta+3}\;E_{1/(\beta+2)}\left[-\gamma_1\;(r-t)^{\beta+2};\;\beta+4\right]\right\}F\,(t)\;dt\,. \end{cases} \quad (42)$$

The determinant

$$\Delta = \left[r - \gamma_1 r^{\beta+3} E_{1/(\beta+2)} \left(-\gamma_1 r^{\beta+2};\, \beta+4\right)\right]^2 - \frac{1}{2}\left[1 - \gamma_1 r^{\beta+2} E_{1/(\beta+2)} \left(-\gamma_1 r^{\beta+2};\, \beta+3\right)\right] \times$$
$$\times \left[r^2 - 2\gamma_1 r^{\beta+4} E_{1/(\beta+2)} \left(-\gamma_1 r^{\beta+2};\, \beta+5\right)\right]$$

of system (42) is always different from zero by virtue of the uniqueness theorem proved above, that is, the constants c_1, c_2, c_3 in (41) are uniquely determined by conditions (9) and system (42).

Thus, the unique solution to problem (38), (9) for any $\lambda \in \left(-\frac{m}{2};\, \frac{m}{2}\right]$ is obtained by formula (41), where the constants c_1, c_2, c_3 are uniquely determined by (42).

Next, by relations (8) and (16) for $\lambda = -\frac{m}{2}$ in view of conditions (9), obtain

$$\tau(x) = \frac{1}{2r^2}\Bigg\{2\,(r-x)^2\ \varphi_1(0) + 2\,x\ (2r-x)\ \varphi_2(0) + 2rx\ (x-r)\,\varphi_3(0) +$$
$$+ (r-x)^2 \int_0^r t^2 \left[f\,(t,\,0) + 2^{1-\beta}\ (1-\beta)^{-\beta}\ t^{\beta}\,\psi'\,(t)\right] dt -$$
$$- r^2 \int_x^r (t-x)^2 \left[f\,(t,\,0) + 2^{1-\beta}\ (1-\beta)^{-\beta}\ t^{\beta}\,\psi'\,(t)\right] dt\Bigg\}.$$

Once the function $\tau(x)$ is obtained, the second desired function $\nu(x)$, depending on λ, can be obtained using relations (8), (12), (14) or (16). Then the regular solution to Problem 1 in the domain Ω_1 is defined as the solution to the Cauchy problem (6)-(7) for equation (2) and is written out according to one of the formulas (11), (13) or (15). And in the domain Ω_2 we arrive at the initial-boundary value problem (4), (6) for equation (3), the solution of which is written out in the monograph of T.D. Dzhuraev. Note that the conditions (34), (35), (36) listed in Theorem 2 ensure the regularity of the obtained solution in the domain Ω.

Conclusion

In the work in the mixed domain one boundary value problem for the model equation of parabolic-hyperbolic type of the third order is investigated. Theorems of existence and uniqueness of a regular solution of the problem under study are proved. To prove the uniqueness theorem the method of energy integrals is applied together with the method of A.M. Nakhushev. To prove the existence theorem the method of integral equations is applied. In terms of the Mittag-Leffler function the solution of the problem is found and written out in explicit form.

Conflict of Interest

There is no conflict of interest

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