MATHEMATICS

https://doi.org/10.31489/2025M3/4-21 *Research article*

Classification and reduction to canonical form of linear differential equations partial of the sixth-order with non-multiple characteristics

A.T. Abdukodirov*, T.A. Tulkinboev

Fergana State University, Fergana, Uzbekistan
(E-mail: abdurashid1976@mail.ru, tulqinjon98@mail.ru)

This paper studies the problems of classification and reduction to canonical form of linear partial differential equations of the sixth-order with non-multiple characteristics and constant coefficients. Considering that with the growth of the order of the equation or the increase in the number of independent variables, the problems of classification and reduction to canonical form become more complicated. The article first provides a general formula for the coefficients of the new equation obtained after the transformation of variables, and then formulates and proves three lemmas that play an important role in finding the canonical form of the equation. The classification problems are considered and the corresponding canonical types of equations are found by a new method in four cases in which the equation with partial derivatives of the sixth-order has: 1) six different real characteristics; 2) four different real roots and two complex-conjugate characteristics; 3) two real roots and four different complex-conjugate characteristics; 4) six different complex-conjugate characteristics and, consequently, the corresponding theorem is proved.

Keywords: a sixth-order partial differential equation, hyperbolic differential operator, elliptic differential operator, classification of differential equations, canonical form of differential equations, non-multiple characteristics, multiple characteristics, real characteristics, complex characteristics, equations of characteristics.

2020 Mathematics Subject Classification: 35G05.

Introduction

In order to achieve meaningful outcomes in the study of boundary or initial value problems for partial differential equations, it is essential to begin by identifying the type of equation and deriving its corresponding canonical form. This classification and transformation play a fundamental role in understanding the general properties of the solutions, ensuring the correct formulation of boundary value problems, informing the selection of suitable solution methods, and facilitating the analysis of both direct and inverse problems. Furthermore, in certain cases, establishing the canonical form may enable the derivation of a general solution or the reduction of the order of the equations .

Therefore, the comprehensive classification, identification of the equation type, and derivation of the corresponding canonical form represent a task of great importance in the theory of differential equations, carrying not only theoretical relevance but also practical significance.

*Corresponding author. *E-mail: abdurashid1976@mail.ru*
Received: 31 October 2024; *Accepted:* 6 June 2025.

The classification and determination of the canonical forms of second-order partial differential equations are well known. A comprehensive treatment of the classification and canonical form reduction for third- and fourth-order equations was provided in [1] and [2], respectively. Further investigations into fifth-order equations were conducted in [3], while the study presented in [4] addressed the derivation of canonical forms for n-th order partial differential equations involving two independent variables.

A significant number of studies have been devoted to the investigation of boundary value problems for high-order partial differential equations. For example, in [5] and [6], initial-boundary value problems for high even-order partial differential equations are analyzed. In [7], a completely new numerical method is proposed for solving general linear and nonlinear high-order partial differential equations. In [8], an initial-boundary value problem for a high-order partial differential equation in the multidimensional case is studied.

However, to this day, the issues of complete classification and determination of canonical forms of linear partial differential equations of sixth and higher orders remain unstudied. Although sixth-order partial differential equations do indeed arise in applied problems (for example, wave motion in water with surface tension is described by a sixth-order equation [9]), significant research has also been devoted to the study of boundary value problems for sixth-order partial differential equations. For instance, in [10] and [11], sixth-order partial differential equations are analyzed with respect to the Painlevé property and the behavior of their solutions. In [12], the reduction of equations describing orthotropic bodies to a sixth-order partial differential equation and its analysis is presented. In [13], a nonlocal inverse boundary value problem for a sixth-order partial differential equation with additional integral conditions is investigated.

Therefore, the wide range of applications involving sixth-order partial differential equations underscores the need for a comprehensive investigation into their full classification and reduction to canonical forms.

It should be noted that the classification and determination of canonical forms of partial differential equations are carried out based on the classification of the roots of the corresponding algebraic equations. As the order of the equation increases or the number of independent variables grows, the problems of classification and reduction to canonical form become increasingly complex. The complete classification of second-order partial differential equations and the determination of their corresponding canonical forms have been studied in three cases; for third-order equations — in four cases; and for fourth- and fifth-order partial differential equations — in nine and twelve cases, respectively.

Based on the above analysis, it can be concluded that the classification of sixth-order linear partial differential equations is fundamentally influenced by the quantity and multiplicity of real and complex roots of the corresponding sixth-degree algebraic equations.

For sixth-degree linear algebraic equations, one of the following scenarios invariably applies:

1) six distinct real roots;
2) four distinct real roots accompanied by one pair of complex conjugates;
3) two distinct real roots along with two distinct pairs of complex conjugates;
4) three distinct pairs of complex conjugate roots;
5) one double real root plus four distinct real roots;
6) two double real roots and two distinct real roots;
7) three double real roots;
8) one double root, one triple root, and one simple real root;
9) one triple root together with three distinct real roots;
10) two triple real roots;
11) one double root and one quadruple real root;
12) one quadruple root with two distinct real roots;
13) one quintuple root alongside one simple real root;

14) one sextuple real root;
15) one double root, two distinct real roots, and one pair of complex conjugates;
16) one double real root and two distinct pairs of complex conjugates;
17) one double real root and one double pair of complex conjugates;
18) one triple root, one simple real root, and one pair of complex conjugates;
19) one quadruple real root and one pair of complex conjugates;
20) two distinct real roots and two distinct double pairs of complex conjugates;
21) two distinct pairs of complex conjugates plus two distinct double pairs of complex conjugates;
22) two distinct double real roots and one pair of complex conjugates;
23) three distinct double pairs of complex conjugate roots.

Consequently, the comprehensive classification and reduction to canonical form of sixth-order equations can be systematically explored through exactly 23 distinct cases, each corresponding to one of the possible root structures of sixth-degree algebraic equations.

In this study, within the scope of the article, we focus on the classification and reduction to canonical form of sixth-order linear partial differential equations possessing non-multiple characteristics.

1 *Main part*

In some domain Ω of the plane xOy, we consider the sixth-order partial differential equation with two independent variables, linear with respect to the highest derivatives:

$$L[u] = \sum_{k=0}^{6} A_k \frac{\partial^6 u}{\partial x^{6-k} \partial y^k} = F, \tag{1}$$

where A_k $(k = \overline{0,6})$ are given constants, and F is a continuous function depending on x, y, u and its partial derivatives with respect to x, y up to the fifth order inclusive, where $\sum\limits_{k=0}^{6} A_k^2 \neq 0$.

Using the transformation of variables $\xi = \xi(x,y)$, $\eta = \eta(x,y)$, allowing for the inverse transformation, that is, fulfilling condition $J = \xi_x \eta_y - \xi_y \eta_x \neq 0$, from (1), we obtain

$$M\,[u] = \sum_{k=0}^{6} a_k \frac{\partial^6 u}{\partial \xi^{6-k} \partial \eta^k} = F_1, \tag{2}$$

where F_1 is a function depending on ξ, η, u and its partial derivatives with respect to ξ, η up to the fifth order inclusive, and a_k are new coefficients that are linearly dependent on A_k, $k = \overline{0,6}$.

Taking into account the notation

$$f\,(z_x, z_y) = A_0 z_x^6 + A_1 z_x^5 z_y + A_2 z_x^4 z_y^2 + A_3 z_x^3 z_y^3 + A_4 z_x^2 z_y^4 + A_5 z_x z_y^5 + A_6 z_y^6,$$

the coefficients a_k $(k = \overline{0,6})$ of equation (2) can be written as

$$a_k = \frac{1}{k!} \left(\eta_x \frac{\partial}{\partial \xi_x} + \eta_y \frac{\partial}{\partial \xi_y} \right)^k f\,(\xi_x, \xi_y) \equiv \frac{1}{(6-k)!} \left(\xi_x \frac{\partial}{\partial \eta_x} + \xi_y \frac{\partial}{\partial \eta_y} \right)^{6-k} f\,(\eta_x, \eta_y)\,. \tag{3}$$

Let us choose variables ξ and η such that equation (1) has a canonical form and so that the largest coefficients of equation (2) vanish. Since, from formula (3) it is clear that all coefficients of equation (2) are related to the function $f\,(z_x, z_y)$ and its partial derivatives with respect to the arguments, we will consider an equation with partial derivatives of the first order:

$$A_0 z_x^6 + A_1 z_x^5 z_y + A_2 z_x^4 z_y^2 + A_3 z_x^3 z_y^3 + A_4 z_x^2 z_y^4 + A_5 z_x z_y^5 + A_6 z_y^6 = 0. \tag{4}$$

Let $z = \varphi(x, y)$ be a particular solution of this equation. If we set $\xi = \varphi(x, y)$, then the coefficient a_0 will obviously be equal to zero. Thus, the above-mentioned problem of choosing new independent variables will be related to the solution of equation (4), and the solution of equation (4) is related by the general integral of the following ordinary differential equation

$$A_0(dy)^6 - A_1(dy)^5 dx + A_2(dy)^4(dx)^2 - A_3(dy)^3(dx)^3 + A_4(dy)^2(dx)^4 - A_5 dy(dx)^5 + A_6(dx)^6 = 0. \quad (5)$$

Equation (5) is called characteristics equation for equation (1), and its integrals are called characteristics. Dividing both parts of (5) by $(dx)^6$ and introducing the notation $t = dy/dx$, we have the following algebraic equation

$$A_0 t^6 - A_1 t^5 + A_2 t^4 - A_3 t^3 + A_4 t^2 - A_5 t + A_6 = 0. \quad (6)$$

Considering $t = dy/dx$, we can see that finding the general integral of the ordinary differential equation (5) is connected with the roots (algebraic with respect to t ($t = dy/dx$) of the equation (6).

Similarly, as in [4], we will prove the following three lemmas, which play an important role in finding the canonical form of equation (1):

Lemma 1. If the function $z = \varphi(x, y)$ is a solution to equation (4), then the relation $\varphi(x, y) = \text{const}$ is a general integral of the ordinary differential equation (5).

Proof. Since the function $z = \varphi(x, y)$ is a solution to equation (4), then the equality

$$A_0\varphi_x^6 + A_1\varphi_x^5\varphi_y + A_2\varphi_x^4\varphi_y^2 + A_3\varphi_x^3 Z_y^3 + A_4\varphi_x^2\varphi_y^4 + A_5\varphi_x\varphi_y^5 + A_6\varphi_y^6 = 0$$

is an identity in the domain where the solution is considered. Dividing both sides of the last equation by φ_y^6, we obtain the following identity:

$$A_0\left(-\frac{\varphi_x}{\varphi_y}\right)^6 - A_1\left(-\frac{\varphi_x}{\varphi_y}\right)^5 + A_2\left(-\frac{\varphi_x}{\varphi_y}\right)^4 - A_3\left(-\frac{\varphi_x}{\varphi_y}\right)^3 + A_4\left(-\frac{\varphi_x}{\varphi_y}\right)^2 - A_5\left(-\frac{\varphi_x}{\varphi_y}\right) + A_6 = 0. \quad (7)$$

It is known that if a function y, determined from an implicit relation $\varphi(x, y) = \text{const}$, satisfies equation (5), then $\varphi(x, y) = \text{const}$ is a general integral of the ordinary differential equation (5). Let $y = f(x, C)$ be this function. Then

$$\frac{dy}{dx} = -\left[\frac{\varphi_x(x, y)}{\varphi_y(x, y)}\right]_{y=f(x,C)}. \quad (8)$$

Here, the square brackets and the index $y = f(x, C)$ indicate that on the righthand side of equality (8) the variable y is not an independent variable, but has a value equal to $f(x, C)$. It follows that $y = f(x, C)$ satisfies equation (5), since

$$A_0\left(\frac{dy}{dx}\right)^6 - A_1\left(\frac{dy}{dx}\right)^5 + A_2\left(\frac{dy}{dx}\right)^4 - A_3\left(\frac{dy}{dx}\right)^3 + A_4\left(\frac{dy}{dx}\right)^2 - A_5\left(\frac{dy}{dx}\right) + A_6 =$$

$$= \Bigg[A_0\left(-\frac{\varphi_x}{\varphi_y}\right)^6 - A_1\left(-\frac{\varphi_x}{\varphi_y}\right)^5 + A_2\left(-\frac{\varphi_x}{\varphi_y}\right)^4 - A_3\left(-\frac{\varphi_x}{\varphi_y}\right)^3 +$$

$$+A_4\left(-\frac{\varphi_x}{\varphi_y}\right)^2 - A_5\left(-\frac{\varphi_x}{\varphi_y}\right) + A_6\Bigg]_{y=f(x,C)} = 0,$$

by virtue of (7) the expression in square brackets is equal to zero for all values of x, y, and not only for $y = f(x, C)$.

Lemma 2. If $\varphi(x,y) = \text{const}$ is a k-fold ($k \leq 6$) general integral of equation (5), then for $z = \varphi(x,y)$ the function $f(z_x, z_y)$ and all its derivatives with respect to z_x, z_y up to and including $(k-1)$ order are equal to zero.

Proof. Let $\varphi(x,y) = \text{const}$ be a k-fold general integral of equation (5), and $t_1, t_2, \dots, t_6$ be the roots of equation (6), where t_1 ($t_1 = -\varphi_x/\varphi_y$) is the corresponding k-fold root of equation (6). Then, based on the corollary of Bezout's theorem, equation (6) can be written in the form

$$A_0 (t - t_1)^k \prod_{j=k+1}^{6} (t - t_j) = 0. \tag{9}$$

If we consider $t = dy/dx$, the equation (9) takes the form

$$A_0 (dy - t_1 dx)^k \prod_{j=k+1}^{6} (dy - t_j dx) = 0.$$

Taking this into account, the function $f(z_x, z_y)$ and the equation (4) can be written as $f(z_x, z_y) = A_0 (z_x + t_1 z_y)^k \prod_{j=k+1}^{6} (z_x + t_j z_y)$ and $A_0 (z_x + t_1 z_y)^k \prod_{j=k+1}^{6} (z_x + t_j z_y) = 0$, respectively. Therefore, for $z = \varphi(x,y)$, we have

$$f(z_x, z_y) = A_0 (z_x + t_1 z_y)^k \prod_{j=k+1}^{6} (z_x + t_j z_y) = 0.$$

It easily follows from this that all derivatives of the function $f(z_x, z_y)$ with respect to z_x, z_y up to $(k-1)$ order inclusive for $z = \varphi(x,y)$ are equal to zero.

Lemma 3. When transforming variables $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ that allow inverse transformation, the number and multiplicity of real and complex roots of equation (6) are invariant, and the identity holds $\tilde{D}_6 = J^{30} D_6$, where

$$D_6 = A_0^{10} \prod_{6 \geq i > j \geq 1} (t_i - t_j)^2 \tag{10}$$

is the discriminant of the equation (6), and

$$\tilde{D}_6 = a_0^{10} \prod_{6 \geq i > j \geq 1} (\mu_i - \mu_j)^2 \tag{11}$$

is the discriminant of the following equation

$$a_0 \mu^6 - a_1 \mu^5 + a_2 \mu^4 - a_3 \mu^3 + a_4 \mu^2 - a_5 \mu + a_6 = 0 \quad (\mu = d\eta/d\xi), \tag{12}$$

where $t_1, t_2, \dots, t_6$ and $\mu_1, \mu_2, \dots, \mu_6$ are the roots of the equations (6) and (12), respectively.

Proof. As shown above, when transforming the variables $\xi = \xi(x,y)$, $\eta = \eta(x,y)$, the equation (1) with the condition $J = \xi_x \eta_y - \xi_y \eta_x \neq 0$, was transformed into equation (2). By introducing the notation $t = dy/dx$, $\mu = d\eta/d\xi$ into the equations of the characteristics for equations (1) and (2), algebraic equations (6) and (12) were found, respectively. Then, taking into account $t = dy/dx$ and $\mu = d\eta/d\xi$, we have the following relation

$$\mu = \frac{d\eta(x,y)}{d\xi(x,y)} = \frac{\eta_x dx + \eta_y dy}{\xi_x dx + \xi_y dy} = \frac{\eta_x + \eta_y (dy/dx)}{\xi_x + \xi_y (dy/dx)} = \frac{\eta_x + \eta_y t}{\xi_x + \xi_y t}. \tag{13}$$

From (13), we find the relation between the roots μ_i and t_i of the equations (6) and (12) in the form $\mu_i = (\eta_x + \eta_y \cdot t_i) / (\xi_x + \xi_y \cdot t_i)$. It follows that the number and multiplicity of the real and complex roots of equations (6) and (12) are the same. That is, when transforming variables $\xi = \xi(x,y), \eta = \eta(x,y)$, allowing for an inverse transformation, the number and multiplicity of the real and complex roots of equation (6) are invariant. In addition, we have

$$\mu_k - \mu_j = J(t_k - t_j)\left[(\xi_x + t_k\xi_y)(\xi_x + t_j\xi_y)\right]^{-1}, \quad k,j = \overline{1,6}.$$

Using these equalities, from (11), we find

$$\tilde{D} = a_0^{10} J^{30} \left[(\xi_x + t_1\xi_y) \cdot (\xi_x + t_2\xi_y) \cdot \ldots \cdot (\xi_x + t_6\xi_y)\right]^{-10} \prod_{6 \geq k > j \geq 1} (t_k - t_j)^2 .$$

From here, opening the brackets inside the square bracket and taking into account equality (10), we obtain

$$\tilde{D} = a_0^{10} J^{30} \cdot D\left[\xi_x^6 + (t_1 + t_2 + \ldots + t_6) \cdot \xi_x^5\xi_y + \ldots + t_1 \cdot t_2 \cdot \ldots \cdot t_6\xi_y^6\right]^{-10} A_0^{-10}. \tag{14}$$

On the other hand, according to Vieta's formulas, the following equalities hold:

$$t_1 + t_2 + \ldots + t_6 = \frac{A_1}{A_0}, \quad t_1t_2 + t_1t_3 + \ldots + t_5t_6 = \frac{A_2}{A_0}, \ldots, t_1 \cdot \ldots \cdot t_6 = \frac{A_6}{A_0}. \tag{15}$$

Based on (15), equality (14) takes the form

$$\tilde{D} = a_0^{10} J^{30} D\left(A_0\xi_x^6 + A_1\xi_x^5\xi_y + \ldots + A_6\xi_y^6\right)^{-10}.$$

Since, according to formula (3), $a_0 = A_0\xi_x^6 + A_1\xi_x^5\xi_y + \ldots + A_6\xi_y^6$, then from the latter it follows that $\tilde{D} = J^{30}D$. From this equality, by virtue of $J \neq 0$, it follows that when transforming variables, the sign of the discriminant D is invariant.

Without loss of generality, we can assume [4] that condition $A_0 > 0$ is also satisfied.

As is well established from the corollary to the Fundamental Theorem of Algebra, any polynomial of degree n over the field of complex numbers possesses exactly n roots, counted with their multiplicities. Accordingly, equation (6) has exactly six roots — real and/or complex conjugates — taking multiplicities into account.

Given that the algebraic equation (6) presents 23 possible root configurations, the corresponding partial differential equation (1) may be analyzed in all these cases. Nevertheless, owing to limitations of space, the present study will concentrate solely on the four cases in which equation (6) has exclusively simple (non-repeated) roots.

1. Let equation (6) have six different real roots $t_1 = \lambda_1$, $t_2 = \lambda_2$, $t_3 = \lambda_3$, $t_4 = \lambda_4$, $t_5 = \lambda_5$, $t_6 = \lambda_6$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 > \lambda_6$. Then, equation (5) has six different real general integrals:

$$\Psi_1(x,y) = y - \lambda_1 x = \text{const}, \quad \Psi_2(x,y) = y - \lambda_2 x = \text{const}, \quad \Psi_3(x,y) = y - \lambda_3 x = \text{const},$$

$$\Psi_4(x,y) = y - \lambda_4 x = \text{const}, \quad \Psi_5(x,y) = y - \lambda_5 x = \text{const}, \quad \Psi_6(x,y) = y - \lambda_6 x = \text{const}.$$

If we take into account (15), then equation (1) can be written as:

$$A_0\left[\frac{\partial^6 u}{\partial x^6} + (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6)\frac{\partial^6 u}{\partial x^5 \partial y} + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_5\lambda_6)\frac{\partial^6 u}{\partial x^4 \partial y^2} + \ldots \right.$$
$$\left. \ldots + (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6)\frac{\partial^6 u}{\partial y^6}\right] = F.$$

Using first-order differential operators of the form $\left(\frac{\partial}{\partial x}+\lambda_i\frac{\partial}{\partial y}\right)$, the last equation can be formally written as:

$$A_0\left[\prod_{k=1}^{6}\left(\frac{\partial}{\partial x}+\lambda_k\frac{\partial}{\partial y}\right)\right]u=F. \tag{16}$$

By introducing the following notations $\frac{(\lambda_1-\lambda_5)}{(\lambda_1-\lambda_6)}=\mu_1$, $\frac{(\lambda_2-\lambda_5)}{(\lambda_2-\lambda_6)}=\mu_2$, $\frac{(\lambda_3-\lambda_5)}{(\lambda_3-\lambda_6)}=\mu_3$, $\frac{(\lambda_4-\lambda_5)}{(\lambda_4-\lambda_6)}=\mu_4$, let us change the variables by

$$\xi=\left(1+\sqrt{\mu_3\mu_4}\right)y-\left(\lambda_5+\lambda_6\sqrt{\mu_3\mu_4}\right)x,\quad \eta=\left(1-\sqrt{\mu_3\mu_4}\right)y-\left(\lambda_5-\lambda_6\sqrt{\mu_3\mu_4}\right)x. \tag{17}$$

Then, taking (17) into account, we have

$$\frac{\partial}{\partial x}=\xi_x\frac{\partial}{\partial \xi}+n_x\frac{\partial}{\partial \eta}=-\left(\lambda_5+\lambda_6\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial \xi}-\left(\lambda_5-\lambda_6\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial \eta},$$
$$\frac{\partial}{\partial y}=\xi_y\frac{\partial}{\partial \xi}+n_y\frac{\partial}{\partial \eta}=\left(1+\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial \xi}+\left(1-\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial \eta}.$$

Substituting these expressions of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ into the equation (16), we obtain

$$\left[\left(\lambda_1-\lambda_5+\sqrt{\mu_3\mu_4}\left(\lambda_1-\lambda_6\right)\right)\frac{\partial}{\partial \xi}+\left(\lambda_1-\lambda_5-\sqrt{\mu_3\mu_4}\left(\lambda_1-\lambda_6\right)\right)\frac{\partial}{\partial \eta}\right]\times$$
$$\times\left[\left(\lambda_2-\lambda_5+\sqrt{\mu_3\mu_4}\left(\lambda_2-\lambda_6\right)\right)\frac{\partial}{\partial \xi}+\left(\lambda_2-\lambda_5-\sqrt{\mu_3\mu_4}\left(\lambda_2-\lambda_6\right)\right)\frac{\partial}{\partial \eta}\right]\times$$
$$\times\left[\left(\lambda_3-\lambda_5+\sqrt{\mu_3\mu_4}\left(\lambda_3-\lambda_6\right)\right)\frac{\partial}{\partial \xi}+\left(\lambda_3-\lambda_5-\sqrt{\mu_3\mu_4}\left(\lambda_3-\lambda_6\right)\right)\frac{\partial}{\partial \eta}\right]\times$$
$$\times\left[\left(\lambda_4-\lambda_5+\sqrt{\mu_3\mu_4}\left(\lambda_4-\lambda_6\right)\right)\frac{\partial}{\partial \xi}+\left(\lambda_4-\lambda_5-\sqrt{\mu_3\mu_4}\left(\lambda_4-\lambda_6\right)\right)\frac{\partial}{\partial \eta}\right]\times$$
$$\times\left[\sqrt{\mu_3\mu_4}\left(\lambda_5-\lambda_6\right)\left(\frac{\partial}{\partial \xi}-\frac{\partial}{\partial \eta}\right)\right]\left[\left(\lambda_6-\lambda_5\right)\left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)\right]u=F_1.$$

Let us divide both sides of the last equation by

$$-\sqrt{\mu_3\mu_4}\left(\lambda_1-\lambda_6\right)\left(\lambda_2-\lambda_6\right)\left(\lambda_3-\lambda_6\right)\left(\lambda_4-\lambda_6\right)\left(\lambda_5-\lambda_6\right)^2(\neq 0).$$

Then, we have

$$\left[\left(\mu_1+\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial \xi}+\left(\mu_1-\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial \eta}\right]\times\left[\left(\mu_2+\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial \xi}+\left(\mu_2-\sqrt{\mu_3\mu_4}\right)\frac{\partial}{\partial \eta}\right]\times$$
$$\times\sqrt{\mu_3}\left[\left(\sqrt{\mu_3}+\sqrt{\mu_4}\right)\frac{\partial}{\partial \xi}+\left(\sqrt{\mu_3}-\sqrt{\mu_4}\right)\frac{\partial}{\partial \eta}\right]\times\sqrt{\mu_4}\left[\left(\sqrt{\mu_4}+\sqrt{\mu_3}\right)\frac{\partial}{\partial \xi}+\left(\sqrt{\mu_4}-\sqrt{\mu_3}\right)\frac{\partial}{\partial \eta}\right]\times$$
$$\times\left(\frac{\partial^2 u}{\partial \xi^2}-\frac{\partial^2 u}{\partial n^2}\right)=F_2, \tag{18}$$

where $F_2=F_1/\left\{-\sqrt{\mu_3\mu_4}\left(\lambda_1-\lambda_6\right)\left(\lambda_2-\lambda_6\right)\left(\lambda_3-\lambda_6\right)\left(\lambda_4-\lambda_6\right)\left(\lambda_5-\lambda_6\right)^2\right\}$.

And the equation (18) can be rewritten as:

$$\left[\frac{\partial}{\partial\xi}+\frac{\left(\mu_1-\sqrt{\mu_3\mu_4}\right)}{\left(\mu_1+\sqrt{\mu_3\mu_4}\right)}\frac{\partial}{\partial\eta}\right]\times\left[\frac{\partial}{\partial\xi}+\frac{\left(\mu_2-\sqrt{\mu_3\mu_4}\right)}{\left(\mu_2+\sqrt{\mu_3\mu_4}\right)}\frac{\partial}{\partial\eta}\right]\times$$

$$\times\left[\frac{\partial^2}{\partial\xi^2}-\frac{\left(\sqrt{\mu_3}-\sqrt{\mu_4}\right)^2}{\left(\sqrt{\mu_3}+\sqrt{\mu_4}\right)^2}\frac{\partial^2}{\partial\eta^2}\right]\times\left(\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\eta^2}\right)=F_3$$

or

$$\left(\frac{\partial}{\partial\xi}+c_1\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+c_2\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^2}{\partial\xi^2}-b^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\eta^2}\right)=F_3,$$

where $c_1=\dfrac{\left(\mu_1-\sqrt{\mu_3\mu_4}\right)}{\left(\mu_1+\sqrt{\mu_3\mu_4}\right)},\quad c_2=\dfrac{\left(\mu_2-\sqrt{\mu_3\mu_4}\right)}{\left(\mu_2+\sqrt{\mu_3\mu_4}\right)},\quad b^2=\dfrac{\left(\sqrt{\mu_3}-\sqrt{\mu_4}\right)^2}{\left(\sqrt{\mu_3}+\sqrt{\mu_4}\right)^2},$
$F_3=F_2/\left\{\sqrt{\mu_3\mu_4}\left(\mu_1+\sqrt{\mu_3\mu_4}\right)\left(\mu_2+\sqrt{\mu_3\mu_4}\right)\left(\sqrt{\mu_3}+\sqrt{\mu_4}\right)^2\right\}.$

Example 1. Consider the following sixth-order partial differential equation:

$$u_{xxxxxx}+6u_{xxxxxy}-10u_{xxxxyy}-100u_{xxxyyy}-111u_{xxyyyy}+94u_{xyyyyy}+120u_{yyyyyy}=0. \tag{19}$$

The characteristic equation corresponding to the equation (19) has the form

$$(dy)^6-6(dy)^5(dx)-10(dy)^4(dx)^2+100(dy)^3(dx)^3-111(dy)^2(dx)^4-94(dy)(dx)^5+120(dx)^6=0.$$

It is easy to verify that this equation has six different real roots for $t=dy/dx$:

$$t_1=1,\ t_2=-1,\ t_3=2,\ t_4=3,\ t_5=-4,\ t_6=5.$$

Then, equation (19) can be written as follows:

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-4\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+5\frac{\partial}{\partial y}\right)u=0. \tag{20}$$

After the transformation $\xi=(1+\sqrt{7})y+(4-5\sqrt{7})x$, $\eta=(1-\sqrt{7})y+(4+5\sqrt{7})x$, we obtain $\frac{\partial}{\partial x}=\left(4-5\sqrt{7}\right)\frac{\partial}{\partial\xi}+\left(4+5\sqrt{7}\right)\frac{\partial}{\partial\eta}$, $\frac{\partial}{\partial y}=\left(1+\sqrt{7}\right)\frac{\partial}{\partial\xi}+\left(1-\sqrt{7}\right)\frac{\partial}{\partial\eta}$. Considering these, from (20), we have

$$\left(\frac{\partial}{\partial\xi}+c_1\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+c_2\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^2}{\partial\xi^2}-b^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\eta^2}\right)=0,$$

where $c_1=\frac{(5+4\sqrt{7})}{(5-4\sqrt{7})}$, $c_2=\frac{(3+6\sqrt{7})}{(3-6\sqrt{7})}$, $b^2=\left(\frac{28+11\sqrt{7}}{3\sqrt{7}}\right)^2$.

2. Let the equation (6) have four different real roots $t_1=\lambda_1$, $t_2=\lambda_2$, $t_3=\lambda_3$, $t_4=\lambda_4$ and two complex conjugate roots $t_5=\alpha+\beta i$, $t_6=\alpha-\beta i$. Then the equation (5) has four different real and two different complex conjugate general integrals:

$\Psi_1(x,y)=y-\lambda_1 x=\text{const}$, $\Psi_2(x,y)=y-\lambda_2 x=\text{const}$, $\Psi_3(x,y)=y-\lambda_3 x=\text{const}$,

$\Psi_4(x,y)=y-\lambda_4 x=\text{const}$, $\varphi(x,y)=y-\alpha x-i\beta x=\text{const}$, $\varphi^*(x,y)=y-\alpha x+i\beta x=\text{const}$, where $\alpha,\beta\in R$ and $\beta\neq 0$.

If we take into account (15), then the equation (1) can be written as:

$$A_0\left[\frac{\partial^6 u}{\partial x^6}+(\lambda_1+\lambda_2+\lambda_3+\lambda_4+t_5+t_6)\frac{\partial^6 u}{\partial x^5\partial y}+(\lambda_1\lambda_2+\lambda_1\lambda_3+\ldots+t_5t_6)\frac{\partial^6 u}{\partial x^4\partial y^2}+\ldots\right.$$

$$\left....+(\lambda_1\lambda_2\lambda_3\lambda_4t_5t_6)\frac{\partial^6 u}{\partial y^6}\right]=F.$$

Hence, similarly to equation (16), we have

$$A_0\left\{\prod_{k=1}^{4}\left(\frac{\partial}{\partial x}+\lambda_k\frac{\partial}{\partial y}\right)\right\}\left(\frac{\partial^2 u}{\partial x^2}+2\alpha\frac{\partial^2 u}{\partial x\partial y}+\left(\alpha^2+\beta^2\right)\frac{\partial^2 u}{\partial y^2}\right)=F. \tag{21}$$

Let us change the variables by the following formulas

$$\xi=y-\alpha x,\quad \eta=\beta x. \tag{22}$$

Then $\frac{\partial}{\partial x}=\xi_x\frac{\partial}{\partial \xi}+n_x\frac{\partial}{\partial \eta}=-\alpha\frac{\partial}{\partial \xi}+\beta\frac{\partial}{\partial \eta},\quad \frac{\partial}{\partial y}=\xi_y\frac{\partial}{\partial \xi}+n_y\frac{\partial}{\partial \eta}=\frac{\partial}{\partial \xi}.$

Substituting these expressions of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ into the equation (21), we have

$$A_0\beta^2\left\{\prod_{k=1}^{4}\left[\left(\lambda_k-\alpha\right)\frac{\partial}{\partial \xi}+\beta\frac{\partial}{\partial \eta}\right]\right\}\left(\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}\right)=F_1. \tag{23}$$

Assume that $\left(\lambda_1-\alpha\right)\left(\lambda_2-\alpha\right)\left(\lambda_3-\alpha\right)\left(\lambda_4-\alpha\right)\neq 0$. Then, dividing both parts of the equality (23) by $A_0\beta^2\left(\lambda_1-\alpha\right)\left(\lambda_2-\alpha\right)\left(\lambda_3-\alpha\right)\left(\lambda_4-\alpha\right)$ and introducing the notation

$$\mu_5=\frac{\beta}{\lambda_1-\alpha},\quad \mu_6=\frac{\beta}{\lambda_2-\alpha},\quad \mu_7=\frac{\beta}{\lambda_3-\alpha},\quad \mu_8=\frac{\beta}{\lambda_4-\alpha},$$

we obtain

$$\left(\frac{\partial}{\partial \xi}+\mu_5\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi}+\mu_6\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi}+\mu_7\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi}+\mu_8\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}\right)=F_4,$$

or

$$\left(\frac{\partial^2}{\partial \xi^2}+\left(\mu_5+\mu_6\right)\frac{\partial^2}{\partial \xi\partial \eta}+\mu_5\mu_6\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial \xi^2}+\left(\mu_7+\mu_8\right)\frac{\partial^2}{\partial \xi\partial \eta}+\mu_7\mu_8\frac{\partial^2}{\partial \eta^2}\right)\times$$
$$\times\left(\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}\right)=F_4, \tag{24}$$

where $F_4=F_1/\left[A_0\beta^2\left(\lambda_1-\alpha\right)\left(\lambda_2-\alpha\right)\left(\lambda_3-\alpha\right)\left(\lambda_4-\alpha\right)\right]$.

If $\mu_5=-\mu_6,\ \mu_7=-\mu_8$, then the equation (24) takes the form

$$\left(\frac{\partial^2}{\partial \xi^2}-\mu_5^2\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial \xi^2}-\mu_7^2\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}\right)=F_4.$$

Let $\mu_5\neq-\mu_6,\ \mu_7\neq-\mu_8$, then $\frac{\partial^2}{\partial \xi^2}+\left(\mu_5+\mu_6\right)\frac{\partial^2}{\partial \xi\partial \eta}+\mu_5\mu_6\frac{\partial^2}{\partial \eta^2}$ and $\frac{\partial^2}{\partial \xi^2}+\left(\mu_7+\mu_8\right)\frac{\partial^2}{\partial \xi\partial \eta}+$ $+\mu_7\mu_8\frac{\partial^2}{\partial \eta^2}$ are hyperbolic differential operators, since $\left(\mu_5+\mu_6\right)^2-4\mu_5\mu_6=\left(\mu_5-\mu_6\right)^2>0$ and $\left(\mu_7+\mu_8\right)^2-4\mu_7\mu_8=\left(\mu_7-\mu_8\right)^2>0$.

To further simplify equation (24), we make a change of variables by

$$s=s(\xi,\eta),\quad t=t(\xi,\eta) \tag{25}$$

and $J=s_\xi t_\eta-s_\eta t_\xi\neq 0$, then $\frac{\partial}{\partial \xi}=s_\xi\frac{\partial}{\partial s}+t_\xi\frac{\partial}{\partial t}, \frac{\partial}{\partial \eta}=s_\eta\frac{\partial}{\partial s}+t_\eta\frac{\partial}{\partial t}$. Taking this into account, from equation (24), we obtain a new equation in the following form

$$\left(\left(s_\xi+\mu_5 s_\eta\right)\frac{\partial}{\partial s}+\left(t_\xi+\mu_5 t_\eta\right)\frac{\partial}{\partial t}\right)\left(\left(s_\xi+\mu_6 s_\eta\right)\frac{\partial}{\partial s}+\left(t_\xi+\mu_6 t_\eta\right)\frac{\partial}{\partial t}\right)\times$$

$$\times\left(\left(s_\xi^2+(\mu_7+\mu_8)\,s_\xi s_\eta+\mu_7\mu_8 s_\eta^2\right)\frac{\partial^2}{\partial s^2}+\left(t_\xi^2+(\mu_7+\mu_8)\,t_\xi t_\eta+\mu_7\mu_8 t_\eta^2\right)\frac{\partial^2}{\partial t^2}+\right.$$
$$\left.+\left(2\left(s_\xi t_\xi+\mu_7\mu_8 s_\eta t_\eta\right)+(\mu_7+\mu_8)\left(s_\xi t_\eta+s_\eta t_\xi\right)\right)\frac{\partial^2}{\partial s\partial t}\right)\times$$
$$\times\left(\left(s_\xi^2+s_\eta^2\right)\frac{\partial^2 u}{\partial s^2}+2\left(s_\xi t_\xi+s_\eta t_\eta\right)\frac{\partial^2 u}{\partial s\partial t}+\left(t_\xi^2+t_\eta^2\right)\frac{\partial^2 u}{\partial t^2}\right)=F_5, \tag{26}$$

where F_5 is a function depending on s,t,u and its partial derivatives with respect to s,t up to the fifth order inclusive.

To make equation (26) simpler, we take a replacement for (25) as

$$s=\eta+\mu_0\xi,\quad t=\mu_0\eta-\xi, \tag{27}$$

where μ_0 is one of two solutions of the equation

$$\mu^2-\frac{2\left(1-\mu_7\mu_8\right)}{\mu_7+\mu_8}\mu-1=0, \tag{28}$$

that is, $\mu_0=\dfrac{1-\mu_7\mu_8}{\mu_7+\mu_8}+\sqrt{\dfrac{(1-\mu_7\mu_8)^2}{(\mu_7+\mu_8)^2}+1}>0$ or $\mu_0=\dfrac{1-\mu_7\mu_8}{\mu_7+\mu_8}-\sqrt{\dfrac{(1-\mu_7\mu_8)^2}{(\mu_7+\mu_8)^2}+1}<0$. In this case, $s_\xi=t_n=\mu_0, s_n=-t_\xi=1$ and therefore $J=s_\xi t_n-s_n t_\xi=\mu_0^2+1\neq 0$. In addition, the equalities $2(s_\xi t_\xi+\mu_7\mu_8 s_\eta t_\eta)+(\mu_7+\mu_8)(s_\xi t_\eta+s_\eta t_\xi)=2(-\mu_0+\mu_7\mu_8\mu_0)+(\mu_7+\mu_8)\times(\mu_0^2-$ $-1)=\left[\mu^2-\dfrac{2(1-\mu_7\mu_8)}{\mu_7+\mu_8}\mu-1\right](\mu_7+\mu_8)=0$, $s_\xi t_\xi+s_\eta t_\eta=-\mu_0+\mu_0=0$ are valid. If we take into account these equalities, then equation (26) takes the form

$$\left((\mu_0+\mu_5)\frac{\partial}{\partial s}+(-1+\mu_5\mu_0)\frac{\partial}{\partial t}\right)\left((\mu_0+\mu_6)\frac{\partial}{\partial s}+(-1+\mu_6\mu_0)\frac{\partial}{\partial t}\right)\times$$
$$\times\left[\left(\mu_0^2+(\mu_7+\mu_8)\,\mu_0+\mu_7\mu_8\right)\frac{\partial^2}{\partial s^2}+\left(1-(\mu_7+\mu_8)\,\mu_0+\mu_7\mu_8\mu_0^2\right)\frac{\partial^2}{\partial t^2}\right]\times$$
$$\times\left(\left(\mu_0^2+1\right)\frac{\partial^2 u}{\partial s^2}+\left(1+\mu_0^2\right)\frac{\partial^2 u}{\partial t^2}\right)=F_5.$$

Then, dividing both parts and both sides of the last equation by $\left(\mu_0^2+1\right)(\mu_0+\mu_5)(\mu_0+\mu_6)(\mu_0+\mu_7)(\mu_0+\mu_8)$ and introducing the notations $c_3=\dfrac{(\mu_5\mu_0-1)}{(\mu_0+\mu_5)}$, $c_4=\dfrac{(\mu_6\mu_0-1)}{(\mu_0+\mu_6)}$, $b_1^2=\dfrac{(1-\mu_7\mu_0)(\mu_8\mu_0-1)}{(\mu_0+\mu_7)(\mu_0+\mu_8)}$, we come

$$\left(\frac{\partial}{\partial s}+c_3\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s}+c_4\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial s^2}-b_1^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_6, \tag{29}$$

where $F_6=F_5/\left[\left(\mu_0^2+1\right)(\mu_0+\mu_5)(\mu_0+\mu_6)(\mu_0+\mu_7)(\mu_0+\mu_8)\right]$.

Let us prove that $b_1^2>0$. Introducing the notations $\nu_1=\dfrac{\mu_0\mu_7-1}{\mu_0+\mu_7}$, $\nu_2=\dfrac{\mu_0\mu_8-1}{\mu_0+\mu_8}$ and taking into account that μ_0 is one of the two solutions of equation (28), that is, $\mu_0^2+\mu_0\dfrac{2\left(\mu_7\mu_8-1\right)}{(\mu_7+\mu_8)}-$ $-1=0$, we have $\nu_1+\nu_2=\dfrac{\mu_0\mu_7-1}{\mu_0+\mu_7}+\dfrac{\mu_0\mu_8-1}{\mu_0+\mu_8}=\dfrac{\mu_0^2+\mu_0\dfrac{2\left(\mu_7\mu_8-1\right)}{(\mu_7+\mu_8)}-1}{(\mu_0+\mu_7)(\mu_0+\mu_8)}(\mu_7+\mu_8)=0$. Then $b_1^2=-\dfrac{(\mu_0\mu_7-1)(\mu_0\mu_8-1)}{(\mu_0+\mu_7)(\mu_0+\mu_8)}=-\nu_1\nu_2=\dfrac{1}{4}\left[(\nu_1+\nu_2)^2-4\nu_1\nu_2\right]=\dfrac{1}{4}(\nu_1-\nu_2)^2>0.$

It is easy to verify that if one of the expressions $\lambda_k - \alpha (k = 1, 2, 3, 4)$ is equal to zero, that is, for example $\lambda_1 - \alpha = 0$, then the equation (23) takes the form

$$\left\{ \prod_{k=2}^{4} \left[(\lambda_k - \alpha) \frac{\partial}{\partial \xi} + \beta \frac{\partial}{\partial \eta} \right] \right\} \left(\frac{\partial^3 u}{\partial \xi^2 \partial \eta} + \frac{\partial^3 u}{\partial \eta^3} \right) = F_1 / \left(A_0 \beta^3 \right)$$

and this equation, as in case (29), after changing variables by $s = \eta + \mu_0 \xi, t = \mu_0 \eta - \xi$, can be brought to the form

$$\left(\frac{\partial}{\partial s} + \mu_0 \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial s} + c_4 \frac{\partial}{\partial t} \right) \left(\frac{\partial^2}{\partial s^2} - b_1^2 \frac{\partial^2}{\partial t^2} \right) \left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} \right) = F_7,$$

where $F_7 = F_1 / \left[A_0 \beta^3 (\lambda_2 - \alpha)(\lambda_3 - \alpha)(\lambda_4 - \alpha) \left(\mu_0^2 + 1 \right) (\mu_0 + \mu_6)(\mu_0 + \mu_7)(\mu_0 + \mu_8) \right]$.

Example 2. Consider the following sixth-order partial differential equation:

$$u_{xxxxxx} + 24 u_{xxxxxy} + 239 u_{xxxxyy} + 1264 u_{xxxyyy} + 3663 u_{xxyyyy} + 5240 u_{xyyyyy} + 2625 u_{yyyyyy} = 0. \quad (30)$$

The characteristic equation corresponding to the equation (30) has the form

$$(dy)^6 - 24(dy)^5(dx) + 239(dy)^4(dx)^2 - 1264(dy)^3(dx)^3 + 3663(dy)^2(dx)^4 - 5240(dy)(dx)^5 + 2625(dx)^6 = 0.$$

It is easy to verify that this equation has four different real roots and two complex conjugate roots for $t = dy/dx$:

$$t_1 = 5,\ t_2 = 3,\ t_3 = 1,\ t_4 = 7,\ t_5 = 4 + 3i,\ t_6 = 4 - 3i.$$

Then, equation (30) can be written as follows:

$$\left(\frac{\partial}{\partial x} + 5 \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + 3 \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + 7 \frac{\partial}{\partial y} \right) \left(\frac{\partial^2}{\partial x^2} + 8 \frac{\partial^2}{\partial x \partial y} + 25 \frac{\partial^2}{\partial y^2} \right) u = 0. \quad (31)$$

After the transformation $\xi = y - 4x$, $\eta = 3x$, we obtain: $\frac{\partial}{\partial x} = -4 \frac{\partial}{\partial \xi} + 3 \frac{\partial}{\partial \eta}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}$. Considering these from (31), we have

$$\left(\frac{\partial^2}{\partial \xi^2} - 9 \frac{\partial^2}{\partial \eta^2} \right) \left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} \right) \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) u = 0.$$

3. Let equation (6) have two different simple real roots and four complex conjugate roots:

$$t_1 = \lambda_1, t_2 = \lambda_2, t_3 = \delta + \gamma i, t_4 = \delta - \gamma i, t_5 = \alpha + \beta i, t_6 = \alpha - \beta i,$$

where $\lambda_1, \lambda_2, \alpha, \beta, \delta, \gamma \in R$ such that $\beta \neq 0, \gamma \neq 0, (\alpha - \delta)^2 + (|\beta| - |\gamma|)^2 \neq 0$.

Then the equation (5) has one real and four different complex conjugate general integrals

$$\Psi_1(x,y) = y - \lambda_1 x = \text{ const}, \ \Psi_2(x,y) = y - \lambda_2 x = \text{ const},$$
$$\Psi_3(x,y) = y - \delta x - i\gamma x = \text{ const}, \ \Psi_4(x,y) = y - \delta x + i\gamma x = \text{ const},$$
$$\varphi(x,y) = y - \alpha x - i\beta x = \text{ const}, \ \varphi^*(x,y) = y - \alpha x + i\beta x = \text{ const}.$$

Using the same reasoning as when obtaining equation (21), equation (1) can be written as

$$A_0 \left(\frac{\partial}{\partial x} + \lambda_1 \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + \lambda_2 \frac{\partial}{\partial y} \right) \left(\frac{\partial^2}{\partial x^2} + 2\delta \frac{\partial^2}{\partial x \partial y} + \left(\delta^2 + \gamma^2 \right) \frac{\partial^2}{\partial y^2} \right) \times$$
$$\times \left(\frac{\partial^2 u}{\partial x^2} + 2\alpha \frac{\partial^2 u}{\partial x \partial y} + \left(\alpha^2 + \beta^2 \right) \frac{\partial^2 u}{\partial y^2} \right) = F. \quad (32)$$

Let us consider the substitution (22). Then, from equation (32), similarly to (23), we obtain the equation

$$A_0\beta^2\left[(\lambda_1-\alpha)\frac{\partial}{\partial\xi}+\beta\frac{\partial}{\partial\eta}\right]\left[(\lambda_2-\alpha)\frac{\partial}{\partial\xi}+\beta\frac{\partial}{\partial\eta}\right]\times$$

$$\times\left[\left(\alpha^2+\delta^2+\gamma^2-2\alpha\delta\right)\frac{\partial^2}{\partial\xi^2}+2\beta(\delta-\alpha)\frac{\partial^2}{\partial\xi\partial\eta}+\beta^2\frac{\partial^2}{\partial\eta^2}\right]\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}\right)=F_1. \tag{33}$$

Hence, dividing both parts of (33) by $A_0\beta^2\left(\lambda_1-\alpha\right)\left(\lambda_2-\alpha\right)\left[(\alpha-\delta)^2+\gamma^2\right](\neq 0)$ and introducing the notations $\mu_5=\dfrac{\beta}{\lambda_1-\alpha}$, $\mu_6=\dfrac{\beta}{\lambda_2-\alpha}$, $\mu_9=\dfrac{\beta(\delta-\alpha)+\gamma\beta i}{(\alpha-\delta)^2+\gamma^2}=\delta_1+\gamma_1 i$, $\mu_{10}=\dfrac{\beta(\delta-\alpha)-\gamma\beta i}{(\alpha-\delta)^2+\gamma^2}=$ $=\delta_1-\gamma_1 i$, $\dfrac{\beta(\delta-\alpha)}{(\alpha-\delta)^2+\gamma^2}=\delta_1$, $\dfrac{\gamma\beta}{(\alpha-\delta)^2+\gamma^2}=\gamma_1$, we have an equation in the form

$$\left(\frac{\partial^2}{\partial\xi^2}+(\mu_5+\mu_6)\frac{\partial^2}{\partial\xi\partial\eta}+\mu_5\mu_6\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2}{\partial\xi^2}+(\mu_9+\mu_{10})\frac{\partial^2}{\partial\xi\partial\eta}+\mu_9\mu_{10}\frac{\partial^2}{\partial\eta^2}\right)\times$$

$$\times\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}\right)=F_8, \tag{34}$$

where $F_8=F_1/A_0\beta^2\left[\left(\lambda_1-\alpha\right)\left(\lambda_2-\alpha\right)\left((\alpha-\delta)^2+\gamma^2\right)\right]$.

Moreover, since $\left(\mu_9+\mu_{10}\right)^2-4\left(\mu_9\mu_{10}\right)=\left(\mu_9-\mu_{10}\right)^2=-\gamma_1^2<0$, then $\frac{\partial^2}{\partial\xi^2}+\left(\mu_9+\mu_{10}\right)\frac{\partial^2}{\partial\xi\partial\eta}+$ $+\mu_9\mu_{10}\frac{\partial^2}{\partial\eta^2}$ is an elliptic differential operator.

If $\mu_9=-\mu_{10}$, then at $\delta_1=0$ equation (34) takes the form

$$\left(\frac{\partial}{\partial\xi}+\mu_5\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+\mu_6\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^2}{\partial\xi^2}+\gamma_1^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}\right)=F_9,$$

where $F_9=F_1/A_0\beta^2\left[\left(\lambda_1-\alpha\right)\left(\lambda_2-\alpha\right)\gamma^2\right]$.

To further simplify equation (34) for $\delta_1\neq 0$, we choose the substitution (27) as a change of variables, where μ_0 is one of the two solutions of equation

$$\mu^2-\frac{2\left(1-\mu_9\mu_{10}\right)}{\mu_9+\mu_{10}}\mu-1=0,$$

then, similarly to case 2, we obtain the equation

$$\left(\frac{\partial}{\partial s}+c_3\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s}+c_4\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial s^2}+b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_{10},$$

where $b_2^2=\dfrac{\left(\mu_0\mu_9-1\right)\left(\mu_0\mu_{10}-1\right)}{\left(\mu_0+\mu_9\right)\left(\mu_0+\mu_{10}\right)}$, $F_{10}=F_5/\left[\left(\mu_0^2+1\right)\left(\mu_0+\mu_6\right)\left(\mu_0+\mu_9\right)\left(\mu_0+\mu_{10}\right)\right]$.

Let us prove that $b_2^2>0$. Introducing the notations $\nu_3=\dfrac{\mu_0\mu_9-1}{\mu_0+\mu_9}$, $\nu_4=\dfrac{\mu_0\mu_{10}-1}{\mu_0+\mu_{10}}$ and taking into account that is, we have $\nu_3+\nu_4=\dfrac{\mu_0^2+\frac{2\mu_0(\mu_9\mu_{10}-1)}{(\mu_9+\mu_{10})}-1}{\left(\mu_0+\mu_9\right)\left(\mu_0+\mu_{10}\right)}\left(\mu_9+\mu_{10}\right)$, $\mu_0^2+\dfrac{2\mu_0\left(\mu_9\mu_{10}-1\right)}{\left(\mu_9+\mu_{10}\right)}-1=0$, $\nu_3+\nu_4=0$. Then $b_2^2=\dfrac{\left(\mu_0\mu_9-1\right)\left(\mu_0\mu_{10}-1\right)}{(\mu_0+\mu_9)(\mu_0+\mu_{10})}=\nu_3\nu_4=\frac{1}{4}\left[4\nu_3\nu_4-\left(\nu_3+\nu_4\right)^2\right]=-\frac{1}{4}(\nu_3-\nu_4)^2=$ $=-\dfrac{1}{4}\left(\dfrac{2\gamma_1 i(1+\mu_0)^2}{\left(\mu_0+\delta_1\right)^2+\gamma_1^2}\right)^2>0.$

Example 3. Consider the following sixth-order partial differential equation:

$$u_{xxxxxx} + 11u_{xxxxxy} + 60u_{xxxxyy} + 130u_{xxxyyy} - 51u_{xxyyyy} - 781u_{xyyyyy} - 650u_{yyyyyy} = 0. \quad (35)$$

The characteristic equation corresponding equation (35) has the form

$$(dy)^6 - 11(dy)^5(dx) + 60(dy)^4(dx)^2 - 130(dy)^3(dx)^3 - 51(dy)^2(dx)^4 + 781(dy)(dx)^5 - 650(dx)^6 = 0.$$

It is easy to verify that this equation has two different simple real roots and four complex conjugate roots for $t = dy/dx$:

$$t_1 = 1,\ t_2 = -2,\ t_3 = 3 + 4i,\ t_4 = 3 - 4i,\ t_5 = 3 + 2i,\ t_6 = 3 - 2i.$$

Then, equation (35) can be written as follows:

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial x^2} + 6\frac{\partial^2}{\partial x \partial y} + 25\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + 6\frac{\partial^2}{\partial x \partial y} + 13\frac{\partial^2}{\partial y^2}\right) u = 0. \quad (36)$$

After the transformation $\xi = y - 3x$, $\eta = 2x$, we obtain: $\frac{\partial}{\partial x} = -3\frac{\partial}{\partial \xi} + 2\frac{\partial}{\partial \eta}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}$. Considering these, from (36), we have

$$\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} - \frac{2}{5}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2}{\partial \xi^2} + \frac{1}{4}\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) = 0.$$

4. If equation (6) has six different complex conjugate roots:

$$t_1 = \sigma + \zeta i, \quad t_2 = \sigma - \zeta i, \quad t_3 = \delta + \gamma i, \quad t_4 = \delta - \gamma i, \quad t_5 = \alpha + \beta i, \quad t_6 = \alpha - \beta i,$$

where $\alpha, \beta, \delta, \gamma, \sigma, \zeta \in R$ such that $\beta \neq 0, \gamma \neq 0, \zeta \neq 0$,

$$\left[(\alpha - \delta)^2 + (|\beta| - |\gamma|)^2\right]\left[(\alpha - \sigma)^2 + (|\beta| - |\zeta|)^2\right]\left[(\delta - \sigma)^2 + (|\zeta| - |\gamma|)^2\right] \neq 0,$$

then after replacing (22), from equation (1), similarly to equation (33), we obtain the equation

$$A_0\beta^2\left[\left(\alpha^2 + \sigma^2 + \zeta^2 - 2\alpha\sigma\right)\frac{\partial^2}{\partial \xi^2} + 2\beta(\sigma - \alpha)\frac{\partial^2}{\partial \xi \partial \eta} + \beta^2\frac{\partial^2}{\partial \eta^2}\right] \times$$

$$\times\left[\left(\alpha^2 + \delta^2 + \gamma^2 - 2\alpha\delta\right)\frac{\partial^2}{\partial \xi^2} + 2\beta(\delta - \alpha)\frac{\partial^2}{\partial \xi \partial \eta} + \beta^2\frac{\partial^2}{\partial \eta^2}\right]\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) = F_1. \quad (37)$$

Hence, dividing both parts of (37) by $A_0\beta^2\left[(\alpha - \sigma)^2 + \zeta^2\right]\left[(\alpha - \delta)^2 + \gamma^2\right] (\neq 0)$ and introducing the notations $\mu_{11} = \dfrac{\beta(\sigma - \alpha) + \zeta\beta i}{(\alpha - \sigma)^2 + \zeta^2} = \sigma_1 + \zeta_1 i$, $\mu_{12} = \dfrac{\beta(\sigma - \alpha) - \zeta\beta i}{(\alpha - \sigma)^2 + \zeta^2} = \sigma_1 - \zeta_1 i$, $\mu_9 = \dfrac{\beta(\delta - \alpha) + \gamma\beta i}{(\alpha - \delta)^2 + \gamma^2} = \delta_1 + \gamma_1 i$, $\mu_{10} = \dfrac{\beta(\delta - \alpha) - \gamma\beta i}{(\alpha - \delta)^2 + \gamma^2} = \delta_1 - \gamma_1 i$, $\dfrac{\beta(\sigma - \alpha)}{(\alpha - \sigma)^2 + \zeta^2} = \sigma_1$, $\dfrac{\zeta\beta}{(\alpha - \sigma)^2 + \zeta^2} = \zeta_1$, $\dfrac{\beta\,(\delta - \alpha)}{(\alpha - \delta)^2 + \gamma^2} = \delta_1$, $\dfrac{\gamma\beta}{(\alpha - \delta)^2 + \gamma^2} = \gamma_1$, we have the following equation

$$\left(\frac{\partial^2}{\partial \xi^2} + (\mu_{11} + \mu_{12})\frac{\partial^2}{\partial \xi \partial \eta} + \mu_{11}\mu_{12}\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial \xi^2} + (\mu_9 + \mu_{10})\frac{\partial^2}{\partial \xi \partial \eta} + \mu_9\mu_{10}\frac{\partial^2}{\partial \eta^2}\right) \times$$

$$\times\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) = F_{11}, \quad (38)$$

where $F_{11} = F_1/A_0\beta^2\left[\left((\alpha-\sigma)^2+\zeta^2\right)\left((\alpha-\delta)^2+\gamma^2\right)\right]$.

Moreover, since $(\mu_{11}+\mu_{12})^2 - 4(\mu_{11}\mu_{12})^2 = (\mu_{11}-\mu_{12})^2 = -\zeta_1^2 < 0$, $(\mu_9+\mu_{10})^2 - 4(\mu_9\mu_{10})^2 = (\mu_9-\mu_{10})^2 = -\gamma_1^2 < 0$, then $\frac{\partial^2}{\partial\xi^2} + (\mu_{11}+\mu_{12})\frac{\partial^2}{\partial\xi\partial\eta} + \mu_{11}\mu_{12}\frac{\partial^2}{\partial\eta^2}$ and $\frac{\partial^2}{\partial\xi^2} + (\mu_9+\mu_{10})\frac{\partial^2}{\partial\xi\partial\eta} + \mu_9\mu_{10}\frac{\partial^2}{\partial\eta^2}$ are elliptic differential operators.

If $\mu_{11} = -\mu_{12}$, $\mu_9 = -\mu_{10}$, that is, when $\sigma_1 = 0, \delta_1 = 0$, equation (38) takes the form

$$\left(\frac{\partial^2}{\partial\xi^2} + \zeta_1^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2}{\partial\xi^2} + \gamma_1^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2}\right) = F_{11}.$$

To further simplify equation (38) for $\sigma_1 \neq 0$ and $\delta_1 \neq 0$, we introduce the substitution (27), where μ_0 is one of the two solutions of equation

$$\mu^2 - \frac{2(1-\mu_9\mu_{10})}{\mu_9+\mu_{10}}\mu - 1 = 0,$$

then, similarly to case 3 , we obtain the equation

$$\left[\frac{\partial^2}{\partial s^2} + (c_5+c_6)\frac{\partial^2}{\partial s\partial t} + c_5c_6\frac{\partial^2}{\partial t^2}\right]\left(\frac{\partial^2}{\partial s^2} + b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right) = F_{12},$$

where $c_5 = [\mu_{11}\mu_0 - 1]/[\mu_{11}+\mu_0]$, $c_6 = [\mu_{12}\mu_0 - 1]/[\mu_{12}+\mu_0]$,
$F_{12} = F_5/\left[\left(\mu_0^2+1\right)(\mu_0+\mu_9)(\mu_0+\mu_{10})(\mu_0+\mu_{11})(\mu_0+\mu_{12})\right]$.

Since $(c_5+c_6)^2 - 4c_5c_6 = (c_5-c_6)^2 = \left(\frac{2\zeta_1 i(1+\mu_0)^2}{(\mu_0+\sigma_1)^2+\zeta_1^2}\right)^2 < 0$ and $b_2^2 > 0$, then the differential operators $\frac{\partial^2}{\partial s^2} + (c_5+c_6)\frac{\partial^2}{\partial s\partial t} + c_5c_6\frac{\partial^2}{\partial t^2}$ and $\frac{\partial^2}{\partial s^2} + b_2^2\frac{\partial^2}{\partial t^2}$ the last equation are elliptic.

Example 4. Consider the following sixth-order partial differential equation:

$$u_{xxxxxx} + 24u_{xxxxxy} + 254u_{xxxxyy} + 1504u_{xxxyyy} + 5233u_{xxyyyy} + 10120u_{xyyyyy} + 8500u_{yyyyyy} = 0. \quad (39)$$

The characteristic equation corresponding to equation (39) has the form

$$(dy)^6 - 24(dy)^5(dx) + 254(dy)^4(dx)^2 - 1504(dy)^3(dx)^3 + 5233(dy)^2(dx)^4 - 10120(dy)(dx)^5 + 8500(dx)^6 = 0.$$

It is easy to verify that this equation has six different complex conjugate roots for $t = dy/dx$:

$$t_1 = 4+i,\ t_2 = 4-i,\ t_3 = 4+2i,\ t_4 = 4-2i,\ t_5 = 4+3i,\ t_6 = 4-3i.$$

Then, equation (39) can be written as follows:

$$\left(\frac{\partial^2}{\partial x^2} + 8\frac{\partial^2}{\partial x\partial y} + 17\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + 8\frac{\partial^2}{\partial x\partial y} + 20\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + 8\frac{\partial^2}{\partial x\partial y} + 25\frac{\partial^2}{\partial y^2}\right)u = 0. \quad (40)$$

After the transformation $\xi = y - 4x$, $\eta = 3x$ we obtain: $\frac{\partial}{\partial x} = -4\frac{\partial}{\partial\xi} + 3\frac{\partial}{\partial\eta}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial\xi}$. Considering these from (40), we have

$$\left(\frac{\partial^2}{\partial\xi^2} + 9\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2}{\partial\xi^2} + \frac{9}{4}\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)u = 0.$$

Thus, we have proved the following

Theorem 1. Let one of the following statements be true with respect to equation (6):
1) has six different real roots;
2) has four different real roots and two complex conjugate roots;
3) has two real roots and four different complex conjugate roots;
4) has six different complex conjugate roots.

Then, in the domain Ω, equation (1) can be reduced to the one of the following canonical forms

1) $\left(\frac{\partial}{\partial\xi}+c_1\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+c_2\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^2}{\partial\xi^2}-b^2\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2 u}{\partial\xi^2}-\frac{\partial^2 u}{\partial\eta^2}\right)=F_3;$

2) $\left(\frac{\partial}{\partial s}+c_3\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s}+c_4\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial s^2}-b_1^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_6;$

3) $\left(\frac{\partial}{\partial s}+c_3\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s}+c_4\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial s^2}+b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_{10};$

4) $\left[\frac{\partial^2}{\partial s^2}+(c_5+c_6)\frac{\partial^2}{\partial s\partial t}+c_5c_6\frac{\partial^2}{\partial t^2}\right]\left(\frac{\partial^2}{\partial s^2}+b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}\right)=F_{12}.$

It should be noted that equations (1), similar to equations of hyperbolic and/or elliptic type, possess simple (non-repeated) real and/or complex characteristics. Consequently, in all cases considered above, the discriminant is non-zero. Moreover, the canonical forms of equation (1) may contain both hyperbolic and/or elliptic differential operators.

Remark 1. The classification and reduction to canonical form of sixth-order linear partial differential equations with multiple real characteristics are studied in ten distinct cases. Analogously, the following theorem can be proven:

Theorem 2. Assume that equation (6) exhibits one of the following root configurations:
1) one double root and four distinct real roots;
2) one double root and one quadruple real root;
3) two triple real roots;
4) one quintuple root and one simple real root;
5) one sextuple real root;
6) two double roots and two distinct real roots;
7) three double real roots;
8) one double root, one triple root, and one simple real root;
9) one triple root and three distinct real roots;
10) one quadruple root and two distinct real roots.

Then, in the domain Ω, equation (1) can be reduced to one of the following canonical forms corresponding to the root structures:

1) $\left(\frac{\partial}{\partial\xi}+c_1\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+c_2\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^4 u}{\partial\xi^4}-\frac{\partial^4 u}{\partial\xi^2\partial\eta^2}\right)=F_3;$

2) $\frac{\partial^6}{\partial\xi^4\partial\eta^2}u=F_1/\left(\lambda_1-\lambda_2\right)^6;$

3) $\frac{\partial^6}{\partial\xi^3\partial\eta^3}u=F_1/\left\{-\left(\lambda_1-\lambda_2\right)^6\right\};$

4) $\frac{\partial^6}{\partial\xi\partial\eta^5}u=F_1/\left\{-\left(\lambda_1-\lambda_2\right)^6\right\};$

5) $\frac{\partial^6 u}{\partial\eta^6}=F_1/\left(\lambda_1-\lambda_2\right)^6;$

6) $\left(\frac{\partial}{\partial\xi}+c_3\frac{\partial}{\partial\eta}\right)^2\left(\frac{\partial^4 u}{\partial\xi^4}-\frac{\partial^4 u}{\partial\xi^2\partial\eta^2}\right)=F_5;$

7) $\left(\frac{\partial}{\partial\xi}+c_4\frac{\partial}{\partial\eta}\right)^2\frac{\partial^4 u}{\partial\xi^2\partial\eta^2}=F_6;$

8) $\frac{\partial^6 u}{\partial\xi^4\partial\eta^2}+c_4\frac{\partial^6 u}{\partial\xi^3\partial\eta^3}=F_7;$

9) $\left(\frac{\partial}{\partial\xi}+c_5\frac{\partial}{\partial\eta}\right)\left(\frac{\partial^5 u}{\partial\xi^2\partial\eta^3}+c_6\frac{\partial^5 u}{\partial\xi\partial\eta^4}\right)=F_8;$

10) $\frac{\partial^6 u}{\partial\xi^2\partial\eta^4} + c_7\frac{\partial^6 u}{\partial\xi\partial\eta^5} = F_9.$

Remark 2. The classification and reduction to canonical form of sixth-order linear partial differential equations with multiple and complex characteristics are studied in nine distinct cases. Analogously, the following theorem can be proven:

Theorem 3. Assume that equation (6) exhibits one of the following root structures:

1) one double real root, two distinct real roots, and one pair of complex conjugate roots;
2) two distinct double real roots and one pair of complex conjugate roots;
3) one triple real root, one simple real root, and one pair of complex conjugate roots;
4) one quadruple real root and one pair of complex conjugate roots;
5) one double real root and two distinct double pairs of complex conjugate roots;
6) two distinct real roots and two distinct double pairs of complex conjugate roots;
7) one double real root and two distinct pairs of complex conjugate roots;
8) two distinct pairs of complex conjugate roots and two distinct double pairs of complex conjugate roots;
9) two distinct triple pairs of complex conjugate roots.

Then, in the domain Ω, equation (1) can be reduced to one of the following canonical forms corresponding to the root structures:

1) $\left(\frac{\partial}{\partial s} + c_1\frac{\partial}{\partial t}\right)^2\left(\frac{\partial^2}{\partial s^2} - b^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right)= F_4$; 2) $\left(\frac{\partial^2}{\partial s^2} - b_1^2\frac{\partial^2}{\partial t^2}\right)^2\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right)= F_6$;

3) $\left(\frac{\partial}{\partial s} + c_2\frac{\partial}{\partial t}\right)\left(\frac{\partial^5 u}{\partial s^2\partial t^3} + \frac{\partial^5 u}{\partial t^5}\right)= F_8$; 4) $\left(\frac{\partial^6 u}{\partial s^2\partial t^4} + \frac{\partial^6 u}{\partial t^6}\right)= F_{10}$; 5) $\left(\frac{\partial^3}{\partial s^2\partial t} + \frac{\partial^3}{\partial t^3}\right)^2 u= F_{12}$;

6) $\left(\frac{\partial^2}{\partial s^2} - b_1^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}\right)^2 u= F_{14}$; 7) $\left(\frac{\partial}{\partial s} + c_1\frac{\partial}{\partial t}\right)^2\left(\frac{\partial^2}{\partial s^2} + b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right)= F_{16}$;

8) $\left(\frac{\partial^2}{\partial s^2} + b_2^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}\right)^2 u= F_{18}$; 9) $\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)^3 u = F_{19}$.

It should be noted that equation (1), like parabolic-type equations, exhibits multiple characteristics. Consequently, in all the cases discussed above, the discriminant is zero ($D = 0$). Nonetheless, the canonical forms of equation (1) may still involve both hyperbolic and/or elliptic differential operators.

Conclusion

In this paper, we prove a theorem on the canonical forms of equation (1) and three lemmas that play an important role in finding the canonical form of the equation (1).

Arguing similarly, we can find canonical forms of equation (1) in cases with multiple characteristics, provided that the coefficients of the equation (1) are sufficiently smooth functions.

We can give a number of examples when only finding the canonical form of an equation helps to obtain serious results. Considering the canonical form of the equation (1), when studying some boundary value problems, we can use potential theory or the Green or Riman function method. Therefore, the found canonical forms of linear differential equations with partial derivatives of the sixth-order with non-multiple characteristics and with constant coefficients allow us to correctly formulate and systematically study correct boundary value problems for such equations. These problems are the subject of further research.

From the canonical form of the equation (1), obtained in the first case considered above, it is clear that if the function F_3 does not depend on the unknown function u and its derivatives, then it is possible to find a general solution to equation (1).

Based on the proposed method for finding the canonical form of the equation (1), it is possible to study the problems of classification and reduction to canonical form of differential equations of higher-order with partial derivatives.

Acknowledgments

The authors express their gratitude to Professor A.K. Urinov for discussing the results of this article.

We would also like to express our sincere gratitude to Olga Ivanovna the responsible secretary of the journal "Bulletin of the Karaganda University. Mathematics Series", for her valuable comments, which significantly improved the content of this article.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Dzhuraev, T.D., & Popelek, Y. (1991). Classification and reduction to canonical form of third-order partial differential equations. *Differential Equations, 27*(10), 1734–1745.

2 Dzhuraev, T.D., & Sopuev, A. (1994). On the classification and reduction to canonical form of fourth-order partial differential equations. *Uzbek Mathematical Journal, 3*, 7–21 [in Russian].

3 Urinov, A.K., & Abdukodirov, A.T. (2012). Kanonicheskie vidy differentsialnykh uravnenii s chastnymi proizvodnymi piatogo poriadka [Canonical types of differential equations with partial derivatives of the fifth order]. *Materialy vtorogo Mezhdunarodnogo Rossiisko-Uzbekskogo Simpoziuma «Uravneniia smeshannogo tipa i rodstvennye problemy analiza i informatiki» — Proceedings of the Second International Russian-Uzbek Symposium "Equations of mixed type and related problems of analysis and informatics"*, 251–254 [in Russian].

4 Urinov, A.K., & Abdukodirov, A.T. (2023). O kanonicheskikh vidakh differentsialnykh uravnenii s chastnymi proizvodnymi n-ogo poriadka [On canonical forms of n^{th}-order partial differential equations]. *Biulleten Instituta Matematiki — Bulletin of the Institute of Mathematics, 6*(5), 157–164 [in Russian].

5 Karimov, Sh.T. (2018). The cauchy problem for the degenerated partial differential equation of the high even order. *Siberian Electronic Mathematical Reports, 15*, 853–862. https://doi.org/10.17377/semi.2018.15.073

6 Urinov, A.K., & Azizov, M.S. (2022). O razreshimosti nelokalnykh nachalno-granichnykh zadach dlia odnogo differentsialnogo uravneniia v chastnykh proizvodnykh vysokogo chetnogo poriadka [On the solvability of nonlocal initial-boundary value problems for a partial differential equation of high even order]. *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Kompiuternye Nauki — Bulletin of Udmurt University. Mathematics. Mechanics. Computer sciences, 32*(2), 240–255. https://doi.org/10.35634/vm220206

7 Gao, X.-W., Zhu, Y.-M., & Pan, T. (2023). Finite line method for solving high-order partial differential equations in science and engineering. *Partial Differential Equations in Applied Mathematics, 7*, 100477. https://doi.org/10.1016/j.padiff.2022.100477

8 Sabitov, K.B., Kasimov, Sh.G., & Madrakhimov, U.S. (2020). Nachalno-granichnaia zadacha dlia uravneniia v chastnykh proizvodnykh vysokogo poriadka v mnogomernom sluchae [Initial boundary value problem for a high order partial differential equation in multidimensional case]. *Vestnik Rossiiskogo gosudarstvennogo gumanitarnogo universiteta. Seriia: Informatika. Informatsionnaia bezopasnost. Matematika — Bulletin of Russian State University for the Humanities. Computer Science. Information Security. Mathematics Series, 1*, 75–101 [in Russian].

9 Schneider, G., & Wayne, C.E. (2001). Kawahara dynamics in dispersive media. *Physica D: Nonlinear Phenomena, 152-153*, 384–394. https://doi.org/10.1016/S0167-2789(01)00181-6

10 Kulesh, E.E., & Martynov, I.P. (2018). Ob odnom differentsialnom uravnenii v chastnykh proizvodnykh shestogo poriadka [One six-order partial differential equation]. *Izvestiia Natsionalnoi akademii nauk Belarusi. Seriia fiziko-matematicheskikh nauk — Proceedings of the National Academy of Sciences of Belarus, Physics and Mathematics series, 54*(1), 7–19 [in Russian]. https://doi.org/10.29235/1561-2430-2018-54-1

11 Kulesh, E.E., & Martynov, I.P. (2018). O svoistvakh reshenii odnogo differentsialnogo uravneniia v chastnykh proizvodnykh shestogo poriadka [On the properties of solutions of one partial differential equation of the sixth order.] *Vestnik Grodnenskogo gosudarstvennogo universiteta imeni Yanki Kupaly. Matematika. Seriia 2. Fizika. Informatika. Vychislitelnaia tekhnika i upravlenie — Bulletin of the Yanka Kupala State University of Grodno. Series 2. Mathematics. Physics. Computer science, computer engineering and control, 8*(2), 19–25 [in Russian].

12 Revenko, V.P., & Bakulin, V.N. (2021). The Reduction of Orthotropic Bodies Equations to Solving the Partial Differential Equation of the Sixth Order and Its Investigation. *Lobachevskii Journal of Mathematicsm, 42*(8), 2030–2036. https://doi.org/10.1134/S1995080221080266

13 Faradzhev, A.S. (2022). Ob odnoi nelokalnoi obratnoi kraevoi zadache dlia uravneniia s chastnymi proizvodnymi shestogo poriadka s dopolnitelnymi integralnymi usloviiami [On a Nonlocal Inverse Boundary Value Problem for a Sixth-Order Partial Differential Equation with Additional Integral Conditions.] *Sbornik statei po materialam XVII Vserossiiskoi s mezhdunarodnym uchastiem nauchno-tekhnicheskoi konferentsii — Collection of articles based on the materials of the XVII All-Russian Conference with International Participation of a Scientific and Technical Conference* [in Russian].

*Author Information**

Abdurashid Tolibjonovich Abdukodirov (*corresponding author*) — Candidate of Sciences of Physics and Mathematics, Associate professor, Fergana State University, 19 Murabbiylar street, Fergana, 150100, Uzbekistan; e-mail: *abdurashid1976@mail.ru*

Tulkinjon Azizjon ugli Tulkinboev — Teacher, Fergana State University, 19 Murabbiylar street, Fergana, 150100, Uzbekistan; e-mail: *tulqinjon98@mail.ru*

*Authors' names are presented in the order: first name, middle name, and last name.