Bulletin of the Karaganda University. Mathematics Series, No. 2(118), 2025, pp. 122–136

https://doi.org/10.31489/2025M2/122-136 *Research article*

Spectral analysis of second order quantum difference operator over the sequence space l_p $(1 < p < \infty)$

N. Kalita[1,*], A.J. Dutta[2]

[1] *Gauhati University, Guwahati, India;*
[2] *Pragjyotish College, Guwahati, India*
(E-mail: nkalita@gauhati.ac.in, amar_iasst@yahoo.co.in)

In this article, we study the spectrum, fine spectrum and boundedness property of second order quantum difference operator Δ_q^2 $(0 < q < 1)$ over the class of sequence l_p $(1 < p < \infty)$, the p^{th} summable sequence space. The second order quantum difference operator Δ_q^2 is a lower triangular triple band matrix $\Delta_q^2(1, -(1+q), q)$. We also determine the approximate point spectrum, defect spectrum, compression spectrum, and Goldberg classification of the operator on the class of sequence. We obtained the results by solving an infinite system of linear equations and computing the inverse of a lower triangular infinite matrix. We also provide appropriate examples along with graphical representations where necessary.

Keywords: spectrum, difference operator, infinite matrices, triple-band matrix.

2020 Mathematics Subject Classification: 47A10, 47B37, 15A18.

Introduction

Spectral theory of bounded linear operators on Banach or Hilbert spaces holds a significant place in different branches of Mathematics due to its many applications. The fundamental principle of the modern spectral theorem is that certain linear operators on infinite dimensional spaces can be represented in a "diagonal" matrix form. From this diagonal form, we can determine the spectrum of the operator. The spectrum of an operator can be classified into three parts: the point spectrum, the continuous spectrum, and the residual spectrum. These three disjoint parts together are referred to as the "fine spectrum".

In operator theory, one of the most important linear operators is the difference operator. The spectrum of this operator and its different forms on various sequence spaces have been studied by many researchers. In recent times, researchers have started analyzing the spectrum of the quantum version of some well-known operators, one of which is the difference operator. In our study, we analyze the spectrum of a second order quantum difference operator. The $q-$analog of the second order difference operator is defined as $(\Delta_q^2 u)_k = u_k - (1+q)u_{k-1} + qu_{k-2}$, for all $k \in \mathbb{N}$ and any term with negative indices are zero. The matrix representation of this operator is given below

$$\Delta_q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ -(1+q) & 1 & 0 & 0 & \dots \\ q & -(1+q) & 1 & 0 & \dots \\ 0 & q & -(1+q) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The n^{th} order $q-$difference operator is defined as $(\Delta_q^n u)_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i}_q q^{\binom{i}{2}} u_{n+k-i}$, which was introduced by Bustoz and Gordillo [1].

*Corresponding author. *E-mail: nkalita@gauhati.ac.in*
Received: 25 October 2024; *Accepted:* 17 March 2025.

The initial study of the spectrum of the difference operator Δ, $(\Delta y_k = y_k - y_{k+1})$ was conducted by Altay and Basar [2], Kayaduman and Furkan [3], and Akhmedov and Basar [4, 5] in the spaces c_o, c, ; l_1, bv and l_p, bv_p respectively. The spectrum of the second-order difference operator Δ^2 was studied by Dutta and Baliarsingh [6] over the space c_0. After this the operator Δ was generalized to $B(r,s)$, $B(r,s)(y_k) = (ry_k + sy_{k-1})$. The spectrum of this operator was studied by Altay and Basar [7], Kayaduman et al. [8], Bilgic and Furkan [9], Dutta and Tripathy [10] over the spaces c_0, c; l_1, bv; l_p, bv_p $(1 \leq p < \infty)$ and cs respectively. $B(r,s)$ was further generalized to $B(r,s,t)$, $B(r,s,t)(y_k) = (ry_k + sy_{k-1} + ty_{k-2})$. The spectrum of this operator was studied by Bilgic and Furkan [11]; Furkan et al. [12, 13] over the spaces l_1, bv, c_0, c and l_p, bv_p respectively. Srivastava and Kumar [14, 15]; Akhmedov and El-Shabrawy [16] studied the spectrum of the generalised difference operator Δ_v, where $\Delta_v(y_k) = (v_k y_k - v_{k-1}y_{k-1})$ over the sequence spaces c_0, l_1 and c, l_p respectively. Akhmedov and El-Shabrawy [17, 18]; Dutta and Baliarsingh [19] obtained the spectrum of the operator Δ_{ab}, where $\Delta_{ab}(y_k) = (a_k y_k + b_{k-1}y_{k-1})$ over the sequence spaces c_0, c and l_p, bv_p respectively. Panigrahi and Srivastava [20, 21] also analyzed the spectrum of Δ^2_{uv}, where $\Delta^2_{uv}(y_k) = (u_k y_k - v_{k-1}y_{k-1} + u_{k-2}y_{k-2})$ and Δ^2_{uvw}, where $\Delta^2_{uvw}(y_k) = (u_k y_k + v_{k-1}y_{k-1} + w_{k-2}y_{k-2})$ over the sequence spaces c_0 and l_1 respectively. Then, Altundağ and Abay [22] studied on the fine spectrum of generalized upper triangular triple-band matrices $(\Delta^2_{uvw})^t$ where the transpose of matrix operator Δ^2_{uvw} over the sequence space l_1. Patra and Srivastava [23] considered a new generalized difference operator $A(p_1, p_2;\ q_1, q_2;\ r_1, r_2)$ and determined its spectrum over the sequence space l_p $(1 \leq p < \infty)$. The operators mentioned above can be expressed using a lower triangular band matrix. Spectral analysis of the quantum versions of some well-known operators has been conducted in recent years. The spectrum of $q-$Cesàro matrix was studied by Yildirim [24], Durna and Turkay [25] over the sequence space c_0 and c respectively. Yaying et al. [26, 27] studied the spectrum of second order $q-$difference operator over the sequence space c_0, l_1 respectively. Spectrum of weighted $q-$difference operator was studied by Yaying et al. [28] over the sequence space c_0.

$q-$Analog: A $q-$analog of a number, a theorem, an identity or an expression is a generalization that involves a new parameter q and it reduced to the original number, theorem, identity as the limit $q \longrightarrow 1^-$. In the 19^{th} century, the basic hypergeometric series became the first $q-$analog to be extensively studied. In recent research of many areas of Mathematics like combinatorics, approximation theory, difference and integral equations, etc., $q-$calculus have been used extensively.

The $q-$analog $[m]_q$ of m for $q \in (0,1)$ can be determined as

$$[m]_q = \begin{cases} \sum_{k=0}^{m-1} q^k, & m = 1, 2, 3, \ldots, \\ 0, & m = 0. \end{cases}$$

One might observe that $[m]_q = m$ whenever $q \longrightarrow 1^-$. The $q-$analog $\binom{m}{k}_q$ of binomial coefficient $\binom{m}{k}$ can be determined as

$$\binom{m}{q}_q = \begin{cases} \dfrac{[m]_q!}{[m-k]_q![k]_q!}, & m \geq k, \\ 0, & k > m, \end{cases}$$

where the $q-$analog of the factorial, i.e., $q-$factorial, is defined as

$$[m]_q! = \begin{cases} \prod_{k=1}^{m} [k]_q, & m = 1, 2, 3, \ldots, \\ 1, & m = 0. \end{cases}$$

The $q-$analog of some specific binomials such as $\binom{0}{0} = \binom{m}{0} = \binom{m}{m} = 1$, also $\binom{m}{m-k}_q = \binom{m}{k}_q$. For an in-depth study of quantum calculus, we refer to the book [29].

1 Some Definitions and Preliminaries

Consider $M : U \longrightarrow V$ be a bounded linear operator, in which U and V are Banach spaces, the following collections

$$
\begin{aligned}
& R(M) = \{v \in V : v = Mu,\ u \in U\} \\
\text{and} \quad & B(U,V) = \{M : U \longrightarrow V : M \text{ is continuous and linear}\}
\end{aligned}
$$

are termed as the range of the operator M and the set of all bounded linear operators from U to V respectively. The adjoint operator M^* of M is defined from V^* to U^*, where V^* and U^* represent the dual space of V and U respectively. Again, it is defined as $(M^*f)(u) = f(Mu)$, for all $f \in V^*$ and $u \in U$.

Let $M : D(M) \longrightarrow U$, where $D(M)$ denotes the domain of M. From M we can get an operator,

$$M_\mu = M - \mu I,$$

where $\mu \in \mathbb{C}$ and I is the identity operator. A regular value $\mu \in \mathbb{C}$ of M is such that M_μ is invertible, and its inverse (M_μ^{-1}) is bounded and defined on a set A and call it the resolvent operator of M, where A is dense in U. The collections of such μ is called the resolvent set and is denoted by $\rho(M,U)$. In the complex plane $\mathbb{C}$, the compliment of $\rho(M,U)$ is denoted by $\sigma(M,U)$, and is called the spectrum of M.

Further, $\sigma(M,U)$ is classified into three disjoint subsets, namely, the point spectrum $\sigma_p(M,U)$, the continuous spectrum $\sigma_c(M,U)$, and the residual spectrum $\sigma_r(M,U)$. In point spectrum, M_μ^{-1} does not exist for any $\mu \in \sigma_p(M,U)$, while in continuous spectrum, M_μ^{-1} exist but unbounded for every $\mu \in \sigma_c(M,U)$, and also defined on a set that is dense in U. On the other hand, in the residual spectrum, M_μ^{-1} exists but may or may not be bounded for $\mu \in \sigma_r(M,U)$ and is not dense in U.

There are more subdivisions of the spectrum of a bounded operator such as approximate point spectrum $\sigma_{ap}(M,U)$, defect spectrum $\sigma_\delta(M,U)$ and compression spectrum $\sigma_{co}(M,U)$, which are defined as follows:

- $\sigma_{ap}(M,U) = \{\mu \in \mathbb{C} : (M - \mu I) \ \text{ is not bounded below}\}$;
- $\sigma_\delta(M,U) = \{\mu \in \mathbb{C} : (M - \mu I) \ \text{ is not surjective}\}$;
- $\sigma_{co}(M,U) = \{\mu \in \mathbb{C} : \overline{R(M - \mu I)} \neq U\}$.

2 Goldberg's classification of spectrum

A detailed classification of the spectrum of an operator was given by Goldberg [30]. This classification is based on the nature of the set $R(M_\mu)$ and the inverse M_μ^{-1}.

If $M \in B(U,U)$, then there are three possibilities for $R(M_\mu)$:

(P) $R(M_\mu) = U$,

(Q) $\overline{R(M_\mu)} = U$, but $R(M_\mu) \neq U$,

(R) $\overline{R(M_\mu)} \neq U$

and three possibilities for M_μ^{-1}:

(*1*) Exist and continuous,

(*2*) Exist but discontinuous,

(*3*) Does not exist.

Combination of the possibilities P, Q, R and 1, 2, 3 leads to nine different states. They are identified as P_1, P_2, P_3, Q_1, Q_2, Q_3, R_1, R_2, and R_3.

If $M_\mu \in P_1$ or $M_\mu \in Q_1$, then $\mu \in \rho(M,X)$. If $M_\mu \in R_2$, then M_μ^{-1} exists and is unbounded, and $\overline{R(M_\mu)} \neq X$ and we can write $\mu \in R_2\sigma(M,X)$. We can summarize this classification in the following Table 1.

Table 1

Goldberg's classification of spectrum of linear operator

		1	2	3
		M_μ^{-1} exist and bounded	M_μ^{-1} exist and unbounded	M_μ^{-1} does not exist
P	$R(M_\mu)=U$	$\mu\in\rho(M,U)$	-	$\mu\in\sigma_p(M,U)$ $\mu\in\sigma_{ap}(M,U)$
Q	$\overline{R(M_\mu)}=U$	$\mu\in\rho(M,U)$	$\mu\in\sigma_c(M,U)$ $\mu\in\sigma_\delta(M,U)$ $\mu\in\sigma_{ap}(M,U)$	$\mu\in\sigma_p(M,U)$ $\mu\in\sigma_\delta(M,U)$ $\mu\in\sigma_{ap}(M,U)$
R	$\overline{R(M_\mu)}\neq U$	$\mu\in\sigma_r(M,U)$ $\mu\in\sigma_\delta(M,U)$ $\mu\in\sigma_{co}(M,U)$	$\mu\in\sigma_r(M,U)$ $\mu\in\sigma_\delta(M,U)$ $\mu\in\sigma_{co}(M,U)$ $\mu\in\sigma_{ap}(M,U)$	$\mu\in\sigma_p(M,U)$ $\mu\in\sigma_\delta(M,U)$ $\mu\in\sigma_{co}(M,U)$ $\mu\in\sigma_{ap}(M,U)$

Proposition 1. ([31], p. 28) Spectral and sub-spectral relationships of an operator M and its adjoint operator M^* are provided below.

(a) $\sigma(M^*,Y^*)=\sigma(M,Y)$,
(b) $\sigma_{ap}(M^*,Y^*)=\sigma_\delta(M,Y)$,
(c) $\sigma_\delta(M^*,Y^*)=\sigma_{ap}(M,Y)$,
(d) $\sigma_p(M^*,Y^*)=\sigma_{co}(M,Y)$,
(e) $\sigma(M,Y)=\sigma_{ap}(M,Y)\bigcup\sigma_p(M^*,Y^*)=\sigma_p(M,Y)\bigcup\sigma_{ap}(M^*,Y^*)$.

Lemma 1. ([30], p. 60) The adjoint operator M^* of M is onto if and only if M has a bounded inverse.

Lemma 2. ([30], p. 59) The bounded linear operator $M:U\longrightarrow V$ has dense range if and only if M^* is one to one.

Throughout this work, the aforementioned spaces c_0, c, l_1, l_p, bv, bv_p, cs and l_∞ represent the spaces of all null, convergent, absolutely summable, $p-$absolutely summable, bounded variation, $p-$bounded variation, convergent series, and bounded sequences, respectively.

Before going to the main results we state a remark, "if z is a complex number, then $\sqrt{z}$ means the square root of z with non-negative real part. If $Re\sqrt{z}=0$, then $\sqrt{z}$ means the square root of z with $Im(z)\geq 0$".

3 Spectrum of $\Delta_q^2(1,-(1+q),q)$ on l_p

Theorem 1. $\Delta_q^2\in B(l_p)$ with $(1+(1+q)^p+q^p)^{1/p}\leq ||\Delta_q^2||_{l_p}\leq 2(q+1)$ for $0<q<1$, where $1<p<\infty$.

Proof. The linearity of Δ_q^2 is straightforward to prove, so it is omitted. Now, consider $e^{(0)}=(1,0,0,\ldots)$ in l_p. Then, $(\Delta_q^2)e^{(0)}=(1,-(1+q),q,0,0,\ldots)$ and it is obtained that

$$\frac{||(\Delta_q^2)e^{(0)}||_{l_p}}{||e^{(0)}||_{l_p}}=(1+(1+q)^P+q^p)^{1/p}.$$

From this we get, $(1+(1+q)^P+q^p)^{1/p}\leq||(\Delta_q^2)e^{(0)}||$. Again for any $u=(u_k)\in l_p$ and using the

Minkowaski inequality, we get

$$
\begin{aligned}
||\Delta_q^2 u||_{l_p} &= \left(\sum_k |qu_{k-1} + (-(1+q))u_k + u_{k+1}|^p\right)^{1/p} \\
&\leq \left(\sum_k |qu_{k-1}|^p\right)^{1/p} + \left(\sum_k |(1+q)u_k|^p\right)^{1/p} + \left(\sum_k |u_{k+1}|^p\right)^{1/p} \\
&= \left(q^p \sum_k |u_{k-1}|^p\right)^{1/p} + \left((1+q)^p \sum_k |u_k|^p\right)^{1/p} + \left(\sum_k |u_{k+1}|^p\right)^{1/p} \\
&= (q + (1+q) + 1)\, ||u||_{l_p} \\
&= 2(1+q)||u||_{l_p}.
\end{aligned}
$$

As a result, we get $(1 + (1+q)^p + q^p)^{1/p} \leq ||\Delta_q^2||_{l_p} \leq 2(q+1)$.

Theorem 2. The point spectrum $\sigma_p(\Delta_q^2, l_p) = \phi$ (the empty set).

Proof. We prove this theorem by the method of contradiction. Consider $\sigma_p(C_1(q), l_p) \neq \phi$. Then for any $0 \neq u \in l_p$ with $(\Delta_q^2)u = \lambda u$, we get the following equalities:

$$
\begin{aligned}
u_0 &= \lambda u_0, \\
-(1+q)u_0 + u_1 &= \lambda u_1, \\
qu_0 - (1+q)u_1 + u_2 &= \lambda u_2, \\
qu_1 - (1+q)u_2 + u_3 &= \lambda u_3, \\
&\vdots \\
qu_{m-2} - (1+q)u_{m-1} + u_m &= \lambda u_m, \\
&\vdots
\end{aligned}
$$

If u_m is the first non zero entry of the sequence $u = (u_m)$, then from the above equations we get $\lambda = 1$. Putting the value of $\lambda = 1$ in the proceeding equation, we get $u_m = 0$, which contradicts our assumption. Thus, $\sigma_p(\Delta_q^2, l_p) = \phi$.

Lemma 3. ([32], p. 126) The matrix $A = (a_{nk})$ defines a bounded linear operator $T \in B(l_1)$, mapping l_1 to itself, if and only if the supremum of l_1 norms of the columns of A is bounded.

Lemma 4. ([32], p. 126) The matrix $A = (a_{nk})$ defines a bounded linear operator $T \in B(l_\infty)$, mapping l_∞ to itself, if and only if the supremum of l_1 norms of the rows of A is bounded.

Lemma 5. ([33], p. 174, Theorem 9) If $1 < p < \infty$ and $A \in (l_1, l_1) \cap (l_\infty, l_\infty)$. Then $A \in (l_p, l_p)$.

Theorem 3. $\sigma(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$.

Proof. We consider

$$
S = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}
$$

and let $\mu \notin S$. So, we get $\mu \neq 1$, and this implies that $(\Delta_q^2 - \mu I)$ has an inverse. Now,

$$(\Delta_q^2 - \mu I) = \begin{bmatrix} 1-\mu & 0 & 0 & 0 & 0 & \dots \\ -(1+q) & 1-\mu & 0 & 0 & 0 & \dots \\ q & -(1+q) & 1-\mu & 0 & 0 & \dots \\ 0 & q & -(1+q) & 1-\mu & 0 & \dots \\ 0 & 0 & q & -(1+q) & 1-\mu & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $(\Delta_q^2 - \mu I)$ is a lower triangular matrix, its inverse can be obtained easily and has been given below:

$$(\Delta_q^2 - \mu I)^{-1} = \begin{bmatrix} m_1 & 0 & 0 & 0 & \dots \\ m_2 & m_1 & 0 & 0 & \dots \\ m_3 & m_2 & m_1 & 0 & \dots \\ m_4 & m_3 & m_2 & m_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$\begin{aligned} m_1 &= \frac{1}{1-\mu}, \\ m_2 &= \frac{q+1}{(1-\mu)^2}, \\ m_3 &= \frac{(q+1)^2 - q(1-\mu)}{(1-\mu)^3}, \\ &\vdots \end{aligned}$$

here the sequence (m_k) satisfies the following recurrence relation

$$m_k = \frac{(q+1)m_{k-1} - q m_{k-2}}{1-\mu}, \text{ for } k \geq 3.$$

From this recurrence relation, we get the characteristic equation as

$$(1-\mu)w^2 - (1+q)w + q = 0,$$

whose solutions are:

$$\begin{aligned} w_1 &= \frac{(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}}{2(1-\mu)}, \\ w_2 &= \frac{(1+q) - \sqrt{(1+q)^2 - 4(1-\mu)q}}{2(1-\mu)}. \end{aligned}$$

From elementary calculations on recurrence sequence, we get

$$m_k = \frac{w_1^k - w_2^k}{\sqrt{(1+q)^2 - 4(1-\mu)q}}. \tag{1}$$

Now, we can proceed to the proof in two cases.

Case 1: If $(1+q)^2 = 4q(1-\mu)$, then we get

$$m_k = \left(\frac{2k}{1+q}\right)\left[\frac{(1+q)}{2(1-\mu)}\right]^k.$$

It can be easily proved that $(m_k) \in l_p$ if $\left|\dfrac{(1+q)}{2(1-\mu)}\right| < 1$. So, $\mu \notin S$ implies that $(m_k) \in l_p$.

Case 2: If $(1+q)^2 \neq 4q(1-\mu)$. Since $\mu \notin S$, we have $|w_1| < 1$. Again, using the inequality $|1-\sqrt{z}| \leq |1+\sqrt{z}|$ for any $z \in \mathbb{C}$, we get

$$\left|\frac{(1+q)}{2(1-\mu)} - \frac{\sqrt{(1+q)^2 - 4(1-\mu)q}}{2(1-\mu)}\right| \leq \left|\frac{(1+q)}{2(1-\mu)} + \frac{\sqrt{(1+q)^2 - 4(1-\mu)q}}{2(1-\mu)}\right|$$
$$\implies |w_2| \leq |w_1| < 1.$$

Using this in equation (1), it is obtained that $(m_k) \longrightarrow 0$ as $k \longrightarrow \infty$. Now

$$\begin{aligned} ||(\Delta_q^2 - \mu I)^{-1}||_{(l_1:l_1)} &= \sup_{k\in\mathbb{N}} \sum_{i=k}^{\infty} |m_i| = \sum_{i=1}^{\infty} |m_i| \\ &\leq \frac{1}{|\sqrt{(1+q)^2 - 4(1-\mu)q}|}\left(\sum_{i=1}^{\infty} |w_1|^i + \sum_{i=1}^{\infty} |w_2|^i\right) < \infty. \end{aligned}$$

Since $|w_1| < 1$ and $|w_2| < 1$, it follows that $(\Delta_q^2 - \mu I)^{-1} \in (l_1, l_1)$. Again, since $(m_k) \in l_1$, the supremum of l_1 norms of the rows of $(\Delta_q^2 - \mu I)^{-1}$ is finite. This results in $(\Delta_q^2 - \mu I)^{-1} \in (l_\infty, l_\infty)$. Now, using Lemma 5, we get $(\Delta_q^2 - \mu I)^{-1} \in (l_1, l_1) \cap (l_\infty, l_\infty) \implies (\Delta_q^2 - \mu I)^{-1} \in (l_p, l_p)$. It proves that $\sigma(\Delta_q^2, l_p) \subseteq S$.

Now consider $\mu \in S$. If $\mu = 1$, then we get

$$(\Delta_q^2 - \mu I) = (\Delta_q^2 - I) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ -(1+q) & 0 & 0 & 0 & 0 & \cdots \\ q & -(1+q) & 0 & 0 & 0 & \cdots \\ 0 & q & -(1+q) & 0 & 0 & \cdots \\ 0 & 0 & q & -(1+q) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now $(\Delta_q^2 - I)u = \theta \implies u = \theta$. So, we get $(\Delta_q^2 - I) : l_p \longrightarrow l_p$ is one-one but not onto. This implies $(\Delta_q^2 - I)$ is not invertible.

If we take μ from S other than 1, then it is obtained from *Case 1* that $\left|\dfrac{(1+q)}{2(1-\mu)}\right| \geq 1$. It implies $(\Delta_q^2 - \mu I)^{-1} \notin B(l_p)$. Again, from *Case 2*, it is obtained that $|w_1| \geq 1$ and the inequality $|w_1| > |w_2|$, which directly implies $(w_k) \nrightarrow 0$, and so $\sum_{i=1}^{\infty} |w_i|^p$ diverges. Consider $v = (1, 0, 0, \ldots) \in l_p$. Then $(\Delta_q^2 - \mu I)^{-1}v = (m_1, m_2, m_3, \ldots)$, which doesn't belong to l_p. Therefore, $(\Delta_q^2 - \mu I)^{-1} \notin B(l_p)$ and this proves that $S \subseteq \sigma(\Delta_q^2, l_p)$. Finally, it is obtained $\sigma(\Delta_q^2, l_p) = S$.

Lemma 6. ([34], p. 215) For any $A \in B(l_p)$ $(1 < p < \infty)$, the adjoint operator $A^* \in B(l_q)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and can be represented by the transpose of A matrix.

Theorem 4. $\sigma_p((\Delta_q^2)^*, l_p^* \cong l_q) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| < \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}.$

Proof. Consider the set $S_1 = \left\{\mu \in \mathbb{C} : |2(1-\mu)| < \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$, and let $\mu \in S_1$. Now, solving $(\Delta_q^2)^* u = \mu u$, for $\theta \neq u \in l_p^* \cong l_q$, i.e.,

$$\begin{bmatrix} 1 & -(1+q) & q & 0 & 0 & 0 & \dots \\ 0 & 1 & -(1+q) & q & 0 & 0 & \dots \\ 0 & 0 & 1 & -(1+q) & q & 0 & \dots \\ 0 & 0 & 0 & 1 & -(1+q) & q & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix} = \mu \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix},$$

we get the following linear equations

$$\begin{aligned} & u_0 - (1+q)u_1 + qu_2 = \mu u_0, \\ & u_1 - (1+q)u_2 + qu_3 = \mu u_1, \\ & u_2 - (1+q)u_3 + qu_4 = \mu u_2, \\ & \vdots \\ & u_k - (1+q)u_{k+1} + qu_{k+2} = \mu u_k, \\ & \vdots \end{aligned}$$

Now, if we take $\mu = 1$, then we get an eigenvector $(1, 0, 0, \dots)$ corresponding to $\mu = 1$. We consider $\mu \in S_1$ other than 1. The above linear equations can also be expressed in terms of u_1 and u_0 as

$$\begin{aligned} u_2 &= \frac{(1+q)}{q} u_1 - \frac{1-\alpha}{q} u_0, \\ u_3 &= \frac{(1+q)^2 - q(1-\mu)}{q^2} u_1 - \frac{(1-\mu)(1+q)}{q^2}, \\ &\vdots \\ u_k &= \frac{m_k (1-\mu)^k}{q^{k-1}} u_1 - \frac{m_{k-1}(1-\mu)^k}{q^{k-1}} u_0, \quad k \geq 2, \end{aligned} \tag{2}$$

in which (m_k) comes from equation (1). Now, we can find the eigenvector (u_k), for $\mu \neq 1$. Here we can make a choice for u_0 and u_1. Let $u_0 = 1$ and $u_1 = \dfrac{2(1-\mu)}{(1+q) + \sqrt{(1+q)^2 - 4q(1-\mu)}}$.

Already, we have obtained that w_1 and w_2 are roots of the characteristic equation $(1-\mu)w^2 - (1+q)w + q = 0$. So, we get $w_1 \cdot w_2 = \dfrac{q}{1-\mu}$, and $w_1 - w_2 = \dfrac{\sqrt{(1+q)^2 - 4q(1-\mu)}}{1-\mu}$.

It can also be seen that $u_1 = \frac{1}{w_1}$. Using these facts in the relation of the sequence (u_k), we get

$$\begin{aligned}
u_k &= \frac{m_k(1-\mu)^k}{q^{k-1}}u_1 - \frac{m_{k-1}(1-\mu)^k}{q^{k-1}}u_0 \\
&= \left(\frac{1-\mu}{q}\right)^{k-1}(1-\mu)(-m_{k-1}u_0 + m_k u_1) \\
&= \frac{1}{(w_1w_2)^{k-1}}(1-\mu)(-m_{k-1} + m_k\frac{1}{w_1}) \\
&= \frac{1}{(w_1w_2)^{k-1}}(1-\mu)\left(\frac{-w_1^{k-1}+w_2^{k-1}}{\sqrt{(1+q)^2-4q(1-\mu)}} + \frac{w_1^k - w_2^k}{\sqrt{(1+q)^2-4q(1-\mu)}}w_1^{-1}\right) \\
&= \frac{1}{w_1^{k-1}w_2^{k-1}}\frac{1-\mu}{\sqrt{(1+q)^2-4q(1-\mu)}}(-w_1^{k-1}+w_2^{k-1}+w_1^{k-1}-w_2^kw_1^{-1}) \\
&= \frac{1}{w_1^{k-1}w_2^{k-1}}\left(\frac{1}{w_1-w_2}\right)w_2^{k-1}\left(1-\frac{w_2}{w_1}\right) \\
&= \frac{1}{w_1^{k-1}}\frac{1}{w_1} \\
&= \frac{1}{w_1^k} \\
&= u_1^k.
\end{aligned}$$

Finally, we obtained that the sequence $(u_k) = u_1^k$, for all $k \geq 2$. If we consider $w_1 = w_2$, i.e., for the case $(1+q)^2 = 4q(1-\mu)$, we get the same result. Thus, it is clearly seen that the sequence $u = (u_k) \in l_p^*$, since $|u_1| < 1$. As a result, $S_1 \subseteq \sigma_p((\Delta_q^2)^*, l_q)$.

Now, we consider $\mu \notin S_1$, i.e.,

$$\begin{aligned}
|2(1-\mu)| &\geq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right| \\
\implies &\left|\frac{1}{w_1}\right| \geq 1 \\
\implies &|w_1| \leq 1.
\end{aligned}$$

Here, we have to show that $\mu \notin \sigma_p((\Delta_q^2)^*, l_q)$. From equation (2), it is obtained that

$$\frac{u_{k+1}}{u_k} = \left(\frac{1-\mu}{q}\right)\left(\frac{m_k}{m_{k-1}}\right)\left(\frac{-u_0 + \dfrac{m_{k+1}}{m_k}u_1}{-u_0 + \dfrac{m_k}{m_{k-1}}u_1}\right).$$

Based on the roots w_1 and w_2, we will consider three cases.

Case 1: $|w_2| < |w_1| \leq 1$.

For this case, we get $(1+q)^2 \neq 4q(1-\mu)$ and

$$\begin{aligned}\lim_{k\to\infty}\frac{m_k}{m_{k-1}} = \lim_{k\to\infty}\frac{m_{k+1}}{m_k} &= \lim_{k\to\infty}\frac{w_1^{k+1}-w_2^{k+1}}{w_1^k-w_2^k}\\ &= \lim_{k\to\infty}\frac{w_1^{k+1}\left[1-\left(\frac{w_2}{w_1}\right)^{k+1}\right]}{w_1^k\left[1-\left(\frac{w_2}{w_1}\right)^{k}\right]}\\ &= w_1.\end{aligned}$$

Again, from equation (2), we get

$$u_k = \left(\frac{1-\mu}{q}\right)^{k-1}(1-\mu)(-m_{k-1}u_0+m_ku_1). \tag{3}$$

If $-u_0+w_1u_1=0$, then from equation (3), we find $(u_k)=\left(\frac{u_0}{w_1^k}\right)$, which doesn't belong to l_q as $|w_1|\leq 1$. Otherwise,

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|^q = \frac{1}{|w_1|^q|w_2|^q}|w_1|^q = \frac{1}{|w_2|^q} > 1.$$

Case 2: $|w_2|=|w_1|<1$.

For this case, we get $(1+q)^2 = 4q(1-\mu)$ and, using the formula

$$m_k = \left(\frac{2k}{1+q}\right)\left[\frac{1+q}{2(1-\mu)}\right]^k, \text{ for all } k\geq 1,$$

we get that

$$\lim_{k\to\infty}\left|\frac{m_k}{m_{k-1}}\right|^q = \left|\frac{1+q}{2(1-\mu)}\right|^q = |w_1|^q$$

which leads to

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right|^q = \frac{1}{|w_1|^q|w_2|^q}|w_1|^q = \frac{1}{|w_2|^q} > 1.$$

This implies that (u_k) doesn't belong to l_q.

Case 3: $|w_1|=|w_2|=1$.

For this case, we get $(1+q)^2 = 4q(1-\mu)$ and $\left|\dfrac{1+q}{2q}\right| = 1$. Assume that $\mu\in\sigma_p((\Delta_q^2)^*, l_q)$, then there exists $\theta \neq u \in l_q$. Now, rewriting equation (2), we get

$$\begin{aligned}
u_k &= \frac{m_k(1-\mu)^k}{q^{k-1}}u_1 - \frac{m_{k-1}(1-\mu)^k}{q^{k-1}}u_0 \\
&= \frac{\left(\frac{2k}{1+q}\right)\left(\frac{1+q}{2(1-\mu)}\right)^k(1-\mu)^k}{q^{k-1}}u_1 - \frac{\left(\frac{2(k-1)}{1+q}\right)\left(\frac{1+q}{2(1-\mu)}\right)^{k-1}(1-\mu)^k}{q^{k-1}}u_0 \\
&= \frac{k(1+q)^{k-1}}{(2q)^{k-1}}u_1 - \frac{(k-1)\left(\frac{1+q}{2}\right)^{k-2}(1-\mu)}{q^{k-1}}u_0 \\
&= \frac{k(1+q)^{k-1}}{(2q)^{k-1}}u_1 - \frac{(k-1)\left(\frac{1+q}{2}\right)^{k-2}\frac{(1+q)^2}{4q}}{q^{k-1}}u_0 \\
&= \frac{k(1+q)^{k-1}}{(2q)^{k-1}}u_1 - \frac{(k-1)(1+q)^k}{(2q)^k}u_0 \\
&= \left(\frac{1+q}{2q}\right)^{k-1}\left[ku_1 - (k-1)\frac{1+q}{2q}u_0\right].
\end{aligned}$$

Since $\lim_{k\to\infty} u_k = 0 \implies \lim_{k\to\infty}\left[ku_1 - (k-1)\frac{1+q}{2q}u_0\right] = 0$, and we must have $u_0 = u_1 = 0$. Consequently, it implies $u = \theta$, a contradiction. So, we get $\mu \notin \sigma_p((\Delta_q^2)^*, l_q)$. Thus, $\sigma_p((\Delta_q^2)^*, l_q) \subseteq S_1$, and hence $\sigma_p((\Delta_q^2)^*, l_q) = S_1$.

Theorem 5. The residual spectrum:

$$\sigma_r(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| < \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}.$$

Proof. From Lemma 2, we get $\sigma_r(\Delta_q^2, l_p) = \sigma_p((\Delta_q^2)^*, l_q) \backslash \sigma_p(\Delta_q^2, l_p)$. Now, applying the Theorems 2 and 4, we get the required result.

Theorem 6. The continuous spectrum:

$$\sigma_c(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| = \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}.$$

Proof. We have $\sigma(\Delta_q^2, l_p) = \sigma_p(\Delta_q^2, l_p) \cup \sigma_r(\Delta_q^2, l_p) \cup \sigma_c(\Delta_q^2, l_p)$ and the corresponding sets are pairwise disjoint. Now, applying Theorems 3, 4 and 5, we get the required result.

Theorem 7. $P_3\sigma(\Delta_q^2, l_p) = Q_3\sigma(\Delta_q^2, l_p) = R_3\sigma(\Delta_q^2, l_p) = \phi$.

Proof. From Table 1, we get $\sigma_p(\Delta_q^2, l_p) = P_3\sigma(\Delta_q^2, l_p) \cup Q_3\sigma(\Delta_q^2, l_p) \cup R_3\sigma(\Delta_q^2, l_p)$. Again, from Theorem 2, we get $\sigma_p(\Delta_q^2, l_p) = \phi$, and consequently we get the required result.

Theorem 8. The operator Δ_q^2 satisfies the following relations:

(a) $Q_2\sigma(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| = \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$,

(b) $R_2\sigma(\Delta_q^2, l_p) \supseteq \left\{\mu \in \mathbb{C} : |2(1-\mu)| < \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\} \backslash \{1\}$,

(c) $R_1\sigma(\Delta_q^2, l_p) \subseteq \{1\}$.

Proof. We have from Table 1 that $\sigma_c(\Delta_q^2, l_p) = Q_2\sigma(\Delta_q^2, l_p)$. Now, using Theorem 6, we get the result of (a).

Again, from Theorem 3, if for any $\mu \in \sigma_r(\Delta_q^2, l_p) \backslash \{1\}$ then the operator $(\Delta_q^2 - \mu I)^{-1} \notin \Delta_q^2(l_p)$. From this, we get $\sigma_r(\Delta_q^2, l_p) \backslash \{1\} \subseteq R_2\sigma(\Delta_q^2, c_o)$ and $R_1\sigma(\Delta_q^2, l_p) \subseteq \{1\}$.

Theorem 9. For the operator Δ_q^2 the following results hold.

(a) $\sigma_{ap}(\Delta_q^2, l_p) \supseteq \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\} \backslash \{1\}$,

(b) $\sigma_{ap}((\Delta_q^2)^*, l_p^*) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$,

(c) $\sigma_\delta(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$,

(d) $\delta_{co}(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| < \left|(1+q) + \sqrt{(1+q)^2 - 4(1-\mu)q}\right|\right\}$.

Proof. (a) From Table 1, we get

$$\sigma_{ap}(\Delta_q^2, l_p) = \sigma(\Delta_q^2, l_p) \backslash R_1\sigma(\Delta_q^2, l_p).$$

Now, applying the Theorems 3 and 8, we get the required result.

(b) The result in (b) is obtained from the relation (e) in Proposition 1.

(c) The result in (c) is obtained from the relation (b) in Proposition 1.

(d) The result in (d) is obtained from the relation (d) in Proposition 1.

4 *Example*

Taking particular values for $q \in (0,1)$, we construct some examples of spectrum of Δ_q^2 that are given below.

(i) If $q = \frac{1}{2}$, then the spectrum of $\Delta_{1/2}^2$ is given by

$$\sigma(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|\frac{3}{2} + \sqrt{\frac{9}{4} - 2(1-\mu)}\right|\right\}.$$

(ii) If $q = \frac{1}{4}$, then the spectrum of $\Delta_{1/4}^2$ is given by

$$\sigma(\Delta_q^2, l_p) = \left\{\mu \in \mathbb{C} : |2(1-\mu)| \leq \left|\frac{5}{4} + \sqrt{\frac{25}{16} - (1-\mu)}\right|\right\}.$$

The graphical representation of the spectra for these examples is presented in Figures 1 and 2.

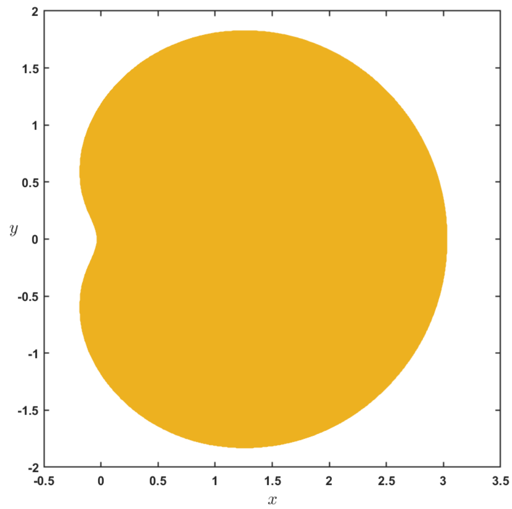

Figure 1. Spectrum of $\Delta_{1/2}^2$

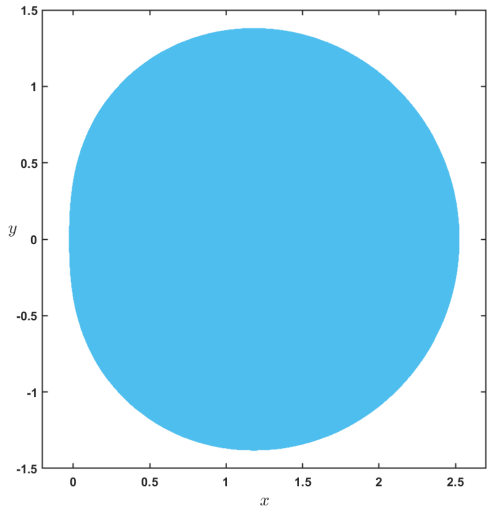

Figure 2. Spectrum of $\Delta^2_{1/4}$

Conclusion

In our study, we have determined the spectrum and fine spectrum of $q-$analog of second order difference operator. This operator reduces to second order difference operator when $q \to 1$. Like a generalized difference operator, we have a generalized quantum difference operator. Spectral analysis of generalized quantum difference operator can also be done in different sequence spaces.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Bustoz, J., & Gordillo, L.F. (2005). q-Hausdorff summability. *Journal of Computational Analysis and Applications, 7*(1), 35–48.

2 Altay, B., & Basar, F. (2004). On the fine spectrum of the difference operator Δ on c_0 and c. *Information Sciences, 168*(1–4), 217–224. https://doi.org/10.1016/j.ins.2004.02.007

3 Kayaduman, K., & Furkan, H. (2006). The fine spectra of the difference operator Δ over the sequence space ℓ_1 and bv. *International Mathematical Forum, 1*(24), 1153–1160. http://dx.doi.org/10.12988/imf.2006.06093

4 Akhmedov, A.M., & Basar, F. (2006). On the fine spectra of the difference operator Δ over the sequence space ℓ_p $(1 \leq p < \infty)$. *Demonstratio Mathematica, 39*(3), 585–596. https://doi.org/10.1515/dema-2006-0313

5 Akhmedov, A.M., & Basar, F. (2007). The fine spectra of the difference operator Δ over the sequence space bvp $(1 \leq p < \infty)$. *Acta Mathematica Sinica, English Series, 23*(10), 1757–1768. https://doi.org/10.1007/s10114-005-0777-0

6 Dutta, S., & Baliarsingh, P. (2013). On the spectrum of 2nd order generalized difference operator Δ^2 over the sequence space c_0. *Boletim da Sociedade Paranaense de Matematica, 31*(2), 235–244. http://dx.doi.org/10.5269/bspm.v31i2.17928

7 Altay, B., & Basar, F. (2005). On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c. *International Journal of Mathematics and Mathematical Sciences, 2005*(18), 3005–3013. https://doi.org/10.1155/IJMMS.2005.3005

8 Furkan, H., Bilgic, H., & Kayaduman, K. (2006). On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence space ℓ_1 and bv. *Hokkaido Mathematical Journal, 35*(4), 893–904.

9 Bilgic, H., & Furkan, H. (2008). On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bvp $(1 < p < \infty)$. *Nonlinear Analysis: Theory, Methods and Applications, 68*(3), 499–506. https://doi.org/10.1016/j.na.2006.11.015

10 Dutta, A.J., & Tripathy, B.C. (2013). Fine spectrum of the generalized difference operator $B(r,s)$ over the class of convergent series. *International Journal of Analysis, 2013*, 630436. http://dx.doi.org/10.1155/2013/630436

11 Bilgic, H., & Furkan, H. (2007). On the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces ℓ_1 and bv. *Mathematical and Computer Modelling, 45*(7–8), 883–891. https://doi.org/10.1016/j.mcm.2006.08.008

12 Furkan, H., Bilgic, H., & Altay, B. (2007). On the fine spectrum of the operator $B(r,s,t)$ over c_0 and c. *Computers and Mathematics with Applications, 53*(6), 989–998. https://doi.org/10.1016/j.camwa.2006.07.006

13 Furkan, H., Bilgic, H., & Basar, F. (2010). On the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces ℓ_p and bv_p, $(1 < p < \infty)$. *Computers and Mathematics with Applications, 60*(7), 2141–2152. https://doi.org/10.1016/j.camwa.2010.07.059

14 Srivastava, P.D., & Kumar, S. (2009). On the fine spectrum of the generalized difference operator Δ_v over the sequence space c_0. *Communications in Mathematical Analysis, 6*(1), 8–21.

15 Srivastava, P.D., & Kumar, S. (2012). Fine spectrum of the generalized difference operator Δ_v on the sequence space ℓ_1. *Thai Journal of Mathematics, 8*(2), 221–223.

16 Akhmedov, A.M., & El-Shabrawy, S.R. (2011). On the fine spectrum of the operator Δ_v over the sequence spaces c and ℓ_p $(1 < p < \infty)$. *Applied Mathematics and Information Sciences, 5*(3), 635–654.

17 El-Shabrawy, S.R. (2014). Spectra and fine spectra of certain lower triangular double-band matrices as operators on c_0. *Journal of Inequalities and Applications, 2014*(1), 241. https://doi.org/10.1186/1029-242X-2014-241

18 Akhmedov, A.M., & El-Shabrawy, S.R. (2011). On the fine spectrum of the operator Δ_{ab} over the sequence space c. *Computers and Mathematics with Applications, 61*(10), 2994–3002. https://doi.org/10.1016/j.camwa.2011.03.085

19 Dutta, S., & Baliarsingh, P. (2014). On the spectrum of difference operator Δ_{ab} over the sequence space ℓ_p and bv_p $(1 < p < \infty)$. *Mathematical Sciences Letters, 3*(2), 115–120. https://dx.doi.org/10.12785/msl/030207

20 Panigrahi, B.L., & Srivastava, P.D. (2012). Spectrum and fine spectrum of generalized second order difference operator Δ^2_{uv} on sequence space c_0. *Thai Journal of Mathematics, 9*(1), 57–74.

21 Panigrahi, B.L., & Srivastava, P.D. (2012). Spectrum and fine spectrum of generalized second order forward difference operator Δ^2_{uvw} on sequence space ℓ_1. *Demonstratio Mathematica, 45*(3), 593–609. https://doi.org/10.1515/dema-2013-0404

22 Altundag, S., & Abay, M. (2016). On the fine spectrum of generalized upper triangular triple-

band matrices $(\Delta^2_{uvw})^t$ over the sequence space ℓ_1. *Filomat, 30*(5), 1363–1373. https://doi.org/10.2298/FIL1605363A

23 Patra, A., & Srivastava, P.D. (2020). Spectrum and fine spectrum of generalized lower triangular triple band matrices over the sequence space ℓ_p. *Linear and Multilinear Algebra, 68*(9), 1797–1816. https://doi.org/10.1080/03081087.2018.1561822

24 Yildirim, M.E. (2020). The spectrum and fine spectrum of q-Cesaro matrices with $0 < q < 1$ on c_0. *Numerical Functional Analysis and Optimization, 41*(3), 361–377. https://doi.org/10.1080/01630563.2019.1633666

25 Turkay, M.E., & Durna, N. (2021). The spectrum of q-Cesaro matrices on c and its various spectral decomposition for $0 < q < 1$. *Operators and Matrices, 15*, 795–813. http://doi.org/10.7153/oam-2021-15-55

26 Yaying, T., Hazarika, B., Tripathy, B.C., & Mursaleen, M. (2022). The spectrum of second order quantum difference operator. *Symmetry, 14*(3), 557. https://doi.org/10.3390/sym14030557

27 Alotaibi, A., Yaying, T., & Mohiuddine, S.A. (2022). Sequence space and spectrum of q-difference operator of second order. *Symmetry, 14*(6), 1155. https://doi.org/10.3390/sym14061155

28 Yaying, T., Hazarika, B., Baliarsingh, P., & Mursaleen, M. (2024). Cesaro q-difference sequence spaces and spectrum of weighted q-difference operator. *Bulletin of the Iranian Mathematical Society, 50*(2), 23. https://doi.org/10.1007/s41980-024-00862-3

29 Kak, V., & Cheung, P. (2002). *Quantum Calculus.* Springer, New York.

30 Goldberg, S. (2006). *Unbounded Linear Operators — Theory and Applications.* Dover Publications, New York.

31 Appell, J., Pascale, E., & Vignoli, A. (2004). *Nonlinear Spectral Theory.* Walter de Gruyter. https://doi.org/10.1515/9783110199260

32 Wilansky, A. (1984). *Summability through Functional Analysis.* North Holland, Amsterdam.

33 Maddox, I.J. (1970). *Elements of Functional Analysis.* Cambridge University Press.

34 Taylor, R.B. (1989). *Introduction to Functional Analysis.* Wiley, New York.

*Author Information**

Nabadeep Kalita (*corresponding author*) — Research Scholar, Department of Mathematics, Gauhati University, Guwahati, 781014, Assam, India; e-mail: *nkalita@gauhati.ac.in*

Amar Jyoti Dutta — Assistant Professor, Department of Mathematics, Pragjyotish College, Guwahati, 781009, Assam, India; e-mail: *amar_iasst@yahoo.co.in*; https://orcid.org/0000-0003-2588-452X

*The author's name is presented in the order: First, Middle, and Last Names.