https://doi.org/10.31489/2025M2/60-75 *Research article*

Approximation of fixed points for enriched $B_{\gamma,\mu}$ mapping using a new iterative algorithm in CAT(0) space

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In this paper, we define enriched $B_{\gamma,\mu}$ mapping in CAT(0) space and derive some fixed point results for such mapping with the help of averaged mapping. Our results extend some existing results. A new iterative algorithm is developed in CAT(0) space to approximate the fixed point of enriched $B_{\gamma,\mu}$ mapping. Δ-convergence and strong convergence of this iterative algorithm for enriched $B_{\gamma,\mu}$ mapping is proved.

Keywords: CAT(0) space, $B_{\gamma,\mu}$ mapping, averaged mapping, fixed point.

2020 Mathematics Subject Classification: 47H09, 47H10, 47J26.

Introduction

Fixed point theory is an active and vibrant area of research, which serves as a powerful tool in various fields of mathematics and other allied areas. Many implications across a diverse range of fields such as engineering, economics, dynamical system, differential equation, integral equation etc., can be formulated as a fixed point problem. In 1922, Banach proved a fundamental theorem [1] in metric fixed point theory known as Banach contraction theorem, and over the past century, this theorem has been generalized by various prominent authors considering different aspects.

In 2008, Suzuki [2] introduced a new class of mappings on a nonempty subset of a Banach space by proposing a condition (C). Later many authors generalized this class of mappings in different ways and derived different fixed point theorems and convergence results of different iteration schemes. These generalizations often involve relaxing the assumptions or considering more general settings, leading to broader applicability and deeper theoretical understanding.

In 2018, Patir et al. [3] introduced a new class of generalized nonexpansive mappings which is wider than the class of mappings satisfying Suzuki (C) condition. They proved some fixed point results as well as some properties of this class of mappings. In 2019, Berinde [4] introduced the class of enriched nonexpansive mappings in Hilbert space and approximated the fixed point of such mappings using Krasnoselskii iteration. Using the technique of enriching a mapping, many authors generalized and introduced several new classes of mappings with different aspects.

In 2024, Dashputre et al. [5] introduced SJR-iteration to approximate fixed point of generalized α-nonexpansive mapping in CAT(0) spaces and established strong and Δ-convergence theorems for such mapping. In the same year, Kim [6] introduced the concept of sequentially admissible mapping and sequentially admissible perturbation with the construction of a new iteration process corresponding to sequentially admissible mappings. They proved convergence results for the Mann type iterative method using uniformly L-Lipschitzian, sequentially admissible perturbation of asymptotically demicontractive mappings. Moreover, convergence result concerning Ishikawa type iterative method using uniformly

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Received: 8 October 2024; *Accepted:* 18 March 2025.

L-Lipschitzian, sequentially admissible perturbation of asymptotically hemicontractive mappings to a fixed point in CAT(0) spaces was also established.

Although the fixed point theory in linear spaces (for example, Banach spaces and Hilbert spaces) have been developed extensively because of the linearity and convexity of the underlying spaces, but due to the unavailability of convex structure in metric space, it seemed impossible to extend the results of Banach space into metric space. Keeping in mind this situation, Reich et al. [7] introduced hyperbolic metric space using geodesic segment and Menger convexity [8]. This class of metric space includes all normed vector spaces, Hadamard manifolds, CAT(0) space, Hilbert balls, and the cartesian product of Hilbert balls, etc. CAT(0) space is a non-linear example of hyperbolic metric space.

Fixed point theory in CAT(0) space was first studied by Kirk [9] in the year 2003, where it was proved that every nonexpansive mapping defined on a bounded closed, convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single valued and multi-valued mappings in CAT(0) space have been rapidly developed with contributions from various prominent researchers (refer to [10–13]).

Motivated by this, in this paper, we define a new class of mappings called enriched $B_{\gamma,\mu}$ mappings in CAT(0) space. Some fixed point results are derived for such mappings. We also define a new iterative algorithm in CAT(0) space using enriched $B_{\gamma,\mu}$ mappings. The Δ-convergence and strong convergence results for this iterative algorithm are developed. This convergence is demonstrated graphically with the help of a numerical example.

1 Preliminaries

For a metric space (X,d), let $x,y \in X$ with $d(x,y)=m$. A geodesic path from x to y is a mapping $c:[0,m]\to X$ such that $c(0)=x$, $c(m)=y$, which is an isometry. A geodesic segment is the image of a geodesic path. A metric space (X,d) is termed a geodesic metric space if it satisfies the property that every pair of points in X can be connected by a geodesic segment. (X,d) is called a uniquely geodesic space if there exists exactly one geodesic segment connecting every two points.

In a geodesic metric space (X,d), a geodesic triangle $\Delta(x_1,x_2,x_3)$ is formed by three points (vertices) x_1,x_2,x_3 in X and a geodesic segment connecting each pair of these vertices. For a geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d), a comparison triangle is a triangle $\bar{\Delta}(x_1,x_2,x_3):=\Delta(\bar{x_1},\bar{x_2},\bar{x_3})$ in $\mathbb{R}^2$ that satisfies the property $d_{\mathbb{R}^2}(\bar{x_i},\bar{x_j})=d(x_i,x_j)$ for $i,j\in\{1,2,3\}$.

A geodesic space is referred to as a Cartan, Alexandrov, and Toponogov (0) space, in short, CAT(0) space with curvature bound 0, if it satisfies the CAT(0) inequality. That is, for each geodesic triangle $\Delta(x_1,x_2,x_3)$ in X and its corresponding comparison triangle $\bar{\Delta}:=\Delta(\bar{x_1},\bar{x_2},\bar{x_3})$ in $\mathbb{R}^2$, the inequality

$$d(x,y)\le d_{\mathbb{R}^2}(x',y')$$

holds for all $x,y\in\Delta$ and $x',y'\in\bar{\Delta}$.

Similarly, for any integer k, one can define CAT(k) space by comparing it with another space. CAT(0) space is uniquely geodesic space. For $k\in[0,1]$, the notation $(1-k)x\oplus ky$ denotes the unique point z on the geodesic segment from x to y with $d(z,x)=kd(x,y)$ and $d(z,y)=(1-k)d(x,y)$. Suppose (X,d) is a CAT(0) space and $x,y,z\in(X,d)$ with $k\in[0,1]$. Then

$$d((1-k)x\oplus ky,z)\le(1-k)d(x,z)+d(y,z)\quad([14]).$$

For detailed discussion on CAT(0) space, one may refer to [15, 16].

Now we recall some basic definitions and key results.

For a bounded sequence $\{x_n\}$ in a nonempty closed convex subset C of a CAT(0) space (X,d), define a functional $r(.,\{x_n\}):X\to\mathbb{R}^+$ by

$$r(x,\{x_n\})=\limsup_{n\to\infty} d(x,\{x_n\}),\ x\in X.$$

The asymptotic radius of the sequence $\{x_n\}$ with respect to C is defined by

$$r(\{x_n\}) = \inf_{x \in C} r(x, \{x_n\}).$$

The asymptotic center of the sequence $\{x_n\}$ with respect to C is defined by

$$A(\{x_n\}) = \{y \in C : r(y, \{x_n\}) = r(\{x_n\})\}.$$

Definition 1. [17] In a CAT(0) space (X, d), a sequence $\{x_n\}$ is said to be Δ-convergent to $x \in X$, if for every subsequence $\{z_n\}$ of $\{x_n\}$ x serves as the unique asymptotic center of $\{z_n\}$. It is denoted by $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and x is referred to as the Δ-limit of $\{x_n\}$.

A bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{z_n\})$ for every subsequence $\{z_n\}$ of $\{x_n\}$. In Banach space, every bounded sequence contains a regular subsequence [18].

Lemma 1. [19] Suppose $\{x_n\}$ is a sequence in a CAT(0) space (X, d) and $\{x_n\}$ is Δ-convergent to $x \in X$. Let $y \in X$ be such that $y \neq x$. Then

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

In a Banach space, the above condition is known as Opial property [20].

Definition 2. [21] For a closed convex subset C of a CAT(0) space (X, d), a bounded sequence $\{x_n\}$ in C converges weakly to $q \in C$ if and only if $\Phi(q) = \inf_{x\in C} \Phi(x)$, where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$, $x \in C$.

Note that $\{x_n\}$ converges weakly to q if and only if $A(\{x_n\}) = \{q\}$ (refer to [17]).

Nanjaras and Panyanak [14] established the following relation between Δ-convergence and weak convergence in CAT(0) space.

Lemma 2. [14] For a bounded sequence $\{x_n\}$ in a CAT(0) space (X, d), let C be a closed convex subset of X which contains $\{x_n\}$. Then

(i) $\Delta\text{-}\lim_{n\to\infty} x_n = x$ implies $\{x_n\}$ converges weakly to x.

(ii) The converse of (i) is true if $\{x_n\}$ is regular.

Lemma 3. [22] For a closed convex subset C of a CAT(0) space (X, d) and a bounded sequence $\{x_n\}$ in C, the asymptotic center of $\{x_n\}$ is in C.

Lemma 4. [14] In a CAT(0) space, every bounded sequence has a Δ-convergent subsequence.

Lemma 5. [23] For a closed convex subset C of a CAT(0) space (X, d) and a bounded sequence $\{x_n\}$ in C, the asymptotic center $A(\{x_n\})$ contains exactly one point.

Lemma 6. [24] In a complete CAT(0) space (X, d) and $x \in X$, suppose $\{t_n\}$ is a sequence in $[p, q]$ for some $p, q \in (0, 1)$ and $\{u_n\}$, $\{v_n\}$ are sequences in X satisfying

$$\limsup_{n\to\infty} d(u_n, x) \leq r,$$

$$\limsup_{n\to\infty} d(v_n, x) \leq r$$

and

$$\lim_{n\to\infty} d(t_n v_n \oplus (1 - t_n)u_n, x) = r$$

for some $r \geq 0$. Then $\lim\limits_{n\to\infty} d(u_n, v_n) = 0$.

We recall that (refer to [25]) for a nonempty subset C of a metric space (X, d), a mapping T on C is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in C.$$

T is a quasi-nonexpansive mapping if

$$d(Tx, y) \leq d(x, y) \text{ for all } x \in C \text{ and for } z \in F(T) \neq \emptyset,$$

where $F(T)$ denotes the set of all fixed points of T.

T is a Suzuki nonexpansive mapping if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in C.$$

In 2020, Berinde et al. [26] defined enriched nonexpansive mapping in Banach space as follows:

Let X be a Banach space. A mapping $T : X \to X$ is said to be an enriched nonexpansive mapping if there exists $b \in [0, \infty)$ such that

$$||b(x - y) + Tx - Ty|| \leq (b + 1)||x - y||$$

for all $x, y \in X$.

Generalizing Suzuki nonexpansive mappings, Patir et al. [3] defined the following class of $B_{\gamma,\mu}$ mappings.

Definition 3. [3] For a nonempty subset C of a Banach space X, let $\gamma \in [0, 1]$ and $\mu \in [0, \frac{1}{2}]$ satisfying $2\mu \leq \gamma$. A mapping $T : C \to C$ is a $B_{\gamma,\mu}$ mapping if

$$\gamma||x - Tx|| \leq ||x - y|| + \mu||y - Ty||$$

implies $||Tx - Ty|| \leq (1 - \gamma)||x - y|| + \mu(||x - Ty|| + ||y - Tx||)$ for all $x, y \in C$.

In metric space setting the above definition will reduce to the following.

Definition 4. Let C be a nonempty subset of a metric space (X, d) and $\gamma \in$ [0,1], $\mu \in [0, \frac{1}{2}]$ so that $2\mu \leq \gamma$. A self-mapping $T : C \to C$ is a $B_{\gamma,\mu}$ mapping if

$$\gamma d(x, Tx) \leq d(x, y) + \mu d(y, Ty)$$

implies

$$d(Tx, Ty) \leq (1 - \gamma)d(x, y) + \mu(d(x, Ty) + d(y, Tx)) \text{ for all } x, y \in C.$$

Lemma 7. [3] Let C be a nonempty subset of a Banach space X and T be a $B_{\gamma,\mu}$ mapping on C. Then T is quasi-nonexpansive.

The concept of an averaged mapping appeared in the work of Krasnoselskii [27] in the context of Hilbert space, and the term averaged was given in [28].

Definition 5. [28] Given a mapping $T : X \to X$, where X is a Banach space, the averaged mapping $T_k : X \to X$ for $k \in (0, 1]$ is defined by

$$T_k(x) = (1 - k)x + kTx \text{ for all } x \in X.$$

Lemma 8. [29] For a self-mapping T on a convex subset C of a Banach space X and for any $k \in (0, 1]$, $F(T_k) = F(T)$.

2 Main results

In this section, we define an enriched class of mappings in CAT(0) space that generalizes the class of $B_{\gamma,\mu}$ mappings. We discuss some fixed point properties of this class. Next, we introduce an iterative algorithm in CAT(0) space involving such mappings with convergence properties.

Definition 6. Let (X, d) be a CAT(0) space and C be a nonempty subset of X. Let $\gamma \in [0,1]$, $\mu \in [0, \frac{1}{2}]$ be such that $2\mu \leq \gamma$. A mapping $T : C \to C$ is said to be an enriched $B_{\gamma,\mu}$ mapping if there exists $b \in [0, \infty)$ such that for $k = \frac{1}{b+1}$,

$$\gamma d(x, (1-k)x \oplus kTx) \leq d(x, y) + \mu d(y, (1-k)y \oplus kTy)$$

implies

$$\begin{aligned} d((1-k)x \oplus kTx, (1-k)y \oplus kTy) \leq (1-\gamma)d(x, y) + \mu(d(x, (1-k)y \oplus kTy) \\ + d(y, (1-k)x \oplus kTx)) \quad \text{for all } x, y \in C. \end{aligned}$$

It can be seen that every $B_{\gamma,\mu}$ mapping is an enriched $B_{\gamma,\mu}$ mapping with $b = 0$.

For $b = 0$, $\gamma = \mu = 0$, an enriched $B_{\gamma,\mu}$ mapping reduces to nonexpansive mapping. Again for $b = 0$, $\gamma = 1/2$, $\mu = 0$, it reduces to Suzuki nonexpansive mapping.

Example 1. Consider the CAT(0) space $(\mathbb{R}, d)$ with $d(x, y) = |x-y|$ for all $x, y \in \mathbb{R}$. Then $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 1 - 2x$ for $x \in \mathbb{R}$ is an enriched $B_{\gamma,\mu}$ mapping with $b = 2, \gamma = \frac{2}{3}, \mu = 0$. But it is not a $B_{\gamma,\mu}$ mapping.

Example 2. For the CAT(0) space $(\mathbb{R}^2, d)$ with

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \text{ for } x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$$

and $C = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, define $T : C \to C$ such that

$$T(x_1, x_2) = (1 - x_1, 1 - x_2) \text{ for all } (x_1, x_2) \in C.$$

Then T is an enriched $B_{\gamma,\mu}$ mapping with $b = 1$, $\gamma = \frac{1}{2}$, $\mu = 0$.

Lemma 9. Let T be a $B_{\gamma,\mu}$ mapping on a CAT(0) space (X, d) with $F(T) \neq \emptyset$. Then T is quasi-nonexpansive.

Proof. Let $z \in F(T)$. Then

$$\gamma d(z, Tz) = 0 \leq d(z, x).$$

Again, for $x(\neq z) \in X$, by $B_{\gamma,\mu}$ condition, we have

$$\begin{aligned} d(Tx, Tz) &\leq (1-\gamma)d(x, z) + \mu(d(x, Tz) + d(z, Tx)), \\ d(Tx, z) &\leq (1-\gamma)d(x, z) + \mu(d(x, z) + d(z, Tx)), \\ d(Tx, z) &\leq \frac{(1-\gamma+\mu)}{1-\mu} d(x, z) \leq d(x, z). \end{aligned}$$

So, T is quasi-nonexpansive.

Lemma 10. If T is an enriched $B_{\gamma,\mu}$ mapping on a CAT(0) space, then for $k = \frac{1}{b+1}$, the averaged mapping T_k is a $B_{\gamma,\mu}$ mapping.

Lemma 11. For a self-mapping T on a convex subset C of a CAT(0) space (X, d) and for any $k \in (0, 1]$, $F(T_k) = F(T)$, where T_k is the averaged mapping of T.

Proof. Clearly, if $x \in F(T)$, then $x \in F(T_k)$. So, $F(T) \subseteq F(T_k)$.
Let $x \in F(T_k)$. Then $T_k x = x$. Now,

$$\begin{aligned}
&d((1-k)x \oplus kTx, x) = kd(x, Tx),\\
&d(x, x) = kd(x, Tx),\\
&d(x, Tx) = 0,\\
&x \in F(T).
\end{aligned}$$

Lemma 12. For a nonempty subset C of a CAT(0) space (X, d), let T be an enriched $B_{\gamma,\mu}$ mapping on C. If $F(T) \neq \emptyset$, then $F(T)$ is closed.

Proof. Let $p \in \overline{F(T)}$ (the closure of $F(T)$). Then there exists a sequence $\{x_n\} \subseteq F(T)$ such that $x_n \to p$. Since T is an enriched $B_{\gamma,\mu}$ mapping, so, for $k = \frac{1}{b+1}$, using Lemma 10 and Lemma 9, T_k is quasi-nonexpansive.

Now,

$$0 = \lim_{n\to\infty} d(x_n, p) \geq \lim_{n\to\infty} d(x_n, T_k p) = d(T_k p, p).$$

So, $T_k p = p$. Therefore by Lemma 11, $p \in F(T)$. Hence $F(T)$ is closed.

Lemma 13. For a nonempty subset C of a CAT(0) space (X, d), let T be an enriched $B_{\gamma,\mu}$ mapping on C with the averaged mapping T_k, for $k = \frac{1}{b+1}$. Then for all $x, y \in C$, $c \in [0, 1]$

(i) $d(T_k x, T_k^2 x) \leq d(x, T_k x)$;

(ii) at least one of the following ((a) and (b)) holds:

(a) $\frac{c}{2} d(x, T_k x) \leq d(x, y)$;

(b) $\frac{c}{2} d(T_k x, T_k^2 x) \leq d(T_k x, y)$.

The condition (a) implies $d(T_k x, T_k y) \leq (1 - \frac{c}{2}) d(x, y) + \mu(d(x, T_k y) + d(y, T_k x))$, the condition (b) implies $d(T_k^2 x, T_k y) \leq (1 - \frac{c}{2}) d(T_k x, y) + \mu(d(T_k x, T_k y) + d(y, T_k^2 x))$;

(iii) $d(x, T_k y) \leq (3 - \frac{c}{2}) d(x, T_k x) + (1 - \frac{c}{2}) d(x, y) + \mu(2d(x, T_k x) + d(x, T_k y) + d(y, T_k x) + 2d(T_k x, T_k^2 x))$.

Proof. (i) For all $x \in C$

$$\gamma d(x, T_k x) \leq d(x, T_k x) + \mu d(T_k x, T_k^2 x).$$

So, by Lemma 10,

$$\begin{aligned}
d(T_k x, T_k^2 x) &\leq (1-\gamma) d(x, T_k x) + \mu d(x, T_k^2 x)\\
&\leq (1-\gamma) d(x, T_k x) + \mu d(x, T_k^2 x) + \mu d(T_k x, T_k^2 x),
\end{aligned}$$

that is,

$$d(T_k x, T_k^2 x) \leq \frac{1-\gamma+\mu}{1-\mu} d(x, T_k x) \leq d(x, T_k x).$$

(ii) If possible, let $\frac{c}{2} d(x, T_k x) > d(x, y)$ and

$$\frac{c}{2} d(T_k x, T_k^2 x) > d(T_k x, y) \quad \text{for some} \quad x, y \in C.$$

Now,

$$\begin{aligned} d(x,T_kx) \leq & d(x,y)+d(y,T_kx) \\ < & \frac{c}{2}d(x,T_kx)+\frac{c}{2}d(T_kx,T_k^2x) \\ \leq & \frac{c}{2}d(x,T_kx)+\frac{c}{2}d(x,T_kx) \text{ (using (i))} \\ \leq & d(x,T_kx), \end{aligned}$$

that is, $d(x,T_kx) < d(x,T_kx)$, which is impossible. So, at least one of (a) and (b) holds.

If (a) holds, then, $\frac{c}{2}d(x,T_kx) \leq d(x,y)$. So,

$$\frac{c}{2}d(x,T_kx) \leq d(x,y)+\mu d(y,T_ky).$$

Therefore,

$$d(T_kx,T_ky) \leq \left(1-\frac{c}{2}\right)+\mu(d(x,T_ky)+d(y,T_kx)).$$

If (b) holds, then, $\frac{c}{2}d(T_kx,T_k^2x) \leq d(T_kx,y)$.

That is,

$$\frac{c}{2}d(T_k^2x,T_kx) \leq d(T_kx,y).$$

So,

$$d(T_k^2x,T_ky) \leq \left(1-\frac{c}{2}\right)d(T_kx,y)+\mu(d(T_kx,T_ky)+d(y,T_k^2x)).$$

(iii) We assume that (a) holds. Then

$$d(T_kx,T_ky) \leq \left(1-\frac{c}{2}\right)d(x,y)+\mu(d(x,T_ky)+d(y,T_kx)).$$

Now,

$$\begin{aligned} d(x,T_ky) & \leq d(x,T_kx)+d(T_kx,T_ky) \\ & \leq d(x,T_kx)+\left(1-\frac{c}{2}\right)d(x,y)+\mu(d(x,T_ky)+d(y,T_kx)) \\ & \leq \left(3-\frac{c}{2}\right)d(x,T_kx)+\left(1-\frac{c}{2}\right)d(x,y)+\mu(d(x,T_ky)+d(y,T_kx) \\ & +2d(x,T_k^2x)), \text{ since } c\in[0,1] \\ & \leq \left(3-\frac{c}{2}\right)d(x,T_kx)+\left(1-\frac{c}{2}\right)d(x,y)+\mu(d(x,T_ky)+d(y,T_kx) \\ & +2d(x,T_kx)+2d(T_kx,T_k^2x)). \end{aligned}$$

Now, suppose (b) holds. Then

$$d(T_k^2x,T_ky) \leq \left(1-\frac{c}{2}\right)d(T_kx,y)+\mu(d(T_kx,T_ky)+d(y,T_k^2x)).$$

By triangle inequality,

$$\begin{aligned}
d(x,T_ky) &\le d(x,T_kx)+d(T_k^2x,T_kx)+d(T_k^2x,T_ky)\\
&\le d(x,T_kx)+d(x,T_kx)+\left(1-\frac{c}{2}\right)d(T_kx,y)+\mu(d(T_kx,T_ky)+d(y,T_k^2x))\\
&\le 2d(x,T_kx)+\left(1-\frac{c}{2}\right)d(T_kx,x)+\left(1-\frac{c}{2}\right)d(x,y)+\mu(d(T_kx,T_ky)+d(y,T_k^2x))\\
&\le \left(3-\frac{c}{2}\right)d(x,T_kx)+\left(1-\frac{c}{2}\right)d(x,y)+\mu(d(x,T_k^2x)+d(T_kx,T_ky)+d(y,T_k^2x))\\
&\le \left(3-\frac{c}{2}\right)d(x,T_kx)+\left(1-\frac{c}{2}\right)d(x,y)+\mu(d(x,T_kx)+d(T_kx,T_k^2x)+d(T_kx,x)\\
&+d(x,T_ky)+d(y,T_kx)+d(T_kx,T_k^2x))\\
&= \left(3-\frac{c}{2}\right)d(x,T_kx)+\left(1-\frac{c}{2}\right)d(x,y)+\mu(2d(x,T_kx)+d(x,T_ky)\\
&+d(y,T_kx)+2d(T_kx,T_k^2x)).
\end{aligned}$$

Next, we derive the following fixed point result using Δ-convergence.

Theorem 1. For a nonempty subset C of a CAT(0) space (X,d), let T be an enriched $B_{\gamma,\mu}$ mapping on C with the averaged mapping T_k, for $k=\frac{1}{b+1}$. Suppose $\{x_n\}$ is a sequence in C such that
(i) $\{x_n\}$ is Δ-convergent to z,
(ii) $\lim_{n\to\infty} d(T_kx_n,x_n)=0$, and
(iii) $d(z,T_kx_n)\le d(z,x_n)$.
Then $z\in F(T)$.

Proof. Since T is enriched $B_{\gamma,\mu}$ mapping, so, by Lemma 13 (iii), for $c\in[0,1]$,

$$\begin{aligned}
d(x_n,T_kz) &\le \left(3-\frac{c}{2}\right)d(x_n,T_kx_n)+\left(1-\frac{c}{2}\right)d(x_n,z)+\mu(2d(x_n,T_kx_n)+d(x_n,T_kz)\\
&+d(z,T_kx_n)+2d(T_kx_n,T_k^2x_n)).
\end{aligned}$$

Using condition (iii) and Lemma 13(i),

$$\begin{aligned}
d(x_n,T_kz) &\le \left(3-\frac{c}{2}\right)d(x_n,T_kx_n)+\left(1-\frac{c}{2}\right)d(x_n,z)+\mu(2d(x_n,T_kx_n)+d(x_n,T_kz)\\
&+d(z,x_n)+2d(x_n,T_kx_n)).
\end{aligned}\tag{1}$$

Taking $n\to\infty$ on both sides of (1) and using conditions (ii) and (iii), we get

$$(1-\mu)\lim_{n\to\infty} d(x_n,T_kz)\le\left(1-\frac{c}{2}+\mu\right)\lim_{n\to\infty} d(x_n,z),$$

that is,

$$\lim_{n\to\infty} d(x_n,T_kz)\le \lim_{n\to\infty} d(x_n,z),$$

$$\limsup_{n\to\infty} d(x_n,T_kz)\le \limsup_{n\to\infty} d(x_n,z).$$

Since $\{x_n\}$ is Δ-convergent to z, so, if $T_kz\ne z$, then by Lemma 1,

$$\limsup_{n\to\infty} d(x_n,z)<\limsup_{n\to\infty} d(x_n,T_kz)\le\limsup_{n\to\infty} d(x_n,z),\quad \text{which is a contradiction.}$$

Hence $T_kz=z$.

Since $k\in(0,1]$, by Lemma 11, $z\in F(T)$.

Theorem 2. Let (X, d) be a CAT(0) space and C be a nonempty closed convex and bounded subset of X. Let T be an enriched $B_{\gamma,\mu}$ mapping on C with the averaged mapping T_k for $k = \frac{1}{b+1}$. Suppose $\{x_n\}$ is a bounded sequence in C that satisfies

(i) $\lim_{n\to\infty} d(x_n, T_k x_n) = 0$ and

(ii) $\lim_{n\to\infty} d(T_k x_n, T_k^2 x_n) = 0$.

Then $F(T) \neq \emptyset$.

Proof. Since $A(\{x_n\})$ contains exactly one point, let $z \in A(\{x_n\})$. Then by Lemma 3, $z \in C$.

By Lemma 13 (iii), for each $n \in \mathbb{N} \cup \{0\}$ and $c \in [0, 1]$, we have

$$d(x_n, T_k z) \leq \left(3 - \frac{c}{2}\right) d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) d(x_n, z) + \mu(2d(x_n, T_k x_n) + d(x_n, T_k z) + d(z, T_k x_n) + 2d(T_k x_n, T_k^2 x_n)),$$

that is,

$$d(x_n, T_k z) \leq \left(3 - \frac{c}{2}\right) d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) d(x_n, z) + \mu(2d(x_n, T_k x_n) + d(x_n, T_k z)+$$

$$d(z, x_n) + d(x_n T_k x_n) + 2d(T_k x_n, T_k^2 x_n)).$$

So,

$$(1 - \mu) \limsup_{n\to\infty} d(x_n, T_k z) \leq \left(1 - \frac{c}{2} + \mu\right) \limsup_{n\to\infty} d(x_n, z).$$

Therefore,

$$r(T_k z, \{x_n\}) \leq r(z, \{x_n\}).$$

Hence

$$T_k z \in A(\{x_n\}).$$

By uniqueness of asymptotic centers in CAT(0) space, we have $T_k z = z$. So, by Lemma 11, z is a fixed point of T.

3 *A new iterative algorithm in CAT(0) space*

In this section, we develop the following iterative scheme for approximating fixed points of enriched $B_{\gamma,\mu}$ mapping in CAT(0) space.

Let C be a nonempty subset of a CAT(0) space and $x_0 \in C$. Let T be an enriched $B_{\gamma,\mu}$ mapping on C with the averaged mapping T_k for $k = \frac{1}{b+1}$.

For $n \in \mathbb{N} \cup \{0\}$, we define

$$\begin{aligned} x_{n+1} &= (1 - \beta_n) y_n \oplus \beta_n T_k y_n, \\ y_n &= T_k z_n, \\ z_n &= (1 - \alpha_n) x_n \oplus \alpha_n T_k x_n, \end{aligned} \tag{2}$$

where $\alpha_n, \beta_n \in [0, 1]$.

Lemma 14. Let T be an enriched $B_{\gamma,\mu}$ mapping on a nonempty closed and convex subset C of a CAT(0) space (X, d). For $x_0 \in C$, let $\{x_n\}$ be a sequence defined by (2). If $F(T) \neq \emptyset$, then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.

Proof. For $p \in F(T)$ and $k = \frac{1}{b+1}$,

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\beta_n)y_n \oplus \beta_n T_k y_n, p) \\ &\leq (1-\beta_n)d(y_n, p) + \beta_n d(T_k y_n, p). \end{aligned}$$

Since T_k is quasi-nonexpansive,

$$\begin{aligned} d(x_{n+1}, p) &\leq (1-\beta_n)d(y_n, p) + \beta_n d(y_n, p) \\ &\leq d(y_n, p) \\ &= d(T_k z_n, p) \\ &\leq d(z_n, p) \\ &= d((1-\alpha_n)x_n \oplus \alpha_n T_k x_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(T_k x_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Thus, $\{d(x_n, p)\}$ is a monotonically decreasing sequence that is bounded below.

Hence, $\lim_{n\to\infty} d(z_n, p)$ exists for all $p \in F(T)$.

Theorem 3. Let C be a nonempty closed and convex subset of a complete CAT(0) space (X, d) and T be an enriched $B_{\gamma,\mu}$ mapping on C. For $x_0 \in C$, let $\{x_n\}$ be a sequence defined by (2). Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, T_k x_n) = 0$, where T_k is the averaged mapping with $k = \frac{1}{b+1}$.

Proof. Let $F(T) \neq \emptyset$ and $p \in F(T)$. So by Lemma 14, $\lim_{n\to\infty} d(x_n, p)$ exists and thus, $\{x_n\}$ is bounded.

Let $\lim_{n\to\infty} d(x_n, p) = a \geq 0$. For $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} d(x_n, T_k x_n) &\leq d(x_n, p) + d(T_k x_n, p) \\ &\leq 2d(x_n, p). \end{aligned}$$

So, $\lim_{n\to\infty} d(x_n, T_k x_n)$ exists.

Now,

$$d(x_{n+1}, p) \leq d(y_n, p) \leq d(z_n, p) \leq d(x_n, p),$$

that is,

$$a = \limsup_{n\to\infty} d(x_{n+1}, p) \leq \limsup_{n\to\infty} d(y_n, p) \leq \limsup_{n\to\infty} d(z_n, p) \leq \limsup_{n\to\infty} d(x_n, p) = a.$$

Also,

$$a = \liminf_{n\to\infty} d(x_{n+1}, p) \leq \liminf_{n\to\infty} d(y_n, p) \leq \liminf_{n\to\infty} d(z_n, p) \leq \liminf_{n\to\infty} d(x_n, p) = a.$$

Hence

$$\limsup_{n\to\infty} d(z_n, p) = \liminf_{n\to\infty} d(z_n, p) = a.$$

So,

$$\lim_{n\to\infty} d(z_n, p) = a,$$

that is,

$$\lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T_k x_n, p) = a.$$

Also, we have

$$\limsup_{n\to\infty} d(x_n, p) \le a \quad \text{and} \quad \limsup_{n\to\infty} d(T_k x_n, p) \le a.$$

So, by Lemma 6, $\lim_{n\to\infty} d(x_n, T_k x_n) = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, T_k x_n) = 0$. Since $A(\{x_n\})$ contains exactly one point, let $p \in A(\{x_n\})$. By Lemma 13 (iii), for $c \in [0,1]$, we get

$$\begin{aligned} d(x_n, T_k p) \le & \left(3 - \frac{c}{2}\right) d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) d(x_n, p) + \mu(2d(x_n, T_k x_n) + d(x_n, T_k p) \\ & + d(p, T_k x_n) + 2d(T_k x_n, T_k^2 x_n)) \\ \le & \left(3 - \frac{c}{2}\right) d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) d(x_n, p) + \mu(2d(x_n, T_k x_n) + d(x_n, T_k p) \\ & + d(p, T_k x_n) + 2d(x_n, T_k x_n)). \end{aligned}$$

So,

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, T_k p) \le & \left(3 - \frac{c}{2}\right) \limsup_{n\to\infty} d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) \limsup_{n\to\infty} d(x_n, p) \\ & + \mu(2 \limsup_{n\to\infty} d(x_n, T_k x_n) + \limsup_{n\to\infty} d(x_n, T_k p) + \limsup_{n\to\infty} d(p, T_k x_n) \\ & + 2 \limsup_{n\to\infty} d(T_k x_n, T_k^2 x_n)) \\ \le & \left(3 - \frac{c}{2}\right) \limsup_{n\to\infty} d(x_n, T_k x_n) + \left(1 - \frac{c}{2}\right) \limsup_{n\to\infty} d(x_n, p) \\ & + \mu(2 \limsup_{n\to\infty} d(x_n, T_k x_n) + \limsup_{n\to\infty} d(x_n, T_k p) \\ & + \limsup_{n\to\infty} d(p, T_k x_n) + 2 \limsup_{n\to\infty} d(x_n, T_k x_n)), \end{aligned}$$

that is,

$$(1-\mu) \limsup_{n\to\infty} d(x_n, T_k p) \le \left(1 - \frac{c}{2} + \mu\right) \limsup_{n\to\infty} d(x_n, p).$$

Since $2\mu \le \gamma$, taking $c = 2\gamma$, we get

$$\limsup_{n\to\infty} d(x_n, T_k p) \le \limsup_{n\to\infty} d(x_n, p).$$

Therefore,

$$r(T_k p, \{x_n\}) \le r(p, \{x_n\}).$$

Hence

$$T_k p \in A(\{x_n\}).$$

By Lemma 5, $T_k p = p$, that is, $p \in F(T_k)$ and by Lemma 11, $p \in F(T)$. Hence, $F(T) \neq \emptyset$.

In view of the above theorem, we can say that the sequence $\{x_n\}$ defined by (2), is Δ-convergent to a fixed point of T.

The next result deals with the strong convergence of the iterative algorithm (2) to a fixed point.

Theorem 4. Let C be a nonempty closed and convex subset of a complete CAT(0) space (X, d). Let T be an enriched $B_{\gamma,\mu}$ mapping on X and for $x_0 \in C$, $\{x_n\}$ be a sequence defined by (2). Let $F(T) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) = \inf\{d(x,p); p \in F(T)\}$.

Proof. Let $\{x_n\}$ be convergent to $p \in F(T)$. Then $\lim_{n\to\infty} d(x_n, p) = 0$. Now,

$$0 \leq d(x_n, F(T)) = inf\{d(x_n, p) : p \in F(T)\} \leq d(x_n, p).$$

Therefore, $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Hence

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Conversely, let

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

By Lemma 14, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and $\{d(x_n, p)\}$ is monotonically decreasing. Thus,

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Consider a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, p_k) < \frac{1}{2^k} \text{ for all } k \in \mathbb{N} \text{ and for } \{p_k\} \subseteq F(T).$$

Since $\{d(x_n, p)\}$, $p \in F(T)$ is monotonically decreasing, so, for each k,

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

Now,

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

Hence $\{p_k\}$ is a Cauchy sequence in $F(T)$. Since $F(T)$ is closed, $\{p_k\}$ converges to some $p \in F(T)$.

Now,

$$d(x_{n_k}, p) \leq d(x_{n_k}, p_k) + d(p_k, p).$$

Taking $k \to \infty$, we get

$$\lim_{n\to\infty} d(x_{n_k}, p) = 0.$$

Since, $\lim_{n\to\infty} d(x_n, p)$ exists, so, $\lim_{n\to\infty} d(x_n, p) = 0$.

Hence $\{x_n\}$ converges to $p \in F(T)$.

We recall from [30] that a mapping T on a nonempty convex subset C of a CAT(0) space (X, d) with $F(T) \neq \emptyset$ satisfies the condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $f(d(x, F(T))) \leq d(x, Tx)$ for all $x \in C$. Here we use this condition (I) to prove the next strong convergence result.

Theorem 5. For a nonempty closed and convex subset C of a complete CAT(0) space (X, d), let T be an enriched $B_{\gamma,\mu}$ mapping on C with the averaged mapping T_k for $k = \frac{1}{b+1}$. Let $F(T) \neq \emptyset$ and for $x_0 \in C$, $\{x_n\}$ be a sequence defined by (2). If T_k satisfies condition (I) for a self-mapping f on $[0, \infty)$, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since $F(T) \neq \emptyset$, by Theorem 3, $\lim_{n\to\infty} d(x_n, T_k x_n) = 0$.
Also, by condition (I),

$$\lim_{n\to\infty} f(d(x_n, F(T_k))) \leq \lim_{n\to\infty} d(x_n, T_k x_n) = 0.$$

So,

$$\lim_{n\to\infty} f(d(x_n, F(T_k))) = 0.$$

Therefore,

$$\lim_{n\to\infty} (d(x_n, F(T_k))) = 0.$$

As $k \in (0, 1]$, so,

$$\lim_{n\to\infty} (d(x_n, F(T))) = 0.$$

Hence $\{x_n\}$ converges strongly to a fixed point of T.

We demonstrate the above theorem by the following example.

Example 3. Consider the CAT(0) space $X = [0, 1]$ with

$$d(x, y) = \begin{cases} x + y, & \text{if } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$$

Let $C = [0, \frac{1}{3}]$ and $T : C \to C$ be defined by $T(x) = 1 - 2x$, $x \in C$. Then T is an enriched $B_{\gamma,\mu}$ mapping with $b = 1$, $\gamma = \frac{2}{3}$, $\mu = 0$.

For $k = \frac{1}{2}$, the averaged mapping T_k is given by $T_k(x) = \frac{1-x}{2}$, $x \in C$.

Clearly, $F(T_k) = F(T) = \{\frac{1}{3}\} \neq \emptyset$.

We take f as the identity mapping on $[0, \infty)$. Then T_k satisfies condition (I) with respect to f.

So, by Theorem 5, for $x_0 \in C$, the sequence $\{x_n\}$ defined by (2) converges strongly to a fixed point of T_k.

Taking $x_0 = 0.05$, $x_0 = 0.1$, $x_0 = 0.2$ and $x_0 = 0.25$, we see the convergence of the iterative scheme as follows:

n	$x_0 = 0.05$	$x_0 = 0.1$	$x_0 = 0.2$	$x_0 = 0.25$
1	0.342188	0.340625	0.337500	0.335938
2	0.333057	0.333105	0.333203	0.333252
3	0.333342	0.333340	0.333337	0.333336
4	0.333333	0.333333	0.333333	0.333333
5	0.333333	0.333333	0.333333	0.333333
6	0.333333	0.333333	0.333333	0.333333

In Figure 1, the blue, purple, red, and green dotted lines represent the sequences defined by the iterative algorithm (2), when $x_0 = 0.05$, $x_0 = 0.1$, $x_0 = 0.2$ and $x_0 = 0.25$ respectively. It is seen that each sequence converges to the fixed point $\frac{1}{3}$.

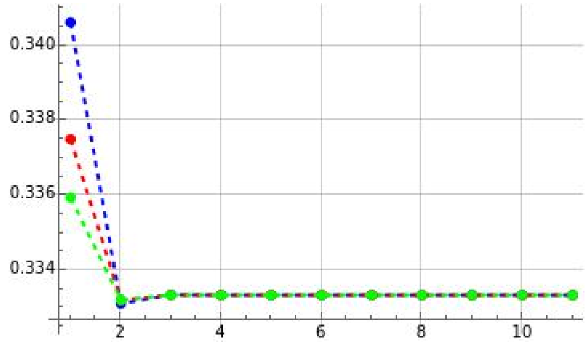

Figure 1. Convergence of the iteration scheme (2) with different initial points

Conclusion

We have established some fixed point results for enriched $B_{\gamma,\mu}$ mapping in CAT(0) space. Also, we introduced a new iteration scheme for such mappings in CAT(0) space and proved weak and strong convergence results of this iteration scheme. In 2022, Tufa et al. [31] constructed an iterative scheme to approximate the fixed point of a countable family of quasi-nonexpansive non-self-mapping in complete CAT(0) space. In this context, the investigation of common fixed points for a countable family of enriched $B_{\gamma,\mu}$ mappings is a scope of future study. Moreover, the comparison of the rate of convergence of our derived iteration scheme with some existing iteration schemes is another aspect for future discussion.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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