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Hyper-Generalized Weakly Symmetric Para-Sasakian Manifolds and Their Geometric Properties

B. Thangjam, M.S. Devi*

Mizoram University, Aizawl, India
(E-mail: bidyabatit@gmail.com, devi_saroja@rediffmail.com)

This paper examines para-Sasakian manifolds that satisfy a hyper-generalized weakly symmetric curvature condition. The conditions under which such a manifold with a hyper-generalized weakly symmetric curvature condition satisfies the η-Einstein manifold are established. Furthermore, the geometric behavior of a hyper-generalized weakly symmetric para-Sasakian manifold admitting quarter-symmetric metric connection is analyzed.

Keywords: hyper-generalized, para-Sasakian manifold, Quarter-symmetric metric connection.

2020 Mathematics Subject Classification: 53C05, 53C25.

Introduction

The study of symmetric and generalized symmetric manifolds play an important role in differential geometry due to their rich structures and diverse applications in both mathematics and theoretical physics. Among the various extensions of, symmetric spaces, the concept of weakly symmetric manifolds, which was first introduced by Tamassy and Binh [1], has been a focal point. This concept has been studied by many geometers (for details, please see [2–9]).

A non-flat, n-dimensional Riemannian manifold (M^n, g) $n > 2$ is called a generalized weakly symmetric manifold [10] if its curvature tensor R of type $(0, 4)$ is non zero and satisfies the condition:

$$\begin{aligned}(\nabla_U R)(V,X,Y,Z) &= A(U)R(V,X,Y,Z) + B(V)R(U,X,Y,Z) + B(X)R(V,U,Y,Z)\\ &+ D(Y)R(V,X,U,Z) + D(Z)R(V,X,Y,U) + \alpha(U)G(V,X,Y,Z)\\ &+ \beta(V)G(U,X,Y,Z) + \beta(X)G(V,U,Y,Z) + \gamma(Y)G(V,X,U,Z)\\ &+ \gamma(Z)G(V,X,Y,U),\end{aligned}$$

where $G(X,\ Y)Z = g(Y,\ Z)X - g(X,\ Z)Y$, and $A,\ B,\ D,\ \alpha,\ \beta,\ \gamma$ are non zero 1-forms defined as follows: $A(X) = g(X,\theta_1)$, $B(X) = g(X,\phi_1)$, $D(X) = g(X,\pi_1)$, $\alpha(X) = g(X,\theta_2)$, $\beta(X) = g(X,\phi_2)$ and $\gamma(X) = g(X,\pi_2)$, in which $\theta_1,\ \phi_1,\ \pi_1,\ \theta_2,\ \phi_2$ and π_2 are associated vector fields of $A,\ B,\ D,\ \alpha,\ \beta$ and γ respectively.

Keeping in tune with Shaikh and Patra [11], we shall call a Riemannian manifold of dimension n, hyper-generalized weakly symmetric if it admits the equation:

$$\begin{aligned}(\nabla_U R)(V,X,Y,Z) &= A(U)R(V,X,Y,Z) + B(V)R(U,X,Y,Z) + B(X)R(V,U,Y,Z)\\ &+ D(Y)R(V,X,U,Z) + D(Z)R(V,X,Y,U)\\ &+ \alpha(U)(g\wedge S)(V,X,Y,Z) + \beta(V)(g\wedge S)(U,X,Y,Z)\\ &+ \beta(X)(g\wedge S)(V,U,Y,Z) + \gamma(Y)(g\wedge S)(V,X,U,Z)\\ &+ \gamma(Z)(g\wedge S)(V,X,Y,U),\end{aligned}$$

*Corresponding author. *E-mail: devi_saroja@rediffmail.com*
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where

$$(g \wedge S)(V,X,Y,Z) = g(V,Z)S(X,Y) + g(X,Y)S(V,Z) - g(V,Y)S(X,Z) - g(X,Z)S(V,Y), \tag{1}$$

and A, B, D, α, β, γ are non zero 1-forms defined as follows: $A(X) = g(X,\theta_1)$, $B(X) = g(X,\phi_1)$, $D(X) = g(X,\pi_1)$, $\alpha(X) = g(X,\theta_2)$, $\beta(X) = g(X,\phi_2)$ and $\gamma(X) = g(X,\pi_2)$.

The $(H(GWS))_n$ manifolds exhibit various properties, aligning with different types of symmetric spaces.

In recent years, Blaga et al. [12] studied hyper generalized pseudo Q-symmetric semi-Riemannian manifold. And Bakshi et al. [13] investigated the existence of hyper-generalized weakly symmetric Lorentzian para-Sasakian manifold.

The concept of an almost para-contact structure was introduced by Sato [14] as an analogue to the almost contact structure, which has been widely studied in differential geometry. While an almost contact manifold is always of odd dimension, an almost para-contact manifold can exist in both odd and even dimensions, making it more versatile structure in geometric analysis.

Kaneyuki and Williams [15] further developed this idea by investigating the almost para-contact structure on pseudo-Riemannian manifolds. Recently, there has been a growing interest in almost para-contact geometry, in particular, para-Sasakian geometry, due to its connections with the theory of para-Kähler manifolds. The study of almost para-contact and para-Sasakian structures is also growing traction because of their applications in pseudo-Riemannian geometry and mathematical physics. Almost para-contact structures facilitate the exploration of new geometric invariants and curvature conditions that difer significantly from their contact counter parts. In recent year, Bulut and İnselöz [16] studied para-Sasaki-like Riemannian manifolds with generalized symmetric metric connection. Extending this, we study para-Sasakian manifold whose curvature tensor satisfies the hyper-generalized weakly symmetric condition.

Golab [17] extended the concept of semi-symmetric connection by introducing the notion of a quarter-symmetric connection within differentiable manifolds equipped with an affine connection. This idea was subsequently explored further by researchers such as Mondal and De [18], Rastogi [19, 20], Yano and Imai [21], among others.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M,g) is defined as a quarter-symmetric connection [17], if its torsion tensor T of the connection $\tilde{\nabla}$, given by

$$T(U,V) = \tilde{\nabla}_U V - \tilde{\nabla}_V U - [U,V],$$

satisfies the condition

$$T(U,V) = \eta(V)\phi U - \eta(U)\phi V,$$

where η is a 1-form and ϕ is a (1,1)-tensor field.

In special case, where $\phi U = U$, the quarter-symmetric connection reduces to the semi-symmetric connection [22, 23], thus showing that the concept of a quarter-symmetric connection generalizes the idea of the semi-symmetric connection.

When a quarter-symmetric connection $\tilde{\nabla}$ satisfies

$$(\tilde{\nabla}_U g)(V,W) = 0,$$

it is known as a quarter-symmetric metric connection. Otherwise it is referred to as a quarter-symmetric non metric connection.

The structure of this paper is as follows: in Section 2, we define para-Sasakian manifold and present some known results of para-Sasakian manifold. Then in the next section, we discuss hyper-generalized para-Sasakian manifold and derive some relations of the 1-forms. In the next two sections, we discuss the conditions under which a hyper-generalized weakly symmetric para-Sasakian manifold admitting Codazzi type of Ricci tensor and recurrent tensor becomes an η-Einstein manifold.

1 Para-Sasakian Manifold

Consider M as an n-dimensional almost para-contact metric manifold admitting an almost para-contact metric structure (ϕ, ξ, η, g), where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric. Then [24]

$$\phi^2 U = U - \eta(U)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi U) = 0,$$

$$g(\phi U, \phi V) = -g(U, V) + \eta(U)\eta(V), \quad g(\phi U, V) = -g(U, \phi V), \quad g(U, \xi) = \eta(U),$$

for all vector fields U, V on TM.

An almost para-contact manifold is called a para-Sasakian manifold if

$$(\nabla_U \phi)V = -g(U, V)\xi + \eta(V)U,$$

$$d\eta = 0 \quad \text{and} \quad \nabla_U \xi = -\phi U.$$

In a para-Sasakian manifold equipped with the structure (ϕ, ξ, η, g), the following relations also hold [24]

$$(\nabla_U \eta)V = g(U, \phi V),$$

$$\eta(R(U, V)W) = g(U, W)\eta(V) - g(V, W)\eta(U),$$

$$R(\xi, U)V = -g(U, V)\xi + \eta(V)U,$$

$$R(U, V)\xi = \eta(U)V - \eta(V)U,$$

$$S(U, \xi) = -(n-1)\eta(U),$$

$$Q\xi = -(n-1)\xi,$$

for all $U, \ V \in TM$, where S is the Ricci tensor, R is a Riemannian curvature tensor.

Let us consider a quarter-symmetric metric connection $\tilde{\nabla}$ on a para-Sasakian manifold [25] given by

$$\tilde{\nabla}_U V = \nabla_U V + \eta(V)\phi U - g(\phi U, V)\xi. \tag{2}$$

The curvature tensor $\tilde{R}$ associated with the quarter-symmetric connection relates to the curvature tensor R of the Levi-Civita connection by [26]:

$$\begin{aligned} \tilde{R}(U, V)W = R(U, V)W + 3g(\phi U, W)\phi V - 3g(\phi V, W)\phi U + [\eta(U)V - \eta(V)U]\eta(W) \\ - [g(V, W)\eta(U) - g(U, W)\eta(V)]\xi, \end{aligned} \tag{3}$$

which yields

$$\tilde{S}(V, W) = S(V, W) + 2g(V, W) - (n+1)\eta(V)\eta(W) - 3tr\phi g(\phi V, W),$$

where $\tilde{S}$ and S are Ricci tensor of $\tilde{\nabla}$ and ∇, respectively.

Definition 1. A para-Sasakian manifold M is said to be an η-Einstein manifold, if its Ricci tensor S is of the form

$$S(X, \ Y) = \alpha g(X, \ Y) + \beta \eta(X) \eta(Y)$$

for any vector fields X and Y, where α and β are constants. If $\beta = 0$, then the manifold M^{2n+1} is an Einstein manifold.

2 Hyper-Generalized Weakly Symmetric Para-Sasakian Manifolds

A para-Sasakian manifold is considered to be hyper-generalized weakly symmetric if it satisfies the following curvature condition [11]

$$\begin{aligned}(\nabla_U R)(V,X,Y,W) &= A(U)R(V,X,Y,W) + B(V)R(U,X,Y,W) + B(X)R(V,U,Y,W)\\ &+ D(Y)R(V,X,U,W) + D(W)R(V,X,Y,U)\\ &+ \alpha(U)(g\wedge S)(V,X,Y,W) + \beta(V)(g\wedge S)(U,X,Y,W)\\ &+ \beta(X)(g\wedge S)(V,U,Y,W) + \gamma(Y)(g\wedge S)(V,X,U,W)\\ &+ \gamma(W)(g\wedge S)(V,X,Y,U),\end{aligned} \tag{4}$$

where $g\wedge S$ is defined in (1).

By applying (1) and then contracting it with U and V in (4), we obtain

$$\begin{aligned}(\nabla_U S)(V,W) &= A(U)S(V,W) + B(V)S(U,W) + D(W)S(U,V) + B(R(U,V)W)\\ &+ D(R(U,W)V) + \alpha(U)[(n-2)S(V,W) + rg(V,W)]\\ &+ \beta(V)[(n-2)S(U,W) + rg(U,W)] + \gamma(W)[(n-2)S(U,V) + rg(U,V)]\\ &+ \tilde{\beta}(U)g(V,W) + \beta(U)S(V,W) - \beta(V)S(U,W) - \tilde{\beta}(V)g(U,W)\\ &+ \tilde{\gamma}(U)g(V,W) + \gamma(U)S(V,W) - \tilde{\gamma}(W)g(V,U) - \gamma(W)S(V,U).\end{aligned} \tag{5}$$

By further contracting the above equation, we get

$$\begin{aligned}dr(U) &= rA(U) + 2\tilde{B}(U) + 2\tilde{D}(U) + 2(n-1)r\alpha(U) + 2(n-2)[\tilde{\beta(U)} + \tilde{\gamma}(U)]\\ &+ 2r[\beta(U) + \gamma(U)],\end{aligned} \tag{6}$$

where $\tilde{B}(U) = S(U,\ \phi_1)$, $\tilde{D}(U) = S(U,\pi_1)$, $\tilde{\beta}(U) = S(U,\ \phi_2)$ and $\tilde{\gamma}(U) = S(U,\ \pi_2)$. Assuming that the scalar curvature of a hyper-generalized weakly symmetric para-Sasakian manifold is a non-zero constant (to avoid flat manifold and to preserve the para-Sasakian Structure), equation (6) reduces to

$$\begin{aligned}&r[A(U) + 2(n-1)\alpha(U) + 2\beta(U) + 2\gamma(U)]\\ &= -2\tilde{B}(U) - 2\tilde{D}(U) - 2(n-2)[\tilde{\beta(U)} + \tilde{\gamma}(U)].\end{aligned} \tag{7}$$

This leads to the following result:

Theorem 1. Let M be a hyper-generalized weakly symmetric para-Sasakian manifold with non-zero constant scalar curvature. The 1-forms are then related by the equation (7).

Putting $V=\xi$ in (5), we get

$$\begin{aligned}(\nabla_U S)(\xi,W) &= -(n-1)A(U)\eta(W) + B(\xi)S(U,W) - (n-1)D(W)\eta(U) + B(\xi)g(U,W)\\ &- \eta(W)B(U) + D(W)\eta(U) - \eta(W)D(U) - (n-1)(n-2)\alpha(U)\eta(W)\\ &+ r\eta(W)\alpha(U) + (n-2)\beta(\xi)S(U,W) + r\beta(\xi)g(U,W) - \tilde{\beta}(\xi)g(U,W)\\ &+ r\gamma(W)\eta(U) + \tilde{\beta}(U)\eta(W) - (n-1)\beta(U)\eta(W) - \beta(\xi)S(U,W) + \tilde{\gamma}(U)\eta(W)\\ &- (n-1)\gamma(U)\eta(W) - \tilde{\gamma}(W)\eta(U) + (n-1)\gamma(W)\eta(U)\\ &- (n-1)(n-2)\gamma(W)\eta(U).\end{aligned} \tag{8}$$

Setting $W = \xi$ in (5), we have

$$\begin{aligned}(\nabla_U S)(V,\xi) &= -(n-1)A(U)\eta(V) - (n-1)B(V)\eta(U) + D(\xi)S(U,V) + B(V)\eta(U)\\ &- B(U)\eta(V) + D(\xi)g(U,V) - D(U)\eta(V) - (n-1)(n-2)\alpha(U)\eta(V)\\ &+ r\alpha(U)\eta(V) - (n-1)(n-2)\beta(V)\eta(U) + r\beta(V)\eta(U) + r\gamma(\xi)g(U,V)\\ &+ (n-2)\gamma(\xi)S(U,V) + \tilde{\beta}(U)\eta(V) - (n-1)\beta(U)\eta(V) - \tilde{\beta}(V)\eta(U)\\ &+ (n-1)\beta(V)\eta(U) + \tilde{\gamma}(U)\eta(V) - (n-1)\gamma(U)\eta(V) - \tilde{\gamma}(\xi)\eta(V)\\ &- \gamma(\xi)S(U,V).\end{aligned} \tag{9}$$

Replacing $V = W$ in the above equation, we obtain

$$\begin{aligned}(\nabla_U S)(W,\xi) &= -(n-1)A(U)\eta(W) - (n-1)B(W)\eta(U) + D(\xi)S(U,W) + B(W)\eta(U)\\ &- B(U)\eta(W) + D(\xi)g(U,W) - D(U)\eta(W) - (n-1)(n-2)\alpha(U)\eta(W)\\ &+ r\alpha(U)\eta(W) - (n-1)(n-2)\beta(W)\eta(U) + r\beta(W)\eta(U) + r\gamma(\xi)g(U,W)\\ &+ (n-2)\gamma(\xi)S(U,W) + \tilde{\beta}(U)\eta(W) - (n-1)\beta(U)\eta(W) - \tilde{\beta}(W)\eta(U)\\ &+ (n-1)\beta(W)\eta(U) + \tilde{\gamma}(U)\eta(W) - (n-1)\gamma(U)\eta(W) - \tilde{\gamma}(\xi)\eta(W)\\ &- \gamma(\xi)S(U,W).\end{aligned} \tag{10}$$

Comparing (8) and (10), we have

$$\begin{aligned}&[B(\xi) + (n-3)\beta(\xi) - D(\xi) - (n-1)\gamma(\xi)]S(U,W)\\ &= [(n-2)D(Z) + (n-2)^2\gamma(W) - r\gamma(W) - (n-2)B(W) + (n-1)(n-3)\beta(W) + r\beta(W)\\ &- \tilde{\beta}(W)]\eta(U) + [\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]g(U,W).\end{aligned} \tag{11}$$

Putting $U = \xi$ in (11), we get

$$\begin{aligned}&(n-2)D(W) + (n-2)^2\gamma(W) - r\gamma(W) - (n-2)B(W) + (n-1)(n-3)\beta(W)\\ &+ r\beta(W) - \tilde{\beta}(W) = [(n-1)B(\xi) + (n-1)(n-3)\beta(\xi) - (n-1)D(\xi)\\ &- (n-1)^2\gamma(\xi)\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]\eta(W).\end{aligned} \tag{12}$$

Substituting (12) in (11), we deduce

$$\begin{aligned}&[B(\xi) + (n-3)\beta(\xi) - D(\xi) - (n-1)\gamma(\xi)]S(U,W)\\ &= [(n-1)B(\xi) + (n-1)(n-3)\beta(\xi) - (n-1)D(\xi)\\ &- (n-1)^2\gamma(\xi)\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]\eta(U)\eta(W)\\ &+ [\tilde{\beta}(\xi) - (r+1)\beta(\xi) + r\gamma(\xi) - \tilde{\gamma}(\xi)]g(U,W).\end{aligned}$$

Thus, we can state the following theorem:

Theorem 2. A hyper-generalized para-Sasakian manifold is an η-Einstein manifold if

$$B(\xi) + (n-3)\beta(\xi) \neq D(\xi) + (n-1)\gamma(\xi).$$

We can also observe that,

$$(\nabla_X S)(Y,\xi) = -(n-1)g(X,\phi Y) + S(Y,\phi X). \tag{13}$$

From (9) and (13), we derive

$$\begin{aligned}
&-(n-1)g(U,\phi V)+S(V,\phi U)\\
&=-(n-1)A(U)\eta(V)-(n-1)B(V)\eta(U)+D(\xi)S(U,V)+B(V)\eta(U)\\
&-B(U)\eta(V)+D(\xi)g(U,V)-D(U)\eta(V)-(n-1)(n-2)\alpha(U)\eta(V)\\
&+r\alpha(U)\eta(V)-(n-1)(n-2)\beta(V)\eta(U)+r\beta(V)\eta(U)+r\gamma(\xi)g(U,V)\\
&+(n-2)\gamma(\xi)S(U,V)+\tilde{\beta}(U)\eta(V)-(n-1)\beta(U)\eta(V)-\tilde{\beta}(V)\eta(U)\\
&+(n-1)\beta(V)\eta(U)+\tilde{\gamma}(U)\eta(V)-(n-1)\gamma(U)\eta(V)-\tilde{\gamma}(\xi)\eta(V)-\gamma(\xi)S(U,V).
\end{aligned} \tag{14}$$

Setting $U=V=\xi$ in (14) yields

$$\begin{aligned}
&(n-1)[A(\xi)+B(\xi)+D(\xi)+(n-2)\alpha(\xi)+(n-2)\beta(\xi)+(n-2)\gamma(\xi)]\\
&-r[\alpha(\xi)+\beta(\xi)+\gamma(\xi)]=0.
\end{aligned} \tag{15}$$

The above equation expresses a relationship between these functions along ξ (reeb vectors), indicating how the curvature properties interact under hyper-generalized weak symmetric. Hence, the following theorem can be stated:

Theorem 3. In para-Sasakian manifold with hyper-generalized weakly symmetric curvature condition, the relations (15) holds.

3 Codazzi Type Of Ricci Tensor

A Ricci tensor is of Codazzi type when its covariant derivative is symmetric with respect to its indices, i.e, for any vector fields U,V,W it satisfies

$$(\nabla_U S)(V,W)=(\nabla_W S)(V,U), \tag{16}$$

where ∇ is the Levi-Civita connection of the manifold's metric.

In view of (5) and (16), we have

$$\begin{aligned}
0&=[A(U)-D(U)+(n-2)\alpha(U)-(n-4)\gamma(U)+\beta(U)]S(V,W)\\
&+[D(W)+(n-4)\gamma(W)-A(W)-(n-2)\alpha(W)-\beta(W)]S(V,U)\\
&+[r\alpha(U)+\tilde{\beta}(U)+2\tilde{\gamma}(U)-r\gamma(U)]g(V,W)\\
&+[r\gamma(W)-\tilde{\gamma}(W)-r\alpha(W)-\tilde{\beta}(W)-\tilde{\gamma}(W)]g(V,U)\\
&+B(R(U,V)W)+2D(R(U,W)V)-B(R(W,V)U).
\end{aligned} \tag{17}$$

Now by substituting $W=\xi$ in (17), we obtain

$$\begin{aligned}
0&=-(n-1)[A(U)-D(U)+(n-2)\alpha(U)-(n-4)\gamma(U)+\beta(U)]\eta(V)\\
&+[D(\xi)+(n-4)\gamma(\xi)-A(\xi)-(n-2)\alpha(\xi)-\beta(\xi)]S(V,U)\\
&+[r\alpha(U)+\tilde{\beta}(U)+2\tilde{\gamma}(U)-r\gamma(U)]\eta(V)\\
&+[r\gamma(\xi)-\tilde{\gamma}(\xi)-r\alpha(\xi)-\tilde{\beta}(\xi)-\tilde{\gamma}(\xi)]g(V,U)\\
&-B(V)\eta(U)+B(U)\eta(V)-2D(U)\eta(V)+2D(\xi)g(U,V)\\
&+B(\xi)g(V,U)-B(V)\eta(U).
\end{aligned} \tag{18}$$

Putting $U=\xi$ in (18), we get

$$B(V)=B(\xi)\eta(V). \tag{19}$$

Replacing $V = \xi$ in (18), we deduce

$$\begin{aligned}&(n-1)[A(U) - D(U) + (n-2)\alpha(U) - (n-4)\gamma(U) + \beta(U)]\\&-[r\alpha(U) + \tilde{\beta}(U) + 2\tilde{\gamma}(U) - r\gamma(U)] - B(U) + 2D(U)\\&= \{-(n-1)[D(\xi) + (n-4)\gamma(\xi) - A(\xi) - (n-2)\alpha(\xi) - \beta(\xi)]\\&+[r\gamma(\xi) - 2\tilde{\gamma}(\xi) - r\alpha(\xi) - \tilde{\beta}(\xi) - B(\xi) + D(\xi)]\}\eta(U).\end{aligned} \tag{20}$$

Using (19) and (20) in (18), we obtain

$$\begin{aligned}&[D(\xi) + (n-4)\gamma(\xi) - A(\xi) - (n-2)\alpha(\xi) - \beta(\xi)]S(U,V)\\&= \{-(n-1)[D(\xi) + (n-4)\gamma(\xi) - A(\xi) - (n-2)\alpha(\xi) - \beta(\xi)]\\&+[r\gamma(\xi) - 2\tilde{\gamma}(\xi) - r\alpha(\xi) - \tilde{\beta}(\xi) - B(\xi) + D(\xi)]\}\eta(U)\eta(V)\\&-[r\gamma(\xi) - 2\tilde{\gamma}(\xi) - r\alpha(\xi) - \tilde{\beta}(\xi) - B(\xi) + D(\xi)]g(U,V).\end{aligned}$$

From this, we can present the following theorem:

Theorem 4. A hyper-generalized weakly symmetric para-Sasakian manifold is η-Einstein if it admits Codazzi type of Ricci tensor, provided that

$$D(\xi) + (n-4)\gamma(\xi) = A(\xi) + (n-2)\alpha(\xi) + \beta(\xi).$$

4 Recurrent Ricci Tensor

If the hyper-generalized weakly symmetric para-Sasakian manifold has a recurrent Ricci tensor then

$$(\nabla_U S)(V,W) = \lambda(U)S(V,W). \tag{21}$$

In view of (5) and (21), we have

$$\begin{aligned}\lambda(U)S(V,W) &= A(U)S(V,W) + B(V)S(U,W) + D(W)S(U,V) + B(R(U,V)W)\\&+ D(R(U,W)V) + \alpha(U)[(n-2)S(V,W) + rg(V,W)]\\&+\beta(V)[(n-2)S(U,W) + rg(U,W)] + \gamma(W)[(n-2)S(U,V) + rg(U,V)]\\&+\tilde{\beta}(U)g(V,W) + \beta(U)S(V,W) - \beta(V)S(U,W) - \tilde{\beta}(V)g(U,W)\\&+\tilde{\gamma}(U)g(V,W) + \gamma(U)S(V,W) - \tilde{\gamma}(W)g(V,U) - \gamma(W)S(V,U).\end{aligned} \tag{22}$$

Placing $W = \xi$ in (22), we obtain

$$\begin{aligned}-(n-1)\lambda(U)\eta(V) &= -(n-1)A(U)\eta(V) - (n-1)B(V)\eta(U) + D(\xi)S(U,V)\\&+ B(V)\eta(U) - B(U)\eta(V) + D(\xi)g(U,V) - D(U)\eta(V)\\&-(n-1)(n-2)\alpha(U)\eta(V) + r\alpha(U)\eta(V) + \tilde{\beta}(U)\eta(V)\\&+ r\beta(V)\eta(U) + (n-2)\gamma(\xi)S(U,V) + r\gamma(\xi)g(U,V)\\&-(n-1)\beta(U)\eta(V) + (n-1)\beta(V)\eta(U) - \tilde{\beta}(V)\eta(U)\\&+\tilde{\gamma}(U)\eta(V) - (n-1)\gamma(U)\eta(V) - \tilde{\gamma}(\xi)g(V,U) - \gamma(\xi)S(V,U)\\&-(n-1)(n-2)\beta(V)\eta(U).\end{aligned} \tag{23}$$

Putting $U = \xi$ in (23), we have

$$\begin{aligned}&-(n-2)B(V) - (n-1)(n-3)\beta(V) + r\beta(V) - \tilde{\beta}(V)\\&= [-(n-1)\lambda(\xi) + (n-1)A(\xi) + (n-1)D(\xi) + B(\xi) + (n-1)(n-2)\alpha(\xi) - r\alpha(\xi)\\&-(n-1)(n-2)\gamma(\xi) - r\gamma(\xi) - \tilde{\beta}(\xi) + (n-1)\beta(\xi)]\eta(V).\end{aligned} \tag{24}$$

Considering $V = \xi$ in (23), we get

$$\begin{aligned}
&(n-1)A(U) - (n-1)\lambda(U) + B(U) + D(U) + (n-1)(n-2)\alpha(U) \\
&- r\alpha(U) + (n-1)\beta(U) - \tilde{\beta}(U) - \tilde{\gamma}(U) + (n-1)\gamma(U) \\
&= [-(n-2)\beta(\xi) - (n-2)D(\xi) - (n-1)(n-2)\beta(\xi) + r\beta(\xi) \\
&- (n-1)^2\gamma(\xi) + r\gamma(\xi) - \tilde{\beta}(\xi) - \tilde{\gamma}(\xi)]\eta(U).
\end{aligned} \tag{25}$$

Now, setting $U = V = \xi$ in (23), we deduce

$$\begin{aligned}
&-(n-1)[A(\xi) + B(\xi) + D(\xi) + (n-2)\alpha(\xi) + (n-2)\beta(\xi) + (n-2)\gamma(\xi) - \lambda(\xi)] \\
&+ r[\alpha(\xi) + \beta(\xi) + \gamma(\xi)] = 0.
\end{aligned} \tag{26}$$

Putting the value of (24), (25), (26) in (23), we obtain

$$\begin{aligned}
&[D(\xi) + (n-3)\gamma(\xi)]S(U,V) \\
&= [B(\xi) + (n-3)D(\xi) - (n-1)(n-3)\gamma(\xi) + (n-1)\beta(\xi) + r\gamma(\xi)]\eta(U)\eta(V) \\
&- [D(\xi) + (r-1)\gamma(\xi)]g(U,V).
\end{aligned}$$

Theorem 5. In a hyper-generalized weakly symmetric para-Sasakian manifold, if the Ricci tensor is recurrent then the manifold becomes an η-Einstein, provided that

$$D(\xi) \neq (n-3)\gamma(\xi).$$

5 *Hyper-generalized Weakly Symmetric Para-Sasakian Manifold Admitting a Quarter-Symmetric Metric Connection*

For a $[(H(GWS))_n, \tilde{\nabla}]$ [11], we have

$$\begin{aligned}
(\nabla_U \tilde{R})(V,X,Y,W) &= A(U)\tilde{R}(V,X,Y,W) + B(V)\tilde{R}(U,X,Y,W) + B(X)R(V,U,Y,W) \\
&+ D(Y)\tilde{R}(V,X,U,W) + D(W)\tilde{R}(V,X,Y,U) \\
&+ \alpha(U)(g \wedge \tilde{S})(V,X,Y,W) + \beta(V)(g \wedge \tilde{S})(U,X,Y,W) \\
&+ \beta(X)(g \wedge \tilde{S})(V,U,Y,W) + \gamma(Y)(g \wedge \tilde{S})(V,X,U,W) \\
&+ \gamma(W)(g \wedge \tilde{S})(V,X,Y,U),
\end{aligned} \tag{27}$$

for all $X, Y, U, V, W \in TM$. Making use of (2), we can find

$$\begin{aligned}
(\tilde{\nabla}_U \tilde{R})(V,X,Y,W) &= (\nabla_U \tilde{R})(V,X,Y,W) + \eta(\tilde{R}(V,X)Y)g(\phi U,W) - \eta(V)\tilde{R}(\phi U,X,Y,W) \\
&- \eta(X)\tilde{R}(V,\phi U,Y,W) - \eta(Y)\tilde{R}(V,X,\phi U,W) - \eta(Y)\tilde{R}(V,X,Y,\phi U) \\
&+ g(\phi U,V)\tilde{R}(\xi,X,Y,W) + g(\phi U,X)\tilde{R}(V,\xi,Y,W) \\
&+ g(\phi U,Y)\tilde{R}(V,X,\xi,W) + g(\phi U,W)\tilde{R}(V,X,Y,\xi).
\end{aligned}$$

Now, using (3) in the foregoing equation, we have

$$
\begin{aligned}
(\tilde{\nabla}_U\tilde{R})(V,X,Y,W) &= (\nabla_U R)(V,X,Y,W) + 3(\nabla_U g)(\phi V,Y)g(\phi X,W) \\
&+ 3g(\phi V,Y)(\nabla_U g)(\phi X,W) - 3(\nabla_U g)(\phi X,Y)g(\phi V,W) \\
&- 3g(\phi X,Y)\nabla_U g)(\phi V,W) + (\nabla_U\eta)(V)g(X,W)\eta(Y) \\
&+ \eta(V)g(X,W)(\nabla_U\eta)(Y) - (\nabla_U\eta)(X)g(V,W)\eta(Y) \\
&- \eta(X)g(V,W)(\nabla_U\eta)(W) - g(X,Y)(\nabla_U\eta)(V)\eta(W) \\
&- g(X,Y)\eta(V)(\nabla_U\eta)(W) + g(V,Y)(\nabla_U\eta)(X)\eta(W) \\
&+ g(V,Y)\eta(X)(\nabla_U\eta)(W) - [\eta(R(V,X,Y)) - \eta(V)g(X,Y) \\
&+ \eta(X)g(V,Y)]g(\phi U,W) - \eta(V)[R(\phi U,X,Y,W) \\
&+ 3g(\phi U,\phi Y)g(\phi X,W) - 3g(\phi V,Y)g(\phi V,\phi W) - \eta(X)g(\phi U,W)\eta(Y) \\
&- \eta(X)g(\phi U,Y)\eta(W)] - \eta(X)[R(V,\phi U,Y,W) \\
&+ 3g(\phi V,Y)g(\phi U,\phi W) - 3g(\phi U,\phi Y)g(\phi V,W) + \eta(V)g(\phi U,W)\eta(Y) \\
&- \eta(V)g(\phi U,Y)\eta(W)] - \eta(Y)[R(V,X,\phi U,W) + 3g(\phi V,\phi U)g(\phi X,W) \\
&- 3g(\phi X,\phi U)g(\phi V,W) - \eta(V)g(X,\phi U)\eta(W) - \eta(X)g(V,\phi U)\eta(W)] \\
&- \eta(W)[R(V,X,\phi U) + 3g(\phi V,Y)g(\phi X,\phi U) \\
&- 3g(\phi X,Y)g(\phi V,\phi U) - \eta(V)g(X,\phi U)\eta(Y) - \eta(X)g(V,\phi U)\eta(Y)] \\
&+ g(\phi U,V)[R(\xi,X,Y,W) + g(X,W)\eta(Y) - g(X,Y)\eta(W)] \\
&+ g(\phi U,X)[R(V,\xi,Y,W) + g(X,W)\eta(V) - g(V,W)\eta(X)] \\
&+ g(\phi U,Y)[R(V,X,\xi,W) + g(X,W)\eta(V) - g(V,W)\eta(Y)] \\
&+ g(\phi U,W)[R(V,X,Y,\xi) - g(X,Y)\eta(V) + g(V,Y)\eta(X)].
\end{aligned} \tag{28}
$$

A hyper-generalized weakly symmetric para-Sasakian manifold with quarter-symmetric metric connection $\tilde{\nabla}$ simplifies to a hyper-generalized weakly symmetric para-Sasakian manifold with a Riemannian metric connection ∇, provided the following condition is satisfied:

$$(\tilde{\nabla}_U\tilde{R})(V,X,Y,W) = (\nabla_U\tilde{R})(V,X,Y,W). \tag{29}$$

Using (3), (27) and (28) in (29), it yields

$$A(\xi) + B(\xi) + D(\xi) = 0,$$

for $U = X = Y = \xi$. From the above, we can state the following:

Theorem 6. A $[(H(GWS))_n, \tilde{\nabla}]$ is a $[(H(GWS))_n, \nabla]$, if the following relations hold

$$A(\xi) + B(\xi) + D(\xi) = 0.$$

Conclusion

In this paper, we have explored the geometric properties of hyper-generalized weakly symmetric (H(GWS)) para-Sasakian manifolds. We have shown that, under certain conditions, (H(GWS)) para-Sasakian manifolds can reduce to simpler forms, such as η-Einstein manifolds, when specific relations between their 1-forms are satisfied. The study also highlights the role of quarter-symmetric metric connections, showing how these connections influence the manifold's curvature and overall geometric behavior. This work not only advances the understanding of para-Sasakian geometry but also provides a framework for exploring further extensions and applications of weakly symmetric manifolds in differential geometry and theoretical physics.

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Author Contributions

Bidyabati Thangjam: Conceptualization, Methodology, Formal Analysis, Writing-Original draft, Writing-Review and Editing, Validation.

M.S. Devi: Methodology, Review, Validation.

Conflict of Interest

The authors declare no conflict of interest.

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Author Information[*]

Bidyabati Thangjam — Research Scholar, Department of Mathematics and Computer Sciences, Mizoram University, Aizawl-796004, India; e-mail: *bidyabatit@gmail.com*; https://orcid.org/0000-0001-9283-7892

Mayanglambam Saroja Devi (*corresponding author*) — Doctorate, Assistant Professor, Research Scholar, Department of Mathematics and Computer Sciences, Mizoram University, Aizawl-796004, India; e-mail: *devi_saroja@rediffmail.com*; https://orcid.org/0000-0002-6925-774X

[*]The author's name is presented in the order: First, Middle, and Last Names.