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Liftings from Lorentzian α-Sasakian manifolds to tangent bundles

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The goal of the current study is to investigate the complete lift of Lorentzian α-Sasakian manifolds to the tangent bundle TM. We also examine the complete lift of the different four types of Lorentzian α-Sasakian manifolds and find that (TM, g^C) is an η-Einstein manifold in each instance. In order to show that a Lorentzian α-Sasakian manifold exists on TM, a non-trivial example by means of partial differential equations is built in the final section.

Keywords: complete lift, tangent bundle, mathematical operators, Weyl conformal curvature tensor, conharmoic curvature tensor, projective curvature tensor, concircular curvature tensor, trans-Sasakian manifolds, Lorentzian α-Sasakian manifolds, partial differential equations.

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Introduction

The geometry of trans-Sasakian manifolds received significant contributions from Blair and Oubina [1]. An almost contact metric manifold $(\mathsf{M}, \phi, \xi, \eta, g)$ with dim $= (2n+1)$ was defined due to Blair [2]. The geometry of the almost Hermitian manifold $(\bar{\mathsf{M}}, J, G)$ helps to determine the geometry of the almost contact metric manifold $(\mathsf{M}, \phi, \xi, \eta, g)$, that offers various structures on M (Sasakian, quasi-Sasakian, Kenmotsu, etc.) [1, 3–6], where $\bar{\mathsf{M}} = \mathsf{M} \times R$, J represents the almost complex structure and G stands for the Hermitian metric. A structure $(\phi, \xi, \eta, g, \alpha, \beta)$ on M is referred to as a trans-Sasakian structure where α, β are smooth functions using the structure in the class W_4 on $(\bar{\mathsf{M}}, J, G)$. On the nearly Hermitian manifold $(\bar{\mathsf{M}}, J, G)$, sixteen different types of structures are known to exist [7].

It is noted that cosymplectic [2], β-Kenmotsu [6] and α-Sasakian [6] are trans-Sasakian structures of type $(0, 0), (0, \beta)$ and $(\alpha, 0)$ respectively. De and Tripathi [8] have explored trans-Sasakian manifolds and achieved outstanding findings. Lorentzian α-Sasakian manifolds were the subject of Yildiz and Murathan's research [9]. Yoldas developed a few classes of generalized recurrent α-cosymplectic manifolds [10–12]. We cite [13, 14] for additional research on the aforementioned subject.

Assuming that $(\mathsf{M}, g), n = \dim \mathsf{M} > 3$ is connected semi Riemannian manifold of class C^∞ and represent by ∇ its Levi-Civita connection. We write the Riemannian-Christoffel curvature tensor R, the Weyl conformal curvature tensor $\mathcal{C}$, the conharmonic curvature tensor K [13], the projective curvature tensor P and the concircular curvature tensor $\tilde{\mathcal{C}}$ of (M, g) by

$$R(s_1, s_2)s_3 = \nabla_{s_1}\nabla_{s_2}s_3 - \nabla_{s_2}\nabla_{s_1}s_3 - \nabla_{[s_1,s_2]}s_3,$$

$$\begin{aligned}\mathcal{C}(s_1, s_2)s_3 &= R(s_1, s_2)s_3 \\ &+ \frac{1}{n-2}[S(s_1, s_3)s_2 - S(s_2, s_3)s_1 + g(s_1, s_3)Qs_2 - g(s_2, s_3)Qs_1] \\ &- \frac{\tau}{(n-1)(n-2)}[g(s_1, s_3)s_2 - g(s_2, s_3)s_1],\end{aligned}$$

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$$K(s_1,s_2)s_3 = R(s_1,s_2)s_3 - \frac{1}{n-2}[S(s_2,s_3)s_1 - S(s_1,s_3)s_2 + g(s_2,s_3)Qs_1 - g(s_1,s_3)Qs_2],$$

$$P(s_1,s_2)s_3 = R(s_1,s_2)s_3 - \frac{1}{n-2}[g(s_2,s_3)Qs_1 - g(s_1,s_3)Qs_2],$$

$$\tilde{C}(s_1,s_2)s_3 = R(s_1,s_2)s_3 - \frac{\tau}{n(n-2)}[g(s_1,s_3)s_2 - g(s_2,s_3)s_1],$$

respectively. In this scenario, Q is the Ricci operator as defined by $S(s_1,s_2) = g(Qs_1,s_2)$, $s_1,s_2,s_3 \in \Im_0^1(\mathsf{M})$, S is the Ricci tensor and $\tau = tr(S)$ is the scalar curvature.

The fundamental characteristics of curvature tensors and the idea of the liftings of tensor fields and connections to their tangent bundle was developed in [15]. In their study, Dida and Hathout [16] looked into Ricci soliton structures on tangent bundles of Riemannian manifolds. Numerous scholars have examined several connections and geometric structures on the tangent bundle and established their fundamental geometric features [17–22].

These works serve as our inspiration as we investigate the complete lift of Lorentzian α-Sasakian manifolds to tangent bundle TM. Additionally, we investigate the complete lift of ϕ-conformally flat, ϕ-conharmonically flat, ϕ-projectively flat and ϕ-concircularly flat Lorentzian α-Sasakian manifolds and derive (TM, g^C) as an η^C-Einstein manifold in each instance where g^C is the Lorentzian metric.

Notations. The notations below appear in several places throughout the text: Both $\Im_a^b(\mathsf{M})$ and $\Im_a^b(\mathsf{TM})$ stand for the set of all tensor fields of type (a,b) [23, 24].

1 *Preliminaries*

If there is a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η, and a Lorentzian metric g that satisfy $\forall s_1,s_2 \in \Im_0^1(\mathsf{M})$ [1, 2, 8, 25]

$$\eta(\xi) = -1, \tag{1}$$
$$\phi^2 = I + \eta \otimes \xi, \tag{2}$$
$$g(\phi s_1, \phi s_2) = g(s_1,s_2) + \eta(s_1)\eta(s_2), \tag{3}$$
$$g(s_1,\xi) = \eta(s_1), \tag{4}$$
$$\phi\xi = 0, \quad \eta(\phi s_1) = 0,$$

then the differentiable manifold M is said to be a Lorentzian α-Sasakian manifold.

The following relationships are also true in a Lorentzian α-Sasakian manifold M [26–30]

$$\nabla_{s_1}\xi = -\alpha\phi s_1, \tag{5}$$
$$(\nabla_{s_1}\eta)s_2 = -\alpha g(\phi s_1, s_2), \tag{6}$$

and M becomes η-Einstein if its Ricci tensor S is given by

$$S(s_1,s_2) = ag(s_1,s_2) + b\eta(s_1)\eta(s_2), \;\; \forall s_1,s_2 \in \Im_0^1(\mathsf{M}), \tag{7}$$

in above equations ∇ represents the covariant differentiation operator w.r.t. g and a, b are functions on M.

For curvature tensor R, we have [8]

$$\begin{aligned}
R(\xi, s_1)s_2 &= \alpha^2(g(s_1,s_2)\xi + \eta(s_2)s_1), &(8)\\
R(s_1,s_2)\xi &= \alpha^2(\eta(s_2)s_1 + \eta(s_1)s_2), &(9)\\
R(\xi,s_1)\xi &= \alpha^2(\eta(s_1)\xi + s_1), &(10)\\
S(s_1,\xi) &= (n-1)\alpha^2\eta(s_1), &(11)\\
Q\xi &= (n-1)\alpha^2\xi, &(12)\\
S(\xi,\xi) &= -(n-1)\alpha^2, &(13)\\
S(\phi s_1,\phi s_2) &= S(s_1,s_2) + (n-1)\alpha^2\eta(s_1)\eta(s_2), &(14)
\end{aligned}$$

where Q stands for the Ricci operator with $S(s_1,s_2) = g(Qs_1,s_2)$.

2 The complete lift from a Lorentzian α-Sasakian manifold to its tangent bundle

Let us consider a local coordinate system $(x^i), i = 1,...,n$ on differentiable manifold M and let (x^i,y^i), $i = 1,...,n$ be an induced local coordinate system on tangent bundle TM. If $s_1 = {s_1}^i\frac{\partial}{\partial x^i}$ is a local vector field on M, then its vertical and complete lifts in the term of partial differential equations are

$$\begin{aligned}
{s_1}^V &= {s_1}^i\frac{\partial}{\partial y^i},\\
{s_1}^C &= {s_1}^i\frac{\partial}{\partial x^i} + \frac{\partial {s_1}^i}{\partial x^j}y^j\frac{\partial}{\partial y^i}.
\end{aligned}$$

Let η, s_1 and ϕ, respectively, represent 1-form, vector field, and a tensor field of type (1,1). Denote the complete and vertical lifts of η, s_1 and ϕ by $\eta^C, {s_1}^C, \phi^C$ and $\eta^V, {s_1}^V, \phi^V$, respectively. Then by using mathematical operators on η, s_1 and ϕ, we have [15, 31]

$$\begin{aligned}
\eta^V({s_1}^C) = \eta^C({s_1}^V) = \eta(s_1)^V,\ \eta^C({s_1}^C) = \eta(s_1)^C,\\
\phi^V{s_1}^C = (\phi s_1)^V,\ \phi^C{s_1}^C = (\phi s_1)^C,\\
[s_1,s_2]^V = [{s_1}^C,{s_2}^V] = [{s_1}^V,{s_2}^C],\ [s_1,s_2]^C = [{s_1}^C,{s_2}^C],\\
\nabla^C_{{s_1}^C}{s_2}^C = (\nabla_{s_1}s_2)^C, \qquad \nabla^C_{{s_1}^C}{s_2}^V = (\nabla_{s_1}s_2)^V,
\end{aligned}$$

∇^C is used for the complete lift of ∇ on TM.

Using the complete lift on (1)–(7), we conclude

$$\begin{aligned}
\eta^C(\xi^C) &= \eta^V(\xi^V) = 0,\ \ \eta^C(\xi^V) = \eta^V(\xi^C) = -1, &(15)\\
(\phi^2)^C &= I + \eta^C\otimes\xi^V + \eta^V\otimes\xi^C, &(16)\\
g^C((\phi s_1)^C,(\phi s_2)^C) &= g^C({s_1}^C,{s_2}^C) + \eta^C({s_1}^C)\eta^V({s_2}^C)\\
&+ \eta^V({s_1}^C)\eta^C({s_2}^C),\\
g^C({s_1}^C,\xi^C) &= \eta^C({s_1}^C),\\
\phi^C\xi^C &= \phi^V\xi^V = \phi^C\xi^V = \phi^V\xi^C = 0, &(17)\\
\eta^C(\phi s_1)^C &= \eta^V(\phi s_1)^V = \eta^C(\phi s_1)^V = \eta^V(\phi s_1)^C = 0, &(18)
\end{aligned}$$

$\forall {s_1}^C, {s_2}^C \in \Im^1_0(\mathsf{TM})$, and

$$\begin{aligned}
\nabla^C_{{s_1}^C}\xi^C &= -\alpha(\phi s_1)^C,\\
(\nabla^C_{{s_1}^C}\eta^C){s_2}^C &= -\alpha g^C((\phi s_1)^C,{s_2}^C).
\end{aligned}$$

When the complete lift of Ricci tensor S holds for

$$S^C({s_1}^C, {s_2}^C) = ag^C({s_1}^C, {s_2}^C) + b\{\eta^C({s_1}^C)\eta^V({s_2}^C) + \eta^V({s_1}^C)\eta^C({s_2}^C)\},$$

then the Lorentzian α-Sasakian manifold M on TM is thought to be η^C-Einstein.

Taking the complete lift on (8)–(14), we infer

$$\begin{aligned}
R^C(\xi^C, {s_1}^C){s_2}^C &= \alpha^2\{g^C({s_1}^C, {s_2}^C)\xi^V + g^C({s_1}^V, {s_2}^C)\xi^V \\
&- \eta^C({s_2}^C){s_1}^V + \eta^V({s_2}^C){s_1}^C\}, \\
R^C({s_1}^C, {s_2}^C)\xi^C &= \alpha^2\{\eta^C({s_2}^C){s_1}^V + \eta^V({s_2}^C){s_1}^C \\
&- \eta^C({s_1}^C){s_2}^V + \eta^V({s_1}^C){s_2}^C\}, \\
R^C(\xi^C, {s_1}^C)\xi^C &= \alpha^2\{\eta^C({s_1}^C)\xi^V + \eta^V({s_1}^C)\xi^C + {s_1}^C\}, \\
S^C({s_1}^C, \xi^C) &= (n-1)\alpha^2\eta^C({s_1}^C), \\
(Q\xi)^C &= (n-1)\alpha^2\xi^C, \\
S^C(\xi^C, \xi^C) &= -(n-1)\alpha^2, \\
S^C((\phi s_1)^C, (\phi s_2)^C) &= S^C({s_1}^C, {s_2}^C) + (n-1)\alpha^2\{\eta^C({s_1}^C)\eta^V({s_2}^C) \\
&+ \eta^V({s_1}^C)\eta^C({s_2}^C)\},
\end{aligned}$$

where $S^C({s_1}^C, {s_2}^C) = g^C((Qs_1)^C, {s_2}^C)$ and $S^({s_1}^V, {s_2}^C) = g^C((Qs_1)^V, {s_2}^C)$.

3 Main Results

Definition 1. Consider a differentiable manifold $(\mathsf{M^n}, g)$ with $n > 3$. Then M is said to be

- ϕ-conformally flat if
$$\phi^2\mathcal{C}(\phi s_1, \phi s_2)\phi s_3 = 0, \tag{19}$$
- ϕ-conharmonically flat if
$$\phi^2 K(\phi s_1, \phi s_2)\phi s_3 = 0,$$
- ϕ-projectively flat provided
$$\phi^2 P(\phi s_1, \phi s_2)\phi s_3 = 0,$$
- ϕ-concircularly flat if
$$\phi^2\tilde{\mathcal{C}}(\phi s_1, \phi s_2)\phi s_3 = 0.$$

Theorem 1. Let M^n be ϕ-conformally flat Lorentzian α-Sasakian manifold and denote by TM its tangent bundle. Then (TM, g^C) is an η^C-Einstein manifold, g^C being the Lorentzian metric of TM.

Proof. For the given assumptions, we see that

$$\phi^2\mathcal{C}(\phi s_1, \phi s_2)\phi s_3 = 0 \Longleftrightarrow g(\mathcal{C}(\phi s_1, \phi s_2)\phi s_3, \phi s_4) = 0, \tag{20}$$

$\forall s_1, s_2, s_3, s_4 \in \Im^1_0(\mathsf{M})$.

Complete lifts on (1) produce

$$\begin{aligned}
\mathcal{C}^C({s_1}^C, {s_2}^C){s_3}^C &= R^C({s_1}^C, {s_2}^C){s_3}^C \\
&+ \frac{1}{n-2}[S^C({s_1}^C, {s_3}^C){s_2}^V + S^C({s_1}^V, {s_3}^C){s_2}^C \\
&- S^C({s_2}^C, {s_3}^C){s_1}^V - S^C({s_2}^V, {s_3}^C){s_1}^C \\
&+ g^C({s_1}^C, {s_3}^C)(Qs_2)^V + g^C({s_1}^V, {s_3}^C)(Qs_2)^C \\
&- g^C({s_2}^C, {s_3}^C)(Qs_1)^V - g^C({s_2}^V, {s_3}^C)(Qs_1)^C] \\
&- \frac{\tau}{(n-1)(n-2)}[g^C({s_1}^C, {s_3}^C){s_2}^V + g^C({s_1}^V, {s_3}^C){s_2}^C \\
&- g^C({s_2}^C, {s_3}^C){s_1}^V - g^C({s_2}^V, {s_3}^C){s_1}^C].
\end{aligned}$$

In view of (20) and (21) ϕ-conformally flat means

$$\begin{aligned}
& (g(R(\phi s_1,\phi s_2)\phi s_3,\phi s_4))^C \\
= & \frac{1}{n-2}[g^C((\phi s_2)^C,(\phi s_3)^C)S^C((\phi s_1)^V,(\phi s_4)^C) \\
+ & g^C((\phi s_2)^V,(\phi s_3)^C)S^C((\phi s_1)^C,(\phi s_4)^C) \\
- & g^C((\phi s_1)^C,(\phi s_3)^C)S^C((\phi s_2)^V,(\phi s_4)^C) \\
- & g^C((\phi s_1)^V,(\phi s_3)^C)S^C((\phi s_2)^C,(\phi s_4)^C) \\
+ & S^C((\phi s_2)^C,(\phi s_3)^C)g^C((\phi s_1)^V,(\phi s_4)^C) \\
+ & S^C((\phi s_2)^V,(\phi s_3)^C)g^C((\phi s_1)^C,(\phi s_4)^C) \\
- & S^C((\phi s_1)^C,(\phi s_3)^C)g^C((\phi s_2)^V,(\phi s_4)^C) \\
- & S^C((\phi s_1)^V,(\phi s_3)^C)g^C((\phi s_2)^C,(\phi s_4)^C)] \\
- & \frac{\tau}{(n-1)(n-2)}[g^C((\phi s_2)^C,(\phi s_3)^C)g^C((\phi s_1)^V,(\phi s_4)^C) \\
+ & g^C((\phi s_2)^V,(\phi s_3)^C)g^C((\phi s_1)^C,(\phi s_4)^C) \\
- & g^C((\phi s_1)^C,(\phi s_3)^C)g^C((\phi s_2)^V,(\phi s_4)^C) \\
- & g^C((\phi s_1)^V,(\phi s_3)^C)g^C((\phi s_2)^C,(\phi s_4)^C)]. \qquad (21)
\end{aligned}$$

Let $\{\varepsilon_i : i=1,...,n-1,\xi\}$ represent a local orthonormal basis. Then $\{(\phi\varepsilon_i)^C : i=1,...,n-1,(\phi\xi)^C\}$ is in TM.

Setting $s_1=s_4=\varepsilon_i$ in (21) produces

$$\begin{aligned}
& \sum_{i=1}^{n-1} g(R(\phi\varepsilon_i,\phi s_2)\phi s_3,\phi\varepsilon_i) \\
= & \frac{1}{n-2}\sum_{i=1}^{n-1}[g^C((\phi s_2)^C,(\phi s_3)^C)S^C((\phi\varepsilon_i)^V,(\phi\varepsilon_i)^C) \\
+ & g^C((\phi s_2)^V,(\phi s_3)^C)S^C((\phi\varepsilon_i)^C,(\phi\varepsilon_i)^C) \\
- & g^C((\phi\varepsilon_i)^C,(\phi s_3)^C)S^C((\phi s_2)^V,(\phi\varepsilon_i)^C) \\
- & g^C((\phi\varepsilon_i)^V,(\phi s_3)^C)S^C((\phi s_2)^C,(\phi\varepsilon_i)^C) \\
+ & S^C((\phi s_2)^C,(\phi s_3)^C)g^C((\phi\varepsilon_i)^V,(\phi\varepsilon_i)^C) \\
+ & S^C((\phi s_2)^V,(\phi s_3)^C)g^C((\phi\varepsilon_i)^C,(\phi\varepsilon_i)^C) \\
- & S^C((\phi\varepsilon_i)^C,(\phi s_3)^C)g^C((\phi s_2)^V,(\phi\varepsilon_i)^C) \\
- & S^C((\phi\varepsilon_i)^V,(\phi s_3)^C)g^C((\phi s_2)^C,(\phi\varepsilon_i)^C)] \\
- & \frac{\tau}{(n-1)(n-2)}\sum_{i=1}^{n-1}[g^C((\phi s_2)^C,(\phi s_3)^C)g^C((\phi\varepsilon_i)^V,(\phi\varepsilon_i)^C) \\
+ & g^C((\phi s_2)^V,(\phi s_3)^C)g^C((\phi\varepsilon_i)^C,(\phi\varepsilon_i)^C) \\
- & g^C((\phi\varepsilon_i)^C,(\phi s_3)^C)g^C((\phi s_2)^V,(\phi\varepsilon_i)^C) \\
- & g^C((\phi\varepsilon_i)^V,(\phi s_3)^C)g^C((\phi s_2)^C,(\phi\varepsilon_i)^C)]. \qquad (22)
\end{aligned}$$

In view of (15), (16), (17), (18) and (19), we infer

$$\sum_{i=1}^{n-1}(g(R(\phi\varepsilon_i,\phi s_2)\phi s_3,\phi\varepsilon_i))^C = S^C((\phi s_2)^C,(\phi s_3)^C) + g^C((\phi s_2)^C,(\phi s_3)^C), \quad (23)$$

$$\sum_{i=1}^{n-1} S^C((\phi\varepsilon_i)^C,(\phi\varepsilon_i))^C = \tau-(n-1)\alpha^2, \quad (24)$$

$$\sum_{i=1}^{n-1}[g^C((\phi\varepsilon_i)^C,(\phi s_3)^C)S^C((\phi s_2)^V,(\phi\varepsilon_i)^C) + g^C((\phi\varepsilon_i)^V,(\phi s_3)^C)S^C((\phi s_2)^C,(\phi\varepsilon_i)^C) = S^C((\phi s_2)^C,(\phi s_3)^C) \quad (25)$$

$$\sum_{i=1}^{n-1} g^C((\phi\varepsilon_i)^C,(\phi\varepsilon_i))^C = n-1, \quad (26)$$

and

$$\sum_{i=1}^{n-1}[g^C((\phi\varepsilon_i)^C,(\phi s_3)^C)g^C((\phi s_2)^V,(\phi\varepsilon_i)^C) + g^C((\phi\varepsilon_i)^V,(\phi s_3)^C)g^C((\phi s_2)^C,(\phi\varepsilon_i)^C) = g^C((\phi s_2)^C,(\phi s_3)^C). \quad (27)$$

Making use of (23)–(27) the equation (22) can be expressed as

$$S^C((\phi s_2)^C,(\phi s_3)^C) = L_1 g^C((\phi s_2)^C,(\phi s_3)^C), \quad (28)$$

where $L_1 = [\frac{\tau}{n-1}-(n-1)\alpha^2-(n-2)]$. Using (17) and (19), the equation (28) becomes

$$S^C({s_2}^C,{s_3}^C) = L_1 g^C({s_2}^C,{s_3}^C) + L_1\{\eta^C({s_2}^C)\eta^V({s_3}^C)+\eta^V({s_2}^C)\eta^C({s_3}^C)\}.$$

Thus (TM, g^C) is an η^C-Einstein manifold.

On the similar devices of Theorem 4.1 and using definition 1, we have

Theorem 2. Let M^n, $(n>3)$ be ϕ-conharmonically flat Lorentzian α-Sasakian manifold and TM be its tangent bundle. Then (TM, g^C) is an η^C-Einstein manifold.

Theorem 3. For any ϕ-projectively flat Lorentzian α-Sasakian manifold M^n $(n>3)$, (TM, g^C) is an η^C-Einstein manifold.

Theorem 4. For any ϕ-concircularly flat Lorentzian α-Sasakian manifold M^n $(n>3)$, (TM, g^C) will be an η^C-Einstein manifold.

4 Example

Assume a differentiable manifold $\mathsf{M} = \{(u,v,w) : u,v,w \in \Re^3, w>0\}$ and denote the L.I. vector fields on M by $\varsigma_1,\varsigma_2,\varsigma_3$ given by [32]

$$\varsigma_1 = \varsigma^{-w}\frac{\partial}{\partial v}, \varsigma_2 = \varsigma^{-w}(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}), \varsigma_3 = \alpha\frac{\partial}{\partial w} = \xi.$$

Further for 1-form η on M, one can write

$$g(\varsigma_1,\varsigma_2) = g(\varsigma_1,\varsigma_3) = g(\varsigma_2,\varsigma_3) = 0, \quad g(\varsigma_1,\varsigma_1) = g(\varsigma_2,\varsigma_2) = 1, \quad g(\varsigma_3,\varsigma_3) = -1$$

and

$$\eta(s_3) = g(s_3, \varsigma_3), \quad s_3 \in \Im_0^1(\mathsf{M}),$$

where g is the Lorentzian metric.

Suppose ϕ stands for $(1,1)$-tensor field satisfying

$$\phi\varsigma_1 = \varsigma_1, \quad \phi\varsigma_2 = \varsigma_2, \quad \phi\varsigma_3 = 0.$$

With the linearity of ϕ, one concludes $\eta(\varsigma_3) = -1$, $\phi^2\zeta_3 = s_3 + \eta(s_3)\varsigma_3$ and $g(\phi s_1, \phi s_2) = g(s_1, s_2) + \eta(s_1)\eta(s_2)$.

Thus, (for $\varsigma_3 = \xi$) M becomes Lorentzian almost paracontact metric manifold with Lorentzian almost paracontact metric structure (ϕ, ξ, η, g) on M.

Also,

$$[\varsigma_1, \varsigma_2] = 0, \quad [\varsigma_1, \varsigma_3] = \alpha\varsigma_1, \quad [\varsigma_2, \varsigma_3] = \alpha\varsigma_2.$$

The Koszul's formula is written as

$$\begin{aligned} 2g(\nabla_{\varsigma_1}\varsigma_2, s_3) &= Xg(s_2, s_3) + s_2 g(s_3, s_1) - \varsigma_3 g(s_1, s_2) \\ &- g(s_1, [s_2, s_3]) + g(s_2, [s_3, s_1]) + g(s_3, [s_1, s_2]), \end{aligned}$$

and we have [32]

$$\nabla_{\varsigma_1}\varsigma_1 = \alpha\varsigma_3, \quad \nabla_{\varsigma_1}\varsigma_3 = \alpha\varsigma_1, \quad \nabla_{\varsigma_2}\varsigma_2 = \alpha\varsigma_3, \quad \nabla_{\varsigma_2}\varsigma_3 = \alpha\varsigma_2, \tag{29}$$

$$\nabla_{\varsigma_1}\varsigma_2 = \nabla_{\varsigma_3}\varsigma_1 = \nabla_{\varsigma_3}\varsigma_2 = \nabla_{\varsigma_2}\varsigma_1 = \nabla_{\varsigma_3}\varsigma_3 = 0. \tag{30}$$

We can easily verify that

$$\begin{aligned} \nabla_{s_1}\xi &= -\alpha\phi s_1, \\ (\nabla_{s_1}\eta)s_2 &= -\alpha g(\phi s_1, s_2). \end{aligned}$$

Hence, M is a "Lorentzian α-Sasakian manifold".

Let us denote the complete and vertical lifts of $\varsigma_1, \varsigma_2, \varsigma_3$ on $T\mathsf{M}$ by $\varsigma_1^C, \varsigma_2^C, \varsigma_3^C$ and $\varsigma_1^V, \varsigma_2^V, \varsigma_3^V$. Further, suppose that g^C be the complete lift of a Riemannian metric g on TM holding

$$\begin{aligned} g^C({s_1}^V, \varsigma_3^C) &= (g^C(s_1, \varsigma_3))^V = (\eta(s_1))^V, \\ g^C({s_1}^C, \varsigma_3^C) &= (g^C(s_1, \varsigma_3))^C = (\eta(s_1))^C, \\ g^C(\varsigma_3^C, \varsigma_3^C) &= -1, \quad g^V({s_1}^V, \varsigma_3^C) = 0, \quad g^V(\varsigma_3^V, \varsigma_3^V) = 0 \end{aligned} \tag{31}$$

and so on.

Consider the $(1,1)$-tensor field ϕ and its complete and vertical lifts ϕ^C and ϕ^V by

$$\phi^V(\varsigma_1^V) = \varsigma_1^V, \quad \phi^C(\varsigma_1^C) = \varsigma_1^C.$$

$$\phi^V(\varsigma_2^V) = \varsigma_2^V, \quad \phi^C(\varsigma_2^C) = \varsigma_2^C,$$

$$\phi^V(\varsigma_3^V) = \phi^C(\varsigma_3^C) = 0.$$

Above equation produces

$$(\phi^2 X)^C = {s_1}^C + \eta^V(s_1)\varsigma_3^C + \eta^C(s_1)\varsigma_3^V, \tag{32}$$

$$\begin{aligned} g^C((\phi\varsigma_1)^C, (\phi\varsigma_2)^C) &= g^C(\varsigma_1^C, \varsigma_2^C) + (\eta(\varsigma_1))^C(\eta(\varsigma_2))^V \\ &+ (\eta(\varsigma_1))^V(\eta(\varsigma_2))^C. \end{aligned}$$

Thus, for $\varsigma_3 = \xi$ in (31)–(32), the structure $(\phi^C, \xi^C, \eta^C, g^C)$ is a Lorentzian almost paracontact metric structure on TM.

The Koszul's formula for ∇^C can be viewed as

$$\begin{aligned} 2g^C(\nabla^C_{\varsigma_1^C}\varsigma_2^C, s_3^C) &= X^C g^C(s_2^C, s_3^C) + s_2^C g^C(s_3^C, s_1^C) - \varsigma_3^C g^C(s_1^C, s_2^C) \\ &- g^C(s_1^C, [s_2^C, s_3^C]) + g^C(s_2^C, [s_3^C, s_1^C]) + g^C(s_3^C, [s_1^C, s_2^C]). \end{aligned}$$

Taking the complete lift on (29) and (30), we conclude

$$\begin{aligned} \nabla^C_{\varsigma_1^C}\varsigma_1^C &= \alpha\varsigma_3^C, \ \nabla^C_{\varsigma_1^C}\varsigma_3^C = \alpha\varsigma_1^C, \ \nabla^C_{\varsigma_2^C}\varsigma_2^C = \alpha\varsigma_3^C, \ \nabla^C_{\varsigma_2^C}\varsigma_3^C = \alpha\varsigma_2^C, \\ \nabla^C_{\varsigma_1^C}\varsigma_2^C &= \nabla^C_{\varsigma_3^C}\varsigma_1^C = \nabla^C_{\varsigma_3^C}\varsigma_2^C = \nabla^C_{\varsigma_2^C}\varsigma_1^C = \nabla^C_{\varsigma_3^C}\varsigma_3^C = 0. \end{aligned}$$

We can easily verify that

$$\begin{aligned} \nabla^C_{s_1^C}\xi^C &= -\alpha\phi^C s_1^C, \\ (\nabla^C_{s_1^C}\eta^C)s_2^C &= -\alpha g^C((\phi s_1)^C, s_2^C). \end{aligned}$$

Hence, $(\phi^C, \xi^C, \eta^C, g^C, \mathsf{TM})$ is a Lorentzian α-Sasakian manifold.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

All authors contributed equally to this work.

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