Bulletin of the Karaganda University. Mathematics Series, No. 3(119), 2025, pp. 85–96

https://doi.org/10.31489/2025M3/85-96 *Research article*

Some Generalized Fractional Hermite-Hadamard-Type Inequalities for $m-$Convex Functions

M. Bilal[1,2,3,*], A.R. Khan[3]

[1] *Govt. Dehli Inter Science College, Hussainabad, Karachi, Pakistan;*
[2] *Institute of Business Management, Korangi Creek, Karachi, Pakistan;*
[3] *University of Karachi, University Road, Karachi-75270, Pakistan*
(E-mail: mbilalfawad@gmail.com, mbilal.vf@iobm.edu.pk, asifrk@uok.edu.pk)

Fractional Hermite-Hadamard-type inequalities represent a significant area of study in convex analysis due to their extensive applications in mathematical and applied sciences. These inequalities provide powerful tools for estimating the integral mean of a convex function in terms of its values at the endpoints of a given interval. In this paper, we focus on the development and refinement of fractional Hermite-Hadamard-type inequalities for the class of twice differentiable m-convex functions. Utilizing advanced analytical techniques, such as Hölder's inequality and the power mean integral inequality, we derive new bounds that generalize existing results in the literature. These findings not only extend classical inequalities to a broader class of convex functions but also provide sharper and more versatile estimations. The presented results are expected to have significant implications in various mathematical domains, including fractional calculus, optimization, and mathematical modeling. This work contributes to the ongoing efforts to generalize and refine integral inequalities by incorporating fractional operators and broader convexity assumptions, offering a deeper understanding of the behavior of m-convex functions under fractional integration.

Keywords: integral inequality, fractional Hermite-Hadamard inequality, convex functions, $m-$convex functions, twice differentiable functions, Euler Beta function, Hölder's integral inequality, power mean integral inequality.

2020 Mathematics Subject Classification: 26A06, 26A51, 26D07, 26D99.

"All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."

– Hardy

Introduction

Let $\xi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function. Then

$$\xi\left(\frac{\varpi_1+\varpi_2}{2}\right) \leq \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2} \xi(\zeta)d\zeta \leq \frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}$$

is known in the literature as Hermite-Hadamard dual inequality [1]. If ξ is concave, then both inequalities hold in the reserved direction. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it easily follows from well-known Jensen's inequality.

*Corresponding Author. *E-mail: mbilalfawad@gmail.com*

The following research is supported by the Higher Education Commission of Pakistan under Indigenous PhD Fellowship for 5000 Scholars, HEC, (Phase-II) (PIN: 518-108309-2PS5-088).

Received: 31 August 2024; *Accepted:* 20 May 2025.

Hermite-Hadamard type inequalities play a significant role in the study of convex functions and have attracted considerable attention in mathematical analysis and its applications. These inequalities provide valuable estimates for the average value of a convex function in terms of its endpoint evaluations. Over the years, various generalizations and extensions have been developed to encompass broader classes of functions, including s-convex, h-convex, and m-convex functions and many more. For further study related to the topic we refer [2–4] to the interested readers.

The concept of m-convexity, introduced as a generalization of classical convexity, is particularly useful in optimization theory, economics, and applied analysis [5]. In 1984, Toader defined the class of $m-$convex functions [6] as:

Definition 1. A function $\xi : [0, \varpi_2] \to \mathbb{R}$ is called $m-$convex, if ξ satisfies

$$\xi(v\zeta_1 + m(1-v)\zeta_2) \leq v\xi(\zeta_1) + m(1-v)\xi(\zeta_2),$$

for all $\zeta_1, \zeta_2 \in [0, \varpi_2]$ and $m, v \in [0, 1]$.

Remark 1. If we put $m = 0$ and $m = 1$ in the above definition then $m-$convexity changes into Star-shaped [1] and classical convexity [7], respectively.

In parallel, the development of fractional calculus the study of integrals and derivatives of arbitrary (non-integer) order – has led to new avenues for generalizing classical inequalities. Fractional integrals, such as the Riemann–Liouville and Hadamard fractional integrals, have proven to be powerful tools in extending integral inequalities to fractional settings (for example see [8–10]).

By combining the frameworks of fractional calculus and m-convexity, researchers have established fractional Hermite-Hadamard type inequalities for m-convex functions, which provide sharper and more generalized bounds than their classical counterparts. These inequalities not only refine existing results but also open up possibilities for applications in diverse fields such as control theory, mathematical physics, signal processing, and differential equations (for further study see [11] and [12]).

Theorem 1. If $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $g \in L_q$, then $\xi g \in L_1$ and

$$\int |\xi(\zeta)g(\zeta)|d\zeta \leq \|\xi\|_p \|g\|_q, \tag{1}$$

where $\xi \in L_p$ if $\|\xi\|_p = \left(\int |\xi(\zeta)|^p d\zeta\right)^{\frac{1}{p}} < \infty$.

The above inequality is known as Hölder's inequality [13].

Remark 2. Note that Cauchy$-$Schwarz inequality would be obtained by taking $p = q = 2$. Also, if we put $q = 1$ and let $p \to \infty$, then we attain,

$$\int |\xi(\zeta)g(\zeta)|d\zeta \leq ||\xi||_\infty ||g||_1,$$

where $||\xi||_\infty$ stands for essential supremum of $|\xi|$, i.e.,

$$||\xi||_\infty = ess \sup_{\forall \zeta} |\xi(\zeta)|.$$

Another representation of Hölder's inequality is known in literature as Power mean integral inequality [14], defined as:

Theorem 2. If ξ and g are real valued functions defined on I with $|\xi|$ and $|\xi||g|^q$ are integrable on I then for $q \geq 1$, we have:

$$\int_{\varpi_1}^{\varpi_2} |\xi(\zeta)||g(\zeta)|d\zeta \leq \left(\int_{\varpi_1}^{\varpi_2} |\xi(\zeta)|d\zeta\right)^{1-\frac{1}{q}} \left(\int_{\varpi_1}^{\varpi_2} |\xi(\zeta)||g(\zeta)|^q d\zeta\right)^{\frac{1}{q}}. \tag{2}$$

Now, we are going to give some necessary definitions and mathematical results related to fractional calculus which will be used further in this article.

Definition 2. [15] Let $\xi \in L[\varpi_1, \varpi_2]$. The Riemann−Liouville integrals $J^{\alpha}_{\varpi_1+}\xi(\zeta)$ and $J^{\alpha}_{\varpi_2-}\xi(\zeta)$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{\varpi_1+}\xi(\zeta) = \frac{1}{\Gamma(\alpha)} \int\limits_{\varpi_1}^{\zeta} (\zeta - v)^{\alpha-1}\xi(v)dv, \quad \zeta > \varpi_1$$

and

$$J^{\alpha}_{\varpi_2-}\xi(\zeta) = \frac{1}{\Gamma(\alpha)} \int\limits_{\zeta}^{\varpi_2} (v - \zeta)^{\alpha-1}\xi(v)dv, \quad \zeta < \varpi_2,$$

respectively, where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u}u^{\alpha-1}du$ is the Gamma function.

Remark 3. Note that if we take $\alpha = 0$, then $J^{0}_{\varpi_1+}\xi(\zeta) = J^{0}_{\varpi_2-}\xi(\zeta) = \xi(\zeta)$ and if we take $\alpha = 1$, then the fractional integrals reduce to the classical one.

In 2013 Bhatti et al. proved the following three distinct results related to fractional Hermite-Hadamard-type inequality for the class of twice differentiable convex functions [16].

Theorem 3. Let $\xi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $|\xi''|$ is a convex function on I. Suppose that $\varpi_1, \varpi_2 \in I^{\circ}$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$, then the below stated inequality for fractional integrals with $\alpha > 0$ holds:

$$\begin{aligned} &\left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2 - \varpi_1)^{\alpha}} [J^{\alpha}_{\varpi_1+}\xi(\varpi_2) + J^{\alpha}_{\varpi_2-}\xi(\varpi_1)] \right| \\ &\quad \leq \frac{\alpha(\varpi_2 - \varpi_1)^2}{2(\alpha+1)(\alpha+2)} \left[\frac{|\xi''(\varpi_1)| + |\xi''(\varpi_2)|}{2} \right] \\ &\quad \leq \frac{(\varpi_2 - \varpi_1)^2}{(\alpha+1)} \beta(2, \alpha+1) \left[\frac{|\xi''(\varpi_1)| + |\xi''(\varpi_2)|}{2} \right], \end{aligned}$$

where β is the Euler Beta function.

Theorem 4. Let $\xi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $p \in \mathbb{R}$, $p > 1$ such that $|\xi''|^{\frac{p}{p-1}}$ is a convex function on I. Suppose that $\varpi_1, \varpi_2 \in I^{\circ}$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$, then the below stated inequality for fractional integrals with $\alpha > 0$ holds:

$$\begin{aligned} &\left| \frac{\xi(\varpi_1) + \xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2 - \varpi_1)^{\alpha}} [J^{\alpha}_{\varpi_1+}\xi(\varpi_2) + J^{\alpha}_{\varpi_2-}\xi(\varpi_1)] \right| \\ &\leq \frac{(\varpi_2 - \varpi_1)^2}{(\alpha+1)} \beta^{\frac{1}{p}}(p+1, \alpha p+1) \left[\frac{|\xi''(\varpi_1)|^q + |\xi''(\varpi_2)|^q}{2} \right]^{\frac{1}{q}}, \end{aligned}$$

where β is the Euler Beta function.

Theorem 5. Let $\xi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $q \geq 1$, $p > 1$ such that $|\xi''|^q$ is a convex function on I. Suppose that $\varpi_1, \varpi_2 \in I^{\circ}$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$, then the below stated inequality for fractional integrals with $\alpha > 0$ holds:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^{\alpha}}[J^{\alpha}_{\varpi_1+}\xi(\varpi_2) + J^{\alpha}_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{\alpha(\varpi_2-\varpi_1)^2}{4(\alpha+1)(\alpha+2)}\left[\left(\frac{2\alpha+4}{3\alpha+9}|\xi''(\varpi_1)|^q + \frac{\alpha+5}{3\alpha+9}|\xi''(\varpi_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{\alpha+5}{3\alpha+9}|\xi''(\varpi_1)|^q + \frac{2\alpha+4}{3\alpha+9}|\xi''(\varpi_2)|^q\right)^{\frac{1}{q}}\right].$$

The structure of this article unfolds as follows: In the subsequent section, we aim to establish three unique outcomes concerning fractional Hermite-Hadamard-type inequalities applicable to the category of twice differentiable $m-$convex functions. Our approach will leverage diverse techniques, encompassing Hölder's and power mean integral inequalities. These findings are anticipated to exhibit a broader scope compared to those presented in [16]. The third section will provide a concluding statement, while the final section will offer insights and future prospects for readers interested in further exploration.

1 Various Estimations of Right Bound of Fractional Hermite-Hadamard-type Inequalities for Twice Differentiable $m-$Convex Functions

In order to prove our main results we need to recall following lemma from [16].

Lemma 1. Let $\xi : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, the interior of I. Assume that $\varpi_1, \varpi_2 \in I^{\circ}$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$, then the below stated identity for fractional integrals with $\alpha > 0$ holds:

$$\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^{\alpha}}[J^{\alpha}_{\varpi_1+}\xi(\varpi_2) + J^{\alpha}_{\varpi_2-}\xi(\varpi_1)]$$
$$= \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int_0^1 v(1-v^{\alpha})[\xi''(v\varpi_1+(1-v)\varpi_2) + \xi''((1-v)\varpi_1+v\varpi_2)]dv,$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$.

Now, we are going to state and prove of our new results related to fractional Hermite-Hadamard-type inequalities for twice differentiable $m-$convex functions.

Theorem 6. Let $\xi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I°, the interior of I. Assume that $\varpi_1, \varpi_2 \in I^{\circ}$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1, \varpi_2]$. If $|\xi''|$ is $m-$convex on I for some $m \in (0,1]$, then the below stated inequality for fractional integrals with $\alpha > 0$ holds:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^{\alpha}}[J^{\alpha}_{\varpi_1+}\xi(\varpi_2) + J^{\alpha}_{\varpi_2-}\xi(\varpi_1)]\right| \leq \frac{\alpha(\varpi_2-\varpi_1)^2}{6(\alpha+1)(\alpha+3)}$$
$$\times\left[|\xi''(\varpi_1)| + |\xi''(\varpi_2)| + m\frac{(\alpha+5)}{2(\alpha+2)}\left(\left|\xi''\left(\frac{\varpi_1}{m}\right)\right| + \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|\right)\right].$$

Proof. By using Lemma 1 and the property of absolute value, we have,

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^{\alpha}}[J^{\alpha}_{\varpi_1+}\xi(\varpi_2) + J^{\alpha}_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int_0^1 |v(1-v^{\alpha})|[|\xi''(v\varpi_1+(1-v)\varpi_2)| + |\xi''((1-v)\varpi_1+v\varpi_2)|]dv. \quad (3)$$

As we have $|\xi''|$ is a $m-$convex function, so we can take

$$\left|\xi''\left(v\varpi_1+(1-v)\varpi_2\right)\right| \le v\left|\xi''\left(\varpi_1\right)\right| + m(1-v)\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|$$

and

$$\left|\xi''\left((1-v)\varpi_1+v\varpi_2\right)\right| \le m(1-v)\left|\xi''\left(\frac{\varpi_1}{m}\right)\right| + v|\xi''(\varpi_2)|.$$

Utilizing the above two results, (3) becomes

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\le \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int_0^1 \left[v^2(1-v^\alpha)|\xi''(\varpi_1)| + mv(1-v)(1-v^\alpha)\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|\right.$$
$$\left. + mv(1-v)(1-v^\alpha)\left|\xi''\left(\frac{\varpi_1}{m}\right)\right| + v^2(1-v^\alpha)|\xi''(\varpi_2)|\right] dv.$$

After arranging and using the following facts the result of Theorem 6 is accomplished.

$$\int_0^1 v^2(1-v^\alpha)dv = \frac{\alpha}{3(\alpha+3)}$$

and

$$\int_0^1 v(1-v^\alpha)(1-v)dv = \frac{\alpha(\alpha+5)}{6(\alpha+2)(\alpha+3)}.$$

Remark 4. The following well-known results would be captured as special cases of our obtained result by varying different values of m and α:

1. If we choose $m=1$ in Theorem 6, then we get first inequality of Theorem 3.
2. If we choose $\alpha=m=1$ in Theorem 6, then we get Hermite-Hadamard-type inequality for twice differentiable convex function [17].

Corollary 1. If we choose $\alpha=1$ in Theorem 6, then we get the following Hermite-Hadamard-type inequality for twice differentiable $m-$convex function:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right|$$
$$\le \frac{(\varpi_2-\varpi_1)^2}{48}\left[|\xi''(\varpi_1)| + |\xi''(\varpi_2)| + m\left(\left|\xi''\left(\frac{\varpi_1}{m}\right)\right| + \left|\xi''\left(\frac{\varpi_2}{m}\right)\right|\right)\right].$$

Theorem 7. Let $\xi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I°, the interior of I. Assume that $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1,\varpi_2]$. If $|\xi''|^q$ is $m-$convex on I for some

$m \in (0,1]$ and $q \geq 1$ then the following inequality for fractional integrals with $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ holds:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\beta^{\frac{1}{p}}(p+1, \alpha p+1)$$
$$\times\left[\left(\frac{|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + |\xi''(\varpi_2)|^q}{2}\right)^{\frac{1}{q}}\right],$$

where β is the Euler Beta function.

Proof. By using Lemma 1 and the property of absolute value, we have

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int_0^1 |v(1-v^\alpha)|[|\xi''(v\varpi_1+(1-v)\varpi_2)| + |\xi''((1-v)\varpi_1+v\varpi_2)|]dv. \tag{4}$$

Applying (1) to $\int_0^1 |v(1-v^\alpha)||\xi''(v\varpi_1+(1-v)\varpi_2)|dv$ and $\int_0^1 |v(1-v^\alpha)||\xi''((1-v)\varpi_1+v\varpi_2)|dv$ implies

$$\int_0^1 |v(1-v^\alpha)||\xi''(v\varpi_1+(1-v)\varpi_2)|dv$$
$$\leq \left(\int_0^1 |v(1-v^\alpha)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |\xi''(v\varpi_1+(1-v)\varpi_2)|^q dv\right)^{\frac{1}{q}}$$

and

$$\int_0^1 |v(1-v^\alpha)||\xi''((1-v)\varpi_1+v\varpi_2)|dv$$
$$\leq \left(\int_0^1 |v(1-v^\alpha)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |\xi''((1-v)\varpi_1+v\varpi_2)|^q dv\right)^{\frac{1}{q}}.$$

As we have $|\xi''|^q$ is a $m-$convex function, so we can take

$$\left|\xi''(v\varpi_1+(1-v)\varpi_2)\right|^q \leq v\left|\xi''(\varpi_1)\right|^q + m(1-v)\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q$$

and

$$\left|\xi''((1-v)\varpi_1+v\varpi_2)\right|^q \leq m(1-v)\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + v|\xi''(\varpi_2)|^q.$$

Utilizing the above four results, (4) becomes

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\left(\int_0^1 v^p(1-v^\alpha)^p dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(|\xi''(\varpi_1)|^q\int_0^1 vdv+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int_0^1(1-v)dv\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q\int_0^1(1-v)dv+|\xi''(\varpi_2)|^q\int_0^1 vdv\right)^{\frac{1}{q}}\right].$$

After using the following facts, the result of Theorem 7 is accomplished.

$$\int_0^1 vdv=\int_0^1(1-v)dv=\frac{1}{2}$$

and

$$\int_0^1 v^p(1-v^\alpha)^p dt\leq\int_0^1 v^p(1-v)^{\alpha p}dv=\beta(p+1,\alpha p+1).$$

Remark 5. Following well-known results would be captured as special cases of our obtained result by varying different values of m and α:

1. If we choose $m=1$ in Theorem 7, then we get Theorem 4.
2. If we choose $\alpha=m=1$ in Theorem 7, then we get Theorem 10 of [18].

Corollary 2. Under the assumptions of the Theorem 7,

1. If we put $p=q=2$, then we get the result obtained by using Cauchy− Schwarz integral inequality as:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq\frac{(\varpi_2-\varpi_1)^2}{2^{\frac{3}{2}}(\alpha+1)}\beta^{\frac{1}{2}}(3,2\alpha+1)$$
$$\times\left[\left(|\xi''(\varpi_1)|^2+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^2\right)^{\frac{1}{2}}+\left(m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^2+|\xi''(\varpi_2)|^2\right)^{\frac{1}{2}}\right],$$

   where β is the Euler Beta function.

2. If we put $q=1$ and $p=\infty$, then we get the result involving essential supremum norm as:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2)+J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\leq\frac{(\varpi_2-\varpi_1)^2}{4(\alpha+1)}||X||_\infty\left[|\xi''(\varpi_1)|+m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|+m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|+|\xi''(\varpi_2)|\right],$$

where $||X||_\infty = ess\sup\limits_{v\in[0,1]} \int\limits_0^1 v(1-v)^\alpha$.

3. If we choose $\alpha = 1$, then we get the following Hermite-Hadamard-type inequality for twice differentiable $m-$convex function:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int\limits_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right| \le \frac{(\varpi_2-\varpi_1)^2}{4}\beta^{\frac{1}{p}}(p+1,p+1)$$
$$\times\left[\left(\frac{|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + |\xi''(\varpi_2)|^q}{2}\right)^{\frac{1}{q}}\right].$$

Theorem 8. Let $\xi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I°, the interior of I. Assume that $\varpi_1, \varpi_2 \in I^\circ$ with $\varpi_1 < \varpi_2$ and $\xi'' \in L[\varpi_1,\varpi_2]$. If $|\xi''|^q$ is $m-$convex on I for some $m \in (0,1]$ and $q \ge 1$ then the following inequality for fractional integrals with $\alpha > 0$ holds:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\le \frac{\alpha(\varpi_2-\varpi_1)^2}{4(\alpha+1)(\alpha+2)(3(\alpha+3))^{\frac{1}{q}}}\left[\left(2(\alpha+2)|\xi''(\varpi_1)|^q + m(\alpha+5)\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(m(\alpha+5)\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + 2(\alpha+2)|\xi''(\varpi_2)|^q\right)^{\frac{1}{q}}\right].$$

Proof. By using Lemma 1 and the property of absolute value, we have

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1+}\xi(\varpi_2) + J^\alpha_{\varpi_2-}\xi(\varpi_1)]\right|$$
$$\le \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\int\limits_0^1 |v(1-v^\alpha)|[|\xi''(v\varpi_1+(1-v)\varpi_2)| + |\xi''((1-v)\varpi_1+v\varpi_2)|]dv. \tag{5}$$

Applying (2) to $\int\limits_0^1 |v(1-v^\alpha)||\xi''(v\varpi_1+(1-v)\varpi_2)|dv$ and $\int\limits_0^1 |v(1-v^\alpha)||\xi''((1-v)\varpi_1+v\varpi_2)|dv$ implies

$$\int\limits_0^1 |v(1-v^\alpha)||\xi''(v\varpi_1+(1-v)\varpi_2)|dv$$
$$\le \left(\int\limits_0^1 v(1-v^\alpha)dv\right)^{1-\frac{1}{q}}\left(\int\limits_0^1 v(1-v^\alpha)|\xi''(v\varpi_1+(1-v)\varpi_2)|^q dv\right)^{\frac{1}{q}}$$

and

$$\int\limits_0^1 |v(1-v^\alpha)||\xi''((1-v)\varpi_1+v\varpi_2)|dv$$
$$\le \left(\int\limits_0^1 v(1-v^\alpha)dv\right)^{1-\frac{1}{q}}\left(\int\limits_0^1 v(1-v^\alpha)|\xi''((1-v)\varpi_1+v\varpi_2)|^q dv\right)^{\frac{1}{q}}.$$

Since $|\xi''|^q$ is an $m-$convex function, so we can take

$$\left|\xi''\left(v\varpi_1+(1-v)\varpi_2\right)\right|^q \leq v\left|\xi''\left(\varpi_1\right)\right|^q + m(1-v)\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q$$

and

$$\left|\xi''\left((1-v)\varpi_1+v\varpi_2\right)\right|^q \leq m(1-v)\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + v|\xi''(\varpi_2)|^q.$$

Utilizing the above four results, (5) becomes

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\varpi_2-\varpi_1)^\alpha}[J^\alpha_{\varpi_1^+}\xi(\varpi_2)+J^\alpha_{\varpi_2^-}\xi(\varpi_1)]\right|$$

$$\leq \frac{(\varpi_2-\varpi_1)^2}{2(\alpha+1)}\left(\int_0^1 v(1-v^\alpha)dv\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(|\xi''(\varpi_1)|^q\int_0^1 v^2(1-v^\alpha)dv + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\int_0^1 v(1-v)(1-v^\alpha)dv\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q\int_0^1 v(1-v^\alpha)dv + |\xi''(\varpi_2)|^q\int_0^1 v^2(1-v)dv\right)^{\frac{1}{q}}\right].$$

After arranging and using the following facts the result of Theorem 8 is accomplished.

$$\int_0^1 v(1-v^\alpha)dv = \frac{\alpha}{2(\alpha+2)},$$

$$\int_0^1 v^2(1-v^\alpha)dv = \frac{\alpha}{3(\alpha+3)}$$

and

$$\int_0^1 v(1-v)(1-v^\alpha)dv = \frac{\alpha(\alpha+5)}{6(\alpha+2)(\alpha+3)}.$$

Remark 6. Following well-known results would be captured as special cases of our obtained result by varying different values of m and α:

1. If we choose $m=1$ in Theorem 8, then we get Theorem 5.
2. If we choose $\alpha=m=1$ in Theorem 8, then we get Theorem 8 of [18].

Corollary 3. If we choose $\alpha=1$ in Theorem 8, then we get the following Hermite-Hadamard-type inequality for twice differentiable $m-$convex function:

$$\left|\frac{\xi(\varpi_1)+\xi(\varpi_2)}{2} - \frac{1}{\varpi_2-\varpi_1}\int_{\varpi_1}^{\varpi_2}\xi(\zeta)d\zeta\right| \leq \frac{(\varpi_2-\varpi_1)^2}{24(2)^{\frac{1}{q}}}$$

$$\times\left[\left(|\xi''(\varpi_1)|^q + m\left|\xi''\left(\frac{\varpi_2}{m}\right)\right|^q\right)^{\frac{1}{q}} + \left(m\left|\xi''\left(\frac{\varpi_1}{m}\right)\right|^q + |\xi''(\varpi_2)|^q\right)^{\frac{1}{q}}\right].$$

2 Conclusion

The fractional Hermite-Hadamard inequality stands out as one of the most renowned within the realm of inequalities, boasting numerous generalizations across different classes of convex functions in existing literature. In this article, we present its extension for twice differentiable $m-$convex functions. Section 1 unveils three distinct findings concerning the estimated right bound of the fractional Hermite-Hadamard inequality in an absolute sense for twice differentiable $m-$convex functions. Here, we employ various methodologies, including Hölder's and Power mean integral inequalities. While some of these results are novel, others have been previously documented in the articles [16–18]. The final section is dedicated to providing remarks and offering future avenues of exploration for interested readers.

Now, we are going to summarize the results of Section 1 in Table 1.

Table 1

Result Summary of Section 1

S. No	m	α	**Results**	**Found in**
1	1	$-$	FHHTI for Ordinary Convex Functions	[16]
2	$-$	1	HHTI for $m-$Convex Functions	This Article
3	1	1	HHTI for Ordinary Convex Functions	[17, 18]

In the preceding table, the abbreviations FHHTI and HHTI refer to the Fractional Hermite-Hadamard type inequality and the Hermite-Hadamard type inequality, respectively, while the symbol "$-$" indicates validity for any value.

Now we are going to give some remarks and future ideas related to our stated results.

3 Remarks and Future Ideas

1. All the inequalities given in this article can be stated in the reverse direction for concave functions using the simple relation that ξ is concave if and only if ξ is convex.
2. One may also work on Fejér inequality by introducing weights in fractional Hermite-Hadamard inequality.
3. One may do similar work by using various distinct classes of convex functions.
4. One may try to state all the results given in this article for the discrete case.
5. One may also state all the results given in this article for Multi-dimensions.
6. One can extend this work to time scale domain or Quantum Calculus.
7. One can try to attain this work for Fuzzy theory.
8. One can try to work for finding refined bounds of all results.

Acknowledgments

The following research is supported by the Higher Education Commission of Pakistan under Indigenous PhD Fellowship for 5000 Scholars, HEC, (Phase-II) (PIN: 518-108309-2PS5-088).

Moreover, the authors would like to express sincere appreciation to the anonymous referees for their careful reading, insightful comments, and constructive suggestions, which have greatly improved the quality of this manuscript. The authors also acknowledge the assistance of artificial intelligence tools, which were used for language refinement and formatting support.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Set, E., Özdemir, M.E., Sarikaya, M.Z., & Karakoc, F. (2017). Hermite–Hadamard-type inequalities for (α, m)−convex functions via fractional integrals. *Moroccan J. Pure and Appl. Anal., 3*(1), 15–21. https://doi.org/10.1515/mjpaa-2017-0002

2 Bilal, M., & Khan, A.R. (2021). On Some New Hermite–Hadamard Dual Inequalities. *JMCMS, 16*(7), 93–98. https://doi.org/10.26782/jmcms.2021.07.00008

3 Bilal, M., & Khan, A.R. (2021). New Generalized Hermite–Hadamard Inequalities for p-convex functions in the mixed kind. *EJPAM, 14*(3), 863–880. https://doi.org/10.29020/nybg.ejpam.v14i3.4015

4 Bilal, M., & Khan, A.R. (2024). Refinement of Hermite–Hadamard-type inequalities for convex functions with applications. *J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math., 31*(1), 33–48. https://doi.org/10.7468/jksmeb.2024.31.1.33

5 Lara, T., Quintero, R., & Rosales, E. (2017). m-convexity and functional equations. *Moroccan J. Pure and Appl. Anal., 3*(1), 56–62. https://doi.org/10.1515/mjpaa-2017-0005

6 Toader, G. (1984). Some generalizations of the convexity. *Proc. Colloq. Aprox. Optim.*, 329–338.

7 Alomari, M., & Darus, M. (2008). Hadamard-type inequalities for s-convex functions. *Int. Math. Forum, 3*(40), 1965–1975.

8 Lakhdari, A., Budak, H., Mlaiki, N., Meftah, B., & Abdeljawad, T. (2025). New insights on fractal–fractional integral inequalities: Hermite–Hadamard and Milne estimates. *Chaos, Solitons and Fractals, 193*, 116087. https://doi.org/10.1016/j.chaos.2025.116087

9 Lakhdari, A., Mohsin, B.B., Jarad, F., Xu, H., & Meftah, B. (2024). A parametrized approach to generalized fractional integral inequalities: Hermite–Hadamard and Maclaurin variants. *Journal of King Saud University - Science, 36*(11), 103523. https://doi.org/10.1016/j.jksus.2024.103523

10 Li, H., Meftah, B., Saleh, W., Xu, H., Kiliçman, A., & Lakhdari, A. (2024). Further Hermite–Hadamard-type inequalities for fractional integrals with exponential kernels. *Fractal and Fractional, 8*(6), 345. https://doi.org/10.3390/fractalfract8060345

11 Bayraktar, B., & Kudaev, V.C. (2019). Some new integral inequalities for (s, m)-convex and (α, m)-convex functions. *Bulletin of the Karaganda University. Mathematics Series,* 2(94), 15–25. https://doi.org/10.31489/2019m2/15-25

12 Özcan, S. (2020). Hermite–Hadamard-type inequalities for m-convex and (α, m)-convex functions. *J. Inequal. Appl., 2020*, Article 175. https://doi.org/10.1186/s13660-020-02442-5

13 Khan, A.R., Khan, I.U., & Muhammad, S. (2021). Hermite–Hadamard-type fractional integral inequalities for s-convex functions of mixed kinds. *TMCS, 1*(1), 25–37.

14 Özcan, S., & Işcan, I. (2019). Some new Hermite–Hadamard-type inequalities for s-convex functions and their applications. *Journal of Inequalities and Applications, 2019*, 201. https://doi.org/10.1186/s13660-019-2151-2

15 Noor, M.A., & Awan, M.U. (2013). Some integral inequalities for two kinds of convexities via fractional integrals. *TJMM, 5*(2), 129–136.

16 Dragomir, S.S., Bhatti, M.I., Iqbal, M., & Muddassar, M. (2015). Some new Hermite–Hadamard's type fractional integral inequalities. *JCAA, 18*(4), 655–661.

17 Özdemir, M.E., Avci, M., & Set, E. (2010). On some inequalities of Hermite–Hadamard-type via m-convexity. *Appl. Math. Lett., 23*(9), 1065–1070. https://doi.org/10.1016/j.aml.2010.04.037

18 Hussain, M., Bhatti, M.I., & Iqbal, M. (2009). Hadamard-type inequalities for s-convex functions I. *Punjab University Journal of Mathematics, 41*, 51–60.

*Author Information**

Muhammad Bilal (*corresponding author*) — Doctor of Mathematics, PhD, Lecturer, Department of Mathematics, Govt. Dehli Inter Science College, Hussainabad, Karachi, Pakistan; Adjunct Faculty Member, Department of Mathematics and Statistics, Institute of Business Management, Korangi Creek, Karachi, Pakistan; e-mail: *mbilalfawad@gmail.com, mbilal.vf@iobm.edu.pk*; https://orcid.org/0000-0002-5518-6370

Asif Raza Khan — Doctor of Mathematics, PhD, Associate Professor, Department of Mathematics, University of Karachi-75270, Karachi, Pakistan; e-mail: *asifrk@uok.edu.pk*; https://orcid.org/0000-0002-4700-4987

*Authors' names are presented in the order: first name, middle name, and last name.