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Almost quasi-Urbanik structures and theories

B.Sh. Kulpeshov[1,2,*], S.V. Sudoplatov[3]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Kazakh British Technical University, Almaty, Kazakhstan;*
[3] *Sobolev Institute of Mathematics, Novosibirsk, Russia*
(E-mail: b.kulpeshov@kbtu.kz, sudoplat@math.nsc.ru)

The notions of almost quasi-Urbanik structures and theories, and studied possibilities for the degrees of quasi-Urbanikness, both for existential and universal cases were introduced. Links of these characteristics and their possible values are described. These values for structures of unary predicates, equivalence relations, linearly ordered, preordered and spherically ordered structures and theories as well as for strongly minimal ones, and for some natural operations including disjoint unions and compositions of structures and theories were studied. A series of examples illustrates possibilities of these characteristics.

Keywords: almost quasi-Urbanik structure, almost quasi-Urbanik theory, degree of quasi-Urbanikness.

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Introduction

The property of quasi-Urbanikness allows to clarify and describe structural properties in various classes of structures and theories, including strongly minimal ones [1, 2]. These properties can be classified using natural semantic and syntactic characteristics. A series of results on these characteristics are obtained in general [3], for abelian groups [4], for variations of rigidity in general [5] and for ordered structures [6], etc.

In the present paper we continue to study related characteristics introducing the notions of almost quasi-Urbanik structures and theories, and their existential and universal degrees. Possibilities of these degrees are described both in general and for a series of natural structures and theories including structures and theories of unary predicates, equivalence relations, ordered structures and theories, strongly minimal structures and theories, disjoint unions and compositions of structures and theories. We illustrate possibilities of degrees by a series of examples.

The paper is organized as follows. The notions of almost quasi-Urbanik structures and theories, degrees and their spectra are described in general, for unary predicates, and equivalence relations are described in Section 1. In Section 2, degrees of quasi-Urbanikness are described for ordered theories including spherically ordered and some preordered ones. Degrees of quasi-Urbanikness and links for dimensions are studied in Section 3. In Sections 4 and 5, we describe possibilities of degrees of quasi-Urbanikness for disjoin unions and E-definable compositions, respectively. In Section 6, we discuss some general operators transforming a given structure into quasi-Urbanik one.

*Corresponding author. *E-mail: b.kulpeshov@kbtu.kz*

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1 Almost quasi-Urbanik structures, their theories and degrees

Let L be a countable first-order language. Throughout we consider L-structures and their complete elementary theories; and we use standard model-theoretic notions and notations [7–10].

Following [1], a theory T is called *strongly minimal* if for any formula $\varphi(x,\bar{a})$ of language obtained by adding parameters of $\overline{a}$ (in a model $\mathcal{M}\models T$) to the language of T, either $\varphi(x,\bar{a})$, or $\neg\varphi(x,\bar{a})$ has finitely many solutions.

Following [11], for $n\in\omega\setminus\{0\}$ and a set A, an element b is called *n-algebraic* over A if $a\in\mathrm{acl}(A)$ and it is witnessed by a formula $\varphi(x,\overline{a})$, for $\overline{a}\in A$, with at most n solutions. The set of all n-algebraic elements over A is denoted by $\mathrm{acl}_n(A)$. If $A=\mathrm{acl}_n(A)$, then A is called *n-algebraically* closed. A type p is *n-algebraic* if it is realized by at most n tuples only, i.e., $\deg(p)\leq n$. The complete n-algebraic types $p(x)\in S(A)$ are exactly ones of the form $\mathrm{tp}(a/A)$, where a is n-algebraic over A, i.e., with $\deg(a/A)\leq n$. Here $\deg(a/A)=k\leq n$ defines the *n-degree* $\deg_n(a/A)$ of $\mathrm{tp}(a/A)$ and of a over A. If $\mathrm{acl}(A)=\mathrm{acl}_n(A)$ then minimal such n is called the *degree of algebraization* over the set A and it is denoted by $\deg_{\mathrm{acl}}(A)$. If that n does not exist, then we put $\deg_{\mathrm{acl}}(A)=\infty$. The supremum of values $\deg_{\mathrm{acl}}(A)$ with respect to all sets A of given theory T is denoted by $\deg_{\mathrm{acl}}(T)$ and called the *degree of algebraization* of the theory T.

Following [2], theories T with $\deg_{\mathrm{acl}}(T)=1$, i.e., with defined $\mathrm{cl}_1(A)$ for any set A of T, are called *quasi-Urbanik*, and the models $\mathcal{M}$ of T are *quasi-Urbanik*, too.

Remark 1. Notice that if a structure $\mathcal{M}$ is quasi-Urbanik it does not guarantee that its theory $T=\mathrm{Th}(\mathcal{M})$ is quasi-Urbanik, too. Indeed, let $\mathcal{M}$ be a strongly minimal structure consisting of infinitely many two-element equivalence classes $E(a)$. Marking one element a in each E-class by a constant c_a, we obtain a syntactically rigid structure $\mathcal{M}'$, with definable $b\in E(a)\setminus\{a\}$ by formulae $E(x,c_a)\wedge\neg x\approx c_a$. At the same time $\mathcal{M}'$ has an elementary extension $\mathcal{N}$ with some unmarked E-classes. These E-classes fail the quasi-Urbanikness of T.

Definition 1. A theory T is called *almost quasi-Urbanik*, if some expansion of T by finitely many constants is quasi-Urbanik, and the models $\mathcal{M}$ of T are *almost quasi-Urbanik*, too. If a finite set A of constants produces a quasi-Urbanik expansion T_A of T then we say that A *witnesses* that T is almost quasi-Urbanik.

The least cardinality of the witnessing set A is called the *quasi-Urbanik $\exists$-degree* of T and it is denoted by $\deg^{\exists}_{\mathrm{qU}}(T)$. If these finite sets A do not exist, we put $\deg^{\exists}_{\mathrm{qU}}(T)=\infty$. The minimal cardinality $n\in\omega$ such that each set A of cardinality n produces the quasi-Urbanik theory T_A is called the *quasi-Urbanik $\forall$-degree* of T and it is denoted by $\deg^{\forall}_{\mathrm{qU}}(T)$. If such n does not exist, then we put $\deg^{\forall}_{\mathrm{qU}}(T)=\infty$. Similarly it is transformed to the models $\mathcal{M}$ of T with quasi-Urbanik $\exists$-degrees $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M})$ and $\forall$-degrees $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M})$.

Clearly, for any theory T, $\deg^{\exists}_{\mathrm{qU}}(T)=0$ iff $\deg^{\forall}_{\mathrm{qU}}(T)=0$, and iff T is quasi-Urbanik. Thus, by the definition, any quasi-Urbanik theory is almost quasi-Urbanik.

Besides, for any theory T,

$$\deg^{\exists}_{\mathrm{qU}}(T)\leq\deg^{\forall}_{\mathrm{qU}}(T)\qquad(1)$$

implying that if T is not almost quasi-Urbanik then $\deg^{\exists}_{\mathrm{qU}}(T)=\deg^{\forall}_{\mathrm{qU}}(T)=\infty$, and vice versa.

The following example shows that the difference in the inequality (1) can be arbitrary:

Example 1. Let $\mathcal{M}$ be a structure of an equivalence relation E, $T=\mathrm{Th}(\mathcal{M})$. Clearly, T is quasi-Urbanik iff $\mathcal{M}$ has 0, 1 or infinitely many one-element E-classes, 0 or infinitely many two-element E-classes, and does not have finite E-classes with at least three elements, producing $\deg^{\exists}_{\mathrm{qU}}(T)=\deg^{\forall}_{\mathrm{qU}}(T)=0$. In particular, $\mathcal{M}$ with zero, one or infinitely many one-element E-classes, zero or infinitely many two-element E-classes, and without n-element E-classes, for $n\geq 3$, is quasi-Urbanik.

A finite value $\deg^{\exists}_{\mathrm{qU}}(T)$ means that we can collect a finite set A containing $m-1$ elements in singletons $E(a)$, if there are $m\in\omega\setminus\{0,1\}$ these singletons, a finite set B containing m elements in pairwise distinct two-element E-classes $E(b)$, if there are $m \in \omega \setminus \{0\}$ these E-classes, and a finite set C containing $n-1$ elements in each E-class $E(a)$ of finite cardinality $n \geq 3$, obtaining $\deg^{\exists}_{\rm qU}(T) = |A| + |B| + |C|$. Here $\deg^{\exists}_{\rm qU}(T) = |C|$ if there are 0, 1 or infinitely many one-element E-classes, and there are 0 or infinitely many two-element E-classes.

At the same time, if $\mathcal{M}$ is not quasi-Urbanik, then $\deg^{\forall}_{\rm qU}(T)$ is finite iff $\mathcal{M}$ is finite. In such a case if $\mathcal{M}$ contains k singletons $E(a)$ and m two-element E-classes, then $\deg^{\forall}_{\rm qU}(T) = k-1$, if $\mathcal{M}$ consists of E-singletons, and $\deg^{\forall}_{\rm qU}(T) = k+2m-1$, if $\mathcal{M}$ consists of one-element and two-element E-classes, and if $\mathcal{M}$ contains n-element E-classes, for $n \geq 3$, then $\deg^{\forall}_{\rm qU}(T) = |M|-1$.

In view of Example 1 we have the following theorem describing possibilities of quasi-Urbanik degrees:

Theorem 1. For any $\mu, \nu \in (\omega\setminus\{0\}) \cup \{\infty\}$ with $\mu \leq \nu$ there is a theory $T_{\mu,\nu}$ such that $\deg^{\exists}_{\rm qU}(T_{\mu,\nu}) = \mu$ and $\deg^{\forall}_{\rm qU}(T_{\mu,\nu}) = \nu$.

For a theory T we denote by $\deg_{2,{\rm qU}}(T)$ the pair $\big(\deg^{\exists}_{\rm qU}(T), \deg^{\forall}_{\rm qU}(T)\big)$ of quasi-Urbanik degrees for T.

In view of the inequality (1) and Theorem 1 the set

$$\mathrm{DEG}_{2,{\rm qU}} = \{(0,0)\} \cup \{(\mu,\nu) \in ((\omega \setminus \{0\}) \cup \{\infty\})^2 \mid \mu \leq \nu\} \qquad (2)$$

collects the spectrum of all possibilities for $\deg_{2,{\rm qU}}(T)$.

For a family $\mathcal{T}$ of theories we denote by $\mathrm{DEG}_{2,{\rm qU}}(\mathcal{T})$ the restriction of $\mathrm{DEG}_{2,{\rm qU}}$ to the family of theories in $\mathcal{T}$:

$$\mathrm{DEG}_{2,{\rm qU}}(\mathcal{T}) = \{\deg_{2,{\rm qU}}(T) \mid T \in \mathcal{T}\}.$$

The operator $\mathrm{DEG}_{2,{\rm qU}}(\cdot)\colon \mathcal{T} \mapsto \mathrm{DEG}_{2,{\rm qU}}(\mathcal{T})$ is monotone: indeed, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ then we have $\mathrm{DEG}_{2,{\rm qU}}(\mathcal{T}_1) \subseteq \mathrm{DEG}_{2,{\rm qU}}(\mathcal{T}_2)$. Hence, if $\mathrm{DEG}_{2,{\rm qU}}(\mathcal{T}_1) = \mathrm{DEG}_{2,{\rm qU}}$ and $\mathrm{DEG}_{2,{\rm qU}}(\mathcal{T}_1) = \mathrm{DEG}_{2,{\rm qU}}$ then

$$\mathrm{DEG}_{2,{\rm qU}}(\mathcal{T}_2) = \mathrm{DEG}_{2,{\rm qU}}.$$

A natural *question* arises on a description of spectra $\mathrm{DEG}_{2,{\rm qU}}(\mathcal{T})$ for various families of theories. Below we will give partial answers to this question.

Similarly to theories, for any structure $\mathcal{M}$, $\deg^{\exists}_{\rm qU}(\mathcal{M}) = 0$ iff $\deg^{\forall}_{\rm qU}(\mathcal{M}) = 0$, and iff $\mathcal{M}$ is quasi-Urbanik. Thus, by the definition any quasi-Urbanik structure is almost quasi-Urbanik.

Besides, for any structure $\mathcal{M}$,

$$\deg^{\exists}_{\rm qU}(\mathcal{M}) \leq \deg^{\forall}_{\rm qU}(\mathcal{M}) \qquad (3)$$

implying that if $\mathcal{M}$ is not almost quasi-Urbanik then $\deg^{\exists}_{\rm qU}(\mathcal{M}) = \deg^{\forall}_{\rm qU}(\mathcal{M}) = \infty$. Example 1 also illustrates that the difference in the inequality (3) can be arbitrary.

For any model $\mathcal{M}$ of a theory T we have:

$$\deg^{\exists}_{\rm qU}(\mathcal{M}) \leq \deg^{\exists}_{\rm qU}(T) \qquad (4)$$

and

$$\deg^{\forall}_{\rm qU}(\mathcal{M}) \leq \deg^{\forall}_{\rm qU}(T). \qquad (5)$$

Indeed, if a set A of constants produces quasi-Urbanik theory T_A then its model $\mathcal{M}_A$ is quasi-Urbanik, too. At the same time, as the following example shows, the inequalities (4) and (5) can be strict.

Example 2. Let $\mathcal{M}$ be a strongly minimal structure of an equivalence relation E consisting of infinitely many n-element E-classes such that there is an E-class $E(a)$ elements of which are not marked by constants and all elements in $M \setminus E(a)$ are marked by constants. We have $\deg^{\exists}_{\rm qU}(\mathcal{M}) = n-1$, witnessed by the set $E(a) \setminus \{a\}$, whereas $\deg^{\exists}_{\rm qU}({\rm Th}(\mathcal{M})) = \infty$ since new E-classes in strict elementary extensions of $\mathcal{M}$ fail the quasi-Urbanikness.

If $\mathcal{M}_0$ is an elementary substructure of $\mathcal{M}$ which does not contain $E(a)$ then $\mathcal{M}_0$ is quasi-Urbanik, with $\deg^{\forall}_{\rm qU}(\mathcal{M}_0) = \deg^{\exists}_{\rm qU}(\mathcal{M}_0) = 0$, whereas for $T = {\rm Th}(\mathcal{M}) = {\rm Th}(\mathcal{M}_0)$, $\deg^{\forall}_{\rm qU}(T) = \deg^{\exists}_{\rm qU}(T) = \infty$.

Example 2 illustrates that there are (almost) quasi-Urbanik structures theories of which are not almost quasi-Urbanik.

The list of inequalities (4) and (5) is extended by the following:

$$\deg^{\exists}_{\rm qU}(\mathcal{M}) \leq \deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M}). \qquad (6)$$

Indeed, the inequality (6) holds for any structure $\mathcal{M}$ since ${\rm dcl}(A) = M$ implies that ${\rm dcl}(B) = M = {\rm acl}(B)$ for any $B \supseteq M$, i.e. $\mathcal{M}_A$ is quasi-Urbanik, with $\deg^{\exists}_{\rm qU}(\mathcal{M}) \leq |A| = \deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$, where A witnesses the value $\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$.

The inequality (6) can be arbitrarily strict since any unar $\mathcal{M}$ with a successor function $s(x)$ is quasi-Urbanik, with $\deg^{\exists}_{\rm qU}(\mathcal{M})$, whereas that unar can have arbitrarily many connected components. The finite number of connected components equals $\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M})$, and if there are infinitely many connected components, then $\deg^{\exists\text{-synt}}_{\rm rig}(\mathcal{M}) = \infty$.

Similarly to the inequality (6) we have the inequality:

$$\deg^{\forall}_{\rm qU}(\mathcal{M}) \leq \deg^{\forall\text{-synt}}_{\rm rig}(\mathcal{M}).$$

The following assertion shows that if a theory T is almost quasi-Urbanik, with finite degrees, then the equalities in (4) and (5) hold:

Proposition 1. If a theory T is almost quasi-Urbanik, then each model $\mathcal{M}$ of T is almost quasi-Urbanik, too, with $\deg^{\exists}_{\rm qU}(\mathcal{M}) = \deg^{\exists}_{\rm qU}(T)$. Moreover, if $\deg^{\forall}_{\rm qU}(T)$ is finite, then we have $\deg^{\forall}_{\rm qU}(\mathcal{M})$ is finite, too, with $\deg^{\forall}_{\rm qU}(\mathcal{M}) = \deg^{\forall}_{\rm qU}(T)$.

Proof. Let $\deg^{\exists}_{\rm qU}(T) = n \in \omega$. Then T admits an expansion T_A by a set A of constants, with minimal cardinality n, such that T_A is quasi-Urbanik: $\deg^{\exists}_{\rm qU}(T_A) = 0$. Since a model $\mathcal{M}$ of T is expansible till a model $\mathcal{M}_A$ of T_A and the property $\deg^{\exists}_{\rm qU}(T) = n$ is expressed syntactically containing a description that $(n-1)$-element sets do not produce the quasi-Urbanikness, it is satisfies in $\mathcal{M}$ implying $\deg^{\exists}_{\rm qU}(\mathcal{M}) = n$. Similar arguments witness that if $\deg^{\forall}_{\rm qU}(T) = n \in \omega$, then $\deg^{\forall}_{\rm qU}(\mathcal{M}) = \deg^{\forall}_{\rm qU}(T)$ for any $\mathcal{M} \models T$. □

Let Σ_1 be a signature of both unary predicate symbols and constant symbols.

The following theorem describes the behavior of almost quasi-Urbanikness of theories in the signature Σ_1.

Theorem 2. Let T be a theory of a signature Σ_1, $\mathcal{M} \models T$. Then the following conditions hold:

1) T is quasi-Urbanik iff each algebraic 1-type over $\emptyset$ has a unique realization;

2) T is almost quasi-Urbanik iff T has finitely many algebraic 1-types $p_1, \ldots, p_n$ over $\emptyset$ with at least two realizations; here $\deg^{\exists}_{\rm qU}(T) = \sum\limits_{i=1}^{n}(|p_i(\mathcal{M})| - 1)$;

3) $\deg^{\forall}_{\rm qU}(T) > 0$ is finite iff $\mathcal{M}$ is finite and has an algebraic 1-type $p \in S(\emptyset)$ with at least two realizations; here $\deg^{\forall}_{\rm qU}(T) = |M| - 1$.

Proof. Without loss of generality we assume that constant symbols are replaced by unary predicates with unique solutions. In view of the signature Σ_1 there are no links between elements and algebraic sets are defined by Boolean combinations of given unary predicates such that these Boolean combinations have finitely many solutions. Thus, the quasi-Urbanikness means that these complete Boolean combinations defining $\mathrm{acl}(\emptyset)$ defines singletons producing Item 1.

The almost quasi-Urbanikness of T means that algebraic 1-types p become definable after fixing all their realizations except one for each type p. It confirms Item 2.

If $\mathrm{deg}^{\forall}_{\mathrm{qU}}(T) \in \omega \setminus \{0\}$, then T is almost quasi-Urbanik and is not quasi-Urbanik. Using 1) and 2) we find an algebraic 1-type $p \in S(\emptyset)$ with at least two realizations. Now $\mathcal{M}$ is finite and the $(|M|-1)$-element subsets of M confirm the value $\mathrm{deg}^{\exists}_{\mathrm{qU}}(T)$ since the smaller quantity can not cover universally $|p(\mathcal{M})| - 1$ realizations of p. □

In view of the equality (2) and Theorem 2, we have the following:

Corollary 1. Let $\mathcal{T}$ be the family of theories in signatures of the form Σ_1. Then

$$\mathrm{DEG}_{2,\mathrm{qU}}(\mathcal{T}) = \mathrm{DEG}_{2,\mathrm{qU}}.$$

Clearly, any theory of a finite structure is almost quasi-Urbanik. At the same time, as the following example shows, there are almost quasi-Urbanik theories of infinite structures which are not quasi-Urbanik.

Example 3. Let $\mathcal{M}$ be a countable structure of an equivalence relation E with one two-element E-class $E_0 = \{a, b\}$ and two infinite E-classes E_a and E_b such that $\mathcal{M}$ is supplied by a binary relation $R = \{(a, a') \mid a' \in E_a\} \cup \{(b, b') \mid b' \in E_b\}$. For $T = \mathrm{Th}(\mathcal{M})$, we have $\mathrm{deg}^{\forall}_{\mathrm{qU}}(T) = 1$ since $\mathrm{dcl}(\emptyset) = \emptyset$, $E_0 = \mathrm{acl}(\emptyset)$, and $M = \mathrm{dcl}(\{d\})$ for any element $d \in M$, producing $\mathrm{dcl}(A) = \mathrm{acl}(A)$ for any nonempty $A \subseteq M$. $\mathrm{deg}^{\exists}_{\mathrm{qU}}(T) = 1$, too. Thus, $\mathrm{deg}_{2,\mathrm{qU}}(T) = (1, 1)$. The theory T is ω-categorical and ω-stable with Morley rank 1 and Morley degree 2: $\mathrm{MR}(T) = 1$, $\mathrm{deg}(T) = 2$.

Below we will show that natural values $\mathrm{deg}^{\forall}_{\mathrm{qU}}(T) \geq 1$ can not be realized in the class of strongly minimal theories, i.e. Morley characteristics in Example 3 are minimally possible.

2 Spectra of almost quasi-Urbanikness for ordered structures and their theories

Example 4. Let $\mathcal{M}$ be a structure of an equivalence relation E expanded by a linear order on the quotient $\mathcal{M}/E$, i.e., $\mathcal{M}$ is a preordered set by a preorder $\leq$ such that maximal antichains form the equivalence relation E such that elements in distinct E-classes are $\leq$-comparable. Besides, $E = \leq \cap \geq$.

Clearly, $T = \mathrm{Th}(\mathcal{M})$ is quasi-Urbanik iff E has either one-element or infinite E-classes. Moreover, any linearly ordered structure $\mathcal{M}$ is quasi-Urbanik, i.e. $\mathrm{deg}_{2,\mathrm{qU}}(T) = (0, 0)$.

Since elements of each E-class $E(a)$ are connected by automorphisms over sets of elements in other E-classes, the possibilities of values $\mathrm{deg}^{\exists}_{\mathrm{qU}}(\mathcal{M})$ and $\mathrm{deg}^{\forall}_{\mathrm{qU}}(\mathcal{M})$ repeat ones in Example 1. Here, if $\mathcal{M}$ has finitely many one-element E-classes then all these E-classes are contained in $\mathrm{dcl}(\emptyset)$.

In particular, $\mathrm{Th}(\mathcal{M})$ is almost quasi-Urbanik with $\mathrm{deg}^{\exists}_{\mathrm{qU}}(\mathrm{Th}(\mathcal{M})) > 0$ iff $\mathcal{M}$ has a finite E-class with at least two elements and there are finitely many these E-classes, and $\mathrm{deg}^{\exists}_{\mathrm{qU}}(\mathrm{Th}(\mathcal{M})) \in \omega \setminus \{0\}$ iff $\mathcal{M}$ is quasi-Urbanik, or $\mathcal{M}$ is finite with some E-class containing at least two elements.

In view of Example 4 we have the following modification of Theorem 1:

Theorem 3. Let $\mathcal{T}_{\mathrm{po}}$ be the family of theories of preordered structures. Then

$$\mathrm{DEG}_{2,\mathrm{qU}}(\mathcal{T}_{\mathrm{po}}) = \mathrm{DEG}_{2,\mathrm{qU}}.$$

Definition 2. [12,13]. The following generalization of linear and circular orders produces an *n-ball*, or *n-spherical*, or *n-circular* order relation, for $n \geq 2$, which is described by an n-ary relation K_n satisfying the following conditions:

(nso1) for any even permutation σ on $\{1, 2, \ldots, n\}$,

$$\forall x_1, \ldots, x_n \left(K_n(x_1, x_2, \ldots, x_n) \to K_n\left(x_{\sigma(1)}, x_{\sigma(2)} \ldots, x_{\sigma(n)}\right)\right);$$

(nso2) $$\forall x_1, \ldots, x_n \Bigg((K_n(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \wedge$$

$$\wedge K_n(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)) \leftrightarrow \bigvee_{1 \leq k < l \leq n} x_k \approx x_l \Bigg)$$

for any $1 \leq i < j \leq n$;

(nso3) $$\forall x_1, \ldots, x_n \Bigg(K_n(x_1, \ldots, x_n) \to$$

$$\to \forall t \left(\bigvee_{i=1}^{n} K_n(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) \right) \Bigg);$$

(nso4) $$\forall x_1, \ldots, x_n (K_n(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \vee$$

$$\vee K_n(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)), \ 1 \leq i < j \leq n.$$

The axioms above produce linear orders K_2 and circular orders K_3.

Structures $\mathcal{M} = \langle M, K_n \rangle$ with n-spherical orders K_n are called *n-spherically ordered sets*, or *n-spherical orders*, too. If a structure $\mathcal{M}$ contains a n-spherical order, then $\mathcal{M}$ is called a *n-spherically ordered structure*, or simply *spherically ordered structure* if n is known.

An n-spherically ordered set $\langle A, K_n \rangle$, where $n \geq 2$, is called *dense* if it contains at least two elements and for each $(a_1, a_2, a_3, \ldots, a_n) \in K_n$ with $a_1 \neq a_2$ there is $b \in A \setminus \{a_1, a_2, \ldots, a_n\}$ such that

$$\models K_n(a_1, b, a_3, \ldots, a_n) \wedge K_n(b, a_2, a_3, \ldots, a_n).$$

Following [14], n-spherical orders K_n on infinite sets M witness the strict order property producing unstable structures $\langle M, K_n \rangle$, since fixing $n-2$ distinct coordinates $a_1, \ldots, a_{n-2}$ in the relation K_n, we obtain a linear order on $M \setminus \{a_1, \ldots, a_{n-2}\}$.

As any linearly ordered structure is quasi-Urbanik, we obtain the following:

Theorem 4. Any n-spherically ordered structure $\mathcal{M}$ has an almost quasi-Urbanik theory T with $\deg^{\forall}_{\rm qU}(T) \leq n - 2$.

Remark 2. The inequality in Theorem 4 can be strict and can produce the equality. Indeed, if $\mathcal{M}$ is a dense spherical order, then $\mathcal{M}$ is quasi-Urbanik, with $\deg^{\forall}_{\rm qU}({\rm Th}(\mathcal{M})) = 0$, since ${\rm dcl}(A) = {\rm acl}(A) = A$ since the theory ${\rm Th}(\mathcal{M})$ has quantifier elimination [13] without possibilities to define new algebraic elements, outside A.

At the same time if $\mathcal{M}'$ is an expansion of $\mathcal{M}$ by a unary predicate P_m containing $m > 1$ elements, then we have to fix $n-2$ arbitrary elements in M producing a quasi-Urbanik expansion that implies $\deg^{\forall}_{\rm qU}({\rm Th}(\mathcal{M}')) = n - 2$. Here the value $\deg^{\exists}_{\rm qU}({\rm Th}(\mathcal{M}'))$ can vary depending on m: it is equal $m - 1$ for $m - 1 < n - 2$, and equals $n - 2$ if $m - 1 \geq n - 2$.

In view of the equality (2), Theorem 4 and Remark 2, we have:

Theorem 5. Let $\mathcal{T}_{\rm so}$ be the family of theories of spherically ordered structures. Then

$${\rm DEG}_{2,{\rm qU}}(\mathcal{T}_{\rm so}) = \{(0,0)\} \cup \{(m,n) \mid m, n \in \omega \setminus \{0\}, m \leq n\} =$$

$$= {\rm DEG}_{2,{\rm qU}} \setminus \{(\mu, \infty) \mid \mu \in (\omega \setminus \{0\}) \cup \{\infty\}\}.$$

The following examples illustrate possibilities for $\deg_{2,\mathrm{qU}}(T)=(m,n)$ for extensions of spherically ordered structures by two new elements.

Example 5. Let $\mathcal{M}_{1,2}$ be a countable structure of an equivalence relation E with one two-element E-class $E_0=\{a_1,a_2\}$ and one infinite E-class E_1 such that E_1 is supplied by a binary relation $<$ of dense linear order without endpoints, and $\mathcal{M}$ is supplied by a ternary relation R_3 consisting of triples (a_1,b_1,b_2) and $(a_2.b_2.b_1)$ for any $b_1<b_2$. For the theory $T_{1,2}=\mathrm{Th}(\mathcal{M}_{1,2})$, we have $\deg^{\exists}_{\mathrm{qU}}(T_{1,2})=1$ and $\deg^{\forall}_{\mathrm{qU}}(T_{1,2})=2$ since $\mathrm{dcl}(\emptyset)=\emptyset$, $E_0=\mathrm{acl}(\emptyset)$, $E_0=\mathrm{dcl}(\{a_i\})=\mathrm{dcl}(\{a_i,b_1\})=\mathrm{dcl}(\{b_1,b_2\})$, $i=1,2$, $b_1,b_2\in E_1$, $b_1\neq b_2$, producing $\mathrm{dcl}(A)=\mathrm{acl}(A)$ for any $A\subseteq M_{1,2}$ with $|A|\geq 2$. Thus $\deg_{2,\mathrm{qU}}(T_{1,2})=(1,2)$.

Now we modify the theory $T_{1,2}$ replacing in $\mathcal{M}_{1,2}$ the linear order $<$ by a dense n-spherical order K_n [13], $n\geq 3$, reduced to the strict one K^*_n, i.e. the reduction of the spherical order to the set of tuples with pairwise distinct coordinates. The relation K_n divides the set of n-tuples with pairwise distinct coordinates in E_1 into two parts such that the complement $\overline{K_n}$ of K_n in E_1^n equals the set of odd permutations of tuples in K^*_n. Instead of R_3 we consider the $(n+1)$-ary relation R_{n+1} collecting tuples $(a_1,\bar b)$, $\bar b\in K^*_n$, and tuples $(a_2,\bar b)$, $\bar b\in\overline{K_n}$. For the obtained structure $\mathcal{M}_{1,n}$ and its theory $T_{1,n}$, we have $\deg^{\exists}_{\mathrm{qU}}(T_{1,n})=1$ and $\deg^{\forall}_{\mathrm{qU}}(T_{1,n})=n$ since $\mathrm{dcl}(\emptyset)=\emptyset$, $E_0=\mathrm{acl}(\emptyset)$, $E_0=\mathrm{dcl}(\{a_i\})=\mathrm{dcl}(\bar b)$, $i=1,2$, $\bar b\in E_1$ with pairwise distinct coordinates, $l(\bar b)=n$, producing $\mathrm{dcl}(A)=\mathrm{acl}(A)$ for any $A\subseteq M_{1,n}$ with $|A|\geq n$. Thus, $\deg_{2,\mathrm{qU}}(T_{1,n})=(1,n)$.

3 Spectra of almost quasi-Urbanikness and relative dimensions of algebraic closures for strongly minimal theories

Theorem 6. For any strongly minimal theory T either $\deg^{\forall}_{\mathrm{qU}}(T)=0$ or $\deg^{\forall}_{\mathrm{qU}}(T)=\infty$.

Proof. Let $\deg^{\forall}_{\mathrm{qU}}(T)=m>0$, $m\in\omega$. Taking a big saturated model $\mathcal{M}$, we find an algebraic set $A\subset M$ with $|A|=n>1$ such that A is definable by a complete formula $\varphi(x,\bar a)$ such that for any m-tuple $\bar b\subset M$, A is divided into singletons by formulae $\psi_i(x,\bar b)$, $i=1,\ldots,n$. Since the tuples $\bar b$ are arbitrary, we can fix $m-1$ coordinates and take one mobile coordinate, say m-th one, realizing the unique non-algebraic type $p(y)$ over A. Moreover, as the model $\mathcal{M}$ is big enough, elements of A are connected by $\bar a$-automorphisms and realizations of $p(y)$ are connected by A-automorphisms. Since A is finite and its elements are connected by $\bar a$-automorphisms we can connect elements of A by a fixed formula ψ_i with infinitely many realizations of $p(y)$ such that ψ_i-images with respect to these realizations are unique. As T is strongly minimal, $\mathcal{M}$ can not be divided into two infinite definable parts. Thus, ψ_i-preimages of elements of A should be intersected contradicting the uniqueness of ψ_i-images. $\square$

In view of the equality (2), Theorem 6 and strongly minimal realizations of quasi-Urbanik degrees in Example 1, we obtain the following:

Corollary 2. Let $\mathcal{T}_{\mathrm{sm}}$ be the family of strongly minimal theories. Then

$$\mathrm{DEG}_{2,\mathrm{qU}}(\mathcal{T}_{\mathrm{sm}})=\{(0,0)\}\cup\{(\mu,\infty)\mid\mu\in(\omega\setminus\{0\})\cup\{\infty\}\}.$$

Example 6. Let $\mathcal{M}=\langle M,s^1\rangle$ be a structure, where $s(x)$ is the successor function, and $Th(\mathcal{M})$ has the following axioms:

$$A_1:=\forall z\exists!\,t\ s(z)=t,$$

$$A_2:=\exists x_1\exists x_2[x_1\neq x_2\wedge\forall y_1\ s(y_1)\neq x_1\wedge\forall y_2\ s(y_2)\neq x_2\wedge\forall t(t\neq x_1\wedge t\neq x_2\to\exists z\ s(z)=t)],$$

$$A_3:=\forall x_1\forall x_2\forall t_1\forall t_2[s(x_1)=t_1\wedge s(x_2)=t_2\to(x_1\neq x_2\leftrightarrow t_1\neq t_2)].$$

Thus, M consists of two disjoint copies of $\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. It can be established that $\mathrm{Th}(\mathcal{M})$ is a strongly minimal theory. Further, we have: $\mathrm{dcl}(\emptyset)=\emptyset$, $\mathrm{acl}(\emptyset)=M$, and $\mathrm{dcl}(A) = \mathrm{acl}(A)$ for any non-empty $A \subseteq M$. Thus, $\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)$ is infinite, $Th(\mathcal{M})$ is almost quasi-Urbanik and non-quasi-Urbanik.

We say that a set A is *definably independent* if $a \notin \mathrm{dcl}(A \setminus \{a\})$ for any $a \in A$. Denote by $\dim_{dcl}(A)$ the cardinality of maximal definably independent subset of A.

Observe that in Example 6 $\dim_{dcl}(\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)) = 1$.

Consider for every $m \geq 2$ the following sentence:

$$B_m := \exists x_1 \ldots \exists x_m [\wedge_{1 \leq i < j \leq m} x_i \neq x_j \wedge \wedge_{i=1}^{m} \forall y\, s(y) \neq x_i \wedge \forall t (\wedge_{i=1}^{m} t \neq x_i \rightarrow \exists z\, s(z) = t)].$$

If we consider the structure $\mathcal{M} = \langle M, s^1 \rangle$ with axioms A_1, B_m and A_3, then we have $\dim_{dcl}(\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)) = m - 1$.

Consider for every $m \geq 2$ the following sentence:

$$C_m := \exists x_1 \ldots \exists x_m [\wedge_{1 \leq i < j \leq m} x_i \neq x_j \wedge \wedge_{i=1}^{m} \forall y\, s(y) \neq x_i].$$

If we consider the structure $\mathcal{M} = \langle M, s^1 \rangle$ with axioms A_1, A_3 and $\{C_m \mid m \geq 1\}$, then we lose the strong minimality, and $Th(\mathcal{M})$ is an ω-stable quasi-Urbanik theory of Morley rank 2 with $\mathrm{acl}(\emptyset) = \emptyset$.

Thus, we have the following proposition:

Proposition 2. For every natural $m \geq 1$ there exists an almost quasi-Urbanik strongly minimal theory such that $\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)$ is infinite and $\dim_{dcl}(\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)) = m$.

The following example shows that if T is not almost quasi-Urbanik, then $\dim_{dcl}(\mathrm{acl}(\emptyset) \setminus \mathrm{dcl}(\emptyset)) = m$ can be infinite.

Example 7. Let $\mathcal{M} = \langle M, E^2 \rangle$ be a strongly minimal structure, where E is an equivalence relation partitioning M into infinitely many n-element E-classes for some $n \geq 3$. Let $\mathcal{M}'$ be an expansion of $\mathcal{M}$ by marking exactly one element from each E-class by a constant. Then we have that both $\mathrm{acl}_{\mathcal{M}'}(\emptyset) \setminus \mathrm{dcl}_{\mathcal{M}'}(\emptyset)$ and $\dim_{dcl}(\mathrm{acl}_{\mathcal{M}'}(\emptyset) \setminus \mathrm{dcl}_{\mathcal{M}'}(\emptyset))$ are infinite, but $\mathrm{Th}(\mathcal{M}')$ is not almost quasi-Urbanik, cf. Example 1. At the same time, following Example 5, the theory $\mathrm{Th}(\mathcal{M}')$ admits a cyclification producing a quasi-Urbanik expansion.

The following example produces a similar effect, as in Example 7, in the class of simple unstable theories.

Example 8. Consider a predicate language L consisting of two unary predicate symbols P and Q, two binary predicate symbols E and R, expanded by countably many constant symbols c_n, $n \in \omega$. We construct a countable structure $\mathcal{M}$ with $M = P \dot{\cup} Q$, where P and Q are countable, such that E is an equivalence relation dividing P on three-element E-classes E_n with $c_n \in E_n$, $n \in \omega$, and having one-element E-classes on Q. Now we interpret R as a random symmetric binary relation connecting each element of Q with one element in each $E(c_n) \setminus \{c_n\}$, $n \in \omega$.

We have $\mathrm{dcl}_{\mathcal{M}}(\emptyset) = \{c_n \mid n \in \omega\}$, $\mathrm{acl}_{\mathcal{M}}(\emptyset) = \bigcup\limits_{n \in \omega} E(c_n)$, with infinite $\mathrm{acl}_{\mathcal{M}}(\emptyset) \setminus \mathrm{dcl}_{\mathcal{M}}(\emptyset)$. Thus the structure $\mathcal{M}$ is not quasi-Urbanik. At the same time $\mathcal{M}$ is almost quasi-Urbanik, since for any element $a \in Q$ the expansion $\mathcal{M}'$ of $\mathcal{M}$ by the constant c_a for this element allows to define all elements of P: $\mathrm{dcl}_{\mathcal{M}'}(\emptyset) = \mathrm{acl}_{\mathcal{M}'}(\emptyset) = P \cup \{c_a\}$. Moreover, for any $A \subseteq M$, $\mathrm{dcl}_{\mathcal{M}'}(A) = \mathrm{acl}_{\mathcal{M}'}(A) = \mathrm{dcl}_{\mathcal{M}'}(\emptyset) \cup (Q \cap A)$.

Finally we observe that the theory $\mathrm{Th}(\mathcal{M})$ is not almost quasi-Urbanik, since $\mathcal{M}$ has elementary extensions with three-element E-classes which are not marked by constants, and finitely many new constants can not reduce algebraic closures to definable ones for these E-classes.

4 Degrees of quasi-Urbanikness for disjoint unions of structures and their theories

In this section we describe possibilities for degrees of quasi-Urbanikness for disjoint unions of structures and their theories. This description correlates with similar description for degrees of rigidity [5].

Definition 3. [15] The *disjoint union* $\bigsqcup\limits_{n\in\omega}\mathcal{M}_n$ of pairwise disjoint structures $\mathcal{M}_n$ for pairwise disjoint predicate languages Σ_n, $n\in\omega$, is the structure of language $\bigcup\limits_{n\in\omega}\Sigma_n\cup\{P_n^{(1)}\mid n\in\omega\}$ with the universe $\bigsqcup\limits_{n\in\omega}M_n$, $P_n=M_n$, and interpretations of predicate symbols in Σ_n coinciding with their interpretations in $\mathcal{M}_n$, $n\in\omega$. The *disjoint union of theories* T_n for pairwise disjoint languages Σ_n accordingly, $n\in\omega$, is the theory

$$\bigsqcup_{n\in\omega}T_n\rightleftharpoons\mathrm{Th}\left(\bigsqcup_{n\in\omega}\mathcal{M}_n\right),$$

where $\mathcal{M}_n\models T_n$, $n\in\omega$.

Clearly, the theory $\bigsqcup\limits_{n\in\omega}T_n$ does not depend on choice of models $\mathcal{M}_n\models T_n$. Besides, the notion of disjoint union admits reductions to finitely many structures and theories, obtaining the structures $\mathcal{M}_1\sqcup\ldots\sqcup\mathcal{M}_n$ and their theories $T_1\sqcup\ldots\sqcup T_n$.

Theorem 7. For any disjoint predicate structures $\mathcal{M}_1$ and $\mathcal{M}_2$ the following conditions hold:

1. $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_1\sqcup\mathcal{M}_2)=\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_1)+\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_2)$, in particular, $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_1\sqcup\mathcal{M}_2)$ is finite iff $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_1)$ and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}_2)$ are finite.

2. $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1\sqcup\mathcal{M}_2)=0$ iff $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1)=0$ and $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_2)=0$, i.e. $\mathcal{M}_1\sqcup\mathcal{M}_2$ is quasi-Urbanik iff $\mathcal{M}_1$ and $\mathcal{M}_2$ are quasi-Urbanik.

3. If $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1\sqcup\mathcal{M}_2)>0$ then it is finite iff $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1)>0$ is finite and $\mathcal{M}_2$ is finite, or $\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_2)>0$ is finite and $\mathcal{M}_1$ is finite. Here,

$$\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1\sqcup\mathcal{M}_2)=\max\{|M_1|+\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_2),|M_2|+\deg^{\forall}_{\mathrm{qU}}(\mathcal{M}_1)\}.$$

Proof word by word repeats the proof of Theorem 2 in [5] replacing degrees of rigidity by degrees of quasi-Urbanikness. □

Theorem 7 immediately implies the following corollaries.

Corollary 3. For any disjoint predicate structures $\mathcal{M}_1$ and $\mathcal{M}_2$ and their theories T_1 and T_2, respectively, the following conditions hold:

1. $\deg^{\exists}_{\mathrm{qU}}(T_1\sqcup T_2)=\deg^{\exists}_{\mathrm{qU}}(T_1)+\deg^{\exists}_{\mathrm{qU}}(T_2)$, in particular, $\deg^{\exists}_{\mathrm{qU}}(T_1\sqcup T_2)$ is finite iff $\deg^{\exists}_{\mathrm{qU}}(T_1)$ and $\deg^{\exists}_{\mathrm{qU}}(T_2)$ are finite.

2. $\deg^{\forall}_{\mathrm{qU}}(T_1\sqcup T_2)=0$ iff $\deg^{\forall}_{\mathrm{qU}}(T_1)=0$ and $\deg^{\forall}_{\mathrm{qU}}(T_2)=0$, i.e. $T_1\sqcup T_2$ is quasi-Urbanik iff T_1 and T_2 are quasi-Urbanik.

3. If $\deg^{\forall}_{\mathrm{qU}}(T_1\sqcup T_2)>0$, then it is finite iff $\deg^{\forall}_{\mathrm{qU}}(T_1)>0$ is finite and $\mathcal{M}_2$ is finite, or $\deg^{\forall}_{\mathrm{qU}}(T_2)>0$ is finite and $\mathcal{M}_1$ is finite. Here,

$$\deg^{\forall}_{\mathrm{qU}}(T_1\sqcup T_2)=\max\{|M_1|+\deg^{\forall}_{\mathrm{qU}}(T_2),|M_2|+\deg^{\forall}_{\mathrm{qU}}(T_1)\}.$$

Corollary 4. Let $\mathcal{T}$ be the family of all theories of form $T_1\sqcup T_2$. Then $\mathrm{DEG}_{2,\mathrm{qU}}(\mathcal{T})=\mathrm{DEG}_{2,\mathrm{qU}}$.

5 *Degrees of quasi-Urbanikness for compositions of structures and their theories*

Recall the notions of composition for structures and theories.

Definition 4. [16] Let $\mathcal{M}$ and $\mathcal{N}$ be structures of relational languages $\Sigma_{\mathcal{M}}$ and $\Sigma_{\mathcal{N}}$ respectively. We define the *composition* $\mathcal{M}[\mathcal{N}]$ of $\mathcal{M}$ and $\mathcal{N}$ satisfying the following conditions:

1) $\Sigma_{\mathcal{M}[\mathcal{N}]}=\Sigma_{\mathcal{M}}\cup\Sigma_{\mathcal{N}}$;

2) $M[N]=M\times N$, where $M[N]$, M, N are universes of $\mathcal{M}[\mathcal{N}]$, $\mathcal{M}$, and $\mathcal{N}$ respectively;

3) if $R\in\Sigma_{\mathcal{M}}\setminus\Sigma_{\mathcal{N}}$, $\mu(R)=n$, then $((a_1,b_1),\ldots,(a_n,b_n))\in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1,\ldots,a_n)\in R_{\mathcal{M}}$;

4) if $R \in \Sigma_{\mathcal{N}} \setminus \Sigma_{\mathcal{M}}$, $\mu(R) = n$, then $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $a_1 = \ldots = a_n$ and $(b_1, \ldots, b_n) \in R_{\mathcal{N}}$;

5) if $R \in \Sigma_{\mathcal{M}} \cap \Sigma_{\mathcal{N}}$, $\mu(R) = n$, then $((a_1, b_1), \ldots, (a_n, b_n)) \in R_{\mathcal{M}[\mathcal{N}]}$ if and only if $(a_1, \ldots, a_n) \in R_{\mathcal{M}}$, or $a_1 = \ldots = a_n$ and $(b_1, \ldots, b_n) \in R_{\mathcal{N}}$.

The theory $T = \mathrm{Th}(\mathcal{M}[\mathcal{N}])$ is called the *composition* $T_1[T_2]$ of the theories $T_1 = \mathrm{Th}(\mathcal{M})$ and $T_2 = \mathrm{Th}(\mathcal{N})$.

By the definition, the composition $\mathcal{M}[\mathcal{N}]$ is obtained replacing each element of $\mathcal{M}$ by a copy of $\mathcal{N}$.

Definition 5. [16]. The composition $\mathcal{M}[\mathcal{N}]$ is called *E-definable* if $\mathcal{M}[\mathcal{N}]$ has an $\emptyset$-definable equivalence relation E E-classes of which are universes of the copies of $\mathcal{N}$ forming $\mathcal{M}[\mathcal{N}]$.

Proposition 3. Let $\mathcal{M}[\mathcal{N}]$ be an E-definable composition consisting of copies $\mathcal{N}_i$, $i \in I$, of the structure $\mathcal{N}$, $A \subseteq M[N]$. Then:

1) $\mathrm{acl}_{\mathcal{M}[\mathcal{N}]}(A) = \bigcup\limits_i \mathrm{acl}_{\mathcal{N}_i}(A \cap N_i) \cup \bigcup\limits_{N_j/E \in \mathrm{acl}_{\mathcal{M}[\mathcal{N}]/E}(A/E)} \mathrm{acl}_{\mathcal{N}_j}(\emptyset)$;

2) $\mathrm{dcl}_{\mathcal{M}[\mathcal{N}]}(A) = \bigcup\limits_i \mathrm{dcl}_{\mathcal{N}_i}(A \cap N_i) \cup \bigcup\limits_{N_j/E \in \mathrm{dcl}_{\mathcal{M}[\mathcal{N}]/E}(A/E)} \mathrm{dcl}_{\mathcal{N}_j}(\emptyset)$.

Proof. 1. By the definition of E-definable composition, formulae define both E-classes by means of the language for $\mathcal{M}$ and subsets of E-classes by means of the language for $\mathcal{N}$. Therefore formulae in the language for $\mathcal{M}[\mathcal{N}]$ define both algebraic sets of E-classes in the quotient $\mathcal{M}[\mathcal{N}]/E$ and algebraic sets inside copies $\mathcal{N}_i$ of $\mathcal{N}$. Thus $\mathrm{acl}_{\mathcal{M}[\mathcal{N}]}(A)$ is composed by algebraic sets inside copies $\mathcal{N}_i$ containing elements of A and defined by restrictions $A \cap N_i$, and by algebraic sets of E-classes with respect to the quotient A/E in $\mathcal{M}[\mathcal{N}]/E$. In the latter case defining finitely many E-classes containing copies $\mathcal{N}_j$, we collect $\mathrm{acl}_{\mathcal{N}_j}(\emptyset)$, obtaining the required equality.

2. We repeat the arguments above replacing algebraic closures by definable ones. □

The following theorem describes possibilities of $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}])$ with respect to characteristics of given predicate structures $\mathcal{M}$ and $\mathcal{N}$.

Theorem 8. For any E-definable composition $\mathcal{M}[\mathcal{N}]$ the following conditions hold:

1) if $\mathcal{N}$ is finite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = 0$, then $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \deg^{\exists}_{\mathrm{qU}}(\mathcal{M})$;

2) if $\mathcal{N}$ is finite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) > 0$, then $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) \cdot |M|$ for finite $\mathcal{M}$ and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \infty$ for infinite $\mathcal{M}$; these equalities stay valid for infinite $\mathcal{N}$ with positive natural $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N})$;

3) if $\mathcal{N}$ is infinite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = 0$, then $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = 0$;

4) if $\mathcal{N}$ is infinite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = \infty$, then $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \infty$.

Proof. 1. Let $\mathcal{N}$ be finite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = 0$. It implies that for any set $A \subseteq N$, its algebraic closure equals definable one. Now taking elements in each copy of $\mathcal{N}$ laying in $\mathcal{M}[\mathcal{N}]$ such that these copies correspond to elements in $\mathcal{M}$ witnessing $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M})$, we obtain algebraic closures composed by copies of $\mathcal{N}$ correspondent to elements of algebraic closures in $\mathcal{M}$ and reduced to definable ones. Thus, using algebraic sets in algebraic closures described in Proposition 3, we obtain $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \deg^{\exists}_{\mathrm{qU}}(\mathcal{M})$.

2. Let $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = n \in \omega \setminus \{0\}$. Since copies of $\mathcal{N}$ in $\mathcal{M}[\mathcal{N}]$ become quasi-Urbanik marking independently appropriate n elements, we have to mark these elements by constants all together obtaining $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = n \cdot |M|$. If $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = \infty$, then each copy of $\mathcal{N}$ in $\mathcal{M}[\mathcal{N}]$ can not become quasi-Urbanik after marking by finitely many constants implying $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = \infty$.

3. Let $\mathcal{N}$ be infinite and $\deg^{\exists}_{\mathrm{qU}}(\mathcal{N}) = 0$. Then algebraic closures are definable inside any copy of $\mathcal{N}$ in $\mathcal{M}[\mathcal{N}]$ and, as $\mathcal{N}$ be infinite, this property is preserved for links between distinct copies of $\mathcal{N}$ in $\mathcal{M}[\mathcal{N}]$ in view of Proposition 3. Thus $\deg^{\exists}_{\mathrm{qU}}(\mathcal{M}[\mathcal{N}]) = 0$.

4. We repeat arguments for Item 2 with infinite $\mathcal{N}$: if $\deg^{\exists}_{\rm qU}(\mathcal{N}) = \infty$, then no finite set can transform an algebraic closure into a definable one implying $\deg^{\exists}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \infty$. □

Remark 3. It is shown in [16] that E-definable compositions $\mathcal{M}[\mathcal{N}]$ uniquely define theories ${\rm Th}(\mathcal{M}[\mathcal{N}])$ by theories ${\rm Th}(\mathcal{M})$ and ${\rm Th}(\mathcal{N})$ and types of elements in copies of $\mathcal{N}$ are defined by types in these copies and types for connections between these copies.

In view of Theorem 8 and Remark 3 we have the following:

Corollary 5. For any E-definable composition $\mathcal{M}[\mathcal{N}]$ and the theories T_1 of $\mathcal{M}$, T_2 of $\mathcal{N}$, and $T_1[T_2]$ of $\mathcal{M}[\mathcal{N}]$ the following conditions hold:

1) if $\mathcal{N}$ is finite and $\deg^{\exists}_{\rm qU}(T) = 0$, then $\deg^{\exists}_{\rm qU}(T_1[T_2]) = \deg^{\exists}_{\rm qU}(T_1)$;

2) if $\mathcal{N}$ is finite and $\deg^{\exists}_{\rm qU}(T_2) > 0$, then $\deg^{\exists}_{\rm qU}(T_1[T_2]) = \deg^{\exists}_{\rm qU}(T_2) \cdot |M|$ for finite $\mathcal{M}$ and $\deg^{\exists}_{\rm qU}(T_1[T_2]) = \infty$ for infinite $\mathcal{M}$; these equalities stay valid for infinite $\mathcal{N}$ with positive natural $\deg^{\exists}_{\rm qU}(T_2)$;

3) if $\mathcal{N}$ is infinite and $\deg^{\exists}_{\rm qU}(T_2) = 0$, then $\deg^{\exists}_{\rm qU}(T_1[T_2]) = 0$;

4) if $\mathcal{N}$ is infinite and $\deg^{\exists}_{\rm qU}(T_2) = \infty$, then $\deg^{\exists}_{\rm qU}(T_1[T_2]) = \infty$.

The following theorem describes possibilities of $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}])$ with respect to characteristics of given predicate structures $\mathcal{M}$ and $\mathcal{N}$.

Theorem 9. For any E-definable composition $\mathcal{M}[\mathcal{N}]$ the following conditions hold:

1) if $\mathcal{N}$ is finite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) = 0$, then $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{M}) \cdot |N|$;

2) if $\mathcal{N}$ is finite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) > 0$, then $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{N}) + (|M| - 1)|N|$ for finite $\mathcal{M}$ and $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \infty$ for infinite $\mathcal{M}$;

3) if $\mathcal{N}$ is infinite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) = 0$, then $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = 0$;

4) if $\mathcal{N}$ is infinite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) > 0$, then $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{N})$ for $|M| = 1$ and $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \infty$ for $|M| > 1$.

Proof. 1. Let $\mathcal{N}$ be finite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) = 0$. Then the value $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}])$ is reduced to the value $\deg^{\forall}_{\rm qU}(\mathcal{M})$, where each element in a set witnessing this value is replaced by a copy of $\mathcal{N}$. Since all elements of these copies are involved to witness $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}])$, we obtain $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{M}) \cdot |N|$.

2. Let $\mathcal{N}$ be finite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) > 0$. In such a case each copy of $\mathcal{N}$ $\mathcal{M}[\mathcal{N}]$ should contain copies of sets witnessing $\deg^{\forall}_{\rm qU}(\mathcal{N})$. Moreover, since elements of one copy can not reduce algebraic closures to definable ones in other copies, the set witnessing the value $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}])$ has to contain all elements in all copies of $\mathcal{N}$ besides one. Thus, $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{N}) + (|M| - 1)|N|$ for finite $\mathcal{M}$ and $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \infty$ for infinite $\mathcal{M}$.

3. If $\mathcal{N}$ is infinite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) = 0$, then neither links between elements of $\mathcal{M}$ nor links between elements of $\mathcal{N}$ can give algebraic sets which are not reduced to definable ones. In view of Proposition 3, we obtain $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = 0$.

4. Let $\mathcal{N}$ be infinite and $\deg^{\forall}_{\rm qU}(\mathcal{N}) > 0$. If $|M| = 1$, then $\mathcal{M}[\mathcal{N}]$ is reduced to $\mathcal{N}$ implying $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \deg^{\forall}_{\rm qU}(\mathcal{N})$. Otherwise algebraic closures in $\mathcal{M}[\mathcal{N}]$ are reduced to algebraic closures inside copies $\mathcal{N}_i$ of $\mathcal{N}$ and finite possibility of $\deg^{\forall}_{\rm qU}(\mathcal{N}_i)$ is witnessed by arbitrary subsets in other copies of $\mathcal{N}$ which are infinite. Then in any case, finite or infinite $\deg^{\forall}_{\rm qU}(\mathcal{N}_i)$, we obtain $\deg^{\forall}_{\rm qU}(\mathcal{M}[\mathcal{N}]) = \infty$. □

In view of Theorem 8 and Remark 3 we have the following:

Corollary 6. For any E-definable composition $\mathcal{M}[\mathcal{N}]$ and the theories T_1 of $\mathcal{M}$, T_2 of $\mathcal{N}$, and $T_1[T_2]$ of $\mathcal{M}[\mathcal{N}]$, the following conditions hold:

1) if $\mathcal{N}$ is finite and $\deg^{\forall}_{\mathrm{qU}}(T_2) = 0$, then $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = \deg^{\forall}_{\mathrm{qU}}(T_1) \cdot |N|$;

2) if $\mathcal{N}$ is finite and $\deg^{\forall}_{\mathrm{qU}}(T_2) > 0$, then $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = \deg^{\forall}_{\mathrm{qU}}(T_2) + (|M| - 1)|N|$ for finite $\mathcal{M}$ and $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = \infty$ for infinite $\mathcal{M}$;

3) if $\mathcal{N}$ is infinite and $\deg^{\forall}_{\mathrm{qU}}(T_2) = 0$, then $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = 0$;

4) if $\mathcal{N}$ is infinite and $\deg^{\forall}_{\mathrm{qU}}(T_2) > 0$, then $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = \deg^{\forall}_{\mathrm{qU}}(T_2)$ for $|M| = 1$ and $\deg^{\forall}_{\mathrm{qU}}(T_1[T_2]) = \infty$ for $|M| > 1$.

6 *Quasi-Urbanikization*

Definition 6. An expansion $\mathcal{M}'$ of a structure $\mathcal{M}$ is called a *quasi-Urbanikization* if $\mathcal{M}'$ is quasi-Urbanik. If $T' = \mathrm{Th}(\mathcal{M}')$ is quasi-Urbanik for a quasi-Urbanikization $\mathcal{M}'$ of $\mathcal{M}$, then T' is a *quasi-Urbanikization* of the theory $\mathrm{Th}(\mathcal{M})$.

Remark 4. Let $\mathcal{M}'$ be a *namization*, or a *constantization* of a structure $\mathcal{M}$, i.e. naming each element of $\mathcal{M}$ by constants. Clearly, $\mathcal{M}'$ is a quasi-Urbanikization of $\mathcal{M}$ whereas this property does not guarantee it for the theory $\mathrm{Th}(\mathcal{M}')$, as illustrated in Example 2.

Here, if $\mathcal{M}$ is finite, then any its namization $\mathcal{M}'$ produces a quasi-Urbanikization $\mathrm{Th}(\mathcal{M}')$ of the theory $\mathrm{Th}(\mathcal{M})$.

Remark 5. Let $\mathcal{M}$ be an infinite structure of an equivalence relation E each E-class of which contains n elements. We expand $\mathcal{M}$ by a unary predicate R, choosing unique element in each E-class, and by unary function f forming a cycle on each E-class $E(a)$ and including all elements of $E(a)$. Thus we obtain a quasi-Urbanikization both for $\mathcal{M}$ and for the theory $T = \mathrm{Th}(\mathcal{M})$. The operator producing that quasi-Urbanikization is called the *R-cyclification* of the structure $\mathcal{M}$ and its theory T.

It is essential here that $\mathcal{M}$ is infinite since the considered cyclification preserves $\mathrm{acl}(\emptyset)$ which is not equal to $\mathrm{dcl}(\emptyset) = \emptyset$ for $|M| \in \omega \setminus \{0, 1\}$.

More generally, we can define cyclifications for algebraic $\overline{a}$-complete formulae $\varphi(x, \overline{a})$, introducing $(l(\overline{a})+2)$-ary predicates $R'(x, y, \overline{z})$ such that $R'(x, y, \overline{a})$ defines a cycle on $\varphi(\mathcal{M}, \overline{a})$ of length $|\varphi(\mathcal{M}, \overline{a})|$, as the R-cyclification for E-classes above.

These possibilities of quasi-Urbanikization can be considered as variations of almost quasi-Urbanikness.

In view of Remarks 4 and 5, we have the following:

Proposition 4. Any structure $\mathcal{M}$ admits a quasi-Urbanikization, i.e. $\mathcal{M}$ has a quasi-Urbanik expansion $\mathcal{M}'$.

A natural *question* arises on the possibility of quasi-Urbanikization of an arbitrary theory T.

Conclusion

We introduced the notions of almost quasi-Urbanik structures and theories, and studied possibilities for the degrees of quasi-Urbanikness, both for existential and universal cases. Links of these characteristics and their possible values are described. We studied these values for linearly ordered, preordered and spherically ordered structures and theories as well as for strongly minimal ones, and for some natural operations including disjoint unions and compositions of structures and theories. A series of examples illustrates possibilities of these characteristics. It would be interesting to continue this research, describing possible values of degrees for natural classes of structures and their theories.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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*Author Information**

Beibut Shaiykovich Kulpeshov (*corresponding author*) — Doctor of physical and mathematical sciences, Corresponding Member of National Academy of Sciences of the Republic of Kazakhstan, Professor, Principal Researcher of Institute of Mathematics and Mathematical Modeling, Professor of Kazakh-British Technical University, 28 Shevchenko street, Almaty, 050010, Kazakhstan; e-mail: *b.kulpeshov@kbtu.kz*; https://orcid.org/0000-0002-4242-0463

Sergey Vladimirovich Sudoplatov — Doctor of physical and mathematical sciences, Professor, Principal Researcher of Sobolev Institute of Mathematics, 4 Academician Koptyug Avenue, Novosibirsk, 630090, Russia; e-mail: *sudoplat@math.nsc.ru*; https://orcid.org/0000-0002-3268-9389

*The author's name is presented in the order: First, Middle and Last Names.