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Computational of the eigenvalues of the fractional Sturm-Liouville problem

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We study the asymptotic distribution for eigenvalues of fourth-order fractional Sturm-Liouville with Dirichlet boundary condition. In this work, we use the inverse Laplace transform method and the Asymptotic formula of the Mittag-Leffler function to get an analytical solution of the fractional Sturm-Liouville problems. When the fractional-order approaches 1, our results agree with the classical ones of fourth-order differential equations.

Keywords: Fractional Sturm-Liouville, Asymptotic formula, Laplace transform, Mittag-Leffler functions, Eigenvalues.

2020 Mathematics Subject Classification: Primary: 26A33; Secondary: 34A08, 33E12.

Introduction

The Sturm-Liouville Problems (SLPs), or eigenvalue problems, for ordinary differential equations play a very important role in theory and applications. These problems have been used to describe a large number of physical, biological and chemical phenomena. Among others, we can refer to the Sturm-Liouville analytical model of dirt transport in industrial washing of wool, which was developed by Caunce et al. [1], the one-dimensional heat and mass diffusion modeling software provided by Barouh and Mikhailov [2] as well as a set of boundary value models [3–6]. However, there are many phenomena in nature that cannot be characterized by classical derivative models [7, 8].

Fractional calculus is a theory that unifies and generalizes the notions of integer order differentiation and integration to any real or complex order. Various types of fractional derivative definitions were introduced in history, which the most popular definitions of fractional derivatives among them are Grünwald-Letnikov, Riemann-Liouville, Dzherbashyan-Caputo, Riesz-Fischer, two-scale fractal derivative [9] which is conformable with the traditional differential derivatives and a new fractional derivative with non-local and no-singular kernel is Atangana-Baleanu's fractional derivative [10] which is presented and applied to solve the fractional heat transfer model. Over the last decade, it has been demonstrated that many systems in science and engineering can be modeled more accurately by employing fractional order rather than integer order derivatives [11–16]. Along with developing the research area of fractional differential equations and applications, many studies have focused on the class of well-known fractional Sturm-Liouville problems (FSLPs). These types of FSLPs, due to their importance, have been a subject of numerous investigations, especially in various areas of science and in engineering fields, for example, chemistry, electricity, mechanics, biology, control theory, and economics [17, 18].

Since it is generally challenging to find analytical solutions for these problems and also FSLPs contain the composition of the left and right-sided derivative. Consequently, several numerical methods have been devoted to seeking approximate solutions, such as the Adomain decomposition method [19], Homotopy analysis method [20] and Fourier series [21].

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On the other hand, recently the FSLPs were studied by Kilmek and Agrawal in [22] and Rivero et al. in [23]. We refer the reader for the higher order of SLPs to [24–26]. Similarly, Dehghan and Mingarelli [27] obtained for the first time, asymptotic formulas of eigenvalues and eigenfunctions of 2α-order FSLP. Also Erdal Bas et al. investigated the conformable SLPs by spectral analysis in [28] and SLPs with a new generalized fractional derivative in [29], and Mortazaasl H. introduced two classes of conformable fractional Sturm-Liouville problem [30]. Significant research on fractional derivatives and numerical solutions of Sturm-Liouville problems can be mentioned by Babak Shiri [31, 32]. Moreover, Jafari et al. in [33], studied the SLPs with a generalized fractional derivative.

It should be noted that since finding analytical solutions for transcendental function is a challenging task, after studying, we realized that we can apply the asymptotic form of Mittag-Leffler's function to get the roots. The asymptotic behavior of Mittag-Leffler functions plays a very important role in the interpretation of the solution of various problems of physics connected with fractional reaction, fractional relaxation, fractional diffusion, and fractional reaction-diffusion, and so forth, in complex systems. The asymptotic expansion of $E_{\alpha,\beta}(z)$ is based on the integral representation of the Mittag-Leffler function in the form

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_C \frac{s^{\alpha-\beta}}{s^\alpha - z} e^s ds, \quad \Re(\alpha) > 0, \quad \Re(\beta) > 0, \quad s, \alpha, \beta \in \mathbb{C}, \tag{1}$$

where the path of integration C is a loop which starts and ends at $-\infty$ and encircles the circular disc $|s| \leq |z|^{\frac{1}{\alpha}}$ in the positive sense and $-\pi \leq \arg s \leq \pi$ on C (this curve is also the Hankel path). The integral representation (1) is used to obtain the asymptotic expansion of the Mittag-Leffler function at infinity [34].

In this work, the inverse Laplace transform method and the Asymptotic formula of the Mittag-Leffler function are applied to obtain analytical solutions of FSLPs. Using the introduced method, we obtained eigenvalues of the fractional Sturm-Liouville problems in three features. The results, show the simplicity and efficiency of this method. This paper's aim is to get an asymptotic formula for the eigenvalues of fourth-order FSLPs.

The paper is organized as follows: In Section 1, we have introduced some necessary definitions and preliminaries of fractional calculus theory. Three illustrative features are discussed in Section 2. The last section includes our conclusion.

1 *Preliminaries*

In this section, we recall some definitions and properties of fractional calculus theory used in this paper. The reader can refer for details to [35–37].

Definition 1. Let $\alpha \in \mathbb{R}$ with $\alpha \notin \mathbb{N}$ and $\alpha > 0$. The left and the right Riemann-Liouville fractional integrals $I^\alpha_{a^+}$ and $I^\alpha_{b^-}$ of order α are defined by

$$I^\alpha_{a^+} f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad x \in (a, b], \tag{2}$$

and

$$I^\alpha_{b^-} f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (\tau-x)^{\alpha-1} f(\tau) d\tau, \quad x \in [a, b).$$

$\Gamma(.)$ denotes the Euler Gamma function. The following property can be easily obtained.

Property 1. We have $I^\alpha_{a^+} C = \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} C$ and $I^\alpha_{b^-} C = \frac{(b-x)^\alpha}{\Gamma(\alpha+1)} C$. C is a constant.

Definition 2. The left and the right Caputo fractional derivatives ${}^cD^{\alpha}_{a^+}$ and ${}^cD^{\alpha}_{b^-}$ of order α are defined by

$$ {}^cD^{\alpha}_{a^+}f(x) := \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-\tau)^{-\alpha} f'(\tau)d\tau, \quad x > a, \tag{3}$$

and

$$ {}^cD^{\alpha}_{b^-}f(x) := \frac{-1}{\Gamma(1-\alpha)}\int_x^b (\tau-x)^{\alpha} f'(\tau)d\tau, \quad x < b,$$

where f is differentiable and $0 \le \alpha < 1$.

Definition 3. The left and the right Riemann-Liouville fractional derivatives $D^{\alpha}_{a^+}$ and $D^{\alpha}_{b^-}$ of order $0 \le \alpha < 1$ are defined by

$$D^{\alpha}_{a^+}f(x) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-\tau)^{-\alpha} f(\tau)d\tau, \quad x > a, \tag{4}$$

and

$$D^{\alpha}_{b^-}f(x) := \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b (\tau-x)^{-\alpha} f(\tau)d\tau, \quad x < b. \tag{5}$$

Property 2. If $0 < \gamma < 1$ and $g \in AC[a,b]$ and $h \in L^q(a,b)(1 \le q < \infty)$. Then we have

$$\int_a^b g(x)D^{\gamma}_{a^+}h(x)dx = \int_a^b h(x){}^cD^{\gamma}_{a^+}g(x)dx + g(x)I^{1-\gamma}_{a^+}h(x)\mid_{x=a}^{x=b}.$$

Property 3. Let $0 < \alpha < \beta$, then the following identities hold:

$$\begin{aligned}
I^{\alpha}_{a^+}(x-a)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1},\\
D^{\alpha}_{a^+}(x-a)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1},\\
I^{\alpha}_{b^-}(b-x)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1},\\
D^{\alpha}_{b^-}(b-x)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-\alpha-1}.
\end{aligned}$$

Property 4. For $a \le x < b$, $0 < \alpha < 1$, we have

$$\begin{aligned}
I^{\alpha}_{a^+}(b-x)^{\alpha-1} &= \frac{(b-x)^{2\alpha-1}}{\Gamma(\alpha)}\int_{\frac{b-a}{b-x}}^1 (1-w)^{\alpha-1}w^{\alpha-1}dw,\\
&= -\frac{(b-x)^{2\alpha-1}}{\Gamma(\alpha)}\Big(B\Big(\frac{b-a}{b-x};\alpha,\alpha\Big) - B(1;\alpha,\alpha)\Big),
\end{aligned}$$

where $B(z;\alpha,\beta)$ is the "Incomplete Beta function" defined by

$$B(z;\alpha,\beta) = \int_0^z w^{\alpha-1}(1-w)^{\beta-1}dw.$$

Property 5. If $\gamma > 0$ and $g \in L^q(a,b)$ $(1 \le q \le \infty)$, then the following equalities

$$\begin{aligned}
D^{\gamma}_{a^+} \circ I^{\alpha}_{a^+}g(x) = g(x),\\
D^{\gamma}_{b^-} \circ I^{\gamma}_{b^-}g(x) = g(x),
\end{aligned}$$

hold on $[a,b]$.

Property 6. If $0 < \alpha < 1$, $f \in L^1(a,b)$ and $I_{a^+}^{1-\alpha} f, I_{b^-}^{1-\alpha} f \in AC[a,b]$, then the following equalities

$$I_{a^+}^{\alpha} \circ D_{a^+}^{\alpha} f(x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} I_{a^+}^{1-\alpha} f(x) \mid_{x=a},$$
$$I_{b^-}^{\alpha} \circ D_{b^-}^{\alpha} f(x) = f(x) - \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} I_{b^-}^{1-\alpha} f(x) \mid_{x=b},$$

hold on $[a,b]$.

Property 7. Let $\Re(\alpha) > 0$ and $f(x) \in L^{\infty}(a,b)$ or $f(x) \in C[a,b]$. If $\Re(\alpha) \notin \mathbb{N}$ or $\alpha \in \mathbb{N}$, then

$$^{c}D_{a^+}^{\alpha} \circ I_{a^+}^{\alpha} f(x) = f(x),$$
$$^{c}D_{b^-}^{\alpha} \circ I_{b^-}^{\alpha} f(x) = f(x).$$

Property 8. Let $0 < \alpha \leq 1$. If $f \in AC[a,b]$, then

$$I_{a^+}^{\alpha} \circ {}^{c}D_{a^+}^{\alpha} f(x) = f(x) - f(a),$$
$$I_{b^-}^{\alpha} \circ {}^{c}D_{b^-}^{\alpha} f(x) = f(b) - f(x).$$

Next, we will review the Mittag-Leffler function. The function $E_{\alpha}(z)$ defined by

$$E_{\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (z \in \mathbb{C}, \Re(\alpha) > 0),$$

was introduced by Mittag-Leffler [36]. The generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C},$$

where $\beta \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Definition 4. The Laplace transform $\mathcal{L}$ of a function $f(x)$, is the function $F(s)$ which is defined by

$$F(s) = \mathcal{L}\{f(x)\} := \int_0^{\infty} e^{-sx} f(x) dx,$$

where $x \geq 0$ and s is the frequency parameter.

Definition 5. If $\mathcal{L}\{f(x)\} = F(s)$ then $f(x)$ is The inverse Laplace transform of $F(s)$ that is given by the complex integral

$$f(x) = \mathcal{L}^{-1}\{F(s)\} := \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{sx} F(s) ds.$$

Definition 6. Convolution of two functions $f(x)$ and $g(x)$ over on a finite rang $[0,x]$ is defined by

$$(f * g)(x) = \int_0^x f(s) g(x-s) ds, \qquad f, g : [0, \infty) \to \mathbb{R}.$$

Property 9. For $\Re(\alpha) > -1$, then

$$\begin{aligned} \mathcal{L}\{t^p\} &= \frac{\Gamma(p+1)}{s^{p+1}}, \\ \mathcal{L}^{-1}\{s^p\} &= \frac{1}{s^{p+1}\Gamma(p)}. \end{aligned} \tag{6}$$

Property 10. Suppose $f(t)$ is a differentiable function of exponential order, then

$$\mathcal{L}\{f'(t)\} = s\{f(t)\} - f(0).$$

Property 11. $\mathcal{L}\{f * g\}(x) = \mathcal{L}\{f(x)\} \cdot \mathcal{L}\{g(x)\}$.

Property 12. $\mathcal{L}^{-1}\{\frac{s^{\alpha-\beta}}{s^{\alpha}+\lambda}\} = x^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})$.

Property 13. According to the definition of the left fractional integral (2), we have

$$I_{0^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\left(f(x) * \frac{1}{x^{1-\alpha}}\right).$$

So, by (6) and Property 11, we get

$$\begin{aligned}\mathcal{L}\{I_{0^+}^{\alpha} f(x)\} &= \frac{1}{\Gamma(\alpha)} \cdot \mathcal{L}\{f(x)\}\mathcal{L}\{\frac{1}{x^{1-\alpha}}\} \\ &= \frac{1}{s^{\alpha}}\mathcal{L}\{f(x)\}.\end{aligned}$$

Property 14. According to the definition of the left Caputo fractional derivative (3) and Properties 13 and 10, we have, for $0 < \alpha < 1$,

$$\mathcal{L}\{{}^{c}D_{0^+}^{\alpha} f(x)\} = s^{\alpha}\mathcal{L}\{f(x)\} - s^{\alpha-1}f(0).$$

Property 15. According to the definition of the left Riemann-Liouville fractional derivative (4), for $0 < \alpha < 1$ by using Properties 13 and 10, we can write

$$\mathcal{L}\{D_{0^+}^{\alpha} f(x)\} = s^{\alpha}\mathcal{L}\{f(x)\} - I_{0^+}^{1-\alpha} f(x)\mid_{x=0} \mathcal{L}\{D I_{0^+}^{1-\alpha} f(x)\}.$$

2 Eigenvalues of fourth order FSLP

In this section, we consider three features of a differently defined fourth order fractional Sturm-Liouville operator. This operator is a composition of right Caputo fractional derivative with a left Riemann-Liouville fractional derivative as follows:

Feature 3.1.

$$ {}^{c}D_{b^-}^{\alpha} \circ D_{a^+}^{\alpha} \circ {}^{c}D_{b^-}^{\alpha} \circ D_{a^+}^{\alpha} y(x) = 0, \qquad 0 < \alpha < 1. \tag{7}$$

Applying the right fractional integral on (7), and using Property 8, we obtain

$$D_{a^+}^{\alpha} \circ {}^{c}D_{b^-}^{\alpha} \circ D_{a^+}^{\alpha} y(x) - D_{a^+}^{\alpha} \circ {}^{c}D_{b^-}^{\alpha} \circ D_{a^+}^{\alpha} y(t)|_{x=b} = 0.$$

Now, by taking the left fractional integral of the above equation and also by using the Properties 1 and 6, we get

$$\begin{aligned}{}^{c}D_{b^-}^{\alpha} \circ D_{a^+}^{\alpha} y(x) &- I_{a^+}^{1-\alpha c}D_{b^-}^{\alpha} \circ D_{a^+}^{\alpha} y(x)|_{x=a}\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \\ &- D_{a^+}^{\alpha} \circ {}^{c}D_{b^-}^{\alpha} \circ D_{a^+}^{\alpha} y(x)|_{x=b}\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} = 0.\end{aligned}$$

Again by taking the right and left fractional integral and using Properties 1 and 6, we get it right away

$$\begin{aligned}y(x) &= I_{a^+}^{1-\alpha} y(t)|_{x=a}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} + D_{a^+}^{\alpha} y(x)|_{x=b}\frac{(x-a)^{\alpha}}{\Gamma(\alpha)} \\ &+ I_{a^+}^{1-\alpha c}D_{b^-}^{\alpha} y(x)|_{x=a}\phi_1(x;a,b,\alpha) + D_{a^+}^{\alpha} \circ {}^{c}D_{b^-}^{\alpha} \circ D_{a^+}^{\alpha} y(x)|_{x=b}\phi_2(x;a,b,\alpha),\end{aligned} \tag{8}$$

where

$$\phi_1(x;a,b,\alpha) = \frac{1}{\Gamma^3(\alpha)} \int_a^x \frac{(\tau-a)^{2\alpha-1}}{(x-\tau)^{1-\alpha}} \left(B\Big(\frac{b-a}{\tau-a};\alpha,\alpha\Big) - B(1;\alpha,\alpha)\right) d\tau,$$

and

$$\phi_2(x;a,b,\alpha) = \frac{1}{\Gamma^2(\alpha)\Gamma(\alpha+1)} \int_a^x \frac{(\tau-a)^{2\alpha-1}}{(x-\tau)^{1-\alpha}} \left(B\Big(\frac{b-a}{\tau-a};\alpha+1,\alpha\Big) - B(1;\alpha+1,\alpha)\right) d\tau.$$

It is worth noting that as α approach 1, (7) reduce $y^{(4)} = 0$, and (8) becomes

$$y(x) = y(a) + y'(b)(x-a) + y''(a)\,\phi_1(x;a,b,1) + y'''(b)\,\phi_2(x;a,b,1),$$

where $\phi_1(x;a,b,1)$ and $\phi_2(x;a,b,1)$ are polynomials of degrees 2 and 3, respectively, in terms of the variable x.

So, we can say fundamental solution is $\left\{ \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}, \frac{(x-a)^{\alpha}}{\Gamma(\alpha)}, \phi_1(x;a,b,\alpha), \phi_2(x;a,b,\alpha) \right\}$, since all four of them satisfy the equation (7) separately and one can see that their Wronskian is not identically zero in $[a,b]$, having discontinuities at a and b. Associated to (7) is another similar but quite different composition.

Feature 3.2.

$$D^{\alpha}_{b-} \circ {}^{c}D^{\alpha}_{a+} \circ D^{\alpha}_{b-} \circ {}^{c}D^{\alpha}_{a+} y(x) = 0, \qquad 0 < \alpha < 1. \tag{9}$$

Applying the right fractional integral on (9) and using Property 6, we have

$${}^{c}D^{\alpha}_{a+} \circ D^{\alpha}_{b-} \circ {}^{c}D^{\alpha}_{a+} y(x) - I^{1-\alpha c}_{b-} D^{\alpha}_{a+} \circ D^{\alpha}_{b-} \circ {}^{c}D^{\alpha}_{a+} y(x)|_{x=b} \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} = 0.$$

Now, by taking the left fractional on the above equation and using Properties 8 and 4, also introducing the function $\psi(t;a,b,\alpha)$ by

$$\psi(x;a,b,\alpha) = \frac{(b-x)^{2\alpha-1}}{\Gamma^2(\alpha)} \left(B\Big(\frac{b-a}{b-x};\alpha,\alpha\Big) - B(1;\alpha,\alpha)\right),$$

we obtain

$$D^{\alpha}_{b-} \circ {}^{c}D^{\alpha}_{a+} y(x) - D^{\alpha}_{b-} \circ {}^{c}D^{\alpha}_{a+} y(x)|_{x=a} - I^{1-\alpha c}_{b-} D^{\alpha}_{a+} \circ D^{\alpha}_{b-} \circ {}^{c}D^{\alpha}_{a+} y(x)|_{x=b} \cdot \psi(x;a,b,\alpha) = 0.$$

Again, by taking the right fractional on the above equation, we have

$$\begin{aligned} {}^{c}D^{\alpha}_{a+} y(t) \quad &- \quad I^{1-\alpha c}_{b-} D^{\alpha}_{b-} y(x)|_{x=b} \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} - D^{\alpha}_{b-}{}^{c}D^{\alpha}_{b-} y(x)|_{x=a} \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \\ &- \quad I^{1-\alpha c}_{b-} D^{\alpha}_{a+} \circ D^{\alpha}_{b-} \circ {}^{c}D^{\alpha}_{a+} y(x)|_{x=b} \cdot \xi(x;a,b,\alpha) = 0, \end{aligned}$$

where

$$\xi(x;a,b,\alpha) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} \psi(\tau;a,b,\alpha) d\tau.$$

Finally

$$\begin{aligned} y(x) \quad &= \quad y(a) + I^{1-\alpha c}_{b-} D^{\alpha}_{b-} y(x)|_{x=b} \frac{(b-x)^{2\alpha-1}}{\Gamma^2(\alpha)} \cdot \left(B\Big(\frac{b-a}{b-x};\alpha,\alpha\Big) - B(1;\alpha,\alpha)\right) \\ &+ \quad D^{\alpha}_{b-} \circ {}^{c}D^{\alpha}_{a+} \circ y(x)|_{x=b} \frac{(b-x)^{2\alpha}}{\Gamma^2(\alpha)} \cdot \left(B\Big(\frac{b-a}{b-x};\alpha+1,\alpha\Big) - B(1;\alpha+1,\alpha)\right) \\ &+ \quad I^{1-\alpha c}_{b-} D^{\alpha}_{a+} \circ D^{\alpha}_{b-} \circ {}^{c}D^{\alpha}_{a+} y(x)|_{x=b} \cdot \int_a^x (x-\tau)^{\alpha-1} \xi(\tau;a,b,\alpha) d\tau. \end{aligned}$$

We see that $\psi(a;a,b,\alpha)$ and consequently $\xi(x;a,b,\alpha)=0$. Also if $\frac{1}{2}<\alpha<1$, we have

$$\lim_{x\to b}\psi(x;a,b,\alpha)=\frac{(b-a)^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}.$$

The functions $\psi(x;a,b,\alpha)$ and consequently $\xi(x;a,b,\alpha)$ have a discontinuity at $x=b$, when $0<\alpha\le\frac{1}{2}$.

Feature 3.3. For $\frac{3}{4}<\alpha\le 1$, consider the following fractional eigenvalue problem on $[0,1]$

$$-{}^cD^{\alpha}_{0+}\circ D^{\alpha}_{0+}\circ {}^cD^{\alpha}_{0+}\circ D^{\alpha}_{0+}y(x)=\lambda y(x), \tag{10}$$

with the boundary conditions

$$\begin{aligned}
&a)\quad I^{1-\alpha}_{0+}y(x)|_{x=0}=0,\\
&b)\quad I^{1-\alpha}_{0+}y(x)|_{x=1}=0,\\
&c)\quad I^{1-\alpha C}_{0+}D^{\alpha}_{0+}\circ D^{\alpha}_{0+}y(x)|_{x=0}=0,\\
&d)\quad I^{1-\alpha C}_{0+}D^{\alpha}_{0+}\circ D^{\alpha}_{0+}y(x)|_{x=1}=0.
\end{aligned} \tag{11}$$

By taking Laplace transformation on both side (10) and using Properties 7 and 8, we have

$$\begin{aligned}
\mathcal{L}(y(x)) &= \frac{s^{3\alpha}}{s^{4\alpha}+\lambda}I^{1-\alpha}_{0+}y(x)|_{x=0}+\frac{s^{3\alpha-1}}{s^{4\alpha}+\lambda}D^{\alpha}_{0+}y(x)|_{x=0}\\
&+ \frac{s^{\alpha}}{s^{4\alpha}+\lambda}I^{1-\alpha c}_{0+}D^{\alpha}_{0+}\circ D^{\alpha}_{0+}y(x)|_{x=0}\\
&+ \frac{s^{\alpha-1}}{s^{4\alpha}+\lambda}D^{\alpha}_{0+}\circ {}^cD^{\alpha}_{0+}\circ D^{\alpha}_{0+}y(x)|_{x=0}.
\end{aligned}$$

Now by taking inverse Laplace transformation in order to get $y(x)$, one can easily see that

$$\begin{aligned}
y(x)= \;& c_1x^{\alpha-1}E_{4\alpha,\alpha}(-\lambda x^{4\alpha})+c_2x^{\alpha}E_{4\alpha,\alpha+1}(-\lambda x^{4\alpha})+c_3x^{3\alpha-1}E_{4\alpha,3\alpha}(-\lambda x^{4\alpha})\\
&+c_4x^{3\alpha}E_{4\alpha,3\alpha+1}(-\lambda x^{4\alpha}).
\end{aligned} \tag{12}$$

Remark 1. When α approaches 1, equation (10) turns into $-y^{(4)}=\lambda y$ and its fundamental set of solution is

$$\left\{\cos\left(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}}\right)\cosh\left(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}}\right),\frac{\cos(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})\sinh(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})}{\sqrt{2}\sqrt[4]{\lambda}x}+\frac{\sin(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})\cosh(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})}{\sqrt{2}\sqrt[4]{\lambda}x},\frac{\sin(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})\sinh(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})}{\sqrt{\lambda}x^2},\frac{\sin(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}}-\frac{\pi}{4})\sinh(\frac{\sqrt[4]{\lambda}x}{\sqrt{2}})}{(\sqrt{\lambda}x)^3}\right\}.$$

Now by imposing the boundary conditions (11) on (12) and with the following formula

$$D^{\gamma}_{0+}\left\{x^{\beta-1}E_{\alpha,\beta}(-\lambda x^{\alpha})\right\}=x^{\beta-\gamma-1}E_{\alpha,\beta-\gamma}(-\lambda x^{\alpha}),$$

finally we have

$$\begin{cases}
c_2xE_{4\alpha,2}(-\lambda x^{4\alpha})+c_4x^{-2\alpha+1}E_{4\alpha,2\alpha+2}(-\lambda x^{4\alpha})|_{x=1}=0,\\
c_2x^{-2\alpha+1}E_{4\alpha,-2\alpha+2}(-\lambda x^{4\alpha})+c_4xE_{4\alpha,2}(-\lambda x^{4\alpha}))|_{x=1}=0.
\end{cases}$$

Now in order to obtain eigenvalues of above system, the coefficient determinant of boundary conditions must be zero, i.e,

$$E^2_{4\alpha,2}(-\lambda)-E_{4\alpha,-2\alpha+2}(-\lambda)E_{4\alpha,2\alpha+2}(-\lambda)=0. \tag{13}$$

This is characteristic equations for eigenvalues, and we note that $E_{4\alpha,2}(-\lambda)$, $E_{4\alpha,-2\alpha+2}(-\lambda)$ and $E_{4\alpha,2\alpha+2}(-\lambda)$ are entire functions of order $\frac{1}{4\alpha}$. For solving (13), from the Mittag-Leffler integral representation (1), we have

$$E_{4\alpha,2}(-\lambda) = \frac{1}{2\pi i}\int_C \frac{s^{4\alpha-2}}{s^{4\alpha}+\lambda}e^s ds.$$

For solving this integral, we use Cauchy's residue theorem

$$s^{4\alpha}+\lambda = 0 \implies s_k = \lambda^{\frac{1}{4\alpha}} e^{i\left(\frac{2k\pi+\pi}{4\alpha}\right)}.$$

On the other hand

$$\frac{3}{4} < \alpha \leq 1 \implies \frac{(2k+1)\pi}{4} < \frac{(2k+1)\pi}{4\alpha} \leq \frac{(2k+1)\pi}{3}.$$

Acceptable poles are

$$s_{-2} = \lambda^{\frac{1}{4\alpha}} e^{-i\left(\frac{3\pi}{4\alpha}\right)} \quad , \quad s_{-1} = \lambda^{\frac{1}{4\alpha}} e^{-i\left(\frac{\pi}{4\alpha}\right)} \quad , \quad s_0 = \lambda^{\frac{1}{4\alpha}} e^{i\left(\frac{\pi}{4\alpha}\right)} \quad , \quad s_1 = \lambda^{\frac{1}{4\alpha}} e^{i\left(\frac{3\pi}{4\alpha}\right)}.$$

Thus

$$E_{4\alpha,2}(-\lambda) = \frac{1}{2\pi i}\int_C \frac{s^{4\alpha-2}}{s^{4\alpha}+\lambda}e^s ds = \frac{1}{2\pi i}\left\{2\pi i \sum_{i=-2}^{1} \frac{e^{s_i}}{4\alpha s_i}\right\} = \frac{1}{4\alpha}\sum_{i=-2}^{1}\frac{e^{s_i}}{s_i}$$

$$= \frac{1}{4\alpha}\left\{ \frac{e^{\lambda^{\frac{1}{4\alpha}} e^{i\frac{-3\pi}{4\alpha}}}}{\lambda^{\frac{1}{4\alpha}} e^{i\frac{-3\pi}{4\alpha}}} + \frac{e^{\lambda^{\frac{1}{4\alpha}} e^{i\frac{-\pi}{4\alpha}}}}{\lambda^{\frac{1}{4\alpha}} e^{i\frac{-\pi}{4\alpha}}} + \frac{e^{\lambda^{\frac{1}{4\alpha}} e^{i\frac{\pi}{4\alpha}}}}{\lambda^{\frac{1}{4\alpha}} e^{i\frac{\pi}{4\alpha}}} + \frac{e^{\lambda^{\frac{1}{4\alpha}} e^{i\frac{3\pi}{4\alpha}}}}{\lambda^{\frac{1}{4\alpha}} e^{i\frac{3\pi}{4\alpha}}} \right\}.$$

Finally

$$E_{4\alpha,2}(-\lambda) = \left\{ \begin{aligned} & \frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{3\pi}{4\alpha}\right))}}{2\alpha\lambda^{\frac{1}{4\alpha}}} \cos\left(\lambda^{\frac{1}{4\alpha}} \sin\frac{3\pi}{4\alpha} - \frac{3\pi}{4\alpha}\right) \\ & + \frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{\pi}{4\alpha}\right)}}{2\alpha\lambda^{\frac{1}{4\alpha}}} \cos\left(\lambda^{\frac{1}{4\alpha}} \sin\frac{\pi}{4\alpha} - \frac{\pi}{4\alpha}\right) \right\}. \end{aligned}\right. \tag{14}$$

In similarly on $E_{4\alpha,2\alpha+2}(-\lambda)$ and $E_{4\alpha,-2\alpha+2}(-\lambda)$, we have

$$E_{4\alpha,2\alpha+2}(-\lambda) = \frac{\lambda^{\frac{-2\alpha-1}{4\alpha}}}{2\alpha}\left\{ \begin{aligned} & e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{3\pi}{4\alpha}\right)} \sin\left(\lambda^{\frac{1}{4\alpha}} \sin\frac{3\pi}{4\alpha} - \frac{3\pi}{4\alpha}\right) \\ & - e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{\pi}{4\alpha}\right)} \sin\left(\lambda^{\frac{1}{4\alpha}} \sin\frac{\pi}{4\alpha} - \frac{\pi}{4\alpha}\right) \right\}, \end{aligned}\right. \tag{15}$$

and

$$E_{4\alpha,-2\alpha+2}(-\lambda) = \frac{\lambda^{\frac{2\alpha-1}{4\alpha}}}{2\alpha}\left\{ \begin{aligned} & e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{3\pi}{4\alpha}\right)} \sin\left(\lambda^{\frac{1}{4\alpha}} \sin\frac{3\pi}{4\alpha} - \frac{3\pi}{4\alpha}\right) \\ & - e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{\pi}{4\alpha}\right)} \sin\left(\lambda^{\frac{1}{4\alpha}} \sin\frac{\pi}{4\alpha} - \frac{\pi}{4\alpha}\right) \right\}. \end{aligned}\right. \tag{16}$$

With substitution (14), (15) and (16) in (13), we get

$$\left\{ \frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{3\pi}{4\alpha}\right)}}{2\alpha\lambda^{\frac{1}{4\alpha}}} \cos\left(\lambda^{\frac{1}{4\alpha}} \sin\frac{3\pi}{4\alpha} - \frac{3\pi}{4\alpha}\right) + \frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{\pi}{4\alpha}\right)}}{2\alpha\lambda^{\frac{1}{4\alpha}}} \cos\left(\lambda^{\frac{1}{4\alpha}} \sin\frac{\pi}{4\alpha} - \frac{\pi}{4\alpha}\right) \right\}^2$$
$$-\left\{ \frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{3\pi}{4\alpha}\right)}}{2\alpha\lambda^{\frac{1}{4\alpha}}} \sin\left(\lambda^{\frac{1}{4\alpha}} \sin\frac{3\pi}{4\alpha} - \frac{3\pi}{4\alpha}\right) - \frac{e^{\lambda^{\frac{1}{4\alpha}}\cos\left(\frac{\pi}{4\alpha}\right)}}{2\alpha\lambda^{\frac{1}{4\alpha}}} \sin\left(\lambda^{\frac{1}{4\alpha}} \sin\frac{\pi}{4\alpha} - \frac{\pi}{4\alpha}\right) \right\}^2 = 0,$$

which implies

$$\frac{e^{\lambda^{\frac{1}{4\alpha}}\cos(\frac{3\pi}{4\alpha})}}{2\alpha}\left[\sin\left(\frac{\pi}{4}-\lambda^{\frac{1}{4\alpha}}\sin\frac{3\pi}{4\alpha}+\frac{3\pi}{4\alpha}\right)\right]+\frac{e^{\lambda^{\frac{1}{4\alpha}}\cos(\frac{3\pi}{4\alpha})}}{2\alpha}\left[\sin\left(\frac{\pi}{4}+\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha}-\frac{\pi}{4\alpha}\right)\right]=0,$$

we denote above transcendental equation with $h_\alpha(-\lambda)=f_\alpha(-\lambda)+g_\alpha(-\lambda)$, with

$$f_\alpha(-\lambda)=\frac{e^{\lambda^{\frac{1}{4\alpha}}\cos(\frac{3\pi}{4\alpha})})}{2\alpha}\left[\sin\left(\frac{\pi}{4}-\lambda^{\frac{1}{4\alpha}}\sin\frac{3\pi}{4\alpha}+\frac{3\pi}{4\alpha}\right)\right],$$

and

$$g_\alpha(-\lambda)=\frac{e^{\lambda^{\frac{1}{4\alpha}}\cos(\frac{3\pi}{4\alpha})})}{2\alpha}\left[\sin\left(\frac{\pi}{4}+\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha}-\frac{\pi}{4\alpha}\right)\right]. \tag{17}$$

It is obvious that

$$g_n=\left(\frac{n\pi+\frac{\pi}{4\alpha}-\frac{\pi}{4}}{\sin\frac{\pi}{4\alpha}}\right)^{4\alpha},\quad n=-1,0,1,\ldots,$$

are positive zeros of $g_\alpha(-\lambda)$ and $g_\alpha(0)=\frac{1}{2\alpha}\sin(\frac{\pi}{4}-\frac{\pi}{4\alpha})$, and $f_\alpha(0)=\frac{1}{2\alpha}\sin(\frac{\pi}{4}-\frac{\pi}{4\alpha})$.

In order to get positive eigenvalues, since $\cos(\frac{3\pi}{4\alpha})$ is negative as long as $\frac{3}{4}<\alpha\leq 1$, then for all sufficiently large λ, $f_\alpha(-\lambda)\to o$, thus

$$h_\alpha(-\lambda)\simeq g_\alpha(-\lambda).$$

Now from $h_\alpha(-\lambda)=0$, it can be concluded that

$$\sin\left(\frac{\pi}{4}+\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha}-\frac{\pi}{4\alpha}\right)=0.$$

Finally from there we obtain the following asymptotic formula

$$\lambda_n\sim\left(\frac{n\pi-\frac{\pi}{4\alpha}+\frac{\pi}{4}}{\sin\frac{\pi}{4\alpha}}\right)^{4\alpha},\quad n\to\infty. \tag{18}$$

A glance at (17) shows that the set of all λ such that

$$\pi(1+2n)<\sin\left(\frac{\pi}{4}+\lambda^{\frac{1}{4\alpha}}\sin\frac{\pi}{4\alpha}-\frac{\pi}{4\alpha}\right)<2\pi(1+n),$$

implies

$$\left(\frac{2n\pi+\frac{3\pi}{4}+\frac{\pi}{4\alpha}}{\sin\frac{\pi}{4\alpha}}\right)^{4\alpha}<\lambda_n(\alpha)<\left(\frac{2n\pi+\frac{7\pi}{4}+\frac{\pi}{4\alpha}}{\sin\frac{\pi}{4\alpha}}\right)^{4\alpha},\qquad n=0,1,2,\ldots$$

Remark 2. As the last discussion, the asymptotic formula (18) is the generalization of classic one i.e. as $\alpha\to 1^-$, this corresponds exactly with the well known classical asymptotic estimate $\lambda_n\sim(\frac{2n\pi}{\sqrt{2}})^4$ as $n\to\infty$.

Remark 3. The uniqueness of the answer is obtained by the fixed point theorem. For this case, as well as the convergence of 10 with boundary conditions (11), we refer the reader to ([30], 3.2).

For the equation (10), Table 1 and Figures 1, 2 show the eigenvalues and eigenfunctions (EFs) for different $\alpha=0.75$, $\alpha=0.86$ and $\alpha=1$, respectively.

Table 1

The eigenvalues λ_n

α=0.76	α= 0.86	α=1.0
λ_1=13334.95	λ_1=83.59	λ_1=97.41
λ_2=192878.31	λ_2=654.87	λ_2= 1558.54
λ_3= 979175.49	λ_3=3098.92	λ_3=7890.14
	λ_4=6255.26	λ_4= 24936.72
		λ_5=60880.68
		$\vdots$

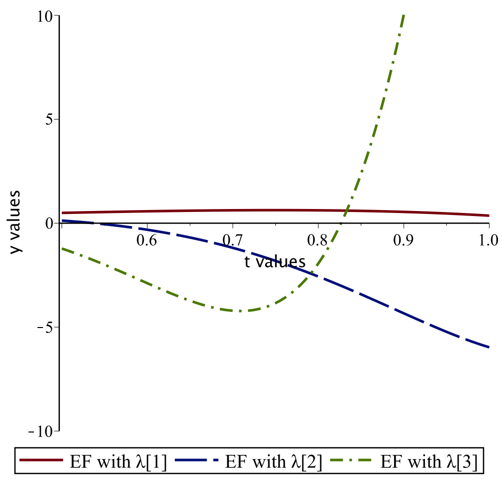

Figure 1.

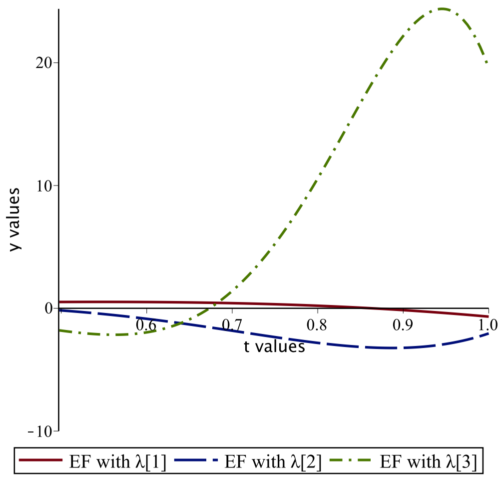

Figure 2.

Here, for the purpose of comparison, we present an example from reference [38].

Example 1. Consider the following fourth order fractional eigenvalue problem

$$D^{\alpha}y(x) = \lambda y(x), \quad x \in (0, 1), \tag{19}$$

with the boundary conditions:

$$y(0) = y'(0) = 0, \; y(1) = y'(1) = 0, \tag{20}$$

where $3 < \alpha = \frac{p}{q} \leq 4$.

By the Laplace transform of the Caputo derivative, we have:

$$y(x) = AE_{\alpha,1}(\lambda x^{\alpha}) + BxE_{\alpha,2}(\lambda x^{\alpha}) + Cx^2E_{\alpha,3}(\lambda x^{\alpha}) + Dx^3E_{\alpha,4}(\lambda x^{\alpha}).$$

Notice that the first derivative of y is given by

$$y'(x) = \frac{A}{\alpha}E_{\alpha,0}(\lambda x^{\alpha}) + BE_{\alpha,1}(\lambda x^{\alpha}) + CxE_{\alpha,2}(\lambda x^{\alpha}) + Dx^2E_{\alpha,3}(\lambda x^{\alpha}).$$

By applying the boundary conditions, to obtain the non-trivial eigenvalues, we set the determinant of the coefficient matrix equal to zero, thus we get

$$(E_{\alpha,3}(\lambda))^2 - E_{\alpha,2}(\lambda)E_{\alpha,4}(\lambda) = 0,$$

a result that was obtained in [38] using a different method, while we applied the asymptotic method introduced in this paper in order to determine the eigenvalues.

Conclusions

In this article, the eigenvalues and the eigenfunctions were derived by studying three features of fractional Sturm-Liouville equations of mixed Riemann-Liouville and Caputo fractional derivatives type. The focus of the paper is on the asymptotic distribution of the eigenvalues obtained from transcendental equation under the assumption the potential function is considered to be zero.

On the other hand in order to get eigenvalues, we studied transcendental equation based on asymptotic behavior of the Mittag-Leffler function rather than numerical method. Also our results showed that it is consistent with the classical one as $\alpha \to 1^-$.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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