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Approximation of a singular boundary value problem for a linear differential equation

R. Uteshova[1,*], Y. Kokotova[2]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan*
(E-mail: r.uteshova@math.kz, kokotovae@mail.ru)

This paper addresses the approximation of a bounded (on the entire real axis) solution of a linear ordinary differential equation, where the matrix approaches zero as $t \to \mp\infty$ and the right-hand side is bounded with a weight. We construct regular two-point boundary value problems to approximate the original problem, assuming the matrix and the right-hand side, both weighted, are constant in the limit. An approximation estimate is provided. The relationship between the well-posedness of the singular boundary value problem and the well-posedness of an approximating regular problem is established.

Keywords: linear differential equation, bounded solution, singular boundary value problem, approximation, well-posedness, parameterization method.

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Introduction

In many fields of applied mathematics, systems of ordinary differential equations that involve singularities or are defined over an infinite interval frequently occur. Numerous studies (see, for example, [1–8]) have explored the existence of bounded solutions for these types of problems and the approximation of these solutions.

In the present paper, we consider the differential equation

$$\frac{dx}{dt} = A(t)x + f(t), \quad x \in \mathbb{R}^n, \quad t \in (-\infty, \infty), \tag{1}$$

where the matrix function $A(t)$ is continuous on $\mathbb{R}$ and $\|A(t)\| := \max_j \sum_{k=1}^{n} |a_{jk}(t)| \leq \alpha(t)$. We assume that $\alpha(t) > 0$ is a continuous function such that

$$\int_{-\infty}^{0} \alpha(t)dt = \infty, \quad \lim_{t\to-\infty} \alpha(t) = 0, \quad \int_{0}^{\infty} \alpha(t)dt = \infty, \quad \lim_{t\to\infty} \alpha(t) = 0.$$

As is known (see, e.g. [9]), the above assumption implies that equation (1) has a bounded solution not for any function $f(t)$ continuous and bounded on the whole axis. For this reason, in [10] the existence and uniqueness of a bounded solution of equation (1) was investigated under the assumption that $f(t)$ is continuous and bounded with a weight.

We will use the following notation:

*Corresponding author. *E-mail: r.uteshova@math.kz*

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$\widetilde{C}(\mathbb{R},\mathbb{R}^n)$ is the space of continuous and bounded functions $x:\mathbb{R}\to\mathbb{R}^n$ equipped with the norm $\|x\|_1=\sup\limits_{t\in\mathbb{R}}\|x(t)\|$;

$\widetilde{C}_{1/\alpha}(\mathbb{R},\mathbb{R}^n)$ is the space of functions $f:\mathbb{R}\to\mathbb{R}^n$ that are continuous and bounded with the weight $1/\alpha(t)$, i.e. $f(t)/\alpha(t)\in\widetilde{C}(\mathbb{R},\mathbb{R}^n)$, equipped with the norm $\|f\|_\alpha=\sup\limits_{t\in\mathbb{R}}\|f(t)/\alpha(t)\|$.

Problem 1 is the problem of finding a bounded on the whole axis solution of equation (1) with $f(t)\in\widetilde{C}_{1/\alpha}(\mathbb{R},\mathbb{R}^n)$.

We say that Problem 1 is well-posed with constant K if it has a unique solution $x(t)\in\widetilde{C}(\mathbb{R},\mathbb{R}^n)$ for any $f(t)\in\widetilde{C}_\alpha(\mathbb{R},\mathbb{R}^n)$, and

$$\|x\|_1\leq K\|f\|_\alpha,$$

where K is a constant independent of $f(t)$.

In [10], Problem 1 was studied by the parameterization method [11] with nonuniform partition $\mathbb{R}=\bigcup\limits_{s=-\infty}^{\infty}[t_{s-1},t_s)$. For a fixed number $\theta>0$, the partition points $t_s\in\mathbb{R}$, $s\in\mathbb{Z}$, are determined as

$$t_0=0,\qquad \int\limits_{t_{s-1}}^{t_s}\alpha(t)dt=\theta.$$

Let $\widetilde{h}(\theta)$ denote a bilaterally infinite sequence of partition step sizes $h_s(\theta)=t_s-t_{s-1}$, $s\in\mathbb{Z}$, i.e. $\widetilde{h}(\theta)=(\ldots,h_s(\theta),h_{s+1}(\theta),\ldots)$. We will use the following spaces:

m_n is the space of bilaterally infinite sequences of $\lambda_s\in\mathbb{R}^n$ equipped with the norm

$$\|\lambda\|_2=\|(\ldots,\lambda_s,\lambda_{s+1},\ldots)\|_2=\sup_s\|\lambda_s\|,\quad s\in\mathbb{Z};$$

$L(m_n)$ is the space of bounded linear operators mapping m_n to itself, equipped with the induced norm;

$m_n(\widetilde{h}(\theta))$ is the space of bounded bilaterally infinite sequences of functions $x_s(t)$, each of which is continuous and bounded on its domain $[t_{s-1},t_s)$, equipped with the norm

$$\|x[t]\|_3=\|(\ldots,x_s(t),x_{s+1}(t),\ldots)\|_3=\sup_s\sup_{t\in[t_{s-1},t_s)}\|x_s(t)\|,\quad s\in\mathbb{Z}.$$

Well-posedness criteria for Problem 1 were obtained in [10] in terms of a bilaterally infinite block-diagonal matrix $Q_{\nu,\widetilde{h}(\theta)}:m_n\to m_n$ of the form

$$Q_{\nu,\widetilde{h}(\theta)}=\left\|\begin{array}{ccccccc}\ldots&\ldots&\ldots&\ldots&\ldots&\ldots&\ldots\\ \ldots&0&I+D_{\nu,s}(h_s(\theta))&-I&0&0&\ldots\\ \ldots&0&0&I+D_{\nu,s+1}(h_{s+1}(\theta))&-I&0&\ldots\\ \ldots&\ldots&\ldots&\ldots&\ldots&\ldots&\ldots\end{array}\right\|,$$

where $D_{\nu,s}(h_s(\theta))=\int\limits_{t_{s-1}}^{t_s}A(\tau_1)d\tau_1+\ldots+\int\limits_{t_{s-1}}^{t_s}A(\tau_1)\ldots\int\limits_{t_{s-1}}^{\tau_{\nu-1}}A(\tau_\nu)d\tau_\nu\ldots d\tau_1$, $s\in\mathbb{Z}$, and I is the identity matrix of order n.

1 Statement of the problem of approximation. A criterion for the well-posedness of Problem 1

In this paper we consider the issue of approximation of Problem 1 by regular two-point boundary value problems. For this purpose, we pose the following problem.

Problem 2. For a given $\varepsilon > 0$ find numbers T_1, $T_2 > 0$, real $n \times n$ matrices B, C, and vector $d \in R^n$, such that a solution $x_{T_1,T_2}(t)$ of the two-point boundary value problem

$$\frac{dx}{dt} = A(t)x + f(t), \qquad t \in (-T_1, T_2), \tag{2}$$

$$Bx(-T_1) + Cx(T_2) = d \tag{3}$$

satisfies the inequality

$$\max_{t\in[-T_1,T_2]} ||x_{T_1,T_2}(t) - x^*(t)|| < \varepsilon,$$

where $x^*(t)$ is a solution of Problem 1.

Problem 2 is considered under the following assumptions.

Assumption 1. $\lim\limits_{t\to\mp\infty} \frac{A(t)}{\alpha(t)} = A_{(\mp)}$, and Re $\xi_j^{\mp} \neq 0$, where $\xi_j^{\mp}$ are the eigenvalues of the matrices $A_{(\mp)}$, $j = 1, 2, \ldots, n$.

Assumption 2. $\lim\limits_{t\to\mp\infty} \frac{f(t)}{\alpha(t)} = f_{(\mp)}$.

We introduce the following functions:

$$\delta_1^-(T) := \sup_{t\in(-\infty,-T]} \left\| \frac{A(t)}{\alpha(t)} - A_{(-)} \right\|, \quad \delta_1^+(T) := \sup_{t\in[T,\infty)} \left\| \frac{A(t)}{\alpha(t)} - A_{(+)} \right\|,$$

$$\delta_2^-(T) := \sup_{t\in(-\infty,-T]} \left\| \frac{f(t)}{\alpha(t)} - f_{(-)} \right\|, \quad \delta_2^+(T) := \sup_{t\in[T,\infty)} \left\| \frac{f(t)}{\alpha(t)} - f_{(+)} \right\|.$$

Obviously, $\delta_r^{\mp}(T) \to 0$ as $T \to \infty$, $r = 1, 2$.

There exist nonsingular real $n \times n$ matrices $S_{(\mp)}$ that transform the matrices $A_{(\mp)}$ into the real Jordan canonical form [12]

$$\widetilde{A}_{(\mp)} = S_{(\mp)} A_{(\mp)} S_{(\mp)}^{-1} = \left\| \begin{array}{cc} A_{11}^{\mp} & 0 \\ 0 & A_{22}^{\mp} \end{array} \right\|, \tag{4}$$

where $A_{11}^{\mp}$ and $A_{22}^{\mp}$ consist of generalized Jordan blocks associated with the eigenvalues of $A_{(\mp)}$ that have negative and positive real parts, the numbers of which we denote by $n_1^{\mp}$ and $n_2^{\mp}$, respectively. We form the $n \times n$ matrices

$$P_1 = \left\| \begin{array}{cc} I_{n_1} & 0 \\ 0 & 0 \end{array} \right\|, \qquad P_2 = \left\| \begin{array}{cc} 0 & 0 \\ 0 & I_{n_2} \end{array} \right\|,$$

where I_{n_r} are the identity matrices of orders n_r, $r = 1, 2$.

The following statement establishes the interrelation between the well-posedness of Problem 1 and that of a two-point boundary value problem.

Theorem 1. Under Assumption 1, Problem 1 is well-posed if and only if:

(i) $n_1^- = n_1^+ = n_1$ and $n_2^- = n_2^+ = n_2$;

(ii) there exist T_0^1, $T_0^2 > 0$ such that for any $T_1 > T_0^1$, $T_2 > T_0^2$ the boundary value problem (2), (3) with $B = -P_1 S_{(-)}$ and $C = P_2 S_{(+)}$, is well-posed with a constant K_1 independent of T_1, T_2.

Proof. Necessity. Let Assumption 1 be fulfilled and let Problem 1 be well-posed. Then, by Theorem 3 [10], there exist $\theta_0 > 0$ such that the matrix $Q_{1,\widetilde{h}(\theta)}$ has an inverse for all $\theta \in (0, \theta_0]$, and the estimate $\|Q_{1,\widetilde{h}(\theta)}^{-1}\|_{L(m_n)} \leq \gamma/\theta$ holds, where γ is a constant independent of $\widetilde{h}(\theta)$. For a fixed $\theta > 0$ we choose T_1 and T_2, so that $t_{-N_1} = -T_1$ and $t_{N_2} = T_2$, and construct the matrix $Q_{1,\widetilde{h}(\theta)}$. In this matrix we then replace $A(t)$ by $\alpha(t)A_{(-)}$ in the block rows numbered $-N_1, -N_1 - 1, \ldots,$ and by $\alpha(t)A_{(+)}$ in the block rows numbered $N_2, N_2+1, \ldots$, and denote the resulting matrix by Q_{θ,T_1,T_2}. Assumption 1 implies that $\|Q_{1,\widetilde{h}(\theta)} - Q_{\theta,T_1,T_2}\|_{L(m_n)} \le \max\{\delta_1^-(T_1), \delta_1^+(T_2 - h_N(\theta))\}\theta$. Hence, by the theorem on small perturbations of boundedly invertible linear operators, if we choose T_0^1, T_0^2 satisfying $\gamma \max\{\delta_1^-(T_0^1), \delta_1^+(T_0^2 - h_N(\theta))\} \le 1/2$, we obtain that the matrix $Q_{\theta,T_1,T_2} : m_n \to m_n$ has an inverse for all $T_1 \ge T_0^1$ and $T_2 \ge T_0^2$, and the estimate

$$\|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \le \frac{\gamma_{T_1,T_2}}{\theta} \le \frac{2\gamma}{\theta}$$

holds. Here $\gamma_{T_1,T_2} = \dfrac{\gamma}{1 - \gamma \max\{\delta_1^-(T_1), \delta_1^+(T_2)\}} \to \gamma$ as $T_1 \to \infty$, $T_2 \to \infty$.

We form a bilaterally infinite matrix $D = \mathrm{diag}(d_{ss})$, where $d_{ss} = S_{(-)}$ for $s = 0, -1, -2, \ldots$, and $d_{ss} = S_{(+)}$ for $s = 1, 2, \ldots$. The matrix $\widetilde{Q}_{\theta,T_1,T_2} = DQ_{\theta,T_1,T_2}D^{-1}$ has a bounded inverse and

$$\|\widetilde{Q}^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \le \|D^{-1}\|_{L(m_n)} \|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \|D\|_{L(m_n)} \le \zeta_1 \gamma_{T_1,T_2} \zeta_2/\theta.$$

Here $\zeta_1 = \|D^{-1}\|_{L(m_n)} = \max\left(\|S^{-1}_{(-)}\|, \|S^{-1}_{(+)}\|\right)$ and $\zeta_2 = \|D\|_{L(m_n)} = \max\left(\|S_{(-)}\|, \|S_{(+)}\|\right)$. In the matrix $\widetilde{Q}_{\theta,T_1,T_2}$ the block rows numbered $s : s \le -N_1$, $s \ge N_2$, are of the form

$$\left\| \begin{matrix} \ldots & 0 & I + \left\| \begin{matrix} A_{11}^{\mp} & 0 \\ 0 & A_{22}^{\mp} \end{matrix} \right\| \theta & -I & 0 & \ldots \end{matrix} \right\|.$$

Rearranging the blocks in $\widetilde{Q}_{\theta,T_1,T_2}$, we obtain the matrix

$$M_{\theta,T_1,T_2} = \left\| \begin{matrix} M_{11}(\theta) & 0 & 0 & 0 & 0 \\ 0 & M_{22}(\theta) & M_{23}(\theta) & 0 & 0 \\ M_{31}(\theta) & 0 & M_{33}(\theta) & 0 & M_{35}(\theta) \\ 0 & 0 & M_{43}(\theta) & M_{44}(\theta) & 0 \\ 0 & 0 & 0 & 0 & M_{55}(\theta) \end{matrix} \right\|.$$

The one-sided infinite matrices $M_{kk}(\theta)$, $k = 1, 2, 4, 5$, are of the form

$$M_{11}(\theta) = \left\| \begin{matrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & I_{n_1^-} + A_{11}^-\theta & -I_{n_1^-} & 0 \\ \ldots & 0 & 0 & I_{n_1^-} + A_{11}^-\theta & -I_{n_1^-} \end{matrix} \right\|,$$

$$M_{22}(\theta) = \left\| \begin{matrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & I_{n_2^-} + A_{22}^-\theta & -I_{n_2^-} & 0 \\ \ldots & 0 & 0 & I_{n_2^-} + A_{22}^-\theta & -I_{n_2^-} \\ \ldots & 0 & 0 & 0 & I_{n_2^-} + A_{22}^-\theta \end{matrix} \right\|,$$

$$M_{44}(\theta) = \left\| \begin{matrix} -I_{n_1^+} & 0 & 0 & 0 & \ldots \\ I_{n_1^+} + A_{11}^+\theta & -I_{n_1^+} & 0 & 0 & \ldots \\ 0 & I_{n_1^+} + A_{11}^+\theta & -I_{n_1^+} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{matrix} \right\|,$$

$$M_{55}(\theta) = \left\| \begin{matrix} I_{n_2^+} + A_{22}^+\theta & -I_{n_2^+} & 0 & 0 & \ldots \\ 0 & I_{n_2^+} + A_{22}^+\theta & -I_{n_2^+} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{matrix} \right\|.$$

The matrix $M_{33}(\theta)$ of dimension $[(N_1+N_2-1)n+n_1^-+n_2^+]\times(N_1+N_2)n$ is of the form

$$M_{33}(\theta)=\left\|\begin{array}{ccccccc} -P_1^{(-)} & 0 & 0 & \dots & 0 & 0 & 0\\ I+\widetilde{A}_{-N_1+1}(\theta) & -I & 0 & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & I+\widetilde{A}_{N_2-1}(\theta) & -I & \\ 0 & 0 & 0 & \dots & 0 & 0 & P_2^{(+)}(I+\widetilde{A}_{(+)}\theta) \end{array}\right\|,$$

where $P_1^{(-)}=(I_{n_1^-},0)$ is a matrix of dimension, $n_1^-\times n$, $P_2^{(+)}=(0,I_{n_2^+})$ is a matrix of dimension $n_2^+\times n$,

$$\widetilde{A}_p(\theta)=\begin{cases} S_{(-)}\int\limits_{t_{p-1}}^{t_p}A(t)dtS_{(-)}^{-1}, & p=-N_1+1,-N_1+2,\dots,1,0,\\ S_{(+)}\int\limits_{t_{p-1}}^{t_p}A(t)dtS_{(+)}^{-1}, & p=1,2,\dots,N_2-1. \end{cases}$$

In the block row of $M_{33}(\theta)$ corresponding to $p=0$, the term $-I$ is replaced by $-S_{(-)}S_{(+)}^{-1}$.

The off-diagonal nonzero blocks of the matrix M_{θ,T_1,T_2} satisfy the relations

$$\|M_{31}(\theta)\|=\|I_{n_1^-}+A_{11}^-\theta\|,\quad \|M_{23}(\theta)\|=1,\quad \|M_{43}(\theta)\|=\|I_{n_1^+}+A_{11}^+\theta\|,\quad \|M_{35}(\theta)\|=1.$$

Due to the invertibility of $\widetilde{Q}_{\theta,T_1,T_2}$, the matrix M_{θ,T_1,T_2} is also invertible, and its inverse satisfies the estimate

$$\|M^{-1}_{\theta,T_1,T_2}\|_{L(m_n)}=\|\widetilde{Q}^{-1}_{\theta,T_1,T_2}\|_{L(m_n)}\le\frac{\zeta_1\gamma_{T_1,T_2}\zeta_2}{\theta}=\frac{\widetilde{\gamma}_{T_1,T_2}}{\theta}.$$

Following the proof scheme in [13], we establish the invertibility of the matrices $M_{kk}(\theta)$, $k=\overline{1,5}$, and the estimates

$$\|[M_{kk}(\theta)]^{-1}\|\le\left[\max_{r=1,2}\left(\|S_{r,\mp}\|,\|S_{r,\mp}^{-1}\|\right)\right]^2\frac{2}{\xi\theta}=\frac{\beta}{\theta},\quad k=1,2,4,5, \tag{5}$$

$$\|[M_{33}(\theta)]^{-1}\|\le\frac{\widetilde{\gamma}_{T_1,T_2}}{\theta}. \tag{6}$$

Here $\xi=\min\left\{|\mathrm{Re}\;\xi_j^{\mp}|,j=1,2,\dots,n\right\}$ and $S_{r,\mp}$ $(r=1,2)$ are nonsingular complex matrices of order $n_r^{\mp}$ reducing $A_{rr}^{\mp}$ to Jordan form with the eigenvalues on the diagonal and $\xi/4$ or zeros on the superdiagonal.

Since the matrix $M_{33}(\theta)$ of dimension $[(N_1+N_2-1)n+n_1^-+n_2^+]\times(N_1+N_2)n$ is invertible, it follows that $n_1^-+n_2^+=n$. In view of the structure of the matrices $\widetilde{A}_{(\mp)}$, we also have $n_1^-+n_2^-=n_1^++n_2^+=n$. Hence, $n_1^-=n_1^+=n_1$, $n_2^-=n_2^+=n_2$.

By rearranging of terms in the matrix $M_{33}(\theta)$, we obtain the invertible matrix

$$N_{33}(\theta)=\left\|\begin{array}{ccccccc} -P_1 & 0 & 0 & \dots & 0 & 0 & P_2(I+\widetilde{A}_{(+)}\theta)\\ I+\widetilde{A}_{-N_1+1}(\theta) & -I & 0 & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 0 & I+\widetilde{A}_{N_2-1}(\theta) & -I \end{array}\right\|,$$

inverse of which, by (6), satisfies the estimate

$$\|[N_{33}(\theta)]^{-1}\|=\|[M_{33}(\theta)]^{-1}\|\le\frac{\widetilde{\gamma}_{T_1,T_2}}{\theta}\le\frac{2\widetilde{\gamma}}{\theta}.$$

Let D_{N_1,N_2} denote the block diagonal matrix consisting of blocks D numbered $s = -N_1, -N_1+1, \dots, N_2-2, N_2-1$. By premultiplying each but the first block row of $N_{33}(\theta)D_{N_1,N_2}$ with $S_{(-)}^{-1}$ or $S_{(+)}^{-1}$, respectively, we obtain the matrix $V_1(\theta)$. Its inverse satisfies the estimate

$$\|[V_1(\theta)]^{-1}\| \le \max(1,\zeta_1)\zeta_2\|[N_{33}(\theta)]^{-1}\| \le \frac{2\widetilde{\gamma}\max(1,\zeta_1)\zeta_2}{\theta} = \frac{\gamma_1}{\theta},$$

where γ_1 is independent of T_1 and T_2. Hence, by following the proof scheme of Theorem 3 in [13] and considering the specifics of our partitioning, it can be shown that for all $T_1 \ge T_0^1$ and $T_2 \ge T_0^2$, the two-point boundary value problem (2), (3) with $B = -P_1S_{(-)}$ and $C = P_2S_{(+)}$ is well-posed with constant K_1 independent of T_1 and T_2.

Sufficiency. Let conditions (i) and (ii) be fulfilled and let $\widetilde{Q}_1(\theta)$ denote the matrix $N_{33}(\theta)$ with the first block row scaled by $\theta > 0$. Then, adapting Theorem 3 in [13] to our partitioning, we obtain that for any $\varepsilon > 0$ there exists $\theta_1 = \theta_1(\varepsilon) > 0$ such that the matrix $\widetilde{Q}_1(\theta)$ is invertible for all $\theta \in (0,\theta_1]$, and

$$\|[\widetilde{Q}_1(\theta)]^{-1}\| \le \frac{(1+\varepsilon)\zeta_1\zeta_2K_1}{\theta} \le \frac{(1+\varepsilon)K_1}{\theta}. \tag{7}$$

The invertibility of $\widetilde{Q}_1(\theta)$ implies that of $M_{33}(\theta)$. Taking into account the bounded invertibility of the matrices $M_{kk}(\theta)$, $k = 1,2,4,5$, and the structure of the matrix M_{θ,T_1,T_2}, we obtain that the last one has a bounded inverse. Let us show that

$$\|M_{\theta,T_1,T_2}^{-1}\|_{L(m_n)} \le \frac{\widetilde{\gamma}}{\theta}, \tag{8}$$

where $\widetilde{\gamma}$ is constant independent of θ. To this end, we consider the equation

$$M_{\theta,T_1,T_2}\mu = b, \qquad \mu, b \in m_n, \tag{9}$$

which can be rewritten as the system

$$M_{11}(\theta)\mu^{(1)} = b^{(1)},$$

$$M_{22}(\theta)\mu^{(2)} + M_{23}(\theta)\mu^{(3)} = b^{(2)}, \tag{10}$$

$$M_{31}(\theta)\mu^{(1)} + M_{33}(\theta)\mu^{(3)} + M_{35}(\theta)\mu^{(5)} = b^{(3)}, \tag{11}$$

$$M_{43}(\theta)\mu^{(3)} + M_{44}(\theta)\mu^{(4)} = b^{(4)}, \tag{12}$$

$$M_{55}(\theta)\mu^{(5)} = b^{(5)}.$$

Here $\mu = (\mu^{(1)},\mu^{(2)},\mu^{(3)},\mu^{(4)},\mu^{(5)})$ and $b = (b^{(1)},b^{(2)},b^{(3)},b^{(4)},b^{(5)})$.

The bounded invertibility of $M_{11}(\theta)$, $M_{55}(\theta)$ and estimate (5) imply the existence of $\mu^{(1)} = [M_{11}(\theta)]^{-1}b^{(1)}$ and $\mu^{(5)} = [M_{55}(\theta)]^{-1}b^{(5)}$, as well as the estimates

$$\|\mu^{(1)}\| \le \frac{\beta}{\theta}\|b^{(1)}\|, \qquad \|\mu^{(5)}\| \le \frac{\beta}{\theta}\|b^{(5)}\|. \tag{13}$$

Let us now multiply by θ the first (from the bottom) block row in equation (10), the first (of dimension n_1) and the last (of dimension n_2) block rows in (11), and the first (from the top) block row in (12). We denote the matrices transformed in this way by $M_{22,\theta}$, $M_{23,\theta}$, $M_{31,\theta}$, $M_{33,\theta}$, $M_{35,\theta}$, $M_{43,\theta}$, $M_{44,\theta}$, the vectors by $b_\theta^{(2)}, b_\theta^{(3)}, b_\theta^{(4)}$ and the equations by $(10)'$, $(11)'$ and $(12)'$. Substituting the obtained sequences $\mu^{(1)}$ and $\mu^{(5)}$ into (11)$'$, we determine $\mu^{(3)}$. Taking into account $\|M_{33,\theta}^{-1}\| = \|[\widetilde{Q}_1(\theta)]^{-1}\|$ and estimate (7), we obtain

$$\|\mu^{(3)}\| = \|M_{33,\theta}^{-1}\left\{b_\theta^{(3)} - M_{31,\theta}[M_{11}(\theta)]^{-1}b^{(1)} - M_{35,\theta}[M_{55}(\theta)]^{-1}b^{(5)}\right\}\| \leq$$
$$\leq \frac{(1+\varepsilon)\widetilde{K}_1}{\theta}\left[\|b_\theta^{(3)}\| + (1+\zeta\theta)\beta\|b^{(1)}\| + \beta\|b^{(5)}\|\right] \leq \frac{(1+\varepsilon)\widetilde{K}_1}{\theta}[1+(2+\zeta\theta)\beta]\max_{k=1,3,5}\|b^{(k)}\|, \tag{14}$$

where $\zeta = [\max(\zeta_1, \zeta_2)]^2$. The one-sided infinite matrices $M_{22,\theta}$ and $M_{44,\theta}$ have bounded inverses, and

$$\|M_{22,\theta}^{-1}\| \leq \beta\frac{\xi}{2}\max\left(\frac{2}{\xi}, 1\right)\frac{1}{\theta}, \quad \|M_{44,\theta}^{-1}\| \leq \beta\frac{\xi}{2}\max\left(\frac{2}{\xi}, 1\right)\frac{1}{\theta}.$$

Substituting $\mu^{(3)}$ into (10) and (12), we determine $\mu^{(2)}$ and $\mu^{(4)}$ and obtain the estimates

$$\|\mu^{(2)}\| \leq \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)(\|b_\theta^{(2)}\| + \theta\|\mu^{(3)}\|) \leq$$
$$\leq \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)\{1 + (1+\varepsilon)\widetilde{K}_1[1+(2+\zeta\theta)\beta]\}\max_{k=1,2,3,5}\|b^{(k)}\|, \tag{15}$$

$$\|\mu^{(4)}\| \leq \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)(\|b_\theta^{(4)}\| + (1+\zeta\theta)\theta\|\mu^{(3)}\|) \leq$$
$$\leq \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)\{1 + (1+\varepsilon)\widetilde{K}_1(1+\zeta\theta)[1+(2+\zeta\theta)\beta]\}\max_{k=2,3,4,5}\|b^{(k)}\|. \tag{16}$$

Thus, for any $b \in m_n$ equation (9) has a unique solution $\mu \in m_n$, and, by (13)–(16), the estimate

$$\|\mu\|_2 \leq \frac{K}{\theta}\|b\|_2$$

holds, where

$$K = \max\{\beta,\ (1+\varepsilon)\widetilde{K}_1[1+(2+\zeta\theta)\beta],\ (\beta\xi/2)\max(2/\xi, 1)[1+(1+\varepsilon)\widetilde{K}_1](1+2\beta+\zeta\beta\theta)]\}.$$

Hence, for any $\varepsilon_1 > 0$ choosing $\theta_2 = \theta_2(\varepsilon_1) > 0$ small enough, we obtain that estimate (8) with $\widetilde{\gamma} = \widetilde{K} + \varepsilon_1 = (\beta\xi/2)\max(2/\xi, 1)[1 + \widetilde{K}_1(1+2\beta)] + \varepsilon_1$ is valid for all $\theta \in (0, \theta_2]$. Under condition (ii) the constant K_1 does not depend of T_1 and T_2, as well as the constant $\widetilde{\gamma} = \widetilde{K} + \varepsilon_1$. Thus, taking into account the estimates

$$\|\widetilde{Q}_{1,\theta} - \widetilde{Q}_{\theta,T_1,T_2}\|_{L(m_n)} \leq \delta_1(T_1, T_2 - h_N(\theta_2))\theta, \quad \|\widetilde{Q}_{\theta,T_1,T_2}^{-1}\|_{L(m_n)} = \|M_{\theta,T_1,T_2}^{-1}\|_{L(m_n)} \leq \frac{\widetilde{K}+\varepsilon_1}{\theta},$$

and choosing T_0^1 and T_0^2 such that $(\widetilde{K}+\varepsilon_1)\zeta\delta_1(T_0^1, T_0^2 - h_N(\theta_2)) \leq 1/2$, we obtain that $\widetilde{Q}_{1,\theta}$ is invertible and $\|\widetilde{Q}_{1,\theta}^{-1}\|_{L(m_n)} \leq 2\widetilde{\gamma}/\theta$. It follows then that

$$\|\widetilde{Q}_{1,\theta}\|_{L(m_n)} \leq \|D^{-1}\|_{L(m_n)}\|\widetilde{Q}_{1,\theta}^{-1}\|_{L(m_n)}\|D\|_{L(m_n)} \leq 2\zeta\widetilde{\gamma}/\theta.$$

Thus, by Theorem 3 in [10], Problem 1 is well-posed for $\nu = 1$. This finishes the proof.

Application of Theorem 1 allows one to obtain effective well-posedness criteria for Problem 1. But condition (ii) somewhat narrows the scope of application, since it becomes necessary to check the well-posedness of the two-point boundary value problem for all T_1 and T_2. However, if we repeat the proof of the sufficiency part of Theorem 1 setting $T_1^0 = T_0^1$, $T_2^0 = T_0^2$ and using the introduced numbers β, ξ, ζ, and then pass in the right part of the inequality to the limit, we establish the following statement.

Theorem 2. Let Assumption 1 hold and the following conditions be met:

(i) $n_1^- = n_1^+ = n_1$ and $n_2^- = n_2^+ = n_2$;

(ii) there exist $T_1^0, T_2^0 > 0$ such that the boundary value problem

$$\frac{dx}{dt} = A(t)x + f(t), \quad t \in (-T_1^0, T_2^0), \tag{17}$$

$$-P_1 S_{(-)} x(-T_1^0) + P_2 S_{(+)} x(T_2^0) = d \tag{18}$$

is well-posed with a constant K_1 satisfying the inequality $\widetilde{K}\zeta\delta_1(T_1^0, T_2^0) < 1$ with

$$\widetilde{K} = (\beta\xi/2)\max(2/\xi, 1)[1 + (1 + 2\beta)K_1\zeta].$$

Then Problem 1 is well-posed with the constant $K = \widetilde{K}\zeta/[1 - \widetilde{K}\zeta\delta_1(T_1^0, T_2^0)]$.

2 *An approximating regular boundary value problem and the estimate for the approximation*

The following theorem provides an approximating two-point boundary value problem and the estimate for the approximation.

Theorem 3. Under Assumptions 1 and 2, let Problem 1 be well-posed with constant K. Then for all $T_1 \geq T_0^1$ and $T_2 \geq T_0^2$, where $T_0^1, T_0^2 > 0$ are some constants determined by $K\max(\delta_1^-(T_0^1), \delta_1^+(T_0^2)) < 1$, the boundary value problem

$$\frac{dx}{dt} = A(t)x + f(t), \qquad t \in (-T_1, T_2), \tag{19}$$

$$P_1 S_{(-)} A_{(-)} x(-T_1) + P_2 S_{(+)} A_{(+)} x(T_2) = -P_1 S_{(-)} f_{(-)} - P_2 S_{(+)} f_{(+)} \tag{20}$$

has a unique solution $x_{T_1,T_2}(t)$, and

$$\max_{t\in[-T_1,T_2]} \|x_{T_1,T_2}(t) - x^*(t)\| \leq$$
$$\leq \frac{K}{1 - K\max(\delta_1^-(T_1), \delta_1^+(T_2))}[K\|f\|_\alpha \max(\delta_1^-(T_1), \delta_1^+(T_2)) + \max(\delta_2^-(T_1), \delta_2^+(T_2))], \tag{21}$$

where $x^*(t)$ is the solution of Problem 1.

Proof. We choose $\theta > 0$ and, applying the parameterization method, obtain that the solution $(\lambda^*, u^*(t)) \in m_n \times m_n(\widetilde{h}(\theta))$ of the boundary value problem with parameter (2)–(5) in [10] satisfies the equation

$$\left[I + \int_{t_{s-1}}^{t_s} A(t)dt\right]\lambda_s^* + \lambda_{s+1}^* = -\int_{t_{s-1}}^{t_s} f(t)dt - \int_{t_{s-1}}^{t_s} A(t)u_s^*(t)dt, \quad s \in Z. \tag{22}$$

By Theorem 3 in [10], for any $\varepsilon > 0$ there exists $\overline{\theta} = \overline{\theta}(\varepsilon)$, such that the estimate $\|Q_{1,\widetilde{h}(\theta)}^{-1}\|_{L(m_n)} \leq \frac{(1+\varepsilon)K}{\theta}$ holds for all $\theta \in (0, \overline{\theta}]$, and, in addition,

$$\left\|\int_{t_{s-1}}^{t_s} A(t)u_s^*(t)dt\right\| \leq c\theta^2, \quad s \in Z,$$

where $c = [1 + (1+\varepsilon)K]e^{\overline{\theta}}\|f\|_\alpha$, then the last term in (22) can be neglected for θ small enough. Let us separate the system (22) into three parts. Replacing $A(t), f(t)$ by $\alpha(t)A_{(-)}, \alpha(t)f_{(-)}$ for $s: s \le N_1$, and by $\alpha(t)A_{(+)}, \alpha(t)f_{(+)}$ for $s: s \ge N_2$, we obtain

$$(I + A_{(-)}\theta)\lambda_{r_1} - \lambda_{r_1+1} = -f_{(-)}\theta, \qquad r_1 = -N_1, -N_1 - 1, \dots, \tag{23}$$

$$\left[I + \int\limits_{t_{r_2-1}}^{t_{r_2}} A(t)dt\right]\lambda_{r_2} + \lambda_{r_2+1} = -\int\limits_{t_{r_2-1}}^{t_{r_2}} f(t)dt, \qquad r_2 = -N_1 + 1, \dots, N_2 - 1, \tag{24}$$

$$(I + A_{(+)}\theta)\lambda_{r_3} - \lambda_{r_3+1} = -f_{(+)}\theta, \qquad r_3 = N_2, N_2 + 1, \dots \tag{25}$$

We rewrite this system in the form

$$Q_{\theta,T_1,T_2}\lambda = -F_{\theta,T_1,T_2}. \tag{26}$$

If we choose $\varepsilon > 0$ to satisfy the inequality, then, by the theorem on small perturbations of boundedly invertible operators, it follows that the matrix Q_{θ,T_1,T_2} is invertible, and its inverse satisfies the estimate

$$\|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \le \frac{(1+\varepsilon)K}{[1 - (1+\varepsilon)K\max(\delta_1^-(T_1), \delta_1^+(T_2))]\theta}. \tag{27}$$

Hence, by Assumptions 1 and 2, we obtain the estimate for the difference between λ^* and the solution λ_{T_1,T_2} of equation (26):

$$\begin{aligned} \|\lambda_{T_1,T_2} - \lambda^*\|_2 &\le \|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)}\|F_{\theta,T_1,T_2} - F_1(\widetilde{h}(\theta)) + [F_1(\widetilde{h}(\theta)) + Q_{\theta,T_1,T_2}\lambda^*]\|_2 = \\ &= \|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)}\|F_{\theta,T_1,T_2} - F_1(\widetilde{h}(\theta)) + [Q_{1,\widetilde{h}(\theta)}\lambda^* + G_1(u^*, \widetilde{h}(\theta)) - Q_{\theta,T_1,T_2}\lambda^*]\|_2 \le \\ &\le \frac{(1+\varepsilon)K[\max(\delta_2^-(T_1), \delta_2^+(T_2)) + K\|f\|_\alpha\max(\delta_1^-(T_1), \delta_1^+(T_2)) + c\theta]}{1 - (1+\varepsilon)K\max(\delta_1^-(T_1), \delta_1^+(T_2))}. \end{aligned} \tag{28}$$

The components of λ_{T_1,T_2} numbered with $s: s \le N_1$ and $s \ge N_2$ satisfy equations (23) and (25), respectively. Hence, the corresponding components of the vector $\mu_{T_1,T_2} = D\lambda_{T_1,T_2}$ solve the equations

$$(I + \widetilde{A}_{(-)}\theta)\mu_{r_1} - \mu_{r_1+1} = -S_{(-)}f_{(-)}\theta, \qquad r_1 = -N_1, -N_1 - 1, \dots,$$

$$(I + \widetilde{A}_{(+)}\theta)\mu_{r_3} - \mu_{r_3+1} = -S_{(+)}f_{(+)}\theta, \qquad r_3 = N_2, N_2 + 1, \dots$$

Then, taking into account the decomposibility of the matrices $\widetilde{A}_{(-)}$ and $\widetilde{A}_{(+)}$, we obtain that $P_1^{(-)}\mu_{r_1}$ and $P_2^{(+)}\mu_{r_3}$ satisfy the equations

$$(I_{n_1^-} + A_{11}^-\theta)P_1^{(-)}\mu_{r_1} - P_1^{(-)}\mu_{r_1+1} = -P_1^{(-)}S_{(-)}f_{(-)}\theta, \tag{29}$$

$$(I_{n_2^+} + A_{22}^+\theta)P_2^{(+)}\mu_{r_3} - P_2^{(+)}\mu_{r_3+1} = -P_2^{(+)}S_{(+)}f_{(+)}\theta. \tag{30}$$

In the proof of Theorem 1 it was shown that the matrices $M_{11}(\theta)$ and $M_{55}(\theta)$ have bounded inverses. Thus, the one-sided infinite systems (29) and (30) have the unique solutions

$$P_1^{(-)}\mu_{-N_1+1} = P_1^{(-)}\mu_{-N_1+2} = \dots = -[A_{11}^-]^{-1}P_1^{(-)}S_{(-)}f_{(-)},$$

$$P_2^{(+)}\mu_{N_2} = P_2^{(+)}\mu_{N_2+1} = \dots = -[A_{22}^+]^{-1}P_2^{(+)}S_{(+)}f_{(+)}.$$

Returning to the variable λ, we obtain

$$A_{11}^{-}P_1^{(-)}S_{(-)}\lambda_{-N_1+1} = -P_1^{(-)}S_{(-)}f_{(-)}, \quad A_{22}^{+}P_2^{(+)}S_{(+)}\lambda_{N_2} = -P_2^{(+)}S_{(+)}f_{(+)}.$$

Then, in view of (4), we have

$$\begin{aligned} A_{11}^{-}P_1^{(-)}S_{(-)}\lambda_{-N_1+1} =& P_1^{(-)}\widetilde{A}_{(-)}S_{(-)}\lambda_{-N_1+1} = P_1^{(-)}S_{(-)}A_{(-)}S_{(-)}^{-1}S_{(-)}\lambda_{-N_1+1} \\ =& P_1^{(-)}S_{(-)}A_{(-)}\lambda_{-N_1+1} = -P_1^{(-)}S_{(-)}f_{(-)}, \\ P_2^{(+)}S_{(+)}A_{(+)}\lambda_{N_2} =& -P_2^{(+)}S_{(+)}f_{(+)}. \end{aligned}$$

These equations together with (24) constitute a closed system in parameters λ_{-N_1+1}, λ_{-N_1+2}, $\ldots$, λ_{N_2-1}, λ_{N_2}. If estimate (27) holds, the boundary value problem (17), (18) is well-posed for all $T_1 \geq T_0^1$, $T_2 \geq T_0^2$. Taking into account that (18) multiplied by $\left\| \begin{matrix} -A_{11}^{-} & 0 \\ 0 & A_{22}^{+} \end{matrix} \right\|$ yields the left-hand side of the boundary condition (20), we obtain that problem (19), (20) is well-posed for all $T_1 \geq T_0^1$, $T_2 \geq T_0^2$.

Let x_{T_1,T_2} be a solution of problem (19), (20), and let $[\lambda_{T_1,T_2}]_{N_1,N_2}$ be the vector composed of those components of $\lambda_{T_1,T_2} \in m_n$ that are numbered $s = -N_1+1, -N_1+2, \ldots, N_2-1, N_2$. Since

$$\max_{s} \sup_{t\in[t_{s-1},t_s)} \|x_{T_1,T_2} - [\lambda_{T_1,T_2}]_{N_1,N_2}\| \leq c_1\theta,$$

where c_1 is a constant independent of θ, we obtain, in view of (28), the following estimate:

$$\max_{t\in[-T_1,T_2]} \|x_{T_1,T_2}(t) - x^*(t)\| \leq \|[\lambda_{T_1,T_2}]_{N_1,N_2} - [\lambda^*]_{N_1,N_2}\| + (c+c_1)\theta \leq$$

$$\leq \frac{(1+\varepsilon)K[K\|f\|_\alpha \max(\delta_1^-(T_1),\delta_1^+(T_2)) + \max(\delta_2^-(T_1),\delta_2^+(T_2)) + c\theta]}{1-(1+\varepsilon)K\max(\delta_1^-(T_1),\delta_1^+(T_2))} + (c+c_1)\theta.$$

Passing to the limit as $\theta \to 0$, we obtain (21). Theorem 3 is proved.

Conclusion

By approximating Problem 1 with a two-point boundary value problem and utilizing well-known results, we developed an approximate method for finding the bounded solution. The form of matrices P_1 and P_2 indicates that the approximating problem involves separated boundary conditions. Theorem 2 allows one to establish the well-posedness of the singular boundary value problem (Problem 1) using the well-posedness constant K_1 of the two-point boundary value problem, the eigenvalues $\xi_j^{\mp}$ of the limit matrices $A_{\mp}$, and the nonsingular matrices $S_{(\mp)}$. This approach provides a robust framework for addressing similar singular problems.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Author Information[*]

Roza Uteshova (*corresponding author*) — Candidate of sciences in physics and mathematics, Associate Professor, Institute of Mathematics and Mathematical Modeling, 125 Pushkin Street, Almaty, 050010, Kazakhstan; e-mail: *r.uteshova@math.kz*; https://orcid.org/0000-0002-8809-9310

Yelena Kokotova — Candidate of sciences in physics and mathematics, K. Zhubanov Aktobe Regional University, 34 Moldagulova Avenue, Aktobe, 030000, Kazakhstan; e-mail: *kokotovae@mail.ru*; https://orcid.org/0009-0007-9625-3334

[*]The author's name is presented in the order: First, Middle and Last Names.