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On interval Riemann double integration on time scales

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This article explores the theory of Riemann double integration for functions whose values are intervals in the framework of time scale calculus. We define the Riemann double Δ-integral and Riemann double ∇-integral for interval valued functions, namely interval Riemann $\Delta\Delta$-integral and interval Riemann $\nabla\nabla$-integral. Some key theorems in the article discuss the uniqueness of the integral, the equality of the interval Riemann double integral to the Riemann double integral when function is degenerate, necessary and sufficient conditions for integrability, proving integrability of a function without knowing the actual value of the integral. Additionally the relationship between the interval Riemann double integral and Riemann double integral for two interval-valued functions is estableshed via Hausdorff-Pompeiu distance. Elementary properties of the integral such as linearity property, subset property and others are established. Using the concept of generalized Hukuhara difference, alternate definitions of the interval Riemann $\Delta\Delta$-integral and interval Riemann $\nabla\nabla$-integral are formulated and theorems proving the equivalence of the integrals defined in both approaches are established. Theorems proving the equivalence of interval Riemann Δ- and ∇-integrals previously defined in both approaches are also shown.

Keywords: interval valued functions, Hausdorff-Pompeiu distance, Riemann $\Delta\Delta$-integral, Riemann $\nabla\nabla$-integral, generalized Hukuhara difference, interval Riemann $\Delta\Delta$-integral, interval Riemann $\nabla\nabla$-integral, time scales.

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Introduction and Motivation

S. Hilger in 1988, as part of his Würzburg doctoral degree [1], introduced the theory of measure chain calculus (which came to be known as the time scale calculus); transcripts later published in 1990, [2]. Time scale calculus unifies and extends discrete and continuous calculus; the theory proves immensely useful when dealing with hybrid models [3]. As theoretical framework, Hilger formulated three axioms [2] (also view [4; 1997]); any set, say $\boldsymbol{T}$, that satisfied these axioms were called time scales. By nature any closed subset $\boldsymbol{T}$ of $\mathbb{R}$ is a time scale, an excerpt "*...any closed subset of* $\mathbb{R}$ *bears the structure of a measure chain in a natural manner.*" [2] concludes this.

Hilger introduced two operators [2]. The forward jump operator denoted by σ and the backward jump operator denoted by ρ. Mapping $\sigma : \boldsymbol{T} \to \boldsymbol{T}$ such that $\sigma(t) = \inf\{u \in \boldsymbol{T} : u > t\}$. Similarly, mapping $\rho : \boldsymbol{T} \to \boldsymbol{T}$ such that $\rho(t) = \sup\{u \in \boldsymbol{T} : u < t\}$.

Using the notion of forward jump operator, Hilger in [2] formulated the delta derivative (Δ-derivative). A decade later in 2000, C. D. Ahlbrandt et al. [5] introduced a notion of derivative, which they called the alpha derivative, consisting both the Δ-derivative and another derivative called the nabla derivative (∇-derivative) as special cases. This ∇-derivative was formulated using the notion of backward jump operator, officially named so in 2002 by F. M. Atici et al. [6].

Integrations of the Δ-derivative and ∇-derivative are extensively discussed in literature, including for the Riemann integration. The Riemann integral for real valued functions on time scales was formulated by S. Sailer [7], using the concept of Darboux sum definition of the integral; and by

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G.S. Guseinov et al., using the concept of Riemann sum definition of the integral [8, 9]. The Riemann double Δ-integral for real valued functions was defined by M. Bohner et al. [10, 11].

Below we give the definition of Riemann double Δ-integral (Riemann $\Delta\Delta$-integral) and Riemann double ∇-integral (Riemann $\nabla\nabla$-integral) for real valued functions as defined in [10].

Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ be two given time scales and put $\boldsymbol{T}_1 \times \boldsymbol{T}_2 = \{(\hat{t}, \check{t}) : \hat{t} \in \boldsymbol{T}_1, \check{t} \in \boldsymbol{T}_2\}$.

The intervals on which integrals are defined, i.e., intervals on time scale $\boldsymbol{T}$ are defined as assuming $v \leq w$ [11]:

$[v, w]_{\boldsymbol{T}} = \{t \in \boldsymbol{T} : v \leq t \leq w\}$; $(v, w)_{\boldsymbol{T}} = \{t \in \boldsymbol{T} : v < t < w\}$;

$[v, w)_{\boldsymbol{T}} = \{t \in \boldsymbol{T} : v \leq t < w\}$; $(v, w]_{\boldsymbol{T}} = \{t \in \boldsymbol{T} : v < t \leq w\}$.

For clarity E, F will represent partitions for the Δ-integral and G, H will represent partitions for the ∇-integral.

Let $[v, w]_{\boldsymbol{T}}$ and $[r, s]_{\boldsymbol{T}}$ be closed intervals on $\boldsymbol{T}$ such that $[v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}} = \{(\hat{t}, \check{t}) : \hat{t} \in [v, w]_{\boldsymbol{T}}, \check{t} \in [r, s]_{\boldsymbol{T}}\}$. We partition the intervals as $[v = \hat{t}_0 < \hat{t}_1 < \ldots < \hat{t}_p = w]$, $p \in \mathbb{N}$ and $[r = \check{t}_0 < \check{t}_1 < \ldots < \check{t}_q = s]$, $q \in \mathbb{N}$; $\mathcal{P}([v, w]_{\boldsymbol{T}})$ will denote the collection of all possible partitions of $[v, w]_{\boldsymbol{T}}$ and $\mathcal{P}([r, s]_{\boldsymbol{T}})$ will denote the collection of all possible partitions of $[r, s]_{\boldsymbol{T}}$.

Let $\mathrm{E} = \{v = \hat{t}_0 < \ldots < \hat{t}_p = w\} \in \mathcal{P}([v, w]_{\boldsymbol{T}})$ and $\mathrm{F} = \{r = \check{t}_0 < \ldots < \check{t}_q = s\} \in \mathcal{P}([r, s]_{\boldsymbol{T}})$. Subintervals are taken to be of the form $[\hat{t}_{e-1}, \hat{t}_e)_{\boldsymbol{T}}$ for $1 \leq e \leq p$ and $[\check{t}_{f-1}, \check{t}_f)_{\boldsymbol{T}}$ for $1 \leq f \leq q$, which we will call the $\Delta\Delta$-subintervals. From each of these $\Delta\Delta$-subintervals we choose $\hat{\vartheta}_e \in [\hat{t}_{e-1}, \hat{t}_e)_{\boldsymbol{T}}$ and $\check{\vartheta}_f \in [\check{t}_{f-1}, \check{t}_f)_{\boldsymbol{T}}$ arbitrarily and call it the $\Delta\Delta$-tags. We define the mesh of E as, mesh-(E) $= \max_{1 \leq e \leq p}(\hat{t}_e - \hat{t}_{e-1}) > 0$. For some $\delta > 0$, E_δ will represent a Δ-partition of $[v, w]_{\boldsymbol{T}}$ with mesh δ satisfying the property: for each $e = 1, 2, \ldots, p$ we have either- $(\hat{t}_e - \hat{t}_{e-1}) \leq \delta$ or $(\hat{t}_e - \hat{t}_{e-1}) > \delta \wedge \rho(\hat{t}_e) = \hat{t}_{e-1}$ (here $\wedge$ stands for "and"). Again, mesh-(F) $= \max_{1 \leq f \leq q}(\check{t}_f - \check{t}_{f-1}) > 0$. For some $\delta > 0$, F_δ will represent a Δ-partition of $[r, s]_{\boldsymbol{T}}$ with mesh δ satisfying the property: for each $f = 1, 2, \ldots, q$ we have either $(\check{t}_f - \check{t}_{f-1}) \leq \delta$ or $(\check{t}_f - \check{t}_{f-1}) > \delta \wedge \rho(\check{t}_f) = \check{t}_{f-1}$.

Riemann $\Delta\Delta$-sum, $\mathrm{R}_{\Delta\Delta}(g; \mathrm{E}_\delta; \mathrm{F}_\delta)$, of real valued function "g" evaluated at the $\Delta\Delta$-tags as follows,

$$\mathrm{R}_{\Delta\Delta}(g; \mathrm{E}_\delta; \mathrm{F}_\delta) := \sum_{e=1}^{p} \sum_{f=1}^{q} g\big(\hat{\vartheta}_e, \check{\vartheta}_f\big)(\hat{t}_e - \hat{t}_{e-1})(\check{t}_f - \check{t}_{f-1}).$$

Definition 1. [10] (Riemann $\Delta\Delta$-integral) Let function $g : [v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}} \to \mathbb{R}$ be a real valued function. Function g is said to be Riemann $\Delta\Delta$-integrable if there exists an $I_{\Delta\Delta} \in \mathbb{R}$ on $[v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-partitions E_δ and F_δ, we have $\big|\mathrm{R}_{\Delta\Delta}(g; \mathrm{E}_\delta; \mathrm{F}_\delta) - I_{\Delta\Delta}\big| < \varepsilon$. Here $I_{\Delta\Delta} = \mathrm{R}_{\Delta\Delta} \int_v^w \int_r^s g(\hat{t}, \check{t}) \Delta\hat{t} \Delta\check{t}$, where $\mathrm{R}_{\Delta\Delta} \int_v^w \int_r^s g(\hat{t}, \check{t}) \Delta\hat{t} \Delta\check{t}$ is called the Riemann $\Delta\Delta$-integral.

Let $\mathrm{G} = \{v = \hat{t}_0 < \ldots < \hat{t}_p = w\} \in \mathcal{P}([v, w]_{\boldsymbol{T}})$ and $\mathrm{H} = \{r = \check{t}_0 < \ldots < \check{t}_q = s\} \in \mathcal{P}([r, s]_{\boldsymbol{T}})$. Subintervals are taken to be of the form $(\hat{t}_{e-1}, \hat{t}_e]_{\boldsymbol{T}}$ for $1 \leq e \leq p$ and $(\check{t}_{f-1}, \check{t}_f]_{\boldsymbol{T}}$ for $1 \leq f \leq q$, which we will call the $\nabla\nabla$-subintervals. From each of these $\nabla\nabla$-subintervals we choose $\hat{\xi}_e \in (\hat{t}_{e-1}, \hat{t}_e]_{\boldsymbol{T}}$ and $\check{\xi}_f \in (\check{t}_{f-1}, \check{t}_f]_{\boldsymbol{T}}$ arbitrarily and call it the $\nabla\nabla$-tags. We define the mesh of G as, mesh-(G) $= \max_{1 \leq e \leq p}(\hat{t}_e - \hat{t}_{e-1}) > 0$. For some $\delta > 0$, G_δ will represent a partition of $[v, w]_{\boldsymbol{T}}$ with mesh δ satisfying the property: for each $e = 1, 2, \ldots, p$ we have either $(\hat{t}_e - \hat{t}_{e-1}) \leq \delta$ or $(\hat{t}_e - \hat{t}_{e-1}) > \delta \wedge \sigma(\hat{t}_{e-1}) = \hat{t}_e$. Again, mesh-(H) $= \max_{1 \leq f \leq q}(\check{t}_f - \check{t}_{f-1}) > 0$. For some $\delta > 0$, H_δ will represent a partition of $[r, s]_{\boldsymbol{T}}$ with mesh δ satisfying the property: for each $f = 1, 2, \ldots, q$ we have either $(\check{t}_f - \check{t}_{f-1}) \leq \delta$ or $(\check{t}_f - \check{t}_{f-1}) > \delta \wedge \sigma(\check{t}_{f-1}) = \check{t}_f$.

Riemann $\nabla\nabla$-sum, $\mathrm{R}_{\nabla\nabla}(g; \mathrm{G}_\delta; \mathrm{H}_\delta)$, of real valued function "g" evaluated at the $\nabla\nabla$-tags as follows,

$$\mathrm{R}_{\nabla\nabla}(g; \mathrm{G}_\delta; \mathrm{H}_\delta) := \sum_{e=1}^{p} \sum_{f=1}^{q} g\big(\hat{\xi}_e, \check{\xi}_f\big)(\hat{t}_e - \hat{t}_{e-1})(\check{t}_f - \check{t}_{f-1}).$$

Definition 2. (Riemann $\nabla\nabla$-integral) Let function $g : [v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}} \to \mathbb{R}$ be a real valued function. Function g is said to be Riemann $\nabla\nabla$-integrable if there exists an $I_{\nabla\nabla} \in \mathbb{R}$ on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-partitions G_δ and H_δ, we have $\left|\mathrm{R}_{\nabla\nabla}(g;\mathrm{G}_\delta;\mathrm{H}_\delta) - I_{\nabla\nabla}\right| < \varepsilon$. Here $I_{\nabla\nabla} = \mathrm{R}_{\nabla\nabla}\int_v^w\int_r^s g(\hat{t},\check{t})\nabla\hat{t}\nabla\check{t}$, where $\mathrm{R}_{\nabla\nabla}\int_v^w\int_r^s g(\hat{t},\check{t})\nabla\hat{t}\nabla\check{t}$ is called the Riemann $\nabla\nabla$-integral.

We take a quick look at the theory of interval analysis. R.E. Moore's monograph [12] and [13] played a vital role as a catalyst to the modern era of extensive research on interval analysis. This monograph was the outgrowth of his Stanford PhD thesis titled "Interval arithmetic and automatic error analysis in digital computing" [14]. Intuitively, interval analysis uses closed intervals of real numbers instead of just numbers for calculations. Following we present basic concepts on classical interval analysis, view [13] for more insight.

Let $\mathbb{R}_\mathrm{I}$ denote the class of all non-empty compact intervals of real numbers. $[\mathrm{P}] = [\mathrm{P}^-, \mathrm{P}^+] \in \mathbb{R}_\mathrm{I}$; P^- represents the left end point and P^+ represents the right end point of interval $[\mathrm{P}]$. If $\mathrm{P}^- = \mathrm{P}^+$ then $[\mathrm{P}]$ is said to be degenerate.

Given $[\mathrm{P}], [\mathrm{Q}] \in \mathbb{R}_\mathrm{I}$, some rules of ordinary interval arithmetic are

$$\text{Minkowski addition}: \quad [\mathrm{P}] \oplus [\mathrm{Q}] = \left[\mathrm{P}^- + \mathrm{Q}^-, \mathrm{P}^+ + \mathrm{Q}^+\right].$$

$$\text{Scalar Product}: \quad \text{for } r \in \mathbb{R}, \quad r[\mathrm{P}] = \left\{[r\mathrm{P}^-, r\mathrm{P}^+] \text{ if } r > 0; \quad [0] \text{ if } r = 0; \right.$$
$$\left. [r\mathrm{P}^+, r\mathrm{P}^-] \text{ if } r < 0\right\}.$$

$$\text{Order}: \quad [\mathrm{P}] < [\mathrm{Q}] \text{ implies } \mathrm{P}^+ < \mathrm{Q}^-.$$

$$\text{Subset}: \quad [\mathrm{P}] \subseteq [\mathrm{Q}] \text{ if and only if } \mathrm{Q}^- < \mathrm{P}^- \text{ and } \mathrm{P}^+ < \mathrm{Q}^+.$$

$$\text{Absolute value}: \quad \left|[\mathrm{P}]\right| = \max\left\{\left|\mathrm{P}^-\right|, \left|\mathrm{P}^+\right|\right\}.$$

Reader is referred to [13] and [15] for theory on ordinary interval analysis.

The Hausdorff-Pompeiu distance between intervals [P] and [Q] is defined as

$$\mathsf{s}\big([\mathrm{P}],[\mathrm{Q}]\big) = \max\left\{\left|\mathrm{P}^- - \mathrm{Q}^-\right|, \left|\mathrm{P}^+ - \mathrm{Q}^+\right|\right\}.$$

It is known that $\big(\mathbb{R}_\mathrm{I}, \mathsf{s}\big)$ is a complete metric space. Properties of "s" are

1. $\mathsf{s}\big([\mathrm{P}],[\mathrm{Q}]\big) = 0 \Leftrightarrow [\mathrm{P}] = [\mathrm{Q}]$;
2. $\mathsf{s}\big(\gamma[\mathrm{P}],\gamma[\mathrm{Q}]\big) = |\gamma|\mathsf{s}\big([\mathrm{P}],[\mathrm{Q}]\big)$ for all $\gamma \in \mathbb{R}$;
3. $\mathsf{s}\big([\mathrm{P}] \oplus [\mathrm{R}],[\mathrm{Q}] \oplus [\mathrm{R}]\big) = \mathsf{s}\big([\mathrm{P}],[\mathrm{Q}]\big)$;
4. $\mathsf{s}\big([\mathrm{P}] \oplus [\mathrm{R}],[\mathrm{Q}] \oplus [\mathrm{S}]\big) \leq \mathsf{s}\big([\mathrm{P}],[\mathrm{Q}]\big) + \mathsf{s}\big([\mathrm{R}],[\mathrm{S}]\big)$,

For details on "s" refer [16].

L. Stefanini in [16, 17] details the general limitation of substraction of sets. To partially overcome this situation, M. Hukuhara [18] introduced the H-difference (Hukuhara difference) which was further generalized by L. Stefanini [17], referring to it as the generalized Hukuhara difference. We will denote generalized Hukuhara difference by "$\ominus_{\mathrm{gH}}$" defined as

$$[\mathrm{P}^-, \mathrm{P}^+] \ominus_{\mathrm{gH}} [\mathrm{Q}^-, \mathrm{Q}^+] = [\mathrm{R}^-, \mathrm{R}^+] \Leftrightarrow \begin{cases} \mathrm{P}^- = \mathrm{Q}^- + \mathrm{R}^-, \mathrm{P}^+ = \mathrm{Q}^+ + \mathrm{R}^+, \\ \qquad\qquad \text{or} \\ \mathrm{Q}^- = \mathrm{P}^- - \mathrm{R}^-, \mathrm{Q}^+ = \mathrm{P}^+ - \mathrm{R}^+, \end{cases}$$

so that $[\mathrm{P}^-, \mathrm{P}^+] \ominus_{\mathrm{gH}} [\mathrm{Q}^-, \mathrm{Q}^+] = [\mathrm{R}^-, \mathrm{R}^+]$ is always defined by

$$\mathrm{R}^- = \min\{\mathrm{P}^- - \mathrm{Q}^-, \mathrm{P}^+ - \mathrm{Q}^+\}, \; \mathrm{R}^+ = \max\{\mathrm{P}^- - \mathrm{Q}^-, \mathrm{P}^+ - \mathrm{Q}^+\},$$

i.e., $[\mathrm{P}] \ominus_{\mathrm{gH}} [\mathrm{Q}] = \left[\min\{\mathrm{P}^- - \mathrm{Q}^-, \mathrm{P}^+ - \mathrm{Q}^+\}, \max\{\mathrm{P}^- - \mathrm{Q}^-, \mathrm{P}^+ - \mathrm{Q}^+\}\right]$.

Properties of "$\ominus_{\mathrm{gH}}$" are

1. $[\mathrm{P}] \ominus_{\mathrm{gH}} [\mathrm{P}] = \{0\}$;
2. $\big([\mathrm{P}] \oplus [\mathrm{Q}]\big) \ominus_{\mathrm{gH}} [\mathrm{Q}] = [\mathrm{P}]$; $[\mathrm{P}] \ominus_{\mathrm{gH}} \big([\mathrm{P}] \oplus [\mathrm{Q}]\big) = -[\mathrm{Q}]$;
3. $\mathtt{s}\big([\mathrm{P}], [\mathrm{Q}]\big) = \mathtt{s}\big([\mathrm{P}] \ominus_{\mathrm{gH}} [\mathrm{Q}], [0]\big)$; here $[0] = [0, 0]$;
4. $\mathtt{s}\big([\mathrm{P}], [\mathrm{Q}]\big) = 0 \Leftrightarrow [\mathrm{P}] \ominus_{\mathrm{gH}} [\mathrm{Q}] = \{0\}$.

For more details on properties of "$\ominus_{\mathrm{gH}}$" one may refer [16] and [17].

Let $[v, w]_{\boldsymbol{T}}$ be a closed interval on $\boldsymbol{T}$. Function h is said to be an interval valued function if it assigns a nonempty interval

$$[h(t)] = \big[h(t)^-, h(t)^+\big] = \big\{h : h(t)^- \le h \le h(t)^+\big\},$$

for each $t \in [v, w]_{\boldsymbol{T}}$, where $h^-, h^+ : [v, w]_{\boldsymbol{T}} \to \mathbb{R}$ are real valued functions.

$h : [v, w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ and $t \in [v, w]_{\boldsymbol{T}}$, $l \in \mathbb{R}_{\mathrm{I}}$ is said to be an interval limit of h as t tends to u, denoted by $\lim_{t\to u} h(t) = l$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathtt{s}(h(t), l) < \varepsilon$ for all $|t - u| < \delta$. Here,

$$\lim_{t\to u} h(t) = l \Leftrightarrow \lim_{t\to u} \big(h(t) \ominus_{\mathrm{gH}} l\big) = \{0\},$$

where the interval limits are in the metric "$\mathtt{s}$". For $h(t) = [h^-(t), h^+(t)]$, $\lim_{t\to u} h(t)$ exists if and only if $\lim_{t\to u} h^-(t)$ and $\lim_{t\to u} h^+(t)$ exists as finite numbers. Here,

$$\lim_{t\to u} h(t) = \Big[\lim_{t\to u} h^-(t), \lim_{t\to u} h^+(t)\Big].$$

$h : [v, w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ is said to be interval continuous at $u \in [v, w]_{\boldsymbol{T}}$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathtt{s}([h(t)], [h(u)]) < \varepsilon$ whenever $|t - u| < \delta$. Also, h is interval continuous at $u \in [v, w]_{\boldsymbol{T}}$ if and only if its end points h^- and h^+ are continuous functions at $u \in [v, w]_{\boldsymbol{T}}$. If h is interval continuous at every $t \in [v, w]_{\boldsymbol{T}}$, then we say that h is interval continuous. h is said to be interval bounded, if for $B > 0$, $|[h(t)]| < B$ for all $t \in [v, w]_{\boldsymbol{T}}$.

Reader is referred to [19] and [20].

Integration of functions whose values are intervals (interval valued functions) have garnered much attention in recent years for both continuous calculus and time scale calculus.

For interval valued functions in continuous calculus, the interval Riemann integral was defined by O. Caprani et al. in [15] (also view [13]); the interval Henstock integral was defined by C. Wu et al. in [21]; the interval Henstock-Stieltjes integral was defined by M. E. Hamid [22]; the interval AP-Henstock integral was defined by M. E. Hamid et al. [23]; the interval AP-Henstock-Stieltjes integral was defined by G. S. Eun et al. [24]; and the interval McShane and interval McShane-Stieltjes integrals are defined by C. K. Park [25].

In 2013, V. Lupulescu [19] introduced the notion of interval analysis to the concept of time scale calculus pioneering extensive research that followed soon. He formulated differentiability and integrability for interval valued functions on time scales using generalized Hukuhara difference.

For interval valued function in time scale calculus the interval Riemann integral was defined by D. Zhao et al. [26] (Δ-integral) and by M. Bohner et al. [20] (∇-integral and $\Diamond_\alpha$-integral), the interval Riemann integral defined using the notion of generalized Hukuhara difference was given by V. Lupulescu [19]; the interval Riemann-Stieltjes integral was defined in [27] (Δ-integral and ∇-integral) and interval Riemann-Stieltjes integral using the notion of generalized Hukuhara difference was also defined in the same [27] (Δ-integral and ∇-integral); the interval Henstock integral was defined by W.T. Oh et al. [28] (Δ-integral); the interval Henstock-Stieltjes integral was defined by J.H. Yoon [29] (Δ-integral); the interval McShane integral was defined by M.E. Hamid et al. [30] (Δ-integral); the interval McShane-Stieltjes integral was defined by M.E. Hamid [31] (Δ-integral); and the interval Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral was defined by D.A. Afariogun et al. [32, 33].

Given $\boldsymbol{T}_1 \times \boldsymbol{T}_2 = \{(\hat{t}, \check{t}) : \hat{t} \in \boldsymbol{T}_1, \check{t} \in \boldsymbol{T}_2\}$, and $[P] = [(P_1, P_2)]$, $[Q] = [(Q_1, Q_2)]$, "$\mathtt{s}$" forms a complete metric space defined as [32, 33]

$$\begin{aligned}\mathtt{s}\big([P],[Q]\big) &= \mathtt{s}\big([(P_1,P_2)],[(Q_1,Q_2)]\big)\\ &= \max\left\{\sqrt{(Q_1^- - P_1^-)^2 + (Q_2^- - P_2^-)^2}, \sqrt{(Q_1^+ - P_1^+)^2 + (Q_2^+ - P_2^+)^2}\right\}.\end{aligned}$$

Below we give the definition of interval Riemann Δ-integral and interval Riemann ∇-integral according to D. Zhao et al. [26] and M. Bohner et al. [20], respectively.

We partition $[v,w]_{\boldsymbol{T}}$ as $\mathrm{E} = \{v = t_0 < \ldots < t_p = w\} \in \mathcal{P}([v,w]_{\boldsymbol{T}})$. Δ-subintervals are of the form $[t_{e-1}, t_e)_{\boldsymbol{T}}$; Δ-tags are $\vartheta_e \in [t_{e-1}, t_e)_{\boldsymbol{T}}$ taken arbitrarily. For some $\delta > 0$, E_δ will represent a Δ-partition of $[v,w]_{\boldsymbol{T}}$ with mesh δ.

Definition 3. [26](Interval Riemann Δ-integral) Let function $h : [v,w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann Δ-integrable if there exists an interval $[I_\Delta] \in \mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-partition E_δ, we have $\mathtt{s}\big(\mathrm{IR}_\Delta(h;\mathrm{E}_\delta), [I_\Delta]\big) < \varepsilon$. Here $[I_\Delta] = \mathrm{IR}_\Delta \int_v^w h(t)\Delta t$; $\mathrm{IR}_\Delta(h;\mathrm{E}_\delta) := \sum_{e=1}^{p}[h(\vartheta_e)](t_e - t_{e-1})$.

The set of all interval Riemann Δ-integrable functions on $[v,w]_{\boldsymbol{T}}$ will be denoted by $\{\mathrm{IR}_\Delta[v,w]_{\boldsymbol{T}}\}$.

The interval Riemann Δ-integral defined using the notion of generalized Hukuhara difference was given by V. Lupulescu [19] as

Definition 4. [19] Let function $h : [v,w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann Δ-integrable if there exists an interval $[I_\Delta] \in \mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-partition E_δ, we have $\mathtt{s}\big(\mathrm{IR}_\Delta(h;\mathrm{E}_\delta) \ominus_{gH} [I_\Delta], [0]\big) < \varepsilon$. Here $[I_\Delta] = \mathrm{IR}_\Delta \int_v^w h(t)\Delta t$; $\mathrm{IR}_\Delta(h;\mathrm{E}_\delta) := \sum_{e=1}^{p}[h(\vartheta_e)](t_e - t_{e-1})$.

We formulate a theorem (Theorem 1) which proves the equivalence of Definition 3 (as defined in [26]) and Definition 4 (as defined in [19]) below

Theorem 1. If $h \in \{\mathrm{IR}_\Delta[v,w]_{\boldsymbol{T}}\}$ then, h is interval Riemann Δ-integrable defined using the generalized Hukuhara difference and vice versa.

Proof. Suppose $h \in \{\mathrm{IR}_\Delta[v,w]_{\boldsymbol{T}}\}$ (Definition 3), then $\mathtt{s}\big(\mathrm{IR}_\Delta(h;\mathrm{E}_\delta), [I_\Delta]\big) < \varepsilon$. Hence,

$$\begin{aligned}&\mathtt{s}\Big(\Big[\min\Big\{\mathrm{IR}_\Delta(h^-;\mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+;\mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\}, \max\Big\{\mathrm{IR}_\Delta(h^-;\mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+;\mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\}\Big], [0]\Big)\\ &= \max\Big\{\Big|\min\Big\{\mathrm{IR}_\Delta(h^-;\mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+;\mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\} - 0^-\Big|, \Big|\max\Big\{\mathrm{IR}_\Delta(h^-;\mathrm{E}_\delta) - \mathrm{I}_\Delta^-,\\ &\mathrm{IR}_\Delta(h^+;\mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\} - 0^+\Big|\Big\} = \max\Big\{\Big|\min\Big\{\mathrm{IR}_\Delta(h^-;\mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+;\mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\}\Big|,\\ &\Big|\max\Big\{\mathrm{IR}_\Delta(h^-;\mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+;\mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\}\Big|\Big\} = \Big|\max\Big\{\mathrm{IR}_\Delta(h^-;\mathrm{E}_\delta) - \mathrm{I}_\Delta^-, \mathrm{IR}_\Delta(h^+;\mathrm{E}_\delta) - \mathrm{I}_\Delta^+\Big\}\Big| < \varepsilon.\end{aligned}$$

Thus, if $h \in \{\mathrm{IR}_\Delta[v,w]_{\boldsymbol{T}}\}$ implies h is interval Riemann Δ-integral defined using the generalized Hukuhara difference. The converse is proved similarly.

For the ∇-integral, we partition $[v,w]_{\boldsymbol{T}}$ as $\mathrm{G} = \{v = t_0 < \ldots < t_p = w\} \in \mathcal{P}([v,w]_{\boldsymbol{T}})$. ∇-subintervals are of the form $(t_{e-1}, t_e]_{\boldsymbol{T}}$; ∇-tags are $\xi_e \in (t_{e-1}, t_e]_{\boldsymbol{T}}$ taken arbitrarily. For some $\delta > 0$, G_δ will represent a ∇-partition of $[v,w]_{\boldsymbol{T}}$ with mesh δ.

Definition 5. [20](Interval Riemann ∇-integral) Let function $h : [v,w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann ∇-integrable if there exists an interval $[I_\nabla] \in \mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-partition G_δ, we have $\mathtt{s}\big(\mathrm{IR}_\nabla(h;\mathrm{G}_\delta), I_\nabla]\big) < \varepsilon$. Here $[I_\nabla] = \mathrm{IR}_\nabla \int_v^w h(t)\nabla t$; $\mathrm{IR}_\nabla(h;\mathrm{G}_\delta) := \sum_{e=1}^{p}[h(\xi_e)](t_e - t_{e-1})$.

The set of all interval Riemann ∇-integrable functions on $[v,w]_{\boldsymbol{T}}$ will be denoted by $\{\mathrm{IR}_\nabla[v,w]_{\boldsymbol{T}}\}$.

The interval Riemann ∇-integral defined using the notion of generalized Hukuhara difference is given below

Definition 6. Let function $h : [v,w]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann ∇-integrable if there exists an interval $[I_\nabla] \in \mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-partition G_δ, we have $\mathtt{s}\big(\mathrm{IR}_\nabla(h;\mathrm{G}_\delta) \ominus_{gH} [I_\nabla],[0]\big) < \varepsilon$. Here $[I_\nabla] = \mathrm{IR}_\nabla \int_v^w h(t)\nabla t$; $\mathrm{IR}_\nabla(h;\mathrm{G}_\delta) := \sum_{e=1}^{p}[h(\xi_e)](t_e - t_{e-1})$.

Theorem 2 states the equivalence of Definition 5 (as defined in [20]) and Definition 6; proof of the statement is omitted due to similarity with Theorem 1.

Theorem 2. If $h \in \{\mathrm{IR}_\nabla[v,w]_{\boldsymbol{T}}\}$, then h is interval Riemann ∇-integrable defined using the generalized Hukuhara difference and vice versa.

To the best of our knowledge, Riemann double integral for interval valued functions on time scales has not been discussed in literature. Hence, the primary objective of this paper is to define the interval Riemann $\Delta\Delta$- and $\nabla\nabla$-integrals and establish some fascinating results.

1 Interval Riemann double integration

Partitioning $[v,w]_{\boldsymbol{T}}$ as $\mathrm{E} = \{v = \hat{t}_0 < \ldots < \hat{t}_p = w\} \in \mathcal{P}([v,w]_{\boldsymbol{T}})$ and $[r,s]_{\boldsymbol{T}}$ as $\mathrm{F} = \{r = \breve{t}_0 < \ldots < \breve{t}_q = s\} \in \mathcal{P}([r,s]_{\boldsymbol{T}})$. $\Delta\Delta$-subintervals for $[v,w]_{\boldsymbol{T}}$ and $[r,s]_{\boldsymbol{T}}$ are of the form $[\hat{t}_{e-1},\hat{t}_e)_{\boldsymbol{T}}$ and $[\breve{t}_{f-1},\breve{t}_f)_{\boldsymbol{T}}$ respectively. $\Delta\Delta$-tags are $\hat{\vartheta}_e \in [\hat{t}_{e-1},\hat{t}_e)_{\boldsymbol{T}}$ and $\breve{\vartheta}_f \in [\breve{t}_{f-1},\breve{t}_f)_{\boldsymbol{T}}$ taken arbitrarily. For some $\delta > 0$, E_δ and F_δ will represent Δ-partitions of $[v,w]_{\boldsymbol{T}}$ and $[r,s]_{\boldsymbol{T}}$ respectively with mesh δ.

Interval Riemann $\Delta\Delta$-sum, $\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_\delta;\mathrm{F}_\delta)$, of interval valued function "h" evaluated at the $\Delta\Delta$-tags as follows,

$$\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_\delta;\mathrm{F}_\delta) := \sum_{e=1}^{p}\sum_{f=1}^{q}\big[h(\hat{\vartheta}_e,\breve{\vartheta}_f)\big](\hat{t}_e - \hat{t}_{e-1})(\breve{t}_f - \breve{t}_{f-1}),$$

$$\begin{aligned}\text{i.e.,}\;\; \mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_\delta;\mathrm{F}_\delta) =& \big[h(\hat{\vartheta}_1,\breve{\vartheta}_1)^-\big(\hat{t}_1 - \hat{t}_0\big)\big(\breve{t}_1 - \breve{t}_0\big), h(\hat{\vartheta}_1,\breve{\vartheta}_1)^+\big(\hat{t}_1 - \hat{t}_0\big)\big(\breve{t}_1 - \breve{t}_0\big)\big] \oplus \ldots \oplus \\ &\big[h(\hat{\vartheta}_p,\breve{\vartheta}_q)^-\big(\hat{t}_p - \hat{t}_{p-1}\big)\big(\breve{t}_q - \breve{t}_{q-1}\big), h(\hat{\vartheta}_p,\breve{\vartheta}_q)^+\big(\hat{t}_p - \hat{t}_{p-1}\big)\big(\breve{t}_q - \breve{t}_{q-1}\big)\big].\end{aligned}$$

Here,

$$\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) := \sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^-(\hat{t}_e - \hat{t}_{e-1})(\breve{t}_f - \breve{t}_{f-1}),$$

$$\mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) := \sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^+(\hat{t}_e - \hat{t}_{e-1})(\breve{t}_f - \breve{t}_{f-1}).$$

Definition 7. (Interval Riemann $\Delta\Delta$-integral) Let function $h : [v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann $\Delta\Delta$-integrable if there exists an interval $[I_{\Delta\Delta}] \in \mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any Δ-partitions E_δ and F_δ, we have

$$\mathtt{s}\big(\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_\delta;\mathrm{F}_\delta),[I_{\Delta\Delta}]\big) < \varepsilon.$$

Here $[I_{\Delta\Delta}] = \mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}$, where $\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}$ is called the interval Riemann $\Delta\Delta$-integral.

The set of all interval Riemann $\Delta\Delta$-integrable functions on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ will be denoted by $\{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$.

Example 1. 1. When $\boldsymbol{T} = \mathbb{R}$, the interval Riemann $\Delta\Delta$-integral coincides with the usual interval Riemann double integral in $\mathbb{R}$.

2. When $\boldsymbol{T} = a\mathbb{Z}$, here $a \in \mathbb{R}$ and $v,w,r,s \in a\mathbb{Z}$, if $h \in \{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$, then

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t} = a^2 \cdot \sum_{i=\frac{v}{a}}^{\frac{w}{a}-1}\sum_{j=\frac{r}{a}}^{\frac{s}{a}-1}\left[h(ai,aj)\right]$$

$$= a^2 \cdot \sum_{i=\frac{v}{a}}^{\frac{w}{a}-1}\sum_{j=\frac{r}{a}}^{\frac{s}{a}-1}\Big[h(ai,aj)^-, h(ai,aj)^+\Big].$$

If $a = 1$, $\boldsymbol{T} = \mathbb{Z}$ and

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t} = \sum_{i=v}^{w-1}\sum_{j=r}^{s-1}\left[h(i,j)\right].$$

For the $\nabla\nabla$-integral, we partition $[v,w]_{\boldsymbol{T}}$ as $\mathrm{G} = \{v = \hat{t}_0 < \ldots < \hat{t}_p = w\} \in \mathcal{P}([v,w]_{\boldsymbol{T}})$ and $[r,s]_{\boldsymbol{T}}$ as $\mathrm{H} = \{r = \breve{t}_0 < \ldots < \breve{t}_q = s\} \in \mathcal{P}([r,s]_{\boldsymbol{T}})$. $\nabla\nabla$-subintervals for $[v,w]_{\boldsymbol{T}}$ and $[r,s]_{\boldsymbol{T}}$ are of the form $(\hat{t}_{e-1},\hat{t}_e]_{\boldsymbol{T}}$ and $(\breve{t}_{f-1},\breve{t}_f]_{\boldsymbol{T}}$, respectively. $\nabla\nabla$-tags are $\hat{\xi}_e \in (\hat{t}_{e-1},\hat{t}_e]_{\boldsymbol{T}}$ and $\breve{\xi}_f \in (\breve{t}_{f-1},\breve{t}_f]_{\boldsymbol{T}}$ taken arbitrarily. For some $\delta > 0$, G_δ and H_δ will represent ∇-partitions of $[v,w]_{\boldsymbol{T}}$ and $[r,s]_{\boldsymbol{T}}$ respectively with mesh δ.

Interval Riemann $\nabla\nabla$-sum, $\mathrm{IR}_{\nabla\nabla}(h;\mathrm{G}_\delta;\mathrm{H}_\delta)$, of interval valued function h evaluated at the $\nabla\nabla$-tags as follows,

$$\mathrm{IR}_{\nabla\nabla}(h;\mathrm{G}_\delta;\mathrm{H}_\delta) := \sum_{e=1}^{p}\sum_{f=1}^{q}\left[h(\hat{\xi}_e,\breve{\xi}_f)\right](\hat{t}_e - \hat{t}_{e-1})(\breve{t}_f - \breve{t}_{f-1}),$$

$$\text{i.e., } \mathrm{IR}_{\nabla\nabla}(h;\mathrm{G}_\delta;\mathrm{H}_\delta) = \Big[h(\hat{\xi}_1,\breve{\xi}_1)^-\big(\hat{t}_1-\hat{t}_0\big)\big(\breve{t}_1-\breve{t}_0\big), h(\hat{\xi}_1,\breve{\xi}_1)^+\big(\hat{t}_1-\hat{t}_0\big)\big(\breve{t}_1-\breve{t}_0\big)\Big] \oplus \ldots \oplus \Big[h(\hat{\xi}_p,\breve{\xi}_q)^-\big(\hat{t}_p-\hat{t}_{p-1}\big)\big(\breve{t}_q-\breve{t}_{q-1}\big), h(\hat{\xi}_p,\breve{\xi}_q)^+\big(\hat{t}_p-\hat{t}_{p-1}\big)\big(\breve{t}_q-\breve{t}_{q-1}\big)\Big].$$

Here,

$$\mathrm{IR}_{\nabla\nabla}(h^-;\mathrm{G}_\delta;\mathrm{H}_\delta) := \sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\xi}_e,\breve{\xi}_f)^-(\hat{t}_e - \hat{t}_{e-1})(\breve{t}_f - \breve{t}_{f-1}),$$

$$\mathrm{IR}_{\nabla\nabla}(h^+;\mathrm{G}_\delta;\mathrm{H}_\delta) := \sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\xi}_e,\breve{\xi}_f)^+(\hat{t}_e - \hat{t}_{e-1})(\breve{t}_f - \breve{t}_{f-1}).$$

Definition 8. (Interval Riemann $\nabla\nabla$-integral) Let function $h : [v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann $\nabla\nabla$-integrable if there exists an interval $[I_{\nabla\nabla}] \in \mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-partitions G_δ and H_δ, we have

$$\mathsf{s}\big(\mathrm{IR}_{\nabla\nabla}(h;\mathrm{G}_\delta;\mathrm{H}_\delta), [I_{\nabla\nabla}]\big) < \varepsilon.$$

Here $[I_{\nabla\nabla}] = \mathrm{IR}_{\nabla\nabla}\int_v^w\int_r^s h(\hat{t},\breve{t})\nabla\hat{t}\nabla\breve{t}$, where $\mathrm{IR}_{\nabla\nabla}\int_v^w\int_r^s h(\hat{t},\breve{t})\nabla\hat{t}\nabla\breve{t}$ is called the interval Riemann $\nabla\nabla$-integral.

The set of all interval Riemann $\nabla\nabla$-integrable functions on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ will be denoted by $\{\mathrm{IR}_{\nabla\nabla}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$.

Example 2. 1. When $\boldsymbol{T} = \mathbb{R}$, the interval Riemann $\nabla\nabla$-integral coincides with the usual interval Riemann double integral in $\mathbb{R}$.

2. When $\boldsymbol{T} = a\mathbb{Z}$, here $a \in \mathbb{R}$ and $v, w, r, s \in a\mathbb{Z}$, if $h \in \{\mathrm{IR}_{\nabla\nabla}[v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}}\}$, then

$$\mathrm{IR}_{\nabla\nabla} \int_v^w \int_r^s h(\hat{t}, \breve{t}) \nabla\hat{t}\nabla\breve{t} = a^2 \cdot \sum_{i=\frac{v}{a}+1}^{\frac{w}{a}} \sum_{j=\frac{r}{a}+1}^{\frac{s}{a}} \big[h(ai, aj)\big]$$

$$= a^2 \cdot \sum_{i=\frac{v}{a}+1}^{\frac{w}{a}} \sum_{j=\frac{r}{a}+1}^{\frac{s}{a}} \Big[h(ai, aj)^-, h(ai, aj)^+\Big].$$

If $a = 1$, $\boldsymbol{T} = \mathbb{Z}$ and

$$\mathrm{IR}_{\nabla\nabla} \int_v^w \int_r^s h(\hat{t}, \breve{t}) \nabla\hat{t}\nabla\breve{t} = \sum_{i=v+1}^{w} \sum_{j=r+1}^{s} \big[h(i, j)\big].$$

Following statements and theorems will be given in regard to the $\Delta\Delta$-integral, $\nabla\nabla$-integral versions are omitted due to their similarity.

Remark 1. If $h \in \{\mathrm{IR}_{\Delta\Delta}[v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}}\}$, then the value of integral $[I_{\Delta\Delta}]$ is unique and well-defined.

If $h \in \{\mathrm{IR}_{\Delta\Delta}[v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}}\}$ and h is degenerate, then interval Riemann $\Delta\Delta$-integral (Definition 7) equals Riemann $\Delta\Delta$-integral (Definition 1).

Theorem 3. Let $h : [\hat{t}_0, \sigma(\hat{t}_0)]_{\boldsymbol{T}} \times [\breve{t}_0, \sigma(\breve{t}_0)]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$, then $h \in \{\mathrm{IR}_{\Delta\Delta}[\hat{t}_0, \sigma(\hat{t}_0)]_{\boldsymbol{T}} \times [\breve{t}_0, \sigma(\breve{t}_0)]_{\boldsymbol{T}}\}$ and

$$\mathrm{IR}_{\Delta\Delta} \int_{\hat{t}_0}^{\sigma(\hat{t}_0)} \int_{\breve{t}_0}^{\sigma(\breve{t}_0)} h(\hat{t}, \breve{t}) \Delta\hat{t}\Delta\breve{t} = \Big[h(\hat{t}_0, \breve{t}_0)^- \big(\sigma(\hat{t}_0) - \hat{t}_0\big)\big(\sigma(\breve{t}_0) - \breve{t}_0\big), h(\hat{t}_0, \breve{t}_0)^+ \big(\sigma(\hat{t}_0) - \hat{t}_0\big)\big(\sigma(\breve{t}_0) - \breve{t}_0\big)\Big].$$

Theorem 4. Let $h : [\rho(\hat{t}_0), \hat{t}_0]_{\boldsymbol{T}} \times [\rho(\breve{t}_0), \breve{t}_0]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$, then $h \in \{\mathrm{IR}_{\Delta\Delta}[\rho(\hat{t}_0), \hat{t}_0]_{\boldsymbol{T}} \times [\rho(\breve{t}_0), \breve{t}_0]_{\boldsymbol{T}}\}$ and

$$\mathrm{IR}_{\Delta\Delta} \int_{\rho(\hat{t}_0)}^{\hat{t}_0} \int_{\rho(\breve{t}_0)}^{\breve{t}_0} h(\hat{t}, \breve{t}) \Delta\hat{t}\Delta\breve{t} = \Big[h\big(\rho(\hat{t}_0), \rho(\breve{t}_0)\big)^- \big(\hat{t}_0 - \rho(\hat{t}_0)\big)\big(\breve{t}_0 - \rho(\breve{t}_0)\big), h\big(\rho(\hat{t}_0), \rho(\breve{t}_0)\big)^+$$

$$\big(\hat{t}_0 - \rho(\hat{t}_0)\big)\big(\breve{t}_0 - \rho(\breve{t}_0)\big)\Big].$$

Theorem 5. An interval valued function $h : [v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}} \to \mathbb{R}_{\mathrm{I}}$ is interval Riemann $\Delta\Delta$-integrable on $[v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}}$ if and only if h^- and h^+ are Riemann $\Delta\Delta$-integrable on $[v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}}$ and

$$\mathrm{IR}_{\Delta\Delta} \int_v^w \int_r^s h(\hat{t}, \breve{t}) \Delta\hat{t}\Delta\breve{t} = \left[\mathrm{R}_{\Delta\Delta} \int_v^w \int_r^s h(\hat{t}, \breve{t})^- \Delta\hat{t}\Delta\breve{t}, \mathrm{R}_{\Delta\Delta} \int_v^w \int_r^s h(\hat{t}, \breve{t})^+ \Delta\hat{t}\Delta\breve{t}\right].$$

Proof. If $h \in \{\mathrm{IR}_{\Delta\Delta}[v, w]_{\boldsymbol{T}} \times [r, s]_{\boldsymbol{T}}\}$, then integral $[I_{\Delta\Delta}] = [I^-_{\Delta\Delta}, I^+_{\Delta\Delta}]$ such that for each $\varepsilon > 0$ there exists δ such that

$$\mathbf{s}\big(\mathrm{IR}_{\Delta\Delta}(h; \mathrm{E}_\delta; \mathrm{F}_\delta), [I_{\Delta\Delta}]\big) = \max\left\{\left|\mathrm{IR}_{\Delta\Delta}(h^-; \mathrm{E}_\delta; \mathrm{F}_\delta) - I^-_{\Delta\Delta}\right|, \left|\mathrm{IR}_{\Delta\Delta}(h^+; \mathrm{E}_\delta; \mathrm{F}_\delta) - I^+_{\Delta\Delta}\right|\right\}$$

$$= \max\left\{\left|\sum_{e=1}^{p} \sum_{f=1}^{q} h(\hat{\vartheta}_e, \breve{\vartheta}_f)^- (\hat{t}_e - \hat{t}_{e-1})(\breve{t}_f - \breve{t}_{f-1}) - I^-_{\Delta\Delta}\right|,\right.$$

$$\left.\left|\sum_{e=1}^{p} \sum_{f=1}^{q} h(\hat{\vartheta}_e, \breve{\vartheta}_f)^+ (\hat{t}_e - \hat{t}_{e-1})(\breve{t}_f - \breve{t}_{f-1}) - I^+_{\Delta\Delta}\right|\right\} < \varepsilon,$$

thus,

$$\left|\sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^-(\hat{t}_e-\hat{t}_{e-1})(\breve{t}_f-\breve{t}_{f-1})-I^-_{\Delta\Delta}\right|<\varepsilon \text{ and}$$
$$\left|\sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^+(\hat{t}_e-\hat{t}_{e-1})(\breve{t}_f-\breve{t}_{f-1})-I^+_{\Delta\Delta}\right|<\varepsilon,$$

hence we conclude.

Conversely, let h^-, h^+ be Riemann $\Delta\Delta$-integrable on $[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}$, then there exists $I_1, I_2\in\mathbb{R}$ such that for each $\varepsilon>0$, there exists δ such that

$$\left|\mathrm{R}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta)-I_1\right|<\varepsilon \text{ and } \left|\mathrm{R}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta)-I_2\right|<\varepsilon.$$

Letting $[I_{\Delta\Delta}]=[I_1,I_2]$, we have

$$\max\left\{\left|\mathrm{R}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta)-I_1\right|,\left|\mathrm{R}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta)-I_2\right|\right\}=\max\left\{\left|\sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^-(\hat{t}_e-\hat{t}_{e-1})\right.\right.$$
$$\left.\left.(\breve{t}_f-\breve{t}_{f-1})-I^-_{\Delta\Delta}\right|,\left|\sum_{e=1}^{p}\sum_{f=1}^{q} h(\hat{\vartheta}_e,\breve{\vartheta}_f)^+(\hat{t}_e-\hat{t}_{e-1})(\breve{t}_f-\breve{t}_{f-1})-I^+_{\Delta\Delta}\right|\right\}<\varepsilon,$$

implies $\mathbf{s}\big(\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_\delta;\mathrm{F}_\delta),[I_{\Delta\Delta}]\big)<\varepsilon$ hence we conclude.

Without actually knowing the value of the integral, we can prove the integrability of a function via the criterion of integrability. It is stated as

Theorem 6. An interval valued function $h:[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}\to\mathbb{R}_{\mathrm{I}}$ is interval Riemann $\Delta\Delta$-integrable on $[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}$ if and only if for each $\varepsilon>0$ there exists δ such that any Δ-partitions $\mathrm{E}_{1_\delta},\mathrm{F}_{1_\delta}$ and $\mathrm{E}_{2_\delta},\mathrm{F}_{2_\delta}$ with mesh$<\delta$ implies

$$\mathbf{s}\big(\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_{1_\delta};\mathrm{F}_{1_\delta}),\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_{2_\delta};\mathrm{F}_{2_\delta})\big)<\varepsilon.$$

A function $h:[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}\to\mathbb{R}_{\mathrm{I}}$ is said to be interval continuous at $(\hat{t}_0,\breve{t}_0)\in[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}$ if for each $\varepsilon>0$ there exists $\delta>0$ such that $\mathbf{s}\big([h(\hat{t},\breve{t})],[h(\hat{t}_0,\breve{t}_0)]\big)<\varepsilon$, whenever $\sqrt{(\hat{t}_0-\hat{t})^2+(\breve{t}_0-\breve{t})^2}<\delta$.

Interval boundedness and interval continuity of a function are sufficient conditions for the existence of interval Riemann double integrability.

Theorem 7. Every bounded continuous interval valued function is interval Riemann $\Delta\Delta$-integrable, and

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}=\left[\mathrm{R}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})^-\Delta\hat{t}\Delta\breve{t},\mathrm{R}_{\Delta\Delta}\int_v^w\int_r^s h(\hat{t},\breve{t})^+\Delta\hat{t}\Delta\breve{t}\right].$$

Below we establish a relation between interval Riemann $\Delta\Delta$-integral and Riemann $\Delta\Delta$-integral for two interval valued functions via Hausdorff-Pompeiu distance.

Theorem 8. Let $h_1,h_2\in\big\{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}\big\}$, if given $\mathbf{s}\big([h_1(\hat{t},\breve{t})],[h_2(\hat{t},\breve{t})]\big)$ is Riemann $\Delta\Delta$-integral then,

$$\mathbf{s}\Big(\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t},\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat{t},\breve{t})\Delta\hat{t}\Delta\breve{t}\Big)\le\mathrm{R}_{\Delta\Delta}\int_v^w\int_r^s\mathbf{s}\big([h_1(\hat{t},\breve{t})],[h_2(\hat{t},\breve{t})]\big)\Delta\hat{t}\Delta\breve{t}.$$

Proof. By definition of distance we have,

$$\begin{aligned}
&\mathbf{s}\Big(\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat t,\breve t)\Delta\hat t\Delta\breve t,\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat t,\breve t)\Delta\hat t\Delta\breve t\Big)\\
&=\max\Big\{\Big|\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat t,\breve t)^-\Delta\hat t\Delta\breve t-\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat t,\breve t)^-\Delta\hat t\Delta\breve t\Big|,\\
&\quad\Big|\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat t,\breve t)^+\Delta\hat t\Delta\breve t-\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat t,\breve t)^+\Delta\hat t\Delta\breve t\Big|\Big\}\\
&\le\max\Big\{\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s\big|h_1(\hat t,\breve t)^- - h_2(\hat t,\breve t)^-\big|\Delta\hat t\Delta\breve t,\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s\big|h_1(\hat t,\breve t)^+ - h_2(\hat t,\breve t)^+\big|\Delta\hat t\Delta\breve t\Big\}\\
&\le\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s\max\Big\{\big|h_1(\hat t,\breve t)^- - h_2(\hat t,\breve t)^-\big|,\big|h_1(\hat t,\breve t)^+ - h_2(\hat t,\breve t)^+\big|\Big\}\Delta\hat t\Delta\breve t\\
&=\mathrm{R}_{\Delta\Delta}\int_v^w\int_r^s \mathrm{s}\big([h_1(\hat t,\breve t)],[h_2(\hat t,\breve t)]\big)\Delta\hat t\Delta\breve t.
\end{aligned}$$

Theorem 9. Let $h_1,h_2\in\big\{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}\big\}$ and $\gamma\in\mathbb{R}$, then

1. $\gamma h_1\in\big\{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}\big\}$ and

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s\gamma h_1(\hat t,\breve t)\Delta\hat t\Delta\breve t=\gamma\,\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat t,\breve t)\Delta\hat t\Delta\breve t,$$

2. $h_1+h_2\in\big\{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}\big\}$ and

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s(h_1+h_2)(\hat t,\breve t)\Delta\hat t\Delta\breve t=\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat t,\breve t)\Delta\hat t\Delta\breve t+$$
$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat t,\breve t)\Delta\hat t\Delta\breve t,$$

3. $h_1(\hat t,\breve t)\subseteq h_2(\hat t,\breve t)$

$$\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_1(\hat t,\breve t)\Delta\hat t\Delta\breve t\subseteq\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h_2(\hat t,\breve t)\Delta\hat t\Delta\breve t.$$

Definition 7 and Definition 8 can also be alternatively defined using the generalized Hukuhara difference as

Definition 9. Let function $h:[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}\to\mathbb{R}_{\mathrm{I}}$ be an interval valued function; h is said to be interval Riemann $\Delta\Delta$-integrable if there exists an interval $[I_{\Delta\Delta}]\in\mathbb{R}_{\mathrm{I}}$ on $[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}$ such that for any $\varepsilon>0$ there exists $\delta>0$ hence for any Δ-partitions E_δ and F_δ, we have

$$\mathbf{s}\big(\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_\delta;\mathrm{F}_\delta)\ominus_{\mathrm{gH}}[I_{\Delta\Delta}],[0]\big)<\varepsilon.$$

Here $[I_{\Delta\Delta}]=\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat t,\breve t)\Delta\hat t\Delta\breve t$, where $\mathrm{IR}_{\Delta\Delta}\int_v^w\int_r^s h(\hat t,\breve t)\Delta\hat t\Delta\breve t$ is called the interval Riemann $\Delta\Delta$-integral.

We establish a theorem which proves the equivalence of Definition 7 and Definition 9.

Theorem 10. If $h\in\big\{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}}\times[r,s]_{\boldsymbol{T}}\big\}$ then, h is interval Riemann $\Delta\Delta$-integrable defined using the generalized Hukuhara difference and vice versa.

Proof. Suppose $h \in \{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$ (Definition 7), then $\mathbf{s}\Big(\mathrm{IR}_{\Delta\Delta}(h;\mathrm{E}_\delta;\mathrm{F}_\delta), [I_{\Delta\Delta}]\Big) < \varepsilon$. Hence,

$$\begin{aligned}
&\mathbf{s}\Big(\Big[\min\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\},\\
&\qquad \max\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\}\Big], [0]\Big)\\
&\qquad = \max\Big\{\Big|\min\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\} - 0^-\Big|,\\
&\qquad\quad \Big|\max\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\} - 0^+\Big|\Big\}\\
&\qquad = \max\Big\{\Big|\min\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\}\Big|,\\
&\qquad\quad \Big|\max\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\}\Big|\Big\}\\
&\qquad = \Big|\max\Big\{\mathrm{IR}_{\Delta\Delta}(h^-;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^-_{\Delta\Delta}, \mathrm{IR}_{\Delta\Delta}(h^+;\mathrm{E}_\delta;\mathrm{F}_\delta) - I^+_{\Delta\Delta}\Big\}\Big| < \varepsilon.
\end{aligned}$$

Thus, if $h \in \{\mathrm{IR}_{\Delta\Delta}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$ implies h is interval Riemann $\Delta\Delta$-integrable defined using the generalized Hukuhara difference. The converse is proved similarly.

Definition 8 is alternatively defined using the notion of generalized Hukuhara difference as

Definition 10. Let function $h : [v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}} \to \mathbb{R}_\mathrm{I}$ be an interval valued function; h is said to be interval Riemann $\nabla\nabla$-integrable if there exists an interval $[I_{\nabla\nabla}] \in \mathbb{R}_\mathrm{I}$ on $[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ hence for any ∇-partitions G_δ and H_δ, we have

$$\mathbf{s}\big(\mathrm{IR}_{\nabla\nabla}(h;\mathrm{G}_\delta;\mathrm{H}_\delta) \ominus_{\mathrm{gH}} [I_{\nabla\nabla}], [0]\big) < \varepsilon.$$

Here $[I_{\nabla\nabla}] = \mathrm{IR}_{\nabla\nabla}\int_v^w\int_r^s h(\hat{t},\breve{t})\nabla\hat{t}\nabla\breve{t}$, where $\mathrm{IR}_{\nabla\nabla}\int_v^w\int_r^s h(\hat{t},\breve{t})\nabla\hat{t}\nabla\breve{t}$ is called the interval Riemann $\nabla\nabla$-integral.

We establish a theorem which proves the equivalence of Definition 8 and Definition 10; prove is omitted due to its similarity with Theorem 10.

Theorem 11. If $h \in \{\mathrm{IR}_{\nabla\nabla}[v,w]_{\boldsymbol{T}} \times [r,s]_{\boldsymbol{T}}\}$, then h is interval Riemann $\nabla\nabla$-integrable defined using the generalized Hukuhara difference and vice versa.

Conclusion

This paper explores the theory of Riemann double integration for interval valued functions on time scales and discuss a few fascinating results.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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